ENGINEERING MATHEMATICS-II

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Today's Target

Important Questions

IMPORTANT TOPIC

UNIT 1

Unit 1

- ❖Method of variation of parameters.
- ❖Method of Changing the independent Variable.

Unit 1 (Two Marks)

1. Find the degree and order of the following differential equation $\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0.$

2. Find the roots of Auxiliary equation of the differential equation $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}$

3. Find P.I. of $\frac{d^2y}{dt^2} + 4y = \text{Sin } 2x$

4. Solve $\left(D^6 - D^4\right)y = 0$

5. Solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 4y = 0$

6. Solve $\left(\mathrm{D}^3 + 2\mathrm{D}^2 - 3\mathrm{D}\right)y = e^x$

7. Solve $\left(D^2 + 1\right)^2(\mathrm{D} - 1)y = 0$

8. Solve $(2D - 3)^3 y = 0$ where $\mathrm{D} = \frac{d}{dx}$

Unit 1 (Two Marks)

9. Find the general solution of $(2D+1)^2 y = 0$ where $D = \frac{d}{dt}$

10. Solve Simultaneous equation $\frac{dx}{dt} = 3y$, $\frac{dy}{dt} = 3x$

11. Find P.I. of $\frac{d^2y}{dx^2} = x^2 + 2x - 1$.

12. Solve P.I. of $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$ with the condition $x = 0, y = 5$ & $x = 2, y = 21$ and find the value of y at $x = 1$.

13. Find the differential equations which represent the family of straight line passing through the origin.

14. Solve of $(D+1)^3 y = 2e^{-x}$.

Unit 1 (Two Marks)

15. Calculate the order and degree of the differential equations $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

16. Solve $\frac{d^4y}{dx^4} - w^4y = 0$.

Rules to find Particular Integral (P.I.)-e^{ax}

2. Solve $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = e^{ax} + e^{bx}$

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = e^{ax} + e^{bx}$$

$$[D^2 - (a+b)D + ab]y = e^{ax} + e^{bx}$$

A.E $\Rightarrow$ $m^2 - (a+b)m + ab = 0$

$$m^2 - am - bm + ab = 0$$

$$m(m-a) - b(m-a) = 0$$

$$(m-a)(m-b) = 0$$

$$\boxed{m = a, b}$$

$$C.F = C_1 e^{ax} + C_2 e^{bx}$$

$$P.I \Rightarrow \frac{1}{D^2 - (a+b)D + ab}(e^{ax} + e^{bx})$$

$$\Rightarrow P.I = \frac{1}{D^2 - (a+b)D + ab} e^{ax} + \frac{1}{D^2 - (a+b)D + ab} e^{bx}$$

[Here $f(a) = 0$ & $f(b) = 0$]
So Rules fails

$$\Rightarrow P.I. \Rightarrow \frac{x}{2D - (a+b)} e^{ax} + \frac{x}{2D - (a+b)} e^{bx}$$

$$\Rightarrow P.I \Rightarrow \frac{x}{2a - (a+b)} e^{ax} + \frac{x}{2b - (a+b)} e^{bx}$$

$$P.I = \frac{x}{a-b}\left[e^{ax} - e^{bx}\right]$$

Final Solution

$$y = C.F. + P.I.$$

$$y = C_1 e^{ax} + C_2 e^{bx} + \frac{x}{a-b}\left[e^{ax} - e^{bx}\right]$$

AS

Rules to find Particular Integral (P.I.)

2. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x$$

$$[D^2 + 5D - 6]y = \sin 3x + \cos 2x$$

A.E $\Rightarrow$ $m^2 + 5m - 6 = 0$

$$m^2 + 6m - m - 6 = 0$$

$$m(m+6) - 1(m+6) = 0$$

$$(m+6)(m-1) = 0$$

$$\boxed{m = 1, -6}$$

$$C.F = c_1 e^{x} + c_2 e^{-6x}$$

$$P.I = \frac{1}{D^2+5D-6}(\sin 3x + \cos 2x)$$

$$P.I = \frac{1}{D^2+5D-6}\sin 3x + \frac{1}{D^2+5D-6}\cos 2x$$

$$P.I = \frac{1}{-9+5D-6}\sin 3x + \frac{1}{-4+5D-6}\cos 2x$$

$$P.I = \frac{1}{5D-15}\sin 3x + \frac{1}{5D-10}\cos 2x$$

$$P.I = \frac{1}{5(D-3)}\sin 3x + \frac{1}{5(D-2)}\cos 2x$$

$$P.I = \frac{D+3}{5(D^2-9)}\sin 3x + \frac{D+2}{5(D^2-4)}\cos 2x$$

$$P.I = \frac{D+3}{5(-9-9)}\sin 3x + \frac{D+2}{5(-4-4)}\cos 2x$$

$$P.I = \frac{D+3}{-90}\sin 3x + \frac{D+2}{-40}\cos 2x$$

$$\Rightarrow -\frac{1}{90}[3\cos 3x + 3\sin 3x] - \frac{1}{40}[-2\sin 2x + 2\cos 2x]$$

$$P.I \Rightarrow \frac{-1}{30}[\cos 3x + \sin 3x] - \frac{1}{20}(\cos 2x - \sin 2x)$$

Final Solution

$$y = C.F + P.I.$$

$$y = c_1 e^{x} + c_2 e^{-6x} - \frac{1}{30}[\cos 3x + \sin 3x] - \frac{1}{20}[\cos 2x - \sin 2x]$$

Rules to find Particular Integral (P.I.)

2. Find the P.I. of $(D^2+5D+4)\,y = x^2+7x+9$

$$(D^2+5D+4)y = x^2+7x+9$$

$$P.I. = \frac{1}{(D^2+5D+4)}(x^2+7x+9)$$

$$P.I. = \frac{1}{4\left[1+\left(\frac{5}{4}D+\frac{D^2}{4}\right)\right]}(x^2+7x+9)$$

$$P.I. = \frac{1}{4}\left[1+\left(\frac{5}{4}D+\frac{D^2}{4}\right)\right]^{-1}(x^2+7x+9)$$

$$P.I = \frac{1}{4}\left[1-\left(\frac{5}{4}D+\frac{D^2}{4}\right)+\frac{(-1)(-1-1)}{\lfloor 2}\left(\frac{5}{4}D+\frac{D^2}{4}\right)^2+\cdots\right](x^2+7x+9)$$

$$P.I = \frac{1}{4}\left[1-\frac{5}{4}D-\frac{D^2}{4}+\frac{25}{16}D^2\right](x^2+7x+9)$$

$$P.I = \frac{1}{4}\left[1-\frac{5}{4}D+\frac{21}{16}D^2\right](x^2+7x+9)$$

$$P.I = \frac{1}{4}\left[(x^2+7x+9)-\frac{5}{4}(2x+7)+\frac{21}{16}\times 2\right]$$

$$P.I = \frac{1}{4}\left[x^2+7x+9-\frac{5x}{2}-\frac{35}{4}+\frac{21}{8}\right]$$

$$P.I = \frac{1}{4}\left[x^2+\frac{9x}{2}+\frac{23}{8}\right]$$

Rules to find Particular Integral (P.I.)

1. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x.e^{3x} + \sin 2x$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x\cdot e^{3x} + \sin 2x$$

$$(D^2 - 3D + 2)y = x\cdot e^{3x} + \sin 2x$$

$$A.E \Rightarrow m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$(m-2)(m-1) = 0$$

$$m = 1, 2$$

$$C.F = c_1 e^{x} + c_2 e^{2x}$$

$$P.I = \frac{1}{D^2 - 3D + 2}\left[x\cdot e^{3x} + \sin 2x\right]$$

$$P.I = \frac{1}{D^2 - 3D + 2} e^{3x}\cdot x + \frac{1}{D^2 - 3D + 2}\sin 2x$$

$$\Rightarrow e^{3x}\frac{1}{(D+3)^2 - 3(D+3) + 2}\cdot x + \frac{1}{-4 - 3D + 2}\sin 2x$$

$$\Rightarrow e^{3x}\frac{1}{D^2 + 9 + 6D - 3D - 9 + 2}x + \frac{1}{-3D - 2}\sin 2x$$

$$\Rightarrow e^{3x}\frac{1}{D^2 + 3D + 2}x - \frac{1}{3D + 2}\sin 2x$$

$$\Rightarrow e^{3x}\frac{1}{2\left[1 + \left(\frac{D^2 + 3D}{2}\right)\right]}x - \frac{3D - 2}{9D^2 - 4}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1 + \left(\frac{D^2 + 3D}{2}\right)\right]^{-1}x - \frac{(3D - 2)}{9(-4) - 4}\sin 2x$$

$$\Rightarrow \frac{e^{3x}}{2}\left[1 - \left(\frac{D^2 + 3D}{2}\right) + \frac{(-1)(-1-1)}{\lfloor 2}\left(\frac{D^2 + 3D}{2}\right)^2 + \cdots\right]x + \frac{3D - 2}{40}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1 - \frac{D^2}{2} - \frac{3D}{2} + \frac{9D^2}{4}\right]x + \frac{3D - 2}{40}\sin 2x$$

$$\Rightarrow \frac{e^{3x}}{2}\left[1 - \frac{3D}{2} + \frac{7D^2}{4}\right]x + \frac{3D - 2}{40}\sin 2x$$

$$\Rightarrow \frac{e^{3x}}{2}\left[x - \frac{3}{2}\right] + \frac{1}{40}\left[6\cos 2x - 2\sin 2x\right]$$

$$P.I = \frac{e^{3x}}{4}(2x - 3) + \frac{1}{40}\left[6\cos 2x - 2\sin 2x\right]$$

Final Solution $\Rightarrow y = C.F. + P.I.$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{4}(2x - 3) + \frac{1}{40}\left[6\cos 2x - 2\sin 2x\right]$$

Rules to find Particular Integral (P.I.)

1. Solve $(D^2+2D+1)\,y = x.\cos x$

$(D^2+2D+1)y = x\cdot\cos x$

A.E $\Rightarrow$ $m^2+2m+1=0$

$m^2+m+m+1=0$

$(m+1)(m+1)=0$

$\boxed{m=-1,-1}$

$C.F. = (c_1+c_2x)e^{-x}$

$P.I. \Rightarrow \dfrac{1}{(D^2+2D+1)}x\cos x$

$P.I = \dfrac{1}{(D+1)^2}x\cos x$

$P.I = \text{Real part of } e^{ix}\dfrac{1}{(D+i+1)^2}x$

$= \text{R.P. of } e^{ix}\dfrac{1}{D^2+2(1+i)D+(i+1)^2}x$

$\text{R.P of } e^{ix}\dfrac{1}{D^2+2(1+i)D+(i^2+1+2i)}x$

$\Rightarrow \text{R.P. of } e^{ix}\dfrac{1}{D^2+2(1+i)D+2i}x$

$\Rightarrow \text{R.P of } \dfrac{e^{ix}}{2i}\left[\dfrac{1}{1+\left(\frac{D^2}{2i}+\frac{1+i}{i}D\right)}\right]x$

$\Rightarrow \text{R.P. of } \dfrac{e^{ix}}{2i}\left[1+\left(\frac{D^2}{2i}+\frac{1+i}{i}D\right)\right]^{-1}x$

$\Rightarrow \text{R.P. of } \dfrac{e^{ix}}{2i}\left[1-\frac{D^2}{2i}-\frac{1+i}{i}D\right]x$

$\Rightarrow \text{R.P. of } \dfrac{e^{ix}}{2i}\left[x-\left(\frac{1+i}{i}\right)\right]$

$\Rightarrow \text{R.P of } \dfrac{(\cos x+i\sin x)}{2i}\left[x-\left(\frac{1+i}{i}\right)\right]$

$\Rightarrow \text{R.P of } \dfrac{-i}{2}(\cos x+i\sin x)\left[x+i-1\right]$

$\Rightarrow \text{R.P of } \dfrac{-1}{2}\left[i\cos x-\sin x\right]\left[x+i-1\right]$

$P.I = \dfrac{-1}{2}\left[-\cos x-x\sin x+\sin x\right]$

$P.I = \dfrac{\cos x}{2}+\dfrac{x\sin x}{2}-\dfrac{\sin x}{2}$

$P.I = \dfrac{\cos x}{2}+\dfrac{\sin x}{2}(x-1)$

Final Solution

$y = C.F. + P.I.$

$y = (c_1+c_2x)e^{-x}+\dfrac{\cos x}{2}+\dfrac{\sin x}{2}(x-1)$ Ans

Rules to find Particular Integral (P.I.)

2. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x.\sin x$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\cdot\sin x$$

$$(D^2 - 2D + 1)y = x\cdot\sin x$$

$$A.E \Rightarrow m^2 - 2m + 1$$

$$m^2 - m - m + 1 = 0$$

$$(m-1)(m-1) = 0 \Rightarrow (m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F = (c_1 + c_2 x)e^x$$

$$P.I. = \frac{1}{(D-1)^2} x\sin x$$

$$= \text{Imaginary Part of } e^{ix}\frac{1}{(D+i-1)^2}x$$

$$= I.P \text{ of } e^{ix}\frac{1}{D^2+2D(i-1)+(i-1)^2}x$$

$$\Rightarrow I.P \text{ of } e^{ix}\frac{1}{D^2+2D(i-1)+(i^2-2i+1)}x$$

$$\Rightarrow I.P \text{ of } e^{ix}\frac{1}{D^2+2D(i-1)-2i}x$$

$$\Rightarrow I.P. \text{ of } \frac{e^{ix}}{-2i}\left[\frac{1}{1-\frac{D^2}{2i}-\frac{D}{i}(i-1)}\right]x$$

$$\Rightarrow I.P \text{ of } \frac{e^{ix}}{-2i}\left[\frac{1}{1+\left\{\frac{-D^2}{2i}-\frac{D}{i}(i-1)\right\}}\right]x$$

$$\Rightarrow I.P \text{ of } \frac{e^{ix}}{-2i}\left[1+\left\{\frac{-D^2}{2i}-D(i+1)\right\}\right]^{-1}x$$

$$\Rightarrow I.P \text{ of } \frac{e^{ix}}{-2i}\left[1-\left\{\frac{-D^2}{2i}-D(i+1)\right\}\right]x$$

$$\Rightarrow I.P. \text{ of } \frac{e^{ix}}{-2i}\left[1+\frac{D^2}{2i}+D(i+1)\right]x$$

$$I.P. \text{ of } \frac{e^{ix}}{-2i}(x+i+1)$$

$$I.P \text{ of } \frac{i(\cos x + i\sin x)}{2}(x+i+1)$$

$$I.P \text{ of } \frac{1}{2}[i\cos x - \sin x][x+i+1]$$

$$P.I. = \frac{1}{2}[x\cos x + \cos x - \sin x]$$

Final Solution

$$y = C.F. + P.I.$$

$$y = (c_1 + c_2 x)e^x + \frac{1}{2}[x\cos x + \cos x - \sin x]$$

Numerical on Differential Equation

1. Solve the differenetial equation $(D^2 - 2D + 1)\, y = e^x Sin\, x$ **2016,2017**

Numerical on Differential Equation

2. Solve the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x+2}$ **2022**

Numerical on Differential Equation

3. Solve $(D^2 - 2D + 4)\, y = e^x . Cosx + Sinx . Cos3x$ **2018**

Homogeneous Linear Differential Equations
(Euler-Cauchy Equation)

2. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$$

Put $x = e^z \Rightarrow z = \log x$, Let $\frac{d}{dz} = D$

$$x\cdot\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$D(D-1)y - Dy - 3y = e^{2z}\cdot z$$

$$[D^2 - D - D - 3]y = e^{2z}\cdot z$$

$$[D^2 - 2D - 3]y = e^{2z}\cdot z$$

A.E $\Rightarrow m^2 - 2m - 3$

$$m^2 - 3m + m - 3 = 0$$

$$m(m-3) + 1(m-3) = 0$$

$$(m-3)(m+1) = 0 \Rightarrow \boxed{m = -1, 3}$$

$$C.F = c_1 e^{-z} + c_2 e^{3z}$$

$$C.F = c_1 e^{-\log x} + c_2 e^{3\log x}$$

$$C.F = c_1 e^{\log(\frac{1}{x})} + c_2 e^{\log x^3}$$

$$\boxed{C.F = \frac{c_1}{x} + c_2 x^3}$$

$$P.I = \frac{1}{(D^2 - 2D - 3)} e^{2z}\cdot z$$

$$P.I = e^{2z}\frac{1}{(D+2)^2 - 2(D+2) - 3} z$$

$$P.I = e^{2z}\frac{1}{D^2 + 4 + 4D - 2D - 4 - 3} z$$

$$P.I = e^{2z}\frac{1}{(D^2 + 2D - 3)} z$$

$$P.I = \frac{e^{2z}}{-3}\frac{1}{\left[1 - \frac{D^2}{3} - \frac{2D}{3}\right]} z$$

$$P.I = \frac{e^{2z}}{-3}\left[1 + \left(-\frac{D^2}{3} - \frac{2D}{3}\right)\right]^{-1}\cdot z$$

$$P.I = \frac{e^{2z}}{-3}\left[1 + \frac{D^2}{3} + \frac{2D}{3}\right] z$$

$$P.I = \frac{-e^{2z}}{3}\left[z + \frac{2}{3}\right]$$

$$P.I = -\frac{x^2}{3}\left[\log x + \frac{2}{3}\right]$$

Complete Solution

$$y = C.F. + P.I.$$

$$y = \frac{c_1}{x} + c_2 x^3 - \frac{x^2}{3}\left[\log x + \frac{2}{3}\right]$$

Ans

Legendre's Homogeneous Differential Equations

4. Solve $(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = (2x+3)(2x+4)$ **2011, 2023**

Legendre's Homogeneous Differential Equations

1. Solve $(3x+2)^2 \frac{d^2y}{dx^2} - (3x+2)\frac{dy}{dx} - 12y = 6x$

Let $(3x+2) = e^z \Rightarrow z = \log(3x+2)$

$x = \frac{e^z - 2}{3}$, $\frac{d}{dz} = D$

$(3x+2)^2 \frac{d^2y}{dx^2} = 9D(D-1)y$

$(3x+2)\frac{dy}{dx} = 3Dy$

$9D(D-1)y - 3Dy - 12y = 6\left[\frac{e^z-2}{3}\right]$

$[9D^2 - 9D - 3D - 12]y = 2e^z - 4$

$[9D^2 - 12D - 12]y = 2e^z - 4$

A.E $\Rightarrow$ $9m^2 - 12m - 12 = 0$

$3m^2 - 4m - 4 = 0$

$3m^2 - 6m + 2m - 4 = 0$

$3m(m-2) + 2(m-2) = 0$

$(3m+2)(m-2) = 0$

$\boxed{m = 2, -\frac{2}{3}}$

$C.F = c_1 e^{2z} + c_2 e^{-\frac{2}{3}z}$

$C.F = c_1 e^{2\log(3x+2)} + c_2 e^{-\frac{2}{3}\log(3x+2)}$

$C.F = c_1 e^{\log(3x+2)^2} + c_2 e^{\log(3x+2)^{-2/3}}$

$\boxed{C.F = c_1(3x+2)^2 + c_2(3x+2)^{-2/3}}$

$P.I = \frac{1}{(9D^2 - 12D - 12)}(2e^z - 4)$

$P.I = 2\frac{1}{9D^2-12D-12}e^z - 4\frac{1}{9D^2-12D-12}e^{0z}$

$P.I = 2\frac{1}{9-12-12}e^z - 4\frac{1}{0-0-12}\times 1$

$P.I = \frac{2}{-15}e^z + \frac{4}{12}$

$P.I = \frac{2}{-15}(3x+2) + \frac{1}{3}$

$P.I = -\frac{2}{5}x - \frac{4}{15} + \frac{1}{3}$

$P.I = -\frac{2}{5}x + \frac{1}{15}$

Complete Solution

$y = C.F. + P.I.$

$y = c_1(3x+2)^2 + c_2(3x+2)^{-2/3} - \frac{2}{5}x + \frac{1}{15}$

Simultaneous Differential Equations

6. Solve simultaneous differential equations $\frac{dx}{dt}+\frac{dy}{dt}+2x+y=0, \frac{dy}{dt}+5x+3y=0$ **2020**

Simultaneous Differential Equations

7. Solve simultaneous equations $\frac{dx}{dt} - y = e^t, \frac{dy}{dt} + x = Sin\ t\ ; x(0) = 1, y(0) = 0$ 2023

Simultaneous Differential Equations

8. Solve simultaneous equations $\frac{dx}{dt} + x - 2y = 0, \frac{dy}{dt} + x + 4y = 0\,; x(0) = y(0) = 1$ **2023**

Simultaneous Differential Equations

1. Solve the following Simultaneous differential equations $\frac{dx}{dt} = 3x + 2y,\ \frac{dy}{dt} = 5x + 3y$

$$\frac{dx}{dt} - 3x - 2y = 0 \quad \text{—— (1)}$$

$$\frac{dy}{dt} - 3y - 5x = 0 \quad \text{—— (2)}$$

Put $\frac{d}{dt} = D$

$$(D-3)x - 2y = 0 \quad \text{—— (3)}$$

$$(D-3)y - 5x = 0 \quad \text{—— (4)}$$

Eqn (3) $\times (D-3)$ + Eqn (4) $\times 2$

$$(D-3)^2 x - \cancel{2(D-3)y} = 0$$

$$\cancel{2(D-3)y} - 10x = 0$$

$$(D-3)^2 x - 10x = 0$$

$$[(D-3)^2 - 10]x = 0$$

$$[D^2 + 9 - 6D - 10]x = 0$$

$$[D^2 - 6D - 1]x = 0$$

A.E $\Rightarrow m^2 - 6m - 1 = 0$

$$m = \frac{6 \pm \sqrt{36+4}}{2} = 3 \pm \sqrt{10}$$

$$C.F = e^{3t}[c_1 \cosh\sqrt{10}t + c_2 \sinh\sqrt{10}t]$$

$$P.I = 0$$

Complete solution

$$x = C.F. + P.I.$$

$$\boxed{x = e^{3t}[c_1 \cosh\sqrt{10}t + c_2 \sinh\sqrt{10}t]}$$

Ans

From eqn (1) $\quad 2y = \frac{dx}{dt} - 3x$

$$2y = \frac{d}{dt} e^{3t}[c_1 \cosh\sqrt{10}t + c_2 \sinh\sqrt{10}t] - 3e^{3t}[c_1 \cosh\sqrt{10}t + c_2 \sinh\sqrt{10}t]$$

$$2y = e^{3t}\sqrt{10}[c_1 \sinh\sqrt{10}t + c_2 \cosh\sqrt{10}t] \cancel{+ 3e^{3t}[c_1 \cosh\sqrt{10}t + c_2 \sinh\sqrt{10}t]} \cancel{- 3e^{3t}[c_1 \cosh\sqrt{10}t + c_2 \sinh\sqrt{10}t]}$$

$$y = \frac{\sqrt{10}}{2} e^{3t}[c_1 \sinh\sqrt{10}t + c_2 \cosh\sqrt{10}t]$$

Ans

Simultaneous Differential Equations

10. Solve $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$, $\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \text{Sin } 2t$ 2007 , 2022

Simultaneous Differential Equations

2. Solve the following Simultaneous differential equations $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y$, $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t$

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x - y = 0 \quad \text{—} (1)$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y - 25x = 16e^t \quad \text{—} (2)$$

Let $\frac{d}{dt} = D$

$$(D^2 - 4D + 4)x - y = 0 \quad \text{—} (3)$$

$$(D^2 + 4D + 4)y - 25x = 16e^t \quad \text{—} (4)$$

Eqn (3) $\times (D^2 + 4D + 4)$ + Eqn (4)

$$(D^2 - 4D + 4)(D^2 + 4D + 4)x - \cancel{(D^2 + 4D + 4)y} = 0$$

$$\cancel{(D^2 + 4D + 4)y} - 25x = 16e^t$$

$$(D^2 - 4D + 4)(D^2 + 4D + 4)x - 25x = 16e^t$$

$$(D^4 + \cancel{4D^3} + 4D^2 - \cancel{4D^3} - 16D^2 - \cancel{16D} + 4D^2 + \cancel{16D} + 16 - 25)x = 16e^t$$

$$(D^4 - 8D^2 - 9)x = 16e^t$$

A.E $\Rightarrow m^4 - 8m^2 - 9 = 0$

$$m^4 - 9m^2 + m^2 - 9 = 0$$

$$m^2(m^2 - 9) + 1(m^2 - 9) = 0$$

$$(m^2 - 9)(m^2 + 1) = 0$$

$m^2 - 9 = 0 \Rightarrow m = \pm 3$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$$C.F. = c_1e^{3t} + c_2e^{-3t} + e^{0x}(c_3\cos t + c_4\sin t)$$

$$C.F = c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t$$

$$P.I = \frac{1}{(D^4 - 8D^2 - 9)}16e^t = 16\frac{1}{1-8-9}e^t = -e^t$$

$$x = C.F + P.I = c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t - e^t$$

From eqn (1) $y = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x$

$$y = \frac{d^2}{dt^2}[c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t - e^t]$$

$$- 4\frac{d}{dt}[c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t - e^t]$$

$$+ 4[c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t - e^t]$$

$$y = 9c_1e^{3t} + 9c_2e^{-3t} - c_3\cos t - c_4\sin t - e^t - 12c_1e^{3t} + 12c_2e^{-3t} + 4c_3\sin t - 4c_4\cos t + \cancel{4e^t} + 4c_1e^{3t} + 4c_2e^{-3t} + 4c_3\cos t + 4c_4\sin t - \cancel{4e^t}$$

$$y = c_1e^{3t} + 25c_2e^{-3t} + \cos t(3c_3 - 4c_4) + \sin t(4c_3 + 3c_4) - e^t$$

Simultaneous Differential Equations

12. Solve simultaneous equations $\mathbf{t}\frac{dy}{dt} + x = 0, \mathbf{t}\frac{dx}{dt} + y = 0$; $x(0) = 1, y(-1) = 0$ **2022**

Solution by Changing Independent Variable

13. By changing the independent variable, solve the differential equation $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$. **2023**

Solution by Changing Independent Variable

14. By changing independent variable, solve differential equation $x\frac{d^2y}{dx^2}+(4x^2-1)\frac{dy}{dx}-4x^3y=2x^3$. 2006, 2013,2022,2023.

Solution by Changing Independent Variable

15. By changing independent variable, solve differential equation $\frac{d^2y}{dx^2} - Cot\ x\frac{dy}{dx}$ - $ySin^2x$ = Cos x-Cos^3x.

2011, 2020

Solution by Changing Independent Variable

3. Solve the differential equation $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$

$$\frac{d^2y}{dx^2}+\frac{1}{1+x}\cdot\frac{dy}{dx}+\frac{1}{(1+x)^2}y=\frac{4\cos\log(1+x)}{(1+x)^2}$$

Compare By

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

$$P=\frac{1}{1+x},\quad Q=\frac{1}{(1+x)^2}$$

$$R=\frac{4\cos\log(1+x)}{(1+x)^2}$$

Let $\boxed{\left(\frac{dz}{dx}\right)^2=\frac{1}{(1+x)^2}}\Rightarrow\boxed{\frac{dz}{dx}=\frac{1}{1+x}}$

$\boxed{z=\log(1+x)}\Rightarrow\boxed{\frac{d^2z}{dx^2}=\frac{-1}{(1+x)^2}}$

$$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{-1}{(1+x)^2}+\frac{1}{(1+x)}\cdot\frac{1}{1+x}}{\frac{1}{(1+x)^2}}=0$$

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{1/(1+x)^2}{1/(1+x)^2}=1$$

$$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{4\cos\log(1+x)/(1+x)^2}{1/(1+x)^2}$$

$$R_1=4\cos\log(1+x)=4\cos z$$

Reduced Equation is

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$$

$$\frac{d^2y}{dz^2}+y=4\cos z$$

$$(D^2+1)y=4\cos z$$

A.E $\Rightarrow m^2+1=0\Rightarrow m^2=-1$, $\boxed{m=\pm i}$

$$C.F=e^{0z}[c_1\cos z+c_2\sin z]$$

$$C.F=c_1\cos z+c_2\sin z$$

$$P.I.=\frac{1}{D^2+1}4\cos z=4\frac{1}{D^2+1}\cos z$$

$$P.I=4\times z\frac{1}{2D}\cos z=2z\frac{1}{D}\cos z$$

$$\boxed{P.I=2z\sin z}$$

Complete Solution $y=C.F.+P.I.$

$$y=c_1\cos z+c_2\sin z+2z\sin z$$

$$y=c_1\cos\log(1+x)+c_2\sin\log(1+x)$$
$$+2\log(1+x)\sin\log(1+x)$$

Method of Variation of Parameters

17. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y$ = Cosec x. 2023

Method of Variation of Parameters

18. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = \tan x$. 2019, 2023

Method of Variation of Parameters

19. Solve by the method of variation of parameters $(D^2-1)y = 2\left[1-e^{-2x}\right]^{-1/2}$ **2011, 2022.**

Method of Variation of Parameters

20. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \text{Sin}(e^x)$ **2020**

Method of Variation of Parameters

21. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \text{Sin}(e^{-x})$ **2020**

Method of Variation of Parameters

22. Solve the differential equation $\left(D^2 + 2D + 2\right)y = e^{-x}Sec^3x$ 2017

Method of Variation of Parameters

23. Solve the differential equation $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3 \log x$ **2023**

Method of Variation of Parameters

3. Solve by the method of variation of parameters $x^2y'' + xy' - y = x^2e^x$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$$

Let $x = e^z \Rightarrow z = \log x$

Let $\frac{d}{dx} = D$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y, \quad x\frac{dy}{dx} = Dy$$

Now

$$D(D-1)y + Dy - y = x^2e^x$$

$$[D^2 - D + D - 1]y = x^2e^x$$

$$[D^2 - 1]y = x^2e^x$$

A.E $\Rightarrow m^2 - 1 = 0 \Rightarrow \boxed{m = \pm 1}$

$$C.F = Ae^z + Be^{-z}$$

$$C.F = Ae^{\log x} + Be^{-\log x}$$

$$C.F = A\cdot x + B\cdot\frac{1}{x}$$

$y_1 = x, \quad y_2 = \frac{1}{x}$

Complete Solution

$$\boxed{y = ux + v\frac{1}{x}}$$

Now $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^x$

So $\boxed{R = e^x}$

$$u = \int\frac{-y_2R}{y_1y_2' - y_1'y_2}dx + C_1$$

$$u = \int\frac{-\frac{1}{x}\cdot e^x}{x\left(\frac{-1}{x^2}\right) - 1\left(\frac{1}{x}\right)}dx + C_1$$

$$u = \int\frac{-\frac{e^x}{x}}{-\frac{2}{x}}dx + C_1$$

$$u = \frac{1}{2}\int e^x dx + C_1 = \frac{1}{2}e^x + C_1$$

$$v = \int\frac{y_1R}{y_1y_2' - y_1'y_2}dx + C_2$$

$$v = \int\frac{x\cdot e^x}{-\frac{2}{x}}dx + C_2 = \frac{-1}{2}\int x^2e^x dx + C_2$$

$$v = \frac{-1}{2}\left[x^2e^x - \int 2xe^x dx\right] + C_2$$

$$v = \frac{-1}{2}\left[x^2e^x - 2\left\{xe^x - \int 1\cdot e^x dx\right\}\right] + C_2$$

$$v = \frac{-1}{2}\left[x^2e^x - 2xe^x + 2e^x\right] + C_2$$

$$v = \frac{-1}{2}x^2e^x + xe^x - e^x + C_2$$

$$v = \frac{-1}{2}x^2e^x + e^x(x-1) + C_2$$

Final Solution $y = ux + v \cdot \frac{1}{x}$

$$y = \left[\frac{1}{2}e^x + c_1\right]x + \left[-\frac{1}{2}x^2 e^x + e^x(x-1) + c_2\right]\frac{1}{x}$$

$$y = \cancel{\frac{1}{2}e^x \cdot x} + \underline{c_1 x} \cancel{- \frac{1}{2}x \cdot e^x} + \frac{e^x(x-1)}{x} + \underline{c_2 \cdot \frac{1}{x}}$$

$$y = c_1 x + \frac{c_2}{x} + e^x\left(1 - \frac{1}{x}\right)$$

Ans

IMPORTANT TOPIC

UNIT 2

Previous Year Topic

- ❖ Laplace transform,
- ❖ Existence theorem,
- ❖ Properties of Laplace Transform,
- ❖ Laplace transform of derivatives and integrals…… 04
- ❖ Unit step function…… 02
- ❖ Laplace transform of periodic function---------- 02
- ❖ Inverse Laplace transform,
- ❖ Convolution theorem…… 05
- ❖ Application of Laplace Transform to solve ordinary differential equations and simultaneous differential equations…… 04

Unit 2

- ❖Convolution theorem
- ❖Application of Laplace Transform to solve ordinary differential equations and simultaneous differential equations……
- ❖ Unit step Function, periodic function

Unit 2 (Two Marks)

1. Explain Laplace Transform.
2. Explain first shifting property of the Laplace with example.
3. Find the inverse Laplace Transform of $f(s) = \frac{1}{s^2+2s+2}$
4. Find $L[F(t)]$, If F(t) = $t.Cos\ 2t$.
5. Find Laplace transform of e^{at} .
6. Find Laplace transform of $t^{-1/2}$
7. Write change of scale property of Laplace Transform.
8. Define unit step function .
9. What is Laplace Transform of periodic function.
10. Determine inverse Laplace Transform of $\frac{1}{s-2}$.

Unit 2 (Two Marks)

11. Find Laplace transform of $f(t) = \frac{Sin\ at}{t}$.

12. Find inverse Laplace Transform of $\frac{s+8}{s^2+4s+5}$.

13. Find Laplace Transform of F(t) = $e^t . t^{-1/2}$.

14. . Find the function whose Laplace Transform of $e^{-\pi s}/_{s^2+2}$.

15. If $L\{F\sqrt{t}\} = \frac{e^{-1/s}}{s}$, Find $L\left[e^{-t}.F(3\sqrt{t})\right]$.

16. Find inverse Laplace transform of $\frac{e^{-s}-3e^{-3s}}{s^2}$.

17. Evaluate $\int_0^\infty e^{-2t}\ t.Cos\ 2t\ dt$

Laplace Transform

1. Find the Laplace Transform of $7e^{2t} + 9e^{-2t} + 5\,\text{Cos}t + 7t^3 + 5\,Sin\,3t + 2$

$$F(t) = 7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2$$

$$LF(t) = 7L(e^{2t}) + 9L(e^{-2t}) + 5L(\cos t)$$
$$+ 7L(t^3) + 5L(\sin 3t) + 2L(1)$$

$$= \frac{7}{s-2} + \frac{9}{s+2} + \frac{5s}{s^2+1} + \frac{7\lfloor 3}{s^{3+1}} + \frac{5\times 3}{s^2+9} + \frac{2}{s}$$

$$= \frac{7}{s-2} + \frac{9}{s+2} + \frac{5s}{s^2+1} + \frac{42}{s^4} + \frac{15}{s^2+9} + \frac{2}{s}$$ Ans

Laplace Transform

1. Find the Laplace Transform of $e^{-3t}(\text{Cos } 4t + 3Sin\ 4t)$

$$L\left[e^{-3t}(\cos 4t + 3\sin 4t)\right]$$

$$L[\cos 4t + 3\sin 4t] = L(\cos 4t) + 3L(\sin 4t)$$

$$= \frac{s}{s^2+16} + \frac{3\times 4}{s^2+16}$$

$$= \frac{s}{s^2+16} + \frac{12}{s^2+16}$$

So

$$L\left[e^{-3t}(\cos 4t + 3\sin 4t)\right] = \frac{(s+3)}{(s+3)^2+16} + \frac{12}{(s+3)^2+16}$$

$$= \frac{s+3+12}{s^2+9+6s+16}$$ (By First Shifting Property)

$$= \frac{s+15}{s^2+6s+25}$$ Ans

Second Shifting Property of Laplace Transform

$$= \int_0^\infty e^{-st} F(t)\,dt$$

❖ If $L\{F(t)\} = F(s)$ and $G(t) = \begin{cases} F(t-a) & \boxed{t > a} \\ 0 & \boxed{t < a} \end{cases}$

$L[G(t)] = e^{-as} F(s)$

Solution ⇒

$$F[G(t)] = \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

$$F[G(t)] = 0 + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$F[G(t)] = \int_a^\infty e^{-st} F(t-a)\,dt$$

Let $t - a = u \Rightarrow \boxed{t = a+u} \Rightarrow \boxed{dt = du}$

At $t = a \Rightarrow \boxed{u = 0}$

At $t = \infty \Rightarrow \boxed{u = \infty}$

$$F[G(t)] = \int_0^\infty e^{-s(a+u)} F(u)\,du$$

$$F[G(t)] = e^{-sa} \int_0^\infty e^{-su} F(u)\,du$$

$$F[G(t)] = e^{-sa} \underbrace{\int_0^\infty e^{-st} F(t)\,dt}$$

$$\boxed{F[G(t)] = e^{-sa} \cdot F(s)}$$

Second Shifting Property of Laplace Transform

1. Find $L\{F(t)\}$ if $F(t) = \begin{cases} \sin(t - \frac{\pi}{3}) & t > \frac{\pi}{3} \\ 0 & t < \frac{\pi}{3} \end{cases}$

$L[F(t)] = e^{-as} \cdot F(s)$ $\quad \{a = \frac{\pi}{3}\}$

Solution ⇒

$$L[F(t)] = e^{-\frac{\pi}{3}s} L(\sin t)$$

$$= e^{-\frac{\pi}{3} \cdot s} \times \frac{1}{s^2+1}$$

{Using Second Shifting Property}

Second Shifting Property of Laplace Transform

2. Find $L\{F(t)\}$ if $F(t) = \begin{cases} (t-1)^2 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$

Here $a = 1$

$$L\{F(t)\} = e^{-s} \cdot L[t^2]$$

$$= e^{-s} \frac{\lfloor 2}{s^{2+1}}$$

$$= e^{-s} \cdot \frac{2}{s^3} \quad \text{Ans}$$

Change of Scale Property

1. *If* $L\{Cos^2 t\} = \frac{s^2+2}{s(s^2+4)}$, find $L\{Cos^2 at\} =$

By Change of Scale Property

$$L[\cos^2 t] = \frac{s^2+2}{s(s^2+4)}$$

$$L[\cos^2 at] = \frac{1}{a}\left[\frac{\left(\frac{s}{a}\right)^2+2}{\frac{s}{a}\left\{\left(\frac{s}{a}\right)^2+4\right\}}\right]$$

$$= \frac{1}{\cancel{a}}\left[\frac{\frac{s^2+2a^2}{\cancel{a^2}}}{\frac{s}{\cancel{a}}\left\{\frac{s^2+4a^2}{\cancel{a^2}}\right\}}\right] = \frac{s^2+2a^2}{s(s^2+4a^2)}$$

Laplace Transform of Integrals

3. If $L(t.\sin\omega t) = \frac{2\omega p}{(p^2+\omega^2)^2}$, evaluate (i) $L\{\omega t \cos \omega t + \sin \omega t\}$

(ii) $L\{2\omega \cos \omega t - \omega^2 t \sin \omega t\}$

$$L[t \cdot \sin\omega t] = \frac{2\omega p}{(p^2+\omega^2)^2}$$

$$F(t) = t \cdot \sin\omega t$$

$$F'(t) = \omega t \cos\omega t + \sin\omega t$$

$$F''(t) = -\omega^2 t \sin\omega t + \omega\cos\omega t + \omega\cos\omega t$$

$$F''(t) = -\omega^2 t \sin\omega t + 2\omega\cos\omega t$$

$$F(0) = 0, \quad F'(0) = 0$$

$$L[F'(t)] = p \cdot L[F(t)] - F(0)$$

$$L[\omega t\cos\omega t + \sin\omega t] = p\frac{2\omega p}{(p^2+\omega^2)^2} - 0$$

$$L[\omega t\cos\omega t + \sin\omega t] = \frac{2\omega p^2}{(p^2+\omega^2)^2}$$

$$L[F''(t)] = p^2 L[F(t)] - pF(0) - F'(0)$$

$$L[2\omega\cos\omega t - \omega^2 t\sin\omega t] = p^2\frac{2\omega p}{(p^2+\omega^2)^2} - p\times 0 - 0$$

$$= \frac{2\omega p^3}{(p^2+\omega^2)^2}$$

Laplace Transform of Division by t

3. Find the Laplace Transform of $\frac{\cos at - \cos bt}{t}$.

Find out $L\left[\frac{\cos at - \cos bt}{t}\right] = ?$

Now $L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

Now $L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$

$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\left[\log \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty$

$= \frac{1}{2}\left[\log 1 - \log \frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{2}\log \frac{s^2+a^2}{s^2+b^2}$

$= \frac{1}{2}\log \frac{s^2+b^2}{s^2+a^2}$ Ans

$\int \frac{x}{x^2+a^2} = \frac{1}{2}\log(x^2+a^2)$

$\log m - \log n = \log \frac{m}{n}$

Laplace Transform of Division by t

2. Using Laplace transform evaluate the integral $\int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}dt$ 2022

Laplace Transform

3. Find Laplace transform of the function $f(x) = x^3 Sin\ x$

Hence prove that $\int_0^\infty e^{-x}\, x^3 Sin\ x dx = 0.$ *2023*

Laplace Transform of Division by t

4. Find $L\left[e^{-2t}\int_0^t \frac{Sin\,3t}{t}dt\right]$ 2020

Laplace Transform of Unit Step Function

5. *Express the following function in terms of Heaviside's Unit Step function and find its Laplace transform* $F(t) = \begin{cases} t^2 & 0 < t < 2 \\ 4t & t > 2 \end{cases}$ *2020.*

Laplace Transform of Unit Step Function

6 . Express the following function in terms of Unit Step function and find its Laplace transform $F(t) = \begin{cases} 2t & 0 < t \leq 2 \\ 10 & t > 5 \end{cases}$ 2020.

Laplace Transform of Periodic Function

7. Draw the graph of the periodic function and find Laplace transform of rectangular wave function of period 2π.

$$F(t) = \begin{cases} t & 0 < t < \pi \\ 2\pi - t & \pi < t < 2\pi \end{cases}$$

2002, 2007,2008,2018

Laplace Transform of Periodic Function

8. Find the Laplace transform of the function

$$F(t) = \begin{cases} \sin wt & 0 < t < \frac{\pi}{w} \\ 0 & \frac{\pi}{w} < t < \frac{2\pi}{w} \end{cases}$$

2010,2022

Numerical on First Translation or Shifting property

3. Find the Inverse Laplace Transform of $\frac{s+8}{s^2+4s+5}$

$$L^{-1}\left[\frac{s+8}{s^2+4s+5}\right] = L^{-1}\left[\frac{s+8}{s^2+2\times2\times s+4+1}\right]$$

$$\Rightarrow L^{-1}\left[\frac{s+8}{(s+2)^2+(1)^2}\right] = L^{-1}\left[\frac{(s+2)+6}{(s+2)^2+(1)^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+(1)^2}\right] + L^{-1}\left[\frac{6}{(s+2)^2+(1)^2}\right]$$

By First Shifting Property

$$= e^{-2t}\, L^{-1}\left[\frac{s}{s^2+1^2}\right] + 6e^{-2t}\, L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= e^{-2t}\cdot\cos t + 6e^{-2t}\cdot\frac{1}{1}\sin t = e^{-2t}\left[\cos t + 6\sin t\right]$$

Ans

Second Translation or Shifting Property of Inverse Laplace Transform

2. Find the Inverse Laplace Transform of $\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]$

$$L^{-1}\left[\frac{e^{-s}}{s^2} - 3\frac{e^{-3s}}{s^2}\right]$$

$$\Rightarrow L^{-1}\left[\frac{e^{-s}}{s^2}\right] - 3L^{-1}\left[\frac{e^{-3s}}{s^2}\right]$$

Now $L^{-1}\left[\frac{1}{s^2}\right] = t$

$$L^{-1}\left[\frac{e^{-s}}{s^2}\right] = (t-1)\,u(t-1)$$

$$L^{-1}\left[\frac{e^{-3s}}{s^2}\right] = (t-3)\,u(t-3)$$

$$L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right] = (t-1)\,u(t-1) - 3(t-3)\,u(t-3)$$

Ans

Convolution Theorem

9. Using Convolution theorem find that $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$ 2017

Convolution Theorem

10. Using Convolution theorem find inverse Laplace transform of $\frac{1}{(s^2+a^2)^2}$. 2023

Convolution Theorem

11. State Convolution theorem of Laplace transform. Find inverse Laplace transform of $\frac{1}{s^2(s+1)^2}$. 2023

Convolution Theorem

1. Find the Inverse Laplace Transform by Convolution theorem $\dfrac{s}{(s^2+1)(s^2+4)}$

Find out $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+4)}\right]$

Let $F(s) = \dfrac{1}{s^2+4}$, $G(s) = \dfrac{s}{s^2+1}$

$$F(t) = \frac{1}{2} L^{-1}\left[\frac{2}{s^2+4}\right] = \frac{1}{2}\sin 2t$$

$$G(t) = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

$$L^{-1}[F(s)G(s)] = \int_0^t F(u)G(t-u)\,du$$

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = \int_0^t \frac{1}{2}\sin 2u \cdot \cos(t-u)\,du$$

$[2\sin A \cos B = \sin(A+B) + \sin(A-B)]$

Now $\Rightarrow$

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = \frac{1}{4}\int_0^t 2\sin 2u \cdot \cos(t-u)\,du$$

$$= \frac{1}{4}\int_0^t \left[\sin(2u+t-u) + \sin(2u-t+u)\right]du$$

$$= \frac{1}{4}\int_0^t \left[\sin(u+t) + \sin(3u-t)\right]du$$

$$= \frac{1}{4}\left[-\cos(u+t) - \frac{\cos(3u-t)}{3}\right]_0^t$$

$$= \frac{1}{4}\left[\left(-\cos 2t - \frac{\cos 2t}{3}\right) - \left(-\cos t - \frac{\cos t}{3}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{4}{3}\cos 2t + \frac{4}{3}\cos t\right] = \frac{1}{3}\left[\cos t - \cos 2t\right]$$ Ans

Convolution Theorem

3. Find the Inverse Laplace Transform by Convolution theorem $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Find out $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = ?$

Let $F(s) = \dfrac{s}{s^2+a^2}$

$G(s) = \dfrac{s}{s^2+b^2}$

$F(t) = \cos at, \quad G(t) = \cos bt$

$L^{-1}[F(s)\cdot G(s)] = \int_0^t F(u)G(t-u)\,du$

$L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t \cos au \cdot \cos b(t-u)\,du$

$\{2\cos A\cdot\cos B = \cos(A+B) + \cos(A-B)\}$

$L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \dfrac{1}{2}\int_0^t 2\cos au\cos b(t-u)\,du$

$= \dfrac{1}{2}\int_0^t [\cos(au+bt-bu) + \cos(au-bt+bu)]\,du$

$= \dfrac{1}{2}\int_0^t [\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}]\,du$

$= \dfrac{1}{2}\left[\dfrac{\sin(a-b)u+bt}{a-b} + \dfrac{\sin(a+b)u-bt}{a+b}\right]_0^t$

$= \dfrac{1}{2}\left[\dfrac{\sin at}{a-b} + \dfrac{\sin at}{a+b} - \dfrac{\sin bt}{a-b} + \dfrac{\sin bt}{a+b}\right]$

$= \dfrac{1}{2}\left\{\dfrac{\sin at - \sin bt}{a-b} + \dfrac{\sin at + \sin bt}{a+b}\right]$

$= \dfrac{1}{2}\left[\dfrac{a\sin at + \cancel{b\sin at} - \cancel{a\sin bt} - b\sin bt + a\sin at - \cancel{b\sin at} + \cancel{a\sin bt} - b\sin bt}{a^2-b^2}\right]$

$= \dfrac{1}{2}\left[\dfrac{2a\sin at - 2b\sin bt}{a^2-b^2}\right]$

$= \dfrac{a\sin at - b\sin bt}{a^2-b^2}$

Ans

Solution of Ordinary Linear Differential Equations with Constant Coefficient

14. Using Laplace Transform, solve the following differential equation $\frac{d^2y}{dx^2}$ +y = 6 Cos $2x$, where $y(0)$ =3, $y'(0)$ =1. 2023.

Solution of Ordinary Linear Differential Equations with Constant Coefficient

15. Apply Laplace transform to solve the equation $\frac{d^2y}{dt^2}$ +9y = Cos $2t$. Given that $y(0)$ =1, y($\frac{\pi}{2}$) =-1.

2002, 2017, 2020.

Solution of Ordinary Linear Differential Equations with Constant Coefficient

16. Solve $y''(t) + 4y'(t) + 4y(t) = 6e^{-t}, where\ y(0) = -2,\ y'(0) = 8,$ using Laplace transform.

2007, 2023

Solution of Ordinary Linear Differential Equations with Constant Coefficient

17. Solve the following differential equation by the Laplace transform $\frac{d^3y}{dt^3}$ -3 $\frac{d^2y}{dt^2}$ + 3 $\frac{dy}{dt}$ - y $=e^t t^2$, *where* $y(0) = 1, y'(0) = 0, y''(0) = -2$ 2008, 2018

Solution of Simultaneous Ordinary Differential Equations

1. Use Laplace Transform to solve: $\frac{dx}{dt} + y = \text{Sin } t$, $\frac{dy}{dt} + x = \text{Cos } t$, **given that** $x = 2$, $y = 0$ **at** $t = 0$.

$\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$

$x(0) = 2$, $y(0) = 0$

Now $\frac{dx}{dt} + y = \sin t$

$x' + y = \sin t$

Taking Laplace

$[s\bar{x} - x(0)] + \bar{y} = \frac{1}{s^2+1}$

$s\bar{x} + \bar{y} = \frac{1}{s^2+1} + 2$ —— (1)

Now

$\frac{dy}{dt} + x = \cos t$

$y' + x = \cos t$

Taking Laplace (Both Side)

$[s\bar{y} - y(0)] + \bar{x} = \frac{s}{s^2+1}$

$s\bar{y} + \bar{x} = \frac{s}{s^2+1}$ —— (2)

Eqn (1) × s − Eqn (2)

$s^2\bar{x} + \cancel{s\bar{y}} = \cancel{\frac{s}{s^2+1}} + 2s$

$\bar{x} + \cancel{s\bar{y}} = \cancel{\frac{s}{s^2+1}}$

$\bar{x}(s^2-1) = 2s$

$\bar{x} = \frac{2s}{s^2-1}$

Taking Inverse Laplace

$L^{-1}[\bar{x}] = 2L^{-1}\left[\frac{s}{s^2-1}\right]$

$x = 2\cosh t = 2\left[\frac{e^t + e^{-t}}{2}\right]$

$\boxed{x = e^t + e^{-t}}$ Ans

From eqn (2)

$s\bar{y} = \frac{s}{s^2+1} - \frac{2s}{s^2-1}$

$\bar{y} = \frac{1}{s^2+1} - \frac{2}{s^2-1}$

Taking Inverse Laplace

$L^{-1}[\bar{y}] = L^{-1}\left[\frac{1}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s^2-1}\right]$

$y = \sin t - 2\sinh t = \sin t - 2\left[\frac{e^t - e^{-t}}{2}\right]$

$\boxed{y = \sin t - e^t + e^{-t}}$ Ans

Solution of Simultaneous Ordinary Differential Equations

2. Use Laplace Transform to solve: $\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$, $\frac{dx}{dt} + 2y = e^{-t}$, **given that** $x = 0$, $y = 0$ **at** $t = 0$.

$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$, $\frac{dx}{dt} + 2y = e^{-t}$

$x(0) = 0$, $y(0) = 0$

Now $\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$

$x' + 4y' - y = 0$

Taking Laplace

$[s\bar{x} - x(0)] + 4[s\bar{y} - y(0)] - \bar{y} = 0$

$s\bar{x} + (4s-1)\bar{y} = 0$ —— (1)

$\frac{dx}{dt} + 2y = e^{-t}$

$x' + 2y = e^{-t}$

Taking Laplace

$[s\bar{x} - x(0)] + 2\bar{y} = \frac{1}{s+1}$

$s\bar{x} + 2\bar{y} = \frac{1}{s+1}$ —— (2)

Eqn (1) − Eqn (2)

$\cancel{s\bar{x}} + (4s-1)\bar{y} = 0$

$\cancel{s\bar{x}} + 2\bar{y} = \frac{1}{s+1}$

$\bar{y}(4s-1-2) = -\frac{1}{s+1}$

$\bar{y} = \frac{-1}{(s+1)(4s-3)}$

Now $\frac{-1}{(s+1)(4s-3)} = \frac{A}{s+1} + \frac{B}{4s-3}$

$-1 = A(4s-3) + B(s+1)$

Put $s = -1$

$-1 = A(-4-3) \Rightarrow \boxed{A = \frac{1}{7}}$

Put $s = \frac{3}{4}$

$-1 = B\left(\frac{3}{4}+1\right) \Rightarrow \boxed{B = \frac{-4}{7}}$

So

$\bar{y} = \frac{1}{7(s+1)} - \frac{4}{7(4s-3)}$

$\bar{y} = \frac{1}{7}\left[\frac{1}{s+1} - \frac{1}{(s-\frac{3}{4})}\right]$

Taking Inverse Laplace

$L^{-1}[\bar{y}] = \frac{1}{7}\left[L^{-1}\frac{1}{s+1} - L^{-1}\frac{1}{s-\frac{3}{4}}\right]$

$y = \frac{1}{7}\left[e^{-t} - e^{\frac{3}{4}t}\right]$ Ans

Solution of Simultaneous Ordinary Differential Equations

Put value of $\bar{y}$ in eqn ②

$$s\bar{x} + 2\bar{y} = \frac{1}{s+1}$$

$$s\bar{x} + \frac{2}{7}\left[\frac{1}{s+1} - \frac{1}{\left(s-\frac{3}{4}\right)}\right] = \frac{1}{s+1}$$

$$s\bar{x} = \frac{1}{s+1} - \frac{2}{7}\cdot\frac{1}{s+1} + \frac{2}{7}\cdot\frac{1}{\left(s-\frac{3}{4}\right)}$$

$$s\bar{x} = \frac{5}{7}\cdot\frac{1}{s+1} + \frac{2}{7}\,\frac{1}{\left(s-\frac{3}{4}\right)}$$

$$\bar{x} = \frac{5}{7}\cdot\frac{1}{s(s+1)} + \frac{2}{7}\cdot\frac{1}{s\left(s-\frac{3}{4}\right)}$$

Now $\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

$$1 = A(s+1) + Bs$$

Put $s=0 \Rightarrow \boxed{A=1}$

Put $s=-1 \Rightarrow \boxed{B=-1}$

So $\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1}$

Now $\dfrac{1}{s\left(s-\frac{3}{4}\right)} = \dfrac{A}{s} + \dfrac{B}{s-\frac{3}{4}}$

$$1 = A\left(s-\frac{3}{4}\right) + Bs$$

Put $s=0 \Rightarrow \boxed{A = \frac{-4}{3}}$

Put $s=\frac{3}{4} \Rightarrow \boxed{B = \frac{4}{3}}$

So $\dfrac{1}{s\left(s-\frac{3}{4}\right)} = \dfrac{-4}{3s} + \dfrac{4}{3\left(s-\frac{3}{4}\right)}$

So $\bar{x} = \dfrac{5}{7}\left[\dfrac{1}{s} - \dfrac{1}{s+1}\right] + \dfrac{2}{7}\left[\dfrac{-4}{3}\cdot\dfrac{1}{s} + \dfrac{4}{3}\cdot\dfrac{1}{\left(s-\frac{3}{4}\right)}\right]$

Taking Inverse Laplace

$$L^{-1}[\bar{x}] = \frac{5}{7}\left[L^{-1}\cdot\frac{1}{s} - L^{-1}\frac{1}{s+1}\right] + \frac{8}{21}\left[L^{-1}\frac{1}{s-\frac{3}{4}} - L^{-1}\frac{1}{s}\right]$$

$$x = \frac{5}{7} - \frac{5}{7}e^{-t} + \frac{8}{21}e^{3t/4} - \frac{8}{21} = \frac{8}{21}e^{3t/4} - \frac{5}{7}e^{-t} + \frac{1}{3}$$ Ans

IMPORTANT TOPIC

UNIT 3

Unit 3

- ❖D’ Alembert’s test
- ❖Fourier series, Sine & cosines Series

Unit 3 (Two Marks)

1. Write down the statement of Raabe's Test

2. Find the constant term when $f(x) = |x| + 1$ is expended in Fourier series in the interval (-3,3) .

3. Find Fourier constant a_1 of $f(x) = x^2, -\pi \leq x \leq \pi$.

4. Find Fourier constant term if the function $f(x) = x^2 + x$, expended in Fourier Series defined in $(-1, 1)$.

5. Discuss the convergence of sequence $a_n = \frac{2n}{n^2+1}$.

6. If f(x) = 1 is expanded in a Fourier Sin series$(0, \pi)$ $then$ find the value of b_n .

7. Discuss the convergence of sequence $(1, 2^1, 2^2, 2^3, 2^4 \ldots\ldots\ldots\ldots)$.

8. Discuss the convergence of $u_n = \text{Sin}\frac{1}{n}$.

Unit 3 (Two Marks)

9. Test the series $\sum_{n=1}^{\infty} \frac{1}{n} \text{Sin} \frac{1}{n}$.

10. Find the value of Fourier coefficient a_0 for the function $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

11. Find Fourier constant term, a_0 if the function $f(x) = x.Sinx$, in the interval $(-\pi, \pi)$.

12. Test the convergence of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \infty$.

13. Discuss the convergence of the sequence a_n $where$ $a_n = \frac{n+1}{n}$.

14. Write the Dirichlet's condition for fourier series .

Comparison Test

1. Discuss the convergence of the series $\sum \sqrt{n+1} - \sqrt{n}$ 2023.

D' Alembert's Test or Ratio Test

2. Test the convergence of the following series $\frac{1}{1.2.3}+\frac{x}{4.5.6}+\frac{x^2}{7.8.9}+$.....where x is a real number 2023

D' Alembert's Test or Ratio Test

4. Test the series $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots\ldots\ldots\ldots\ldots\ldots\frac{x^n}{n^2+1}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$u_n = \frac{x^n}{n^2+1}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

Now $\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{x^n}$

$$\frac{u_{n+1}}{u_n} = \frac{x \times \cancel{n^2}\left[1+\frac{1}{n^2}\right]}{\cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}$$

$$\frac{u_{n+1}}{u_n} = \frac{x\left[1+\frac{1}{n^2}\right]}{\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x\left(1+\frac{1}{n^2}\right)}{\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{x}{1} = x$$

If $x>1$, then series is divergent

If $x<1$, then series is Convergent

If $x=1$, then Test fails.

If $x=1$, $u_n = \frac{1^n}{n^2+1} = \frac{1}{n^2+1}$

Now nth term of Auxiliary Series

$$v_n = \frac{1}{n^2}$$

$$\frac{u_n}{v_n} = \frac{1}{n^2+1} \times \frac{n^2}{1} = \frac{1}{1+\frac{1}{n^2}}$$

Now

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = 1$$

which is fixed, finite & non-zero quantity

So we will apply Comparison Test

$p = 2 > 1$

So Series is Convergent.

Finally

Σu_n is Convergent $x \le 1$

Σu_n is Divergent $x > 1$

Raabe's Test

2. Test the Convergence of the series $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}\ldots\ldots$

Series $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+\cdots$

$$u_n=\frac{x^n}{(2n-1)2n}$$

$$u_{n+1}=\frac{x^{n+1}}{[2n+1][2n+2]}$$

$$\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{[2n+1][2n+2]}\times\frac{(2n-1)\times 2n}{x^n}$$

$$\frac{u_{n+1}}{u_n}=\frac{x\times 2n(2n-1)}{(2n+1)(2n+2)}$$

$$\frac{u_{n+1}}{u_n}=\frac{x\times 2\cancel{n}\times\cancel{n}\left(2-\frac{1}{n}\right)}{\cancel{n}\left[2+\frac{1}{n}\right]\times 2\cancel{n}\left[1+\frac{1}{n}\right]}$$

$$\frac{u_{n+1}}{u_n}=\frac{x\left(2-\frac{1}{n}\right)}{\left[2+\frac{1}{n}\right]\left(1+\frac{1}{n}\right)}$$

D Alembert's Test

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{x\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\frac{2x}{2}=x$$

If $x<1$, Σu_n is Convergent

If $x>1$, Σu_n is Divergent

If $x=1$, D Alembert Test fail

For $x=1$ Apply Raabe's Test.

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]$$

$$\lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{2n(2n-1)}-1\right]$$

$$\lim_{n\to\infty}\cancel{n}\left[\frac{4n^2+4n+2n+2-4n^2+2n}{2\cancel{n}(2n-1)}\right]$$

$$\lim_{n\to\infty}\left[\frac{8n+2}{2(2n-1)}\right]$$

$$=\lim_{n\to\infty}\frac{2n\left[4+\frac{1}{n}\right]}{2n\left[2-\frac{1}{n}\right]}=\lim_{n\to\infty}\frac{\left(4+\frac{1}{n}\right)}{\left(2-\frac{1}{n}\right)}$$

$$=\frac{4}{2}=2>1$$

Hence at $x=1$, Series is Convergent.

D' Alembert's Test or Ratio Test

5. Interpret the convergence of the series $1+\frac{x}{2}+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\ldots\ldots\ldots\ldots, x>0$

2020, 2023

D' Alembert's Test or Ratio Test

6. Examine the series for convergence or Divergence $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\ldots\ldots\ldots\ldots$

2022

Fourier Series (Interval 0 to 2π)

2. Expand $f(x) = x.\sin x, 0 < x < 2\pi$, as a Fourier series.

$$f(x) = x\cdot\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cdot\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\, dx$$

$$a_0 = \frac{1}{\pi}\left[x\cdot(-\cos x) - 1\cdot(-\sin x)\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left[(-2\pi + 0) - (0 + 0)\right] = \frac{1}{\pi}(-2\pi)$$

$$\boxed{a_0 = -2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos nx = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\cdot\cos nx\, dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\cdot(2\cos nx\cdot\sin x)dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$a_n = \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right\} - \left\{\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$a_n = \frac{1}{2\pi}\left[\left\{2\pi\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) - (0-0)\right\} - \left\{0 - (0-0)\right\}\right]$$

$$a_n = \frac{1}{\cancel{2\pi}}\left[\cancel{2\pi}\left(\frac{-\cancel{n}+1+\cancel{n}+1}{n^2-1}\right)\right] \Rightarrow \boxed{a_n = \frac{2}{n^2-1}} \quad (n \neq 1)$$

We know that $a_n = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\cdot\cos nx\, dx$

Put $n = 1 \Rightarrow a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\, 2\sin x\cos x\, dx$

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\cdot\sin 2A$$

Fourier Series (Interval 0 to 2π)

2. Expand $f(x) = x.\sin x, 0 < x < 2\pi$, as a Fourier series.

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\cdot\sin 2x$$

$$a_1 = \frac{1}{2\pi}\left[x\cdot\left(\frac{-\cos 2x}{2}\right) - 1\cdot\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left[\left(-\frac{2\pi}{2} + 0\right) - (0 + 0)\right]$$

$$a_1 = \frac{1}{2\pi}(-\pi) = -\frac{1}{2} \Rightarrow \boxed{a_1 = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\cdot\sin nx\,dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin nx\cdot\sin x)\,dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$b_n = \frac{1}{2\pi}\left[x\cdot\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - \left\{\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left[\left\{2\pi(0-0) - \left(\frac{-1}{(n-1)^2} + \frac{1}{(n+1)^2}\right)\right\} + \left(\frac{-1}{(n-1)^2} + \frac{1}{(n+1)^2}\right)\right]$$

$$b_n = \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n-1)^2}} + \cancel{\frac{1}{(n+1)^2}}\right]$$

$$\boxed{b_n = 0} \qquad [n \neq 1]$$

We know that $b_n = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\cdot\sin nx\,dx$

Put $n = 1 \Rightarrow \quad b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin^2 x\,dx$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\,(1 - \cos 2x)\,dx$$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x - \frac{1}{2\pi}\int_0^{2\pi} x\cdot\cos 2x\,dx$$

Fourier Series (Interval 0 to 2π)

2. Expand $f(x) = x.\sin x, 0 < x < 2\pi$, as a Fourier series.

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\cdot dx - \frac{1}{2\pi}\int_0^{2\pi} x\cdot\cos 2x\, dx$$

$$b_1 = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x\cdot\left(\frac{\sin 2x}{2}\right) - 1\cdot\left(-\frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$b_1 = \frac{1}{4\pi}\left[4\pi^2 - 0\right] - \frac{1}{2\pi}\left[\frac{x\cdot\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$b_1 = \pi - \frac{1}{2\pi}\left[\left(0 + \frac{1}{4}\right) - \left(0 + \frac{1}{4}\right)\right]$$

$$b_1 = \pi - \frac{1}{2\pi}\left(\frac{1}{4} - \frac{1}{4}\right)$$

$$\boxed{b_1 = \pi}$$

$$x\cdot\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\cdot\sin nx$$

$$x\cdot\sin x = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cdot\cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx$$

$$x\cdot\sin x = \frac{-2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x + 0$$

$$\boxed{x\cdot\sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x}$$

Ans

Fourier Series (Interval π to - π)

8. Find a Fourier series to represent $f(x) = x - x^2$ in the interval $-\pi \leq x \leq \pi$.

Hence obtain the following relation. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$ 2023

Fourier Series (Interval π to - π)

9. Find a Fourier series expansion of the function $f(x) = x.\mathrm{Sin}x$ in the interval $-\pi \leq x \leq \pi$. Hence deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi-2}{4}$ 2019, 2020

Fourier Series for Discontinuous Functions

1. Find the Fourier Series to represent the function f(x) given by $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$, Prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx + \sum_{n=1}^{\infty} b_n \cdot \sin nx$$

NOW

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x \cdot dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\, dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{(2\pi - x)^2}{-2}\right]_{\pi}^{2\pi}$$

$$a_0 = \frac{1}{2\pi}\left[\pi^2 - 0\right] - \frac{1}{2\pi}\left[0 - \pi^2\right]$$

$$a_0 = \frac{\pi^2}{2\pi} + \frac{\pi^2}{2\pi} = \frac{2\pi^2}{2\pi} = \pi$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x \cdot \cos nx \cdot dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x) \cdot \cos nx \cdot dx$$

$$a_n = \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) + \left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] + \frac{1}{\pi}\left[\left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{(-1)^n}{n^2}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$\boxed{a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]}$$

Fourier Series for Discontinuous Functions

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \cdot \sin nx \cdot dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\sin nx \cdot dx$$

$$b_n = \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$+ \frac{1}{\pi}\left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) + \left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$b_n = \frac{1}{\pi}\left[\left(\frac{-\pi(-1)^n}{n} - 0\right) - (0 - 0)\right]$$

$$+ \frac{1}{\pi}\left[(0 - 0) - \left(\frac{-\pi(-1)^n}{n} - 0\right)\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n + \frac{\pi(-1)^n}{n}\right]$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \cdot \cos nx$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2\cos x}{1^2} - \frac{2\cos 3x}{3^2} - \frac{2\cos 5x}{5^2} - \cdots\right]$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Put $x = 0$

$$x = 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$-\frac{\pi}{2} \times -\frac{\pi}{4} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

Ans

Fourier Series for Discontinuous Functions

2. Find the Fourier Series to represent the function f(x) given by $f(x) =$

$$\begin{cases} x, & -\pi \le x \le 0 \\ -x, & 0 \le x \le \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx + \sum_{n=1}^{\infty} b_n \cdot \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x \cdot dx + \frac{1}{\pi}\int_{0}^{\pi} (-x)\,dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} - \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$a_0 = \frac{1}{2\pi}\left[0 - \pi^2\right] - \frac{1}{2\pi}\left[\pi^2 - 0\right]$$

$$a_0 = \frac{-\pi^2}{2\pi} - \frac{\pi^2}{2\pi} = \frac{-2\pi^2}{2\pi}$$

$$\boxed{a_0 = -\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x \cdot \cos nx \cdot dx + \frac{1}{\pi}\int_{0}^{\pi} (-x)\cos nx \cdot dx$$

$$a_n = \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0} - \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\left(0 + \frac{1}{n^2}\right) - \left(0 + \frac{(-1)^n}{n^2}\right)\right] - \frac{1}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2} + \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[1 - (-1)^n\right]$$

$$\boxed{a_n = \frac{2}{\pi n^2}\left[1 - (-1)^n\right]}$$

Fourier Series for Discontinuous Functions

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x \cdot \sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} (-x) \sin nx \cdot dx$$

$$b_n = \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$- \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[(0-0) - \left(\pi \frac{(-1)^n}{n} - 0\right)\right]$$

$$- \frac{1}{\pi}\left[\left(-\pi \frac{(-1)^n}{n} + 0\right) - (0-0)\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx + \sum_{n=1}^{\infty} b_n \cdot \sin nx$$

$$\boxed{f(x) = -\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[1-(-1)^n\right] \cdot \cos nx}$$

AS

Fourier Series (Interval π to - π)

12. Find the Fourier Series for the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$ Hence deduce that

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$ 2014, 2022

Fourier Series (Interval π to - π)

13. If $f(x) = \begin{cases} lx & 0 \le x \le \frac{\pi}{2} \\ l(\pi - x) & \frac{\pi}{2} \le x \le \pi \end{cases}$ show that $f(x) = \frac{4l}{\pi}\left[\frac{Sinx}{1^2} - \frac{Sin3x}{3^2} + \frac{Sin5x}{5^2} + \ldots\ldots\ldots\ldots\ldots\ldots\right]$ 2020

Fourier Series (Change of Interval)

3. Obtain the Fourier series for $= f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$

Interval $= 2l = 2 - 0 = 2$

$$\boxed{l = 1}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos n\pi x + \sum_{n=1}^{\infty} b_n \cdot \sin n\pi x$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \int_0^2 f(x)\,dx$$

$$a_0 = \int_0^1 \pi x \cdot dx + \int_1^2 \pi(2-x)\,dx$$

$$a_0 = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$a_0 = \frac{\pi}{2}[1-0] + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$a_0 = \frac{\pi}{2} + \pi\left[2 - \frac{3}{2}\right] = \frac{\pi}{2} + \frac{\pi}{2}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\cdot dx$$

$$a_n = \int_0^2 f(x)\cdot\cos n\pi x \cdot dx$$

$$a_n = \int_0^1 \pi x \cdot \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$a_n = \pi\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - \left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+ \pi\left[(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) + \left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$a_n = \pi\left[\left\{0 + \frac{(-1)^n}{n^2\pi^2}\right\} - \left\{0 + \frac{1}{n^2\pi^2}\right\}\right]$$

$$+ \pi\left[\left\{0 - \frac{1}{n^2\pi^2}\right\} - \left\{0 - \frac{(-1)^n}{n^2\pi^2}\right\}\right]$$

$$a_n = \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$a_n = \frac{2\pi}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\boxed{a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\cdot \sin\frac{n\pi x}{l}$$

$$b_n = \int_0^2 f(x)\sin n\pi x$$

$$b_n = \int_0^1 \pi x\cdot \sin n\pi x + \int_1^2 \pi\,(2-x)\sin n\pi x\cdot dx$$

$$b_n = \pi\left[x\cdot\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{n\pi}\right)\right]_0^1$$

$$+\pi\left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) + \left(\frac{-\sin n\pi x}{n\pi}\right)\right]_1^2$$

$$b_n = \pi\left[\left\{\frac{-(-1)^n}{\pi n} + 0\right\} - \{0-0\}\right]$$

$$+\pi\left[\left\{(0-0) - \left(\frac{-(-1)^n}{n\pi} - 0\right)\right\}\right]$$

$$b_n = \pi\left[\cancel{-\frac{(-1)^n}{n\pi}} + \cancel{\frac{(-1)^n}{n\pi}}\right] \Rightarrow \boxed{b_n = 0}$$

So

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cdot\cos n\pi x + \sum_{n=1}^{\infty} b_n\cdot\sin n\pi x$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos n\pi x$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left\{\frac{-2\cos\pi x}{1^2} - \frac{2}{3^2}\cos 3\pi x - \frac{2}{5^2}\cos 5\pi x - \cdots\right\}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$$

Fourier Series(Half Range Series)

15. Expand $f(x) = 2x - 1$ *as a* cosine series in $0 < x < 2$. **2017**

Fourier Series(Half Range Series)

16. Find the half range cosine series for the function $f(x) = (x-1)^2$ *in the interval* 0 to 1 and show

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$$ 2023

Fourier Series(Half Range Series)

17. Find the Fourier half range cosine series for the function $f(x) = \begin{cases} 2t & 0 < t < 1 \\ 2(2-t) & 1 < t < 2 \end{cases}$ 2018, 2023

Fourier Series(Half Range Series)

18. Find half range sine series of $f(x) = \begin{cases} x & 0 < x < 2 \\ 4 - x & 2 < x < 4 \end{cases}$ 2019.

Fourier Series(Half Range Series)

2.If $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$ Show that

i. $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \dots\dots\dots\dots\right]$

ii. $f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{\cos 2x}{2^2} + \frac{\cos 6x}{6^2} + \frac{\cos 10x}{10^2} \dots\dots\dots\dots\right]$

(1) $f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \cdot \sin nx$

$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \cdot dx$

$b_n = \frac{2}{\pi}\int_0^{\pi/2} x \cdot \sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx \cdot dx$

$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) + \left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$

$b_n = \frac{2}{\pi}\left[\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[(0+0) - \left(\frac{-\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]$

$b_n = \frac{2}{\pi}\left[\cancel{\frac{-\pi}{2n}\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \cancel{\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$

$b_n = \frac{2}{\pi} \times \frac{2}{n^2}\sin\frac{n\pi}{2} \Rightarrow \boxed{b_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}}$

So $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$

$f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots\right]$

Fourier Series(Half Range Series)

(ii) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx$

Now $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$a_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x \cdot dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\,dx$

$a_0 = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}$

$a_0 = \frac{1}{\pi}\left[\frac{\pi^2}{4} - 0\right] + \frac{2}{\pi}\left[\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$

$a_0 = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{3\pi^2}{8}\right] = \frac{\pi}{4} + \frac{2}{\pi} \times \frac{\pi^2}{8}$

$a_0 = \frac{\pi}{4} + \frac{\pi}{4} \Rightarrow \boxed{a_0 = \frac{\pi}{2}}$

Now

$a_n = \frac{2}{l}\int_0^l f(x) \cdot \cos\frac{n\pi x}{l}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi/2} x \cdot \cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\cos nx \cdot dx$

$a_n = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) + \left(\frac{-\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$

$a_n = \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \left[\left(0 - \frac{(-1)^n}{n}\right) - \left(\frac{\pi}{2n}\sin\frac{n\pi}{2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right)\right]$

$a_n = \frac{2}{\pi}\left[\cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$

$a_n = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]$

Fourier Series(Half Range Series)

$$a_n = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]$$

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]\cos nx$$

$$a_1 = \frac{2}{\pi \times 1^2}\left[2\times 0 + 1 - 1\right] = 0$$

$$a_2 = \frac{-8}{\pi \cdot 2^2}$$

$$a_3 = 0$$

$$a_4 = 0$$

$$a_5 = 0$$

$$a_6 = \frac{-8}{\pi \cdot 6^2}$$

$$a_7 = 0$$

$$a_8 = 0$$

$$a_9 = 0$$

$$a_{10} = \frac{-8}{\pi \times 10^2}$$

$$f(x) = \frac{\pi}{4} + \left[\frac{-8}{\pi \cdot 2^2}\cos 2x - \frac{8}{\pi \cdot 6^2}\cos 6x - \frac{8}{\pi \times 10^2}\cos 10x + \cdots\right]$$

$$f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{\cos 2x}{2^2} + \frac{\cos 6x}{6^2} + \frac{\cos 10x}{10^2} + \cdots\right]$$

IMPORTANT TOPIC

UNIT 4

Unit 4

- Milne’s Thompson Method
- Mobius transformation

Unit 4 (Two Marks)

1. **Write the Cauchy Riemann Equations in Polar Coordinates systems.**
2. **Write the Cauchy Riemann Equations for $f(z) = u + iv$.**
3. Prove that $f(z) = Sinhz$ is analytic.
4. Define Bilinear transformation.
5. Show that the function $f(z) = |z|^2$ is not analytic at the origin .
6. Define conformal mapping.
7. Find the fixed point of bilinear transformation $w = \frac{3z-4}{z-1}$
8. Prove that $u(x, y) = 2x + y^3 - 3x^2y$ is harmonic function
9. Show that $f(z) = z^3$ is analytic
10. Define Harmonic Function.

Unit 4 (Two Marks)

11. Find the point of invariant of the transformation.

12. For the conformal transformation w $=z^2$, find the coefficient of magnification at z=2+i

13. Define Analytic Function.

14. Prove that $u(x, y) = e^x . Cosy$ is harmonic function.

15. Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

16. Find the image of $|z - 2i| = 2$ $under\ the\ mapping\ w = \frac{1}{z}$.

Limit of a function of Complex Number

1. Check Lim $\left(\frac{z}{\bar{z}}\right)^2$ exist or not at z=0.

$z = x+iy$
$\bar{z} = x-iy$

$$\lim_{z\to z_0} f(z) = \lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2 = \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x+iy}{x-iy}\right)^2$$

(1) Along y-axis $(x=0)$

$$\lim_{z\to 0} f(z) = \lim_{y\to 0}\left(\frac{0+iy}{0-iy}\right)^2 = \lim_{y\to 0}\left(\frac{iy}{-iy}\right)^2$$

$$\lim_{z\to 0} f(z) = \lim_{y\to 0}(-1)^2 = 1$$

(2) Along x-axis $(y=0)$

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\left(\frac{x}{x}\right)^2 = \lim_{x\to 0}(1)^2 = 1$$

(3) Along $y = mx$

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\left(\frac{x+imx}{x-imx}\right)^2$$

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\left(\frac{1+im}{1-im}\right)^2$$

$$\lim_{z\to 0} f(z) = \left(\frac{1+im}{1-im}\right)^2$$

So limit is not unique

Hence Limit does not exist.

Continuity

2. Show that z. $\bar{z}$ is continuous at origin.

$$\lim_{z\to z_0} f(z) = \lim_{z\to 0} z\cdot\bar{z} = \lim_{z\to 0}(x+iy)(x-iy)$$

$$\lim_{z\to z_0} f(z) = \lim_{z\to 0}(x^2+y^2) = \lim_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)$$

At y-Axis $(x=0)$

$$\lim_{z\to z_0} f(z) = \lim_{y\to 0} y^2 = 0$$

At x-Axis $(y=0)$

$$\lim_{z\to z_0} f(z) = \lim_{x\to 0} x^2 = 0$$

At $y = mx$

$$\lim_{z\to z_0} f(z) = \lim_{x\to 0}(x^2+m^2x^2) = \lim_{x\to 0} x^2(1+m^2)$$

$$\lim_{z\to z_0} f(z) = 0$$

So Limit Exist. [Unique & Single, Infinite]

Now

$$f(z) = z\cdot\bar{z} \Rightarrow f(0) = 0$$

$$f(z) = x^2+y^2 \quad f(0) = 0$$

So $f(z)$ is defined

$$\lim_{z\to 0} f(z) = f(0) = f(z) = 0$$

Hence $f(z) = z\cdot\bar{z}$ is continuous at origin.

Differentiability/ Derivative of Function

3. Show that the complex variable function $f(z) = |z|^2$ is differentiable at origin.

$$|z| = \sqrt{x^2+y^2}$$

$$f(z) = |z|^2 = x^2+y^2$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f'(0) = \lim_{z \to 0} \frac{(x^2+y^2) - f(0)}{z - 0}$$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2+y^2}{x+iy}$$

Along y-axis $(x = 0)$

$$f'(0) = \lim_{y \to 0} \frac{y^2}{iy} = \lim_{y \to 0} \frac{y}{i} = 0$$

Along x-axis $(y = 0)$

$$f'(0) = \lim_{x \to 0} \left(\frac{x^2}{x}\right) = \lim_{x \to 0} (x) = 0$$

About $y = mx$

$$f'(0) = \lim_{x \to 0} \frac{x^2+m^2x^2}{x+imx} = \lim_{x \to 0} \frac{x(1+m^2)}{(1+im)}$$

$$f'(0) = \frac{0(1+m^2)}{(1+im)} = 0$$

$f'(0)$ is unique for all possible path

Hence $f(z)$ is differentiable at origin.

Cauchy-Riemann Equation (Cartesian Form)

1. Show that the function $e^x(Cosy + i.Siny)$ is an analytic function, find its derivative.

$f(z) = u + iv = e^x[\cos y + i\sin y]$

$u + iv = \underline{e^x.\cos y} + i\underline{e^x.\sin y}$

So $u = e^x.\cos y$, $v = e^x.\sin y$

$\frac{\partial u}{\partial x} = e^x.\cos y$, $\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial v}{\partial y} = e^x.\cos y$, $\frac{\partial v}{\partial x} = e^x.\sin y$

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$, $\boxed{\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}}$

Hence C-R Equation are Satisfied.

All partial derivative Continuous.

Hence $e^x[\cos y + i\sin y]$ is analytical

Derivative

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = e^x.\cos y + i e^x \sin y$

$f'(z) = e^x[\cos y + i\sin y]$

$f'(z) = e^x.e^{iy} = e^{(x+iy)} = e^z$

$f'(z) = e^z$ Ans

$\{e^{i\theta} = \cos\theta + i\sin\theta\}$

Cauchy-Riemann Equation (Cartesian Form)

5. Find the value of c_1 and c_2 such that the function $f(z) = x^2+c_1y^2-2xy + i(c_2x^2-y^2+2xy)$ is analytic & find $f'(z)$.

$f(z) = (x^2+c_1y^2-2xy) + i(c_2x^2-y^2+2xy)$

$u+iv = (x^2+c_1y^2-2xy) + i(c_2x^2-y^2+2xy)$

$u = (x^2+c_1y^2-2xy)$, $v = (c_2x^2-y^2+2xy)$

$\frac{\partial u}{\partial x} = 2x-2y$, $\frac{\partial u}{\partial y} = 2c_1y-2x$

$\frac{\partial v}{\partial y} = -2y+2x$, $\frac{\partial v}{\partial x} = 2c_2x+2y$

Our function is analytic

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x-2y$ (which is true)

Now

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2c_1y - 2x = -2c_2x - 2y$

Comparison of coefficient of x & y

So $2c_1 = -2 \Rightarrow \boxed{c_1 = -1}$

$-2 = -2c_2 \Rightarrow \boxed{c_2 = 1}$ } Ans

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x-2y) + i(2c_2x+2y)$

$f'(z) = 2x-2y+(2x+2y)i$

$f'(z) = 2x-2y+2xi+2yi$

$f'(z) = 2(x+iy)+2i^2y+2xi$

$f'(z) = 2(x+iy)+2i(x+iy)$

$f'(z) = 2z+2iz = 2z(1+i)$ Ans

Cauchy-Riemann Equation (Cartesian Form)

7. Show that the function $f(z) = u + iv$, where $f(z) = f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

Satisfied the Cauchy –Riemann equation at z=0. Is the function analytic at z=0? Justify the answer.

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$u+iv = \frac{x^3+ix^3-y^3+iy^3}{x^2+y^2}$$

$$u+iv = \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}$$

So $u = \frac{x^3-y^3}{x^2+y^2}$, $v = \frac{x^3+y^3}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h}$$

(At origin)

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}-0}{h} = \lim_{h\to 0} \frac{h^3}{h^3}$$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} 1 = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,0+k)-u(0,0)}{k}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{\frac{-k^3}{k^2}-0}{k} = \lim_{k\to 0} \frac{-k^3}{k^3}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} (-1) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,0)-v(0,0)}{h}$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}-0}{h} = \lim_{h\to 0} \frac{h^3}{h^3}$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} (1) = 1$$

$$\frac{\partial v}{\partial y} = \lim_{h\to 0} \frac{v(0,0+k)-v(0,0)}{k}$$

$$\frac{\partial v}{\partial y} = \lim_{h\to 0} \frac{\frac{k^3}{k^2}-0}{k} = \lim_{h\to 0} \frac{k^3}{k^3} = \lim_{h\to 0} 1 = 1$$

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Hence C-R eqn are satisfied at $z=0$

Cauchy-Riemann Equation (Cartesian Form)

7. Show that the function $f(z) = u + iv$, where $f(z) = f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

Satisfied the Cauchy –Riemann equation at z=0. Is the function analytic at z=0? Justify the answer.

Now

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

So

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}.$$

$$f'(0) = \lim_{z \to 0} \frac{\frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)} - 0}{(x+iy)}$$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Along x-axis $(y = 0)$

$$f'(0) = \lim_{x \to 0} \frac{x^3+ix^3}{x^3} = \lim_{x \to 0} (1+i) = (1+i)$$

Along y-axis $(x = 0)$

$$f'(0) = \lim_{y \to 0} \frac{-y^3+iy^3}{iy^3} = \lim_{y \to 0} \frac{-1+i}{i} \times \frac{i}{i} = \lim_{y \to 0} \frac{-i+i^2}{i^2}$$

$$f'(0) = \frac{-i-1}{-1} = (i+1)$$

Along $y = mx$

$$f'(0) = \lim_{x \to 0} \frac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{(x^2+m^2x^2)(x+imx)} = \lim_{x \to 0} \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

$$f'(0) = \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}$$

If $m=1$

$$f'(0) = \frac{0 + i(2)}{2(1+i)} = \frac{2i}{2(1+i)} = \frac{i}{1+i} \times \frac{1-i}{1-i}$$

$$f'(0) = \frac{i - i^2}{1-i^2} = \frac{i+1}{1+1} = \frac{i+1}{2}$$

$f'(0)$ is not unique Hence the function $f(z)$ is not analytic at $z=0$. Ans

Cauchy-Riemann Equation (Cartesian Form)

2. Show that the function $f(z) = u + iv, where\ f(z) = f(z) = \begin{cases} \frac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is not analytic at the origin even though it satisfies Cauchy Riemann equations at the origin. 2020, 2023

Cauchy-Riemann Equation (Cartesian Form)

3. Show that the function $f(z) = u + iv, where\ f(z) = f(z) = \begin{cases} \frac{x^2y^5(x+iy)}{x^4+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ satisfied Cauchy Riemann Equation at $z = 0$ but function is not differentiable. 2016, 2019

Harmonic Function & Determination of Conjugate Function

1. Show that the function $u = \frac{1}{2}log(x^2 + y^2)$ is harmonic. Find its harmonic conjugate.

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \times \frac{1 \times 2x}{(x^2+y^2)} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \times \frac{1 \times 2y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2) \times 1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2} = 0$$

Hence u is a harmonic function.

Let v is the harmonic conjugate of u

$$v = v(x, y)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

$$dv = \left[\frac{-y}{x^2+y^2}\right]dx + \left[\frac{x}{x^2+y^2}\right]dy \quad \{M \cdot dx + N dy\}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right] = \frac{(x^2+y^2)(-1) + y \times 2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] = \frac{(x^2+y^2)(1) - x \times 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Hence function is Exact.

$$V = \int_{(y=\text{const})} M \cdot dx + \int_{(\text{without x term})} N \cdot dy + C$$

$$v = \int \frac{-y}{x^2+y^2}\,dx + C$$

$$v = -y \int \frac{1}{x^2+y^2}\,dx + C$$

$$v = -y \times \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + C$$

$$\boxed{v = -\tan^{-1}\left(\frac{x}{y}\right) + C}$$

Ans

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Harmonic Function & Determination of Conjugate Function

1. Show that the function $u(x,y) = x^4 - 6x^2y^2 + y^4$ is harmonic. Also find the analytic function $f(z) = u(x,y) + i.v(x,y)$

$u(x,y) = x^4 - 6x^2y^2 + y^4$

$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$

$\frac{\partial u}{\partial y} = -12x^2y + 4y^3, \quad \frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{12x^2} - \cancel{12y^2} - \cancel{12x^2} + \cancel{12y^2} = 0$

Hence u is a harmonic function.

Let v is a conjugate function

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$

$dv = (12x^2y - 4y^3)dx + (4x^3 - 12xy^2)dy$

$dv = 12x^2y\,dx - 4y^3dx + 4x^3dy - 12xy^2dy$

$dv = [12x^2y\,dx + 4x^3dy] - [4y^3dx + 12xy^2dy]$

$dv = d(4x^3y) - d(4xy^3)$

Integration both side

$$\boxed{V = 4x^3y - 4xy^3 + C}$$

$f(z) = u + iv = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3 + c)$

Numerical on Milne's Thomson Method

1. If $u = e^x(xCosy - ySiny)$ is a harmonic function, find an analytic function $f(z) = u + iv$ such that $f(1) = e$

$$u = e^x[x\cos y - y\sin y]$$

$$\frac{\partial u}{\partial x} = e^x[x\cos y - y\sin y] + e^x[\cos y - 0]$$

$$\frac{\partial u}{\partial x} = e^x\{x\cos y - y\sin y + \cos y\} = \phi_1(x,y) \text{ (say)}$$

$$\frac{\partial u}{\partial y} = e^x[-x\sin y - y\cos y - \sin y] = \phi_2(x,y) \text{ (say)}$$

$$\phi_1(z,0) = e^z[z+1]$$

$$\phi_2(z,0) = e^z[0] = 0$$

By Milne & Thomson method

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$$

$$f(z) = \int e^z(z+1)\,dz + c$$

$$f(z) = (z+1)e^z - 1\cdot e^z + c$$

$$f(z) = e^z[z+1-1] + c$$

$$f(z) = e^z\cdot z + c \quad \text{——— (1)}$$

Now $f(1) = e = e^1\cdot 1 + c$

$$e = e + c \Rightarrow \boxed{c = 0}$$

$$\boxed{f(z) = z\cdot e^z}$$ Ans

Numerical on Milne's Thomson Method

2. Find an analytic function whose imaginary part is $e^{-x}(xCosy + ySiny)$.

$$v = e^{-x}[x\cos y + y\sin y]$$

$$\frac{\partial v}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y] = \psi_1(x,y) \text{ (say)}$$

$$\frac{\partial v}{\partial x} = -e^{-x}[x\cos y + y\sin y] + e^{-x}[\cos y] = \psi_2(x,y) \text{ (say)}$$

$$\psi_1(z,0) = e^{-z}[0] = 0$$

$$\psi_2(z,0) = -e^{-z}[z] + e^{-z}[1] = e^{-z}[1-z]$$

Milne's Thomson method

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c$$

$$f(z) = i\int e^{-z}(1-z)\,dz + c$$

$$f(z) = i\left[(1-z)\left(\frac{e^{-z}}{-1}\right) - (-1)\left(\frac{e^{-z}}{+1}\right)\right] + c$$

$$f(z) = i[e^{-z}(z-1) + e^{-z}] + c$$

$$f(z) = i[e^{-z}(z-1+1)] + c$$

$$\boxed{f(z) = i \cdot z \cdot e^{-z} + c}$$

Numerical on Milne's Thomson Method

3. $u+v=\dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ and $f(z)=u+iv$ is an analytic function of $z=x+iy$, find $f(z)$ in terms of z.

$$u+v=\frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$$

$$\left\{\cosh x=\frac{e^{x}+e^{-x}}{2}\right\}$$

$$u+v=\frac{2\sin 2x}{2\cosh 2y-2\cos 2x}$$

$$u+v=\frac{\sin 2x}{\cosh 2y-\cos 2x}$$

Let $f(z)=u+iv$ — (1)

$if(z)=iu-v$ — (2)

(1) + (2)

$f(z)+if(z)=(u-v)+i(u+v)$

$(1+i)f(z)=(u-v)+i(u+v)$

$F(z)=U+iV$

where $F(z)=(1+i)f(z)$

$U=u-v$, $V=u+v$

It means

$$V=\frac{\sin 2x}{\cosh 2y-\cos 2x}$$

$$\frac{\partial V}{\partial y}=\frac{-\sin 2x\times\sinh 2y}{(\cosh 2y-\cos 2x)^2}=\psi_1(x,y) \text{ say}$$

$$\frac{\partial V}{\partial x}=\frac{(\cosh 2y-\cos 2x)2\cos 2x-\sin 2x(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}$$

$$\frac{\partial V}{\partial x}=\frac{2\cosh 2y\cos 2x-2\cos^2 2x-2\sin^2 2x}{(\cosh 2y-\cos 2x)^2}$$

$$\frac{\partial V}{\partial x}=\frac{2\cos 2x\cosh 2y-2}{(\cosh 2y-\cos 2x)^2}=\psi_2(x,y) \text{ (say)}$$

$$\psi_1(z,0)=0$$

$$\psi_2(z,0)=\frac{2\cos 2z-2}{[1-\cos 2z]^2}=\frac{-2[1-\cos 2z]}{[1-\cos 2z]^2}$$

$$\psi_2(z,0)=\frac{-2}{1-\cos 2z}=\frac{-2}{1-1+2\sin^2 z}$$

$$\psi_2(z,0)=\frac{-2}{2\sin^2 z}=-\operatorname{cosec}^2 z$$

Apply Milne's Thomson Method

$$F(z)=\int\psi_1(z,0)\,dz+i\int\psi_2(z,0)\,dz+c$$

$$F(z)=i\int-\operatorname{cosec}^2 z\,dz+c$$

$$F(z)=i\cot z+c$$

$$f(z)=\frac{F(z)}{1+i}=\frac{i\cot z}{1+i}+\frac{c}{1+i}$$

$$f(z)=\frac{i(1-i)}{1-i^2}\cot z+c_1 \qquad \left[c_1=\frac{c}{1+i}\right]$$

$$f(z)=\frac{i+1}{2}\cot z+c_1 \quad \text{Ans}$$

Numerical on Milne's Thomson Method

8. If $u - v = (x - y)(x^2 + 4xy + y^2)$, then find f(z) in terms of z.

Numerical on Milne's Thomson Method

9. Define an analytic function. If $f = u + iv$ is an analytic function find $f(z)$ in terms of z if $u - v = e^x$ (Cos y-Sin y). 2022

Numerical on Milne's Thomson Method

10. If $f(z) = u + iv$ is an analytic function find $f(z)$ in terms of z if $u - v = \frac{e^y - Cosx + Sinx}{Coshy - Cosx}$ when $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$ 2022.

Example of Transformation or Mapping

1. Transformation the rectangular region ABCD in z –plane bounded by $x = 1, x = 3, y = 0, y = 3$ under the transformation $w = z + (2 + i)$.

In Z- Plane

$x = 1$, $x = 3$

$y = 0$, $y = 3$

In w plane

$w = z + (2 + i)$

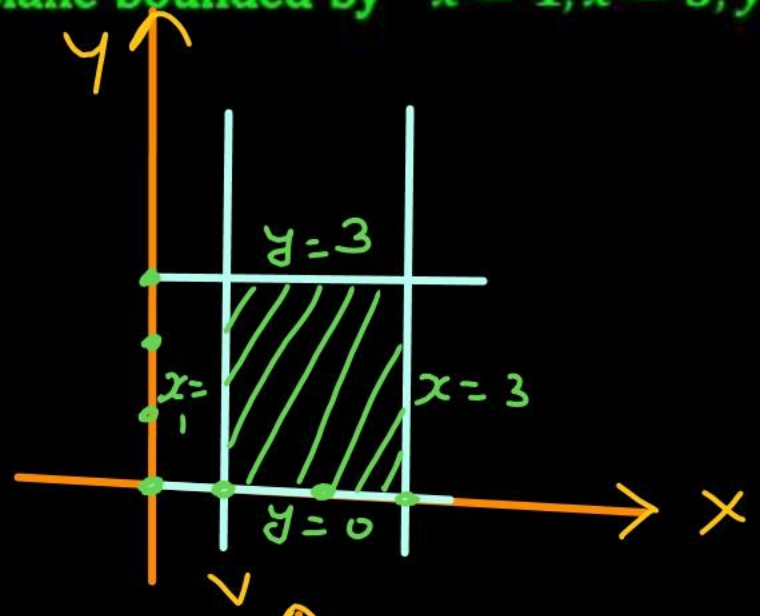
$u + iv = x + iy + 2 + i$

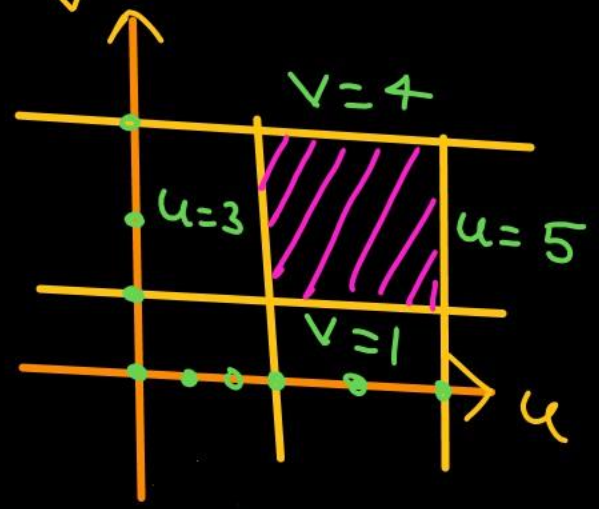
$u + iv = (x+2) + i(y+1)$

So $u = x + 2$

$v = y + 1$

Z-Plane	w- Plane
$x = 1 \Rightarrow$	$u = 3$
$x = 3 \Rightarrow$	$u = 5$
$y = 0 \Rightarrow$	$v = 1$
$y = 3 \Rightarrow$	$v = 4$





Coefficient of Magnification & Angle of Rotation

1. For the conformal mapping $w = z^2$ show that

The coefficient of magnification at $z = 2 + i$ is $2\sqrt{5}$ & angle of rotation is $tan^{-1}0.5$

$w = f(z) = z^2$

$f'(z) = 2z$

$f'(2+i) = 2(2+i) = 4+2i$

Coefficient of magnification =

$|f'(2+i)| = |4+2i|$

$= \sqrt{4^2+2^2} = \sqrt{16+4}$

$= \sqrt{20} = 2\sqrt{5}$ Ans

Angle of rotation

$amp[f'(2+i)]$

$amp(4+2i)$

$= \tan^{-1}\frac{2}{4}$

$= \tan^{-1}\frac{1}{2}$

$= \tan^{-1}0.5$ Ans

Equation of Circle

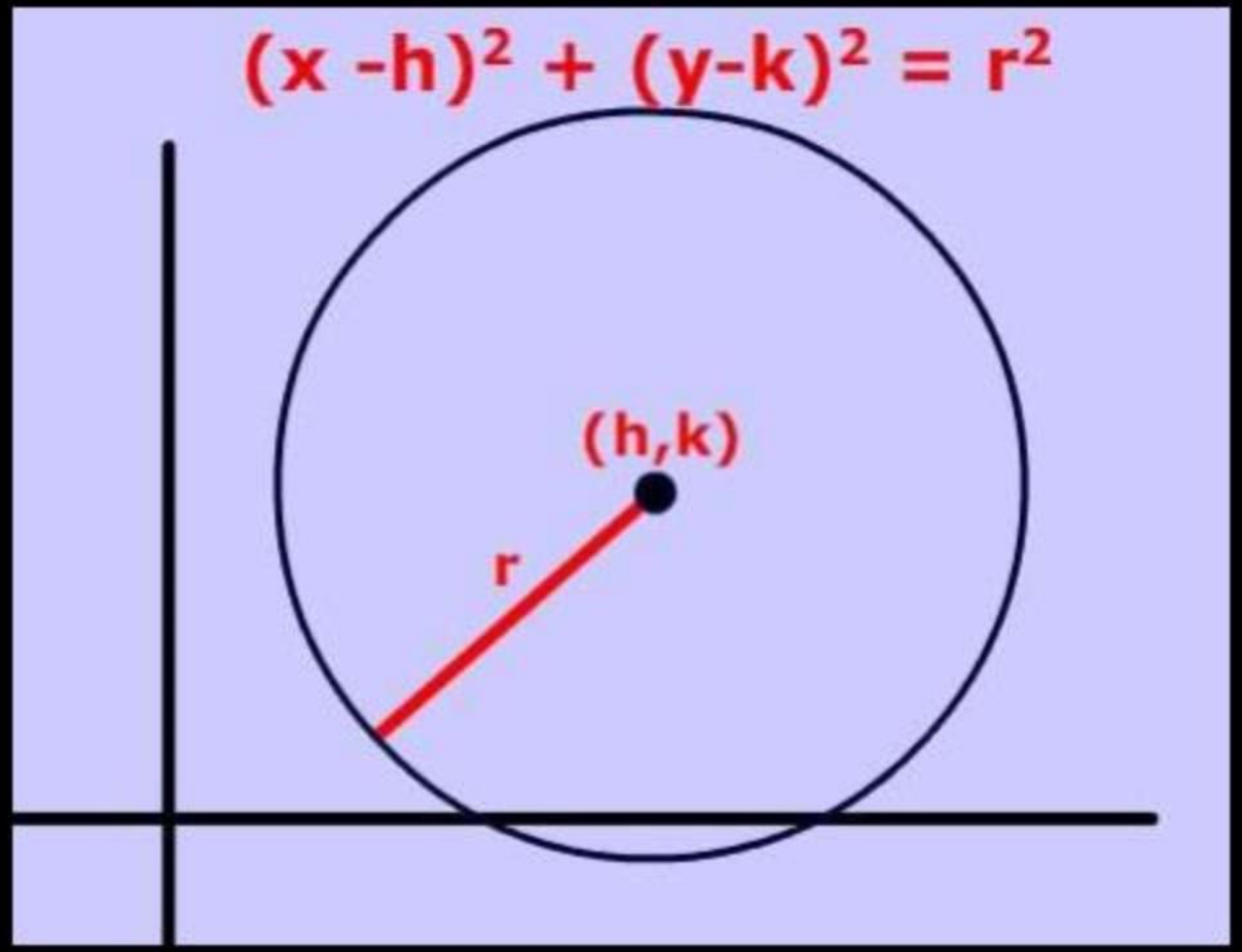
$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

(h, k) → Centre of circle

r = Radius.

At origin point when $h=k=0$

$$\boxed{x^2+y^2=r^2}$$



Theorem on conformal

1. If $u = 2x^2 + y^2$ & $v = \frac{y^2}{x}$ show that u = constant and v = constant cut orthogonally at all intersection but that the transformation $w = u + iv$ is not conformal.

$2x^2 + y^2 = u \Rightarrow 2x^2 + y^2 = k_1$ (say)

Differentiating $\Rightarrow 4x + 2y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-4x}{2y} \Rightarrow \boxed{m_1 = \frac{-2x}{y}}$$

$$V = \frac{y^2}{x} = k_2 \Rightarrow y^2 = k_2 x$$

Differentiating $2y\frac{dy}{dx} = k_2 = \frac{y^2}{x}$

$$\frac{dy}{dx} = \frac{y^2}{2xy} \Rightarrow \boxed{m_2 = \frac{y}{2x}}$$

Now $m_1 \cdot m_2 = -\frac{2x}{y} \times \frac{y}{2x}$

$$\boxed{m_1 \cdot m_2 = -1}$$

Hence two curve cut orthogonally.

$u = 2x^2 + y^2 \Rightarrow \frac{\partial u}{\partial x} = 4x, \quad \frac{\partial u}{\partial y} = 2y$

$V = \frac{y^2}{x} \Rightarrow \frac{\partial v}{\partial x} = \frac{-y^2}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{2y}{x}$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

So $w = u + iv$ is not analytic. Hence transformation is not conformal.

Translation

1. Let a rectangular domain R bounded by $x = 0, y = 0, x = 2, y = 1$. Determine the region R' of w plane into which R is mapped under the transformation $w = z + (1 - 2i)$

In Z-Plane

$x=0,\ y=0,\ x=2,\ y=1$

In w-plane

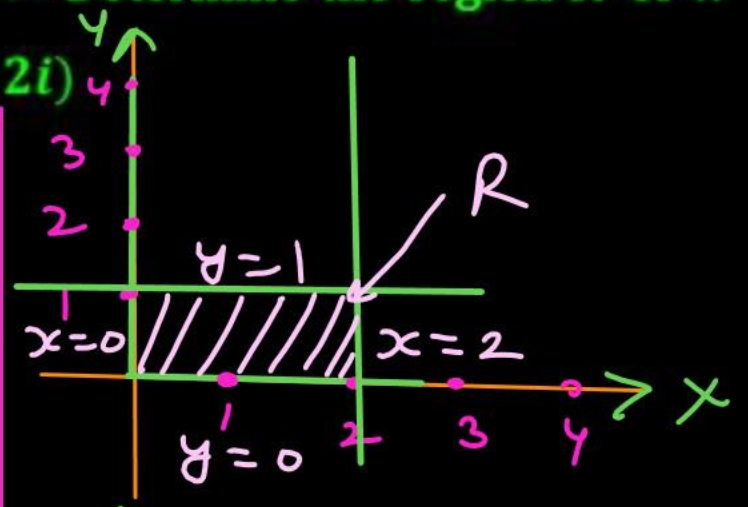
$w = z + (1-2i) = x + iy + 1 - 2i$

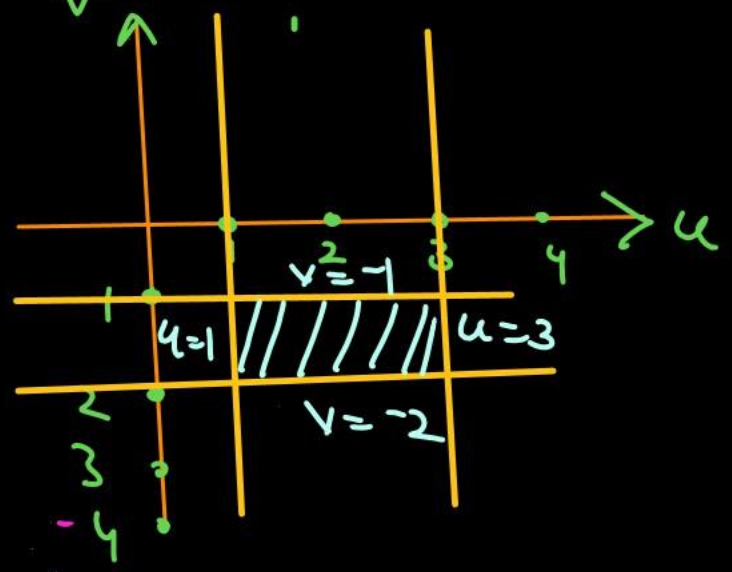
$w = (x+1) + i(y-2)$

$u + iv = (x+1) + i(y-2)$

$$\begin{cases} u = x+1 \\ v = y-2 \end{cases}$$

Z-Plane	W-Plane
$x = 0$	$u = 1$
$y = 0$	$v = -2$
$x = 2$	$u = 3$
$y = 1$	$v = -1$





Translation

2. Find the image of $2x + y - 3 = 0$ *under the transformation* $w = z + 2i$.

In z-plane

$2x+y-3=0$

$2x+y=3$ —— (1)

In w plane

$w = z + 2i$

$u+iv = x+iy+2i$

$u+iv = x+i(y+2)$

$u = x, \quad v = y+2$

$\Downarrow$

$x = u, \quad y = v-2$

Now $2x+y=3$

$2u+v-2=3$

$2u+v=5$ —— (2)

For x-y plane

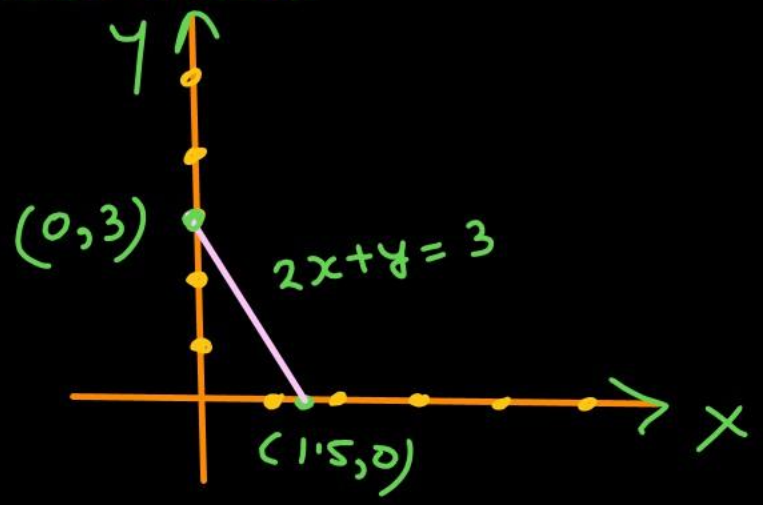
$2x+y=3$

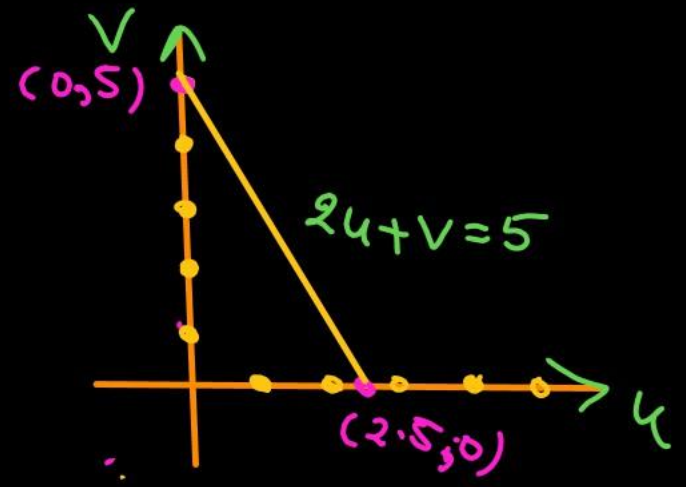
x	0	1.5
y	3	0

For u-v plane

$2u+v=5$

u	0	2.5
v	5	0





Rotation

1. Consider the transformation $w = z.e^{i\pi/4}$ and determine the region R' in w plane corresponding to the triangular region R bounded by the lines $x = 0, y = 0$ *and* $x + y = 1$ *in z plane.*

$x = y$
$x = y$

In Z-plane

$x = 0,\ y = 0,\ x + y = 1$

In w-plane

$w = z \cdot e^{i\frac{\pi}{4}}$

$u + iv = (x+iy)\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$u + iv = (x+iy)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$u + iv = \frac{1}{\sqrt{2}}(x+iy)(1+i)$

$u + iv = \frac{1}{\sqrt{2}}(x + ix + iy - y)$

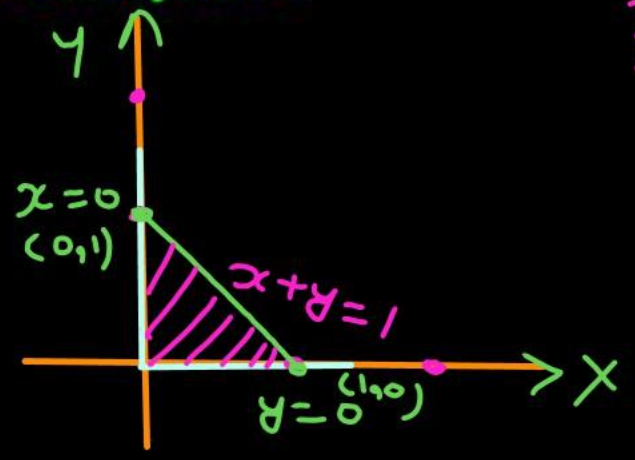
$u + iv = \left(\frac{x-y}{\sqrt{2}}\right) + i\left(\frac{x+y}{\sqrt{2}}\right)$

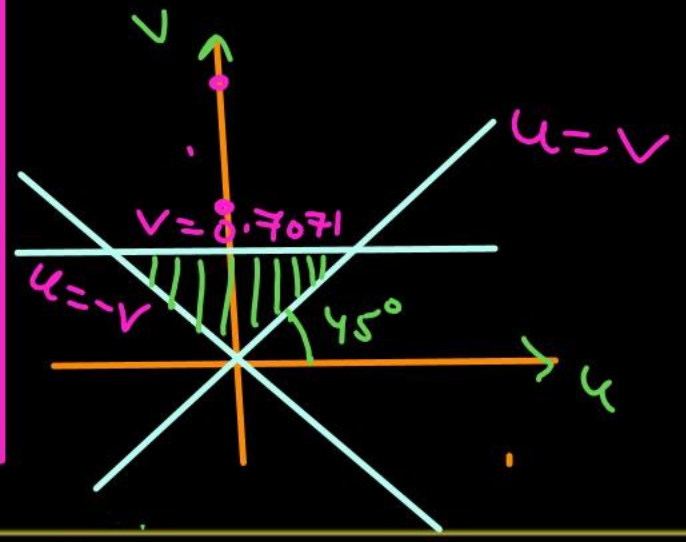
$u = \frac{x-y}{\sqrt{2}},\ v =$

Z-Plane	w = Plane
$x = 0$	$u = \frac{-y}{\sqrt{2}},\ v = \frac{+y}{\sqrt{2}}$ $\boxed{u = -v}$
$y = 0$	$u = \frac{x}{\sqrt{2}},\ v = \frac{x}{\sqrt{2}}$ $\boxed{u = v}$
$x + y = 1$	$v = \frac{1}{\sqrt{2}} = 0.7071$ $\boxed{v = 0.7071}$

For $x + y = 1$

x	0	1
y	1	0





Rotation

2. Determine the region in the w plane in which the rectangle bounded by the lines $x = 0, y = 0, x = 2, y = 1$ is mapped under the transformation $w = \sqrt{2}e^{i\pi/4}.z$.

In Z-Plane

$x=0,\ y=0,\ x=2,\ y=1$

In w-plane

$$w = \sqrt{2}\cdot e^{i\frac{\pi}{4}} z$$

$$u+iv = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] z$$

$$u+iv = \sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] z$$

$$(u+iv) = (1+i)(x+iy)$$

$$u+iv = x+ix+iy-y$$

$$u+iv = (x-y)+i(x+y)$$

$$u = x-y$$

$$v = x+y$$

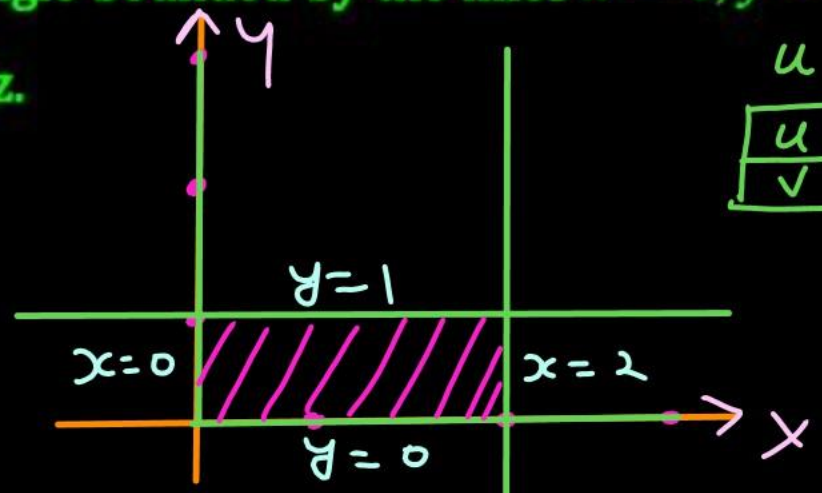
Z-Plane	W-Plane
$x=0$	$u=-y,\ v=+y$ $\boxed{u=-v}$
$y=0$	$u=x,\ v=x$ $\boxed{u=v}$
$x=2$	$u=2-y,\ v=2+y$ $\boxed{u+v=4}$
$y=1$	$u=x-1,\ v=x+1$ $\boxed{v-u=2}$

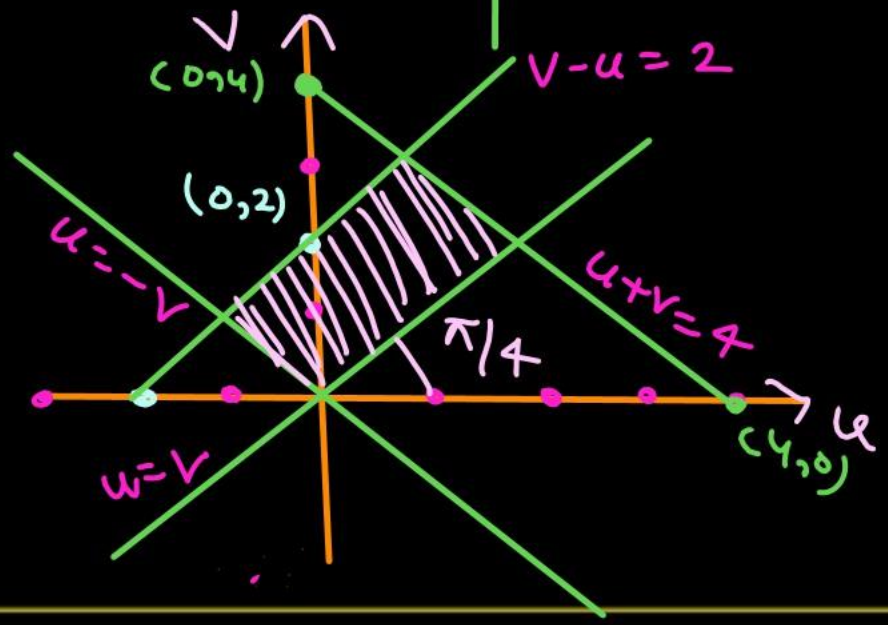
$v-u=2$

u	0	-2
v	2	0

$u+v=4$

u	4	0
v	0	4





Magnification (Stretching)

1. Consider the transformation $w = 2z$ & determine the region R' of w plane into which the triangular region R enclosed by the lines $x = 0, y = 0, x + y = 1$ in z plane is mapped under the map.

In z- Plane

$x = 0$, $y = 0$

$x + y = 1$

x	0	1
y	1	0

In w- plane

$w = 2z$

$u + iv = 2(x + iy)$

$u + iv = 2x + i \cdot 2y$

$u = 2x \Rightarrow x = \frac{u}{2}$

$v = 2y \Rightarrow y = \frac{v}{2}$

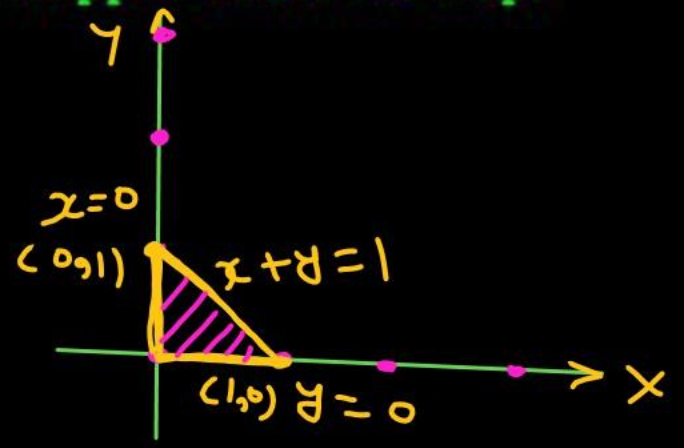
z-Plane	w-Plane
$x = 0$	$u = 0$
$y = 0$	$v = 0$
$x + y = 1$	$u + v = 2$

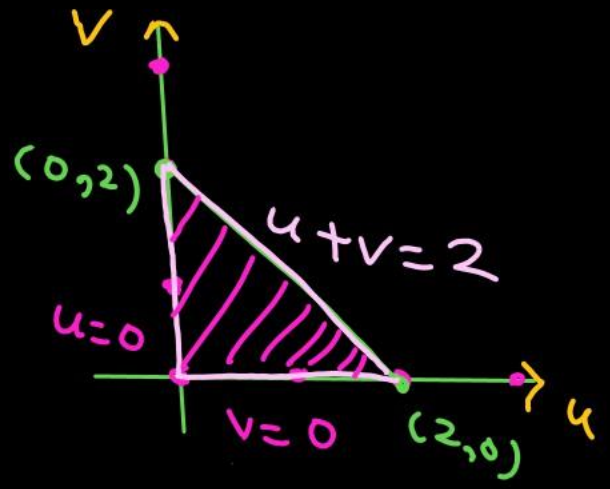
$x + y = 1$

$\frac{u}{2} + \frac{v}{2} = 1$

$\boxed{u + v = 2}$

u	0	2
v	2	0





Magnification (Stretching)

2. Find the image of circle $|z| = 2$ under the transformation $w = 3z$.

In z-plane

$|z| = 2$

$|x+iy| = 2$

$\sqrt{x^2+y^2} = 2$

$x^2+y^2 = 2^2$ —— (1)

It is a eqn of circle with centre (0,0) and radius 2.

In w plane

$w = 3z$

$u+iv = 3(x+iy)$

$u+iv = 3x+i3y$

$u = 3x, \quad v = 3y$

$x = \frac{u}{3}, \quad y = \frac{v}{3}$

From eqn (1)

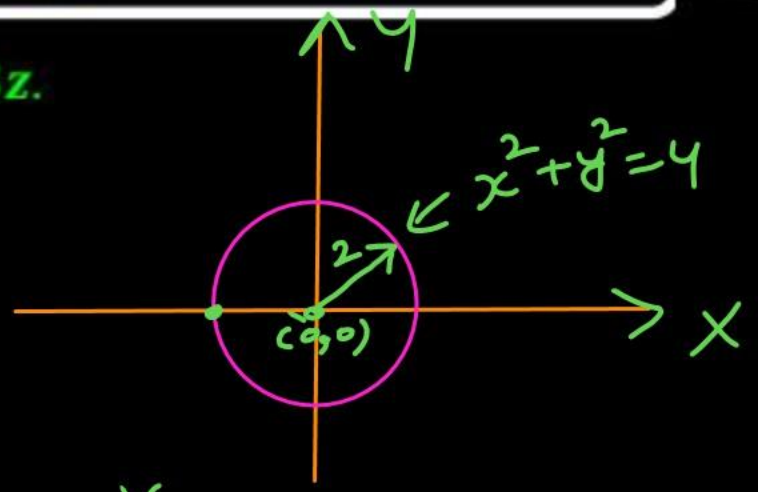
$x^2+y^2 = 4$

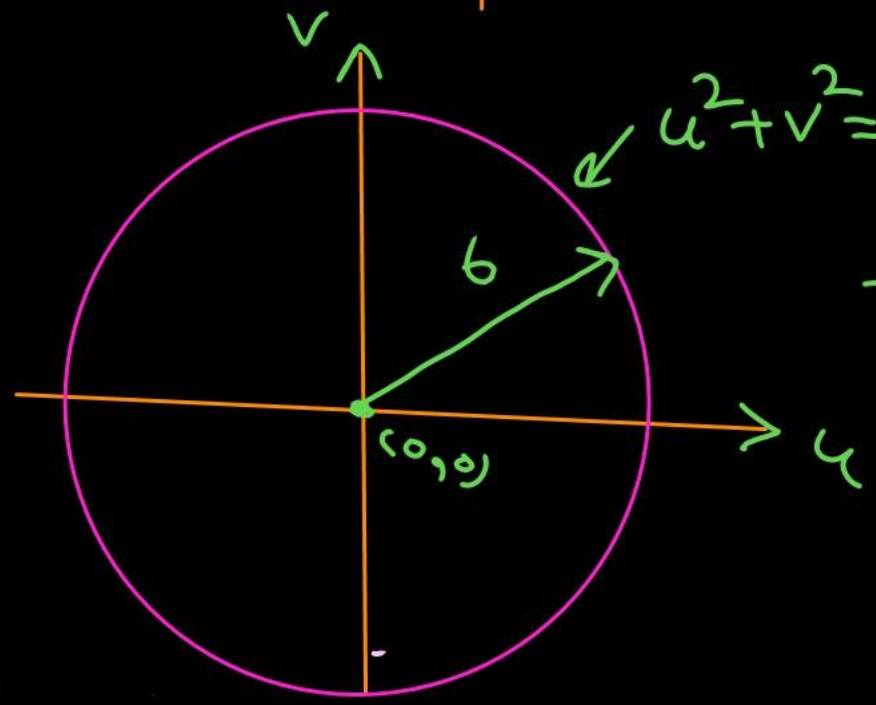
$\left(\frac{u}{3}\right)^2 + \left(\frac{v}{3}\right)^2 = 4$

$\frac{u^2}{9} + \frac{v^2}{9} = 4$

$u^2+v^2 = 36 = 6^2$

It is a eqn of circle with centre (0,0) and radius 6.





Magnification, Translation & Rotation (Mixed Problem)

3. Find the image of the triangle with vertices at i, $1+i$, $1-i$ in the z plane, under the transformation

$$w = 3z + 4 - 2i.$$

In Z-Plane

Vertices are $i, 1+i, 1-i$

Co-ordinates of triangle are $(0,1), (1,1)$ & $(1,-1)$

W - Plane

$$w = 3z + 4 - 2i$$

$$u + iv = 3(x+iy) + 4 - 2i$$

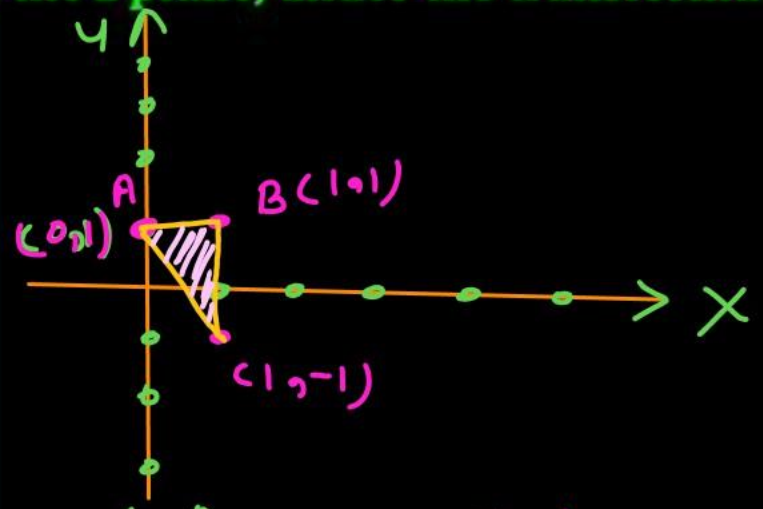
$$u + iv = 3x + i \cdot 3y + 4 - 2i$$

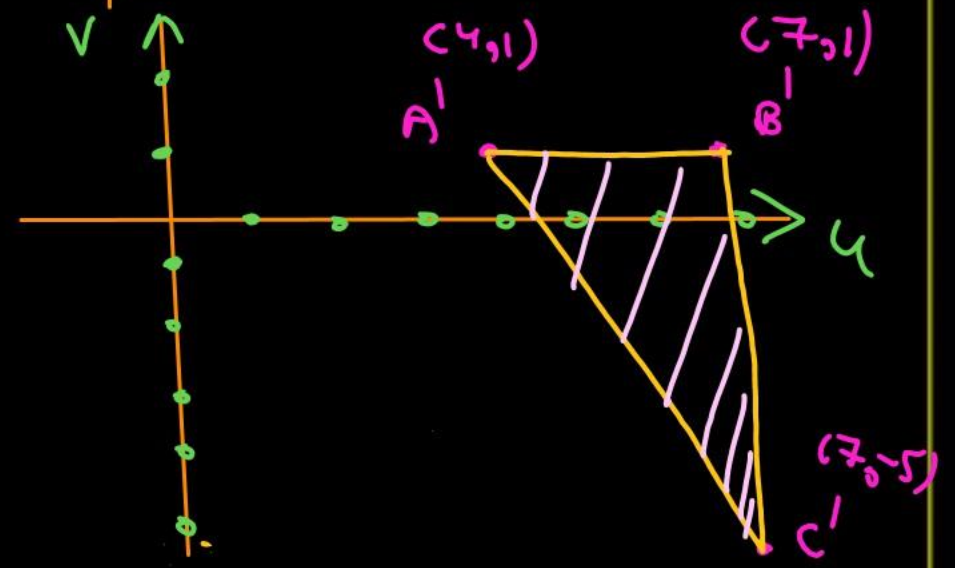
$$u + iv = (3x+4) + i(3y-2)$$

$$u = 3x + 4$$

$$v = 3y - 2$$

Z Plane		W-Plane	
x	y	u	v
0	1	4	1
1	1	7	1
1	-1	7	-5





Magnification, Translation & Rotation (Mixed Problem)

4. Find the image of the triangle with vertices at i, $1+i$, $1-i$ in the z plane, under the transformation

$$w = e^{\frac{5\pi i}{3}}.z - 2 + 4i$$

In z plane

Co-ordinate are

$(0,1)$, $(1,1)$ $(1,-1)$

In w - plane

$$w = e^{\frac{5\pi i}{3}}.z - 2 + 4i$$

$$w = \left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]z - 2 + 4i$$

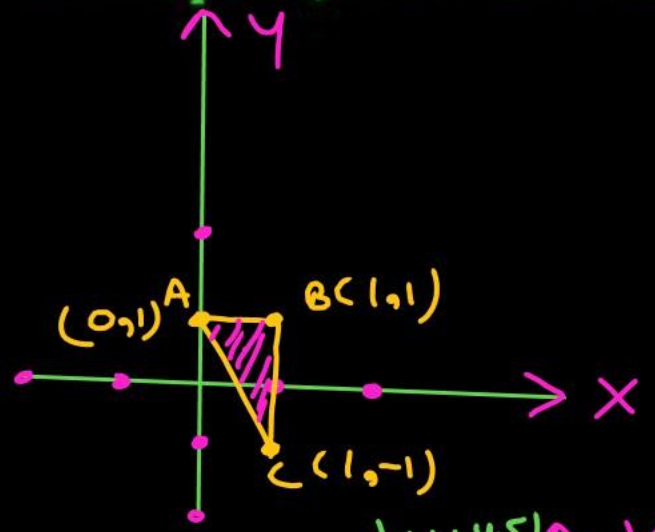
$$w = \left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right](x+iy) - 2 + 4i$$

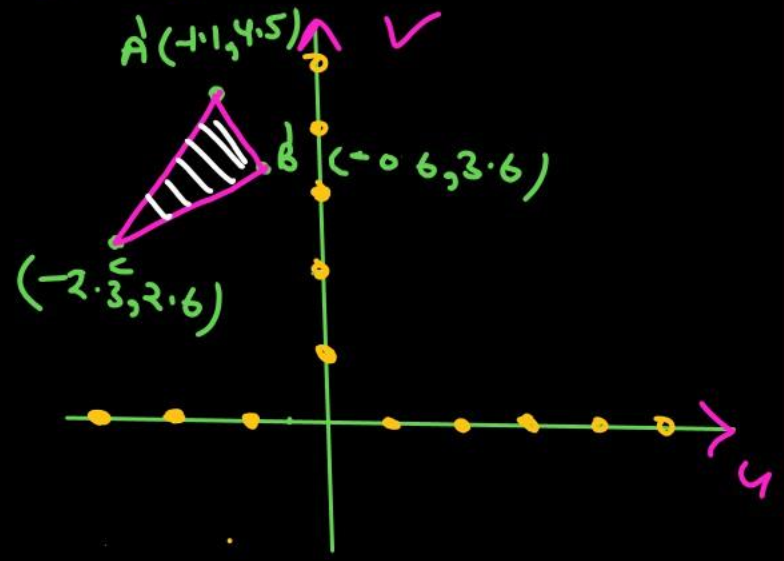
$$u + iv = \left(\frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y\right) + i\left(\frac{\sqrt{3}}{2}x + \frac{y}{2} + 4\right)$$

$$u = \frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y$$

$$v = \frac{\sqrt{3}}{2}x + \frac{y}{2} + 4$$

Z-Plane		W-Plane	
x	y	u	v
0	1	-1.1	4.5
1	1	-0.6	3.6
1	-1	-2.3	2.6





Inversion or Reflection

1. Find the image of $|z-2i| = 2$ under the mapping $w = \frac{1}{z}$.

In z-Plane

$|z-2i| = 2$

$|x+iy-2i| = 2$

$|x+i(y-2)| = 2$

$\sqrt{x^2+(y-2)^2} = 2$

$x^2+(y-2)^2 = 2^2$ — (1)

It is a circle of centre $(0,2)$ & radius 2.

In w plane

$w = \frac{1}{z}$

Put

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

Now eqn

$x^2+(y-2)^2 = 4$

$x^2+y^2-4y+4 = 4$

$x^2+y^2-4y = 0$

$\left[\frac{u}{u^2+v^2}\right]^2 + \left[\frac{-v}{u^2+v^2}\right]^2 - 4\left[\frac{-v}{u^2+v^2}\right] = 0$

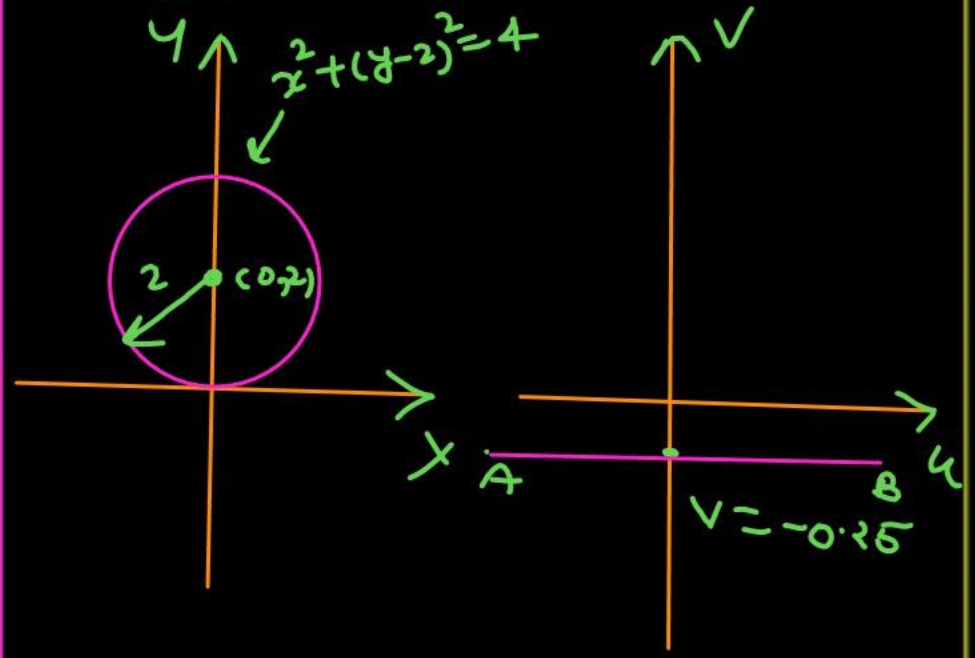
$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{(u^2+v^2)} = 0$

$\frac{u^2+v^2}{(u^2+v^2)^2} + \frac{4v}{(u^2+v^2)} = 0$

$\frac{1}{u^2+v^2} + \frac{4v}{u^2+v^2} = 0$

$\frac{1+4v}{u^2+v^2} = 0 \Rightarrow 1+4v = 0$

$\boxed{v = -\frac{1}{4}} \Rightarrow \boxed{v = -0.25}$



Inversion or Reflection

2. Find the image of infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ *under the transformation* $w = \frac{1}{z}$. Also show the region graphically.

In z plane

$y \ge \frac{1}{4}$, $y \le \frac{1}{2}$

In w plane $\omega = \frac{1}{z}$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

So $\frac{-v}{u^2+v^2} \ge \frac{1}{4}$

$-4v \ge u^2+v^2$

$0 \ge u^2+v^2+4v$

$u^2+v^2+4v \le 0$

$u^2+v^2+4v+4-4 \le 0$

$u^2+(v+2)^2 \le 4$

$u^2+(v+2)^2 \le 2^2$

Centre of circle $= (0,-2)$

Radius $= 2$

Now $\frac{-v}{u^2+v^2} \le \frac{1}{2}$

$-2v \le u^2+v^2$

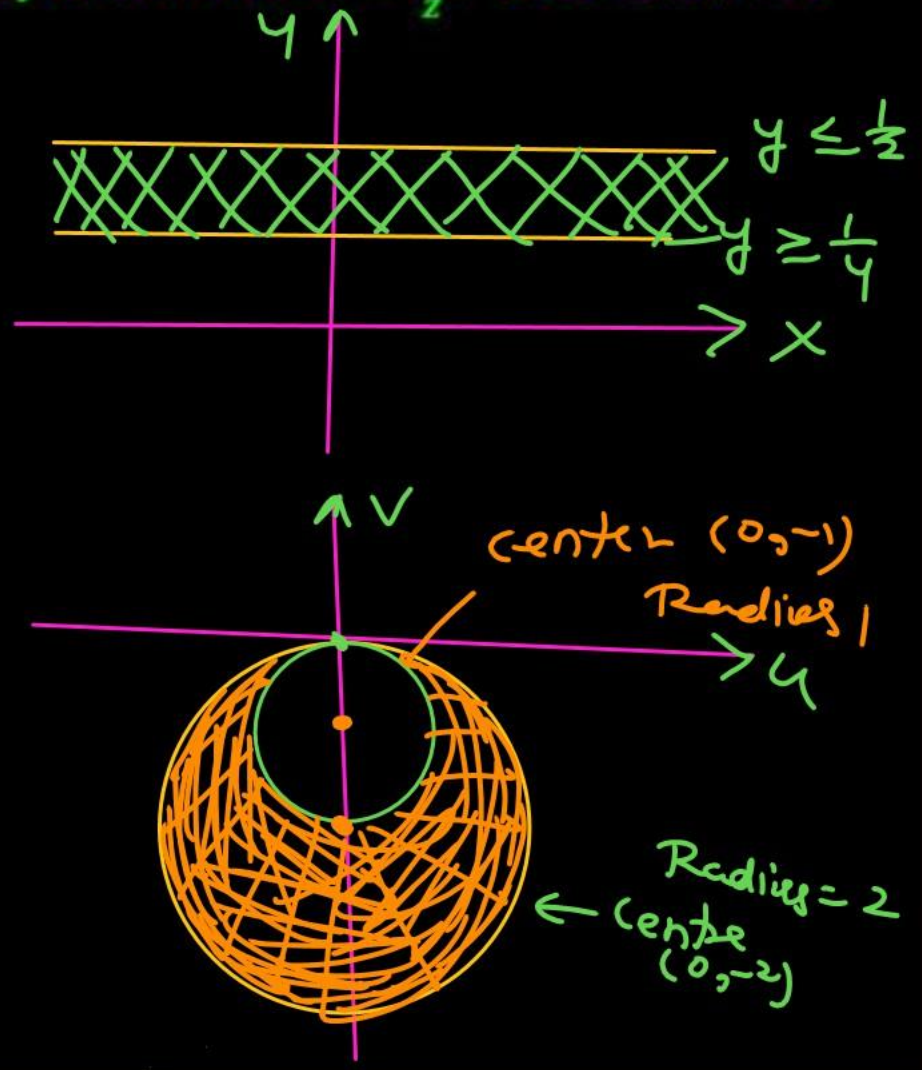
$u^2+v^2+2v \ge 0$

$u^2+v^2+2v+1-1 \ge 0$

$u^2+(v+1)^2 \ge 1^2$

Centre of circle $= (0,-1)$

Radius $= 1$



Inversion or Reflection

3. Find the image of the circle $|z-1| = 1$ *in the complex plane under the mapping* $w = \frac{1}{z}$.

z-Plane

$$|z-1| = 1$$

$$|x+iy-1| = 1$$

$$|(x-1)+iy| = 1$$

$$\sqrt{(x-1)^2+y^2} = 1$$

$$(x-1)^2+y^2 = 1^2$$

Circle of centre $(1,0)$ radius 1.

In w-plane

If $w = \frac{1}{z}$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

$$(x-1)^2+y^2 = 1$$

$$\left(\frac{u}{u^2+v^2}-1\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2 = 1$$

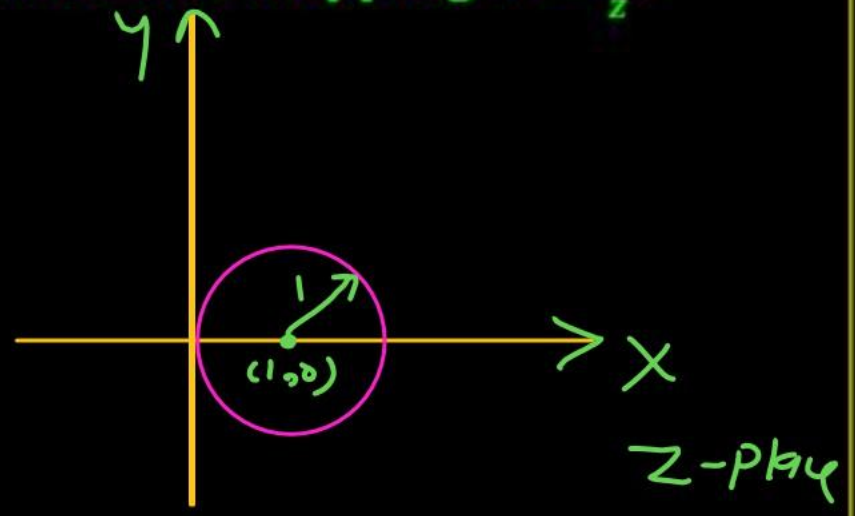
$$\frac{u^2}{(u^2+v^2)^2}+\cancel{1}-\frac{2u}{u^2+v^2}+\frac{v^2}{(u^2+v^2)^2} = \cancel{1}$$

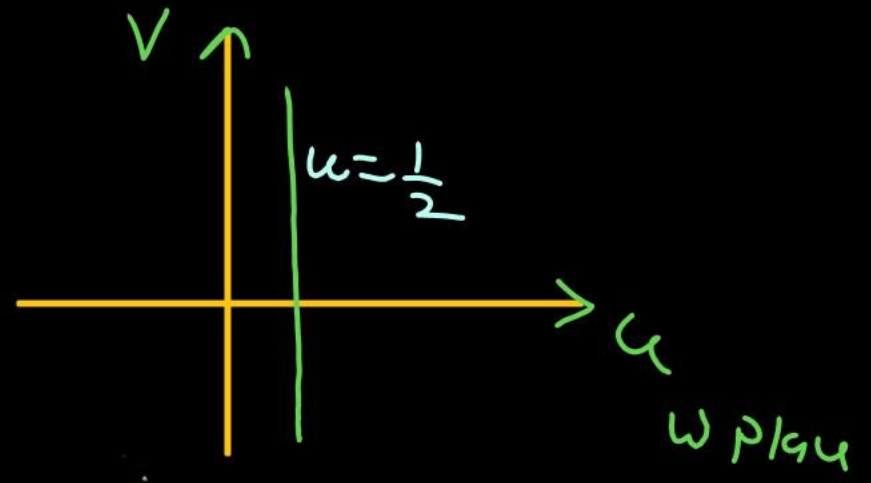
$$\frac{u^2+v^2}{(u^2+v^2)^2}-\frac{2u}{u^2+v^2} = 0$$

$$\frac{1}{u^2+v^2}-\frac{2u}{u^2+v^2} = 0$$

$$\frac{1-2u}{u^2+v^2} = 0$$

$$1-2u = 0 \Rightarrow \boxed{u = \frac{1}{2}}$$





Inversion or Reflection

12. Find the image of the circle $|z-3| = 5$ *under the mapping* $w = \frac{1}{z}$. 2020

Type 1: If all the points of z plane & w plane are finite

1. Find a bilinear transformation which maps the points $z = 0, -i, -1$ into the points $w = i, 1, 0$.

$z_1 = z$, $w_1 = w$

$z_2 = 0$, $w_2 = i$

$z_3 = -i$, $w_3 = 1$

$z_4 = -1$, $w_4 = 0$

By Mobius Transformation $\Rightarrow$

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$\frac{w-i}{+w(1-i)} = \frac{z(-i+1)}{i(-1-z)}$$

$$\frac{w-i}{w-wi} = \frac{-zi+z}{-i-iz}$$

$\Rightarrow (w-i)(-i-iz) = -(-w+wi)(-zi+z)$

$\Rightarrow -wi-wiz+i^2+i^2z = [wzi-wz-i^2wz+wiz]$

$\Rightarrow -wi-wiz-1-z = -[wzi-\cancel{wz}+\cancel{wz}+wzi]$

$\Rightarrow -wi-wiz-1-z = -2wzi$

$\Rightarrow -wi-wiz+2wzi = 1+z$

$= w[-i-iz+2iz] = (1+z)$

$= w[-i+iz] = z+1$

$= -wi[1-z] = z+1$

$= \boxed{w = \frac{z+1}{-i(1-z)}}$ Ans

Type 2: If some points of z plane & w plane are Infinite

2. Find the bilinear transformation which transform the points $z = \infty, i, 0$ into $w = 0, i, \infty$ respectively.

$z_1 = z$, $w_1 = w$

$z_2 = \infty$, $w_2 = 0$

$z_3 = i$, $w_3 = i$

$z_4 = 0$, $w_4 = \infty$

By Mobius Transformation

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

$$\frac{(w_1 - w_2)\left(\frac{w_3}{w_4} - 1\right)}{(w_2 - w_3)\left(1 - \frac{w_1}{w_4}\right)} = \frac{\left(\frac{z_1}{z_2} - 1\right)(z_3 - z_4)}{\left(1 - \frac{z_3}{z_2}\right)(z_4 - z_1)}$$

Now

$$\frac{(w - 0)\left(\frac{i}{\infty} - 1\right)}{(0 - i)\left(1 - \frac{w}{\infty}\right)} = \frac{\left(\frac{z}{\infty} - 1\right)(i - 0)}{\left(1 - \frac{i}{\infty}\right)(0 - z)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$$\frac{w}{i} = \frac{i}{z}$$

$$w = \frac{i^2}{z}$$

$$\boxed{w = \frac{-1}{z}}$$

Ans

Type 2: If some points of z plane & w plane are Infinite

13. Show that the bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = i, w_2 = -1, w_3 = -i$ respectively is $w = \frac{z-i}{iz-1}$. 2023

Type 2: If some points of z plane & w plane are Infinite

14. Find the bilinear transformation which maps the points $z = i, -i, 1$ into $w = 0, 1, \infty$ respectively.

2022

Type 2: If some points of z plane & w plane are Infinite

15. Find the Mobius transformation that maps the points $z = 0, -i, 2i$ into $w = 5i, \infty, \frac{-i}{3}$ respectively.

2019.

Type 3: If the image of figure is also find

3. Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$. Also find the image of the unit circle $|z| = 1$.

$z_1 = z$	$w_1 = w$
$z_2 = 0$	$w_2 = i$
$z_3 = -1$	$w_3 = 0$
$z_4 = i$	$w_4 = \infty$

By Mobius Transformation

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(w_1-w_2)\left(\frac{w_3}{w_4}-1\right)}{(w_2-w_3)\left(1-\frac{w_1}{w_4}\right)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(w-i)\left(\frac{0}{\infty}-1\right)}{(i-0)\left(1-\frac{w}{\infty}\right)} = \frac{(z-0)(-1-i)}{(0+1)(i-z)}$$

$$\boxed{w = \frac{z+1}{z-i}}$$

Now $wz - iw = z+1$

$wz - z = 1 + iw$

$z(w-1) = 1+iw \Rightarrow \boxed{z = \frac{iw+1}{w-1}}$

In z-plane

$|z| = 1 \Rightarrow |x+iy| = 1$

$\sqrt{x^2+y^2} = 1 \Rightarrow \boxed{x^2+y^2 = 1^2}$

It is a circle of center (0,0) and radius 1.

For w-plane

$|z| = 1 \Rightarrow \left|\frac{iw+1}{w-1}\right| = 1$

$\frac{|iw+1|}{|w-1|} = 1 \Rightarrow |iw+1| = |w-1|$

$|i(u+iv)+1| = |u+iv-1|$

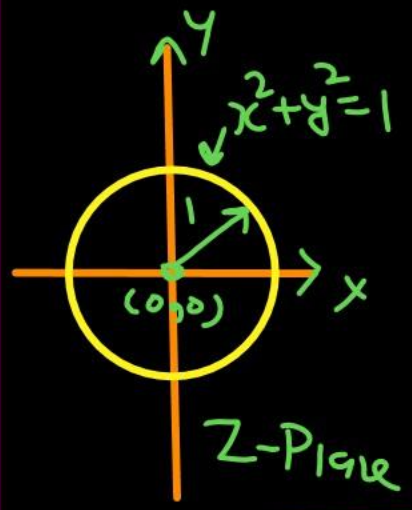
$|iu - v + 1| = |u + iv - 1|$

$|(1-v) + iu| = |(u-1) + iv|$

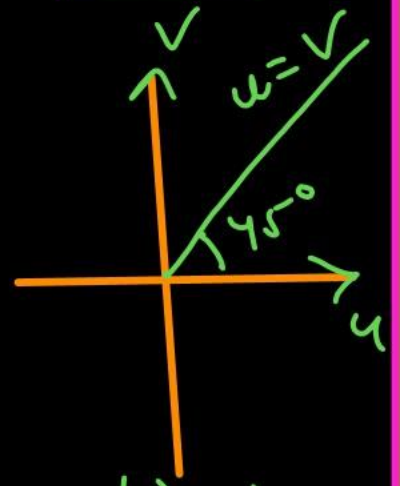
$\sqrt{(1-v)^2 + u^2} = \sqrt{(u-1)^2 + v^2}$

$\cancel{1} + \cancel{v^2} - 2v + \cancel{u^2} = \cancel{u^2} + \cancel{1} - 2u + \cancel{v^2}$

$-2v = -2u \Rightarrow \boxed{u = v}$



Z-Plane



W-plane

Type 3: If the image of figure is also find

4. Find the bilinear transformation which maps the points $z = 1, i, -1$ into $w = 0, 1, \infty$. Show that the transformation maps the interior of the unit circle of the z plane onto the upper half of the w plane.

$z_1 = z$, $w_1 = w$

$z_2 = 1$, $w_2 = 0$

$z_3 = i$, $w_3 = 1$

$z_4 = -1$, $w_4 = \infty$

By Mobius Transformation

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$\Downarrow$

After Calculation

$$\boxed{w = -\frac{i(z-1)}{z+1}}$$

Now $wz + w = -iz + i$

$wz + iz = i - w$

$z(w+i) = i - w$

$$\boxed{z = \frac{i-w}{w+i}}$$

z Plane $|z| < 1$

$$\boxed{x^2 + y^2 < 1}$$

In w plane.

$|z| < 1$

$$\left|\frac{i-w}{w+i}\right| < 1 \Rightarrow \frac{|i-w|}{|w+i|} < 1$$

$|i-w| < |w+i|$

$|i-u-iv| < |u+iv+i|$

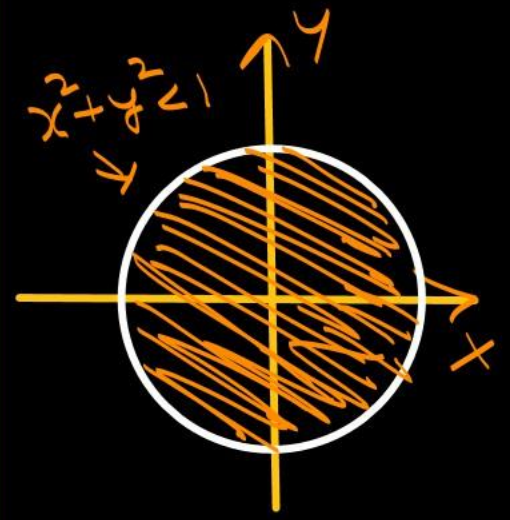
$|-u+i(1-v)| < |u+i(v+1)|$

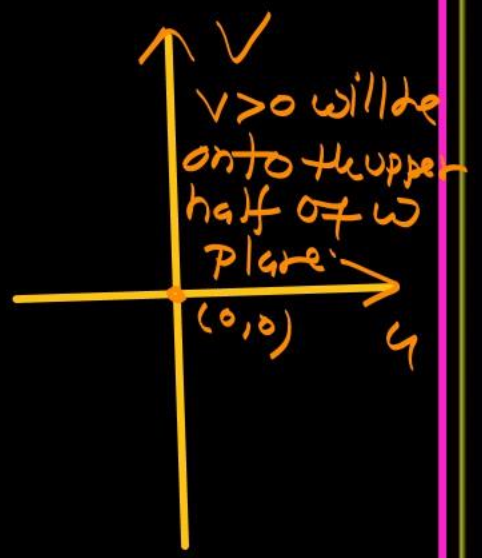
$\sqrt{u^2+(1-v)^2} < \sqrt{u^2+(v+1)^2}$

$\cancel{u^2}+1+\cancel{v^2}-2v < \cancel{u^2}+\cancel{v^2}+1+2v$

$-2v < 2v \Rightarrow 0 < 4v \Rightarrow 0 < v$

$$\boxed{v > 0}$$





Type 4: To find image on z plane & w plane

5. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z plane into the upper half of the w plane. What is the image of circle $|z| = 1$ under the transformation.

$$w = \frac{z}{1-z} \Rightarrow u+iv = \frac{x+iy}{1-x-iy}$$

$$u+iv = \frac{x+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$$

$$u+iv = \frac{x(1-x)+ixy+iy(1-x)-y^2}{(1-x)^2-i^2y^2}$$

$$u+iv = \frac{x(1-x)-y^2+i(xy+y-xy)}{(1-x)^2+y^2}$$

$$u+iv = \frac{x(1-x)-y^2}{(1-x)^2+y^2} + i\frac{y}{(1-x)^2+y^2}$$

So $u = \frac{x(1-x)-y^2}{(1-x)^2+y^2}$

$v = \frac{y}{(1-x)^2+y^2}$

If $y>0$ (Upper half of z plane) then $v>0$ (Upper half of w plane)

$$w = \frac{z}{1-z} \Rightarrow w - wz = z$$

$$w = wz + z \Rightarrow w = z(w+1)$$

$$\boxed{z = \frac{w}{w+1}}$$

In z plane: $|z| = 1$

Circle of center (0,0) with radius 1.

In w plane $|z| = 1$

$$\left|\frac{w}{w+1}\right| = 1 \Rightarrow \frac{|w|}{|w+1|} = 1$$

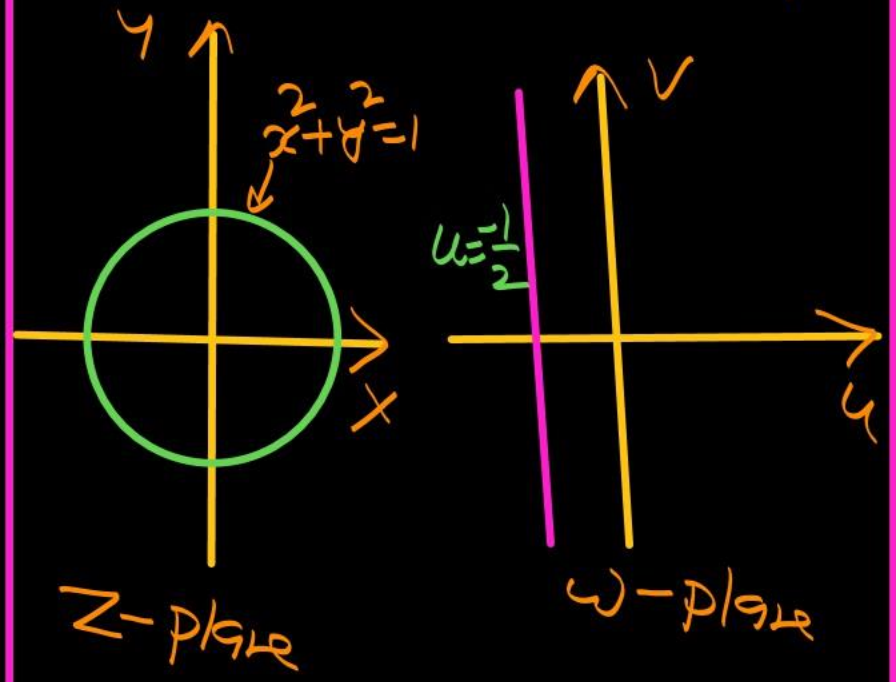
$$|w| = |w+1|$$

$$|u+iv| = |(u+1)+iv|$$

$$\sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$$

$$\cancel{u^2}+\cancel{v^2} = \cancel{u^2}+1+2u+\cancel{v^2}$$

$$0 = 2u+1 \Rightarrow \boxed{u = \frac{-1}{2}}$$



Type 4: To find image on z plane & w plane

18. Show that $w = \frac{i-z}{i+z}$ maps the real axis of the z plane into the circle $|w| = 1$ and the half plane $y > 0$ into the interior of the unit circle $|w| < 1$ in the w plane . 2005 , 2023

Invariant or Fixed Point

6. Find the fixed points of $w = \frac{3z-4}{z-1}$ & find normal form of the following transformation.

$w = \frac{3z-4}{z-1} \Rightarrow$ Put $w = z$

$z = \frac{3z-4}{z-1} \Rightarrow z^2 - z = 3z - 4$

$z^2 - z - 3z + 4 = 0$

$z^2 - 4z + 4 = 0$

$z^2 - 2z - 2z + 4 = 0$

$z(z-2) - 2(z-2) = 0$

$(z-2)(z-2) = 0$

$\boxed{z = 2}$ Ans

Normal form $= \frac{1}{w - \text{fixed point}}$

$= \frac{1}{\frac{3z-4}{z-1} - 2}$

$= \frac{z-1}{3z-4-2z+2}$

$= \frac{z-1}{z-2}$ Ans

Invariant or Fixed Point

6. Find the fixed points of $w = \frac{2z-5}{z+4}$ under the transformation.

$$w = \frac{2z-5}{z+4}, \quad \text{Put } w = z$$

$$z = \frac{2z-5}{z+4}$$

$$z^2 + 4z = 2z - 5$$

$$z^2 + 4z - 2z + 5 = 0$$

$$z^2 + 2z + 5 = 0$$

$$z = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2 \times 1} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$\boxed{z = -1 \pm 2i}$$

Ans

IMPORTANT TOPIC

UNIT 5

Unit 5

- ❖Cauchy integral formula
- ❖Taylor’s and Laurent’s series

Unit 5 (Two Marks)

1. State Cauchy Integral Theorem.

2. Write the statement of generalized Cauchy's Integral formula for nth derivative of an analytic function at the point $z = z_0$.

3. Evaluate $\int_c^2 \frac{e^z}{z+1} dz$ where C is the circle $|z| = 2$.

4. Expand $f(z) = e^{z/z-2}$ in a Laurent series about the point z=2.

5. Write Cauchy Integral Formula

6. Discuss the nature of singularity of $f(z) = \frac{Sin\, z}{z^2}$.

7. Expand $Sin^{-1}z$ in power of z.

8. Evaluate $\int_0^{1+i} (x^2 - iy) dz$ along the path $y = x$

Unit 5 (Two Marks)

9. Find Residue of $f(z) = \frac{Cos\, z}{z(z+5)}$ at z=0 .

10. Evaluate $\int \frac{e^z}{z^2+1} dz$ in the region $|z| = \frac{1}{2}$.

11. Classify the singularity of $f(z) = \frac{e^{1/z}}{z}$

12. . Find the residue of $f(z) = Cotz \;\; at\ its\ pole$

13. Expand $\frac{1}{(z+1)(z+3)}$ in the region $|z| < 1$

14.Discuss singularity and its types

15. Discuss the nature of singularity of $\frac{Cot\pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$

Numerical on Line Integration

1. Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the path (a). $y = x$ and (b) $y = x^2$

(a) Along the line $y = x$

$y = x \Rightarrow dy = dx$

$z = x + iy$, $dz = dx + i\,dy$

$dz = dx + i\,dx = (1+i)\,dx$

Lower Limit $z_1 = 0 = 0 + 0i$

Co-ordinate $= (0,0)$

Upper Limit $z_2 = 1 + i$

Co-ordinate $= (1,1)$

Now

$$\int_0^{1+i}(x^2 - iy)(dx + i\,dy)$$

$$= \int_0^1 (x^2 - ix)(1+i)\,dx$$

$$= (1+i)\int_0^1 (x^2 - ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} - \frac{ix^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3} - \frac{i}{2}\right]$$

$$= \frac{1}{3} - \frac{i}{2} + \frac{i}{3} + \frac{1}{2}$$

$$= \frac{5}{6} - \frac{i}{6} = \frac{5-i}{6} \quad \text{Ans}$$

(b) Along $y = x^2$

$dy = 2x \cdot dx$

$dz = dx + i\,dy = dx + i\,2x\,dx$

$dz = (1 + 2ix)\,dx$

Now

$$\int_0^{1+i}(x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1+2ix)\,dx$$

$$= (1-i)\int_0^1 (x^2 + 2ix^3)\,dx$$

$$= (1-i)\left[\frac{x^3}{3} + \frac{2ix^4}{4}\right]_0^1$$

$$= (1-i)\left[\frac{x^3}{3} + \frac{ix^4}{2}\right]_0^1$$

$$= (1-i)\left[\frac{1}{3} + \frac{i}{2}\right]$$

$$\frac{1}{3} + \frac{i}{2} - \frac{i}{3} + \frac{1}{2}$$

$$= \frac{5}{6} + \frac{i}{6} = \frac{5+i}{6} \quad \text{Ans}$$

Numerical on Line Integration

2. Evaluate the line integral $\int_c z^2 . dz$ where C is the boundary of a triangle with vertices $0, -1+i, 1+i$ clockwise.

$z = x+iy, \quad dz = dx + i\,dy$

Vertices of a triangle

$0 \rightarrow 0+0i \rightarrow (0,0)\ A$

$-1+i \longrightarrow (-1,1)\ B$

$1+i \longrightarrow (1,1)\ C$

Along AB

$(y-0) = \frac{1-0}{-1-0}(x-0) \Rightarrow \boxed{y=-x}$

$\boxed{dy = -dx}$

Along BC

$(y-1) = \frac{1-1}{-1-1}(x-1) \Rightarrow y-1=0 \quad \boxed{y=1}$

$\boxed{dy=0}$

Along CA

$(y-0) = \frac{1-0}{1-0}(x-0) \Rightarrow \boxed{y=x} \Rightarrow \boxed{dy=dx}$

$$\Rightarrow \int_c z^2 dz = \int_{AB} (x+iy)^2 (dx+i\,dy) + \int_{BC} (x+iy)^2 (dx+i\,dy) + \int_{CA} (x+iy)^2 (dx+i\,dy)$$

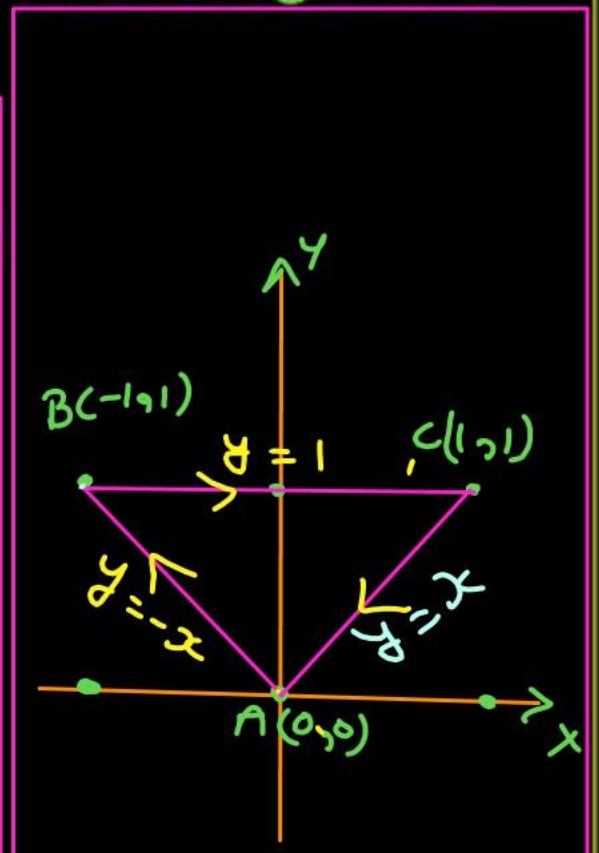
$$\Rightarrow \int_0^{-1} (x-ix)^2 (dx-i\,dx) + \int_{-1}^{1} (x+i)^2 dx + \int_1^0 (x+ix)^2 (dx+i\,dx)$$

$$\Rightarrow (1-i)^3 \int_0^{-1} x^2 dx + \int_{-1}^{1} (x+i)^2 dx + (1+i)^3 \int_1^0 x^2 dx$$

$$= (1-i)^3 \left[\frac{x^3}{3}\right]_0^{-1} + \left[\frac{(x+i)^3}{3}\right]_{-1}^{1} + (1+i)^3 \left[\frac{x^3}{3}\right]_1^0$$

$$= -\frac{(1-i)^3}{3} + \frac{(1+i)^3}{3} - \frac{(-1+i)^3}{3} + (1+i)^3 \left[\frac{-1}{3}\right]$$

$$= \cancel{-\frac{(1-i)^3}{3}} + \cancel{\frac{(1+i)^3}{3}} + \cancel{\frac{(1-i)^3}{3}} - \cancel{\frac{(1+i)^3}{3}} = 0$$



Cauchy's Integral Theorem

1. Evaluate $\frac{1}{2\pi i}\oint_c \frac{z^2-z+1}{z-2}$ dz where C is the region $|z-1|=\frac{1}{2}$

2021-22

Cauchy's Integral Theorem

6. Verify Cauchy theorem by integrating e^{iz} along the boundary of the triangle with the vertices at the point $1+i, -1+i, -1-i$.

Vertices of Triangle

$1+i \Rightarrow (1,1)$ A

$-1+i \Rightarrow (-1,1)$ B

$-1-i \Rightarrow (-1,-1)$ C

$f(z) = e^{iz} = e^{i(x+iy)}$

$f(z)$ is not infinite at any point inside and on the boundary of the region.

So $f(z)$ is analytic inside and on the boundary of the region.

By Cauchy's Integral theorem

$\oint_C e^{iz} \cdot dz = 0$

Along AB $\Rightarrow$ $y=1$, $dy=0$

Along BC $\Rightarrow$ $x=-1$, $dx=0$

Along CA $=$ $y=x$, $dy=dx$

So

$$\int e^{iz} dz = \int_{AB} e^{i(x+iy)}(dx+idy) + \int_{BC} e^{i(x+iy)}(dx+idy)$$

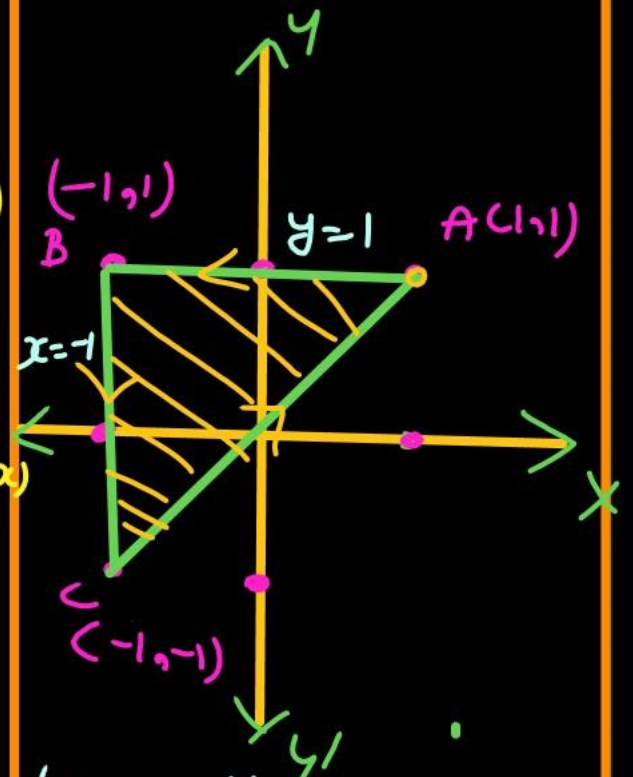
$$+ \int_{CA} e^{i(x+iy)}(dx+idy)$$

$$= \int_1^{-1} e^{i(x+i)} \cdot dx + \int_1^{-1} e^{i(-1+iy)} \cdot i\,dy + \int_{-1}^{1} e^{i(x+ix)}(dx+idx)$$

$$= \int_1^{-1} e^{ix-1} \cdot dx + i\int_1^{-1} e^{-i-y} \cdot dy + (1+i)\int_{-1}^{1} e^{i(1+i)x} \cdot dx$$

$$= \left[\frac{e^{ix-1}}{i}\right]_1^{-1} + i\left[\frac{e^{-i-y}}{-1}\right]_1^{-1} + \left[\frac{e^{i(1+i)x}}{i(1+i)}\right]_{-1}^{+1}(1+i)$$

$$= \frac{1}{ei}\left[e^{-i}+e^{i}\right] - ie^{-i}\left[e^{1}-e^{-1}\right] + \frac{1}{i}\left[e^{i(i+1)} - e^{-i(i+1)}\right]$$



$$(y-y_1) = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Numerical on Cauchy's Integral Formula

5. Evaluate by Cauchy's Integral formula to evaluate $\oint_C \frac{1}{z(z+\pi i)} dz$ where C is the circle $|z+3i| = 1$.

$|z+3i| = 1 \Rightarrow |z-(0-3i)| = 1$

$z_0 = 0-3i \Rightarrow (0,-3)$

It is a eqn of circle of centre $(0,-3)$ & radius 1.

Singular point

$z(z+\pi i) = 0$

$z = 0 = 0+0i \Rightarrow (0,0)$

$z = -\pi i = 0-\pi i \Rightarrow (0,-3.14)$

$z=0$ lie outside the Region &

$z=-\pi i$ lie inside the Region

Apply Cauchy Integral Formula

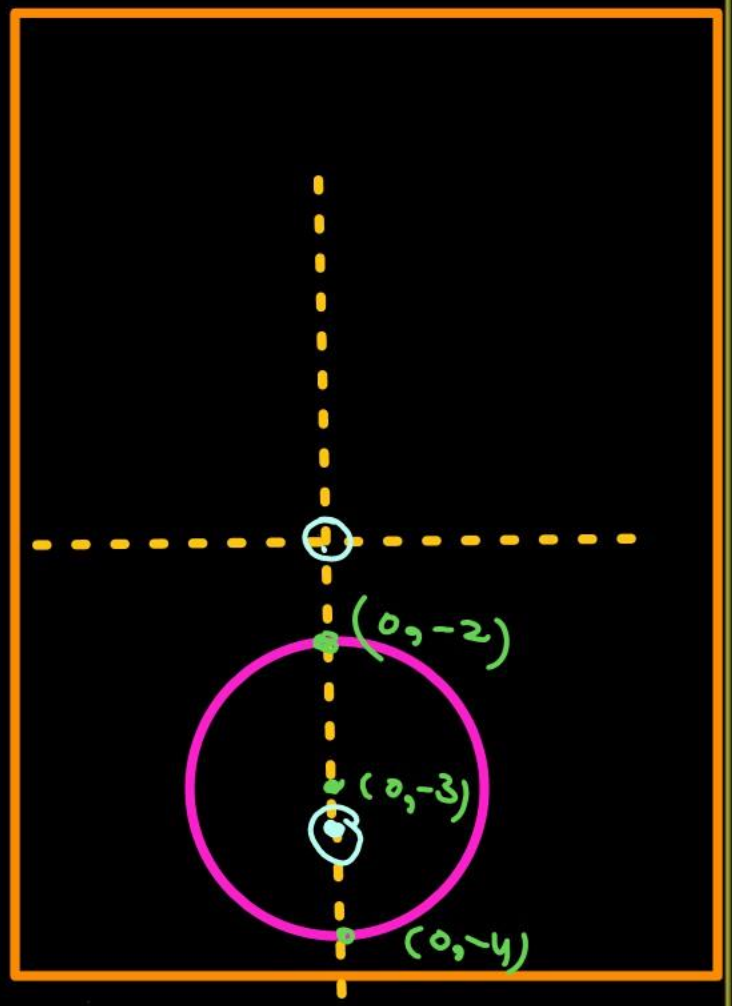
$f(z) = \frac{1}{z}$

$f(-\pi i) = \frac{1}{-\pi i}$

$\Rightarrow \oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

$\oint_C \frac{1}{z(z+\pi i)} = 2\pi i \times \frac{1}{-\pi i}$

$= -2$ Ans



Numerical on Extension of Cauchy's Integral Formula

4. Use Cauchy Integral formula to evaluate $\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz$, where C is the circle |z|=3.

$|z| = 3 \Rightarrow$ Eqn of Circle

Centre $= (0,0)$, Radius $= 3$

Singular Point

$(z-1)(z-2) = 0$

$z = 1, 2$

Co-ordinate of Singular point

$(1,0)$ & $(2,0)$

$z=1$ & 2 lie inside the Circle.

So Apply Cauchy Integral Formula

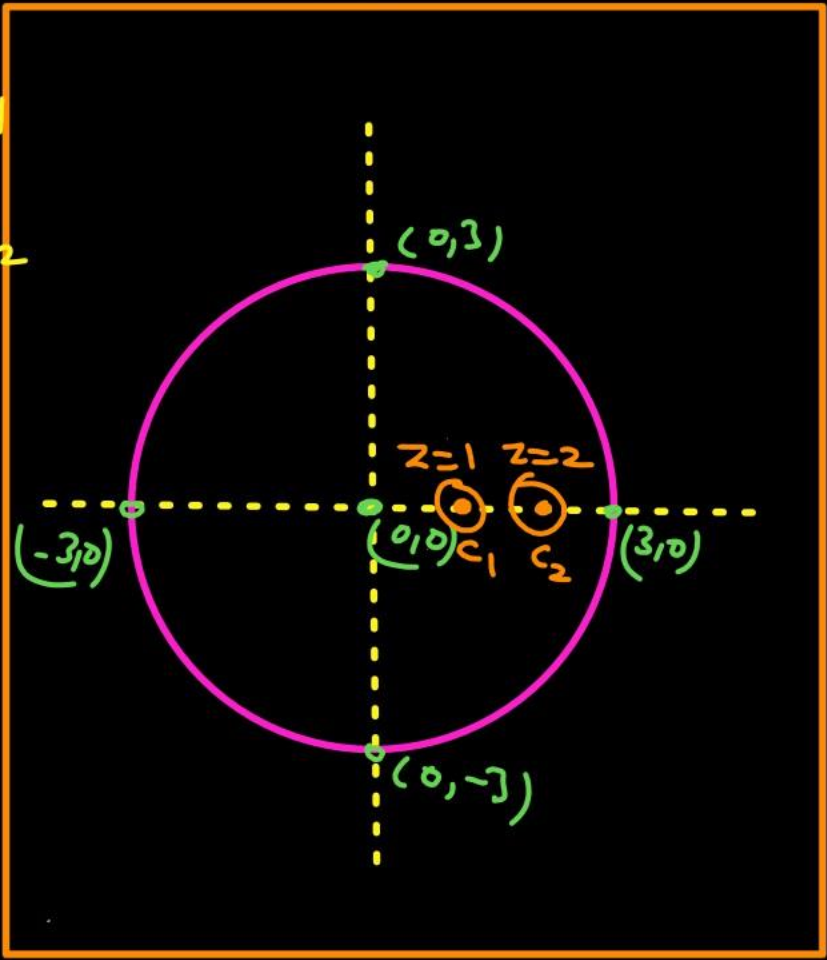
$$\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}}{(z-1)}\,dz + \int_{C_2} \frac{\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}}{(z-2)}\,dz$$

$$= 2\pi i\left[\frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}\right]_{z=1} + 2\pi i\left[\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}\right]_{z=2}$$

$$= 2\pi i\left[\frac{-1}{-1}\right] + 2\pi i\left[\frac{1}{1}\right]$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i \quad \text{Ans}$$



Numerical on Extension of Cauchy's Integral Formula

5. Evaluate $\oint_C \frac{z^2+1}{z^2-1}\,dz$, where C is the circle (i) $|z| = \frac{3}{2}$, (ii) $|z-1| = 1$ (iii) $|z| = \frac{1}{2}$

Singular Point $\Rightarrow z^2-1=0$,

$(z-1)(z+1)=0$, $z = 1, -1$

(i) $|z| = \frac{3}{2} \Rightarrow$ centre $(0,0)$, $R = \frac{3}{2}$

By Cauchy Integral Formula

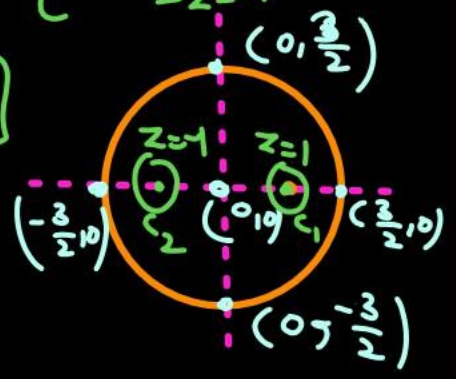
$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_{C_1} \frac{z^2+1/z+1}{(z-1)}dz + \oint_{C_2} \frac{\frac{z^2+1}{z-1}}{z+1}dz$$

$$= 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} + 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1}$$

$$= 2\pi i \times \frac{2}{2} + 2\pi i\left[\frac{2}{-2}\right]$$

$$= 2\pi i - 2\pi i$$

$$= 0 \quad \text{Ans}$$



(ii) $|z-1| = 1 \Rightarrow$ Eqn of circle

centre $(1,0)$, $R = 1$

$z=1$ lie inside the Region &

$z=-1$ lie outside Region

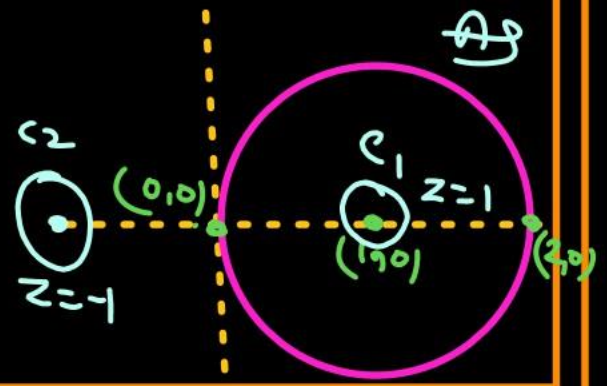
So Apply Cauchy Integral formula

$f(z) = \frac{z^2+1}{z+1}$, $f(1) = 1$

$$\oint_C \frac{z^2+1}{z^2-1} = 2\pi i f(1) = 2\pi i \times 1$$

$$= 2\pi i \times 1 = 2\pi i$$

Ans



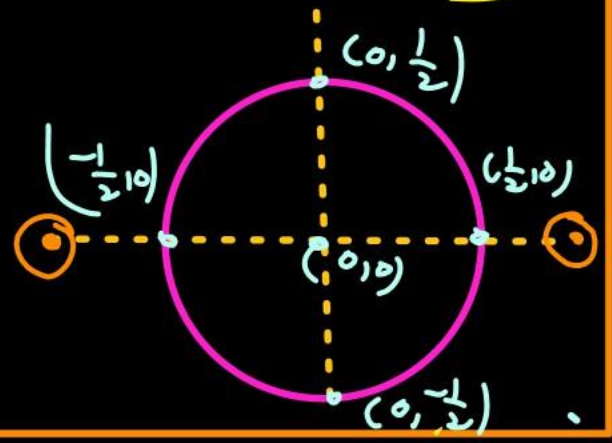
(iii) $|z| = \frac{1}{2} \Rightarrow$ Eqn of circle

centre $(0,0)$, $R = \frac{1}{2}$

$z=1$ & -1 lie outside the Region.

So By Cauchy Integral Theorem

$$\oint_C \frac{z^2+1}{z^2-1}dz = 0 \quad \text{Ans}$$



Numerical on Cauchy's Integral Formula for the Derivative

5. Evaluate $\oint_c \frac{e^z}{z(1-z)^3}\,dz$, about (a)$|z|=\frac{1}{2}$, $(b)|z-1|=\frac{1}{2}$, $(c)|z|=2$. 2021-22

Numerical on Cauchy's Integral Formula for the Derivative

6. Using Cauchy's Integral formula evaluate $\oint_c \frac{Sin2z}{(z+3)(z+1)^2}$ dz, where C is the rectangle with vertices $3 \pm i$ *and* $-2 \pm i$. (2022-23)

Numerical on Cauchy's Integral Formula for the Derivative

7. Using Cauchy's Integral formula evaluate $\oint_c \frac{Sinz}{(z^2+25)^2}$ dz, where C is the circle |z| = 8. (2018-19)

Taylor's Theorem

8. State Taylor's Theorem for complex functions. Find the first three terms of the Taylor's Series Expansion of $f(z) = \frac{1}{z^2+4}$ about $z = -i$.

Numerical on Laurent's Theorem

9. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's Series for the region

(a) $1<|z|<3$, (b) $0<|z+1|<2$ 2019-20

Numerical on Laurent's Theorem

10. Expand $f(z) = \frac{7z-2}{z^3-z^2-2z}$ in the following region

(a) $0<|z|<1$, (b) $1<|z|<2$ © $|z|>2$. 2022-23

Numerical on Laurent's Theorem

1. Expand the following function in a Laurent's Series about the point z=0. $f(z) = \frac{1-Cosz}{z^3}$.

$$f(z) = \frac{1-\cos z}{z^3} = \frac{1}{z^3}\left[1-\cos z\right]$$

$$f(z) = \frac{1}{z^3}\left[1-\left\{1-\frac{z^2}{\lfloor 2}+\frac{z^4}{\lfloor 4}-\frac{z^6}{\lfloor 6}+\cdots\right\}\right]$$

$$f(z) = \frac{1}{z^3}\left[\cancel{1}-\cancel{1}+\frac{z^2}{\lfloor 2}-\frac{z^4}{\lfloor 4}+\frac{z^6}{\lfloor 6}-\cdots\right]$$

$$f(z) = \frac{1}{z^3}\left[\frac{z^2}{\lfloor 2}-\frac{z^4}{\lfloor 4}+\frac{z^6}{\lfloor 6}-\cdots\right]$$

$$f(z) = \frac{1}{z\lfloor 2}-\frac{z}{\lfloor 4}+\frac{z^3}{\lfloor 6}+\cdots$$

It is a Laurent Series because it contain negative power of z. within z = 0.

Numerical on Laurent's Theorem

5. Find the Taylor's & Laurent Series which represent the function $\frac{z^2-1}{(z+2)(z+3)}$ when

(a) $|z| < 2$, (b) $2<|z| < 3$, (c) $|z| > 3$,

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{(z^2+5z+6)}$$

$$f(z) = \frac{(z^2+5z+6)-5z-7}{z^2+5z+6}$$

$$f(z) = 1 - \frac{5z+7}{z^2+5z+6} = 1 - \frac{5z+6}{(z+2)(z+3)}$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(a) $$f(z) = 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

Taylor Series

(b)

(b) $$f(z) = 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

It is a Laurent Series

(c)

$$f(z) = 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right]$$

Laurent Series.

Numerical on Laurent's Theorem

3. Expand $\frac{1}{z^2-3z+2}$ in the region (a) $|z| < 1$, (b) $1<|z| < 2$, (c) $|z| > 2$, (b) $0<|z-1| < 1$

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{z^2-2z-z+2} = \frac{1}{(z-2)(z-1)}$$

Now $\frac{1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}$

$$1 = A(z-1) + B(z-2)$$

Put $z=2 \Rightarrow \boxed{1 = A}$, Put $z=1 \Rightarrow \boxed{B=-1}$

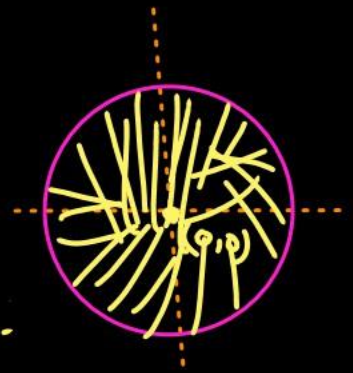
$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

(a) $|z|<1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$f(z) = \frac{1}{-2\left[1-\frac{z}{2}\right]} - \frac{1}{-1(1-z)}$$

$$f(z) = -\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1} + (1-z)^{-1}.$$



$$f(z) = -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] + (1+z+z^2+z^3+\cdots)$$

It is a Laurent series within $|z|<1$.

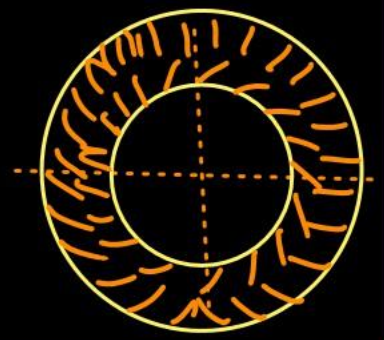
(ii) $1<|z|<2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$f(z) = \frac{1}{-2\left[1-\frac{z}{2}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

It is a Laurent series within $1<|z|<2$.



Numerical on Laurent's Theorem

3. Expand $\frac{1}{z^2-3z+2}$ in the region (a) $|z| < 1$, (b) $1<|z| < 2$, (c) $|z| > 2$, (b) $0<|z-1| < 1$

$|z|>2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

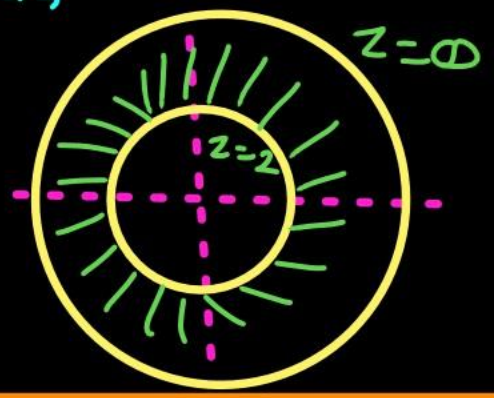
$$f(z) = \frac{1}{z\left[1-\frac{2}{z}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]}$$

$$f(z) = \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$f(z) = \frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

It is a Laurent series within $|z|>2$.

Ans



$0<|z-1|<1$ Let $z-1=t \Rightarrow z=t+1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$ so $0<|t|<1$

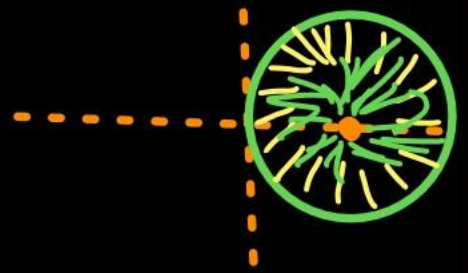
$$f(z) = \frac{1}{t+1-2} - \frac{1}{t} = \frac{1}{t-1} - \frac{1}{t}$$

$$f(z) = -\frac{1}{t} + \frac{1}{t-1} = -\frac{1}{t} - \frac{1}{(1-t)} = \frac{1}{t} - (1-t)^{-1}$$

$$f(z) = -\frac{1}{t} - \left[1+t+t^2+t^3+\cdots\right]$$

$$f(z) = -\frac{1}{(z-1)} - \left[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right]$$

It is a Laurent series within $0<|z-1|<1$



Residue at Pole

1. Determine the pole of the following functions and residue at each pole $f(z) = \frac{1-e^{2z}}{z^4}$.

$$f(z) = \frac{1-e^{2z}}{z^4} = \frac{1}{z^4}\left[1-\left(1+2z+\frac{4z^2}{\lfloor 2}+\frac{8z^3}{\lfloor 3}+\frac{16z^4}{\lfloor 4}+\frac{32z^5}{\lfloor 5}\cdots\right)\right]$$

$$= \frac{1}{z^4}\left[\cancel{1}-1-2z-\frac{4z^2}{\lfloor 2}-\frac{8z^3}{\lfloor 3}-\frac{16z^4}{\lfloor 4}-\frac{32z^5}{\lfloor 5}\cdots\right]$$

$$= -\frac{2}{z^3}-\frac{4}{z^2\lfloor 2}-\frac{8}{z\lfloor 3}-\frac{16}{\lfloor 4}-\frac{32z}{\lfloor 5}\cdots$$

Pole $z^3=0 \Rightarrow z=0$ [Pole of Order 3]

$$\text{Res}[z=0] = \frac{-8}{\lfloor 3} = \frac{-8}{6} = -\frac{4}{3}$$ Ans

Other Method to find Residue at Pole

2. Determine the Pole and Residue at the pole of the function $f(z) = \frac{z}{(z-1)}$.

$f(z) = \frac{z}{z-1}$

For Pole $\Rightarrow z-1=0 \Rightarrow \boxed{z=1} \rightarrow$ Simple Pole

Now $\text{Res}(z=a) = \lim_{z \to a} (z-a) f(z)$

$\text{Res}(z=1) = \lim_{z \to 1} (z-1) \times \frac{z}{(z-1)}$

$= \lim_{z \to 1} z$

$= 1$ Ans

Other Method to find Residue at Pole

3. Determine the Residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its simple pole.

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$$

For Pole $\Rightarrow (z-1)^4(z-2)(z-3) = 0$

$z-2=0 \Rightarrow z=2$ [Simple Pole]

$z-3=0 \Rightarrow z=3$ [Simple Pole]

$(z-1)^4=0 \Rightarrow z=1$ [Pole of Order 4]

(i) For $z=2$

$$\text{Res}(z=2) = \lim_{z\to 2} \cancel{(z-2)} \frac{z^3}{(z-1)^4\cancel{(z-2)}(z-3)}$$

$$= \lim_{z\to 2} \frac{z^3}{(z-1)^4(z-3)}$$

$$= \frac{8}{1\times -1} = -8 \text{ Ans}$$

(ii) At $z=3$

$$\text{Res}(z=3) = \lim_{z\to 3} \cancel{(z-3)} \frac{z^3}{(z-1)^4(z-2)\cancel{(z-3)}}$$

$$\lim_{z\to 3} \frac{z^3}{(z-1)^4(z-2)}$$

$$= \frac{27}{16\times 1}$$

$$= \frac{27}{16} \text{ Ans}$$

Other Method to find Residue at Pole

4. Determine the Residue of a function $f(z) = \frac{z^2}{(z+1)^2(z-2)}$ at its double pole.

$$f(z) = \frac{z^2}{(z+1)^2(z-2)}$$

For Pole $\Rightarrow (z+1)^2(z-2) = 0$

$(z+1)^2 = 0 \Rightarrow z = -1$ [Order 2]

$z-2 = 0 \Rightarrow z = 2$ [Single Pole]

$$Res(z=a) = \frac{1}{\lfloor m-1} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

$$Res(z=-1) = \frac{1}{\lfloor 1} \lim_{z \to -1} \left[\frac{d}{dz} \cancel{(z+1)^2} \times \frac{z^2}{\cancel{(z+1)^2}(z-2)}\right]$$

$$Res(z=-1) = \lim_{z \to -1} \left[\frac{d}{dz} \frac{z^2}{z-2}\right]$$

$$Res(z=-1) = \lim_{z \to -1} \left[\frac{(z-2) \cdot 2z - z^2(1)}{(z-2)^2}\right]$$

$$Res(z=-1) = \frac{(-3)(-2) - 1}{9}$$

$$= \frac{6-1}{9}$$

$$= \frac{5}{9} \text{ Ans}$$

Other Method to find Residue at Pole

5. Determine the Poles and Residue at each pole of the function $f(z) = Cot\ z$.

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

For Pole $\Rightarrow$ $\sin z = 0$

$$\sin z = \sin n\pi$$

$$\boxed{z = n\pi}$$ Simple Pole

For Pole $z = n\pi$

$$Res(z = n\pi) = \left[\frac{\cos z}{\cos z}\right]_{z=n\pi}$$

$$= [1]_{z=n\pi}$$

$$= 1 \text{ Ans}$$

Other Method to find Residue at Pole

6. Determine the Residue of a function $f(z) = \dfrac{z^2}{(z-1)(z-2)(z-3)}$ at infinity.

$$f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$\text{Res}(z=\infty) = \lim_{z\to\infty} [-z f(z)]$$

$$= \lim_{z\to\infty} \left[-z \times \frac{z^2}{(z-1)(z-2)(z-3)}\right] = \lim_{z\to\infty} \left[\frac{-z^3}{(z-1)(z-2)(z-3)}\right]$$

$$= \lim_{z\to\infty} \left[\frac{-1}{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)}\right]$$

$$= \frac{-1}{(1-0)(1-0)(1-0)}$$

$$= -1$$

Other Method to find Residue at Pole

7. Determine the Residue of a function $f(z) = \frac{z^3}{z^2-1}$ at $z = \infty$.

$$f(z) = \frac{z^3}{(z^2-1)} = \frac{z^3}{z^2(1-\frac{1}{z^2})} = z\left(1-\frac{1}{z^2}\right)^{-1}$$

$$f(z) = z\left[1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots\right]$$

$$f(z) = z + \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \cdots$$

$$\text{Res}(z=\infty) = -1$$

Numerical on Residue Theorem

14. State Cauchy residue Theorem and evaluate $\oint_c \frac{dz}{(z+\pi i)}$ where C is the region $|z+3i| = 1$ 2022-23

Numerical on Residue Theorem

15. Find residue at each pole of the function $\frac{4+3z}{(z-2)(z-3)}$ and hence using Cauchy Residue Theorem evalaute integral $\oint_c \frac{4+3z}{(z-2)(z-3)}dz$ where $|z| = 1$ 2021-22

Numerical on Residue Theorem

16.State the residue theorem. Find residue at each pole of the function $\frac{z^3}{(z-2)(z-3)(z-1)^4}$ and evalaute integral

$\oint_c \frac{z^3}{(z-2)(z-3)(z-1)^4} dz$ where $|z| = \frac{5}{2}$ 2022-23

Numerical on Application Cauchy Residue Theorem

1. Using complex integration method, evaluate $\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta$

$$I = \int_0^{2\pi} \frac{1}{5+4\cos\theta} \cdot d\theta$$

Let $|z| = 1$ is a closed curve C.

Put $z = e^{i\theta} \Rightarrow \boxed{d\theta = \frac{dz}{iz}}$

$$\boxed{\cos\theta = \frac{z^2+1}{2z}}$$

$$I = \oint_C \frac{1}{5+4\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{zi}$$

$$I = \oint_C \frac{dz}{5+\frac{2(z^2+1)}{z}} \; \frac{1}{zi}$$

$$I = \frac{1}{i}\oint_C \frac{dz}{5z+2z^2+2}$$

$$I = \frac{1}{i}\oint_C \frac{1}{2z^2+5z+2}\, dz$$

By Cauchy Residue Theorem

$$I = \frac{1}{i}\left[2\pi i(\text{Sum of Residue})\right] \quad \text{— (I)}$$

$$f(z) = \frac{1}{2z^2+5z+2}$$

Pole $\Rightarrow 2z^2+5z+2 = 0$

$$2z^2+4z+z+2 = 0$$

$$(2z+1)(z+2) = 0$$

$2z+1 = 0 \Rightarrow z = \frac{-1}{2}$ (Simple Pole)

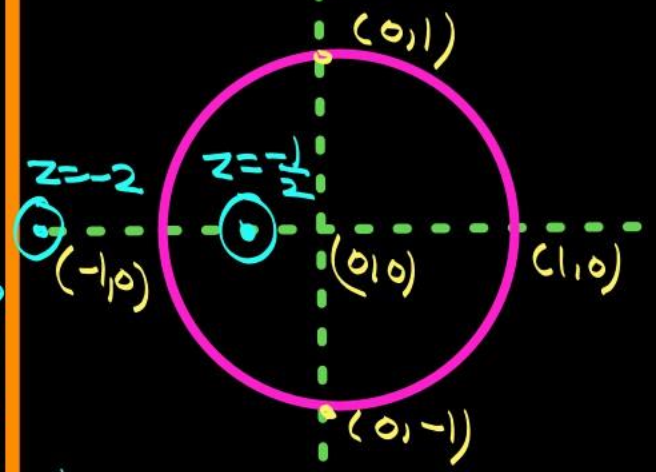
$z+2 = 0 \Rightarrow z = -2$ (Simple Pole)

$$\text{Res}\left(z = -\tfrac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z+\tfrac{1}{2}\right) \cdot \frac{1}{(2z+1)(z+2)}$$

$$= \lim_{z \to -\frac{1}{2}} \cdot \frac{1}{2} \cdot \frac{1}{(z+2)}$$

$$R_1 = \frac{1}{2} \times \frac{1}{\frac{3}{2}} = \frac{1}{3}$$

$$\boxed{R_1 = \frac{1}{3}}$$



So

$$I = \frac{1}{i}\left[2\pi i \times \frac{1}{3}\right] = \frac{2\pi}{3}$$

① If calculate $\int_0^{\pi} \frac{d\theta}{5+4\cos\theta}$

We know that

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{5+4\cos\theta}$$

(when $f(2\pi-\theta) = f(\theta)$)

$$\frac{2\pi}{3} = 2\int_0^{\pi} \frac{d\theta}{5+4\cos\theta}$$

So $$\int_0^{\pi} \frac{d\theta}{5+4\cos\theta} = \frac{\pi}{3}$$

② If Evaluate $\int_{-\pi}^{\pi} \frac{d\theta}{5+4\cos\theta}$

It is a Even function

$$\int_{-\pi}^{\pi} \frac{d\theta}{5+4\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{5+4\cos\theta}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{5+4\cos\theta} = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$$

Numerical on Application Cauchy Residue Theorem

2. Using complex integration method, evaluate $\int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta$ where $a > |b|$. Hence or otherwise

Evaluate: 1) $\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\, d\theta$ 2) $\int_0^{\pi} \frac{1}{a+b\cos\theta}\, d\theta$

Let $I = \int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta$

Sol $\Rightarrow$ $|z| = 1$ is a circle (closed curve)

Put $z = e^{i\theta} \Rightarrow \boxed{d\theta = \frac{dz}{iz}}$

$$\boxed{\cos\theta = \frac{z^2+1}{2z}}$$

$$I = \oint_C \frac{1}{a+b\left[\frac{z^2+1}{2z}\right]} \cdot \frac{dz}{iz} = \oint_C \frac{1}{\frac{2az+bz^2+b}{2z}} \cdot \frac{dz}{iz}$$

$$I = \frac{2}{i} \oint_C \frac{1}{2az+bz^2+b}\, dz = \frac{2}{ib} \oint_C \frac{dz}{z^2+\frac{2a}{b}z+1}$$

By Cauchy Residue Theorem

$$I = \frac{2}{ib}\left[2\pi i\,(\text{Sum of Residue})\right] \quad \text{--- (1)}$$

Now $f(z) = \frac{1}{z^2+\frac{2a}{b}z+1}$

Pole $\Rightarrow z^2+\frac{2a}{b}z+1 = 0 \Rightarrow (z-\alpha)(z-\beta)$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2}$$

$$z = \frac{-\frac{2a}{b} \pm \frac{2}{b}\sqrt{a^2-b^2}}{2} = -\frac{a}{b} \pm \frac{\sqrt{a^2-b^2}}{b}$$

So $\alpha = -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}$

$\beta = -\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}$

$a > |b|$

So $\beta > 1$

$\alpha \cdot \beta = 1 \qquad \left[\alpha\beta = \frac{c}{a}\right]$

$|\alpha\beta| = 1$

$|\alpha||\beta| = 1 \Rightarrow |\alpha| = \frac{1}{|\beta|}$

so $|\alpha| < 1$

Hence only $z = \alpha$ Lie inside the Region.

$$\text{Res}(z=\alpha) = \lim_{z\to\alpha} \frac{(z-\alpha)}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta} = \frac{1}{\alpha-\beta}$$

$$R_1 = \frac{1}{\alpha-\beta} = \frac{1}{\frac{2\sqrt{a^2-b^2}}{b}} = \frac{b}{2\sqrt{a^2-b^2}}$$

So

$$I = \frac{\cancel{2}}{\cancel{i}\cancel{b}}\left[2\pi\cancel{i} \times \frac{\cancel{b}}{\cancel{2}\sqrt{a^2-b^2}}\right] = \frac{2\pi}{\sqrt{a^2-b^2}}$$

So $\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$ Ans

(I) Put $a = \sqrt{2}$, $b = -1$

$$\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{2-1}}$$

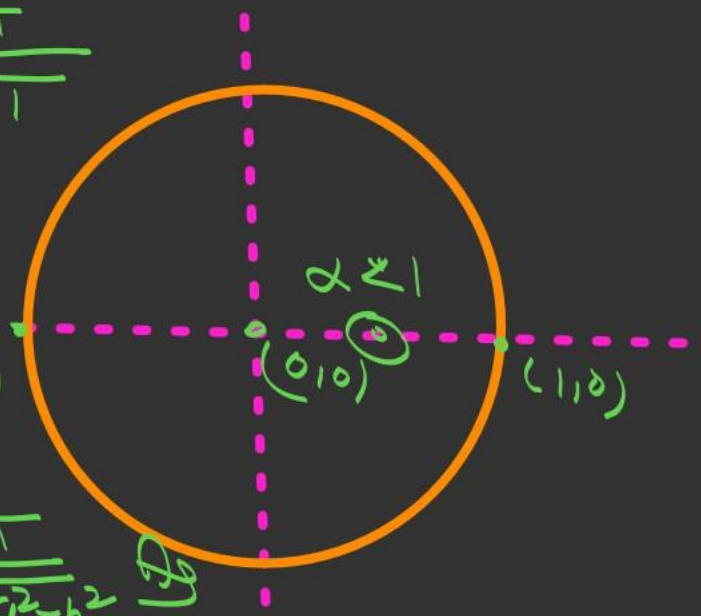
$= 2\pi$ Ans

(II)

$$\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,d\theta = 2\int_0^{\pi} \frac{1}{a+b\cos\theta}\,d\theta$$

$$\frac{2\pi}{\sqrt{a^2-b^2}} = 2\int_0^{\pi} \frac{1}{a+b\cos\theta}\,d\theta$$

$$\int_0^{\pi} \frac{1}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$$ Ans



Numerical on Application Cauchy Residue Theorem

3. Using complex integration method, evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}$$

Let $|z|=1$ is a closed curve

Put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{z^2+1}{2z}$$

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i2\theta} = \underbrace{\cos 2\theta}_{\text{Real Part}} + i\underbrace{\sin 2\theta}_{\text{Imaginary Part}}$$

Hence

$$I = \oint_C \frac{\text{R.P. of } e^{i2\theta}}{\left[5+4\left(\frac{z^2+1}{2z}\right)\right]} \cdot \frac{dz}{iz}$$

$$I = \text{R.P. of } \frac{1}{i}\oint_C \frac{z^2}{2z^2+5z+2}\, dz$$

By Cauchy Residue Theorem

$$I = \text{R.P. of } \frac{1}{i}\left[2\pi i\,(\text{Sum of Residue})\right]$$

Pole $\Rightarrow 2z^2+5z+2=0$

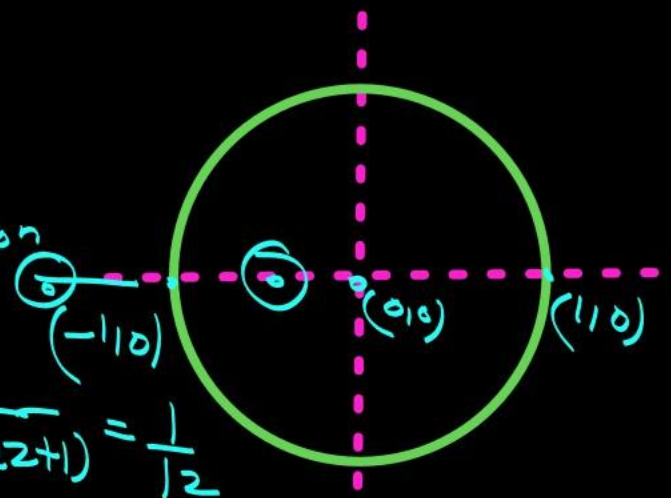
$(z+2)(2z+1)=0$

$z+2=0 \Rightarrow z=-2$

$2z+1=0 \Rightarrow z=-\frac{1}{2}$

$z=-\frac{1}{2}$ lie inside the Region

$$\text{Res}\left(z=-\tfrac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z+\tfrac{1}{2}\right)\cdot\frac{z^2}{(z+2)(2z+1)} = \frac{1}{12}$$



$$I = \text{R.P. of } \frac{1}{i}\left[2\pi i \times \frac{1}{12}\right]$$

$$= \text{R.P. of } \frac{\pi}{6}$$

$$= \frac{\pi}{6} \text{ Ans}$$

Numerical on Application Cauchy Residue Theorem

17.Evaluate the integral by using Contour Integration $\int_0^{2\pi} \frac{1}{5-3Cos\theta}\, d\theta$ $\quad$ **2019 – 20**

Numerical on Application Cauchy Residue Theorem

18.Using Residue Evaluate the integral $\int_0^{2\pi} \frac{1}{5-4Sin\theta} d\theta$ $\quad$ $2022-23$

Numerical on Application Cauchy Residue Theorem

19. Evaluate the integral by using Contour Integration $\int_0^{\pi} \frac{a}{a^2+Sin^2\theta} d\theta \quad a > 0 \quad 2022-23$

Numerical on Application Cauchy Residue Theorem

20. Using Residue theorem to evalaute $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)+(x^2+4)}\, dx \qquad 2018-19$

Thank You For Watching