→ Methods to solve improper integral :-

i) $\int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \lim_{x_1 \to -\infty} \int_{x_1}^{0} f(x)\,dx + \lim_{x_2 \to \infty} \int_{0}^{x_2} f(x)\,dx$$

here, instead of zero, you can choose any real number at which function/integrand is continuous.

also, to be this whole integral $\int_{-\infty}^{\infty} f(x)$ convergent, both half should be convergent indivisually.

ii) $\int_a^b f(x)\,dx$, has a point c as discontinuty, $b > c > a$

$$= \lim_{\phi_1 \to 0} \int_a^{c-\phi_1} f(x)\,dx + \lim_{\phi_2 \to 0} \int_{c+\phi_2}^{b} f(x)\,dx$$

here, also to be this whole integral $\int_a^b f(x)\,dx$ ~~continuous~~ convergent, both half should be convergent indivisually.

iii) $\int_{-\infty}^{a} f(x)\,dx$

$$= \lim_{r \to -\infty} \int_r^a f(x)\,dx$$

iv) $$\int_{b}^{\infty} f(x)\,dx.$$

$$\lim_{r \to \infty} \int_{b}^{r} f(x)\,dx$$

v) $\int_{a}^{b} f(x)\,dx$, if a is point of discontinuity

$$\lim_{\phi \to 0} \int_{a+\phi}^{b} f(x)\,dx$$

vi) $\int_{a}^{b} f(x)\,dx$, if b is point of discontinuity

$$\lim_{\phi \to 0} \int_{a}^{b-\phi} f(x)\,dx$$

NOTE :- If a integral $\int_{a}^{b} f(x)\,dx$ is discontinuous at c, such that

$$b > c > a$$

then we can write it

$$\lim_{r_1 \to 0} \int_{a}^{c-r_1} f(x)\,dx + \lim_{r_2 \to 0} \int_{c+r_2}^{b} f(x)\,dx$$

it it diverses or integral is divergent then, take $r_1 = r_2 = r$

so that

$$\lim_{r\to 0}\int_a^{0-r} f(x)\,dx + \lim_{r\to 0}\int_{0+r}^{b} f(x)\,dx$$

if it's value comes to be finit it will be called Cauchy's principle value and if it also comes to be infinite then value of integral does n't exist.

Q. Find, wheather given integrals are convergent or divergent :-

Q. $\int_0^\infty \frac{dx}{(x+1)(x+2)}$

$$\Rightarrow \lim_{r\to\infty}\int_0^r \frac{dx}{(x+1)(x+2)} = \lim_{r\to\infty}\left\{\int_0^r \frac{dx}{x+1} - \int_0^r \frac{dx}{x+2}\right\}$$

$$= \lim_{r\to\infty}\left\{\left[\log(x+1)\right]_0^r - \left[\log(x+2)\right]_0^r\right\}$$

$$= \lim_{r\to\infty}\left\{\log (r+1) - \log 1 - \log (r+2) + \log 2\right\}$$

$$= \lim_{r\to\infty}\left\{\log\left(\frac{r+1}{r+2}\right) + \log 2\right\}$$

$$= \lim_{r\to\infty}\log\left(\frac{1+1/r}{1+2/r}\right) + \log 2$$

$= \log 1 + \log 2$

$= \log 2$

hence, given integral is convergent and converges to $\log 2$.

Q. $\int_{-1}^{1} \frac{1}{x^3} dx$

→ Given, integral is improper as '0' is the point of discontinuty, where,

$$-1 < 0 < 1.$$

now, $\int_{-1}^{1} \frac{1}{x^3} dx = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{1} \frac{dx}{x^3}$

$$= \lim_{r_1 \to 0} \int_{-1}^{0-r_1} \frac{dx}{x^3} + \lim_{r_2 \to 0} \int_{0+r_2}^{1} \frac{dx}{x^3}$$

$$= \lim_{r_1 \to 0} \left[-\frac{1}{2x^2} \right]_{-1}^{0-r_1} + \lim_{r_2 \to 0} \left[\frac{1}{2x^2} \right]_{0+r_2}^{1}$$

$$= -\frac{1}{0} + 1 + (-1) + \frac{1}{0}$$

$$= -\infty + \infty$$

$$= \infty$$

Given integral is divergent.

Lets take $r_1 = r_2 = r$ (cauchy's theorem)

$$= \lim_{r \to 0} \left[-\frac{1}{2x^2} \right]_{-1}^{0-r} + \lim_{r \to 0} \left[-\frac{1}{2x^2} \right]_{0+r}^{1}$$

$$= -\frac{1}{0} + 1 - 1 + \frac{1}{0}$$

$$= -\infty + \infty$$

$$= \infty$$

So, given integral's value doesn't exist.

Q. $\int_0^{\infty} x e^{-x^2} dx$

→ Given integral is proper, now, we can define it as,

$$\int_0^{\infty} x e^{-x^2} dx = \lim_{r \to \infty} \int_0^{r} x e^{-x^2} dx$$

$$= \frac{1}{2} \lim_{r \to \infty} \int_0^{r} 2x e^{-x^2} dx$$

Let $x^2 = t \Rightarrow dt = 2x dx$

$x \to 0, t \to 0$

$x \to \infty, t \to \infty$

$$= \frac{1}{2} \lim_{r \to \infty} \int_0^r e^{-t}\, dt$$

$$= \frac{1}{2} \lim_{r \to \infty} \left[-e^{-t}\right]_0^r$$

$$= \frac{1}{2} \lim_{r \to \infty} \left(-e^{-r} + e^0\right)$$

$$= \frac{1}{2} \times 1$$

$$= \frac{1}{2}\ \text{Ans}$$

So, given integral is convergent & converges to $1/2$.

Q. $\displaystyle\int_0^{\pi} \frac{\sin x\, dx}{\cos^2 x}$

→ Given integral is improper as $\pi/2$ is a point of discontinuity, where

$$0 < \pi/2 < \pi$$

So, we can write given integral as

$$\int_0^{\pi/2} \frac{\sin x\, dx}{\cos^2 x} + \int_{\pi/2}^{\pi} \frac{\sin x\, dx}{\cos^2 x}$$

$$= \lim_{r_1 \to 0} \int_0^{\pi/2 - r_1} \frac{\sin x\, dx}{\cos^2 x} + \lim_{r_2 \to 0} \int_{\pi/2 + r_2}^{\pi} \frac{\sin x\, dx}{\cos^2 x}$$

$$= \lim_{r_1 \to 0} \int_0^{\pi/2 - r_1} \tan x \cdot \sec x\, dx + \lim_{r_2 \to 0} \int_{\pi/2 + r_2}^{\pi} \tan x \cdot \sec x\, dx$$

$$\cancel{= \lim_{r_1 \to 0} \int_0^{\pi/2 - r}}$$

$$= \lim_{r_1 \to 0} \left[\sec x\right]_0^{\pi/2 - r_1} + \lim_{r_2 \to 0} \left[\sec x\right]_{\pi/2 + r_2}^{\pi}$$

$$= \lim_{r_1 \to 0} \left[\sec(\pi/2 - r_1) - \sec 0\right] + \lim_{r_2 \to 0} \left[\sec \pi - \sec(\pi/2 + r_2)\right]$$

$$= \infty + \infty$$

$$= \infty$$

Given, integral is divergent.

Let $r_1 = r_2 = r$ (Cauchy's theorem)

$$\Rightarrow \lim_{r \to 0} \left[\sec(\pi/2 - r) - \sec 0\right] + \lim_{r_2 \to 0} \left[\sec \pi - \sec(\pi/2 - r)\right]$$

$$= \infty + \infty$$

$$= \infty$$

So, value of given integral doesn't exist.

Q. $\int_0^{\pi} \frac{dx}{1+\cos x}$

→ Given integral is improper as π is the point of discontinuty.

So, given integral can be written as,

$$\lim_{r \to 0} \int_0^{\pi - r} \frac{dx}{1+\cos x} = \lim_{r \to 0} \int_0^{\pi - r} \frac{dx}{1+\cos 2 \cdot x/2}$$

$$= \lim_{r \to 0} \int_0^{\pi - r} \frac{dx}{2\cos^2 x/2}$$

$$= \frac{1}{2} \lim_{r \to 0} \int_0^{\pi - r} \frac{dx}{\cos^2 x/2}$$

$$= \frac{1}{2} \lim_{r \to 0} \int_0^{\pi - r} \sec^2 x/2$$

$$= \frac{1}{2} \lim_{r \to 0} \left[2\tan x/2 \right]_0^{\pi - r}$$

$$= \lim_{r \to 0} \left[\tan\left(\frac{\pi - r}{2}\right) - \tan\frac{0}{2} \right]$$

$$= \infty$$

So, it is divergent