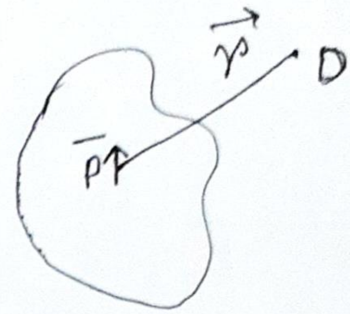
The Potential of a Polarized Object



Suppose we have a piece of polarized material – that is an object containing a lot of microscopic dipoles lined up. The dipole moment per unit volume $\bar{P}$ is given. What will be the potential (field) due to this polarized medium? We can obtain the potential at an external point by integrating the effect of individual dipoles.

For a single dipole $\vec{p}$ we know the potential is

$$V(r) = \frac{1}{4\pi\epsilon_0} \times \frac{\hat{r}.\bar{p}}{r^2}$$

In the present context we have the dipole moment of the bulk medium $\bar{p} = \int_v \bar{P} dv$

Therefore, potential due to the bulk medium is

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\hat{r}.\bar{P}}{r^2} dv$$

In order to obtain more insight we write the above intergal in much more illuminating form.

We know that $\nabla\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$

$$V = \frac{1}{4\pi\epsilon_0}\int_V \bar{P}\cdot\nabla\left(\frac{1}{r}\right) dv$$

From the vector identity relation we have

$$\nabla\cdot\left(\frac{\bar{P}}{r}\right) = \bar{P}\cdot\nabla\left(\frac{1}{r}\right) + \frac{1}{r}(\nabla\cdot\bar{P})$$

$$\therefore \quad V = \frac{1}{4\pi\epsilon_0}\left[\int_V \nabla\cdot\left(\frac{\bar{P}}{r}\right) dv - \int_V \frac{1}{r}(\nabla\cdot\bar{P})\, dv\right].$$

By applying the divergence theorem in the first term of the R.H.S we get.

$$V = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\bar{P}}{r}\cdot d\bar{s} - \int_V \frac{1}{r}(\nabla\cdot\bar{P})\, dv\right]$$

$$= \frac{1}{4\pi\epsilon_0}\oint_S \frac{1}{r}\,\bar{P}\cdot\hat{n}\, ds - \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{r}(\nabla\cdot\bar{P})\, dv.$$

The first term in the R.H.S looks like the potential of a surface charge

$$\boxed{\sigma_b = \bar{P}\cdot\hat{n}}$$

where $\hat{n}$ is the normal unit vector to the surface.

The second term in the R.H.S looks like the potential of volume charge.

$$\boxed{\rho_b = -\nabla\cdot\bar{P}}$$

~~With~~ With these definitions, the overall potential will become

$$V(r) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{r}\,ds + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b}{r}\,dv$$

This means that the potential of a polarised object is the same as that produced by volume charge density $\rho_b = -\nabla\cdot\bar{P}$ plus a surface charge density $\sigma_b = \bar{P}\cdot\hat{n}$. Instead of integrating over a whole volume, we just need to find these bound charges and then calculate the field they produce.

<u>The Electric Displacement</u>

Due to a polarized body the bound charge densities are $\rho_b = -\nabla\cdot\bar{P}$ and $\sigma_b = \bar{P}\cdot\hat{n}$

As the surface charge is loosely bounded on the surface of the body. Therefore, we will take $\sigma_b = \rho_f$, the free charge density. Within the dielectric, then, the total charge density can be written as

$$\rho = \rho_b + \rho_f$$

By applying the Gauss's law ~~we~~ for electrostatics

$$\nabla\cdot\bar{E} = \rho/\epsilon_0$$

$$\therefore\ \epsilon_0(\nabla\cdot\bar{E}) = \rho = \rho_b + \rho_f = -\nabla\cdot\bar{P} + \rho_f$$

$$\text{or, } \nabla\cdot(\epsilon_0\bar{E}) + \nabla\cdot\bar{P} = \rho_f$$

$$\therefore\ \nabla\cdot(\epsilon_0\bar{E} + \bar{P}) = \rho_f$$

The expression in the parentheses, designated by the letter $\vec{D}$

$$\therefore\ \boxed{\bar{D} = \epsilon_0\bar{E} + \bar{P}}$$

is known as the electric displacement.

In terms of $\bar{D}$, Gauss's law reads

$$\boxed{\nabla\cdot\bar{D} = \rho_f}$$

or in integral form $\boxed{\oint \bar{D}\cdot d\bar{s} = Q_f}$

Where Q_f is the total free charge enclosed in the volume.