Maxwell's wave equation for free space

In the free space there is no charge or current. Therefore, in free space $\rho = J = 0$. Hence, Maxwell's four equations read as

(i) $\nabla \cdot \bar{E} = 0$; (ii) $\nabla \cdot \bar{B} = 0$; (iii) $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$; (iv) $\nabla \times \bar{B} = \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$

These equations consitute a set of coupled first-order, partial differential equations for $\bar{E}$ and $\bar{B}$. They can be decoupled by applying the curl to (iii) & (iv)

$$\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = \nabla \times \left(-\frac{\partial \bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \bar{B})$$

Therefor $\nabla(\nabla \cdot E) - \nabla^2 \bar{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2}$

Also, $\nabla \times (\nabla \times \bar{B}) = \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}\right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \bar{E})$

$\therefore \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}$

Since, $\nabla \cdot \bar{B} = 0$ and $\nabla \cdot \bar{E} = 0$ for free space,

$$\therefore \boxed{\nabla^2 \bar{E} = \mu_0 \epsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} \; ; \; \nabla^2 \bar{B} = \mu_0 \epsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}}$$

These are called Maxwell's wave equations for free space.

Solution of Maxwell's wave equation for free space

In vacuum, the components of $\bar{E}$ & $\bar{B}$ satisfy the equation:

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

This is called the wave equation, it describes wave traveling with a velocity v - According to Maxwell's equations, then, empty space supports the propagation of the electromagnetic wave a speed

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \simeq 3.00 \times 10^8 \, m/s$$

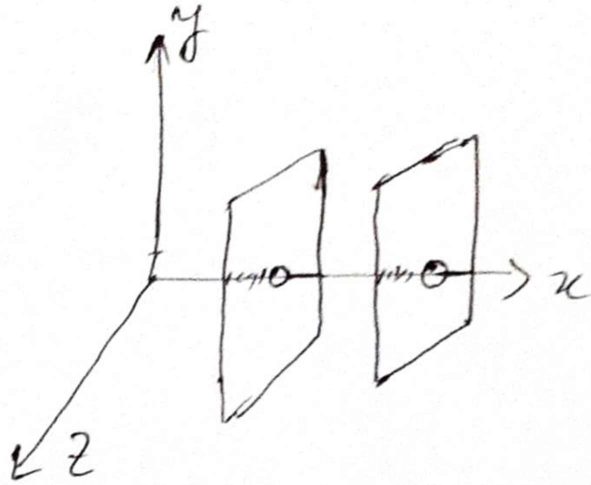
Which is the speed of light in free space.

Maxwell's electromagnetic wave equation of electric field $\bar{E}$ and magnetic field $\bar{B}$ satisfying 3D wave equation is given by

$$\nabla^2 \bar{E} = \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}, \quad \nabla^2 \bar{B} = \frac{1}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2}$$

Where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is speed of light in vacuum.

Suppose for the moment that the waves are traveling in the x-direction and have no y- or z-dependence: these are called plane waves, because



the fields are uniform over every plane perpendicular to the direction of propagation. Therefore, the electric and magnetic field wave we can write

$$\bar{E}(x,t) = \tilde{E}_0 e^{i(\bar{k}\cdot\bar{x}-\omega t)}$$

$$\bar{B}(x,t) = \bar{B}_0 e^{i(\bar{k}\cdot\bar{x}-\omega t)}$$

Here $\tilde{E}_0$ and $\tilde{B}_0$ are the complex amplitudes of electric and magnetic fields, the physical real parts ~~of~~ are $\bar{E}$ and $\bar{B}$ repectively.

The above electromagnetic wave equations are derived from Maxwell's equations. Hence, every solution to Maxwell's equations must obey the wave equations. Therefore, since $\nabla\cdot\bar{E}=0$ and $\nabla\cdot\bar{B}=0$ it follows $\bar{K}\cdot\bar{E}=\bar{K}\cdot\bar{B}=0$. Therefore, the electromagnetic waves are transverse. The electric and magnetic fields are perpendicular to the direction of propagation.

Moreover, Faraday's law gives

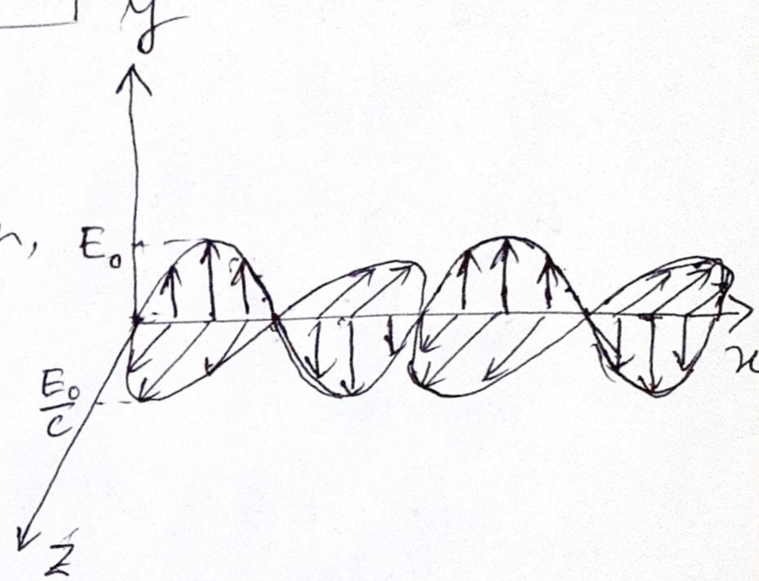
$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$ then we ~~get~~ get

$$\nabla \times \bar{E}_0 e^{i(\bar{k}\cdot\bar{x}-\omega t)} = -\frac{\partial}{\partial t}\bar{B}_0 e^{i(\bar{k}\cdot\bar{x}-\omega t)}$$

or, $$i\bar{k}\times\bar{E} = -(-i\omega)\bar{B}$$

or, $$\boxed{\bar{k}\times\bar{E} = \omega\bar{B}}$$

As the direction of $\bar{k}$ is in the x-direction, more compactly we can write taking the amplitudes of $\bar{E}$ and $\bar{B}$ as



$$\boxed{\bar{B}_0 = \frac{k}{\omega}(\hat{i}\times\bar{E}_0)}$$

Evidently, $\bar{E}$ and $\bar{B}$ are in phase and mutually perpendicular, their real amplitudes are related by

$$\boxed{B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0}$$

If $\bar{E}$ points in the y-direction, the according to the above relation $\bar{B}$ points in z-direction

As the wave is propagating in the x-direction. Therefore, if

$$\bar{E}(x,t) = \bar{E}_0 e^{i(\bar{k}\cdot\bar{x} - \omega t)} \hat{j},$$

then $$\bar{B}(x,t) = \frac{1}{c} \times \bar{E}_0 e^{i(\bar{k}\cdot\bar{x} - \omega t)} \hat{k}.$$

The electromagnetic wave as a whole is said to be polarised along y-direction by convention, we use the direction of $\bar{E}$ to specify the <u>polarisation of an electromagnetic wave</u>.

<u>Electromagnetic wave in a free conducting media</u>

For free space the free charge density ρ_f and free current density J_f be zero, and every property of E.M. wave we derived was based on this assumption. But in case of conductors (sea water, metals etc.) we can not control the flow of charge and in general $\bar{J}_f$ is not zero. In fact according to Ohm's law, the current density is proportional to the electric field:

$$\bar{J}_f = \sigma \bar{E}$$