Q. Show that :-

$$\int_a^b (x-a)^3 (b-x)^2 = \frac{(b-a)^6}{60}$$

→ Let $x = a\cos^2\theta + b\sin^2\theta$

$dx = (-2a\cos\theta\sin\theta + 2b\sin\theta\cos\theta)d\theta$

$= (b-a)\,2\cos\theta\sin\theta\, d\theta$

if $x \to a \Rightarrow a = a\cos^2\theta + b\sin^2\theta$

$\Rightarrow \theta \to 0$

if $x \to b \Rightarrow b = a\cos^2\theta + b\sin^2\theta$

$\Rightarrow \theta \to \pi/2$

$$\Rightarrow LHS = \int_0^{\pi/2} (a\cos^2\theta + b\sin^2\theta - a)^3 (b - a\cos^2\theta - b\sin^2\theta)^2 (b-a)\,2\cos\theta\sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \{(b-a)\sin^2\theta\}^3 \{(b-a)\cos^2\theta\}^2 (b-a) \cos\theta\sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (b-a)^6 \sin^7\theta \cdot \cos^5\theta\, d\theta$$

$$= (b-a)^6\, 2\int_0^{\pi/2} \sin^7\theta \cdot \cos^5\theta \, d\theta$$

$$= (b-a)^6 \frac{\Gamma\frac{7+1}{2}\,\Gamma\frac{5+1}{2}}{\Gamma\frac{7+5+2}{2}}$$

$$= (b-a)^6 \frac{\Gamma 4\,\Gamma 3}{\Gamma 7}$$

$$= \frac{(b-a)^6}{60} = LHS$$

Q. Show that :-

$$\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}$$

$\rightarrow$

$$\frac{B(p,q+1)}{q} = \frac{\Gamma p\,\Gamma q+1}{q\cdot\Gamma p+q+1} = \frac{\Gamma p\cdot q\,\Gamma q}{q\cdot\Gamma (p+q)+1}$$

$$= \frac{\Gamma p\,\Gamma q}{p+q\,\Gamma p+q}$$

$$= \frac{B(p,q)}{p+q} \quad (1)$$

and

$$\frac{\beta(p+1,q)}{p} = \frac{\Gamma p+1\,\Gamma q}{p\cdot\Gamma p+1+q} = \frac{p\,\Gamma p\cdot\Gamma q}{p\,\Gamma(p+q)+1}$$

$$= \frac{\Gamma p\,\Gamma q}{p+q\,\Gamma p+q}$$

$$= \frac{\beta(p,q)}{p+q} \quad \text{-(ii)}$$

from eq (i) & (ii)

$$\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$$

- $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$
- $\beta(m,n)\beta(m+n,l) = \beta(n,l)\cdot\beta(n+l,m) = \beta(l,m)\cdot\beta(l+m,n)$

Q. evaluate :-

$$\int_0^{\pi/2} \sin^9 x\, dx$$

→ Given,

$$\int_0^{\pi/2} \sin^9 x \cos^0 x\, dx$$

$$= \frac{1}{2} \frac{\Gamma\frac{9+1}{2}\,\Gamma\frac{0+1}{2}}{\Gamma\frac{9+0+2}{2}}$$

$$= \frac{1}{2} \frac{\Gamma 5\,\Gamma 1/2}{\Gamma 11/2}$$

$$= \frac{\sqrt{\pi}}{2} \frac{4\times3\times2}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2}}$$

$$= \frac{4\times3\times2}{\cancel{2}\times\frac{9}{\cancel{2}}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}}$$

$$= \frac{4\times3\times2}{\frac{9\times7\times5\times3}{16}}$$

$$= \frac{4\times\cancel{3}\times2\times16}{9\times7\times5\times\cancel{3}}$$

$$= \frac{128}{315} \text{ Ans}$$

Q. $\int_0^{\pi/2} \sin^p x\,dx \times \int_0^{\pi/2} \sin^{p+1} x\,dx = \frac{\pi}{2(p+1)}$

→ In such question where two integral of same limits are multiplied, we can ~~use~~ solve these by solving them indivisually and then multipli them.

So, $\int_0^{\pi/2} \sin^p x \, dx \times \int_0^{\pi/2} \sin^{p+1} x \, dx$

$$\Rightarrow \int_0^{\pi/2} \sin^p x \cos^0 x \, dx \times \int_0^{\pi/2} \sin^{p+1} x \cos^0 x \, dx$$

$$= \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{0+1}{2}}{\Gamma\frac{p+2}{2}} \times \frac{1}{2} \frac{\Gamma\frac{p+2}{2} \, \Gamma\frac{0+1}{2}}{\Gamma\frac{p+3}{2}}$$

$$= \frac{1}{4} \pi \frac{\Gamma\frac{p+1}{2} \, \cancel{\Gamma\frac{p+2}{2}}}{\cancel{\Gamma\frac{p+2}{2}} \, \Gamma\frac{p+3}{2}}$$

$$= \frac{\pi}{4} \frac{\Gamma\frac{p+1}{2}}{\Gamma\left(\frac{p+1}{2}+1\right)} = \frac{\pi}{4} \frac{\Gamma\frac{p+1}{2}}{\frac{p+1}{2} \, \Gamma\frac{p+1}{2}}$$

$$= \frac{\pi}{2(p+1)} = \underline{\underline{LHS}}$$

Q. $\int_0^{\infty} e^{-x^4} dx \times \int_0^{\infty} e^{-x^4} x^2 \, dx = \frac{\pi}{8\sqrt{2}}$

→ Given,

$$\int_0^{\infty} e^{-x^4} dx \times \int_0^{\infty} e^{-x^4} x^2 \, dx$$

$$= \int_0^{\infty} e^{-(x^2)^2} dx \times \int_0^{\infty} e^{-(x^2)^2} x^2 \, dx$$

Let $x^2 = t$

$\Rightarrow \frac{dt}{dx} = 2x$

$\Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$= \int_0^\infty e^{-t^2} \frac{dt}{2\sqrt{t}} \times \int_0^\infty e^{-t^2} \cdot t \cdot \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t^2}\, t^{-1/2}\, dt \times \frac{1}{2}\int_0^\infty e^{-t^2} \cdot t^{1/2}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t^2}\, t^{2\times 1/4 - 1}\, dt \times \frac{1}{2}\int_0^\infty e^{-t^2} \cdot t^{2\times 3/4 - 1}\, dt$$

$$= \frac{1}{2} \times \frac{1}{2}\Gamma 1/4 \times \frac{1}{2} \times \frac{1}{2}\Gamma 3/4$$

$$\left[\int_0^\infty e^{-x^2} x^{2n-1}\, dx = \frac{1}{2}\Gamma n \right]$$

$$= \frac{1}{4}\Gamma 1/4 \times \frac{1}{4}\Gamma 3/4$$

$$= \frac{1}{16}\Gamma 1/4\, \Gamma 3/4$$

$$= \frac{1}{16}\Gamma 1/4\, \Gamma (1 - 1/4) = \frac{1}{16}\frac{\pi}{\sin \pi/4}$$

$$= \frac{1}{16} \frac{\pi}{1/\sqrt{2}}$$

$$= \frac{\sqrt{2}\,\pi}{16} = \frac{\pi}{8\sqrt{2}} = \text{LHS}$$

Q. Show that

$$\int_0^\infty \frac{x^{m-1}\,dx}{(a+bx)^{m+n}} = \frac{1}{a^n b^m} \cdot \beta(m,n) \quad \forall\ m,n>0$$

→ Given -

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$$

Let $bx = at$

$\Rightarrow b\,dx = a\,dt$

$\Rightarrow dx = \frac{a}{b}\,dt$

if $x \to 0$, $t \to 0$

if $x \to \infty$, $t \to \infty$

$$\Rightarrow \int_0^\infty \frac{\left(\frac{a}{b}t\right)^{m-1}}{(a+at)^{m+n}} \frac{a}{b}\,dt$$

$$= \int_0^\infty \frac{\left(\frac{a}{b}\right)^m t^{m-1}\,dt}{\{a(1+t)\}^{m+n}}$$

$$= \frac{\left(\frac{a}{b}\right)^m}{a^{m+n}} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \frac{a^m}{b^m a^m a^n} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \frac{1}{a^m b^m} \beta(m,n) = \text{LHS}$$

$$\left[\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m,n)\right]$$

Q. $\int_0^\infty e^{-x^2} x^\alpha dx = \frac{1}{2}\Gamma\frac{\alpha+1}{2} \; ; \; \alpha > -1$

$\rightarrow$ Given -

$$\int_0^\infty e^{-x^2} x^\alpha dx$$

$$= \int_0^\infty e^{-x^2} x^{2\left(\frac{\alpha+1}{2}\right)-1} dx$$

$$= \frac{1}{2}\Gamma\frac{\alpha+1}{2} \quad \left(\text{since } \int_0^\infty e^{-x^2} x^{2n-1} dx = \frac{1}{2}\Gamma n\right)$$

$2n-1=\alpha$

$2n=\alpha+1$

Q. $\int_0^1 x^p (1-x^q)^n \, dx = \frac{1}{q} \cdot \beta\left(\frac{p+1}{q}, n+1\right)$

$\rightarrow$ Given:-

$$\int_0^1 x^p (1-x^q)^n \, dx$$

let $x^q = t \Rightarrow x = t^{1/q}$

$$\cancel{\Rightarrow \frac{dt}{dx} = \frac{1}{q} x^{q-1}}$$

$$\cancel{\Rightarrow dt = \frac{1}{q} \frac{x^q}{x} dx}$$

$$\Rightarrow dx = \frac{1}{q} t^{1/q-1} \, dt$$

if $x \to 0, t \to 0$

$x \to 1, t \to 1$

$$\Rightarrow \int_0^1 t^{p/q} (1-t)^n \frac{1}{q} t^{1/q-1} \, dt$$

$$\Rightarrow \frac{1}{q} \int_0^1 t^{p/q+1/q-1} (1-t)^{(n+1)-1} \, dt$$

$$= \frac{1}{q} \int_0^1 t^{\frac{p+1}{q}-1} (1-t)^{(n+1)-1} \, dt$$

$$= \frac{1}{q} \left(B\left(\frac{p+1}{q}, n+1\right) \right) \quad = \text{LHS}$$

Γn

Q. evaluate $\int_0^{\pi/2} \sqrt{\tan x}\, dx = ?$

$\rightarrow \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\, dx$

$$\left[\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2} \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} \right]$$

$$\Rightarrow \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\, dx = \frac{1}{2} \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma 1}$$

$$= \frac{1}{2} \frac{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)}{1}$$

$$= \frac{1}{2} \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{\sqrt{2}}$$

Q. $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4} \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$

$\rightarrow$ RHS $= \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Let $x^4 = t$

$$\frac{dt}{dx} = 4x^3$$

$$\Rightarrow dx = \frac{dt}{4x^3}$$

$x \to 0 \Rightarrow t \to 0.$
$x \to 1 \Rightarrow t \Rightarrow 1$

$$\Rightarrow \int_0^1 \frac{dt}{4x^3\sqrt{1-t}}$$

$$= \frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{4}\int_0^1 t^{1/4-1}\,(1-t)^{1/2-1}\,dt$$

$$\left[\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)\right]$$

$$= \frac{1}{4}\beta\left(1/4, 1/2\right)$$

$$= \frac{1}{4}\,\frac{\Gamma 1/4\;\Gamma 1/2}{\Gamma 3/4} = \text{LHS}$$

NOTE:-

- $B(1/2, 1/2) = \dfrac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1/2+1/2} = \{\Gamma 1/2\}^2 = \pi$

- $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\Gamma 1/2 = \frac{\sqrt{\pi}}{2}$

- $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$