Divergence of Electrostatic Field ($\bar{E}$)

In order to obtain the value of divergence of electrostatic field (E) that is $\nabla \cdot \bar{E}$, we first calculate the volume integral of the $\nabla \cdot \bar{E}$ which is $\int_V (\nabla \cdot \bar{E})\, dv$. Now by applying divergence theorem to the above volume integral we get

$$\boxed{\int_V (\nabla \cdot \bar{E})\, dV = \oint_S \vec{E} \cdot \vec{ds}}$$

If we consider a point charge q is inside a sphere of radius r, then the electric flux will cross through the surface of the sphere. Therefor $\oint_S \bar{E} \cdot \vec{ds}$ represents the flux of electrostatic field $\bar{E}$ through the surface S. In case of spherically symmetric surface,

$$\oint_S \bar{E} \cdot \vec{ds} = \int_S \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{r}\right) \cdot \left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right)$$

where $\vec{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$$\therefore \oint \bar{E} \cdot \bar{ds} = \frac{q}{4\pi\epsilon_0} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{q}{4\pi\epsilon_0} \times 2 \times 2\pi = \frac{q}{\epsilon_0}$$

For any closed surface we will take the ~~ets~~ enclosed charge as Q_{enc}. Therefore, the previous relation we can write as

$$\boxed{\oint_S \vec{E} \cdot \vec{ds} = \frac{Q_{enc}}{\epsilon_0}}$$

The ~~eq~~ above relation is called Gauss's law in electrostatics in ~~differential form~~ integral form.

Again $\oint_S \bar{E} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{E}) dv = \frac{Q_{enc}}{\epsilon_0}$

If the volume charge density is ρ then we can write $Q_{enc} = \int_V \rho \, dv$

Therefore, from the above two relations we get $\int_V (\nabla \cdot \bar{E}) dv = \frac{1}{\epsilon_0} \int_V \rho \, dv$

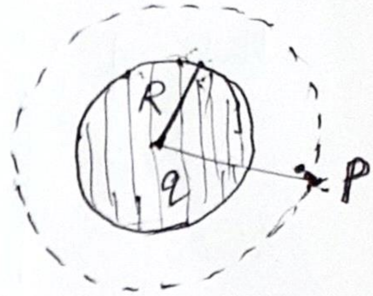
from removing volume integral from both sides we get

$$\boxed{\nabla \cdot \bar{E} = \rho / \epsilon_0}$$

This equation is called Gauss's law in electrostatics in differential form.

Application: Find the electric field outside a uniformly charged solid sphere of radius 'R' and total charge 'q'.

Soln: we have to ~~ca~~ calculate the field at point r distance from the sphere. where $r > R$. Now draw a Gaussian surface around the sphere of radius 'r'. By applying Gauss's law we get



$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$

Here $Q_{enc} = q$

$$\therefore \int_S |\vec{E}|\, ds = \frac{q}{\epsilon_0} = E \int_S ds = E\, 4\pi r^2$$

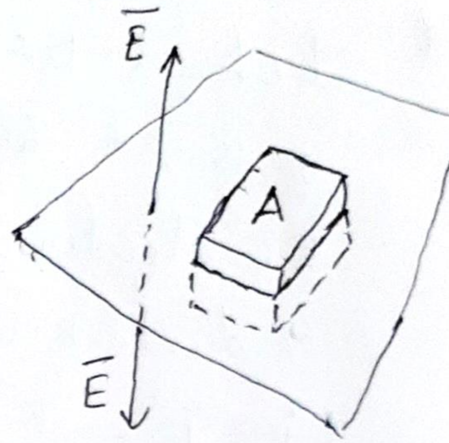
Thus, $$E\, 4\pi r^2 = \frac{q}{\epsilon_0}$$

or, $$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

Application: Find the electric field out of a plane carries a uniform surface charge 'σ'.

Soln: In order to obtain the electric field out of the surface, draw a "Gaussian pillbox" extending equal distance above and below the plane. By applying the Gauss's law to the surface we get

$$\oint_S \bar{E}\cdot\bar{ds} = \frac{Q_{enc}}{\epsilon_0}$$



$$\therefore \oint |\bar{E}|\, ds = 2AE$$

In this case $Q_{enc} = \sigma A$.

A is the area of the pillbox at both sides.

$\therefore$ from the above two equations we get

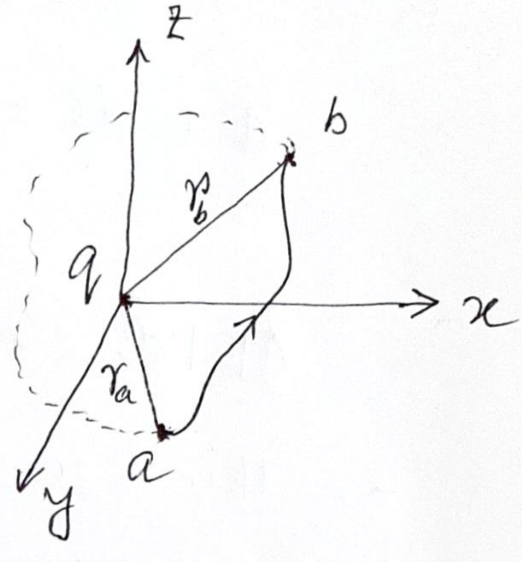
$$2A|\vec{E}| = \frac{\sigma A}{\epsilon_0}$$

$$\therefore \boxed{|\vec{E}| = \frac{\sigma}{2\epsilon_0}} \quad \text{or,} \quad \boxed{\bar{E} = \frac{\sigma}{2\epsilon_0}\hat{n}}$$

Where $\hat{n}$ is a unit vector pointing away from the surface.

Curl of $\bar{E}$ $(\nabla \times \bar{E})$

Let a point charge q is at the origin. Hence, the electric field at a point r distance from origin is $\vec{E} = \frac{1}{4\pi\epsilon_0} \times \frac{q}{r^2}\hat{r}$



We now calculate line integral of this field from some point 'a' to 'b'

Hence, $\int_a^b \bar{E}\cdot\bar{dl}$ required to obtain

In spherical polar coordinate system

$$\bar{dl} = \hat{r}\,dr + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\hat{r}\cdot\hat{r} = 1$$
$$\hat{r}\cdot\hat{\theta} = 0$$
$$\hat{r}\cdot\hat{\phi} = 0$$

Therefore, $\bar{E}\cdot\bar{dl} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr$

Hence, $$\int_a^b \bar{E}\cdot\bar{dl} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\,dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\Bigg|_{r_a}^{r_b}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right).$$

where r_a and r_b are distance from the origin to point a and b respectively.

Here the important thing is that the line integral is plainly independent of path. All it depends on ~~its~~ is the two end points, in fact, all that matter is how far a and ~~b~~ b from the charge. You may have question why this happened? Because, $\bar{E}$ points in the radial direction it cost nothing to move around θ & ϕ. directions. Therefore, any contribution from such a displacement is wiped out by the ~~d~~ dot product $\bar{E}\cdot\bar{dl}$. The integral around a closed path is obviously zero.

$r_a = r_b$

Hence, $\oint \bar{E}\cdot\bar{dl} = 0$ and by applying Stoke's theorem we get $\boxed{\nabla\times\bar{E} = 0}$ As

As $\oiint_s (\nabla\times\bar{E})\cdot d\bar{s} = 0$

As $\nabla\times\bar{E} = 0$, the $\bar{E}$ vector is called conservative or irrotational vector.