Maxwell's Equations

Electrodynamics before Maxwell

So far we have encountered the following laws specifying the divergence and curl of electric and magnetic field;

(i) $\nabla \cdot \bar{E} = \rho/\epsilon_0$ (Gauss' law in electrostatics)

(ii) $\nabla \cdot \bar{B} = 0$ (Gauss law in magnetostatics)

(iii) $\nabla \times \bar{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faradays law)

(iv) $\nabla \times \bar{B} = \mu_0 \vec{J}$ (Ampere's law)

These represents the state of electromagnetic theory before Maxwell's work. But there is a fatal inconsistancy in these formulas. It has to do with the identity of divergence of curl of a vector is zero. ~~I~~ Now if we apply divergence to equation (iii)

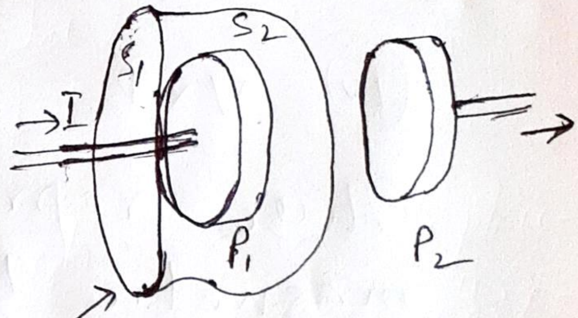
$$\nabla \cdot (\nabla \times \bar{E}) = \nabla \cdot \left(-\frac{\partial \bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \bar{B})$$

In the above equation both the R.H.S & L.H.S are zero. But when we apply the same to the equation (IV) we get into trouble. We get as

$$\nabla\cdot(\nabla\times\bar{B}) = \nabla\cdot(\mu_0\bar{J}) = \mu_0(\nabla\cdot\bar{J})$$

The L.H.S of the above equation is zero, but the R.H.S in general, is not, For steady currents, the $\nabla\cdot\bar{J}$ is zero, but evidently when we go beyond magnetostatics, Ampere's law can not be right.

There is another way to see the Ampere's law is bound to fail for non-steady currents. Let us consider the circuit as shown in the figure, which consist of a parallel plate capacitor being charged (or discharged) through a certain external resistance. If we apply Ampere's law to the contour C and the surface S_1, we find that



$$\oint_C \bar{B}\cdot d\bar{l} = \mu_0 I$$, I current passing through S_1 surface. If on the other hand Ampere's law is applied to the contour C and surface S_2, then I is zero at all points of S_2. As current is not

flowing through the surface S_2, then

$$\oint_C \bar{B}\cdot d\bar{l} = \mu_0 \int_{S_2} \bar{J}\cdot d\bar{s} = 0$$

It is easy to see that the above two equations contradict each other. Further the latter equation can not be wrong. Therefore, it appears that Ampere's equation (iv) requires modification.

First we note that the surface S_2 "cuts" only the electric field. In accordance with Gauss' theorem, the flux of electric field vector $\bar{E}$ through the closed surface is

$$\oint \bar{E}\cdot d\bar{s} = q/\epsilon_0 .$$ Therefore, the current I is

$$I = \frac{dq}{dt} = \epsilon_0 \frac{\partial}{\partial t} \oint \bar{E}\cdot d\bar{s} = \epsilon_0 \int \frac{\partial \bar{E}}{\partial t}\cdot d\bar{s}$$

On the other hand, according to the continuity equation $\nabla\cdot\bar{J} = -\frac{\partial \rho}{\partial t}$, then using Gauss' theorem $\oint \bar{J}\cdot d\bar{s} = -\frac{dq}{dt}$, where $\bar{J}$ is the conduction current density.

Summing up the surface integrals of the above two equations, we obtain

$$\oint \bar{J}\cdot d\bar{s} + \epsilon_0 \oint \frac{\partial \bar{E}}{\partial t}\cdot d\bar{s} = \frac{dq}{dt} - \frac{dq}{dt} = 0$$

Therefore, $\oint \left[\bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}\right]\cdot d\bar{s} = 0$

This equation is similar to the continuity equation for direct current. There is one more term $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$, whose dimension is same as for current density. Maxwell termed this term as the density of displacement current ($\vec{J}_d$). Thus,

$$\vec{J}_d = \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

The sum of the conduction and displacement currents is called the total current. It's density is given by

$$\vec{J}_T = \bar{J} + \vec{J}_d = \bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

Thus the theorem on the line integral of magnetic field $\bar{B}$, which was established for direct currents, can be generalised for an arbitrary case in the following form :

$$\oint \bar{B}\cdot d\bar{l} = \mu_0 I_T = \mu_0 \int \bar{J}_T\cdot d\bar{s} = \int \left[\mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}\right]\cdot d\bar{s}$$

Therefore, the Ampere's law in integral form reads

$$\oint \bar{B}\cdot d\bar{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \left(\frac{\partial \bar{E}}{\partial t}\right)\cdot \vec{ds}$$

It is possible to convert the line integral in the above equation into a surface integral using stoke's theorem

$$\int_S (\nabla \times \bar{B}) \cdot \bar{ds} = \int_S \left[\mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}\right] \cdot \bar{ds}$$

Since the above result is true for any surface, therefore, we can write

$$\boxed{\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}}$$

This is the general form of Ampere's law.
Hence, the Maxwell's equations becomes for time-varying electric and magnetic fields as

(i) $\nabla \cdot \bar{E} = \rho/\epsilon_0$ Gauss' law in electrostatics
(ii) $\nabla \cdot \bar{B} = 0$ Gauss' law in magnetostatics
(iii) $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ Faraday's law
(iv) $\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$ Ampere's law corrected by Maxwell.