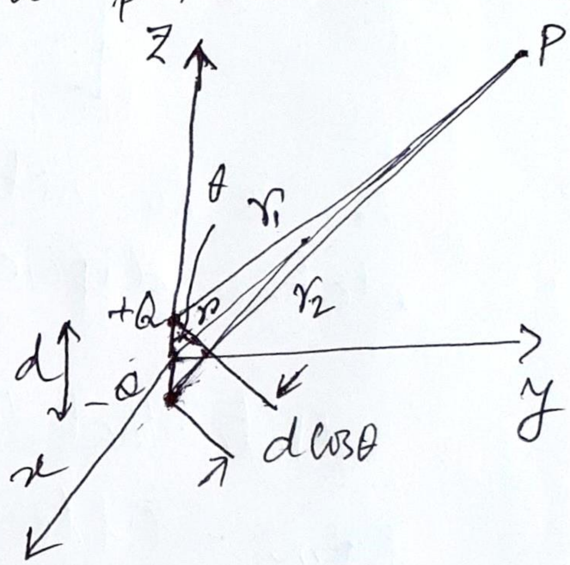
Electric Dipole & Flux Lines

An electric dipole is formed when two point charges of equal magnitude but opposite sign are separated by a small distance.

First we will derive the potential due an electric dipole. Consider the dipole arrangement as shown in the figure. The potential at point $P(r, \theta, \phi)$ is given by



$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$ where r_1 and r_2 are the distances between P and $+Q$ and P and $-Q$ respectively. If $r >> d$, $\underline{r_2 - r_1 = d\cos\theta}$.

Where θ is the angle between d and r. r is the distance of P from the dipole, 'd' is the separation between $+Q$ and $-Q$ charges.

Hence, $r_1 r_2 \simeq r^2$ therefor the potential at P becomes:

$$V = \frac{Q}{4\pi\epsilon_0} \frac{d\cos\theta}{r^2}$$

Now, $d\cos\theta = \vec{d}\cdot\hat{r}$, where $\vec{d} = d\hat{k}$

We have the dipole moment $\vec{p} = Q\vec{d}$

Hence,

$$\boxed{V = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}}$$

It is to note that potential due to an electric dipole varies as $\frac{1}{r^2}$.

Note that the dipole moment $\vec{p}$ is directed from $-Q$ to $+Q$.

If the dipole center is not at the origin but at $\vec{r}'$, the potential becomes.

$$\boxed{V(r) = \frac{\vec{p}\cdot(\vec{r}-\vec{r}')}{4\pi\epsilon_0 |r-r'|^3}}$$

Now we shall calculate the corresponding electric field.

The electric field due to the dipole with center at the origin can be obtained readily from the above potential expression as

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}\right]$$

Here $V = \frac{Q}{4\pi\epsilon_0} \times \frac{d\cos\theta}{r^2}$

$$\therefore \bar{E} = \frac{Qd\cos\theta}{2\pi\epsilon_0 r^3}\hat{r} + \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\hat{\theta}$$

Hence, $\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

where $p = |\vec{p}| = Qd$.

Note that a point charge is a monopole and its electric field varies inversely as r^2 while its potential field varies as inversely as r. But in case of a dipole, the electric field varies inversely as r^3, while its potential varies inversely as r^2. The electric field due to successive higher order multipoles such as quadrupole consisting of two dipoles or an octupole consisting of two quadrupoles vary inversely as $r^4, r^5, r^6 \ldots$ while corresponding potentials vary inversely as $r^3, r^4, r^5 \ldots$

Electric Flux Lines

The idea of electric flux lines or electric lines of force as they are sometimes called was introduced by Michael Faraday in his experimental investigation as a way of visualizing the electric field.

An electric flux line is an imaginary path or line drawn in such a way that its direction at any point is the direction of the electric field at that point.

Electric Flux Density

The flux density due to electric field $\vec{E}$ can be calaulated by using the relation of flux $\phi = \int_s \bar{E} \cdot d\bar{s}$

For practical reasons, however, this quantity is not usually consider to be the most useful flux in electrostatics. The electric field intensity is depend on the medium in which the charge is placed. For this we introduce a new vector field $\vec{D}$ is defined by $\bar{D} = \epsilon_0 \bar{E}$.

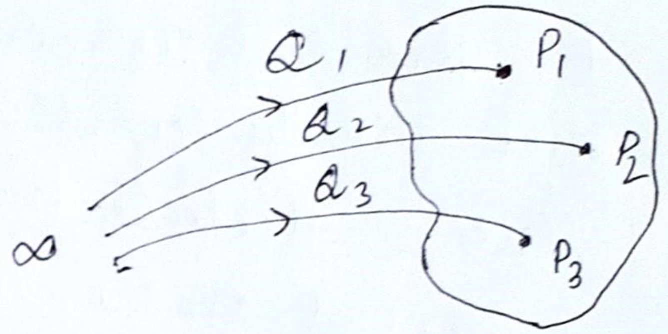
To define electric flux ϕ in terms of D, namely $\phi = \int_s \bar{D} \cdot d\bar{s}$.

In SI units, one line of electric flux originates from +1C and terminates on -1C. Therefore, the electric flux is measured in Coulombs. Hence the vector field $\vec{D}$ is called the electric flux density and is measured in Coulombs per square meter. The flux density is also known as electric displacement.

Energy Density in Electrostatic Fields

To determine the energy present in an assembly, we have to determine the amount of work necessary to assemble them.

Suppose we wish to position three point charges Q_1, Q_2, and Q_3 in initially empty region as shown in the figure bounded by closed line.



No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field. The work done transferring Q from infinity to P_2 is equal to the product of Q and the potential V_{21} at P_2 due to Q_1. Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32}+V_{31})$ where V_{32} and V_{31} are the potentials at P_3 due to Q_2 and Q_1. Hence, the total work done in positioning the three charges is

$$W_E = W_1 + W_2 + W_3 = 0 + Q_2 V_{21} + Q_3(V_{32} + V_{31})$$

If the charges were positioned in reverse order

then $W_E = W_3 + W_2 + W_1$

$$= 0 + Q_2 V_{23} + Q_3 (V_{32} + V_{13})$$

Where V_{23} is the potential at P_2 due to Q_3. V_{12} & V_{13} are respectively the potential at P_1 due to Q_2 and Q_3.

Adding the above two work done gives

$$2 W_E = Q_1(V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$
$$= Q_1 V_1 + Q_2 V_2 + Q_3 V_3 .$$

or, $W_E = \frac{1}{2} (Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$

Where V_1, V_2, V_3 are total potentials at P_1, P_2, P_3 respectively. Therefore, if there are n- point charges, the above equation becomes

$$W_E = \frac{1}{2} \sum_{K=1}^{n} Q_K V_K$$

If instead of point charges, the region has a continuous charge distribution, the summation in the above equation becomes integration, that is

$W_E = \frac{1}{2} \int_L \rho_L V dl$ (for line charge),

$W_E = \frac{1}{2} \int_S \rho_S V ds$ (for surface charge)

$$W_E = \frac{1}{2}\int_V \rho_v V\,dv \quad \text{(for volume charge)}$$

We know that $\rho_v = \epsilon_0(\nabla\cdot\bar{E})$. Hence, for volume charge distribution, the work done to become,

$$W_E = \frac{1}{2}\epsilon_0\int_V (\nabla\cdot\bar{E})V\,dv$$

But for any vector $\vec{A}$ and scalar ϕ, the identity ~~is~~ relation is

$$\nabla\cdot(\phi\vec{A}) = \bar{A}\cdot(\nabla\phi) + \phi(\nabla\cdot\vec{A})$$

$$\text{or,}\quad (\nabla\cdot\bar{A})\phi = \nabla\cdot(\phi\bar{A}) - \bar{A}\cdot(\nabla\phi)$$

Therefore, $W_E = \frac{1}{2}\epsilon_0\int_V \nabla\cdot(V\bar{E})\,dv - \frac{1}{2}\epsilon_0\int_V (\bar{E}\cdot\nabla V)\,dv$

Applying divergence theorem to the first term on the right hand side we get

$$W_E = \frac{1}{2}\epsilon_0\oint_S (V\bar{E})\cdot d\bar{s} - \frac{1}{2}\epsilon_0\int_V (\bar{E}\cdot\nabla V)\,dv$$

We know the V ~~varies as~~ varies as $1/r$ and and $\bar{E}$ as $1/r^2$ for point charge; V varies as $1/r^2$ and $\bar{E}$ as $1/r^3$ for dipoles and so on. Hence, $V\bar{E}$ in the first term on the R.H.S must vary at least as $1/r^3$ while ds varies as r^2. Consequently, the first ~~terms~~

integral must tend to zero as the surface becomes large. Hence,

$$W_E = -\frac{1}{2}\epsilon_0 \int_v (\bar{E}\cdot\nabla V)\,dv$$

and since, $\bar{E} = -\nabla V$

$$\therefore \boxed{W_E = \frac{1}{2}\epsilon_0 \int_v (\bar{E}\cdot\bar{E})\,dv = \frac{1}{2}\int \epsilon_0 E^2 dv}$$

We can define electrostatic energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\epsilon_0 E^2 = \frac{D^2}{2\epsilon_0}$$

As $W_E = \int_v w_E\, dv$.