Energy Stored in Magnetic field

We have derived that the energy stored in electric field is

$$U_E = \frac{1}{2} \epsilon_0 E^2$$

Also we know the relation between $\bar{E}$ & $\bar{B}$ is $\bar{E} = c\bar{B}$

Therefore, energy stored in magnetic field is

$$U_M = \frac{1}{2} \epsilon_0 (c^2 B^2) = \frac{1}{2} \epsilon_0 \times \frac{1}{\mu_0 \epsilon_0} \times B^2 = \frac{1}{2} \frac{B^2}{\mu_0}$$

$$\therefore \boxed{U_M = \frac{1}{2\mu_0} B^2}$$

Electromagnetic energy flow & Poynting Vector:

Poynting Theorem

Work necessary to assemble a static charge charge distribution (against the Coulomb repulsion of like charges is

$$W_E = \frac{\epsilon_0}{2} \int E^2 dv$$ where $\bar{E}$ is resulting electric field.

Likewise, the work required to get currents going against the back emf is

$$W_B = \frac{1}{2\mu_0} \int B^2 dv$$

$\therefore$ The total energy stored in the E-M field is $W_{EB} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) dv$

In order to obtain energy conservation law for electrodynamics, suppose we have some charge and current configuration which, at time time 't', produces fields $\bar{E}$ & $\bar{B}$. In the next instant, dt, the charges move around a bit. According to Lorentz force law, the work done on an element of charge ~~dq~~ dq is

$$dw = \bar{F}.\bar{dl} = dq(\bar{E} + \bar{v} \times \bar{B}).\bar{v}dt = \bar{E}.\bar{v}dq\,dt$$

Now $dq = \rho dv$ and $\rho \bar{v} = \bar{J}$, so the total work done on all the charges in the volume V is given by

$$\frac{dw}{dt} = \int_V \vec{E}.\bar{v}\rho\, dv = \int_V (\bar{E}.\bar{J})\, dv$$

Thus here $\bar{E}.\bar{J}$ is the work done per unit time, per unit volume, - which is to say, the power delivered per unit volume.

$$\bar{E}.\bar{J} = \frac{1}{\mu_0}\bar{E}.(\nabla \times \bar{B}) - \epsilon_0 \bar{E}.\frac{\partial \bar{E}}{\partial t}$$

From the product rule

$$\nabla.(\bar{E} \times \bar{B}) = \bar{B}.(\nabla \times \bar{E}) - \bar{E}.(\nabla \times \bar{B})$$

Invoking Faraday's law

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \quad \text{it follows}$$

$$\bar{E}\cdot(\nabla\times\bar{B}) = -\bar{B}\cdot\frac{\partial\bar{B}}{\partial t} - \nabla\cdot(\bar{E}\times\bar{B})$$

Meanwhile $\bar{B}\cdot\frac{\partial\bar{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2)$ &

$$\bar{E}\cdot\frac{\partial\bar{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

Therefore, $\bar{E}\cdot\bar{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla\cdot(\bar{E}\times\bar{B})$

Hence, work done per unit time becomes

$$\frac{dw}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)dv$$

$$-\frac{1}{\mu_0}\int_V \nabla\cdot(\bar{E}\times\bar{B})\,dv$$

Applying divergence theorem to the second term on the right hand side, we get

$$\boxed{\frac{dw}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)dv - \frac{1}{\mu_0}\oint_S (\bar{E}\times\bar{B})\cdot\bar{ds}}$$

Where S is the surface bounding the volume V.

This is Poynting's theorem; it is the work-energy theorem of electrodynamics. The first integral on the right hand side is the total energy stored in the fields W_{EB}. The second term evidently represents the rate at which energy is carried out of V, across the boundary surface by the electromagnetic fields.

Poynting's theorem says, then, that the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy which flowed out through the surface.

The energy per unit time, per unit area, transported by the fields is called the Poynting Vector:

$$\boxed{\bar{S} = \frac{1}{\mu_0}(\bar{E} \times \bar{B})}$$

Specifically, $\bar{S} \cdot d\bar{s}$ is the energy per unit time crossing the infinitesimal surface $d\bar{s}$, - the energy flux.

Therefore, $\bar{S}$ (the poynting vector) is the energy flux density.

Also the Poynting's theorem can be expressed more compactly as

$$\frac{dw}{dt} = -\frac{dW_{EB}}{dt} - \oint_s \bar{S} \cdot d\bar{s}$$

Of course, the work W done on the charges will increase their mechanical energy. If we let U_M denote the mechanical energy density, so that

$$\frac{dW}{dt} = \frac{d}{dt}\int U_M \, dV$$

and U_{EB} for the energy density of the fields,

$$U_{EB} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

Then we get

$$\frac{d}{dt}\int_V (U_M + U_{EB})\, dV = -\oint_S \bar{S} \cdot d\bar{s}$$

$$= -\int_V (\nabla \cdot \bar{S})\, dV$$

Hence, we can write

$$\boxed{\nabla \cdot \bar{S} = -\frac{\partial}{\partial t}(U_M + U_{EB})}$$

This is the differential version of Poynting's theorem. We might compare it with the continuity equation expressing conservation of charge $\nabla \cdot \bar{J} = -\frac{\partial}{\partial t}\rho$

The charge density is replaced by the total energy density, whereas current density is replaced by Poynting vector.