

# Applications of Galois Theory



Thesis Submitted to  
the Central Department of Mathematics  
Institute of Science and Technology  
Tribhuvan University  
Kathmandu, Nepal

By  
**Sandesh Thakuri**

Submitted for the  
Partial Fulfillment of the Requirement for  
the Master in Science/Arts (M.Sc./M.A.) Degree in Mathematics

April 25, 2024



© 2024  
Sandesh Thakuri  
All Rights Reserved



## DEDICATION

To

My Mom and Dad

**Sharada Thakuri and Prem Bdr. Thakuri.**



## STUDENT'S DECLARATION

This thesis entitled "**Applications of Galois Theory**", which has been submitted to the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University, Nepal for the partial fulfillment of the Master in Science/Arts (M.Sc./M.A.) Degree in Mathematics, is a genuine work that I carried out under my supervisor **Assoc. Prof. Tulasi Prasad Nepal** and that no sources other than those listed in the Bibliography have been used in this work. Moreover, this work has not been published or submitted elsewhere for the requirement of any degree programme.

---

Sandesh Thakuri

Batch: 2077

TU Registration Number: 5-2-37-1874-2016

Date: April 25, 2024



## RECOMMENDATION

This is to recommend that Mr. **Sandesh Thakuri** has prepared this thesis entitled "**Applications of Galois Theory**" for the partial fulfillment of the Master in Science/Arts (M.Sc./M.A.) in Mathematics under my supervision. To my knowledge, this work has not been submitted for any other degree. He has fulfilled all the requirements laid down by the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University (TU), Kirtipur for the submission of the thesis for the partial fulfillment of M.Sc./M.A. Degree in Mathematics.

.....  
Associate Professor Tulasi Prasad Nepal  
**(Supervisor)**  
Central Department of Mathematics  
Tribhuvan University, Kirtipur, Nepal.  
Date: April 25, 2024



## LETTER OF APPROVAL

We certify that the Research Evaluation Committee of the Central Department of Mathematics, Tribhuvan University, Kirtipur approved this research work entitled **“Applications of Galois Theory”** carried out by Mr. **Sandesh Thakuri** in the scope and generality as a thesis in the partial fulfillment for the requirement of the M.Sc./M.A. degree in Mathematics.

.....  
**(External Examiner)**

Dr. Santosh Ghimire  
Pulchowk Campus,  
Institute of Engineering,  
Tribhuvan University,  
Kathmandu, Nepal.  
Date: April 25, 2024

.....  
**(Supervisor)**

Assoc. Prof. Tulasi Prasad Nepal  
Central Department of Mathematics,  
Institute of Science & Technology,  
Tribhuvan University,  
Kirtipur, Kathmandu, Nepal.

Date: April 25, 2024

.....  
**(Head of Department)**

Prof. Dr. Chet Raj Bhatta  
Central Department of Mathematics  
Institute of Science & Technology  
Tribhuvan University, Kathmandu, Kirtipur, Nepal.  
Date: April 25, 2024

## ACKNOWLEDGMENTS

First of all I would like to thank my **supervisor** **Mr. Tulasi Prasad Nepal** (*Associate Professor of Central Department of Mathematics*). Then I am thankful to former HOD of the department Prof. Dr. **Tanka Nath Dhamala** for his valuable support and time.

I am grateful to **Kathmandu Center for Research and Education Chinese Academy of Sciences-TU** for awarding me with **KCRE Excellent Student Thesis Grant 2023**(Grant number: 08092023).

I would like thank Mr. **Santosh Gnawali** for providing me the necessary articles. Then I am thankful to Mr. **Dipak Babu Amgain** for assisting me with the bibliography. Finally I am thankful my friends and family.

---

**Sandesh Thakuri**

Date: April 25, 2024

# ABSTRACT

In this thesis entitled [Applications of Galois Theory](#), first the Galois Theory is introduced and the fundamental theorem of Galois Theory is proved. Then using the frame-work of the fundamental theorem the applications of the Galois theory is explored. First it is applied to compute the Galois Groups of Polynomials of various degrees. Two new examples of polynomials of degree more than are also explained and the concept of Galois group of a single variable polynomial is extended to the Galois group of a multi-variable polynomial. Then it is applied to solve the non-existence of a “formula” to solve a polynomial equation of a degree five or above. Finally theory of Galois field is applied to the computer science in the coding theory and cryptography. In Coding theory it is applied to the Error-Correcting codes and in Cryptography it is applied in Block Cipher algorithm.

# Symbol Conventions

Through out the thesis following Symbol Convention has been used.

- **K** a field.
- **F** an extension field of the field **K**

## Some Standard Symbols

1.  $\mathbb{Z}$  set of integers.
2.  $\mathbb{Q}$  set of rationals.
3.  $\mathbb{R}$  set of reals.
4.  $\mathbb{C}$  set of complex numbers.
5.  $\mathbb{Z}_n$  ring of set of integers modulo  $n$ .
6.  $\mathbb{Z}_p$  finite prime field.
7. AES American Encryption Standard.

# Contents

<b>DEDICATION</b>	iii
<b>STUDENT DECLARATION</b>	iv
<b>RECOMMENDATION</b>	v
<b>LETTER OF APPROVAL</b>	vi
<b>ACKNOWLEDGMENTS</b>	vii
<b>ABSTRACT</b>	viii
<b>Symbol Conventions Used</b>	ix
<b>Contents</b>	x
<b>1 Introduction</b>	1
1.1 Structure of a Field Extension . . . . .	2
1.2 Galois Group . . . . .	3
1.3 Galois Extension . . . . .	3
1.4 Fundamental Theorem of Galois Theory . . . . .	4
1.5 Insights to the Fundamental Theorem . . . . .	6
1.6 Structure of a Galois Extension . . . . .	7
1.7 Separable Extension . . . . .	8
1.8 Galois extension and Splitting Field . . . . .	8
<b>2 Applications to the Galois Groups</b>	9
2.1 Galois Group of a Polynomial . . . . .	9
2.2 Galois Group of Cubic polynomials . . . . .	9
2.3 Galois Group of Quartic polynomials . . . . .	10
2.4 Galois group of Polynomials of degree Five or more . . . . .	12
2.5 Galois Group of Reducible polynomials . . . . .	13
2.6 Galois groups of Multi-variable Polynomials . . . . .	14
2.7 Application to the Classic Problem . . . . .	15

2.8	Outcomes . . . . .	17
<b>3</b>	<b>Applications to the Coding Theory and Cryptography</b>	<b>18</b>
3.1	Galois Field . . . . .	18
3.2	Representation of Galois Fields . . . . .	19
3.3	Error-Correcting Code . . . . .	20
3.4	Syndrome Decoding . . . . .	22
3.5	Cyclic Code . . . . .	24
3.6	Cryptography . . . . .	25
3.7	The Cipher(Algorithm) . . . . .	26
<b>Conclusions</b>		<b>30</b>
<b>REFERENCES</b>		<b>31</b>

## List of Tables

2.1	Roots permutation table. . . . .	11
3.1	Binary representation table of field elements. . . . .	19
3.2	Terms and their meanings. . . . .	26

## List of Figures

1.1	Portrait of Galois . . . . .	1
1.2	Equivalency . . . . .	7
2.1	Subgroup diagram of the Galois group 'G'. . . . .	12
2.2	Intermediate field diagram of the Galois extension $\mathbb{Q}(u, i)$ over $\mathbb{Q}$ . . . . .	12
2.3	Plotted by the “GNU-Octave” . . . . .	13
2.4	Plotted by the “Geogebra” . . . . .	14

# Chapter 1

## Introduction

Galois Theory is a popular and one of the important theory in Abstract Algebra. It's foundation was laid by the French Mathematician *Évariste Galois* by determining the necessary and sufficient condition for solving a polynomial equation by radicals and thereby solving the problem that was open for 350 years old [1].

The core-part of the Galois Theory is the “*Fundamental Theorem of Galois Theory*” [3] which links two main parts of Abstract Algebra; Field Theory and Group Theory. This is a profound result in Abstract Algebra.



Figure 1.1: Portrait of Galois

### 1.0.1 Approaches to the Theory

1. Galois approached this problem by using the group of “permutations of the roots” [1] of a polynomial equation. Only those permutations of the roots are considered which leaves any equation satisfied by the roots unchanged.
2. The modern approach is to use the **field extension** of the underlying field of the polynomial and examine the groups of automorphism of the extension field that fixes the underlying field.

So, the Galois theory is the theory of field extensions.

## 1.1 Structure of a Field Extension

A field  $F$  is an extension field of a field  $K$  if the field  $K$  can be embedded into the field  $F$ . The structure of  $F$  is that it is a vector space over the field  $K$  [3]

Let  $F = K(u)$  be an extension field of the field  $K$ . Then  $F$  is a vector space over  $K$  generated by  $u$ . We have  $u^n \in F$  for all  $n \in \mathbb{Z}$  because  $F$  is a field. As “ $F$  is a vector space over  $K$ ” [3];  $F$  consists of all linear combinations of  $u^n$ 's, and the quotients of these linear combinations. A such linear combinations is:  $a_n u^n + a_{n-1} u^{n-1} + \dots + a_1 u + a_0$  which is given by the polynomial  $f(u)$ , where  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ .

So the structure of the extension field  $F = K(u)$  is:

$$F = \left\{ \frac{f(u)}{g(u)} \mid f, g \in K[x], g(u) \neq 0 \right\} [3].$$

**Theorem 1.** [3] *[Existence of an Extension field] If  $K$  is a field and  $f \in K[x]$  is a polynomial of degree  $n$ , then there exists a simple extension field  $F = K(u)$  of  $K$  such that  $u \in F$  is a root of  $f$ .*

### 1.1.1 Algebraic and Transcendental element

**Theorem 2.** [3] *Let  $F$  be an extension field of a field  $K$ .*

*A map  $\phi : K[X] \rightarrow K[u]$  where  $u \in F$  defined by  $\phi(f(x)) = f(u)$*

*i.e.,  $\phi(a_0 + a_1 x + \dots + a_n x^n) = a_0 + a_1 u + \dots + a_n u^n$  is a ring homomorphism.*

Thus  $K[x]$  and  $K[u]$  are homomorphic.

1. If  $u$  is transcendental over  $K$  then  $K[u]$  is not a field and  $K[x] \cong K[u]$ .
2. If  $u$  is algebraic over  $K$  then,  $K[x] \not\cong K[u]$  because  $\text{Ker } \phi$  is non trivial and we have,
  - i) “ $K(u) \cong K[u]$ ;
  - ii)  $K(u) \cong K[x]/(f)$ , where  $f \in K[x]$  is an irreducible monic polynomial of degree  $n \geq 1$ ;
  - iii)  $[K(u) : K] = n$  and  $\{1_k, u, u^2, \dots, u^{n-1}\}$  is a basis of the vector space  $K(u)$  over  $K$ ” [3].

**Theorem 3.** [3] *[Isomorphism of Extension fields] Let  $K$  be a field.*

*Then 'u' and 'v' are roots of the same irreducible polynomial  $f \in K[x]$  if and only if there is an isomorphism of fields  $K[u] \cong K[v]$  which sends  $u$  onto  $v$  and is the identity on  $K$ .*

## 1.2 Galois Group

Let  $F$  be a field. “The set  $\text{Aut}F$  of all field-automorphisms  $F \rightarrow F$  forms a group under the function composition” [3].

Let  $E$  and  $F$  be the extension fields of a field  $K$ .

If a non-zero field-homomorphism  $\sigma : E \rightarrow F$  is a  $K$ -module homomorphism then  $\sigma(k) = \sigma(k1_E) = k\sigma(1_E) = k1_F = k$ . i.e,  $\sigma$  fixes  $K$  element-wise.

Conversely, if a field homomorphism  $\sigma : E \rightarrow F$  fixes  $K$  element-wise, then  $\sigma$  is a non-zero and for any  $u \in E$ , we have,  $\sigma(ku) = \sigma(k)\sigma(u) = k\sigma(u)$  i.e  $\sigma$  is a  $K$ -module homomorphism [3].

**Definition 1.** [3] A field-automorphism  $\sigma \in \text{Aut}F$  which is also  $K$ -homomorphism is called  $K$ -automorphism. In other words, a field-automorphism  $\sigma \in \text{Aut}F$  that fixes  $K$  element-wise is called  $K$ -automorphism.

**Definition 2.** [3] The group of all  $K$ -automorphisms of  $F$  is called the Galois group of  $F$  over  $K$  and it is denoted by  $\text{Aut}_K^F$ .

## 1.3 Galois Extension

Let  $E$  be an intermediate field and  $H$  be a subgroup of  $\text{Aut}_K^F$ , then:

- i) “ $H' = \{v \in F \mid \sigma(v) = v, \text{ for all } \sigma \in H\}$ ” [3] is an intermediate field of the extension field  $F$  of  $K$ ;
- ii) “ $E' = \{\sigma \in \text{Aut}_K^F \mid \sigma(u) = u, \text{ for all } u \in E\} = \text{Aut}_E^F$ ” [3] is a subgroup of  $\text{Aut}_K^F$ .

The field  $H'$  is called the fixed field of the subgroup  $H$  in  $F$  [3].

### 1.3.1 Fixed Field

We have,

$H' \rightarrow$  fixed field and  $E' \rightarrow \text{Aut}_E^F$ . Let  $\text{Aut}_K^F = G$  then the field fixed by it is  $G'$ . It is not necessary that the field fixed by  $G$  is  $K$  i.e,  $G' = K$ .

**Example 1.** For  $f(x) = x^3 - 2 \in Q[x]$ . Let  $u \in F$  such that  $f(u) = 0$  and let  $F = Q[u]$ . Then  $G = \text{Aut}_Q^{Q(u)} = 1$  so,  $G' = F \neq K$ .

**Definition 3.** [3] Let  $F$  be an extension field of  $K$  such that the fixed field of the Galois group  $\text{Aut}_k^F$  is  $K$  itself. Then  $F$  is said to be a Galois extension of  $K$  or Galois over  $K$ .

## 1.4 Fundamental Theorem of Galois Theory

**Theorem 4.** [3] If  $F$  is a finite dimensional Galois extension of  $K$ , then there is a one-to-one correspondence between the set of all intermediate fields of  $F$  over  $K$  and the set of subgroups of the Galois group  $\text{Aut}_K^F$  such that:

- i) the relative dimension of two intermediate fields is equal to the relative index of the corresponding subgroups. In particular  $\text{Aut}_K^F$  has order  $[F : K]$ ;
- ii)  $F$  is Galois over every intermediate field  $E$ , but  $E$  is Galois over  $K$  if and only if the corresponding subgroup  $E' = \text{Aut}_E^F$  is normal in  $G = \text{Aut}_K^F$ . In this case  $G/E'$  is isomorphic to the Galois group  $\text{Aut}_K^E$  of  $E$  over  $K$ .

We already have a correspondence between the intermediate fields and the subgroup of Galois group.

That is to each intermediate field  $E$ , there is a subgroup  $\text{Aut}_E^F$  and to each subgroup  $H$  there is a fixed field  $H'$ . But this correspondence is one-to-one if and only if for each intermediate field  $E$ , it satisfies  $E'' = E$  and for each subgroup  $H$ , it satisfies  $H'' = H$ .

### 1.4.1 Closed Field and Subgroup

**Definition 4.** [3] Let  $X$  be an intermediate field or subgroup of the Galois group.  $X$  will be called **closed** provided  $X = X''$ .

**Lemma 1.** [3]

- i) If  $F$  is an extension field of  $K$ , then there is one-to-one correspondence between the closed intermediate fields of the extension and the closed subgroups of the Galois group, given by  $E \rightarrow E' = \text{Aut}_E^F$ .
- ii) If  $F$  is a finite dimensional Galois extension of  $K$ , then all intermediate fields and all subgroups of the Galois group are closed and  $\text{Aut}_K^F$  has order  $[F : K]$ .

### 1.4.2 Stable Intermediate

**Definition 5.** [3] An intermediate field  $E$  of  $F$  over  $K$  is said to be stable intermediate if  $\sigma(E) \subseteq E$  for every  $\sigma \in \text{Aut}_K^F$ .

### Lemma 2. [3]

- i) If  $E$  is a stable intermediate field of the extension, then  $E' = \text{Aut}_E^F$  is a normal subgroup of the Galois group  $\text{Aut}_K^F$  ;
- ii) if  $H$  is a normal subgroup of  $\text{Aut}_K^F$ , then the fixed field  $H'$  of  $H$  is a stable intermediate field of the extension

### 1.4.3 Proof of the Fundamental Theorem

*Proof.* From lemma 1 above, one-to-one correspondence between the intermediate fields and the subgroups follows [3]. And from lemma 2 the next two statements of the theorem follows [3].  $\square$

### 1.4.4 Applications

The fundamental theorem links Field theory to Group theory. This allows us to use the tools of Group theory to solve the problems involving field theory. Solving a polynomial equation is a problem of field theory. We can use insights of Group theory to solve this problem of field theory which is discussed in some detail in next part of this thesis [3].

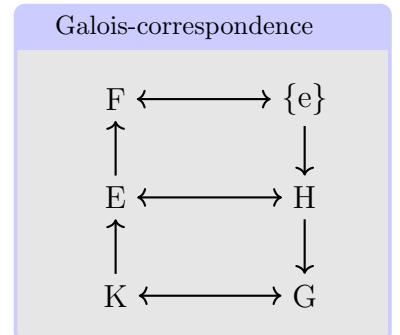
### 1.4.5 Illustration of The Fundamental Theorem

Let  $F$  be an Galois extension of a field  $K$ . Let the towers of the intermediate fields of  $F$  over  $K$  be as follows:

$$K \subset E \subset F$$

Let  $G$  be the Galois group of  $F$  over  $K$ . Then its subsets are

$$\{e\} \subset H \subset G$$



Then the one-to-one correspondence is as shown:

*Remark.* The intermediate fields are getting larger as we go from bottom to top as the fields are getting extended. But the subgroups are getting smaller.

## 1.5 Insights to the Fundamental Theorem

### 1.5.1 Nature of a Number

The nature of a number depends upon the underlying field.

$\sigma : \{a + b\sqrt{2}\} \mapsto \{a - b\sqrt{2}\}$  where  $a, b \in \mathbb{Q}$  is a field-automorphism of  $\mathbb{Q}(\sqrt{2})$  that fixes  $\mathbb{Q}$ . This map  $\sigma$  is also denoted by  $\sqrt{2} \mapsto -\sqrt{2}$ . So, any polynomial equation over  $\mathbb{Q}$  satisfied by the number  $\sqrt{2}$  is also satisfied by the number  $-\sqrt{2}$ . You can fluidly pass between these two numbers and the equation with a rational coefficient will not know. *Hence the two numbers  $\sqrt{2}$  and  $-\sqrt{2}$  are algebraically same over  $\mathbb{Q}$ .*

But the map  $\sqrt{2} \mapsto -\sqrt{2}$  does not fix the field  $\mathbb{Q}(\sqrt{2})$  i.e does not fix itself. The only automorphism of  $\mathbb{Q}(\sqrt{2})$  is the identity map. So, you cannot pass  $\sqrt{2}$  for  $-\sqrt{2}$  for every equation with coefficients in  $\mathbb{Q}(\sqrt{2})$ . *Hence the two numbers  $\sqrt{2}$  and  $-\sqrt{2}$  are not algebraically same over  $\mathbb{Q}(\sqrt{2})$ .*

$$\begin{array}{ccc} \mathbb{Q}(\sqrt{2}) & \longleftrightarrow & \{e\} \\ \uparrow & & \downarrow \\ \mathbb{Q} & \longleftrightarrow & \{i, \sqrt{2} \mapsto -\sqrt{2}\} \end{array}$$

### 1.5.2 Structure of a Field

The structure of a field as an extension field over some field is mirrored in the structure of the “group” of permutations of its elements that keeps the base field fixed. But these permutations are the symmetries of the field. *So, the structure of field extension is equals to its own symmetry.*

The structure of a field is a **complicated** thing; specially if it is infinite. But the structure of a group is rather simple; especially if it is finite. So the Galois theory has fairly simplified the complicated thing in a very insightful and beautiful way.

Also Galois theory gives a new sights of study of fields which is “study of fields via study of its automorphisms” which are permutations of the field. That is

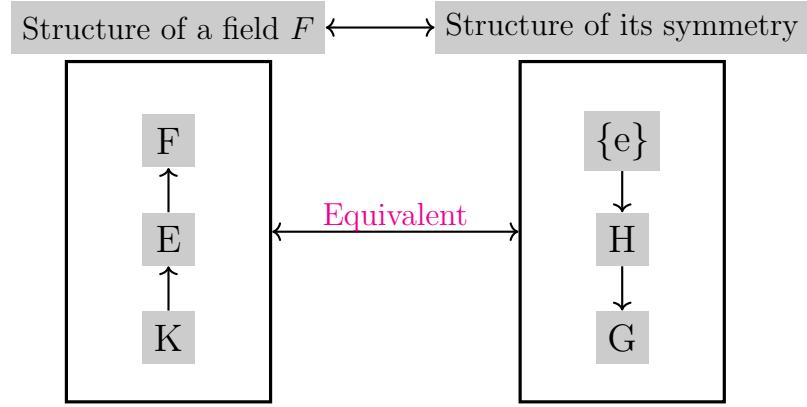


Figure 1.2: Equivalency

Galois extension  $F$  of  $K$  is a field for which the fixed field of the Galois group  $\text{Aut}_K^F$  is  $K$  itself [3].

But for what extension field  $F$  of  $K$  the Galois group keeps the “base field”  $K$  fixed? What is the structure of Galois field extension and how do we construct(obtain) it?

## 1.6 Structure of a Galois Extension

Since, for  $F = K(u)$ , any  $\sigma \in \text{Aut}_K^F$  is “completely determined” [3] by its action on  $u$ . Any algebraic Galois extension of  $K$  is generated by all roots  $u$  of a polynomial  $f \in K[x]$  [3].

**Definition 6.** [3] Such a minimal field  $F$  where a polynomial  $f \in K[x]$  “splits into linear factors” and thus “contains all roots of  $f(x)$ ” is called a splitting field of  $f$  over  $K$ .

Thus, an algebraic Galois extension is going to be characterized by “a splitting field of a polynomial over the base field” [3].

**Theorem 5.** [3] *[Existence of a Splitting field] If  $K$  is a field and  $f \in K[x]$  has degree  $n \geq 1$ , then there exists a splitting field  $F$  of  $f$  with  $[F : K] \leq n!$ .*

### 1.6.1 Algebraic Closure of a Field

“A field  $F$  is said to be algebraically closed if every non-constant polynomial  $f \in F[x]$  has a root in  $F$ ” [3]. For example the field of complex number  $\mathbb{C}$  is algebraically closed.

**Definition 7.** [3] An extension field  $F$  of a field  $K$  is said to be algebraic closure of  $K$  if,

- i)  $F$  is algebraically closed and,
- ii)  $F$  is algebraic over  $K$ .

So,  $\mathbb{C}$  is algebraically closed field but is not an algebraic closure of  $\mathbb{Q}$  because  $\mathbb{C}$  is not algebraic over  $\mathbb{Q}$  [3]. But  $\mathbb{C}$  is an algebraic closure of  $\mathbb{R}$ .

This shows algebraic closure is a special case of a splitting field.

## 1.7 Separable Extension

“An irreducible polynomial  $f \in K[x]$  is said to be separable if in some splitting field of  $f$  over  $K$  every root of  $f$  is a simple root” [3].

An algebraic element  $u \in F$  is said to be “separable over  $K$  provided its irreducible polynomial is separable” [3].

**Definition 8.** [3] If every element of  $F$  is separable over  $K$ , then  $F$  is said to be a separable extension of  $K$ .

### Characteristic of a Separable extension

*Remark.* [3] Every algebraic extension field of a field of characteristic 0 is separable.

If a polynomial  $f \in K[x]$  is separable over  $K$  then it has no multiple roots in any splitting field of  $f$  over  $K$  [3]. “This shows that an irreducible polynomial in  $K[x]$  is separable if and only if its derivative is nonzero” [3]. This holds true for every field of characteristic zero [3].

## 1.8 Galois extension and Splitting Field

**Theorem 6.** [3]  $F$  is algebraic and Galois over  $K$  if and only if  $F$  is separable over  $K$  and  $F$  is a splitting field over  $K$  of a set  $S$  of polynomials in  $K[x]$ .

This proves the **Generalized Fundamental Theorem of Galois Theory**, which states the *Fundamental Theorem of Galois Theory* still holds if the extension field  $F$  is not finite dimensional as well i.e, if “ $F$  is algebraic and Galois over  $K$ ” [3].

# Chapter 2

## Applications to the Galois Groups

### 2.1 Galois Group of a Polynomial

**Definition 9.** [3] The Galois group of a polynomial  $f \in K[x]$  is the group  $\text{Aut}_K^F$ , where  $F$  is a splitting field of  $f$  over  $K$ .

**Theorem 7.** [3] Let  $G$  be a Galois group of a polynomial  $f \in K[x]$ .

- i)  $G$  is isomorphic to a subgroup of some symmetric group  $S_n$ .
- ii) If  $f$  is separable of degree  $n$ , then  $n$  divides  $|G|$  and  $G$  isomorphic to a transitive subgroup of  $S_n$ .

Because the Galois group  $\text{Aut}_K^F$  is a “group of automorphisms of  $F$  which is given by the permutations of the roots” Galois group of a polynomial is identified with the subgroup of  $S_n$ ” [3].

**Corollary 7.1.** [3]

- i) If the degree of  $f$  is 2 then its Galois group  $G \cong \mathbb{Z}_2$ .
- ii) If the degree of  $f$  is 3 then its Galois group  $G$  is either  $S_3$  or  $A_3$ .

### 2.2 Galois Group of Cubic polynomials

**Definition 10.** [3] Let  $K$  be a field with  $\text{char}K \neq 2$  and  $f \in K[x]$  a polynomial of degree  $n$  with  $n$  distinct roots  $u_1, u_2, \dots, u_n$  in some splitting field  $F$  of  $f$  over  $K$ . Let  $\Delta = \prod_{i < j} (u_i - u_j) = (u_1 - u_2)(u_1 - u_3)\dots(u_{n-1} - u_n) \in F$ .

The discriminant of  $f$  is the element  $D = \Delta^2$ .

**Theorem 8.** [3]

- i) The discriminant  $\Delta^2$  of  $f$  actually lies in  $K$ .
- ii) For each  $\sigma \in \text{Aut}_K^F < S_n$ ,  $\sigma$  is an even [resp. odd] if and only if  $\sigma(\Delta) = \Delta$  [resp.  $\sigma(\Delta) = -\Delta$ ].

Since  $\Delta^2 \in K$  and  $\Delta \in F$ , and  $K(\Delta)$  is a stable intermediate; in the “Galois correspondence the subfield  $K(\Delta)$  corresponds to the subgroup  $G \cap A_n$ . In particular,  $G$  consists of even permutations if and only if  $\Delta \in K$ ” [3].

**Corollary 8.1.** [3] If  $f$  is a separable polynomial of degree 3, then the Galois group of  $f$  is  $A_3$  if and only if the discriminant of  $f$  is the square of an element of  $K$ .

**Theorem 9.** [3] Let  $K$  be a field of  $\text{char}K \neq 2, 3$ . If  $f(x) = x^3 + bx^2 + cx + d \in K[x]$  has three distinct roots in some splitting field, then the polynomial  $g(x) = f(x - b/3) \in K[x]$  has the form  $x^3 + p^2 + q$  and the discriminant of  $f$  is  $-4p^3 - 27q^2$ .

## 2.3 Galois Group of Quartic polynomials

**Definition 11.** [3] [Resolvent Cubic of a Quartic] Let  $K, f, F, u_i, V$ , and  $G = \text{Aut}_K^F < S_4$  be as in the preceding paragraph and  $\alpha = u_1u_2 + u_3u_4$ ,  $\beta = u_1u_3 + u_2u_4$ ,  $\gamma = u_1u_4 + u_2u_3$ . The polynomial  $(x - \alpha)(x - \beta)(x - \gamma)$  is called the resolvent cubic of  $f$ . The resolvent cubic is actually a polynomial over  $K$ .

Now under “the Galois correspondence the subfield  $K(\alpha, \beta, \gamma)$  corresponds to the normal subgroup  $V \cap G$ ” [3] because  $K(\alpha, \beta, \gamma)$  is a splitting field of the resolvent cubic whose Galois group is a subgroup of  $S_3$  and only normal subgroup  $N$  of  $S_4$  with  $|N| \leq 6$  is  $V$ , where  $V = \{(1), (12)(34), (13)(24), (14)(23)\}$ . “Hence  $K(\alpha, \beta, \gamma)$  is Galois over  $K$  and  $\text{Aut}_K^{K(\alpha, \beta, \gamma)} = G/(G \cap V)$ ” [3].

*Remark.* [3] If  $K$  is a field and  $f = x^4 + bx^3 + cx^2 + dx + e \in K[x]$ , then the resolvent cubic of  $f$  is the polynomial  $x^3 - cx^2 + (bd - 4e)x - b^2e + 4ce - d^2 \in K[x]$ .

**Theorem 10.** [3] Let  $K$  be a field and  $f \in K[x]$  a separable quartic with Galois Group  $G$ . Let  $\alpha, \beta, \gamma$  be the roots of the resolvent cubic of  $f$  and let  $m = [K(\alpha, \beta, \gamma) : K]$  then,

- i)  $m = 6 \iff G = S_4$ ;
- ii)  $m = 3 \iff G = A_4$ ;
- iii)  $m = 1 \iff G = V$ ;
- iv)  $m = 2 \iff G = D_4$  or  $G = \mathbb{Z}_4$ ; in this case  $G = \mathbb{Z}_4$  if  $f$  is irreducible over  $K(\alpha, \beta, \gamma)$  and  $G = D_4$  otherwise.

This is because we have  $[K(\alpha, \beta, \gamma) : K] = |G/G \cap V|$ .

### 2.3.1 Galois Group of some Polynomials

**Example 2.** [3] The polynomial is  $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ . This polynomial is irreducible and therefore separable over  $\mathbb{Q}$ . The resolvent cubic is  $x^3 + 8x = x(x + 2i\sqrt{2})(x - 2i\sqrt{2})$  and  $\mathbb{Q}(\alpha, \beta, \gamma) = \mathbb{Q}(i\sqrt{2})$  has dimension 2 over  $\mathbb{Q}$ .  $x^4 - 2$  is irreducible over  $\mathbb{Q}(i\sqrt{2})$  because  $\sqrt[4]{2} \notin \mathbb{Q}(i\sqrt{2})$ . Therefore the Galois group  $G \cong D_8$ .

“Let  $F \subset \mathbb{C}$  be a splitting field over  $\mathbb{Q}$  of  $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ . If  $u$  is the positive real fourth root of 2, then the roots of  $f$  are  $u, -u, ui, -ui$ . To find the Galois group  $G = Aut_{\mathbb{Q}}^F$  of  $f$  which is a subgroup of  $S_4$ , we find the permutations of the roots. To do so we must choose an ordering of the roots, say  $u_1 = u, u_2 = ui, u_3 = -u, u_4 = -ui$ . The complex conjugation is an  $\mathbb{R}$ -automorphism of  $\mathbb{C}$  which clearly sends:  $u \mapsto u, -u \mapsto -u, ui \mapsto -ui$  and  $-ui \mapsto ui$ .

This induces a  $\mathbb{Q}$ -automorphism  $\tau$  of  $F = \mathbb{Q}(u, ui)$ . As an element of  $S_4$ ” [3],  $\tau = (24)$ . Now the generator of  $D_8$  containing  $\tau = (24)$  is  $\sigma = (1234)$ . We have “ $F = \mathbb{Q}(u, ui) = \mathbb{Q}(u, i)$ , so every  $\mathbb{Q}$ -automorphism of  $F$  is completely determined by its action on  $u$  and  $i$ . Thus the elements of  $G$  may be described either in terms of  $\sigma$  and  $\tau$  or by their action on  $u$  and  $i$ . The information is summarized in the table” [3]

	(1)	(24) $\tau$	(1234) $\sigma$	(13)(24) $\sigma^2$	(1432) $\sigma^3$	(12)(34) $\sigma\tau$	(13) $\sigma^2\tau$	(14)(32) $\sigma^3\tau$
$u \mapsto$	$u$	$u$	$ui$	$-u$	$-ui$	$ui$	$-u$	$-ui$
$i \mapsto$	$i$	$-i$	$i$	$i$	$i$	$-i$	$-i$	$-i$

Table 2.1: Roots permutation table.

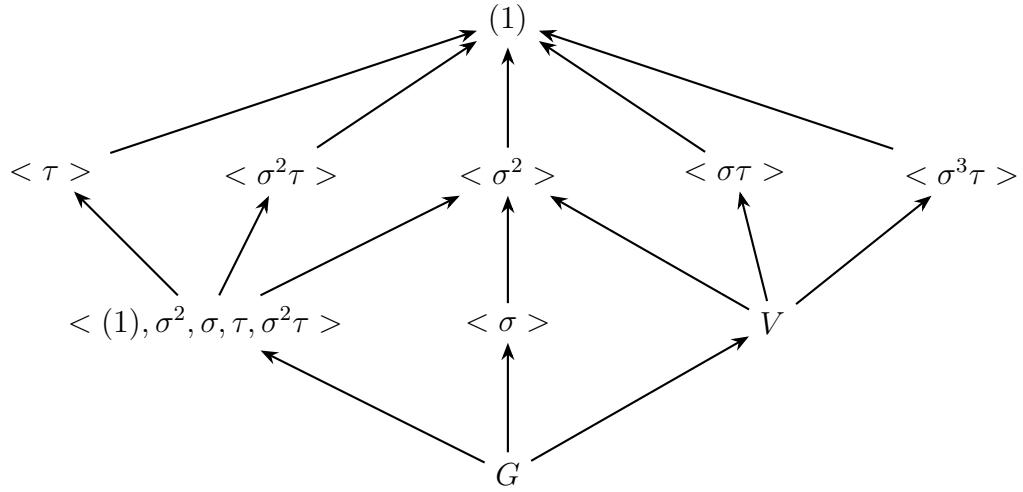


Figure 2.1: Subgroup diagram of the Galois group 'G'.

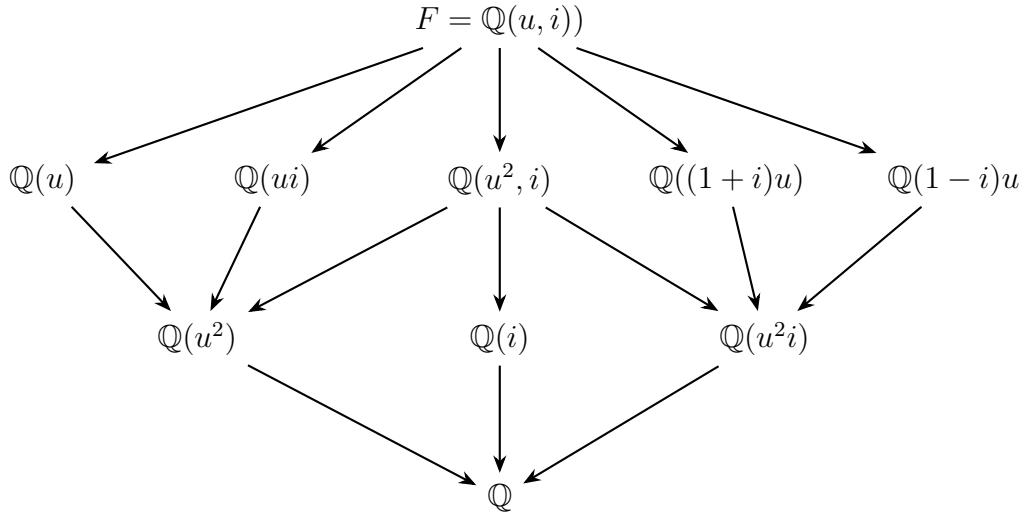


Figure 2.2: Intermediate field diagram of the Galois extension  $\mathbb{Q}(u, i)$  over  $\mathbb{Q}$

## 2.4 Galois group of Polynomials of degree Five or more

“There are very less techniques for computing Galois groups of polynomials of degree greater than 4 over arbitrary fields” [3].

**Theorem 11.** [3] If  $p$  is a prime and  $f$  is an irreducible polynomial of degree  $p$  over  $\mathbb{Q}$  which has precisely two non-real roots, then the Galois group of  $f$  is  $S_p$ .

**Example 3.** The polynomial is  $f(x) = x^5 - 10x + 5 \in \mathbb{Q}[x]$ . Its graph is shown below.

From its graph this polynomial has only three real roots. This polynomial is “irreducible over  $\mathbb{Q}$  by the Eisenstein’s criterion” [3] so by Theorem-11 its Galois group is  $S_5$  which contains  $5! = 120$  elements.

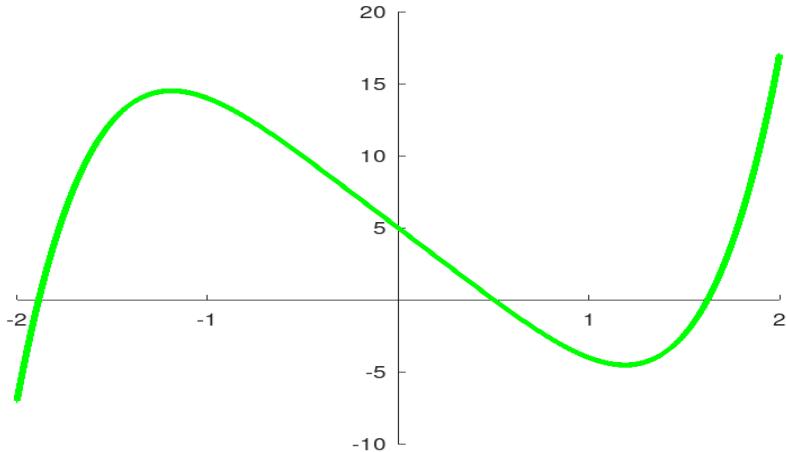


Figure 2.3: Plotted by the “GNU-Octave”

#### GNU-Octave Code for the above plotting

```

1      x=-2:0.000001:2;
2      y=x.^5-10*x+5;
3
4      plot(x,y,"g","linewidth",3);
5      xlabel="x";                      ylabel="y";
6      title="Graph of the quantic y";
7      box off;
8      set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin');

```

**Example 4.** The polynomial is  $f(x) = x^7 - 2x^5 - 4x^3 + 2x^2 + 4x - 2$  which is “irreducible over  $\mathbb{Q}$  by the Eisenstein’s criterion” [3]. Its graph is shown below. The Graph shows this polynomial has exactly five real roots. So exactly two of its roots are complex. Hence by the Theorem-11 its Galois Group is  $S_7$  which contains  $7! = 5040$  elements.

## 2.5 Galois Group of Reducible polynomials

For any polynomial  $f \in K[x]$  we factor  $f$  into irreducibles as  $f_1 f_2 \dots f_k$  and compute the Galois group  $G_i$  of  $f_i$  for each  $i = 1, 2, \dots, k$ . “Then the Galois group  $G$  of  $f$  is isomorphic to a subgroup of  $\prod G_i$ ” [4].

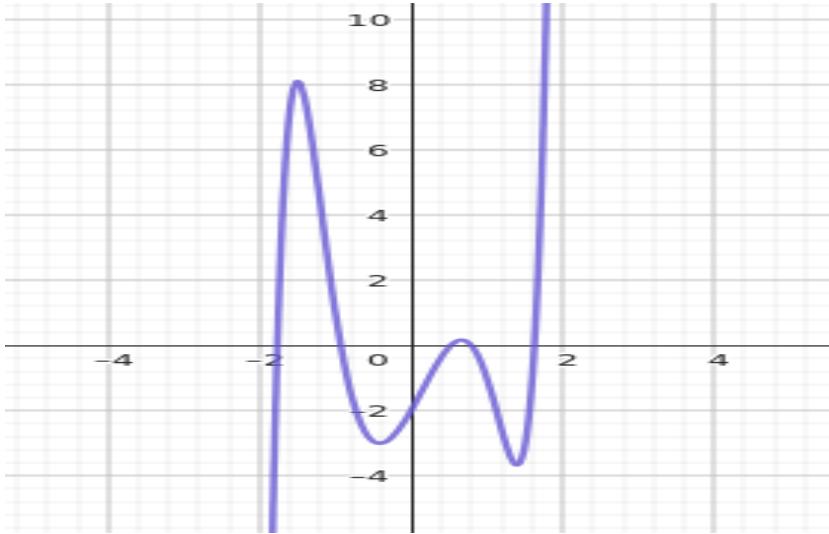


Figure 2.4: Plotted by the “Geogebra”

**Example 5.** The polynomial is  $f(x) = x^4 - 7x^2 + 15$

Here,  $f(x) = (x^2 - 3)(x^2 - 5)$  so it is reducible over  $\mathbb{Q}$ . Let  $f_1(x) = (x^2 - 3)$  and  $f_2(x) = (x^2 - 5)$ . Then  $f_1, f_2$  are both irreducible over  $\mathbb{Q}$ .

The splitting field for  $f_1$  is  $\mathbb{Q}(\sqrt{3})$  so its “Galois group is  $\mathbb{Z}_2$ ” [3]. The splitting field for  $f_2$  is  $\mathbb{Q}(\sqrt{5})$  so its Galois group is also  $\mathbb{Z}_2$ . Now we have the Galois group of  $f$  is a subgroup of  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ .

Since the intersection of  $\mathbb{Q}(\sqrt{3})$  and  $\mathbb{Q}(\sqrt{5})$  is trivial the Galois group of  $f$  is  $G$  itself which is *Klein 4-group*.

**Example 6.** The polynomial is  $f(x) = x^7 - 5x^5 - 10x^3 + 5x^2 + 50x - 25 \in \mathbb{Q}[x]$ . This polynomial factors into irreducibles over  $\mathbb{Q}$  as  $(x^2 - 5)(x^5 - 10x + 5)$ .

“The Galois group of  $x^2 - 5$  is  $\mathbb{Z}_2$ ” [4] and of  $x^5 - 10x + 5$  is  $S_5$  from the example-3. Also the roots of  $x^2 - 5$  are  $\sqrt{5}$  and  $-\sqrt{5}$  which are not the roots of  $x^5 - 10x + 5$  from the its graph figure-4.1, so the intersection of the splitting fields of these two factor polynomials of  $f$  is trivial. Hence the Galois group of  $f$  is  $\mathbb{Z}_2 \times S_5$ .

## 2.6 Galois groups of Multi-variable Polynomials

The Galois group of a polynomial in single variable can be generalized to the Galois group of a multi-variable polynomial.

**Example 7.** The polynomial is  $f(x, y) = x + y \in \mathbb{Q}[x, y]$ . The roots of  $f$  spans all over the complex numbers. Hence its Galois group is a subgroup of the symmetric

group  $S_{|\mathbb{C}|}$ .

**Example 8.** The polynomials in  $\mathbb{Q}[x, y]$  are:

$$y = x^2 + 1 \quad (2.1)$$

$$y = x \quad (2.2)$$

. The roots of these simultaneous polynomials are  $\omega, \omega^2$ . Then the splitting field of this system is  $\mathbb{Q}(\omega)$ . Here the automorphisms of  $\mathbb{Q}(\omega)$  are:

$$\omega \mapsto \omega \text{ and } \omega \mapsto \omega^2.$$

Hence the Galois group of this system is  $\{(1), (12)\} \cong \mathbb{Z}_2$ .

## 2.7 Application to the Classic Problem

1. Is every polynomial equation solvable by the method of radicals?
2. Equivalently, does there exist an explicit "formula" which gives all solutions of a polynomial equation?

If the degree of the polynomial  $f$  is at most four then the answer is **yes** [3].

For example, the quadratic polynomial equation  $ax^2 + bx + c = 0$  has the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### 2.7.1 Formulation of the Classic Problem

The formula by the method of radicals means the "formula involving only field operations and the extraction of  $n$ th roots" [3]. "The existence of a formula means there is a finite sequence of steps, each step being a field operation or the extraction of an  $n$ th root, which yields all solutions of the given polynomial. Performing a field operation leaves the base field unchanged, but the extraction of an  $n$ th root of an element  $c$  in a field  $K$  amounts to constructing an extension field  $K(u)$  with  $u^n \in K$ . Thus the existence of a formula for solving  $f(x) = 0$  would imply the existence of a finite tower of fields

$$K = E_0 \subset E_1 \subset \dots \subset E_n$$

such that  $E_n$  contains a splitting field of  $f$  over  $K$  and for each  $i \geq 1$ ,  $E_i = E_{i-1}(u_i)$  with some positive power of  $u_i$  lying in  $E_{i-1}$ " [3].

“Conversely suppose there exists such a tower of fields and that  $E_n$  contains a splitting field of  $f$ . Then

$$E_n = K(u_1, u_2, \dots, u_n)$$

and each solution is of the form  $f(u_1, \dots, u_n)/g(u_1, \dots, u_n)$  where  $f, g \in K[x_1, \dots, x_n]$ . Thus each solution is expressible in terms of a finite number of elements of  $K$ , a finite number of field operations and  $u_1, \dots, u_n$ . But this amounts to saying that there is a formula for the solutions of the particular given equation” [3].

**Definition 12.** [3] [Radical Extension] An extension field  $F$  of a field  $K$  is a radical extension of  $K$  if  $F = K(u_1, \dots, u_n)$ , some power of  $u_1$  lies in  $K$  and for each  $i \geq 2$ , some power of  $u_i$  lies in  $K(u_1, \dots, u_{i-1})$ .

*Remark.* [3] If  $u_i^m \in K(u_1, \dots, u_{i-1})$  then  $u_i$  is a root of  $x^m - u_i^m \in K(u_1, \dots, u_{i-1})[x]$ . Therefore every radical extension  $F$  of  $K$  is finite dimensional algebraic over  $K$ .

**Definition 13.** [3] The polynomial equation  $f(x) = 0$  is *solvable by radicals* if there exists a radical extension  $F$  of  $K$  and splitting field  $E$  of  $f$  over  $K$  such that  $F \supset E \supset K$ .

**Theorem 12.** [3] If  $F$  is a radical extension of  $K$  and  $E$  is an intermediate field, then  $\text{Aut}_K^E$  is a solvable group.

**Corollary 12.1.** [3] If the polynomial equation  $f(x) = 0$  is solvable by radicals, then the Galois group of  $f$  is a solvable group.

## 2.7.2 Group Theoretic Concepts

Let  $G$  be a group.

**Definition 14.** [3] [Solvable Series] A finite chain of subgroups  $G = G_0 > G_1 > \dots > G_n = e$  such that  $G_{i+1}$  is normal in  $G_i$  for  $0 \leq i < n$  is called a subnormal series of  $G$ .

A subnormal series is a solvable series if each factor group  $G_i/G_{i+1}$  is abelian.

**Definition 15.** [3] [Solvable Group] A group is solvable if and only if it has a solvable series.

If  $F$  is a radical extension of  $K$  then  $F$  is Galois over  $K$  and by the Fundamental Theorem of Galois  $\text{Aut}_K^E$  has a solvable series where  $E$  is an intermediate field. So  $\text{Aut}_K^E$  is solvable.

## 2.8 Outcomes

**Theorem 13.** [3] *The symmetric group  $S_n$  is not solvable for  $n \geq 5$ .*

The polynomial  $f(x) = x^5 - 10x + 5 \in \mathbb{Q}[x]$  has Galois group “ $S_5$ , which is not a solvable group” [3].

The quantic polynomial equations over  $\mathbb{Q}$  are not solvable by radicals. That is there does not exist an explicit formula for solving the quantics.

Moreover, “polynomial equations of degree  $n \geq 5$  are not solvable by radicals” [3].

### 2.8.1 Illustrations

Galois theory gives the precise condition under which a polynomial of degree  $n \geq 5$  is solvable by radicals or not.

**Example 9.** The polynomial is  $g(x) = x^5 - 1 \in \mathbb{Q}[x]$ .

The set of roots of this polynomial equation  $g(x) = 0$  are the fifth roots of unity which forms a group under addition modulo 5. Hence the “Galois group is isomorphic to  $\mathbb{Z}_5$ ” [3]. The group  $\mathbb{Z}_5$  is cyclic and “every cyclic group is solvable” [1]. Hence this polynomial is solvable by radicals.

**Definition 16.** [1] [Cyclotomic Polynomial] The  $n$ th-cyclotomic polynomial is the polynomial  $\Phi_n$  defined as  $\Phi_n = \prod (x - \zeta)$ , where  $\zeta$  is a primitive- $n$ th of unity.

**Theorem 14.** [1] *The Galois group of a  $n$ th-cyclotomic polynomial  $\Phi_n$  of is  $\mathbb{Z}_n$ .*

**Example 10.** The polynomial is  $f(x) = x^{12} - x^{10} + x^8 - x^6 - x^2 + 1 \in \mathbb{Q}$  which is a 58th-cyclotomic polynomial i.e this polynomial  $f(x) = \Phi_{58}$ .

So its Galois group is  $\mathbb{Z}_{58}$ , which is abelian and hence is solvable. Therefore this polynomial equation  $f(x) = 0$  is solvable by radicals.

# Chapter 3

## Applications to the Coding Theory and Cryptography

### 3.1 Galois Field

Galois fields are the finite fields. “They can be completely characterized in terms of splitting fields of some polynomials. It is found that the Galois group of an extension of a finite field by a finite field is cyclic” [3]. We denote Galois field with  $q$  elements by  $GF(q)$ .

**Definition 17.** [3] [Prime Fields] Let  $F$  be a field and let  $P$  be the intersection of all subfields of  $F$ . Then  $P$  is a field with no proper subfields. This field  $P$  is called the Prime subfield of  $F$ .

1. “If  $\text{char}F = p(\text{prime})$ , then  $P \cong \mathbb{Z}_p$ .
2. If  $\text{char}F = 0$ , then  $P \cong \mathbb{Q}$ ” [3].

**Theorem 15.** [3] A finite field  $F$  has  $p^n$  number of elements where  $p \in \mathbb{Z}_+$  is a prime and it has  $p^n$  number of elements if and only if  $F$  is a splitting field of  $x^{p^n} - x$  over  $\mathbb{Z}_p$ .

**Theorem 16.** [3] If  $F$  is a finite dimensional extension field of a finite field  $K$ , then  $F$  is finite and is Galois over  $K$ . The Galois group  $\text{Aut}_K^F$  is cyclic.

## 3.2 Representation of Galois Fields

Basically there are two types of representation of a finite field. These two representations are equivalent.

### 3.2.1 Integer representation

“ $GF(p^n) = \{0, 1, \dots, p-1\} \cup \{p, p+1, \dots, p+p-1\} \cup \dots \cup \{p^{n-1}, p^{n-1}+1, \dots, p^{n-1} + p^{n-2} + \dots + p-1\}$ ” [1].

**Example 11.**  $GF(2) = \{0, 1\}$

$$GF(2^3) = \{0, 1\} \cup \{2, 2+1\} \cup \{2^2, 2^2+1, 2^2+2, 2^2+2+1\} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

But the digits 2, 3, .., 7 of the field  $GF(2^3)$  do not lie on the field  $GF(2)$ . If we look the field  $GF(2^3)$  as an extension field of  $GF(2)$  and write its elements using only the elements of the base field  $GF(2)$  then we have the following representations:

Digits	Expansion	Binary rep..
3	$2+1$	011
4	$2^2 + 2^1 \times 0 + 2^0 \times 0$	100
5	$2^2 + 1$	110

Table 3.1: Binary representation table of field elements.

This is actually **Binary representation** of the field  $GF(2^3)$

### 3.2.2 Polynomial representation

“For a field  $F$  and an irreducible polynomial  $f(x) \in F[x]$  the quotient ring  $F[x]/(f(x))$  is field” [1].

If  $F$  is a finite field consisting of  $p$  elements and  $f(x) \in F[x]$  is irreducible then  $F[x]/(f(x))$  is finite field. This field consists of all polynomials modulo  $f(x)$ . If  $F = GF(2^3)$  then  $f(x) = x^8 + x^7 + \dots + x + 1 \in F[x]$  is irreducible in  $F[x]$ . Since  $F$  has 8 elements which are modulo 8, elements of  $F$  is represented by the elements of the factor ring  $F[x]/(f(x))$  [5].

In the example-5, the number 5 has the representation  $2^2 + 1$ . This gives the polynomial representation  $x^2 + 1 = (1, 0, 1)$ (coefficient of  $x^2$  is 1 of  $x$  is 0 and of constant is 1 ) Now the binary equivalent of 5 is 101.

### 3.2.3 Operations in Galois Field

Let the Galois field be  $GF(p^n)$ . Since the elements of a Galois field can be represented as polynomials the operations are similar to polynomial operations. Let  $f(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$  and  $g(x) = (b_0 + b_1x + \dots + b_{n-1}x^{n-1})$ .

1. Addition

$$f(x) + g(x) \pmod{p}$$

2. Multiplication

$$f(x).g(x) \pmod{p} [5].$$

## 3.3 Error-Correcting Code

The loss of information is inevitable. It cannot be prevented or stopped. So, we need a way of retrieving the lost information or correcting the false information.

1. Paintings gets deteriorated over time and has to be renovated.
2. The data stored in a CD is lost over time [6].

To be able to detect and correct errors during transmission of information in digital system "coding theory" is developed. The idea of coding theory is to append some extra digits to the information and use this to detect and possibly correct the errors during transmission.

**Definition 18.** [6] These codes that can correct themselves are called "Error correcting codes"

In digital system, information are transmitted as strings of 0 and 1. So the fundamental of the coding theory in computer system is the manipulation of strings of binary digits. The proper and complete manipulation of these strings is possible only if the space of the strings is a field. This field is finite so this field is a **Galois field**. This is where the application of Galois theory comes. Another advantage of using field is that the space of code forms a vector space over the base field.

The widely used field for coding in electronically transmitting device is an extension field  $\mathbb{Z}_2$  which is the field  $GF(2)$  consisting of 0 and 1. Recent works has shown that it is possible to extend codes to more general type of numbers called rings. These rings are called "Galois rings" [2].

The non-empty set of symbols for the code  $\mathcal{A}$  called **alphabet**. A finite sequence of elements from  $\mathcal{A}$  is called a **word** over  $\mathcal{A}$ . Let  $\mathcal{A}^*$  be the set of all words over

$\mathcal{A}$ . A subset  $C$  of  $\mathcal{A}$  is called a code. If the cardinality of the alphabet  $\mathcal{A}$  is  $q$  then the code  $C$  is called  $q$ -ary code. For  $q = 2$  it is called binary and for  $q = 3$  it is called terniary [2].

### 3.3.1 Linear Codes

Let  $K = GF(q)$  be a Galois field. Then a finite extension of  $K$  of dimension  $n$  is  $V = GF(q)^n = GF(q^n)$ .

**Definition 19.** [6] A linear code  $C$  is a subspace of  $V$ .

The code  $C$  has dimension  $k \leq n$  and the length  $n$ . It is called a “ $(n, k)$  code” [2].

The usefulness of linear code is that they are vector spaces over the base field so they have a basis. All the code words can be generated with this basis. Instead of storing all  $2^k$  number of code words (for  $k$ -dimensional binary codes), storing only  $k$  basis elements is sufficient which saves massive storage.

Let  $C$  be  $(n, k)$  code which is a subspace of  $V$ .

**Definition 20.** [2] [Generator Matrix] Let  $\{v_1, v_2, \dots, v_k\}$  be a basis of  $C$ . A generator matrix is the  $k \times n$

$$G = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix}$$

**Definition 21.** [2] [Parity check matrix] “The dual code of  $C$  is the set  $C^\perp = \{x \in V \mid x.y = 0 \ \forall y \in V\}$ . The dual code is a code in itself and has dimension  $n - k$ .

The  $C^\perp$  is linear so it has a generator matrix. “A generator matrix  $H$  of  $C^\perp$  is called a parity check matrix” [2].

**Theorem 17.** [6] If  $G = (I_{k \times k}, A_{k \times (n-k)})$  is a generator matrix of an  $(n, k)$  code  $C$  then its parity check matrix is  $H = (I_{(n-k) \times (n-k)}, A')$  where  $A'$  is the transpose of  $A$ .

**Definition 22.** [6] The *Hamming distance* between  $v, w \in V$  is defined by  $d_h(v, w) = |\{i \mid v_i \neq w_i; 1 \leq i \leq n\}|$ .

The *minimum distance* of a code  $C$  is defined as  $\min\{d_h(v, w) \mid d_h(v, w) \neq 0, v, w \in C\}$ .

“The weight of a vector is its distance from zero and the *minimum weight* of a code  $C$  is the minimum weight of all non-zero weights of the vectors in  $C$ ” [2].

**Theorem 18.** [2] A linear code  $C$  with minimum weight  $d$  can correct strings having number of errors up to  $t = \lfloor (d - 1)/2 \rfloor$ .

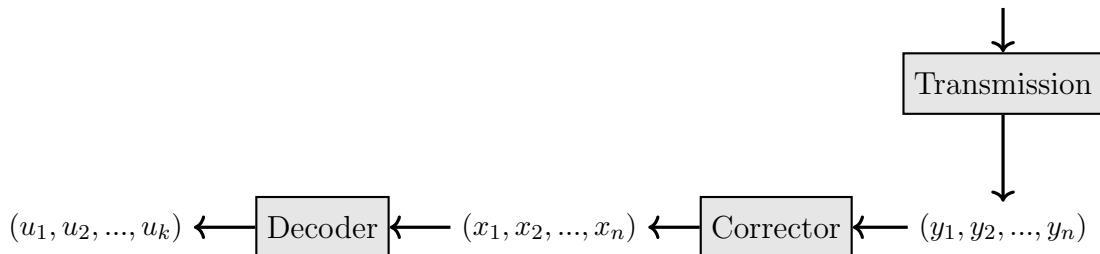
### 3.3.2 Illustration

To apply  $(n, k)$  coding first we need to group our information into blocks of length  $k$ .  $u_1, \dots, u_k, u_k, \dots, u_{2k}, \dots$ . This space has dimension  $k$ . Now these block of codes are encoded separately each to a code of length  $n$  as shown [6].



Mathematically, the encoded vector  $x$  is obtained form the original vector  $u$  using the generator matrix  $G$  by the relation  $x = uG$  [6].

To continue and complete the diagram.



## 3.4 Syndrome Decoding

**Definition 23.** [2] The syndrome of a vector  $y \in V$  is defined as

$$syn(y) = \begin{pmatrix} y.h_1 \\ y.h_2 \\ \vdots \\ y.h_{n-k} \end{pmatrix}, \quad \text{where } \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-k} \end{pmatrix} \text{ is the parity check matrix of } C.$$

Now the code  $C$  is a subgroup of  $V$  under addition. Moreover, it is a normal subgroup of  $V$ .

**Theorem 19.** [2] Two vectors in  $V$  have the same syndrome if and only if they are in the same co-set of  $C$ .

### 3.4.1 Decoding Process

Suppose the signal received is the vector  $y$ .

1. First we determine its syndrome,  $syn(y)$ .
2. Determine the co-set of  $C$  containing  $syn(y)$ , say  $e + C$ .
3. Then  $y = e + x$  for some  $x \in C$ . This implies  $x = y - e$ . Since  $x \in C$ , this  $x$  is the required decoding of  $y$  [2].

This  $e$  is also called "error vector" [2].

**Example 12.** Consider a generator matrix  $G = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ . Then the parity check matrix is  $H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ . And the code generated by  $G$  is  $C = \{(0, 0, 0, 0), (1, 0, 1, 0), (0, 1, 1, 1), (1, 0, 1, 1)\} \subset GF(2^4)$ .

Suppose the received vector is  $y = (1, 1, 1, 0)$ . Then  $y \notin C$  so the information is distorted from the original information. To get the original information:

$$syn(y) = \begin{pmatrix} y \cdot h_1 \\ y \cdot h_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ where } h_1 \text{ is the first row and } h_2 \text{ is the second row of } H.$$

Now if  $e = (0, 1, 0, 0)$  then  $e + C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  so  $y - e = (1, 1, 1, 0) - (0, 1, 0, 0) = (1, 0, 1, 0) \in C$  is the original information [2].

### 3.4.2 Perfect Code

The code  $C \subseteq V$  as of above is perfect if the union of all the spheres of radius  $t$  about its code-words is the vector space  $V$ .

This code is  $C$  is called perfect because every received vector with the number of errors given by  $t$  can be decoded to a code-word of  $C$  [2].

**Example 13.** The code  $C = V$  is a perfect code. This code cannot correct any errors because every possible code word is in the  $C$ . Therefore this perfect code is trivial [2].

**Example 14.** The general binary Hamming code  $H_r$ ,  $r \in \mathbb{N}$  whose parity check matrix  $H$  column consisting of non-zero r-tuples [2].

## 3.5 Cyclic Code

The code  $C$  as of above is cyclic if  $(a_0, a_1, \dots, a_{n-1}) \in C \implies (a_{n-1}, a_0, \dots, a_{n-2}) \in C$ .

Suppose  $C$  is a code over a Galois field  $F = GF(q)$ . Then there exist a correspondence  $\Phi : C \mapsto F[x]/(x^n - 1)$  such that  $\{(a_0, a_1, \dots, a_{n-1}), (a_1, \dots, a_{n-1}, a_0), \dots, (a_{n-1}, a_0, \dots, a_{n-2})\} \mapsto a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$ .

This map  $\Phi$  is homomorphism. This shows that the cyclic code  $C$  can be embedded into the ring  $R_n = F[x]/(x^n - 1)$  [2].

**Theorem 20.** [2]

1. A subset  $S$  of  $R_n$  corresponds to a cyclic code if and only if  $S$  is an ideal of  $R_n$  and
2. if  $S = (g(x))$  if and only if  $g(x)$  divides  $x^n - 1$ .

This theorem determines all cyclic codes. They are ideals of  $R_n$  and these ideals are generated by the polynomials that divides  $x^n - 1$ . Thus cyclic code have a generator polynomial which is computationally simpler than having a generator matrix.

This is where usages of polynomial representation of finite fields come into play and this is where cyclic code has advantage over general linear code.

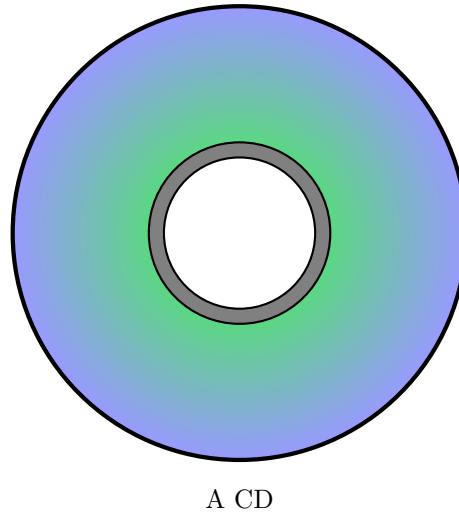
**Example 15.** The divisors of  $x^3 - 1 \in F = GF(2^3)$  are  $1, x + 1, x^2 + x + 1, x^3 - 1$ . For  $g(x) = x + 1$  we have  $F[x]/(g(x)) = \{(0), (1 + x), (1 + x^2), (x + x^2)\}$  so the corresponding cyclic code is  $\{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$  [2].

“Similar to general linear codes which are defined using the generator matrix or the parity-check matrix, cyclic codes are defined using generator polynomial or parity-check polynomial and the advantage of cyclic code is that there exists efficient algebraic decoding algorithm for them” [6].

### 3.5.1 Usages

1. The  $(3, 1)$  binary code is used in the short-range wireless communication system like *Bluetooth<sup>TM</sup>* [7].

2. The Hamming Code (7, 4) is used in memory devices like RAM [6].
3. The Cyclic codes are used in storing data in CDs and DVDs [6].



## 3.6 Cryptography

It is the science of safe-guarding information by converting the original message into something unreadable.

Galois Fields are the life of modern cryptography used in digital communication.

### 3.6.1 Advance Encryption Standard(AES)

The Advance Encryption Standard is a Computer Security Standard for cryptography which is approved by the “Federal Information Processing Standards Publications” of USA which became effective on May 26, 2002. “The AES algorithm is a *symmetric block cipher* that can encrypt and decrypt digital information” [5]. Symmetric key cryptography is used to share information between two parties where the two parties share a secret “key” and a public encryption algorithm.

“In 2000, National Institute of Standards and Technology(NIST) announced the selection of the “Rijndael” block cipher family as the winner of the AES competition and since then AES has been the standard for digital cryptography” [5]. “The Rijndael block cipher was developed by two Belgian cryptographers, **Vincent Rijmen and Joan Daemen**” [5].

The generic algorithm of AES consists of smaller sub-algorithms namely “Sub-Bytes, Shift-Rows, Mix-Columns and Add-Round-Key” [5].

Bit	A binary digit: 0 or 1.
Byte	A sequence of 8 bits.
Block	A sequence of bits of a fixed length. In this Standard, block consists of 128 bits.
Block cipher	A family of permutations of blocks.
Key	The parameter that selects the permutation from the block cipher.

Table 3.2: Terms and their meanings.

## 3.7 The Cipher(Algorithm)

### 3.7.1 The State

First the data is broken into blocks and each block is broken into smaller chunks of a size byte (16 bytes for a block of size 128 bits). This block is then represented in a  $4 \times 4$  matrix whose each entry is a byte of the block. This matrix is called the State.

Suppose the block is  $b_1, b_2, \dots, b_{16}$ . Then the state is  $\begin{bmatrix} b_1 & b_5 & \dots & \dots \\ b_2 & b_6 & \dots & \dots \\ b_3 & b_7 & \dots & \dots \\ b_4 & b_8 & \dots & b_{16} \end{bmatrix}$ .

Mathematical operations are not applicable to the data directly so the significance of this step is to make the data applicable for mathematical operations.

For the 128-bit key encryption the algorithm forms a  $4 \times 4$  matrix with each entry of a size one byte. This matrix can afford to evaluate a data of size 16 byte at a time [5].

### 3.7.2 Sub-Bytes

In this step, first each byte of the matrix is replaced with its multiplicative inverse if it has one. Then it transforms each bytes using an invertible affine transformation,  $x \mapsto Ax + b$  [5].

### 3.7.3 Mathematical Preliminaries

Here 0, 1 are the elements of the finite field  $GF(2)$ . Since a byte consists of eight bits for example 00001111, so each byte in the state i.e each entry in the matrix, is interpreted as one of the 256 elements of a finite field  $GF(2^8)$ . Then

the addition, multiplication operations are performed according to the respective field operations of the field  $GF(2^8)$ .

### 3.7.4 Shift-Rows

In this step entries of a row is shifted to scramble data. Row-n shifted to the left by  $n - 1$  unit. Here,

$1 - 1 = 0$ , so row-1 is left unchanged.  $2 - 1 = 1$ , so row-2 is shifted to the left by 1 unit and row-3 by 2 unit and so on as shown below [5].

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \text{then } A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} & a_{43} \end{bmatrix}$$

is the matrix after Shift-Row

### 3.7.5 Mix-Columns

In this step each column is transformed using a linear transformation,  $c \mapsto Bc$  where  $c$  is a column of the matrix obtained above. Since linear transformation is invertible this step is invertible. Note every step of this algorithm must be invertible to be able to decrypt the data [5].

### 3.7.6 Add-Round-Key

This is the step where the encrypted data gets uniqueness. Each user is assigned an "unique key" and this key is added to the matrix obtained from the last step [5].

### 3.7.7 Algorithms

#### Encryption Algorithm

```

procedure CIPHER(state, key)
    for round from 1 to 9 do
        1)  SUBBYTES(state)
        2)  SHIFTROWS(state)
        3)  MIXCOLUMNS(state)
        4)  ADDROUNDKEY(state, key)
    end for
end procedure

```

### Decryption Algorithm

```

procedure CIPHER(state, key)
    for round from 1 to 9 do
        1) InvADDROUNDKEY(state, key)
        2) InvMIXCOLUMNS(state)
        3) InvSHIFTROWS(state)
        4) InvSUBBYTES(state)
    end for
end procedure

```

#### 3.7.8 Illustration

Let us encrypt the sentence "Fun Cryptography". This consists of exactly 16 characters.

1. First we write the ASCII representation of each character of the sentence as shown below. We do so because the ASCII representation gives the binary representation of each character which has a size of a byte. The ASCII representation of "F" is 70 which is 01000110 in binary.

$$\begin{bmatrix} 70 & 117 & 110 & 32 \\ 67 & 114 & 121 & 112 \\ 116 & 111 & 103 & 114 \\ 97 & 112 & 104 & 121 \end{bmatrix} = \begin{bmatrix} 01000110 & 01110110 & 01101110 & 00010000 \\ 01000011 & 01110010 & 01111001 & 01110000 \\ 01110100 & 01101111 & 01100111 & 01110010 \\ 01100001 & 01110000 & 01101000 & 01111001 \end{bmatrix}$$

2. After performing Sub-Bytes, Shift-Rows, Mix-Columns, we get the following matrix.

$$\begin{bmatrix} 11100111 & 00011000 & 00100100 & 01110000 \\ 00101010 & 10101011 & 00111001 & 01100011 \\ 00010101 & 01100101 & 11110111 & 10100111 \\ 10101011 & 11110110 & 00000011 & 10100100 \end{bmatrix} = \begin{bmatrix} 231 & 24 & 36 & 112 \\ 42 & 171 & 57 & 99 \\ 21 & 101 & 247 & 167 \\ 171 & 246 & 3 & 164 \end{bmatrix}$$

3. We have omitted the Add-Round-Key step just for the sake of simplicity. The matrix obtained at last in step-2 translates to something different from our original sentence.
4. The decryption process is applying the inverse of the encryption process which is the decryption process explained in the above decryption algorithm [5].

# Conclusions

The Galois theory that begun in the 19th century due to the french mathematician *Évariste Galois* is still a relevant field of research today. Over the 200 years this theory has found its development as a linking theory of the two main theories: Group Theory and Field theory. It has found its applications in both pure and applied mathematics; where-ever “Field Theory” has anything to do with.

Many concepts of Abstract algebra, Algebraic number theory, Algebraic geometry, etc rely heavily on Galois theory because they are developed on field extensions, and the computer science relies heavily on Galois field.

# REFERENCES

- [1] Escofier, J. P. (2000) *Galois Theory*. Springer, New York.
- [2] Holdman, G. R. (2019) *Error Correcting Codes Over Galois Rings*. Graduate Dissertation, Department of Mathematics, Whitman college, 345 Boyer Ave. Walla Walla, Washington, U.S.A.
- [3] Hungerford, T. W. (2012) *Algebra*. Springer (India), New Dheli.
- [4] Lenstra, A., Lenstra, H., & Lovasz, L. (1982) Factoring polynomials with rational coefficients. *Mathematische Annalen*, 261, 12.
- [5] National Institute of Standards and Technology. (2001) Advanced Encryption Standard (AES). (*Department of Commerce, Washington, D.C.*), *Federal Information Processing Standards Publication (FIPS) NIST FIPS. 197-upd.1* updated May 9, 2023. doi:10.6028/NIST.FIPS.197-upd1.
- [6] Neubaer, A., Freudenberg, J., & Kuhn, V. (2007) *Coding Theory, Algorithms, Architectures, and Applications*. John Wiley and Sons Ltd, Chichester, West Sussex, England:1-93.
- [7] Sarma, D. (2018) Implementation of Galois Field for Application in Wireless Communication Channels. *MATEC Web of Conferences*, 2010:03012.

# Seminars and Awards

