Finite Difference Method in MATLAB

For PDEs: Heat Equation, Advection-Diffiusion Equation,...

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Contents

One

Finite Difference

In this method we discretize a space into a finite number of points and use these points to find an approximate solution of a PDE.

1.1 Schemes for Discretization

Let $u = u(x,t)$ be a function in two variables.

Let the interval of the variable x is discretized into m number of points; $x_1, x_2, ..., x_m$ and of the variable t into n number of points; $t_1, t_2, ..., t_n$.

Forward Difference of 1^{st} order

$$\frac{\partial u(x_i, t_j)}{\partial x} = \frac{u(x_i + h, t_j) - u(x_i, t_j)}{h} = \frac{V_{i+1}^j - V_i^j}{h}$$

Likewise,

$$\frac{\partial u}{\partial t} = \frac{V_i^{j+1} - V_i^j}{k}$$

Backward Difference of 1^{st} order

$$\frac{\partial u(x_i, t_j)}{\partial x} = \frac{u(x_i, t_j) - u(x_i - h, t_j)}{h} = \frac{V_i^j - V_{i-1}^j}{h}$$

Likewise,

$$\frac{\partial u}{\partial t} = \frac{V_i^j - V_i^{j-1}}{k}$$

Central Difference of 1^{st} order

It is the average of forward difference and backward difference.

$$\frac{\partial u(x_i, t_j)}{\partial x} = \frac{V_{i+1}^j - V_i^j + V_i^j - V_{i-1}^j}{2h} = \frac{V_{i+1}^j - V_{i-1}^j}{2h}$$

Central Difference of 2^{nd} order

$$\frac{\partial^2 u}{\partial x^2} = \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{h^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{V_i^{j+1} - 2V_i^j + V_i^{j-1}}{k^2}$$

Code Conventions:

Variables	Axis	Loop var.	difference	Grid section
x	X	i	h	m
t/y	Y	j	k	n

Two

Heat Equation

Forward time Central space Scheme **(FTCS)** for the heat equation: $u_t = \alpha u_{xx}$.
We have,

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha \frac{\partial^2 u}{\partial x^2} \\
\frac{V_i^{j+1} - V_i^j}{k} &= \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{h^2} \\
V_i^{j+1} - V_i^j &= \frac{\alpha k}{h^2}(V_{i+1}^j - 2V_i^j + V_{i-1}^j) \\
V_i^{j+1} &= V_i^j + c(V_{i+1}^j - 2V_i^j + V_{i-1}^j)
\end{aligned}
$$

where $c = \frac{\alpha k}{h^2}$. This the required scheme.

Question

$$
\begin{aligned}
u_t &= \alpha u_{xx} & 0 < x < 1, \quad t > 0, \quad \alpha = 0.05 \\
BCS: \quad u(0,t) &= u(1,t) = 0 & t > 0 \\
IC: \quad u(x,0) &= sin\pi x & 0 < x < 1
\end{aligned}
$$

2.1 Matlab Code

```
%initialization
L=1;       m=10;    h=L/m;     x=0:h:L;
T=1;       n=10;    k=T/n;     t=0:k:T;

a=0.05;         c=a*k/(h*h);
v=zeros(m+1, n+1);

%checking
if (c<0|c>0.5)
  disp('The FTCS is unstable.');
else

%boundary equations
  v(1,:)=0;     v(m+1,:)=0;
  v(:,1)=sin(pi*x);

%scheme
  for j=1:n
    for i=2:m
      v(i,j+1)=v(i,j)+c*(v(i+1,j)-2*v(i,j)+v(i-1,j));
    endfor
  endfor
endif

%graph
[p,q]=meshgrid(x,t);

surf(p,q,v);
xlabel('x');  ylabel('y');  zlabel('z');
title('Numerical solution of heat equation.');
```

Three

Advection Diffusion Equation

Forward Time Backward Space Central Space Scheme (FTBSCS) for the following advection diffiusion equation:

$$\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}$$

where,
C is concentration of disolved sustances
v is velocity of the fluid
D is diffussion coefficient.

We have,

$$\begin{aligned}
\frac{V_i^{j+1} - V_1^J}{k} + v\frac{V_i^j - V_{i-1}^j}{h} &= D\frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{h^2} \\
V_i^{j+1} - V_1^j &= \frac{Dk}{h^2}(V_{i+1}^j - 2V_i^j + V_{i-1}^j) - \frac{vk}{h}(V_i^j - V_{i-1}^j) \\
V_i^{j+1} &= F(V_{i+1}^j - 2V_i^j + V_{i-1}^j) - G(V_i^j - V_{i-1}^j)
\end{aligned}$$

where,
$F = \frac{Dk}{h^2}$ $\quad G = \frac{vk}{h}$

Question

$$C_t + vC_x = DC_{xx}$$

$$0 \leq t \leq 4000 * 24, \qquad D = 10^{-6} * 3600$$

$$0 \leq x \leq 100, \qquad v = 10^{-7} * 3600$$

$$IC: \quad C(x,0) = 100$$

3.1 Matlab Code

```
%Initialization
L=100;    m=20;   h=L/m;    x=0:h:L;
T=4000*24;  n=20;   k=T/n;  t=0:k:T;

D=(1E-6)*3600;      F=(D*k)/(h*h);
V=(1E-7)*3600;      G=(V*k)/h;

v=zeros(m+1,n+1);

%Initial conditon
v(:,1)=100;

%Scheme
for j=1:n
  for i=2:m
    v(i,j+1)=v(i,j)+F*(v(i+1,j)-2*v(i,j)+v(i-1,j))-G*(v(i,j)-v(i-1,j));
  endfor
endfor

%Graph
[p,q]=meshgrid(x,t);
surf(p,q,v);
xlabel('x';    ylabel('y');    zlabel('z');
title('Numerical solution of Advection-Diffusion equation.');
```

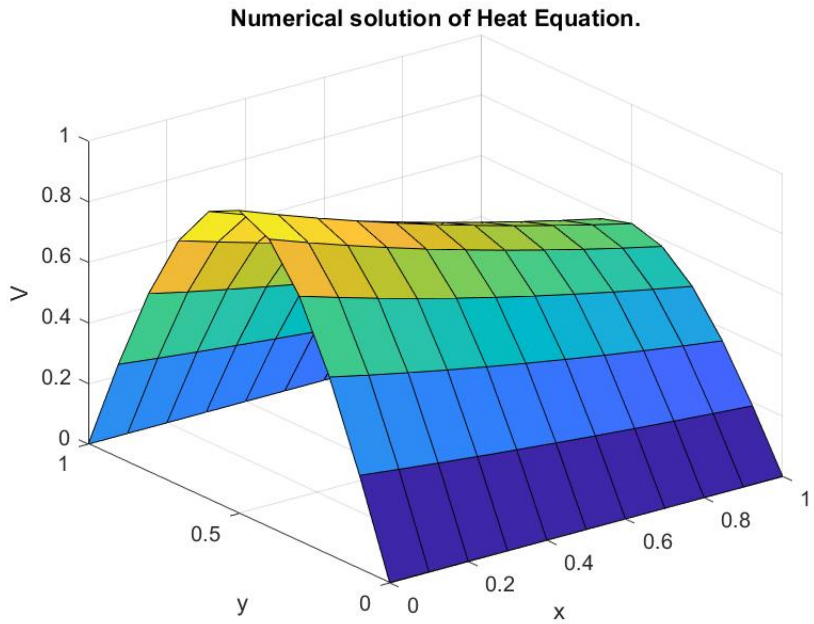


Figure 3.1: Heat Equation

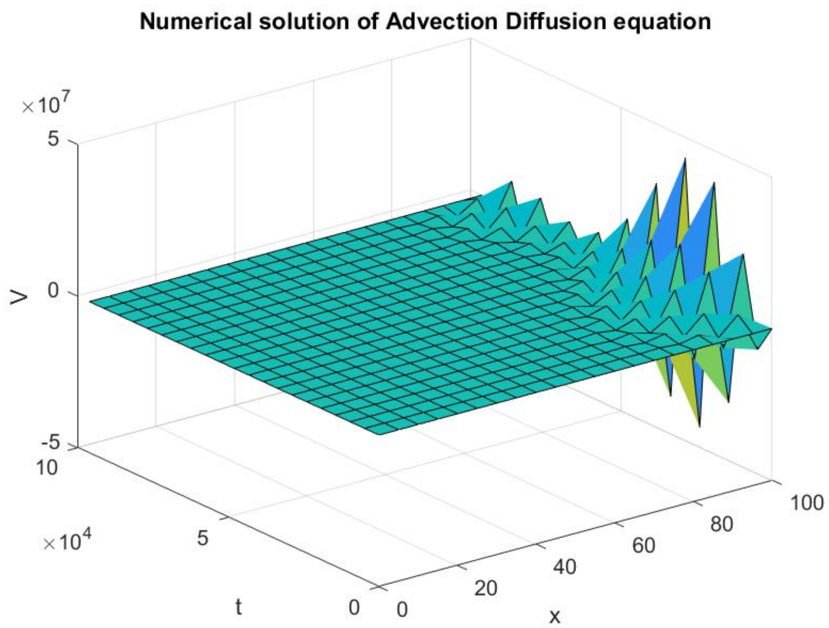


Figure 3.2: Advection Diffiusion equation

Four

Laplace Equation

The Laplace equation is

$$u_{xx} + u_{yy} = 0$$

The Central Space Central Space Scheme (**CSCSS**) for the above equation:

$$\begin{aligned}
\frac{V_{i+1}^{j} - 2V_i^j + V_{i-1}^j}{h^2} + \frac{V_i^{j+1} - 2V_i^j + V_i^{j-1}}{k^2} &= 0 \\
k^2(V_{i+1}^j - 2V_i^j + V_{i-1}^j) + h^2(V_i^{j+1} - 2V_i^j + V_i^{j-1}) &= 0 \\
k^2(V_{i+1}^j + V_{i-1}^j) + h^2(V_i^{j+1} + V_i^{j-1}) &= 2(h^2 + k^2)V_i^j
\end{aligned}$$

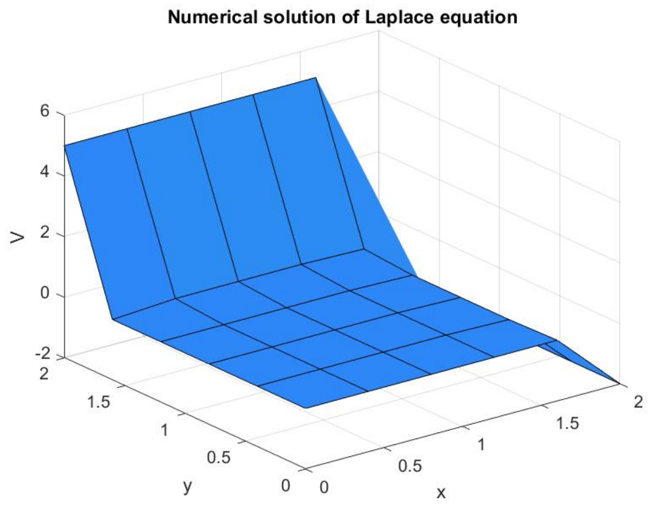
$$V_i^j = \frac{k^2(V_{i+1}^j + V_{i-1}^j) + h^2({}_i^{j+1} + V_i^{j-1})}{2(h^2 + k^2)} \tag{4.1}$$

Question

$$\begin{aligned}
& u_{xx} = u_{yy} & & 0 \le x \le 2, \quad 0 \le y \le 2 \\
BCS: \quad & u(0, y) = 0, \quad u(2, y) = 5 & & 0 \le y \le 2 \\
& u(x, 0) = 0, \quad u(x, 2) = -2 & & 0 \le x \le 2
\end{aligned}$$

4.1 Matlab Code

```
%Initialization
L=2;        m=5;    h=L/m;     x=0:h:L;
B=2;        n=5;    k=B/n;     t=0:k:B;

c=1/(2*(h^2+k^2));
v=zeros(m+1, n+1);

%Boundary Conditions
v(1,:)=0;      v(m+1,:)=5;
v(:,1)=0;      v(:,n+1)=-2;

%Scheme
for j=2:n
  for i=2:m
    v(i,j)=c*((h^2*(v(i,j+1)+v(i,j-1)))+(k^2*(v(i+1,j)+v(i-1,j))));
  endfor
endfor

%Graph
[p,q]=meshgrid(x,t);

surf(p,q,v);
xlabel('x');  ylabel('y');  zlabel('z');
title('Numerical solution of Laplace equation.');
```



Five

Wave Equation

The wave equation is $u_{tt} = a u_{xx}$

The Central Space Central Space Scheme (**CTCSS**) for the above equation:

$$\frac{V_i^{j+1} - 2V_i^j + V_i^{j-1}}{k^2} = a\frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{h^2}$$
$$V_i^{j+1} - 2V_i^j + V_i^{j-1} = \frac{ak^2}{h^2}(V_{i+1}^j - 2V_i^j + V_{i-1}^j)$$
$$V_i^{j+1} = 2(1-c)V_i^j + c(V_{i+1}^j + V_{i-1}^j) - V_i^{j-1} \tag{5.1}$$

where, $c = \frac{ak^2}{h^2}$

Question

$$\begin{aligned} &u_{tt} = a u_{xx} && 0 \le x \le 1, \quad t \ge 0 \\ BCs: \quad &u(0,t) = u(1,t) = 0 && t \ge 0 \\ ICs: \quad &u(x,0) = 10 sin \pi x \quad u_t(x,0) = 0 && 0 \le x \le 1 \end{aligned}$$

5.1 Scheme for Initial Condition

The initial condition $u_t(x, 0) = 0$ also needs a separate scheme as it is in a derivative form.

The Central Time Scheme for u_t is :

$$u_t = \frac{V_i^{j+1} - V_i^{j-1}}{2k}$$
$$\implies 2ku_t = V_i^{j+1} - V_i^{j-1}$$
$$For j = 0$$
$$2ku_t = V_i^1 - V_i^{-1}$$

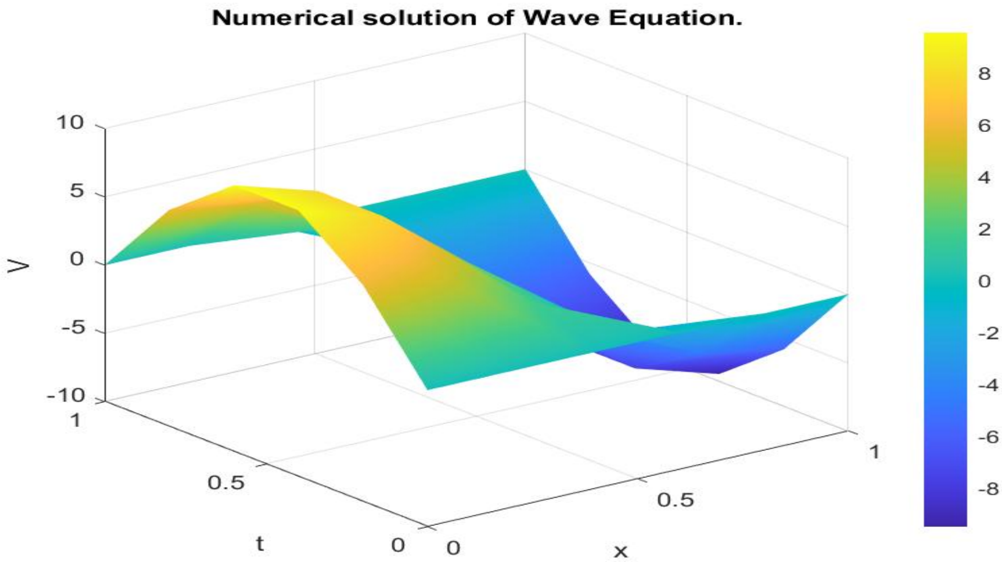
$$\implies V_i^{-1} = V_i^1 - 2ku_t \tag{5.2}$$

Also, Putting $j = 0$ in equation (5.1)

$$V_i^1 = 2(1-c)V_i^0 + c(V_{i+1}^0 + V_{i-1}^0) - V_i^{-1} \tag{5.3}$$

From equation (5.2) and (5.3)

$$V_i^1 = (1-c)V_i^0 + 0.5c(V_{i+1}^0 + V_{i-1}^0) + ku_t \tag{5.4}$$



5.2 Matlab Code

```
%Initialization
L=1;        m=5;        h=L/m;        x=0:h:L;
T=1;        n=5;        k=T/n;        t=0:k:T;

a=1;                        c=a*(k^2/h^2);
V=zeros(m+1,n+1);

%Checking
if (c<=0|c>1)
    disp('The CTCSS is unstable')
else

%Boundary Conditions
    V(1,:)=0;        V(m+1,:)=0;

%Initial Conditions
    V(:,1)=10*sin(pi*x);

    for i=2:m
        V(i,2)=(1-c)*V(i,1)+0.5*c*(V(i+1,1)+V(i-1,1));
    end

%Main Scheme
    for j=2:n
        for i=2:m
            V(i,j+1)=2*(1-c)*V(i,j)+c*(V(i+1,j)+V(i-1,j))-V(i,j-1);
        end
    end

%Graph
    [p,q]=meshgrid(x,t);
    surf(p,q,V,'Edgecolor','none');
    xlabel('x');        ylabel('t');        zlabel('V');
    title('Numerical solution of Wave Equation.');
    shading interp;
end
```