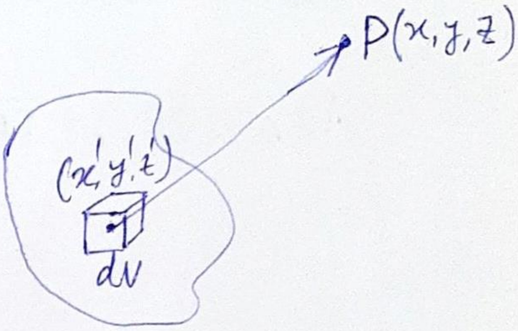
Divergence of Magnetic field

The Biot-Savart law for the general case of volume current reads

$$\bar{B} = \frac{\mu_0}{4\pi}\int \frac{\bar{J} \times \hat{r}}{r^2}\, dv \qquad \vec{J} = \frac{\bar{I}}{d\bar{a}}$$

The above formula gives the magnetic field at a point $P(x,y,z)$ in terms of an integral over the current density $J(x',y',z')$



Applying the divergence to $\bar{B}$ we obtain

$$\nabla \cdot \bar{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\bar{J} \times \frac{\hat{r}}{r^2}\right) dv$$

We know the product rule

$$\nabla \cdot \left(\bar{J} \times \frac{\hat{r}}{r^2}\right) = \frac{\hat{r}}{r^2} \cdot (\nabla \times \bar{J}) - \bar{J} \cdot \left(\nabla \times \frac{\hat{r}}{r^2}\right)$$

$\nabla \times \bar{J} = 0$, because $\bar{J}$ does not depend on (x,y,z)

Also $\nabla \times \left(\frac{\hat{r}}{r^2}\right) = 0$

Therefore, we get $\boxed{\nabla \cdot \bar{B} = 0}$

which is known as differential form of Gauss' law. It tells that the magnetic ~~is~~ field is solenoidal. Comparing it with Gauss' law in electrostatics $\nabla \cdot \bar{E} = \rho/\epsilon_0$, we can conclude that magnetic monopole does not exist.

The Curl of $\bar{B}$

Applying curl to $\bar{B}$ we get

$$\nabla \times \bar{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\bar{J} \times \frac{\hat{r}}{r^2} \right) dv$$

Also, we have from product rule

$$\nabla \times \left(\bar{J} \times \frac{\hat{r}}{r^2} \right) = \bar{J} \left(\nabla \cdot \frac{\hat{r}}{r^2} \right) - (\bar{J} \cdot \nabla) \frac{\hat{r}}{r^2}$$

The second term is zero as $\nabla \cdot \bar{J}$ is zero.

Here $\nabla \cdot (\hat{r}/r^2) = 4\pi \delta^3(r)$, where $\delta^3(r)$ is 3-D Dirac-Delta function,

where $\int \delta^3(r)\, dv = 1$.

$$\therefore \nabla \times \bar{B} = \frac{\mu_0}{4\pi} \int J \left(\nabla \cdot \hat{r}/r^2 \right) dv = \frac{\mu_0}{4\pi} \times 4\pi \int \delta^3(r)\, dv$$

$$= \frac{\mu_0 \bar{J}}{4\pi} \times 4\pi$$

Hence, $\boxed{\nabla \times \bar{B} = \mu_0 \bar{J}}$ which is known as Ampere's law

Ampere's law in differential form

$\nabla \times \bar{B} = \mu_0 \bar{J}$ we can convert it into integral form. By applying stoke's theorem we get

$$\oint (\nabla \times \bar{B}) \cdot \bar{ds} = \oint \bar{B} \cdot \bar{dl} = \mu_0 \int \bar{J} \cdot \bar{ds} = \mu_0 I_{enc}$$

Here $\int \bar{J} \cdot \bar{ds}$ is the total current passing the surface which we call I_{enc}. Thus

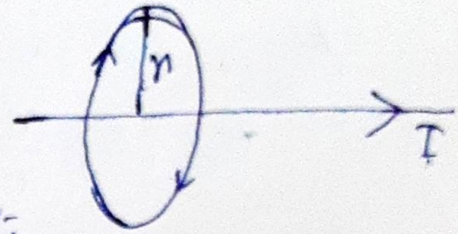
$$\boxed{\oint \bar{B} \cdot \bar{dl} = \cancel{\mu_0 \bar{J}}\ \mu_0 I_{enc}}$$

This is the integral version of Ampere's law

Application of Ampere's law

Find the magnetic field at distant point r from a long straight wire, carrying a steady current I.

Here the direction of $\bar{B}$ is circumferential, circling around the wire.



By symmetry, the magnitude of $|\bar{B}|$ is constant around an "amperian loop" of radius r centered on the wire, so, Ampere's law gives

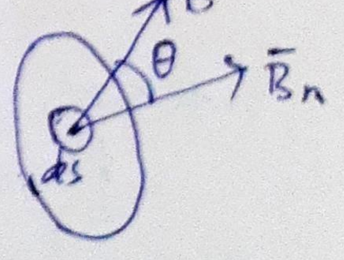
$$\oint \bar{B}.d\bar{l} = B\oint d\bar{l} = B \times 2\pi r = \mu_0 I$$

$$\therefore \bar{B} = \frac{\mu_0 I}{2\pi r}$$

Magnetic Flux

Magnetic flux through a surface is defined as the number of magnetic field lines crossing the surface normally.

Therefore, magnetic flux through a surface ds, where in a magnetic field $\bar{B}$ is given by



$d\phi = B_n ds$, where B_n is the normal component of the magnetic field B

$\therefore B_n = B\cos\theta$

Hence $d\phi = B\cos\theta\, ds = \bar{B}.d\bar{s}$