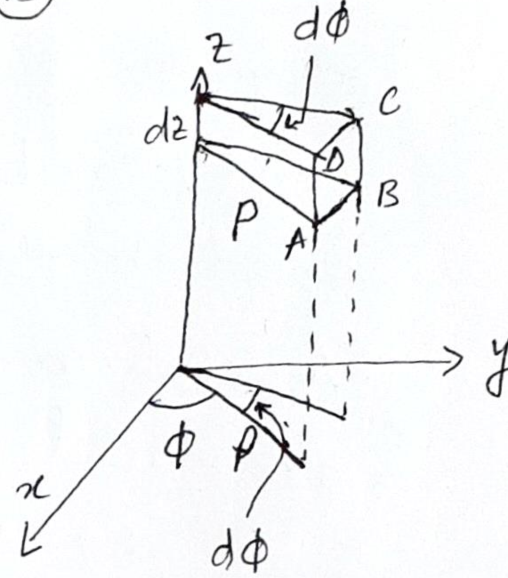
Cylindrical Polar coordinate



The elementary surface area of a cylinder

$$ds = AB \times CB$$
$$= \rho\, d\phi \times dz$$

The elementary volume of the cylinder becomes

$$dv = \rho\, d\rho\, d\phi\, dz$$

$\therefore$ Total volume of the cylinder is

$$V = \iiint_{\rho\,\phi\,z} dv = \int_0^r \rho\, d\rho \times \int_0^{2\pi} d\phi \times \int_0^z dz = \frac{r^2}{2} \times 2\pi \times z$$

$$\therefore \boxed{V = \pi r^2 z}$$

Electrostatic Potential

Electrostatic potential at a point in an electric field $\bar{E}$ is defined as the amount of work done in bringing a unit charge (+ve) from infinity to that point against the electrostatic force. A positive charge always tends to move from higher potential to lower potential.

Thus, the potential at the point is

$dV = -\vec{E}.\vec{dr}$ due to bring +ve charge $\vec{dr}$ distance in $\bar{E}$ electric field.

Also from the fundamental theorem of gradient we have,

$$dV = \nabla V . \vec{dr}$$

Hence, we can write

$$-\vec{E}.\vec{dr} = \nabla V . \vec{dr}$$

Therefore, $\boxed{\vec{E} = -\nabla V}$

This equation says the electric field is the gradient of a scalar potential.

Poisson's equation & Laplace's equation

The electric field is defined in terms of potential (V) as $\bar{E} = -\nabla V$

We also know the divergence of $\bar{E}$ as

$$\nabla . \bar{E} = \rho/\epsilon_0 = \nabla . (-\nabla V) = -\nabla^2 V$$

Hence, we get $\boxed{\nabla^2 V = -\rho/\epsilon_0}$ This is known as Poisson's equation. Now in the region where there is no charge i.e, $\rho = 0$. Hence, Poisson's equation reduced to Laplace's equation as $\boxed{\nabla^2 V = 0}$ ✗

Laplacian Operator (∇^2)

In cartesian coordinate

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Therefore, the Laplacian operator in cartesian coordinate system becomes

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In spherical polar coordinate system

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}.$$

So the Laplacian operator becomes:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

In cylindrial polar coordinate

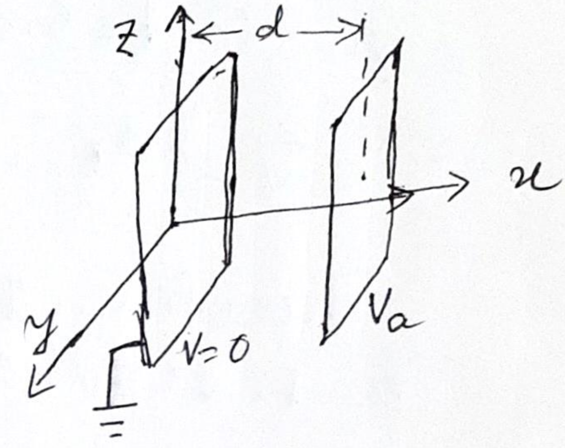
$$\nabla = \frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}.$$

The laplacian operator becomes:

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Application of Laplace's Equation (1D problem)

Potential between the plates of a parallel-plate capacitor:



Consider a parallel-plate capacitor having two plates, one at $x=0$ and other at $x=d$. Potential at the plate at the right is V_a and the other is grounded.

Hence, the potential depends on along x-direction only. Therefore, the Laplace's equation $\nabla^2 V = 0$, becomes in this case

$\frac{\partial^2 V}{\partial x^2} = 0$, Hence, $\frac{\partial V}{\partial x} = C$, and

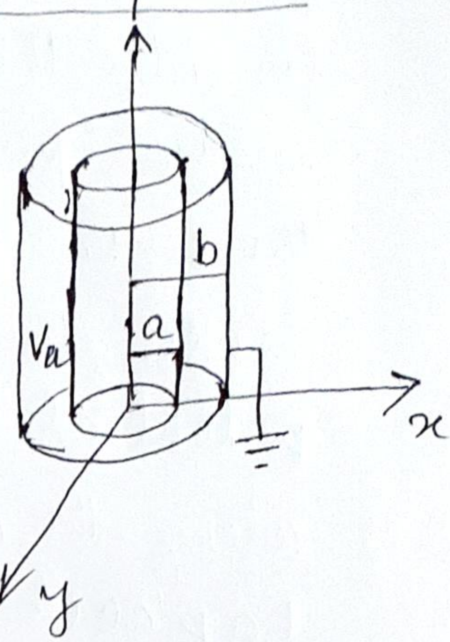
$V = Cx + D$ where C & D are constant of integration.

Applying the boundary condition i.e, at $x=0$, $V=0$ we get $D=0$. And at $x=d$, $V=V_a$

Therefore, $V_a = C \times d$ $\therefore$ $C = \frac{V_a}{d}$

So potential between the plates is $\boxed{V = \frac{V_a}{d} x}$

Potential of coaxial cylindrical capacitor



Consider a cylindrical capacitor of inner radius a, and outer radius b. Potential at inner cylinder is Va and outer cylinder is V=0

As the potential varies along radial direction. Hence, Laplace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \; , \text{ So, } \frac{\partial V}{\partial r} = C/r$$

Again integrating the above equation we get

$V = C \ln r + D$, where C & D are constants.

Applying the boundary condition at $r = b$, $V = 0$

$\therefore \; 0 = C \ln b + D$ or, $D = -C \ln b$

And at $r = a$, $V = Va$, Hence

$$Va = C\ln a - C\ln b = C \ln a/b \text{ or, } C = \frac{Va}{\ln\frac{a}{b}}$$

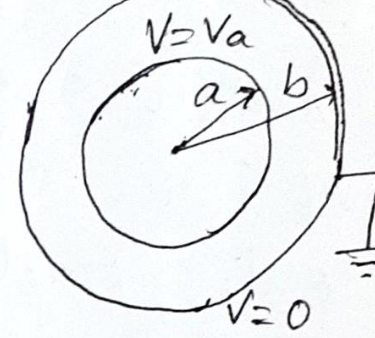
$$\text{So, } \; D = -\frac{Va}{\ln\frac{a}{b}}(\ln b)$$

$$\text{Therefore, } \; V = \frac{Va}{\ln\frac{a}{b}}\ln r - \frac{Va}{\ln\frac{a}{b}}\ln b = Va\frac{\ln(r/b)}{\ln(a/b)}$$

$$\therefore \boxed{V = \frac{Va \ln(r/b)}{\ln(a/b)}}$$

Potential of a concentric spherical capacitor

Consider the potential of the inner sphere is Va and the outer sphere is zero.



Since, the variation of potential exists only along the radial direction, then from the Laplace's equation we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=0$$

Integrating the above equation w.r.t. 'r' we get $V=-\frac{C}{r}+D$ where C & D are constants

The boundary condition at $r=b$, $V=0$ $\therefore D=C/b$.

So, $V=-\frac{C}{r}+\frac{C}{b}=C\left(\frac{1}{b}-\frac{1}{r}\right)$

From the condition $r=a$, $V=Va$, the above ~~eqn~~ equation becomes $Va=C\left(\frac{1}{b}-\frac{1}{a}\right)$ $\therefore C=\frac{Va}{\frac{1}{b}-\frac{1}{a}}$

Therefore, $\boxed{V=\frac{Va}{\left(\frac{1}{b}-\frac{1}{a}\right)}\times\left(\frac{1}{b}-\frac{1}{r}\right)}$

Which is the potential inside a spherical capacitor.