Probability Review and Naïve Bayes

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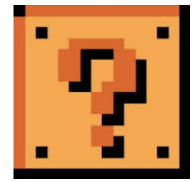
Some slides adapted from Dan Jurfasky and Brendan O'connor

What is Probability?

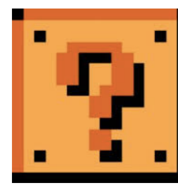
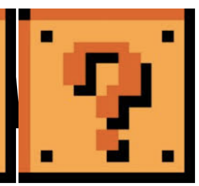
- "The probability the coin will land heads is 0.5"
  - Q: what does this mean?
- 2 Interpretations:
  - Frequentist (Repeated trials)
    - If we flip the coin many times…
  - Bayesian
    - We believe there is equal chance of heads/tails
    - Advantage: events that do not have long term frequencies

Q: What is the probability the polar ice caps will melt by 2050?

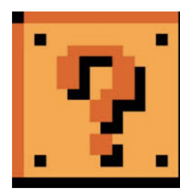
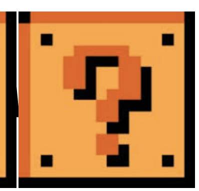
Probability Review

$$\sum_x P(X = x) =$$ 

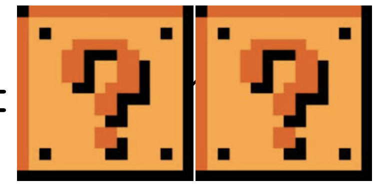
Conditional Probability

$$\frac{P(A, B)}{P(B)} =$$  

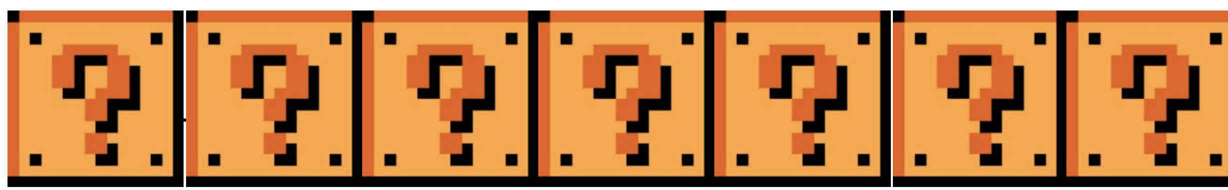
Chain Rule

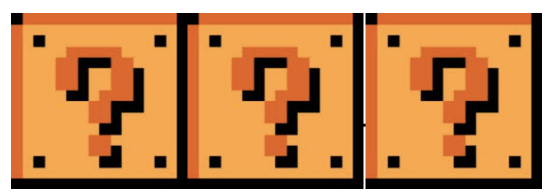
$$P(A|B)P(B) =$$  

Probability Review

$$\sum_x P(X = x, Y) =$$ 

Disjunction / Union:

$$P(A \vee B) =$$ 

Negation: $$P(\neg A) =$$ 

Bayes Rule

Hypothesis (Unknown)

Data (Observed Evidence)

$$P(D|H)$$

Generative Model of How Hypothesis Causes Data

$$H \quad D$$

Bayesian Inferece

$$P(H|D)$$

Bayes Rule tells us how to flip the conditional
Reason about effects to causes
Useful if you assume a generative model for your data

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

Bayes Rule

Bayes Rule tells us how to flip the conditional
Reason about effects to causes
Useful if you assume a generative model for your data

Likelihood

Prior

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

Posterior

Normalizer

Bayes Rule

Bayes Rule tells us how to flip the conditional
Reason about effects to causes
Useful if you assume a generative model for your data

Likelihood

Prior

$$P(H|D) = \frac{P(D|H)P(H)}{\sum_h P(D|H)P(H)}$$

Posterior

Normalizer

Bayes Rule

Bayes Rule tells us how to flip the conditional
Reason about effects to causes
Useful if you assume a generative model for your data

Likelihood

Prior

$$P(H|D) \propto P(D|H)P(H)$$

Posterior

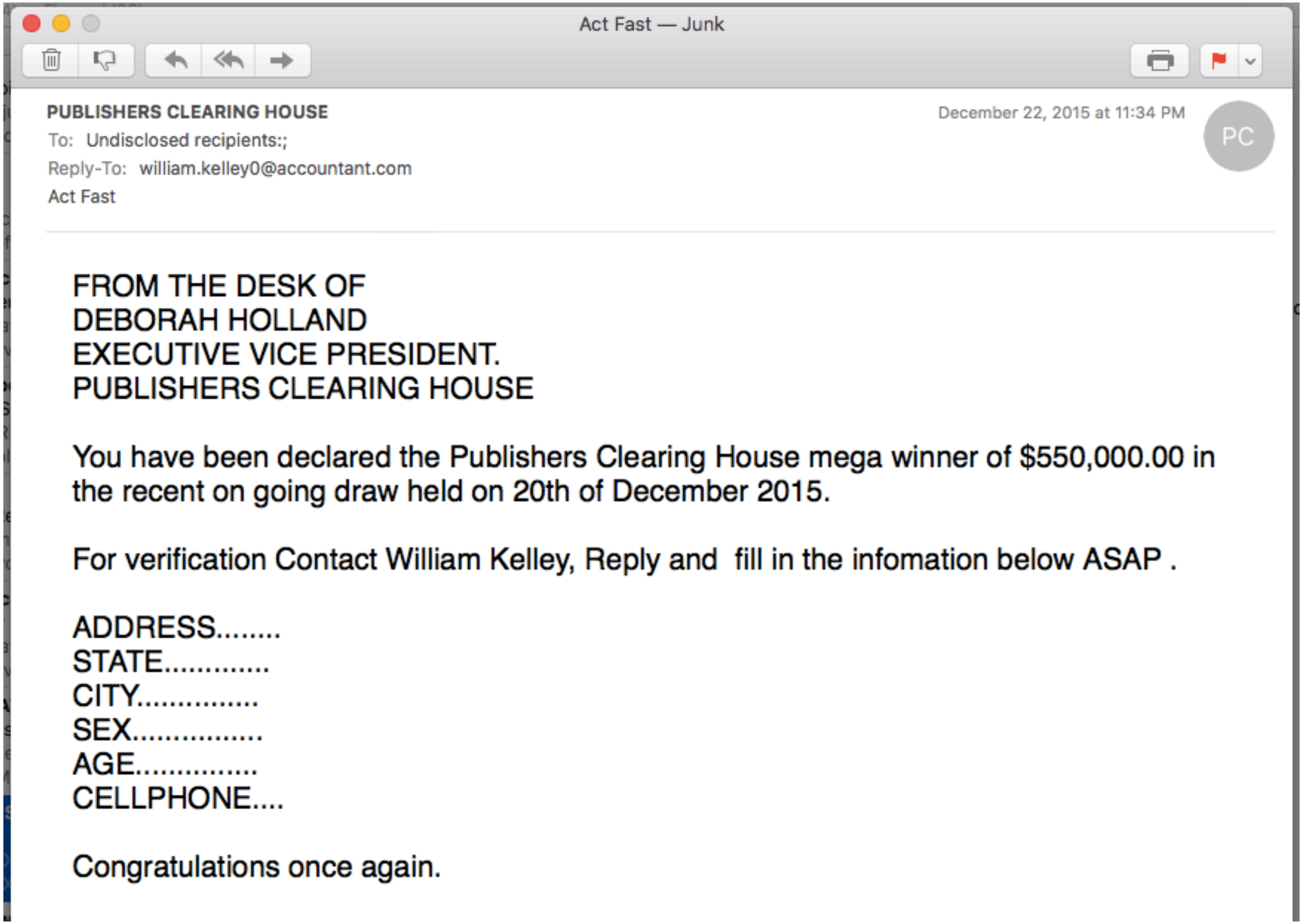
Proportional To
(Doesn't sum to 1)

Bayes Rule Example

- There is a disease that affects a tiny fraction of the population (0.01%)
- Symptoms include a headache and stiff neck
  - 99% of patients with the disease have these symptoms
- 1% of the general population has these symptoms
- Q: assume you have the symptom, what is your probability of having the disease?

Text Classification

Is this Spam?



Act Fast — Junk

PUBLISHERS CLEARING HOUSE

December 22, 2015 at 11:34 PM

PC

To: Undisclosed recipients:;

Reply-To: william.kelley0@accountant.com

Act Fast

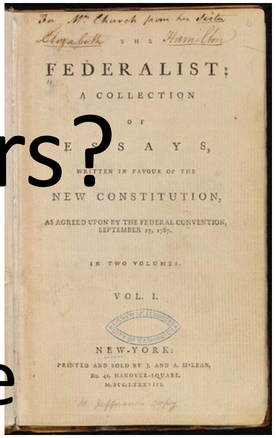
FROM THE DESK OF
DEBORAH HOLLAND
EXECUTIVE VICE PRESIDENT.
PUBLISHERS CLEARING HOUSE

You have been declared the Publishers Clearing House mega winner of $550,000.00 in the recent on going draw held on 20th of December 2015.

For verification Contact William Kelley, Reply and fill in the infomation below ASAP .

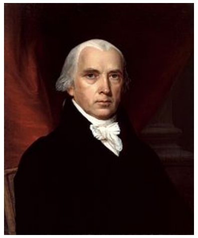
ADDRESS........
STATE.............
CITY...............
SEX................
AGE...............
CELLPHONE....

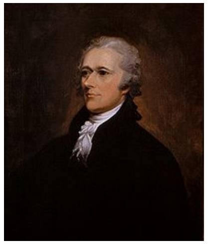
Congratulations once again.



Who wrote which Federalist papers?

- 1787-8: anonymous essays try to convince New York to ratify U.S Constitution: Jay, Madison, Hamilton.
- Authorship of 12 of the letters in dispute
- 1963: solved by Mosteller and Wallace using Bayesian methods


James Madison


Alexander Hamilton

What is the subject of this article?

MEDLINE Article


Brain and Cognition

Syntactic frame and verb bias in aphasia: Plausibility judgments of undergoer-subject sentences

MeSH Subject Category Hierarchy

- Antogonists and Inhibitors
- Blood Supply
- Chemistry
- Drug Therapy
- Embryology
- Epidemiology
- …

Positive or negative movie review?

-  unbelievably disappointing
-  Full of zany characters and richly applied satire, and some great plot twists
-  this is the greatest screwball comedy ever filmed
-  It was pathetic. The worst part about it was the boxing scenes.

Text Classification: definition

- *Input*:
  - a document d
  - a fixed set of classes $C = \{c_1, c_2, \ldots, c_J\}$

- *Output*: a predicted class $c \in C$

Classification Methods: Hand-coded rules

- Rules based on combinations of words or other features
  - spam: black-list-address OR (“dollars” AND “have been selected”)
- Accuracy can be high
  - If rules carefully refined by expert
- But building and maintaining these rules is expensive

Classification Methods: Supervised Machine Learning

- *Input:*
  - a document d
  - a fixed set of classes $C = \{c_1, c_2,\ldots, c_J\}$
  - A training set of m hand-labeled documents $(d_1,c_1),\ldots,(d_m,c_m)$
- *Output:*
  - a learned classifier $\gamma: d \rightarrow c$

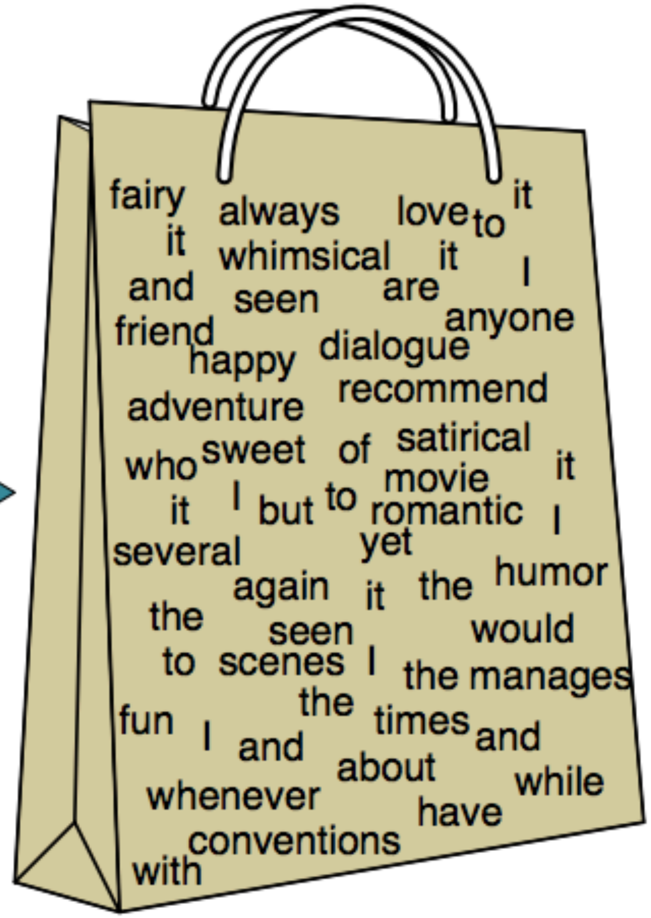
Classification Methods: Supervised Machine Learning

- Any kind of classifier
  - Naïve Bayes
  - Logistic regression
  - Support-vector machines
  - k-Nearest Neighbors
  - …

Naïve Bayes Intuition

- Simple (“naïve”) classification method based on Bayes rule
- Relies on very simple representation of document
  - Bag of words

I love this movie! It's sweet, but with satirical humor. The dialogue is great and the adventure scenes are fun... It manages to be whimsical and romantic while laughing at the conventions of the fairy tale genre. I would recommend it to just about anyone. I've seen it several times, and I'm always happy to see it again whenever I have a friend who hasn't seen it yet!



it	6
I	5
the	4
to	3
and	3
seen	2
yet	1
would	1
whimsical	1
times	1
sweet	1
satirical	1
adventure	1
genre	1
fairy	1
humor	1
have	1
great	1
…	…

Bag of words for document classification

?

Test document

parser
language
label
translation
…

Machine Learning	NLP	Garbage Collection	Planning	GUI
learning	parser	garbage	planning	…
training	tag	collection	temporal	
algorithm	training	memory	reasoning	
shrinkage	translation	optimization	plan	
network…	language…	region…	language…	

Bayes' Rule Applied to Documents and Classes

- For a document *d* and a class *c*

$$P(c \mid d) = \frac{P(d \mid c)P(c)}{P(d)}$$

Naïve Bayes Classifier (I)

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}}\, P(c \mid d)$$

MAP is "maximum a posteriori" = most likely class

$$= \underset{c \in C}{\operatorname{argmax}}\, \frac{P(d \mid c)P(c)}{P(d)}$$

Bayes Rule

$$= \underset{c \in C}{\operatorname{argmax}}\, P(d \mid c)P(c)$$

Dropping the denominator

Naïve Bayes Classifier (II)

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}} P(d \mid c) P(c)$$

$$= \underset{c \in C}{\operatorname{argmax}} P(x_1, x_2, \ldots, x_n \mid c) P(c)$$

Naïve Bayes Classifier (IV)

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}} P(x_1, x_2, \ldots, x_n \mid c) P(c)$$

O(|*X*|^n•|*C*|) parameters

Could only be estimated if a very, very large number of training examples was available.

How often does this class occur?

We can just count the relative frequencies in a corpus

Multinomial Naïve Bayes Independence Assumptions

$$P(x_1, x_2, \ldots, x_n \mid c)$$

- **Bag of Words assumption**: Assume position doesn't matter
- **Conditional Independence**: Assume the feature probabilities $P(x_i \mid c_j)$ are independent given the class c.

Multinomial Naïve Bayes Classifier

$$c_{MAP} = \underset{c \in C}{\operatorname{argmax}} P(x_1, x_2, \ldots, x_n \mid c) P(c)$$

$$c_{NB} = \underset{c \in C}{\operatorname{argmax}} P(c_j) \prod_{x \in X} P(x \mid c)$$

Applying Multinomial Naive Bayes Classifiers to Text Classification

$\text{positions} \leftarrow$ all word positions in test document

$$c_{NB} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j) \prod_{i \in positions} P(x_i \mid c_j)$$

Learning the Multinomial Naïve Bayes Model

- First attempt: maximum likelihood estimates
  - simply use the frequencies in the data

$$\hat{P}(c_j) = \frac{doccount(C = c_j)}{N_{doc}}$$

$$\hat{P}(w_i \mid c_j) = \frac{count(w_i, c_j)}{\sum_{w \in V} count(w, c_j)}$$

Parameter estimation

$$\hat{P}(w_i \mid c_j) = \frac{count(w_i, c_j)}{\sum_{w \in V} count(w, c_j)}$$

fraction of times word w_i appears among all words in documents of topic c_j

- Create mega-document for topic *j* by concatenating all docs in this topic
  - Use frequency of *w* in mega-document

Problem with Maximum Likelihood

- What if we have seen no training documents with the word ***fantastic*** and classified in the topic **positive** (***thumbs-up)***?

- Zero probabilities cannot be conditioned away, no matter the other evidence!

$$c_{MAP} = \operatorname{argmax}_c \hat{P}(c)\prod_i \hat{P}(x_i \mid c)$$

Laplace (add-1) smoothing for Naïve Bayes

$$\hat{P}(w_i \mid c) = \frac{count(w_i, c)}{\sum_{w \in V} \left(count(w, c)\right)}$$

$$= \frac{count(w_i, c) + 1}{\left(\sum_{w \in V} count(w, c)\right) + |V|}$$

Multinomial Naïve Bayes: Learning

- Calculate $P(c_j)$ terms
  - For each c_j in C do

    $docs_j \leftarrow$ all docs with class $= c_j$

$$P(c_j) \leftarrow \frac{|docs_j|}{|\text{total \# documents}|}$$

Multinomial Naïve Bayes: Learning

From training corpus, extract $Vocabulary$

Calculate $P(w_k \mid c_j)$ terms

- $Text_j \leftarrow$ single doc containing all $docs_j$
- For each word w_k in $Vocabulary$

  $n_k \leftarrow$ # of occurrences of w_k in $Text_j$

$$P(w_k \mid c_j) \leftarrow \frac{n_k + \alpha}{n + \alpha |Vocabulary|}$$

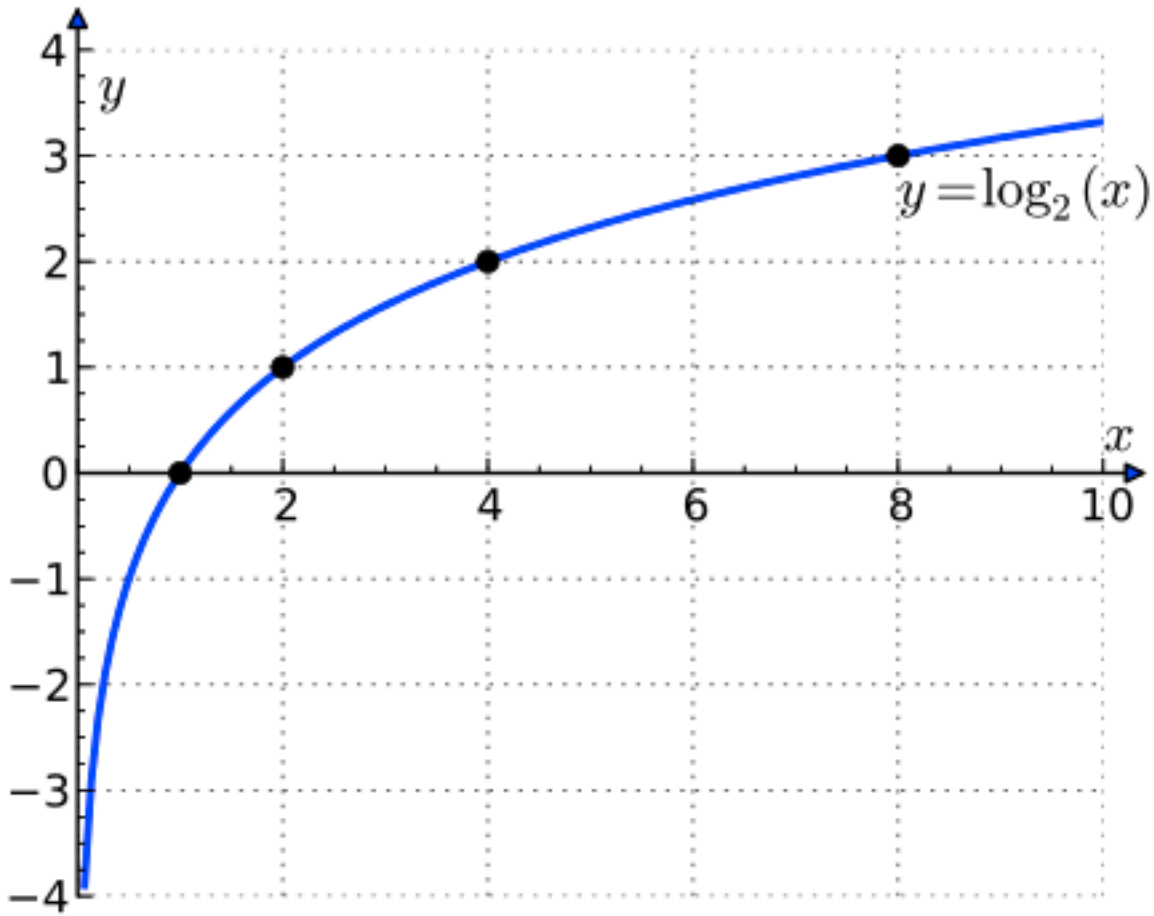
Exercise

Naïve Bayes Classification: Practical Issues

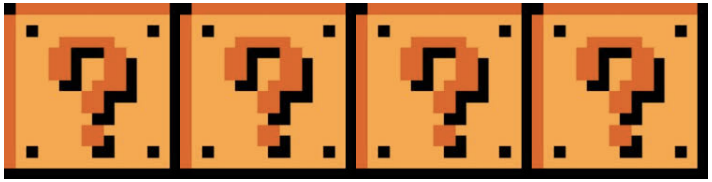
$$
\begin{aligned}
c_{MAP} &= \text{argmax}_c P(c|x_1, \ldots, x_n) \\
&= \text{argmax}_c P(x_1, \ldots, x_n|c)P(c) \\
&= \text{argmax}_c P(c) \prod_{i=1}^{n} P(x_i|c)
\end{aligned}
$$

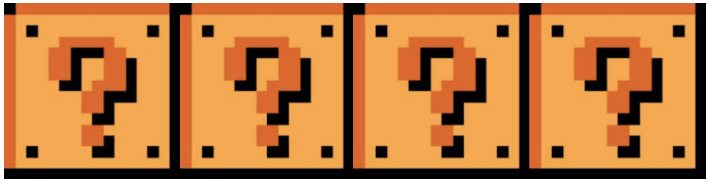
- Multiplying together lots of probabilities
- Probabilities are numbers between 0 and 1
- Q: What could go wrong here?

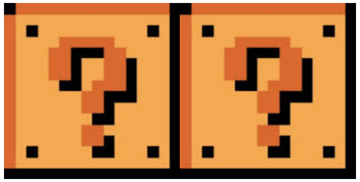
Working with probabilities in log space



Log Identities (review)

$$\log(a \times b) =$$ 

$$\log(\frac{a}{b}) =$$ 

$$\log(a^n) =$$ 

Naïve Bayes with Log Probabilities

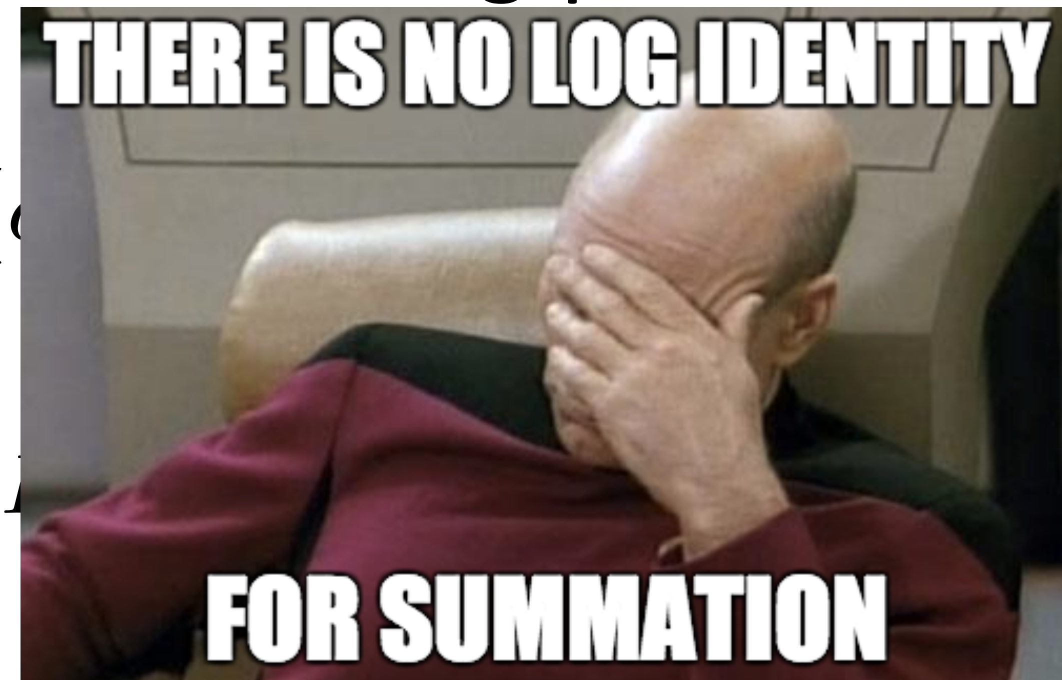
$$
\begin{aligned}
c_{MAP} &= \operatorname{argmax}_c P(c|x_1, \ldots, x_n) \\
&= \operatorname{argmax}_c P(c) \prod_{i=1}^{n} P(x_i|c) \\
&= \operatorname{argmax}_c \log \left(P(c) \prod_{i=1}^{n} P(x_i|c) \right) \\
&= \operatorname{argmax}_c \log P(c) + \sum_{i=1}^{n} \log P(x_i|c)
\end{aligned}
$$

Naïve Bayes with Log Probabilities

$$c_{MAP} = \mathrm{argmax}_c \log P(c) + \sum_{i=1}^{n} \log P(x_i|c)$$

- Q: Why don't we have to worry about floating point underflow anymore?

What if we want to calculate posterior log-probabilities?



$$P(\ldots \prod_{i=1}^{n} P(x_i|c)\Big/ \ldots \prod_{i=1}^{n} P(x_i|c')$$

$$\log \ldots \prod_{i=1}^{n} P(x_i|c)\Big/ \ldots c') \prod_{i=1}^{n} P(x_i|c')$$

$$= \log P(c) + \sum_{i=1} P(x_i|c) - \log \left[\sum_{c'} P(c') \prod_{i=1}^{n} P(x_i|c') \right]$$

Log Exp Sum Trick: motivation

- We have: a bunch of log probabilities.
  - log(p1), log(p2), log(p3), … log(pn)
- We want: log(p1 + p2 + p3 + … pn)
- We could convert back from log space, sum then take the log.
  - If the probabilities are very small, this will result in floating point underflow

Log Exp Sum Trick:

$$\log[\sum_i \exp(x_i)] = x_{max} + \log[\sum_i \exp(x_i - x_{max})]$$