→ Beta function:-

Beta function is used to convert big or complex integral into a small or simpler form ~~~~ so that we can get their values by solving them or by using gamma function.

Beta function is denoted by $B(m,n)$, where $m, n > 0$

It is defined as,

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \quad \forall \; m, n > 0$$

<u>Properties of Beta function:-</u>

i) $B(m,n) = B(n,m) \quad \forall \; m, n > 0$

→ <u>Proof:-</u>

We have defination of beta function as

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \quad \forall \; m, n > 0$$

We can see that it is improper at it's both limits so, we can write it.

$$B(m,n) = \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{0+\delta}^{1-\epsilon} x^{m-1} (1-x)^{n-1} \, dx$$

Let $1 - x = t$

$\Rightarrow dt = -dx$

and,

$x \to 1-\epsilon \quad \Rightarrow t \to \epsilon$

$x \to 0+\delta \quad \Rightarrow t \to 1-\delta$

$$\Rightarrow B(m,n) = \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} -\int_{1-\delta}^{\epsilon} (1-t)^{m-1} \, t^{n-1} \, dt$$

Putting limiting values

$$\Rightarrow B(m,n) = -\int_{1}^{0} (1-t)^{m-1} \, t^{n-1} \, dt$$

[For even fuc. $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$]

$$\Rightarrow B(m,n) = \int_{0}^{1} (1-t)^{m-1} \, t^{n-1} \, dt$$

$$\Rightarrow B(m,n) = \int_{0}^{1} t^{n-1} (1-t)^{m-1} \, dt$$

$$\Rightarrow B(m,n) = B(n,m)$$

proved

ii) $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = B(n,m)$

$\forall \; m,n > 0$

→ Proof :- According to defination of beta function-

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \forall \; m,n > 0$$

but it is improper at its limits, so,

$$\Rightarrow B(m,n) = \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{0+\epsilon}^{1-\delta} x^{m-1} (1-x)^{n-1} dx$$

now,

let, $x = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1$

$dx = -\frac{1}{(1+y)^2} dy$

if $x \to 1-\delta \Rightarrow y \to \frac{\delta}{1-\delta}$

if $x \to 0+\epsilon \Rightarrow y \to \frac{1-\epsilon}{\epsilon}$

$$\Rightarrow B(m,n) = \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} -\int_{\frac{1-\epsilon}{\epsilon}}^{\frac{\delta}{1-\delta}} \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \cdot \frac{1}{(1+y)^2} dy$$

$$= \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} - \int_{\frac{1-\epsilon}{\epsilon}}^{\frac{\delta}{1-\delta}} \left(\frac{1}{1+y}\right)^{m-1} \left(\frac{y}{1+y}\right)^{n-1} \cdot \frac{1}{(1+y)^2}\, dy$$

$$= \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} - \int_{\frac{1-\epsilon}{\epsilon}}^{\frac{\delta}{1-\delta}} \left(\frac{1}{1+y}\right)^{m+1} \cdot y^{n-1} \cdot \left(\frac{1}{1+y}\right)^{n-1} dy$$

$$= \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} - \int_{\frac{1-\epsilon}{\epsilon}}^{\frac{\delta}{1-\delta}} \frac{y^{n-1}}{(1+y)^{m+1} \cdot (1+y)^{n-1}}\, dy$$

$$= \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} - \int_{\frac{1-\epsilon}{\epsilon}}^{\frac{\delta}{1-\delta}} \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$= \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{\frac{\delta}{1-\delta}}^{1-\epsilon/\epsilon} \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

putting limiting values

$$= \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

Replacing ~~y~~ by x we get

$$B(m,n) = \int_{0}^{\infty} \frac{x^{n-1}}{(1+y)^{m+n}}\, dy$$

Same as above

$$B(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\Rightarrow B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(n,m)$$

Relation b/w Beta and Gamma function :-

- $$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \quad \forall\, m,n>0$$

- $$B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma \frac{p+1}{2}\,\Gamma \frac{q+1}{2}}{\Gamma \frac{p+q+2}{2}} \quad \forall\, p,q>-1$$

iii) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta \quad \forall\, m,n>0$

→ According to defination of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \forall\, m,n>0$$

It is improper at it's limits, So, we can write it,

$$\Rightarrow B(m,n) = \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{0+\epsilon}^{1-\delta} x^{m-1} (1-x)^{n-1} dx \quad \forall\, m,n>0$$

now,

let,

$$x = \sin^2\theta$$

$$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$$

if $x \to 1-\delta \Rightarrow \theta \to \sin^{-1}\sqrt{1-\delta}$

if $x \to 0+\epsilon \Rightarrow \theta \to \sin^{-1}\sqrt{\epsilon}$

$$\Rightarrow B(m,n) = \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{\sin^{-1}\sqrt{\epsilon}}^{\sin^{-1}\sqrt{1-\delta}} \sin^{2(m-1)}\theta \, (1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{\sin^{-1}\sqrt{\epsilon}}^{\sin^{-1}\sqrt{1-\delta}} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2 \cdot \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{\sin^{-1}\sqrt{\epsilon}}^{\sin^{-1}\sqrt{1-\delta}} \frac{\sin^{2m-1}\theta}{\sin\theta} \cdot \frac{\cos^{2n-1}\theta}{\cos\theta} \cdot \sin\theta\cos\theta\, d\theta$$

$$= 2 \lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{\sin^{-1}\sqrt{\epsilon}}^{\sin^{-1}\sqrt{1-\delta}} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

putting limiting value

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta \quad \forall\, m,n>0$$

(iv) $B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} \quad \forall\, p,q > -1$

Q. Show that

$$\int_0^{\pi/2} \cos^m x \, dx = \int_0^{\pi/2} \sin^m x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{m+1}{2}}{\Gamma\frac{m+2}{2}} \quad \forall\, m > -1$$

$\rightarrow$ $$\int_0^{\pi/2} \cos^m x \, dx = \int_0^{\pi/2} \sin^0 x \cdot \cos^m x \, dx$$

$$= \frac{1}{2} \frac{\Gamma\frac{0+1}{2}\,\Gamma\frac{m+1}{2}}{\Gamma\frac{m+0+2}{2}}$$

$$\left[\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \, d\theta = \frac{1}{2} \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{m+n+2}{2}} \quad \forall\, m,n > -1\right]$$

$$\Rightarrow \int_0^{\pi/2} \cos^m x \, dx = \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{m+1}{2}}{\Gamma\frac{m+2}{2}}$$

$$\Rightarrow \int_0^{\pi/2} \cos^m x\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{m+1}{2}}{\Gamma\frac{m+2}{2}} \quad \forall m > -1 \qquad -(\mathrm{I})$$

now,

$$\int_0^{\pi/2} \sin^m x\, dx = \int_0^{\pi/2} \sin^m x \cos^0 x\, dx$$

$$= \frac{1}{2} \frac{\Gamma\frac{m+1}{2}\, \Gamma\frac{0+1}{2}}{\Gamma\frac{m+0+2}{2}}$$

$$\left[\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2} \frac{\Gamma\frac{m+1}{2}\, \Gamma\frac{n+1}{2}}{\Gamma\frac{m+n+2}{2}} \quad \forall m, n > -1\right]$$

$$\Rightarrow \int_0^{\pi/2} \sin^m x\, dx = \frac{1}{2} \frac{\Gamma\frac{1}{2}\, \Gamma\frac{m+1}{2}}{\Gamma\frac{m+2}{2}}$$

$$\Rightarrow \int_0^{\pi/2} \sin^m x .\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{m+1}{2}}{\Gamma\frac{m+2}{2}} \qquad -(\mathrm{II})$$

from eq (I) & (II)

$$\int_0^{\pi/2} \cos^m x\, dx = \int_0^{\pi/2} \sin^m x\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{m+1}{2}}{\Gamma\frac{m+2}{2}}$$

$$\forall m > -1$$

Q. Show that:-

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \, dx = \frac{8}{315}$$

→

$$LHS = \int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \, dx$$

$$= \frac{1}{2} \frac{\Gamma\frac{4+1}{2} \, \Gamma\frac{5+1}{2}}{\Gamma\frac{4+5+2}{2}} \quad \left[\int_0^{\pi/2} \sin^p \theta \cdot \cos^q \theta \, d\theta = \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} \right]$$

$$= \frac{1}{2} \frac{\Gamma\frac{5}{2} \, \Gamma\frac{6}{2}}{\Gamma\frac{11}{2}}$$

$$= \frac{1}{2} \frac{\Gamma\frac{5}{2} \, \Gamma 3}{\Gamma\frac{11}{2}}$$

$$= \frac{1}{2} \frac{\left(\frac{3}{2} \times \frac{1}{2} \times \Gamma\frac{1}{2}\right) \left(2 \times 1 \, \Gamma 1\right)}{\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \, \Gamma\frac{1}{2}}$$

$$= \frac{1}{2} \times \frac{\frac{3}{\cancel{2}\cancel{4}} \cancel{\sqrt{\pi}} \cdot \cancel{2}}{\frac{945}{32} \cancel{\sqrt{\pi}}}$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{3} \times \cancel{32} \, \cancel{16} \, 8}{\cancel{945}_{315} \times \cancel{2}} = \frac{8}{315} = RHS$$

Q. Show that:-

$$\int_a^b (x-a)^{m-1} \cdot (b-x)^{n-1} dx = (b-a)^{m+n-1} \cdot \beta(m,n)$$
$$\forall m,n>0$$

$\rightarrow$

$$LHS = \int_a^b (x-a)^{m-1} \cdot (b-x)^{n-1} dx$$

Let $x = a\cos^2\theta + b\sin^2\theta$

$\Rightarrow \frac{dx}{d\theta} = 2a\cos\theta(-\sin\theta) + 2b\sin\theta\cos\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta(b-a)\,d\theta$

If $x \to a$ $\Rightarrow a = a\cos^2\theta + b\sin^2\theta$
$\Rightarrow a(1-\cos^2\theta) = b\sin^2\theta$
$\Rightarrow a\sin^2\theta = b\sin^2\theta$
$\Rightarrow \sin^2\theta(b-a) = 0$
$\Rightarrow \theta \to 0$

If $x \to b$ $\Rightarrow b = a\cos^2\theta + b\sin^2\theta$
$\Rightarrow b(1-\sin^2\theta) = a\cos^2\theta$
$\Rightarrow b\cos^2\theta = a\cos^2\theta$
$\Rightarrow (b-a)\cos^2\theta = 0$
$\Rightarrow \theta \to \pi/2$

hence,

$$LHS = \int_0^{\pi/2} (a\cos^2\theta + b\sin^2\theta - a)^{m-1} (b - a\cos^2\theta - b\sin^2\theta)^{n-1} \, 2\sin\theta\cos\theta(b-a)\,d\theta$$

$$= \int_0^{\pi/2} \{a(\cos^2\theta - 1) + b\sin^2\theta\}^{m-1} \{b(1-\sin^2\theta) - a\cos^2\theta\}^{n-1} \, 2\sin\theta\cos\theta(b-a)\,d\theta$$

$$= \int_0^{\pi/2} \{b\sin^2\theta - a(1-\cos^2\theta)\}^{m-1} \{b\cos^2\theta - a\cos^2\theta\}^{n-1} \, 2\sin\theta\cos\theta(b-a)\,d\theta$$

$$= \int_0^{\pi/2} \{(b-a)\sin^2\theta\}^{m-1} \{(b-a)\cos^2\theta\}^{n-1} \, 2\sin\theta\cos\theta(b-a)\,d\theta$$

$$= 2\int_0^{\pi/2} (b-a)^{m+n-2} \frac{\sin^{2m-1}\theta}{\cancel{\sin\theta}} \cdot \frac{\cos^{2n-1}\theta}{\cancel{\cos\theta}} \cancel{\sin\theta\cos\theta}(b-a)\,d\theta$$

$$= (b-a)^{m+n-1} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

$$= (b-a)^{m+n-1} \beta(m,n) = LHS$$