→ Gamma function :-

Gamma function is used to solve big integal or complex integral but not solving it exactly but converting it in special form of equation, called Gamma function.

We can get value of gamma function without solving the integral.

Gamma function is defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad \forall n > 0$$

If any complex integral is given convert it in form of gamma function and then replace it by Γn.

ex. -

$$\int_0^\infty e^{-x^2} x^{2n-1} dx$$

Convert it / represent it in form of gamma function.

$$\rightarrow \int_0^\infty e^{-x^2} x^{2n-1} dx$$

Let $x^2 = t \Rightarrow dx = \frac{1}{2} \frac{dt}{\sqrt{t}}$

$$= \int_0^\infty e^{-t} (\sqrt{t})^{2n-1} \frac{1}{2} \frac{dt}{\sqrt{t}}$$

$$= \int_0^\infty e^{-t} \frac{(\sqrt{t})^{2n-1}}{\sqrt{t}} \frac{1}{2} \cdot dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} \, t^{n-1} \, dt$$

$$= \frac{1}{2} \Gamma n$$

So, $\int_0^\infty e^{-x^2} x^{2n-1} dx = \frac{1}{2} \Gamma n$

NOTE:- Value of gamma function doesn't depends upon variable x, but only depends upon value of 'n'.

→ **Properties of Γn :-**

- $\Gamma n+1 = n \Gamma n = n!$

→ $n \Gamma n = n \int_0^\infty e^{-x} x^{n-1} dx \quad \forall \, n > 0$

- $\Gamma n+1 = \int_0^\infty e^{-x} x^n dx$

$$= \lim_{\substack{r \to \infty \\ \epsilon \to 0}} \int_{0+\epsilon}^{r} e^{-x} x^{n} \, dx$$

$$= \lim_{\substack{r \to \infty \\ \epsilon \to 0}} \left[x^{n} \int e^{-x} dx - \int \left[\frac{d x^{n}}{dx} \int e^{-x} dx \right] dx \right]_{0+\epsilon}^{r}$$

$$= \lim_{\substack{r \to \infty \\ \epsilon \to 0}} \left[-x^{n} e^{-x} - \int -n x^{n-1} e^{-x} dx \right]_{0+\epsilon}^{r}$$

$$\cancel{= \lim_{\substack{r \to \infty \\ \epsilon \to 0}} \left[-x^{n} e^{-x} \right.}$$

$$= \left[-x^{n} e^{-x} \right]_{0}^{\infty} + \int_{0}^{\infty} n x^{n-1} e^{-x} dx$$

$$= 0 + n \int_{0}^{\infty} x^{n-1} e^{-x} dx$$

$$= n \int_{0}^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = n \Gamma(n)$$

NOTE :- $\Gamma 1 = 1$ and $\Gamma \frac{1}{2} = \sqrt{\pi}$

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n\Gamma((n-1)+1)$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)\Gamma((n-2)+1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)\Gamma((n-3)+1)$$
$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$
$$= n(n-1)(n-2)(n-3)\cdots 1\,\Gamma(1)$$
$$= n(n-1)(n-2)(n-3)\cdots 1$$
$$= n!$$

$$\Rightarrow \Gamma(n+1) = n\Gamma(n) = n!$$

- $$\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi}$$

$$\rightarrow \int_0^\infty e^{-a^2x^2}\,dx$$

Let $a^2x^2 = z$

$$ax = \sqrt{z}$$

$$\Rightarrow a\,dx = \frac{1}{2}z^{-1/2}\,dz$$

$a \to 0,\ z \to 0$
$a \to \infty,\ z \to \infty$

$$\Rightarrow \int_0^\infty e^{-z} \frac{1}{2a} z^{-1/2}\, dz$$

$$= \frac{1}{2a} \int_0^\infty e^{-z}\, z^{1/2 - 1}\, dz$$

$$= \frac{1}{2a} \Gamma 1/2$$

$$\int_0^\infty e^{-a^2x^2} dx = \frac{1}{2a} \sqrt{\pi}$$

Q. Find the value of

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

$\rightarrow$ Since e^{-x^2} is a even function so,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^\infty e^{-x^2} dx$$

$$= 2 \lim_{r \to \infty} \int_0^r e^{-x^2} dx$$

Let $x^2 = t \Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$= 2 \lim_{r\to\infty} \int_0^{r} e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= 2 \times \frac{1}{2} \lim_{r\to\infty} \int_0^{r} e^{-t}\, t^{-1/2}\, dt$$

$$= \lim_{r\to\infty} \int_0^{r} e^{-t}\, t^{1/2-1}\, dt$$

$$= \int_0^{\infty} e^{-t}\, t^{1/2-1}\, dt$$

$$= \Gamma\tfrac{1}{2}$$

$$= \sqrt{\pi}$$

NOTE :- $\Gamma m \, \Gamma 1-m = \dfrac{\pi}{\sin m\pi}$

ex.- $\Gamma \tfrac{1}{3}\, \Gamma \tfrac{2}{3} = ?$

$$\Rightarrow \Gamma \tfrac{1}{3}\, \Gamma 1-\tfrac{1}{3} = \frac{\pi}{\sin \pi/3} = \frac{\pi}{\sqrt{3}/2}$$

$$= \frac{2\pi}{\sqrt{3}}$$

Q. Evaluate :-

i) $\int_0^\infty e^{-x} \cdot x^{3/2}\, dx$

$= \int_0^\infty e^{-x} \cdot x^{5/2-1}\, dx$

$= \Gamma 5/2$

$= \Gamma\left(\frac{3}{2}+1\right)$

$= \frac{3}{2}\Gamma 3/2$

$= \frac{3}{2}\Gamma\left(1/2+1\right)$

$= \frac{3}{2} \times \frac{1}{2}\Gamma 1/2$

$= \frac{3}{4}\sqrt{\pi}$ Ans

ii) show that

$$\int_0^\infty e^{-4x} \cdot x^{3/2}\, dx = \frac{3}{128}\sqrt{\pi}$$

$\rightarrow$ LHS $= \int_0^\infty e^{-4x} \cdot x^{3/2}\, dx$

Let $4x = t$

$\Rightarrow \frac{dt}{dx} = 4 \Rightarrow dx = \frac{dt}{4}$

$$\Rightarrow \int_0^\infty e^{-t} \cdot \left(\frac{t}{4}\right)^{3/2} \frac{dt}{4}$$

$$= \frac{1}{4} \times \left(\frac{1}{4}\right)^{3/2} \int_0^\infty e^{-t}\, t^{3/2}\, dt$$

$$= \frac{1}{4} \times \left(\frac{1}{4}\right)^{3/2} \int_0^\infty e^{-t}\, t^{5/2-1}\, dt$$

$$= \frac{1}{4} \times \frac{1}{8} \int_0^\infty e^{-t}\, t^{5/2-1}\, dt$$

$$= \frac{1}{32} \Gamma 5/2$$

$$= \frac{1}{32} \times \frac{3}{4} \sqrt{\pi}$$

$$\int_0^\infty e^{-4x} \cdot x^{3/2}\, dx = \frac{3}{128} \sqrt{\pi}$$

iii) Show that $\Gamma 1/9 \cdot \Gamma 2/9 \cdots \Gamma 8/9 = \frac{16}{3} \pi^4$

$$\rightarrow \Gamma 1/9 \cdot \Gamma 2/9 \cdot \Gamma 3/9 \cdots \Gamma 8/9$$

$$= \left(\Gamma 1/9 \cdot \Gamma 8/9\right)\left(\Gamma \tfrac{2}{9}\, \Gamma 7/9\right)\left(\Gamma 3/9 \cdot \Gamma 6/9\right)\left(\Gamma 4/9\, \Gamma 5/9\right)$$

$$= \left(\Gamma 1/9\, \Gamma 1-\tfrac{1}{9}\right)\left(\Gamma 2/9\, \Gamma 1-2/9\right)\left(\Gamma 3/9 \cdot \Gamma 1-3/9\right)\left(\Gamma 4/9 \cdot \Gamma 1-4/9\right)$$

$$= \left(\frac{\pi}{\sin \pi/9}\right) \cdot \left(\frac{\pi}{\sin \frac{2\pi}{9}}\right) \cdot \left(\frac{\pi}{\sin 3\pi/9}\right) \cdot \left(\frac{\pi}{\sin \frac{4\pi}{9}}\right)$$

$$= \frac{\pi^4}{\sin \pi/3 \cdot \sin \pi/9 \cdot \sin 2\pi/9 \cdot \sin 4\pi/9}$$

$$= \frac{\pi^4}{\sqrt{3}/2 \cdot \sin(\pi/9) \cdot \sin(\pi/3 - \pi/9) \cdot \sin(\pi/9 + \pi/3)}$$

$$\left[\sin\theta \cdot \sin(60 - \theta) \cdot \sin(\theta + 60^\circ) = \frac{1}{4}\sin 3\theta\right]$$

$$= \frac{\pi^4}{\sqrt{3}/2 \cdot \frac{1}{4}\sin 3 \cdot \pi/9}$$

$$= \frac{\pi^4}{\frac{\sqrt{3}}{2} \times \frac{1}{4} \cdot \frac{\sqrt{3}}{2}}$$

$$= \frac{16\pi^4}{3}$$ Ans

NOTE:- Gamma function is applicable for infinite limits. as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad \forall n > 0$$