→ Reduction formula :-

Reduction formula helps to solve high order proper integral by converting them in simple form.

To obtain reduction formula of of order 'n' like $\int_a^b x^n dx$, we first assume it be equal to an identity like

$$I_n = \int_a^b x^n dx$$

Then, solve this integral (mainly by integration by part) to get reduction formula only in terms of idententity.

→ Reduction formula of $\int \sin^n x \, dx$ and $\int_0^{\pi/2} \sin^n x \, dx$.

• Reduction formula of $\int \sin^n x \, dx$

$\Rightarrow I_n = \int \sin^n x \, dx$

$\Rightarrow I_n = \int \sin^{n-1} x \sin x \, dx$

$= \sin^{n-1} x(-\cos x) - \int \left(\frac{d \sin^{n-1} x}{dx} \int \sin x \, dx \right) dx$

$= \sin^{n-1} x(-\cos x) - \int (n-1) \sin^{n-2} x \cos x \cdot (-\cos x) dx$

$= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x \, dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$

$\Rightarrow I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$

$\Rightarrow I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

$$\Rightarrow I_n + (n-1) I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$$

$$\Rightarrow n I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$$

$$\boxed{\Rightarrow I_n = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2}}$$

- Reduction formula of $\int_0^{\pi/2} \sin^n x \, dx$

Let,

$$J_n = \int_0^{\pi/2} \sin^n x \, dx$$

we know $I_n = \int \sin^n x \, dx$

$$= -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

So, $J_n = \left[I_n \right]_0^{\pi/2}$

$$\Rightarrow J_n = -\left[\frac{\sin^{n-1}x \cos x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \left[I_{n-2} \right]_0^{\pi/2}$$

$$= 0 + \frac{n-1}{n} J_{n-2}$$

$$\boxed{\Rightarrow J_n = \frac{n-1}{n} J_{n-2}}$$

Q. Find the value of

$$\int_0^{\pi/2} \sin^9 x \, dx$$

→ from the reduction of formula of

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

we know,

$$I_n = \frac{n-1}{n} I_{n-2}$$

so, $$I_9 = \int_0^{\pi/2} \sin^9 x \, dx = \frac{8}{9} I_7$$

$$I_7 = \frac{6}{7} I_5$$

$$I_5 = \frac{4}{5} I_3$$

$$I_3 = \frac{2}{3} I_1$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx$$

$$= \Big[-\cos x\Big]_0^{\pi/2}$$

$$= 1$$

$$\Rightarrow I_9 = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1$$

$$= \frac{128}{315} \text{ Ans}$$

→ Reduction formula of $\int \cos^n x\,dx$ and $\int_0^{\pi/2} \cos^n x\,dx$, $n > 1$

• Reduction formula of $\int \cos^n x\,dx$

Let $I_n = \int \cos^n x\,dx$

$\Rightarrow I_n = \int \cos^{n-1} x \cos x\,dx$

$= \cos^{n-1} x \sin x - \int (n-1)\cos^{n-2} x(-\sin x) \cdot \sin x\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx$

$= \cos^{n-1} x \sin x + n-1\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$

$\Rightarrow I_n = \cos^{n-1} x \sin x + n-1\, I_{n-2} - (n-1)\, I_n$

$\Rightarrow n I_n = \cos^{n-1} x \sin x + n-1\, I_{n-2}$

$$\boxed{\Rightarrow I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}}$$

- Reduction formula of $\int_0^{\pi/2} \cos^n x \, dx$

Let $J_n = \int_0^{\pi/2} \cos^n x \, dx = [I_n]_0^{\pi/2}$

$\Rightarrow J_n = \left[\frac{\cos^{n-1}x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}[I_{n-2}]_0^{\pi/2}$

$\Rightarrow J_n = 0 + \frac{n-1}{n} J_{n-2}$

$\boxed{\Rightarrow J_n = \frac{n-1}{n} J_{n-2}}$

→ Reduction formula of $\int \sin^m x \cos^n x \, dx$ and $\int_0^{\pi/2} \sin^m x \cos^n x \, dx \ \forall m>1, n>1.$

- Reduction of $\int \sin^m x \cos^n x \, dx$

$I_{m,n} = \int \sin^m x \cos^n x \, dx$

$= \int \cos^{n-1} x \sin^m x \cos x \, dx$

~~$\Rightarrow I_{m,n} = \cos^{n-1} x \cdot \sin^{m+1} x$~~

$\Rightarrow I_{m,n} = \cos^{n-1} x \int \sin^m x \cos x \, dx - \int \left[(n-1)\cos^{n-2} x \sin x \int \sin^m x \cos x \, dx\right] dx$

$[\cos x \, dx = d \sin x$

$$= \cos^{n-1}x \int \sin^m x \, d\sin x - \int (n-1)\cos^{n-2}x \sin x \int \sin^m x \, d\sin x$$

$$= \cos^{n-1}x \, \frac{\sin^{m+1}x}{m+1} + (n-1)\int \cos^{n-2}x \sin x \, \frac{\sin^{m+1}x}{m+1} \, dx$$

$$= \cos^{n-1}x \, \frac{\sin^{m+1}x}{m+1} + \frac{(n-1)}{m+1}\int \cos^{n-2} \sin^{m+2}x \, dx$$

$$\Rightarrow I_{m,n} = \frac{1}{m+1}\cos^{n-1}x \sin^{m+1}x + \frac{n-1}{m+1} I_{m+2,n-2}$$

$$\boxed{\Rightarrow I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m+2,n-2}}$$

Or there can be another reduction formula or,

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$