Magnetic Scalar and Vector Potentials

We recall that some electrostatic field problems were simplified by relating the electric potential V to the field intensity $\bar{E}$ ($\bar{E} = -\nabla V$). Similarly, we can define a potential associated with magnetostatic field $\bar{B}$. In fact, the magnetic potential could be scalar V_m or vector $\vec{A}$.

In order to define V_m and $\vec{A}$, we recall two important ~~identities~~ divergence & curl of $\bar{B}$.

$$\nabla \times \bar{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \bar{B} = 0$$

If in some region of space, current density $\vec{J} = 0$, then the above two equations will be

$$\nabla \times \bar{B} = 0 \quad \&$$

$$\nabla \cdot \bar{B} = 0$$

The above two relations can be related with two identities

$$\nabla \times (\nabla V_m) = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

Therefore, we may express $\bar{B} = -\nabla V_m$ & hence $\bar{B}$ can be expressed by gradient of a scalar quantity V_m. Where V_m is called the magnetic scalar potential.

Since in this case $\nabla \cdot \bar{B} = 0$ so $\nabla \cdot (-\nabla V_m) = 0$ or $\nabla^2 V_m = 0$, we see V_m satisfies Laplace's equation in space where $\bar{J} = 0$.

Magnetic Vector Potential

We have already discussed that the Gauss' law in magnetostatics that always $\nabla \cdot \bar{B} = 0$. Again we know that divergence of ~~any~~ curl of any vector is zero. For any vector $\vec{A}$, $\nabla \cdot (\nabla \times \vec{A}) = 0$. Therefore we get

$$\nabla \cdot \bar{B} = 0 \quad \&$$
$$\nabla \cdot (\nabla \times \bar{A}) = 0$$

From the above two equations we can say that $\bar{B} = \nabla \times \bar{A}$.

The vector function $\vec{A}$ which satisfies the above equation is known as magnetic vector potential. This vector potential can help to determine $\bar{B}$ at a given point, since $\bar{B}$ is the space derivative of $\vec{A}$. The magnetic vector potential may be defined as a vector, the curl of which gives the magnetic induction produced at any point by a closed-loop carrying current.

We know that $\nabla \times \bar{B} = \mu_0 \bar{J}$

Also, $\bar{B} = \nabla \times \bar{A}$

$$\therefore \nabla \times (\nabla \times \vec{A}) = \mu_0 J$$

$$\text{or, } \nabla (\nabla \cdot \vec{A} - \nabla^2 \vec{A}) = \mu_0 J$$

Here, ~~as~~, $\nabla \cdot \vec{A} = 0$, for steady current as $\nabla \cdot \bar{B} = 0$. Therefore, we get

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

The earlier equation is same as Poisson's equation in electrostatics which is

$$\nabla^2 V = -\rho/\epsilon_0$$

Where V is the electrostatic potential and satisfies

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, dv}{r}$$

In analogy with this we can write

$$\bar{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\, dv}{r} \quad \text{for volume current}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}\, ds}{r} \quad \text{for surface current}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_l \frac{\vec{I}\, dl}{r} \quad \text{for line current.}$$

Problem: A magnetic field $4\times10^{-3}\hat{k}$ tesla exerts a force of $(4\hat{i}+3\hat{j})\times10^{-10}$ N on a particle having charge of 1×10^{-9} C and moving in the xy plane, calculate the velocity of the particle.

Soln: We know the magnetic force

$$\bar{F} = q(\bar{v}\times\bar{B})$$

Here $(4\hat{i}+3\hat{j})\times10^{-10} = 1\times10^{-9}[(v_x\hat{i}+v_y\hat{j})\times4\times10^{-3}\hat{k}]$

$$= 4\times10^{-12}[v_x(-\hat{j})+v_y(\hat{i})]$$

$$\therefore v_x = \frac{3\times10^{-10}}{4\times10^{-12}}\text{ m/s} = -75\text{ m/s}$$

$$v_y = \frac{4\times10^{-10}}{4\times10^{-12}}\text{ m/s} = 100\text{ m/s}$$

Therefore velocity $\bar{v} = (-75\hat{i}+100\hat{j})$ m/s