Ve have not given the formulas for the inverse transforms. They are obtained s above by reversing the arrows and changing the signs.

We give one further example of a family of three transforms. We have aken as an example transforms that start with an update step and a preiction step, which are common to all three. Again, at the end there is a ormalization step.

$$s_{j-1}^{(1)}[n] = s_j[2n] - \frac{1}{3}s_j[2n-1] \tag{3.42}$$

$$d_{j-1}^{(1)}[n] = s_j[2n+1] - \frac{1}{8}(9s_{j-1}^{(1)}[n] + 3s_{j-1}^{(1)}[n+1]) \tag{3.43}$$

$$\text{CDF(3,1)} \qquad s_{j-1}^{(2)}[n] = s_{j-1}^{(1)}[n] + \frac{4}{9}d_{j-1}^{(1)}[n] \tag{3.44}$$

$$\begin{aligned}\text{CDF(3,3)} \qquad s_{j-1}^{(2)}[n] = s_{j-1}^{(1)}[n] + \frac{1}{36}(3d_{j-1}^{(1)}[n-1] \\ + 16d_{j-1}^{(1)}[n] - 3d_{j-1}^{(1)}[n+1])\end{aligned} \tag{3.45}$$

$$\begin{aligned}\text{CDF(3,5)} \qquad s_{j-1}^{(2)}[n] = s_{j-1}^{(1)}[n] - \frac{1}{288}(5d_{j-1}^{(1)}[n-2] - 34d_{j-1}^{(1)}[n-1] \\ - 128d_{j-1}^{(1)}[n] + 34d_{j-1}^{(1)}[n+1] \\ - 5d_{j-1}^{(1)}[n+2])\end{aligned} \tag{3.46}$$

$$d_{j-1}[n] = \frac{\sqrt{2}}{3}d_{j-1}^{(1)}[n] \tag{3.47}$$

$$s_{j-1}[n] = \frac{3}{\sqrt{2}}s_{j-1}^{(2)}[n]\,. \tag{3.48}$$

The above formulas for the CDF(2,x) and CDF(3,x) families have been taken rom the technical report [27]. Further examples can be found there.

Exercises

.1 Verify that the CDF(2,2) transform, defined in (3.13) and (3.14), preerves the first moment, i.e. verify that (3.15) holds.

4. Analysis of Synthetic Signals

The discrete wavelet transform has been introduced in the previous two chapters. The general lifting scheme, as well as some examples of transforms, were presented, and we have seen one application to a signal with just 8 samples. In this chapter we will apply the transform to a number of synthetic signals, in order to gain some experience with the properties of the discrete wavelet transform. We will process some signals by transformation, followed by some alteration, followed by inverse transformation, as we did in Chap. 2 to the signal with 8 samples. Here we use significantly longer signals. As an example, we will show how this approach can be used to remove some of the noise in a signal. We will also give an example showing how to separate slow and fast variations in a signal.

The computations in this chapter have been performed using MATLAB. We have used the toolbox *Uvi_Wave* to perform the computations. See Chap. 14 for further information on software, and Chap. 13 for an introduction to MATLAB and *Uvi_Wave*. At the end of the chapter we give some exercises, which one should try after having read Sect. 13.1.

4.1 The Haar Transform

Our first examples are based on the Haar transform. The one scale direct Haar transform is given by equations (3.28)–(3.31), and its inverse by equations (3.32)–(3.35). We start with a very simple signal, given as a continuous signal by the sine function. More precisely, we take the function $\sin(4\pi t)$, with $0 \leq t \leq 1$. We now sample this signal at 512 equidistant points in $0 \leq t \leq 1$. This gives us a discrete signal $\mathbf{s}_9$. The index 9 comes from the exponent $512 = 2^9$, as we described in Chap. 3. This signal is plotted in Fig. 4.1. We label the entries on the horizontal axis by sample index. Note that due to the density of the sampling the graph looks like the graph of the continuous function.

We want to perform a wavelet transform of this discrete signal. We choose to do this over three scales. If we order the entries in the transformed signal as in Table 2.1, then we get the result shown in Fig. 4.2. The ordering of the entries is $\mathbf{s}_6$, $\mathbf{d}_6$, $\mathbf{d}_7$, $\mathbf{d}_8$. At each index point we have plotted a vertical line of length equal to the value of the coefficient. It is not

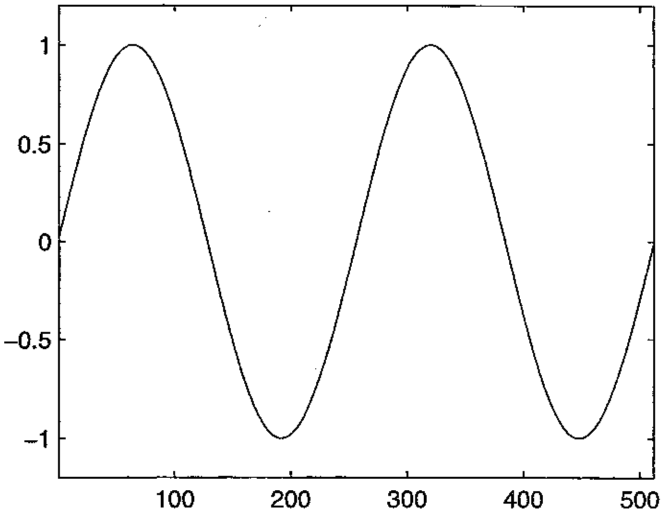

Fig. 4.1. The signal $\sin(4\pi t)$, 512 samples

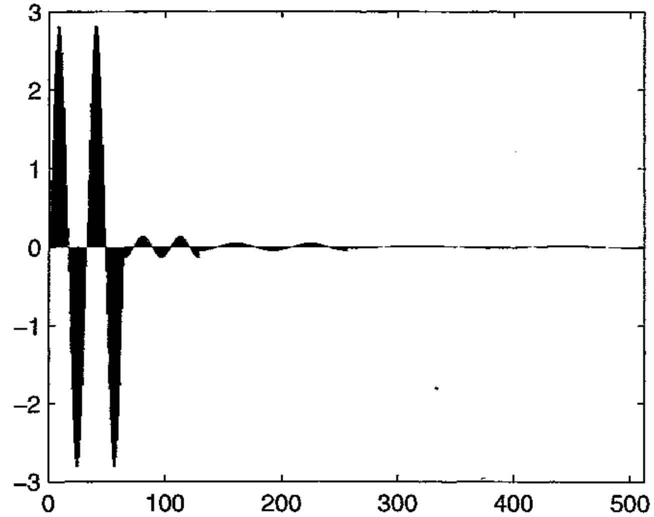

Fig. 4.2. The wavelet coefficients from the DWT of the signal in Fig. 4.1, using the Haar transform

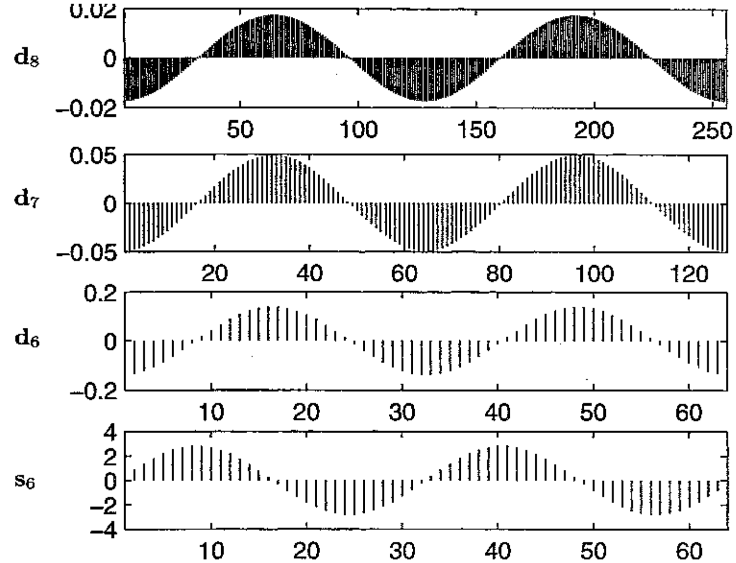

Fig. 4.3. The wavelet coefficients from Fig. 4.2 divided into scales, from the DWT of the signal in Fig. 4.1

one should interpret this graph. In Fig. 4.3 we have plotted the four parts separately. The top plot is of $\mathbf{d}_8$, followed by $\mathbf{d}_7$, $\mathbf{d}_6$, and $\mathbf{s}_6$. Note that each plot has its own axes, with different units. Again, these plots are not easy to interpret. We try a third approach.

We take the transformed signal, $\mathbf{s}_6$, $\mathbf{d}_6$, $\mathbf{d}_7$, $\mathbf{d}_8$, and then replace all entries except one with zeroes, i.e. sequences of the appropriate length consisting entirely of zeroes. For example, we can take $\mathbf{0}_6, \mathbf{d}_6, \mathbf{0}_7, \mathbf{0}_8$, where $\mathbf{0}_8$ is a signal of length $256 = 2^8$ with zero entries. We then invert this signal using the inverse three scale discrete wavelet transform based on (3.32)–(3.35). Schematically, it looks like

$$
\begin{array}{rcl}
W_{\mathrm{a}}^{(3)} : \mathbf{s}_9 & \to & \underbrace{\mathbf{s}_6, \mathbf{d}_6, \mathbf{d}_7, \mathbf{d}_8} \\
 & & \downarrow \\
 & W_{\mathrm{s}}^{(3)} : & \overbrace{\mathbf{0}_6, \mathbf{d}_6, \mathbf{0}_7, \mathbf{0}_8} \quad \to \quad \mathbf{s}_9'
\end{array}
$$

The result $\mathbf{s}_9'$ of this inversion is a signal with the property that if it were transformed with $W_{\mathrm{a}}^{(3)}$, the result would be precisely the signal $\mathbf{0}_6, \mathbf{d}_6, \mathbf{0}_7, \mathbf{0}_8$. Hence $\mathbf{s}_9'$ contains all information on the coefficients on the third scale. The four possible plots are given in Fig. 4.4. The top plot is the inversion of $\mathbf{0}_6, \mathbf{0}_6, \mathbf{0}_7, \mathbf{d}_8$ followed by $\mathbf{0}_6, \mathbf{0}_6, \mathbf{d}_7, \mathbf{0}_8$, and $\mathbf{0}_6, \mathbf{d}_6, \mathbf{0}_7, \mathbf{0}_8$, and finally at the bottom $\mathbf{s}_6, \mathbf{0}_6, \mathbf{0}_7, \mathbf{0}_8$. This representation, where the contributions are separated as described, is called the *multiresolution* representation of the signal

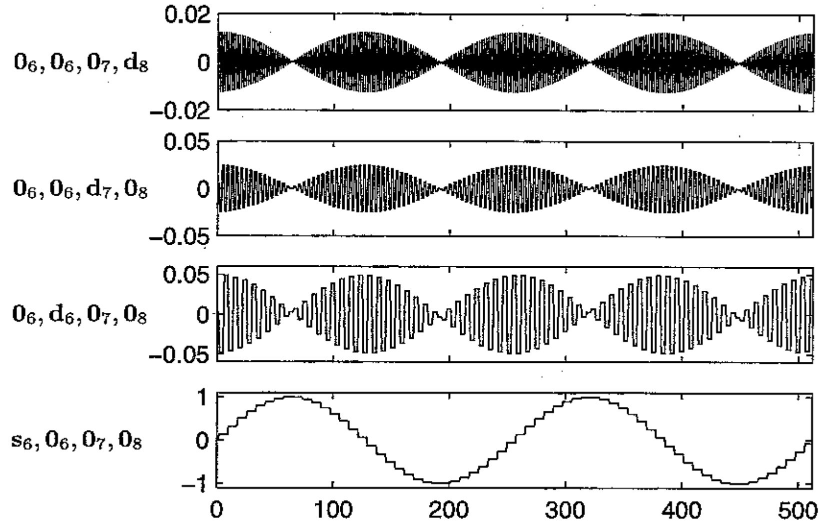

Fig. 4.4. DWT of the signal in Fig. 4.1, Haar transform, multiresolution representation, separate plots

(in this case over three scales). The plots in Fig. 4.4 correspond to our intuition associated with repeated means and differences. The bottom plot in Fig. 4.4 could also have been obtained by computing the mean of 8 successive samples, and then replacing each of these 8 samples by their mean value. Thus each mean value is repeated 8 times in the plot.

If we invert the transform, we will get back the original signal. In Fig. 4.5 we have plotted the inverted signal, and the difference between this signal and the original signal. We see that the differences are of magnitude 10^{-15}, corresponding to the precision of the MATLAB calculations.

We have now presented a way of visualizing the effect of a DWT over a finite number of scales. We will then perform some experiments with synthetic signals. As a first example we add an impulse to our sine signal. We change the value at sample number 200 to the value 2. We have plotted the three scale representation in Fig. 4.6. We see that the impulse can be localized in the component $\mathbf{d}_8$, and in the averaged signal $\mathbf{s}_6$ the impulse has almost disappeared. Here we have used the very simple Haar transform. By using other transforms one can get better results.

Let us now show how we can reduce noise in a signal by processing it in the DWT representation. We take again the sine signal plus an impulse, and add some noise. The signal is given in Fig. 4.7. The multiresolution representation is given in Fig. 4.8 The objective now is to remove the noise from the signals. We will try to do this by processing the signal as follows. In the transformed representation, $\mathbf{s}_6, \mathbf{d}_6, \mathbf{d}_7, \mathbf{d}_8$, we leave unchanged the largest

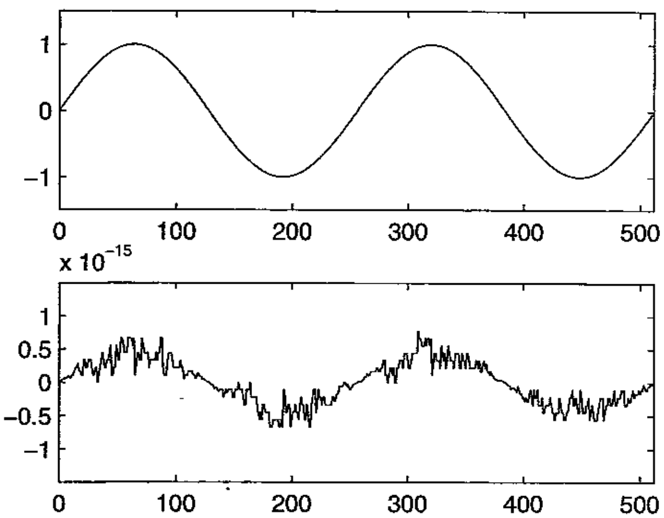

Fig. 4.5. Top: Inverse DWT of the signal in Fig. 4.1. Bottom: Difference between inverted and original signal

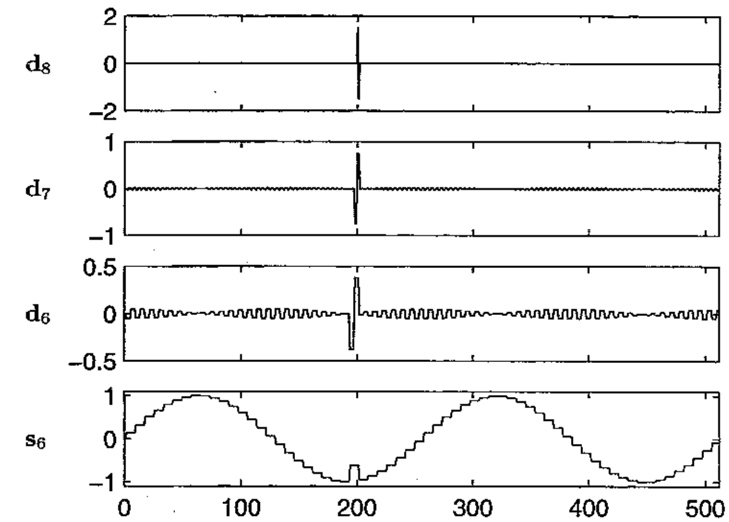

Fig. 4.6. Multiresolution representation of sine plus impulse at 200, Haar transform

10% of the coefficients, and change the remaining 90% to zero. We then apply the inverse transform to this altered signal. The result is shown in Fig. 4.9. We see that it is possible to recognize both the impulse and the sine signal, but the sine signal has undergone considerable changes. The next section

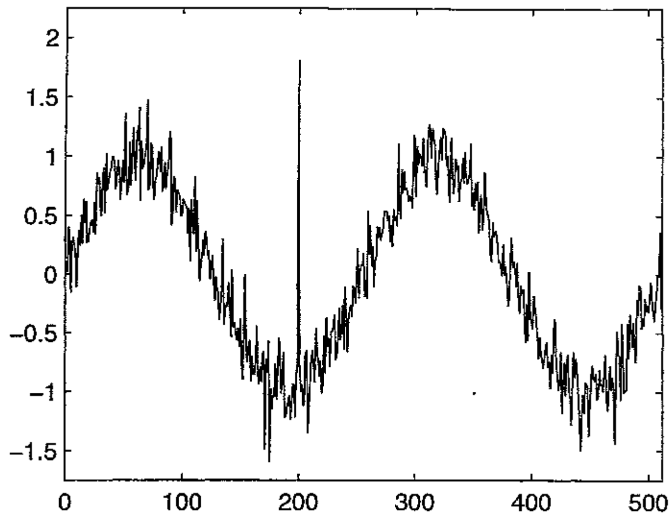

Fig. 4.7. Sine plus impulse at 200 plus noise

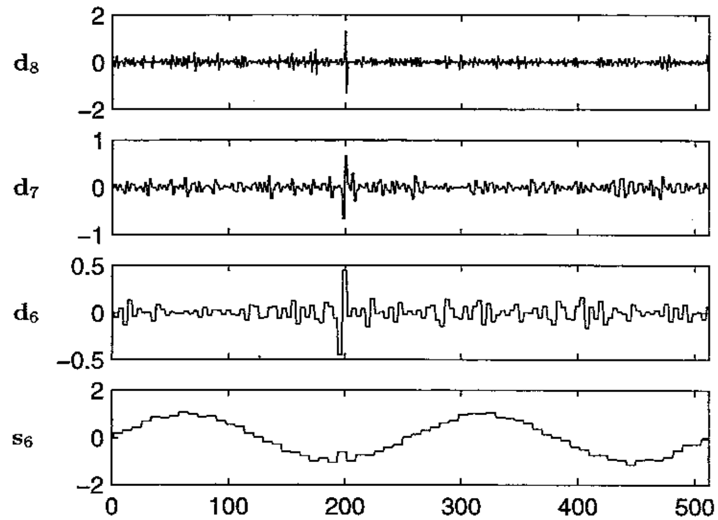

Fig. 4.8. Sine plus impulse at 200 plus noise, Haar transform, multiresolution representation

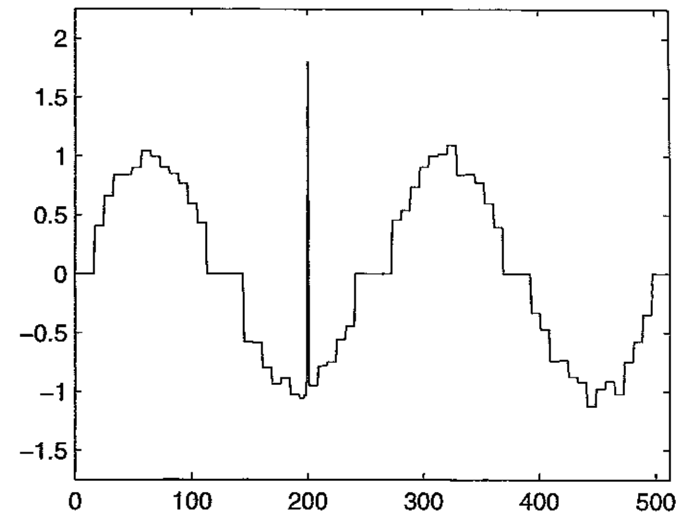

Fig. 4.9. Sine plus impulse at 200 plus noise, reconstruction based on the largest 10% of the coefficients, Haar transform

shows that these results can be improved by choosing a more complicated transform.

4.2 The CDF(2,2) Transform

We will now perform experiments with the DWT based on the building block CDF(2,2), as it was defined in Sect. 3.3. We will continue with the noise reduction example from the previous section. In Fig. 4.10 we have given the multiresolution representation, using the new building block for the DWT. In Fig. 4.11 we have shown reconstruction based on the 15% and the 10% largest coefficients in the transformed signal. The result is much better than the one obtained using the Haar transform. Let us note that there exists an extensive theory on noise removal, including very sophisticated applications of the DWT, but it is beyond the scope of this book.

As a second example we show how to separate fast and slow variations in a signal. We take the function

$$\log(2+\sin(3\pi\sqrt{t})), \qquad 0 \le t \le 1\,, \tag{4.1}$$

and sample its values in 1024 points, at $1/1024$, $2/1024$, ..., $1024/1024$. Then we change the values at $1/1024$, $33/1024$, $65/1024$, etc. by adding 2 to the computed values. This signal has been plotted in Fig. 4.12. We will now try to separate the slow variation and the sharp peaks in the signal. We take

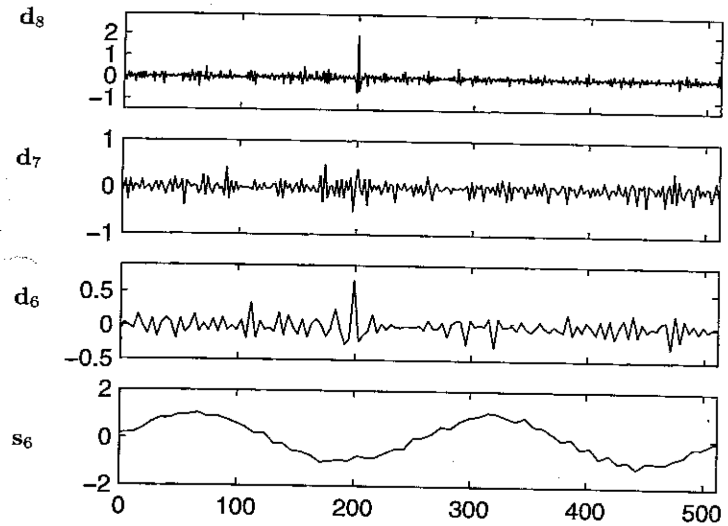

Fig. 4.10. Sine plus impulse at 200 plus noise, multiresolution representation, CDF(2,2), three scales

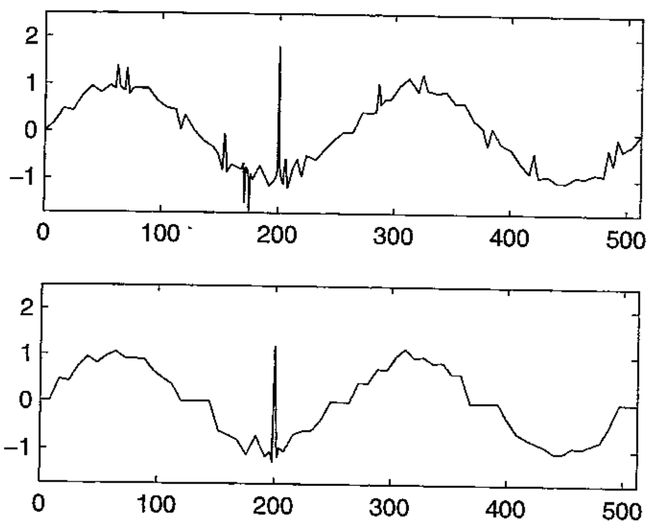

Fig. 4.11. Sine plus impulse at 200 plus noise, CDF(2,2), three scales, top reconstruction based on 15% largest coefficients, bottom based on 10% largest coefficients

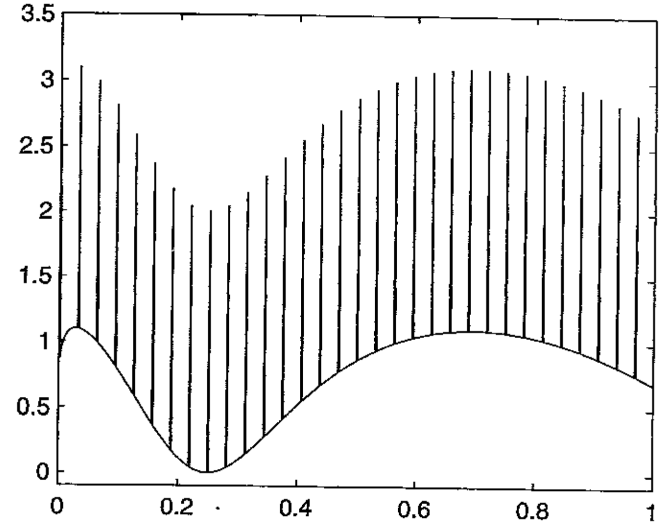

Fig. 4.12. Plot of the function $\log(2+\sin(3\pi\sqrt{t}))$ plus 2 at points 1/1024, 33/1024, 65/1024, etc.

a multiresolution representation over six scales, as shown in Fig. 4.13. We see from the bottom graph in the figure that we have succeeded in removing the sharp peaks in that part of the representation. In Fig. 4.14 we have plotted this part separately. Except close to the end points of the interval, this is the slow variation. We subtract this part from the original signal and obtain Fig. 4.15. In these two figures we have used the variable t on the horizontal axis. Figure 4.15 shows that except for problems at the edges we have succeeded in isolating the rapid variations, without broadening the sharp peaks in the rapidly varying part. This example is not only of theoretical interest, but can also be applied to for example ECG signals.

Exercises

All exercises below require access to MATLAB and *Uvi_Wave* (or some other wavelet toolbox), and some knowledge of their use. You should read Sect. 13.1 before trying to solve these exercises.

4.1 Go through the examples in this chapter, using MATLAB and *Uvi_Wave*.

4.2 Carry out experiments on the computer with noise reduction. Vary the number of coefficients retained, and plot the different reconstructions. Discuss the results.

4.3 Find the multiresolution representation of a chirp (i.e. a signal obtained from sampling $\sin(t^2)$).

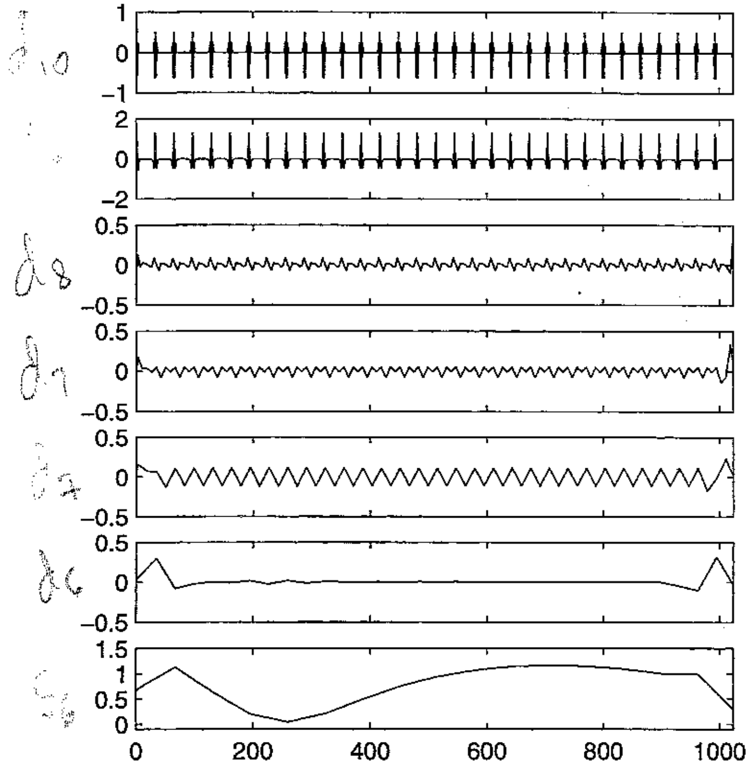

Fig. 4.13. Multiresolution representation of the signal from Fig. 4.12, six scales, CDF(2,2)

4.4 Find the multiresolution representation of a signal obtained by sampling the function

$$f(t) = \begin{cases} \sin(4\pi t) & \text{for } 0 \le t < \frac{1}{4}\,, \\ 1 + \sin(4\pi t) & \text{for } \frac{1}{4} \le t < \frac{3}{4}\,, \\ \sin(4\pi t) & \text{for } \frac{3}{4} \le t \le 1\,. \end{cases}$$

Add noise, and try out noise removal, using both the Haar transform, and the CDF(2,2) transform.

4.5 (For readers with sufficient background in signal analysis.) Try to separate the low and high frequencies in the signal in Fig. 4.12 by a low pass filtering (use for example a low order Butterworth filter). Compare the result to Fig. 4.14. Subtract the low pass filtered signal from the original and compare the result to Fig. 4.15.

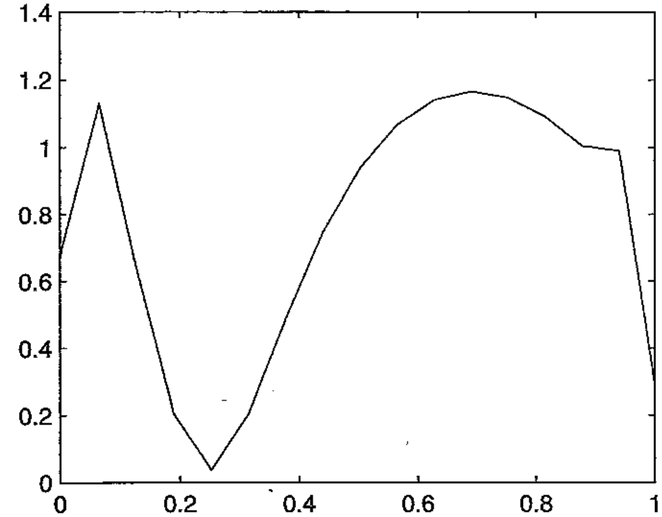

Fig. 4.14. Bottom graph in Fig. 4.13, fitted to $0 \le t \le 1$

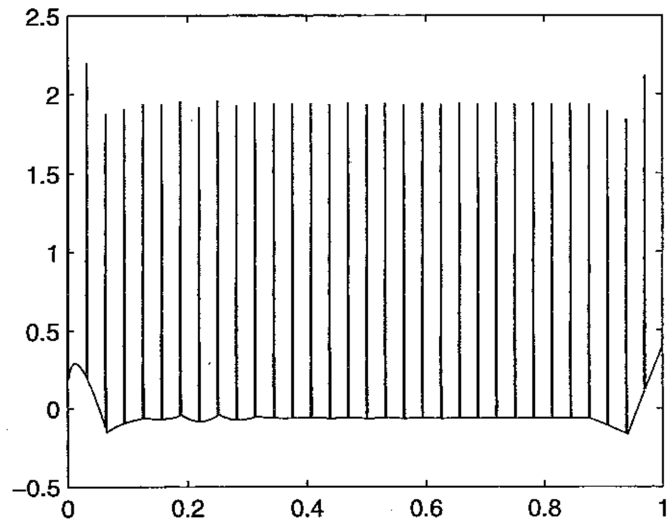

Fig. 4.15. Signal from Fig. 4.12, with slow variations removed