Limit continuity & Derivatives

1. Show that every differential function is continuous at a point but converse may not always true.

⇒

Proof

⇒ Let $f(x)$ is derivable at $x=0$, then ~~f'(a)~~

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{(x-a)} \text{ exist.}$$

Then, we have to show $f(x)$ is continuous at $x=a$

Now,

$$[f(x)-f(a)] = \frac{[f(x)-f(a)]}{(x-a)} \times (x-a)$$

Taking limit as $x\to a$ on both sides.

$$\lim_{x\to a} [f(x)-f(a)] = \lim_{x\to a} \left[\frac{f(x)-f(a)}{(x-a)}\right] \times (x-a)$$

$$= \lim_{x\to a} f(x)-f(a) = f'(a)\times 0 = 0$$

So,

$$\lim_{x\to a} f(x) = f(a)$$

So, the function $f(x)$ is continuous at $x=0$

⇐ Let us consider of function $f(x)=|x|$, use to try to show $f(x)$ is continuous but not differentiable at $x=0$

For R.H.L

$$\lim_{x\to 0^+} f(x) = \lim_{x\to a^+} |x| = 0$$

For L.H.L

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} |x| = 0$$

and at $x=0$, $f(x) = |x| = 0$

So, R.H.L = L.H.L = $f(x)$ at $x=0$ gives the function is continuous at $x=0$

For differentiability

R.H.D

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$\lim_{h \to 0} = \frac{|h|}{h} \Rightarrow \lim_{h \to 0} \frac{h}{h} = 1$$

L.H.D

$$\lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$\lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$\lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1$$

So, R.H.D $\neq$ L.H.D gives the function is not differentiable at $x=0$ where it is continuous at $x=0$.

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2. Examine the continuity and derivability at $x=0$ and $x = \pi/2$ of the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} 1 & \text{when } (-\infty, 0) \\ 1+\sin x & \text{when } x \in [0, \pi/2) \\ 2+(x-\pi/2)^2 & \text{when } x \in [\pi/2, \infty) \end{cases}$$

sol^n

For continuity (at $x=0$)

$$R.H.L = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} 1+\sin x$$

$$= 1+\sin 0 = 1$$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} 1$$

functional value $f(0) = 1+\sin x$

$$= 1+\sin 0$$

$$= 1+0 = 1$$

since, $R.H.L = L.H.L = f(x)$, therefore $f(x)$ is continuous at $x=0$

P.T.O

For differentiability (at $x=0$)

$$R.H.D = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \qquad h>0$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{1+\sin h - 1}{h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \lim_{h\to 0} \cos h$$

$$= 1$$

$$L.H.D = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} \qquad h>0$$

$$= \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{1+\sin(-h)-1}{-h}$$

$$= \lim_{h\to 0} \frac{1-\sin h-1}{-h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \lim_{h\to 0} \cos h$$

$$= 1$$

$\therefore$ R.H.D = L.H.D, so, $f(x)$ is differentiable at $x=0$

P.T.O

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For continuity (at $x = \pi/2$)

$$R.H.L = \lim_{x \to \pi/2^+} f(x)$$

$$= \lim_{x \to \pi/2} 2 + \left(x - \pi/2\right)^2$$

$$= 2 + \left(\frac{\pi}{2} - \pi/2\right)^2$$

$$= 2$$

$$L.H.L = \lim_{x \to \pi/2^-} f(x)$$

$$= \lim_{x \to \pi/2^-} 1 + \sin x$$

$$= 1 + \sin \pi/2$$

$$= 1 + 1$$

$$= 2$$

function value $f(\pi/2) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2$

$$= 2$$

since $R.H.L = L.H.R = f(\pi/2)$, therefore $f(x)$ is continuous at $x = \pi/2$

For differentiability (at $x = \pi/2$)

$$R.H.D = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2} + h\right) - f(\pi/2)}{h}$$

$$= \lim_{h \to 0} \cancel{2 + \left(x - \frac{\pi}{2}\right.} \frac{2 + \left\{\left(\frac{\pi}{2} + h\right) - \frac{\pi}{2}\right\}^2 - 2}{h}$$

P.T.O

$$\lim_{h\to 0} \frac{\left\{\left(\frac{\pi}{2}+h\right)-\frac{\pi}{2}\right\}^2}{h}$$

$$= \lim_{h\to 0} \frac{\left\{\left(\frac{\pi+2h}{2}-\frac{\pi}{2}\right)\right\}^2}{h}$$

$$= \lim_{h\to 0} \frac{\left\{\left(\frac{\cancel{\pi}+2h-\cancel{\pi}}{2}\right)\right\}^2}{h}$$

$$= \lim_{h\to 0} \frac{h^2}{h}$$

$$= 0$$

$$L.H.D = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right)-f\left(\pi/2\right)}{-h}$$

$$= \lim_{h\to 0} \frac{1+\sin\left(\frac{\pi}{2}-h\right)-2}{-h}$$

$$= \lim_{h\to 0} \frac{1+\cos \pi/2 - 2}{-h}$$

$$= \lim_{h\to 0} \frac{+1}{+h}$$

$$\infty$$

R.H.D $\neq$ L.H.D, so, $f(x)$ is not differentiable at $x = \pi/2$

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3. Show that the function $f(x) = \begin{cases} x & \text{for } x < 1 \\ 2-x & \text{for } 1 \le x < 2 \\ -2+3x-x^2 & \text{for } x \ge 2 \end{cases}$

is continuous at $x=1$ but not differentiable at $x=1$.

Sol^n

For continuity

$$R.H.L = \lim_{x \to 1^+} f(x)$$
$$= \lim_{x \to 1^+} 2-x$$
$$= 2-1$$
$$= 1$$

$$L.H.L = \lim_{x \to 1^-} f(x)$$
$$= \lim_{x \to 1^-} x$$
$$= 1$$

Function value $f(1) = 2-x = 2-1$
$= 1$

Since, $R.H.L = L.H.L = f(1)$, so, $f(x)$ is continuous at $x=1$

P.T.O

for differentiability

R.H.D $= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0} \frac{\{2-(1+h)\} - 1}{h}$

$= \lim_{h \to 0} \frac{(2-1-h-1)}{h}$

$= \lim_{h \to 0} \frac{-h}{h}$

$= -1$

L.H.D $= \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$

$= \lim_{h \to 0} \frac{\cancel{1}-h-\cancel{1}}{-h}$

$= \lim_{h \to 0} \frac{h}{h}$

$= 1$

R.H.D $\neq$ L.H.D, so, $f(x)$ is not derivable at $x = 1$.

1. Show that the function $f(x) = \begin{cases} x^2+2 & \text{for } x \le 1 \\ 3x & \text{for } x > 1 \end{cases}$

is continuous but not differentiable at $x = 1$.

Sol

For continuity

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} 3x$$

$$= 3 \times 1$$

$$= 3$$

$$L.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^-} x^2+2$$

$$= 1^2+2$$

$$= 3$$

Function value $f(1) = x^2+2 = 1^2+2 = 3$

Since, $R.H.L = L.H.L = f(1)$, so, $f(x)$ is continuous at $x = 1$.

For differentiability

$$R.H.D = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

P.T.O

$$= \lim_{h \to 0} \frac{3(1+h) - 3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} + 3h - \cancel{3}}{h}$$

$$= 3$$

$$L.H.D = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)^2 + 2 - 3}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)^2 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h)\cancel{(-h)}}{\cancel{(-h)}}$$

$$= 2 - 0$$

$$= 2$$

As $R.H.D \neq L.H.D$, $f(x)$ is not differentiable at $x = 1$.

5. Define continuity and differentiability of a function. show that the function:

$$f(x) = \begin{cases} x^2+2 & \text{for } x \le 1 \\ 3x & \text{for } x > 1 \end{cases}$$

is continuous at $x=1$ but not differentiable at $x=1$.

⇒ A function $y=f(x)$ is said to be continuity at point $x=a$ if its limiting value is equal to its functional value.

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

A function $y=f(x)$ is said to be differentiability at point $x=a$,

if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists

OR,

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} \text{ exists.}$$

For continuity

$$R.H.L = \lim_{x\to 1^+} f(x)$$

$$= \lim_{x\to 1^+} 3x = 3$$

$$L.H.L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} x^2+2$$

$$= 1^2+2 = 3$$

functional value $f(1) = 1^2+2 = 3$

since, $R.H.L = L.H.L = f(1)$, so, $f(x)$ is continuous at $x=1$.

<u>For differentiability</u>

$$R.H.D = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3(1+h) - 3}{h}$$

$$= \lim_{h \to 0} \frac{3 + 3h - 3}{h}$$

$$= 3$$

$$L.H.D = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)^2 + 2 - 3}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)^2 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h)\cancel{(-h)}}{\cancel{(-h)}}$$

$$= 2-0$$

$$= 2$$

As, $R.H.D \neq L.H.D$, so, $f(x)$ is not differentiable at $x=1$.

Higher Order Derivatives

1. State and verify Leibnitz Theorem.

⇒ Let $y = uv$, where u and v are functions of x possessing n^{th} order derivable, then

$$y_n = u_n v + C(n,1)u_{n-1}v_1 + C(n,2)u_{n-2}v_2 + \dots + C(n,r)u_{n-r}v_r + \dots + uv_n$$

where the suffices denotes order derivative.

Proof:-

We use induction method to prove this theorem.

$$y = uv \text{ --- (i)}$$

Diff. eq^n (i), we get

$$y_1 = u_1 v + u v_1 \text{ --- (ii)}$$

Diff. eq^n (ii), we get

$$y_2 = u_2 v + u_1 v_1 + u_1 v_1 + u v_2$$

$$y_2 = u_2 v + 2u_1 v_1 + u v_2 = u_2 v + C(2,1) u_1 v_1 + u v_2$$

$$y_3 = u_3 v + u_2 v_2 + 2[u_2 v_1 + u_1 v_2] + u_1 v_2 + u v_3$$

$$= u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3$$

$$= u_3 v + 3c_1 u_2 v_1 + 3c_2 u_1 v_2 + 3c_3 u v_3$$

Here, we see that the theorem is true for $n=2$ and $n=3$. Let us assume that, the theorem is true for $n=m$

i.e. $$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \dots + {}^mC_r u_{m-r} v_r + \dots + {}^mC_m u v_m$$

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Diff. w.r.t x, we get,

$$y_{m+1} = u_{m+1}v + ({}^mC_1 + 1)u_m v_1 + ({}^mC_2 + {}^mC_1)u_{m-1}v_2 + \cdots + ({}^mC_r + {}^mC_{r-1})u_{m-r+1}v_r + \cdots + uv_{m+1}$$

$$\Rightarrow y_{m+1} = u_{m+1}v + {}^{(m+1)}C_1 u_m v_1 + {}^{(m+1)}C_2 u_{m-1}v_2 + \cdots + {}^{(m+1)}C_r u_{m-r+1}v_r + \cdots + uv_{m+r}$$

By using,

$${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r \text{ and } {}^nC_1 + 1 = {}^{(n+1)}C_1$$

It is true for $n = 1, 2,$ and 3 and is true for $n = m+1$ wherever it is true for $n = m$. So, it is true for all n.

2. If $y = (x^2-1)^n$ show that:

i) $(x^2-1)y_2 + 2(1-n)xy_1 - 2ny = 0$

ii) $(x^2-1)y_{n+1} + 2xy_{n+1} - n(n+1)y_n = 0$

Sol^n

Here,

(i) $$y = (x^2-1)^n \quad \text{— (i)}$$

Diff. eq^n (i) w.r.t x, we get,

$$\frac{dy}{dx} = n(x^2-1)^{n-1} \cdot 2x$$

$$y_1 = 2nx(x^2-1)^{n-1}$$

$$\text{or, } y_1 = \frac{2nx(x^2-1)^n}{(x^2-1)}$$

$$\text{or, } (x^2-1)y_1 = 2nxy$$

Again, diff. w.r.t x, we get,

$$(x^2-1)y_2 + 2xy_1 = 2n(xy_1 + y)$$

or, $(x^2-1)y_2 + 2(n-1)xy_1 - 2ny = 0$

(ii) we have,

$$(x^2-1)y_2 + 2(n-1)xy_1 - 2ny = 0$$

Diff. upto n times w.r.t x, we get,

$$(x^2-1)y_{n+2} + n(2x)y_{n+1} + \frac{n^2-n}{2} \cdot 2y_n$$

$$\Rightarrow -2(n-1)\{xy_{n+1} + ny_n\} - 2ny_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + 2nxy_{n+1} - 2(n-1)xy_{n+1} + 2\left(\frac{n^2-n}{2} - n^2 - n + n\right)y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + xy_{n+1}(2n+2-2n) + \frac{2(n^2-n-n^2)}{2}y_n = 0$$

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$

3. If $\log y = \tan^{-1}x$ prove that $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$

Sol^n

Here,

$$\log y = \tan^{-1}x \quad \text{--- (i)}$$

Diff. eq^n (i) w.r.t x, we get,

$$\frac{d}{dx}\log y = \frac{d}{dx}\tan^{-1}x$$

$$\frac{1}{y} \times y_1 = \frac{1}{1+x^2}$$

$$(1+x^2)y_1 = y$$

Again, diff. w.r.t x, we get,

$$(1+x^2)y_2 + y_1 \cdot 2x = y_1$$

$$(1+x^2)y_2 + (2x-1)y_1 = 0 \quad \text{--- (ii)}$$

Diff. eq^n (ii) up to n times w.r.t x, we get

$$(1+x^2)y_{n+2} + C(n,1)\cdot 2x \cdot y_{n+2-1} + C(n,2)\cdot 2 \cdot y_{n+2-1-1} + (2x-1)y_{n+1} + C(n,1)\cdot 2 \cdot y_{n+1-1} = 0$$

$$\text{or, } (1+x^2)y_{n+2} + 2nx\,y_{n+1} + \frac{n^2-n}{\cancel{2}}\cdot\cancel{2}\,y_n + (2x-1)y_{n+1} + 2n y_n$$

$$\text{or, } (1+x^2)y_{n+2} + 2nx\,y_{n+1} + n^2y_n - ny_n + (2x-1)y_{n+1} + 2ny_n = 0$$

$$\text{or, } (1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$$

5. State Leibnitz theorem. If $y = \log(x + \sqrt{a^2+x^2})$, show that

$$(a^2+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$$

$\Rightarrow$ Let $y = uv$, where u and v are functions of x having n^{th} order derivative then

$$y_n = u_nv + C(n,1)u_{n-1}v_1 + C(n,2)u_{n-2}v_2 + \cdots + C(n,r)u_{n-r}v_r + \cdots + uv_n$$

where the suffices denotes their derivatives

Here,

$$y = \log(x + \sqrt{a^2+x^2}) \quad \text{— (i)}$$

Diff. eq^n (i) w.r.t x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(x + \sqrt{a^2+x^2})$$

$$= \frac{d[\log(x+\sqrt{a^2+x^2})]}{d(x+\sqrt{a^2+x^2})} \times \frac{d}{dx}(x+\sqrt{a^2+x^2})$$

$$= \frac{1}{x+\sqrt{a^2+x^2}} \times \left(1 + \frac{x}{\sqrt{a^2+x^2}}\right)$$

Let $y = \sqrt{a^2+x^2}$

$y = (a^2+x^2)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(a^2+x^2)^{-1/2} \cdot (2x)$

$= \frac{x}{\sqrt{a^2+x^2}}$

$$= \frac{1}{\cancel{(x+\sqrt{a^2+x^2})}} \times \frac{\cancel{(x+\sqrt{a^2+x^2})}}{\sqrt{a^2+x^2}}$$

$$y_1 = \frac{1}{\sqrt{a^2+x^2}}$$

$$y_1 \cdot \sqrt{a^2+x^2} = 1$$

$$\text{or, } (a^2+x^2)y_1^2 = 1 \quad \text{— (ii)}$$

Diff. eq^n (ii) w.r.t x, we get.

$$2y_1y_2(a^2+x^2) + y_1^2 \cdot 2x = 0$$

$$\text{or, } (a^2+x^2)y_2 + xy_1 = 0 \quad \text{— (iii)}$$

Diff. eq^n (iii) up to n times w.r.t x, we get

$$(a^2+x^2).y_{n+2} + C(n,1).2x.y_{n+2-1} + C(n,2).2.y_{n+2-1-1} + xy_{n+1} + C(n,1).1.y_{n+1-1} = 0$$

or, $(a^2+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n^2-n}{2}.2y_n + xy_{n+1} + ny_n = 0$

or, $(a^2+x^2)y_{n+2} + 2nxy_{n+1} + n^2y_n - \cancel{ny_n} + xy_{n+1} + \cancel{ny_n} = 0$

or, $(a^2+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

6. State Leibnitz theorem for successive derivative of the product of two functions. If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)y_2 - xy_1 - 2 = 0$ and hence show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

=> Let $y = uv$ where u and v are function of x having n^{th} order derivative then

$$y_n = u_nv + C(n,1)u_{n-1}v_1 + C(n,2)u_{n-2}v_2 + \dots + C(n,r)u_{n-r}v_r + \dots + uv_n$$

where the suffices denotes their derivatives

Here,

$y = (\sin^{-1}x)^2$ — (i)

Diff. eq^n (i) w.r.t x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^2$$

$$= \frac{d(\sin^{-1}x)^2}{d(\sin^{-1}x)} \times \frac{d(\sin^{-1}x)}{dx}$$

$$y_1 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

or, $y_1 \cdot \sqrt{1-x^2} = 2\sin^{-1}x$

or, $(1-x^2)y_1^2 = 4(\sin^{-1}x)^2$

$(1-x^2)y_1^2 = 4y$

Again, diff. w.r.t x, we get

$2y_1 \cdot y_2(1-x^2) + y_1^2 \cdot (-2x) = 4y_1 = 0$

or, $(1-x^2)y_2 - xy_1 - 2 = 0$ — (ii)

Diff. eq^n (ii) upto n-times w.r.t x, we get

$(1-x^2)y_{n+2} + \cancel{(1-x)}\, C(n,1)\cdot(-2x)\,y_{n+2-1} + C(n,2)\cdot(-2)\,y_{n+2-1-1}$

$- x y_{n+1} - C(n,1)\cdot 1 \cdot y_{n+1-1} = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} + \frac{(n^2-n)}{2}\cdot(-2)y_n - xy_{n+1} - ny_n = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

7. If $y = \sin^{-1}x$, show that

i) $(1-x^2)y_2 - xy_1 = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

Here,

$y = \sin^{-1}x$ — (i)

Diff. eq^n (i) w.r.t x, we get,

$$\frac{dy}{dx} = \frac{d(\sin^{-1}x)}{dx}$$

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$y_1^2 = \frac{1}{1-x^2}$$

$(1-x^2)y_1^2 = 1$

Again, diff. eq^n w.r.t x, we get,

$2y_1 \cdot y_2 (1-x^2) + y_1^2 \cdot (-2x) = 0$

$y_2(1-x^2) - xy_1 = 0$ — (ii)

Diff. eq^n (ii) up to n-times w.r.t x, we get

$(1-x^2)y_{n+2} + c(n,1).(-2x).y_{n+2-1} + c(n,2).(-2).y_{n+2-1-1}$

$- xy_{n+1} - c(n,1).(1).y_{n+1-1} = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} + \frac{n^2-n}{2}.(-2).y_n - xy_{n+1}$

$- ny_n = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$ ##

8) If $\log y = \tan^{-1}x$, show that

i) $(1+x^2)y_2 + (2x-1)y_1 = 0$

ii) $(1+x^2)y_{n+2} + (2nx+2x-1)xy_{n+1} + n(n+1)y_n = 0$

Here,

$\log y = \tan^{-1}x$ — (i)

Diff. eq^n (i) w.r.t x, we get,

$\frac{d}{dx}\log y = \frac{d}{dx}\tan^{-1}x$

$\frac{1}{y} \times y_1 = \frac{1}{1+x^2}$

or, $(1+x^2)y_1 = y$

P.T.O

Again, diff. w.r.t x, we get

$$(1+x^2)y_2 + y_1 \cdot 2x - y_1 = 0$$

$$(1+x^2)y_2 + (2x-1)y_1 = 0 \quad \text{--- (ii)}$$

Diff. eq^n (ii) upto n-times w.r.t x, we get

$$(1+x^2)y_{n+2} + C(n,1) \cdot 2x \cdot y_{n+2-1} + C(n,2) \cdot 2 y_{n+2-1-1} + (2x-1)y_{n+1} + C(n,1) \cdot 2 \cdot y_{n+1-1} = 0$$

or, $(1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n^2-n}{2} \cdot 2y_n + (2x-1)y_{n+1} + ny_n = 0$

or, $(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n^2 y_n - ny_n + ny_n = 0$

or, $(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$

9. If $y = \log(x+\sqrt{a^2+x^2})$ prove that $(a^2+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$

Here,

$$y = \log(x+\sqrt{a^2+x^2}) \quad \text{--- (i)}$$

Diff. eq^n (i) w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(x+\sqrt{a^2+x^2})]$$

$$= \frac{d[\log(x+\sqrt{a^2+x^2})]}{d(x+\sqrt{a^2+x^2})} \times \frac{d(x+\sqrt{a^2+x^2})}{dx}$$

$$= \frac{1}{(x+\sqrt{a^2+x^2})} \times \left(1+\frac{x}{\sqrt{a^2+x^2}}\right)$$

$y = \sqrt{a^2+x^2}$

$y = (a^2+x^2)^{1/2}$

$\frac{dy}{dx} = \frac{x}{\sqrt{a^2+x^2}}$

$$y_1 = \frac{1}{(x+\sqrt{a^2+x^2})} \times \left(\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}\right)$$

$$y_1 = \frac{1}{\sqrt{a^2+x^2}}$$

$$y_1 \cdot \sqrt{a^2+x^2} = 1$$

Again, diff. w.r.t. x, we get,

$$2y_1 \cdot y_2 (a^2+x^2) + y_1^2 \cdot 2x = 0$$

$$(a^2+x^2) y_2 + x y_1 = 0 \quad \text{— (ii)}$$

Diff. eq^n (ii) upto n-times, we get,

$$(a^2+x^2) y_{n+2} + C(n,1) \cdot 2x \cdot y_{n+2-1} + C(n,2) \cdot 2 \cdot y_{n+2-1-1}$$
$$+ x y_{n+1} + C(n,1) \cdot 1 \cdot y_{n+1-1} = 0$$

or, $(a^2+x^2) y_{n+2} + 2nx \cdot y_{n+1} + \frac{n^2-n}{2} \cdot 2 y_n + x y_{n+1}$

$$+ n y_n = 0$$

or, $(a^2+x^2) y_{n+2} + 2nx y_{n+1} + n^2 y_n - \cancel{n y_n} + x y_{n+1} + \cancel{n y_n} = 0$

or, $(a^2+x^2) y_{n+2} + (2n+1) x y_{n+1} + n^2 y_n = 0$

10) If $y = a\cos(\log x) + b\sin(\log x)$ show that

i) $x^2y_2 + xy_1 + y = 0$

ii) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Here,

$$y = a\cos(\log x) + b\sin(\log x) \quad \text{--- (i)}$$

Diff. eqn (i) w.r.t x, we get,

$$\frac{dy}{dx} = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

or, $xy_1 = -a\sin(\log x) + b\cos(\log x)$

Again, diff. w.r.t x, we get,

$$xy_2 + y_1 = -a\cdot\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

or, $x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

or, $x^2y_2 + xy_1 = -y$

or, $x^2y_2 + xy_1 + y = 0 \quad \text{--- (ii)}$

Diff. eqn (ii) upto n-times, w.r.t x, we get,

$$x^2y_{n+2} + C(n,1)\cdot 2x\cdot y_{n+2-1} + C(n,2)\cdot 2\cdot y_{n+2-1-1} + xy_{n+1} + C(n,1)\cdot 1\cdot y_{n+1-1} + y_n = 0$$

or, $x^2y_{n+2} + 2nxy_{n+1} + \frac{n^2-n}{2}\cdot 2\cdot y_n + xy_{n+1} + ny_n + y_n = 0$

or, $x^2y_{n+2} + 2nxy_{n+1} + n^2y_n - \cancel{ny_n} + xy_{n+1} + \cancel{ny_n} + y_n = 0$

or, $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$ #

Indeterminant Forms :-

1) State and verify L-Hospital Theorem. Use it to evaluate ..

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$$

=> Let $f(x)$ and $g(x)$ are two functions such that $f(a)=g(a)=0$ and their derivative $f'(x)$ and $g'(x)$ are continuous on $x=0$, then,

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}=\frac{f(x)}{g(x)}$$

Proof:-

Here $f(x)$ and $g(x)$ are two function such that $f(a)=g(a)=0$ with their first order derivative are continuous at $x=a$

Now,

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}=\lim_{x\to 0}\frac{\frac{f(x)-f(a)}{(x-a)}}{\frac{g(x)-g(a)}{(x-a)}}$$

$$=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$=\lim_{x\to 0}\frac{f(x)-0}{g(x)-0}$$

$$=\lim_{x\to a}\frac{f(x)}{g(x)}$$ verified

Here,

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left[\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin^2 x - x^2}{x^2 \left(\frac{\sin x}{x}\right)^2 \cdot x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin^2 x - x^2}{x^4}\right]$$

$$= \lim_{x\to 0}\frac{2\sin x.\cos x - 2x}{4x^3}$$

$$= \lim_{x\to 0}\frac{\sin 2x - 2x}{4x^3}$$

$$= \lim_{x\to 0}\frac{2\cos 2x - 2}{12x^2}$$

$$= \lim_{x\to 0} - \frac{4\sin 2x}{24x}$$

$$= \lim_{x\to 0}\frac{-8\cos 2x}{24}$$

$$= -8/24$$

$$= -1/3$$

2. What is an indeterminate form? Evaluate

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$$

$\Rightarrow$ The limit of a function of the form $\frac{f(x)}{g(x)}$ as $x\to a$ is the quotient of the limiting values of a numerator and denominator. But when both numerator and denominator approach to zero as $x\to a$, then quotient takes the form $\frac{0}{0}$, which is meaningless expression called Indeterminate form. Similarly, other indeterminate forms are ∞/∞, $\infty-\infty$, 0^0, $1^{\pm\infty}$, ∞^0, $0\times\infty$, etc.

We have,

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$$

Let $y = \left(\frac{\sin x}{x}\right)^{1/x^2}$

Taking log on both sides,

$$\log y = \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$\log y = \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$$

Taking $\lim_{x\to 0}$ on both sides.

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x^2}$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{x}{\sin x} \cdot \frac{\frac{x\cos x - \sin x}{x^2}}{2x}$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^3}$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{x\cos x - x\sin x \cdot \cos x}{6x^2}$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{-\sin x}{6x}$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{-\cos x}{6} = -1/6$$

Taking antilog on both sides.

$$\exp\left\{\log\left(\lim_{x\to 0} y\right)\right. = \exp\left(\frac{-1}{6}\right)$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$$

4. State and prove L-Hospital's rule. Also evaluate: $\lim_{x \to 1} x^{\left(\frac{1}{1-x}\right)}$

Sol^n

Let $y = x^{\left(\frac{1}{1-x}\right)}$

Taking log on both sides

$\log y = \log x^{\left(\frac{1}{1-x}\right)}$

Taking $\lim_{x \to 1}$ on both sides.

$\lim_{x \to 1} \log y = \lim_{x \to 1} \left(\frac{1}{1-x}\right) \log x$

$\log\left(\lim_{x \to 1} y\right) = \lim_{x \to 1} \frac{1/x}{-1}$

$\Rightarrow \left(\log \lim_{x \to 1} y\right) = \cancel{\frac{1}{-1}} \lim_{x \to 1} \frac{1}{-x}$

$\log\left(\lim_{x \to 1} y\right) = -1$

Taking antilog on both sides

$\exp\left\{\log\left(\lim_{x \to 1} y\right)\right\} = \exp(-1)$

$\lim_{x \to 1} y = e^{-1}$

$\lim_{x \to 1} x^{\left(\frac{1}{1-x}\right)} = \frac{1}{e}$ #

5. Evaluate:

$$\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$$

Sol^n

Let $y = \left(\frac{\sin x}{x}\right)^{1/x}$

Taking log on both sides

$$\log y = \log\left(\frac{\sin x}{x}\right)^{1/x}$$

$$\log y = \frac{1}{x} \log\left(\frac{\sin x}{x}\right)$$

Taking $\lim_{x \to 0}$ on both sides

$$\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x}$$

$$\log\left(\lim_{x \to 0} y\right) = \lim_{x \to 0} \frac{x}{\sin x} \cdot \frac{x\cos x - \sin x}{x^2}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{x \sin x}$$

$$= \lim_{x \to 0} \frac{-x\sin x + \cos x - \cos x}{x\cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{-x\sin x}{x\cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{-x\cos x - \sin x}{-x\sin x + \cos x + \cos x}$$

$$\log\left(\lim_{x\to 0} y\right) = 0$$

Taking antilog on both side,

$$\exp\left\{\left(\lim_{x\to 0} y\right)\right\} = \exp(0)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x} = 1$$

Mean Value Theorems:-

1.) State and prove Lagrange's mean value theorem. Interpret it geometrically. Verify it for $f(x) = e^x$ in $[0,1]$.

⇒ If f is a function, continuous on $[a,b]$ and derivable on (a,b). Then there exists at least one point $c(a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Proof:-

Here, f is a function continuous on (a,b) and derivable on (a,b). Let us define a new function F on $[a,b]$ such that $F(x) = f(x) + Ax$ where A is constant so chosen $F(b) = F(a)$

Now,

Since f is continuous on $[a,b]$. So, F is also continuous on $[a,b]$ and from assumption $F(b) = F(a)$. So, F satisfies all the criteria for Rolle's theorem. So, there should exist at least one element $c \in (a,b)$ such that

$$F'(c) = 0 \quad \text{—(i)}$$

Again, we have,

$$F(b) = F(a)$$

$$f(b) + Ab = f(a) + Aa$$

$$f(b) - f(a) = A(a-b)$$

$$A = \frac{f(b) - f(a)}{b-a} \quad \text{—(ii)}$$

P.T.O

Again,

$$F(x) = f(x) + Ax$$
$$F'(x) = f'(x) + A$$
$$F'(c) = f'(c) + A$$
$$f'(c) + A = 0 \quad [\text{from (i)}]$$

$$f'(c) = -A$$
$$f'(c) = (-)(-)\frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$$

Here,
we have,
$f(x) = e^x$ which is continuous in $[0, 1]$
and we have $f'(x) = e^x$ which is differentiable in $(0,1)$.
~~Then, there exists $c \in (0,1)$, such that~~

$\cancel{f'(c) = f}$
$$f(a) = f(0) = e^0 = 1$$
$$f(b) = f(1) = e^1 = e$$

Then, there exist $c \in (0,1)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$
$$= \frac{e-1}{1-0} = e-1 \notin (0,1)$$

Hence, Lagrange's mean value theorem is not verified for $f(x)$ in $[0,1]$.

2. state and prove cauchy's Mean value theorem. Verify it for $f(x) = x^2$ and $g(x) = x$, in $[1,3]$?

⇒ If $f(x)$ and $g(x)$ are two functions continuous on $[a,b]$ and derivable on (a,b) then there exists at least one element $c \in (a,b)$.
such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}, \quad g'(c) \neq 0$$

Proof :-
Here $f(x)$ and $g(x)$ are continuous on $[a,b]$ and derivable on (a,b), then by Lagrange mean value theorem ~~their~~ exist $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{--- (i)}$$

Again, $g(x)$ is continuous on $[a,b]$ and derivable on (a,b) then by Lagrange Mean value Theorem their exists $c \in (a,b)$ such that

$$g'(c) = \frac{g(b)-g(a)}{b-a}$$

Dividing (i) by (ii)

$$\frac{f'(c)}{g'(c)} = \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}}$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

we have,

$f(x) = x^2$ and $g(x) = x$ are polynomial function, so continuous in $[1, 3]$

$f'(x) = 2x$ and $g'(x) = 1$ which are differentiable in $(1, 3)$.

$f(a) = f(1) = 1^2 = 1$

$f(b) = f(3) = 3^2 = 9$

$g(a) = g(1) = 1$

$g(b) = g(3) = 3$

Then, by cauchy's mean value theorem there exists $c \in (1, 3)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{2c}{1} = \frac{9 - 1}{3 - 1}$$

$$2c = \frac{8}{2} = 4$$

$$2c = 4$$

$$c = 2 \in (1, 3)$$

Hence, cauchy's mean value theorem is verified.

3. Ex - 3.1

1) verify Rolle's theorem for

i) $f(x) = x^2 - 5x + 10$ in $[2,3]$

soln,

we have,

$$f(x) = x^2 - 5x + 10$$

which is continuous since it is polynomial function,

$f'(x) = 2x - 5$ which exist in $(2,3)$

$$f(a) = f(2) = 2^2 - 5\times 2 + 10$$
$$= 4$$

$$f(b) = f(3) = 3^2 - 5\times 3 + 10$$
$$= 4$$

Then there exist at least one element $c \in (a,b)$ such that,

$$f'(c) = 0$$
$$2c - 5 = 0$$
$$2c = 5$$
$$c = 5/2 \in (2,3)$$

Hence, Rolle's theorem is verified.

2. Verify Lagrange's mean value theorem for

i) $f(x) = x^2$ in $[1,2]$

soln

$f(x)$ is continuous function in $[1,2]$

$f'(x) = 2x$ which exist in $(1,2)$

$f(a) = f(1) = 1^2 = 1$

$f(b) = f(2) = 2^2 = 4$

Then, there exist $c \in (1,2)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$2c = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

$$2c = 3$$

$$c = 3/2 \in (1,2)$$

Hence, Lagrange's mean value theorem is verified.

ii) $f(x) = x(x-1)(x-2)$ in $[0, 11/2]$

soln

$f(x) = (x^2-x)(x-2)$

$= x^3 - 2x^2 - x^2 + 2x$

$f(x) = x^3 - 3x^2 + 2x$ — (i)

which is continuous in $[0, 11/2]$ since, it is polynomial function.

P.T.O

$f'(x) = 3x^2 - 6x + 2$ which exist in $(0, 11/2)$

$f(a) = f(0) = 0^3 - 2 \cdot 0^2 + 2 \cdot 0 = 0$

$f(b) = f(11/2) = \left(\frac{11}{2}\right)^3 - 3 \cdot \left(\frac{11}{2}\right)^2 + 2 \cdot \frac{11}{2}$

$= \frac{1331}{8} - \frac{3 \times 121}{4} + 11$

$= \frac{1331 - 3 \times 121 \times 2 + 11 \times 8}{8}$

$= \frac{693}{8}$

Then, there exist $c \in (0, 11/2)$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$3c^2 - 6c + 2 = \frac{f(11/2) - f(0)}{\frac{11}{2} - 0}$

$\frac{11}{2}(3c^2 - 6c + 2) = \frac{\frac{693}{8} - 0}{}$

$\cancel{\frac{11}{2}}(3c^2 - 6c + 2) = \frac{\cancel{693}\ 63}{\cancel{8}\ 4}$

or, $12c^2 - 24c + 8 - 63 = 0$

or, $12c^2 - 24c - 55 = 0$

$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$c = \frac{24 \pm \sqrt{(24)^2 + 4 \times 12 \times 55}}{2 \times 12}$

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$$= \frac{24 \pm \sqrt{576+2640}}{24}$$

$$= \frac{24 \pm 56.7}{24}$$

Taking +ve,

$$c = 3.36$$

Taking -ve,

$$c = -1.3625$$

$$c = 3.36 \in (0, \pi/2)$$

Hence, Lagrange mean value theorem is verified.

3. Show that $|\sin b - \sin a| \le |b-a|$, by using Lagrange's mean value theorem

Sol^n

Let $f(x) = \sin x$, then,

$f(x) = \sin x$ is a continuous function in $[a, b]$

$f'(x) = \cos x$ is derivable function in (a, b)

Now,

$$f(a) = \sin a \quad \text{and} \quad f(b) = \sin b$$

Using L.M.V.T, there should a point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

P.T.O

we have,

$-1 \le \cos c \le 1$

$|\cos c| \le 1 \quad [\because -a \le x \le a \Rightarrow |x| \le a]$

$|f'(c)| \le 1$

$$\left| \frac{f(b)-f(a)}{b-a} \right| \le 1$$

$$\left[\frac{|f(b)-f(a)|}{|b-a|} \right] \le 1 \qquad \left[\because \left|\frac{x}{y}\right| = \frac{|x|}{|y|} \right]$$

$|f(b)-f(a)| \le |b-a|$

$|\sin b - \sin a| \le |b-a|$

verified//

13. show that $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$ by using Lagrange mean value theorem.

Soln

Let us consider the interval having lower and upper end point a and b.

$$\frac{b-a}{b} \le \log\left(\frac{b}{a}\right) \le \frac{b-a}{a}$$

$$\therefore \frac{b-a}{b} \le \log b - \log a \le \frac{b-a}{a}$$

$$\frac{1}{b} < \frac{\log b - \log a}{b-a} < \frac{1}{a}$$

Let $f(x) = \log x \Rightarrow f'(x) = 1/x$

Now, using L.M.V.T there exist a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Now,

$c \in (a,b) \Rightarrow a < c < b$

$\Rightarrow a < c$ and $c < b$

$\Rightarrow \frac{1}{a} > \frac{1}{c}$ & $\frac{1}{c} > \frac{1}{b}$

$\Rightarrow \frac{1}{b} < \frac{1}{c} < \frac{1}{a}$

$\Rightarrow \frac{1}{b} < f'(c) < \frac{1}{a}$

$\Rightarrow \frac{1}{b} < f'(c) < \frac{1}{a}$

or, $\frac{1}{b} < \frac{f(b)-f(a)}{b-a} < \frac{1}{a}$

proved//

Asymptotes

1. Define the asymptotes of a curve and classify them. Find the asymptotes of the curve

$$x^2(x-y)^2 - a^2(x^2+y^2) = 0$$

$\Rightarrow$ A straight line is said to be asymptote to a curve if a point $P(x,y)$ on curve, the distance between the point to the straight line to zero if the points tends to infinity.

Types of asymptotes

i) Asymptotes parallel to axes :-

Asymptotes which are parallel to either x-axis or y-axis is said to be asymptotes parallel to axes.

ii) Oblique Asymptotes :-

Asymptotes which are parallel to neither of the axis is said to be oblique asymptotes.

The given curve is

$$x^2(x-y)^2 - a^2(x^2+y^2) = 0$$

$$\Rightarrow x^2(x^2-2xy+y^2) - a^2(x^2+y^2) = 0$$

$$\Rightarrow x^4 - 2x^3y + x^2y^2 - a^2y^2 = 0 \quad \text{--- (i)}$$

This is a curve with degree 4. So it has at most four asymptotes. Here y^4 is absent but x^4 is present.

P.T.O

Equating coefficients of highest powers of y to zero, i.e. coeff. of y^2 to zero, we get,

$$x^2 - a^2 = 0$$
$$x = \pm a$$

Therefore $x = a$ and $x = -a$ are two vertical asymptotes of the given curve. There may be other oblique asymptotes.

Let $y = mx + c$ be other asymptotes of given curve

put $x = 1$ and $y = m$ in eq^n (1)

Here, $\phi_4(m) = 1 - 2m + m^2$

$\phi_3(m) = 0$

$\phi_2(m) = -a^2 - a^2m^2$

To find m we solve

$$\phi_4(m) = 0$$
or, $1 - 2m + m^2 = 0$

or, $(1-m)^2 = 0$

$\therefore m = \pm 1$ (repeated roots of m)

for value of c we use,

$$\frac{c^2}{2}\phi_4''(m) + c\phi_3'(m) + \phi_2(m) = 0$$

or, $\frac{c^2}{2} \cdot 2 + c \cdot 0 + (-a^2 - a^2m^2) = 0$

$c^2 - a^2 - a^2 = 0$

or, $c^2 = 2a^2$

$c = \pm a\sqrt{2}$

$\therefore$ Req. oblique asymptotes are $y = x \pm a\sqrt{2}$

Thus, req. asymptotes of the given curve are

$x = \pm a$ and $y = x \pm a\sqrt{2}$

2. Define asymptotes of a curve. Find the asymptotes of the curve $x^3+y^3=3axy$.

⇒ A straight line is said to be asymptotes to a curve if a point $P(x,y)$ on curve, the distance between the point to the straight line to zero if the points tends to infinity.

The given curve is
$$x^3+y^3=3axy$$
$$\Rightarrow x^3+y^3-3axy=0 \quad \text{— (i)}$$
This is a curve of degree 3. It has at most three asymptotes.
Here x^3 and y^3 both are present. so it can have only oblique asymptotes

Let $y=mx+c$ be the required asymptotes
put $x=1$ and $y=m$ in eqn (i)

$$\phi_3(m)=1+m^3$$
$$\phi_2(m)=-3am$$
$$\phi_1(m)=0$$
To find m, we solve
$$\phi_3(m)=0$$
$$\Rightarrow 1+m^3=0$$
$$\Rightarrow m^3=-1$$
$$\therefore m=-1$$
as other, value of m are imaginary

and

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$

$$= \frac{-(-3am)}{3m^2} = \frac{\cancel{3}a\cancel{m}}{\cancel{3}m^{\cancel{2}}}$$

$$= \cancel{3a} \frac{a}{\cancel{m}} = -a \quad (\text{for } m = -1)$$

Therefore, req. asymptote is

$$y = -x - a$$

$$\therefore \; y + x + a = 0$$

3. Find all the asymptotes of $y^3 + x^2y + 2xy^2 - y + 1 = 0$

Sol

The given eq is

$$y^3 + x^2y + 2xy^2 - y + 1 = 0 \quad \text{--- (i)}$$

Here, the given curve is of degree 3 and x^3 is absent and y^3 is present.

let ~~the as~~ $y = mx + c$ be the oblique asymptote & put $x = 1$ and $y = m$ in eq (i)

$$\phi_3(m) = m^3 + m$$

$$\phi_2(m) = 2m^2$$

$$\phi_1(m) = -m$$

Now,

$$\phi_3(m) = 0$$

$$m^3 + m = 0$$

$$\text{or, } m(m^2 + 1) = 0$$

$$m = 0, \; m = \pm 1$$

Thus, value of m are $0, 1, -1$

for value of c:

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-2m^2}{3m^2+1} = \cancel{\;}\,/3$$

At $m=0$, $c=0$

At $m=1$, $c=-1/2$

At $m=-1$, $c=-1/2$

$\therefore$ $y = 0$

$y = x - \frac{1}{2}$

$2y = 2x - 1$

$2x - 2y = 1$

$y = -x - \frac{1}{2}$

$2y = -2x - 1$

$2x + 2y = -1$

Date ______
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4\. Define asymptotes of the curve with different types. Find asymptotes of $x^3+2x^2y-xy^2-2y^3+4y^2+2xy-5y+6=$

$\Rightarrow$ A straight line is said to be asymptote to a curve if a point $P(x, y)$ on curve, the distance between the point to the straight line to zero if the points tends to infinity.

<u>Types of asymptotes</u>

i) <u>Asymptotes parallel to axes</u>
Asymptotes which are parallel to either x-axis or y-axis is said to be asymptotes parallel to axes.

ii) <u>oblique Asymptotes</u>
Asymptotes which are parallel to neither of the axis is said to be asymptotes ~~p~~ oblique asymptotes.

The given eqn is

$$x^3+2x^2y-xy^2-2y^3+4y^2+2xy-5y+6=0 \quad -(i)$$

Here, eqn (i) is of degree 3 and x^3 and y^3 are present, then asymptotes are oblique and at most 3.

Let the asymptotes are $y = mx + c$
put $x=1$ and $y=m$ in eqn (i)

$$\phi_3(m) = 1 - 2m^3 + 2m - m^2$$

$\phi_2(m) = 4m^2 + 2m$

$\phi_1(m) = -5m + 6$

Now,

$\phi_3(m) = 0$

$1 - 2m^3 + 2m - m^2 = 0$

$-2m^3 + 2m - m^2 + 1 = 0$

$2m^3 - 2m + m^2 - 1 = 0$

or, $2m(m^2-1) + 1(m^2-1) = 0$

$(2m+1)(m^2-1) = 0$

$\therefore m = -1/2,\ 1, -1$

Now,

$c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = \dfrac{-4m^2 + 2m}{-6m^2 - 2m + 2}$

for, $m = -\frac{1}{2}$, $c = \dfrac{-4(-\frac{1}{2})^2 - 2(-\frac{1}{2})}{-6(-\frac{1}{2})^2 - 2(-\frac{1}{2}) + 2} = 0$

$= \cancel{1}\ 0$

for, $m = 1$, $c = \dfrac{-4(1)^2 - 2\times 1}{-6(1)^2 - 2\times 1 + 2} = 1$

for $m = -1$, $c = \dfrac{-4(-1)^2 - 2\times(-1)}{-6(-1)^2 - 2\times(-1) + 2} = 1$

$\therefore y = -\frac{1}{2}x$, $y = x + 1$ and $y = -x + 1$

or, $2y = -x$; $x - y + 1 = 0$; $x + y - 1 = 0$

$x + 2y = 0$

5. Define asymptotes of a curve. Find the asymptotes of $y^3 + x^2y + 2xy^2 - y + 1 = 0$

⇒ A straight line is said to be asymptote to a curve if a point $P(x, y)$ on curve, the distance between the point to the straight line to zero if the points tends to infinity.

The given curve is

$$y^3 + x^2y + 2xy^2 - y + 1 = 0 \quad -(i)$$

Here, eqn (i) is of degree 3 and x^3 is absent and y^3 is present.

Let $y = mx + c$ be the oblique asymptotes.

put $x = 1$ and $y = m$

$$\phi_3(m) = m^3 + m + 2m^2$$

$$\phi_2(m) = 0$$

$$\phi_1(m) = -m + 1$$

Now,

$$\phi_3(m) = 0$$

$$m^3 + m + 2m^2 = 0$$

$$m^3 + m^2 + m^2 + m = 0$$

$$\text{or, } \cancel{m^2(m+1) + m(m+1) = 0}$$

$$\text{or, } \cancel{(m+1)(m^2+m) = 0}$$

$$m(m^2 + m + m + 1) = 0$$

$$m^2 + m + m + 1 = 0$$

$$m(m+1) + 1(m+1) = 0$$

$$\therefore m = 0, -1, -1$$

For $m=0$ (non-repeating value), the value of c is given by

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{0}{3m^2+4m+1} = 0$$

$\therefore$ $y=0$ is the asymptote.

For $m=-1,-1$ (repeated root)

$$\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2''(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2}(6m+4) + c\times 0 - m = 0$$

$$(3m+2)c^2 - m = 0$$

putting $m=-1$, we get,

$$-c^2+1=0$$

$$c^2 = 1$$

$$\Rightarrow c = \pm 1$$

$\therefore$ $y=-x+1$ and $y=-x-1$

or, $x+y-1=0$ $\quad$ $x+y+1=0$ $\}$ A

6. Define asymptotes. Find the asymptotes of the curve. $x^3+3x^2y-4y^3-x+y+3=0$

⇒ A straight line is said to be asymptote to a curve if a point $P(x,y)$ on curve, the distance between the points to the straight line to zero if the points tends to infinity.

The given curve is

$$x^3+3x^2y-4y^3-x+y+3=0 \quad \text{--- (i)}$$

eq^n (1) is of degree 3. x^3 and y^3 are present then asymptotes are oblique and at most 3.

∴ Let asymptotes are $y=mx+c$
put $x=1$ and $y=m$

$\phi_3(m) = 1+3m-4m^3$
$\phi_2(m) = 0$
$\phi_1(m) = -1+m+3$

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Now,

$\phi_3(m) = 0$

$1+3m-4m^3=0$

$4m^3-3m-1=0$

$(2m-1)^2(m+1)=0$

$2m-1=0$

$m=1/2$

$m=-1$

$4m^2 \;\; 4m^2) \; 4m^3-3m-1 \; (m-1$
$4m^3-4m^2$
$4m^2-3m-1$

$3m+1+4m$
$6m+4$

for $m = 1/2$

$$c = \frac{-\phi_2(m)}{\phi'(3)} = \frac{-6}{3-12m} = 0$$

for $m = -1$

$$c = \frac{-\phi_2(m)}{\phi'(m)} = \frac{-6}{3-12m} = 0$$

$$y = \frac{1}{2}x$$

$$2y = x$$

$$x - 2y = 0$$

$$y = -x$$

$$x + y = 0$$

PLANE

1. Define Parabola, Hyperbola and ellipse in terms of eccentricity. Derive the standard equation of parabola, ellipse and Hyperbola.

⇒ The conic section whose eccentricity is equal to 1 is called parabola.

The conic section whose ecentricity is always greater than 1, ~~which can be calculate~~ is called hyperbola.
Ecentricity is calculated using formula

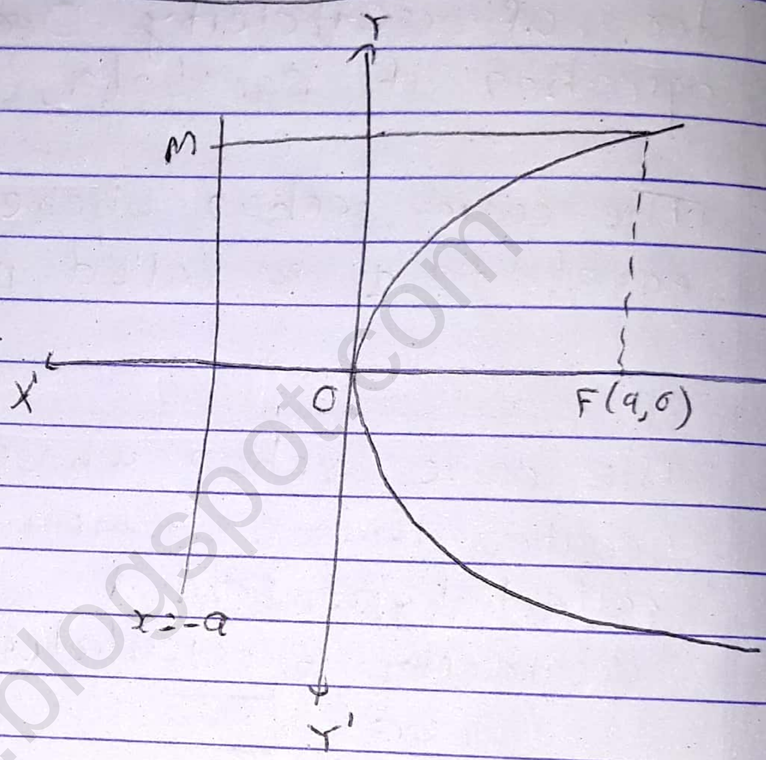
$$e = \frac{\sqrt{a^2+b^2}}{a}$$

The conic section whose ecentricity will always be between 0 and 1 is called ellipse.
Ecentricity ~~is~~ can calculated using formula

when $a>b$, we use $e = \sqrt{a^2-b^2}/a$
when $b>a$, we use $e = \sqrt{b^2-a^2}/b$

P.T.O

standard equation of parabola



Let $P(x,y)$ be any point and $F(a,0)$ be a fixed point and $x=-a$ is the fixed straight line (Directrix). Now, by the definition of parabola.

$$PF = PM$$

$$\sqrt{(x-a)^2+y^2} = 1\,(x+a)$$

squaring on Both side, we get,

$$(x-a)^2+y^2 = (x+a)^2$$

$$x^2-2ax+a^2+y^2 = x^2+2ax+a^2$$

$$\boxed{y^2 = 4ax}$$

which is req. standard eq^n of parabola.

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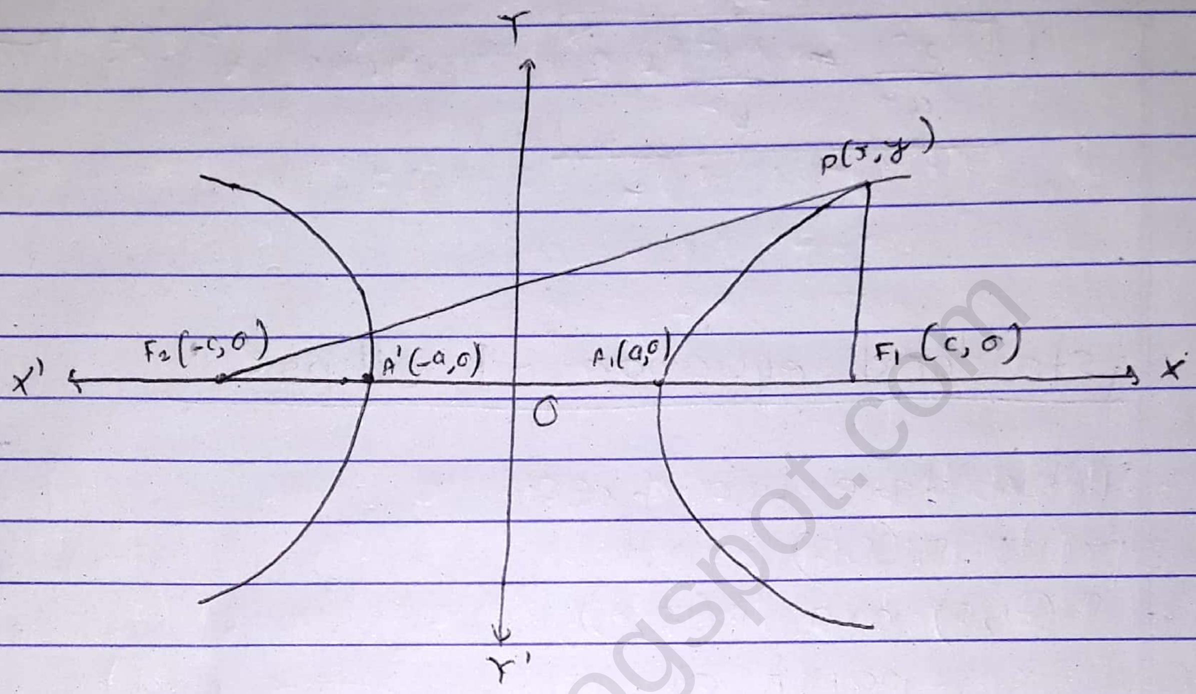
standard equation of hyperbola



let $F_1(c, 0)$ & $F_2(-c, 0)$ are two fixed points (foci) and $A(a, 0)$ & $A'(-a, 0)$ are the vertices of hyperbola. By the definition of hyperbola

$$PF_2 - PF_1 = 2a$$

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a$$

$$\sqrt{(x+c)^2+y^2} = 2a + \sqrt{(x-c)^2+y^2}$$

On squaring, we get,

$$\left\{\sqrt{(x+c)^2+y^2}\right\}^2 = \left\{2a + \sqrt{(x-c)^2+y^2}\right\}^2$$

or, $$x^2+c^2+y^2+2cx = 4a^2+x^2+y^2+c^2-2cx+4a\sqrt{(x-c)^2+y^2}$$

$$4cx - 4a^2 = 4a\sqrt{(x-c)^2+y^2}$$

$$cx - a^2 = a\sqrt{(x-c)^2+y^2}$$

Again, squaring, we get

$$(cx-a^2)^2 = a^2\{x^2+y^2+c^2-2cx\}$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 + a^2y^2 + a^2c^2 - 2a^2cx$$

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$x^2(c^2-a^2) - a^2y^2 = a^2(c^2-a^2)$$

P.T.O

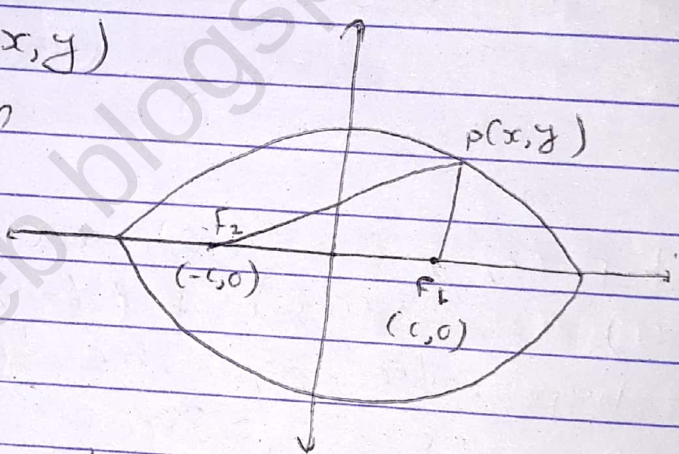
$$\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad \text{where } b^2 = c^2 - a^2$$

standard equation of ellipse

Let us take two fixed points $F_1(c,0)$ and $F_2(-c,0)$ and $P(x,y)$ be any point on the ellipse.
Then by the definition

$PF_1 + PF_2 = 2a$



Let $AA' = 2a$ and $BB' = 2b$

$$PF_1 + PF_2 = 2a$$

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

$$\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

squaring on both sides

$$\left\{\sqrt{(x-c)^2+y^2}\right\}^2 = \left\{2a - \sqrt{(x+c)^2+y^2}\right\}^2$$

$$x^2+c^2+y^2-2xc = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + x^2+c^2+y^2+2cx$$

$$4a\sqrt{(x+c)^2+y^2} = (cx+a^2)$$

Again, squaring both sides

$$a^2(x^2+c^2+y^2+2cx) = c^2x^2 + 2a^2cx + a^4$$

$$a^2x^2 + a^2c^2 + a^2y^2 + 2a^2cx = c^2x^2 + 2a^2cx + a^4$$

P.T.O

$$a^2x^2 + a^2c^2 + a^2y^2 - c^2x^2 - a^4 = 0$$

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

3. ~~Find by~~ using vector method find the equation of a plane through the point $(2, 1, -1)$ and perp. to both the planes $2x + y - z = 3$, $x + 2y + z = 2$.

Sol

The normal to the two given plane are

Let $\vec{p} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{q} = i + 2\vec{j} + \vec{k}$

Let $\vec{R}$ perp. to both vectors is given by

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$i(1+2) - \vec{j}(2+1) + \vec{k}(4-1)$$

$$\vec{R} = 3\vec{i} - 3\vec{j} + 3\vec{k}$$

Let $\vec{S} = (2, 1, -1)$ and $T(x, y, z)$ lies on the plane

$$\vec{ST} = (x-2)\vec{i} + (y-1)\vec{j} + (z+1)\vec{k}$$ ~~lies~~

P.T.O

$$\vec{ST} \cdot \vec{R} = 0$$

$$\left[(x-2)\vec{i} + (y-1)\vec{j} + (z+1)\vec{k}\right] \cdot \left(3\vec{i} - 3\vec{j} + 3\vec{k}\right) = 0$$

or, $3(x-2) - 3(y-1) + 3(z+1) = 0$

$$x - 2 - y + 1 + z + 1 = 0$$

$$x - y + z = 0$$

2. Find by vector method the equation of the plane which passes through the points $(1,1,-1)$, $(2,0,2)$ and $(0,-2,1)$

Soln

Here the plane passes through $A(1,1,-1)$, $B(2,0,2)$ and $C(0,-2,1)$

$$\vec{AB} = \vec{i} - \vec{j} + 3\vec{k}$$

$$\vec{BC} = -2\vec{i} - 2\vec{j} - \vec{k}$$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ -2 & -2 & -1 \end{vmatrix}$$

$$= \vec{i}(1+6) - \vec{j}(-1+6) + \vec{k}(-2-2)$$

$$= 7\vec{i} - 5\vec{j} - 4\vec{k} = \vec{v}$$

Let $P(x,y,z)$ lies on the plane

$$\vec{AP} = (x-1)\vec{i} + (y-1)\vec{j} + (z+1)\vec{k}$$

P.T.O

So, $\vec{v}$ & $\vec{AP}$ are orthogonal,

$\vec{AP} \cdot \vec{v} = 0$

$[(x-1)\vec{i} + (y-1)\vec{j} + (z+1)\vec{k}] \cdot [7\vec{i} - 5\vec{j} - 4\vec{k}] = 0$

$7(x-1) - 5(y-1) - 4(z+1) = 0$

$7x - 7 - 5y + 5 - 4z - 4 = 0$

$7x - 5y - 4z - 6 = 0$

4. Find the equation of the plane through (3, 2, 1) and (1, 2, 3) which is perpendicular to the plane $4x - y + 2z = 7$

Sol^n

Given that the required plane passes through A (1, 2, 3), B (3, 2, 1)

So, $\vec{AB} = (2, 0, -2)$ lies on the plane.

Let $\vec{n} = (a, b, c)$ be a vector perp. to the required plane.

Also, given that the plane $4x - y + 2z = 7$ is perp. to the req. plane. So, (4, -1, 2) is parallel to the required plane.

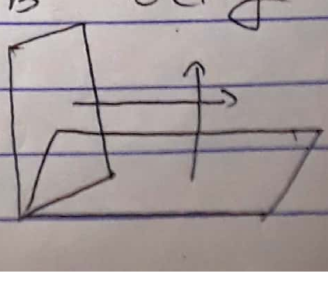
Therefore, (4, -1, 2) is normal to $\vec{n}$.

i.e

$(4, -1, 2) \cdot \vec{n} = 0$

$\Rightarrow 4a - b + 2c = 0$ — (i)

And, $\vec{n}$ is normal to $\vec{AB}$, $\vec{AB}$ being on the plane.



So, $\vec{AB} \cdot \vec{n} = 0$

$\Rightarrow 2a - 2c = 0$

solving eq^n (i) and (ii), we get

$a = c = \frac{b}{6} = k$ (say)

$\Rightarrow a = k,\ b = 6k,\ c = k$

Thus, $\vec{n} = (k, 6k, k)$ is normal to the plane

Hence, the eq^n of plane that passes through A and normal to $\vec{n}$ be

$(\vec{r} - \vec{a}).\vec{n} = 0$ for $\vec{a} = \vec{OA}$

$(x-1, y-2, z-3).(1, 6, 1) = 0$ for $k \neq 0$

$x + 6y + z = 16$

This is req. eq^n of plane.

5. find the equation of plane through $(0, 2, 4)$ and normal to $2\vec{i} + 4\vec{j} + \vec{k}$

Sol^n

Here, the plane passes through $(0, 2, 4)$ and normal to $2\vec{i} + 4\vec{j} + \vec{k} = \vec{v}$

let $P(x, y, z)$ lies on the plane.

so, $\vec{AP} = x\vec{i} + (y-2)\vec{j} + (z-4)\vec{k}$

Here,

$\vec{AP}.\vec{v} = 0$

$[x\vec{i} + (y-2)\vec{j} + (z-4)\vec{k}].[2\vec{i} + 4\vec{j} + \vec{k}] = 0$

$2x + 4(y-2) + z - 4 = 0$

$2x + 4y - 8 + z - 4 = 0$

$2x + 4y + z - 12 = 0$

6. Find the eq^n of line through $(2,-9,5)$ and parallel to the vector $2\vec{i}+5\vec{j}+6\vec{K}$.

sol^n

Here, plane line through $A(2,-9,5)$ parallel to the vector $\vec{v} = 2\vec{i}+5\vec{j}+6\vec{K}$

Let $P(x,y,z)$ lies on the line.

So, $\vec{AP} = (x-2)\vec{i} + (y+9)\vec{j} + (z-5)\vec{k}$

Now, $\vec{AP}$ & $\vec{v}$ are parallel

$\vec{AP} = \lambda\vec{v}$ where λ is scalar.

$(x-2)\vec{i} + (y+9)\vec{j} + (z-5)\vec{k} = 2\lambda\vec{i} + 5\lambda\vec{j} + 6\lambda\vec{k}$

$x-2 = 2\lambda$	$y+9 = 5\lambda$	$z-5 = 6\lambda$
$\frac{x-2}{2} = \lambda$	$\frac{y+9}{5} = \lambda$	$\frac{z-5}{6} = \lambda$

So, $\frac{x-2}{2} = \frac{y+9}{5} = \frac{z-5}{6}$ is req: eq^n of line.

7. Define eccentricity of a conic section and derive the equation of a ellipse in its standard form.

⇒ The constant ratio between the distance from a fixed point and distance from a fixed straight line of a conic section.

P.T.O

$\frac{16}{112}$

8. Find the equation of the tangents to the parabola $y^2 = 7x$ which is perp. to the line $4x+y=0$. Also, find the point of contact.

Sol^n

The given eq^n of parabola is
$$y^2 = 7x \quad \text{—} (1)$$

~~eq^n (1) is tangent~~

Eq^n of tangent on eq^n (1) perp. to $4x+y=0$ is
$$x-4y+k=0 \quad \text{—} (ii)$$

~~where K is s~~

since eq^n (ii) is tangent on (1), then,
$$\left(\frac{x+k}{4}\right)^2 = 7x$$

$$x^2+2kx+k^2 = 112x$$

or, $$x^2+(2k-112)x+k^2=0 \quad \text{—} (iii)$$

which is quad. in x. since eq^n (ii) is tangent of eq^n (1) therefore discriminant of eq^n (3) must be equal to zero.

$$B^2-4AC=0$$
$$(2k-112)^2-4.1.k^2=0$$
$$\cancel{4k^2}-448k-12544-\cancel{4k^2}=0$$
$$448k=12544$$
$$k=28$$

$\therefore$ The req. eq^n of tangent is
$$x-4y+28=0$$

For point of contact solving,

$y^2 = 7x$ and $x - 4y + 28 = 0$

$y = \frac{x+28}{4}$

$\therefore \left(\frac{x+28}{4}\right)^2 = 7x$

$x^2 + 56x + 784 = \cancel{142x}\ 112x$

$\cancel{x^2 + 28x}$

$x^2 - 56x + 784 = 0$

$(x-28)^2 = 0$

$x = 28$

$\therefore y = \frac{28+28}{4} = \frac{\overset{14}{\cancel{56}}}{\cancel{4}} = 14$

$\therefore$ The req. point of contact is $(28, 14)$

9. Find the condition, when the line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find their point of contact.

sol^n

Given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$ — (i)

and ~~th~~ the line is $lx + my + n = 0$

$\Rightarrow y = -\left(\frac{lx+n}{m}\right)$ — (ii)

$\therefore b^2x^2 + \frac{a^2}{m^2}(lx+n)^2 = a^2b^2$

$$b^2x^2 + \frac{a^2}{m^2}(l^2x^2 + 2lnx + n^2) = a^2b^2$$

$$b^2m^2x^2 + a^2l^2x^2 + 2a^2lnx + a^2n^2 - a^2b^2m^2 = 0$$

or, $(b^2m^2 + a^2l^2)x^2 + 2a^2lnx + (a^2n^2 - a^2b^2m^2) = 0$ (iii)

which is quad- in x. So to be (ii) tangent on (i), the discriminant $B^2 - 4AC = 0$

$A = b^2m^2 + a^2l^2$, $B = 2a^2ln$ & $C = a^2n^2 - a^2b^2m^2$

for,

$$B^2 - 4AC = 0$$

$$4a^4l^2n^2 - 4(b^2m^2 + a^2l^2)(a^2n^2 - a^2b^2m^2) = 0$$

$$a^4l^2n^2 - a^2b^2m^2n^2 + a^2b^4m^4 - a^4l^2n^2 + a^4b^2l^2m^2 = 0$$

$$a^2b^4m^4 + a^4b^2l^2m^2 - a^2b^2m^2n^2 = 0$$

$$a^2b^2m^2(b^2m^2 + a^2l^2 - n^2) = 0$$

$\therefore$ ~~$a^2l^2 + a^2l^2 =$~~

$b^2m^2 + a^2l^2 = n^2$ which is req. condition

for point of contact,

$$x = -B/2A = \frac{-2a^2ln}{2(a^2l^2 + b^2m^2)} = \frac{-a^2ln}{n^2} = -\frac{a^2l}{n}$$

and,

$$y = -\left\{\frac{lx+n}{m}\right\} = \cancel{-\frac{1}{m}\left(\frac{-la^2lx}{n^2} + n\right)}$$

$$= -\left\{\frac{l\left(-\frac{a^2l}{n}\right) + n}{m}\right\} = \cancel{-\frac{1}{m}\left(\frac{-l^2a^2}{n^2} + n\right)}$$

$$= -\left\{\frac{-\frac{a^2l^2}{n} + n}{m}\right\} = \cancel{-\frac{1}{m}\left(\frac{-l^2a^2 + n^2}{n}\right)}$$

$$= -\left\{\frac{-a^2l^2 + n^2}{mn}\right\} = \cancel{\frac{n^2 - a^2l^2}{mn}} = \frac{b^2m^2}{mn} = \frac{b^2m}{n}$$

$$= -\frac{b^2m^2}{mn} = \frac{b^2m}{n}$$

∴ Point of contact is $\left(-\frac{a^2 l}{n}, -\frac{b^2 m}{n}\right)$

10. Find the equation of parabola having focus $(-3,0)$ and directrix $x+5=0$

Here, focus is $(-3,0)$ & directrix $x+5=0$
∴ As per the rule of parabola, let (x,y) be any point on parabola.

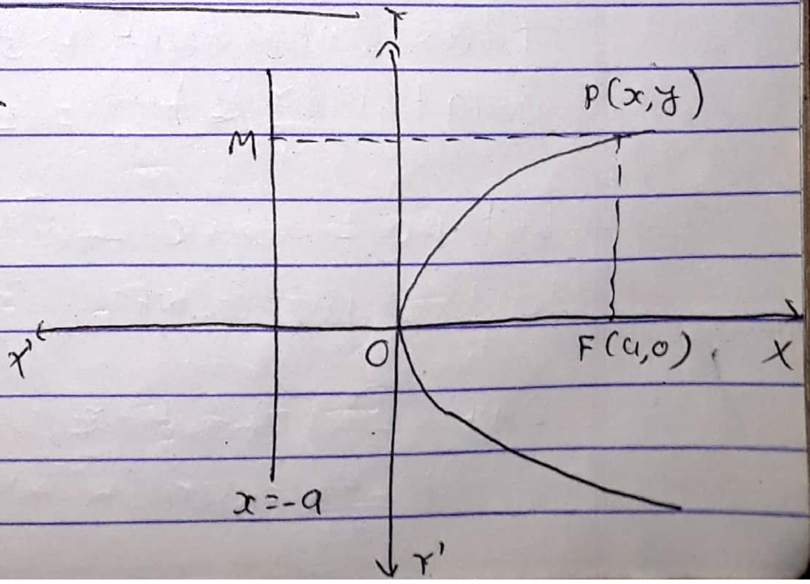
$$\sqrt{(x+3)^2+y^2} = x+5$$

$$\sqrt{x^2+6x+9+y^2} = x+5$$
$$\cancel{x^2}+y^2+6x+9 = \cancel{x^2}+10x+25$$
$$y^2 = 4x+16$$
$y^2 = 4(x+4)$ is req. eq^n of parabola

12. Derive the standard eq^n of parabola $y^2=4ax$. Also find the condition that $y=mx+c$ may be tangent to the parabola.

Sol^n

standard eq^n of parabola



let $P(x,y)$ be any point and $F(a,0)$ be a fixed point and $x=-a$ is the fixed straight line ~~through~~ (Directrix)
now, by the def^n of parabola.

P.T.O

$PF = PM$

$\sqrt{(x-a)^2+y^2} = \pm(x+a)$

squaring on both side, we get

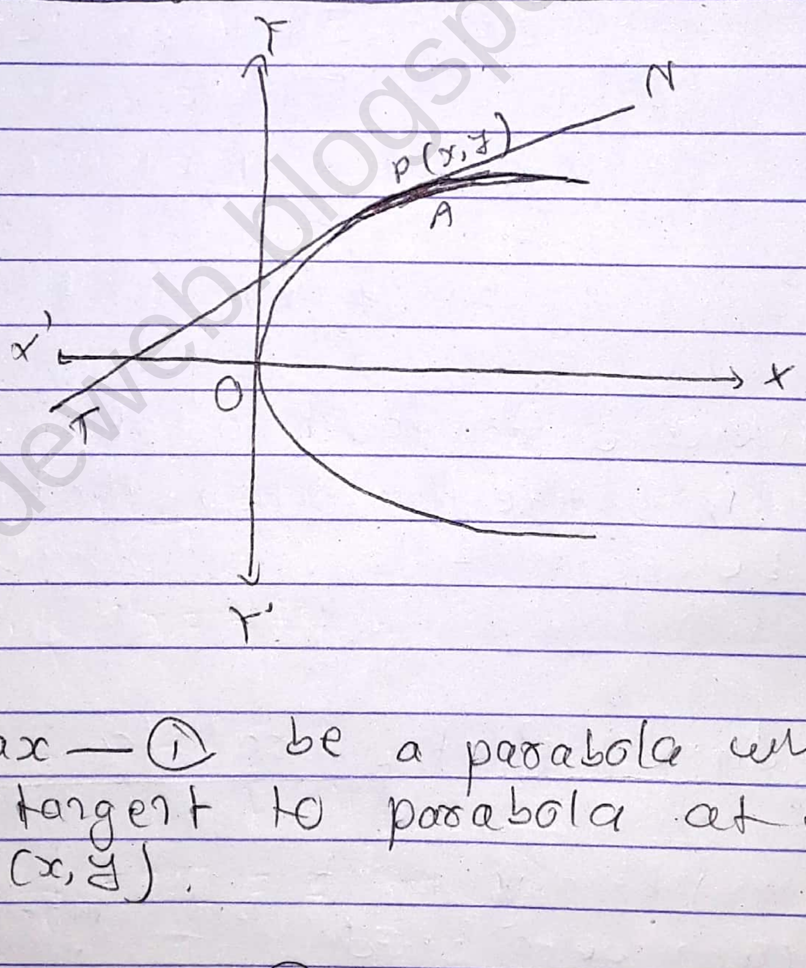
$(x-a)^2+y^2 = (x+a)^2$

$x^2-2ax+a^2+y^2 = x^2+2ax+a^2$

$$\boxed{y^2=4ax}$$

which is standard eqn of parabola.

~~Condition that~~ Equation $y=mx+c$ be tangent to parabola $y^2=4ax$



Let $y^2=4ax$ — (i) be a parabola where TAN is tangent to parabola at the point $P(x,y)$.

Let $y=mx+c$ — (ii) be the eqn of tangent to the parabola $y^2=4ax$, — (iii)

Diff. (i) w.r.t x,

$2y\frac{dy}{dx} = 4a$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{dy}{dx}_{(x_1, y_1)} = \frac{2a}{y_1} \quad (m \text{ say})$$

Again,

$y = mx + c$ gives $c = y - mx$ also passes through (x_1, y_1) so, $c = y_1 - mx_1$

putting value of m & c on eqn (ii)

$$y = \frac{2a}{y_1}x + y_1 - mx_1$$

$$y = \frac{2ax}{y_1} + y_1 - \frac{2a}{y_1}x_1$$

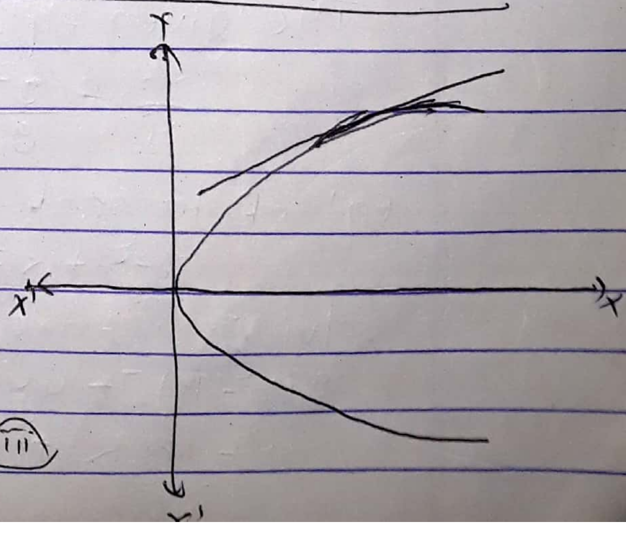
$$yy_1 = 2ax + y_1^2 - 2ax_1$$

$$yy_1 = 2ax + 4ax_1 - 2ax_1$$

$$yy_1 = 2ax + 2ax_1$$

$$\boxed{yy_1 = 2a(x + x_1)}$$

condition for a line $y = mx + c$ is tangent to $y^2 = 4ax$



let the line $y = mx + c$ — (i)

is tangent to parabola

$y^2 = 4ax$ — (ii)

$$(mx + c)^2 = 4ax$$

$$m^2x^2 + 2mcx + c^2 = 4ax$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0 \quad \text{— (iii)}$$

eq^n (iii) is quad. in x. so to (i) tangent to (ii), the eq^n (iii) must be perfect square.

$x^2m^2 + (2mc - 4a)x + c^2 = 0$ must be perfect square.

comparing with $Ax^2 + Bx + C = 0$

$$B^2 - 4AC = 0$$

$$4(mc-2a)^2 - 4m^2c^2 = 0$$

$$m^2c^2 - 4amc + 4a^2 - m^2c^2 = 0$$

$$4amc = 4a^2$$

$$mc = a$$

$c = a/m$ is req. condition.

13. Find the equation of tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, which is parallel to the line $x = y + 4$.

Soln

The given eq^n. of ellipse is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \text{—— (i)}$$

Eq^n of tangent parallel to $x = y + 4$ is

$x - y + k = 0$ —— (ii) where k is scalar

solving (i) and (ii)

$$9(y-k)^2 + 4y^2 = 36$$

$$9y^2 - 18yk + 9k^2 + 4y^2 - 36 = 0$$

which is quad- in y and have most two two equal root since (ii) touches (i)

$$B^2 - 4AC = 0$$

$$(-18K)^2 - 4.13(9K^2 - 36) = 0$$

$$324K^2 - 468K^2 + 1872 = 0$$

$$144K^2 = 1872$$

$$K^2 = 13$$

$$K = \pm\sqrt{3}$$

$\therefore$ The req. eq^n of tangent is

$$x - y \pm \sqrt{13} = 0$$

44. Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the point of contact. Do the same for hyperbola as well.

Sol^n

Given eq^n of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x^2b^2 - a^2y^2 = a^2b^2 \quad \text{— (i)}$$

Given line is

$$lx + my + n = 0$$

$$y = -\frac{(lx+n)}{m} \quad \text{— (ii)}$$

$$\therefore b^2x^2 - \frac{a^2}{m^2}(lx+n)^2 = a^2b^2$$

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$$\Rightarrow b^2m^2x^2 - a^2l^2x^2 - 2a^2lxn - a^2n^2 = a^2m^2b^2$$

$$(b^2m^2 - a^2l^2)x^2 - 2a^2lnx - a^2n^2 - a^2m^2b^2 = 0 \quad (iii)$$

If eqn (ii) is tangent on (i), then the discriminant term of (iii) should be zero.

i.e. $(-2a^2ln)^2 + 4(b^2m^2 - a^2l^2)(a^2n^2 + a^2m^2b^2) = 0$

$$\Rightarrow a^4l^2n^2 + a^2b^2m^2n^2 + a^2b^4m^4 - a^4n^2l^2 - a^4m^2l^2b^2 = 0$$

$$\Rightarrow n^2 + b^2m^2 - a^2l^2 = 0$$

$$\Rightarrow -b^2m^2 + a^2l^2 = n^2$$

This shows that the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, when

$$a^2l^2 - b^2m^2 = n^2$$

For ~~soln~~ point of contact,

$$x = -B/2A = \frac{2a^2ln}{2(b^2m^2 - a^2l^2)} = \frac{a^2ln}{-n^2} = -\frac{a^2l}{n}$$

$$y = -\left\{\frac{l\cdot\left(-\frac{a^2l}{n}\right) + n}{m}\right\}$$

$$= -\left\{\frac{-a^2l^2 + n^2}{mn}\right\}$$

$$= -\left\{\frac{-b^2m^2}{mn}\right\}$$

$$= b^2m/n$$

Point of contact is $\left(-\frac{a^2l}{n}, b^2m/n\right)$

15. Find the vertices and foci, line of symmetry and directrics of the parabola.

(i) $y^2-4y-4x=0$ (ii) $x^2-2x-8y-15=0$

(i) Given eq^n is

$y^2-4y-4x=0$

$y^2-2\cdot2y+2^2-2^2-4x=0$

or, $(y-2)^2-4-4x=0$

or, $(y-2)^2-4(x+1)=0$

or, $(y-2)^2=4(x+1)$ — (i)

comparing eq^n (i) with $(y-k)^2=4a(x-h)$, we get,

$h=-1$, $k=2$, $a=1$

vertex $V(h,k)=V(-1,2)$

foci : $F(h+a,k)=F(0,2)$

line of symmetry be $(y-2)^2=0$

$y=2$

Equation of directrix be,

$x=h-a$

$x=-1-1$

$x+2=0$

(ii) $x^2-2x-8y-15=0$

$x^2-2\cdot1\cdot x+1^2-1^2-8y-15=0$

$(x-1)^2-8y-16=0$

$(x-1)^2=8(y+2)$ — (i)

comparing eq^n (i) with $(x-h)^2=4a(y-k)$, we g—

$h=1$, $k=-2$, $a=2$

vertex $V(h,k)=V(1,-2)$

focus $= f(h, k+a) = f(1, 0)$

line of symmetry be $(x-1)^2 = 0$

$x = 1$

eqⁿ of directrix is

$y = k - a$

$y = -2 - 2$

$y + 4 = 0$

19. A double ordinate of the parabola $y^2 = 2ax$ is $4a$. Prove that the line joining vertex to its ends is at right angle.

Solⁿ

Given parabola is

$y^2 = 2ax$ — (i)

and length of double ordinates $4a$

So,

$2y = 4a$

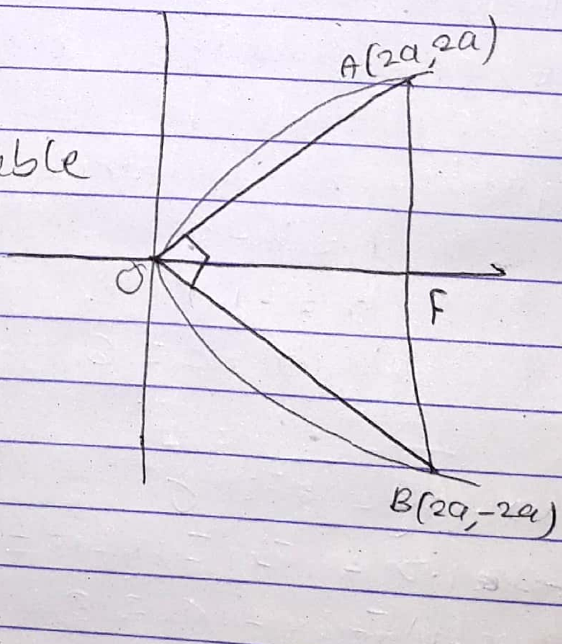
$y = 2a$

from (i)

$y^2 = 2ax$

$4a^2 = 2ax$

$\Rightarrow x = 2a$



So, co-ordinates of $A(2a, 2a)$ and $B(2a, -2a)$

slope of OA $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{2a}{2a} = 1$

P.T.O

slope of OB = $\frac{2a}{-2a} = -1$

slope of OA x slope of OB = -1

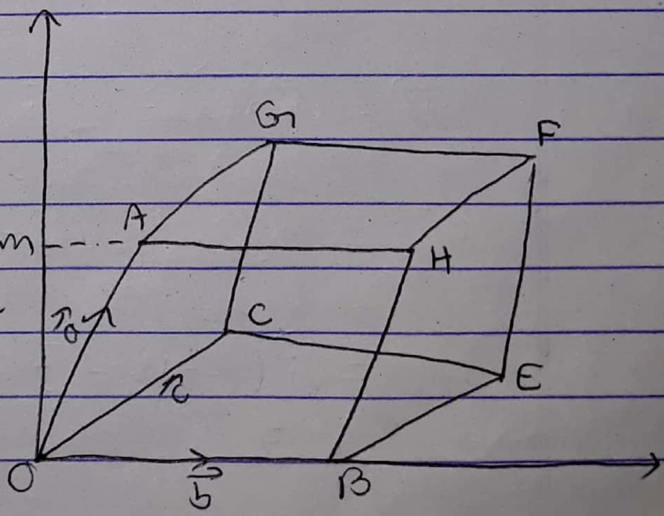
OA and OB are perpendicular.

Vector Geometry

1. Define scalar and vector product of three vectors. Prove that the scalar triple product of three vectors represent the volume of parallelepiped. What conclusion can you draw about these vectors if $\vec{a}\cdot(\vec{b}\times\vec{c})=0$.

⇒ Let $\vec{a}$, $\vec{b}$ & $\vec{c}$ are three vectors then scalar triple product of $\vec{a}$, $\vec{b}$ & $\vec{c}$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ is scalar quantity and is defined as $\vec{a}\cdot[\vec{b}\times\vec{c}]$

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors. Then the vector product of $\vec{a}$ and $\vec{b}\times\vec{c}$ is called vector triple product of $\vec{a}$, $\vec{b}$, $\vec{c}$ and is denoted by $\vec{a}\times(\vec{b}\times\vec{c})$.



Let OABCEFGH is a parallelopiped where $\vec{OA}=\vec{a}$, $\vec{OB}=\vec{b}$ & $\vec{OC}=\vec{c}$ and $\vec{b}\times\vec{c}$ is a vector perp. to $\vec{b}$ and $\vec{c}$ both. Let θ be an angle between $\vec{b}\times\vec{c}$ and $\vec{a}$

Now,

$[\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})=|\vec{b}\times\vec{c}|$

{projection of $\vec{a}$ on $\vec{b}\times\vec{c}$}

$|\vec{b}\times\vec{c}|\ \vec{a}\cdot\cos\theta$

P.T.O

$$|\vec{b}\times\vec{c}| \cdot \frac{(OA)\cdot(OM)}{(OA)}$$

$$|\vec{b}\times\vec{c}|.OM$$

= Area of parallelogram x height of parallelopiped

= volume of parallelopiped.

→ $\vec{a}.(\vec{b}\times\vec{c}) = 0$ means that vectors $\vec{a}, \vec{b}$ $\vec{c}$ are coplanar.

2. Find the vector perpendicular to both the vector $\vec{a}$ on to $\vec{b}$ if $\vec{a} = 3\vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$.

$$\vec{a} = 3\vec{i} - \vec{j} + \vec{k}$$
$$\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$$

$$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \vec{i}(2-1) - \vec{j}(-6-2) + \vec{k}(3+2)$$
$$= \vec{i} + 8\vec{j} + 5\vec{k}$$

$$\hat{n} = \frac{\vec{i} + 8\vec{j} + 5\vec{k}}{3\sqrt{10}}$$

$$= \frac{1}{3\sqrt{10}}(\vec{i} + 8\vec{j} + 5\vec{k})$$

3. If $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$, find unit projection on both vectors $\vec{a}$ & $\vec{b}$.

Soln

$\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$

$\vec{b} = \vec{i} + \vec{j} + \vec{k}$

$\vec{a} \cdot \vec{b} = 1 + 2 + 1 = 4$

$|\vec{b}| = \sqrt{1+1+1} = \sqrt{3}$

Now, vector projection of $\vec{a}$ on to $\vec{b}$ is

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$= \frac{4}{\sqrt{3}}$$ A

5. Show that the vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to the vector $\vec{a}$.

Sol^n

Let $\vec{r} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} + [\vec{a}\ \vec{c}\ \vec{b}]\vec{d} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{d}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{d}\ \vec{b}]\vec{c}$

$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} + \{-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + \{-[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}\} - \{-[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}\}$

$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$

$= 2[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - 2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$

$= 2\{[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}\} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$

$= 2\{(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$

$= 2\{[\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a}\} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$

$= 2[\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - 2[\vec{c}\ \vec{d}\ \vec{b}]\vec{a} - 2[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$

$= 2[\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - 2\{-[\vec{b}\ \vec{d}\ \vec{c}]\vec{a}\} - 2[\vec{c}\ \vec{d}\ \vec{a}]\vec{b}$

$= 2[\vec{b}\ \vec{d}\ \vec{c}]\vec{a}$

$\vec{r} = \lambda\vec{a}$ where $\lambda = 2[\vec{b}\ \vec{d}\ \vec{c}]$ = scalar

Hence $\vec{r}$ is parallel to $\vec{a}$.

6. Find the projection of the vector $3\vec{i}-\vec{j}+\vec{k}$ on the vector $2\vec{i}+\vec{j}+\vec{k}$. Define scalar and vector triple product vectors. Show that $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ if the vectors $\vec{a}$ and $\vec{c}$ are collinear.

Soln

Let $\vec{a} = 3\vec{i}-\vec{j}+\vec{k}$

$\vec{b} = 2\vec{i}+\vec{j}-\vec{k}$

Then,

$\vec{a}\cdot\vec{b} = 6-1-1 = 4$

$|\vec{b}| = \sqrt{2^2+1^2+1^2}$

$= \sqrt{\cancel{6}} = \sqrt{6}$

Now, vector projection of $\vec{a}$ onto $\vec{b}$ is

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{4}{\sqrt{6}}$$

Suppose that,

$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$

$(\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

or, $(\vec{c}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\vec{c} = 0$

$(\vec{c}\cdot\vec{b})\vec{a} = (\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{a} = \frac{(\vec{a}\cdot\vec{b})\,\vec{c}}{(\vec{c}\cdot\vec{b})}$$

$\vec{a} = \lambda\vec{c}$ where $\lambda = \frac{\vec{a}\cdot\vec{b}}{\vec{c}\cdot\vec{b}}$ = some scalar quantity

Thus, $\vec{a}$ and $\vec{c}$ are collinear.

7. Define vector triple product and show that $[\vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a} \;\; \vec{a}\times\vec{b}] = [\vec{a}\;\vec{b}\;\vec{c}]^2$

Sol^n

$$[\vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a} \;\; \vec{a}\times\vec{b}]$$
$$= (\vec{b}\times\vec{c}) \cdot \{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\}$$
$$= (\vec{b}\times\vec{c}) \cdot \{[\vec{c}\;\vec{a}\;\vec{b}]\vec{a} - [\vec{c}\;\vec{a}\;\vec{a}]\vec{b}\}$$
$$= (\vec{b}\times\vec{c}) \cdot \{[\vec{c}\;\vec{a}\;\vec{b}]\vec{a} - 0\} \quad (\because [\vec{c}\;\vec{a}\;\vec{a}] = 0)$$
$$= (\vec{b}\times\vec{c}) \cdot [\vec{c}\;\vec{a}\;\vec{b}]\vec{a}$$
$$\cancel{\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\cdot[\vec{c}}$$
$$= \{(\vec{b}\times\vec{c})\cdot\vec{a}\}\cdot[\vec{c}\;\vec{a}\;\vec{b}]$$
$$= [\vec{a}\;\vec{b}\;\vec{c}]\cdot[\vec{a}\;\vec{b}\;\vec{c}]$$
$$= [\vec{a}\;\vec{b}\;\vec{c}]^2$$

Thus $[\vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a} \;\; \vec{a}\times\vec{b}] = [\vec{a}\;\vec{b}\;\vec{c}]^2$

8. Define scalar triple product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. if $[\vec{a}\;\vec{b}\;\vec{c}]$ denotes scalar triple product then prove that

$$\vec{d} = \frac{[\vec{b}\;\vec{c}\;\vec{d}]}{[\vec{a}\;\vec{b}\;\vec{c}]}\vec{a} + \frac{[\vec{c}\;\vec{a}\;\vec{d}]}{[\vec{a}\;\vec{b}\;\vec{c}]}\vec{b} + \frac{[\vec{a}\;\vec{b}\;\vec{d}]}{[\vec{a}\;\vec{b}\;\vec{c}]}\vec{c}$$

$[\vec{a}\;\vec{b}\;\vec{c}] \neq 0.$

Sol^n

By definition of vector product of four vectors, we have,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\;\vec{b}\;\vec{d}]\vec{c} - [\vec{a}\;\vec{b}\;\vec{c}]\vec{d}$$

and $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{c}\;\vec{d}\;\vec{a}]\vec{b} - [\vec{c}\;\vec{d}\;\vec{b}]\vec{a}$

Therefore,

$$[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a}$$

$$= -[\vec{c}\ \vec{a}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

or, $$[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} + [\vec{c}\ \vec{a}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$$

$$\vec{d} = \frac{[\vec{b}\ \vec{c}\ \vec{d}]}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{a} + \frac{[\vec{c}\ \vec{a}\ \vec{d}]}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{b} + \frac{[\vec{a}\ \vec{b}\ \vec{d}]}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{c}$$

10. Find the volume of a parallelepiped whose concurrent edges are represented by $\vec{i}+\vec{j}+\vec{k}$, $\vec{i}-\vec{j}+\vec{k}$ and $\vec{i}+2\vec{j}-\vec{k}$.

Soln

let $\vec{a} = \vec{i}+\vec{j}+\vec{k}$, $\vec{b} = \vec{i}-\vec{j}+\vec{k}$ and $\vec{c} = \vec{i}+2\vec{j}-\vec{k}$.

Now, the volume of the parallelopiped whose sides are $\vec{a}$, $\vec{b}$ and $\vec{c}$ be

$$V = \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 1(1-2) - 1(-1-1) + 1(2+1)$$

$$= -1+2+3$$

$$= 4 \text{ cubic unit}$$

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11. Define vector triple product. If $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{c} = \vec{i} + 3\vec{j} - 2\vec{k}$. Find $(\vec{a} \times \vec{b}) \times \vec{c}$. Also verify that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

Given,

$$\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$$
$$\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$$
$$\vec{c} = \vec{i} + 3\vec{j} - 2\vec{k}$$

Now,

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & -3 & -2 \end{vmatrix}$$

$$= \vec{i}(-2-3) - \vec{j}(-4+1) + \vec{k}(-6-1)$$
$$= -5\vec{i} + 3\vec{j} - 7\vec{k} = (-5, 3, -7)$$

Then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}(2+3) - \vec{j}(-1+6) + \vec{k}(1+4)$$
$$= 5\vec{i} - 5\vec{j} + 5\vec{k} = (5, -5, 5)$$

Then,

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -5 & 5 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \vec{i}(10-15) - \vec{j}(-10-5) + \vec{k}(15+5)$$
$$= -5\vec{i} + 15\vec{j} + 20\vec{k} = (-5, 15, 20)$$

$$|(\vec{a}\times\vec{b})\times\vec{c}| = \sqrt{(-5)^2+(15)^2+(20)^2} = 5\sqrt{26}$$

Now, we have show that

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$$

$$\therefore \quad \vec{c}\cdot\vec{a} = (1,3,-2)\cdot(1,-2,-3)$$
$$= 1-6+6$$
$$= 1$$

$$\vec{c}\cdot\vec{b} = (1,3,-2)\cdot(2,1,-1)$$
$$= 2+3+2$$
$$= 7$$

Here

$$(\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a} = \cancel{1(1,-2,-3)-7(}$$
$$=1(2,1,-1) - 7(1,-2,-3)$$
$$= (2,1,-1)-(7,-14,-21)$$
$$= (-5,15,20)$$

$$= (\vec{a}\times\vec{b})\times\vec{c}$$

This verifies the scalar vector triple product

12. If $\vec{a} = \vec{i}+2\vec{j}+\vec{k}$, $\vec{b} = \vec{i}+\vec{j}+\vec{k}$, find unit vecto along $\vec{a}\times\vec{b}$.

Soln

$$\vec{a} = \vec{i}+2\vec{j}+\vec{k}$$
$$\vec{b} = \vec{i}+\vec{j}+\vec{k}$$

P.T.O

$$\vec a \times \vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \vec i(2-1) - \vec j(1-1) + \vec k(1-2)$$
$$= \vec i + \vec k$$

$$|\vec a \times \vec b| = \sqrt{1^2+1^2} = \sqrt2$$

Let $\hat n$ is unit vector along $\vec a \times \vec b$.

$$\hat n = \frac{\vec a \times \vec b}{|\vec a \times \vec b|} = \frac{\vec i - \vec k}{\sqrt2} = \frac{1}{\sqrt2}(\vec i - \vec k)$$

13. Define scalar triple product. If the vectors $2\vec i - \vec j + 2\vec k$, $5\vec i + \lambda\vec j + 2\vec k$ and $\vec i + 6\vec k$ are coplanar, find the value of λ.

<u>Sol</u>

~~Given that~~

Let $\vec a = (2,-1,2)$, $\vec b = (5,\lambda,2)$ and $\vec c = (1,0,6)$

Since $\vec a$, $\vec b$ and $\vec c$ are coplanar.

So,

$$\vec a \cdot (\vec b \times \vec c) = 0$$

$$\begin{vmatrix} 2 & -1 & 2 \\ 5 & \lambda & 2 \\ 1 & 0 & 6 \end{vmatrix} = 0$$

$$2(6\lambda - 0) + 1(30-2) + 2(0-\lambda) = 0$$

$$12\lambda + 28 - 2\lambda = 0$$
$$10\lambda = -28$$
$$\lambda = \frac{-28}{10} = -14/5$$

15. Find a set of reciprocal of given set of vectors $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = \vec{i} - \vec{j} - 2\vec{k}$
$\vec{c} = -\vec{i} + 2\vec{j} + 2\vec{k}$

Soln

$$\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$$
$$\vec{b} = \vec{i} - \vec{j} - 2\vec{k}$$
$$\vec{c} = -\vec{i} + 2\vec{j} + 2\vec{k}$$

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= \vec{i}(-6-1) - \vec{j}(-4+1) + \vec{k}(-2-3)$$
$$= -7\vec{i} + 3\vec{j} - 5\vec{k}$$

And,

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= \vec{i}(-2+4) - \vec{j}(2-2) + \vec{k}(2-1)$$
$$= 2\vec{i} + \vec{k}$$

Also,

$$\vec{c} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= i(-2-6) - j(1-4) + \vec{k}(-3-4)$$
$$= -8\vec{i} + 3\vec{j} - 7\vec{k}$$

Then,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= 2(-2+4) - 3(2-2) - 1(2-1)$$
$$= 4 - 0 - 1 = 3$$

Now, the reciprocal of $\vec{a}$, $\vec{b}$ and $\vec{c}$ are

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{2\vec{i} + \vec{k}}{3}$$

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{-8\vec{i} + 3\vec{j} - 7\vec{k}}{3}$$

$$\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{-7\vec{i} + 3\vec{j} - 5\vec{k}}{3}$$

16. Find the cosine angle between $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} - 2\vec{j} + 2\vec{k}$

Sol^n

$\vec{a} = \vec{i} - \vec{j} + \vec{k}$

$\vec{b} = \vec{i} - 2\vec{j} + 2\vec{k}$

$\vec{a} \cdot \vec{b} = 1 + 2 + 2 = 5$

$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$|\vec{b}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

Now,

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$5 = \sqrt{3} \cdot 3 \cos\theta$

$5 = 3\sqrt{3} \cos\theta$

$\cos\theta = \frac{5}{3\sqrt{3}}$

$\theta = \cos^{-1}\left(\frac{5}{3\sqrt{3}}\right)$

18. If $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$, find a unit vector perp. to both $\vec{a}$ and $\vec{b}$.

Sol^n

$\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \vec{i}(1+2) - \vec{j}(2+1) + \vec{k}(4-1)$$
$$= 3\vec{i} - 3\vec{j} + 3\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{3^2+3^2+3^2} = 3\sqrt{3}$$

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{1}{3\sqrt{3}}(3\vec{i} - 3\vec{j} + 3\vec{k})$$
$$= \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{k})$$

13. If $\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$ are any two vectors. Find $\vec{a}\times\vec{b}$ and check either $(\vec{a}\times\vec{b})\cdot\vec{a}$ is equal to $(\vec{a}\times\vec{b})\cdot\vec{b}$ or not? If yes, trace out the conclusion.

Soln

$$\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$$
$$\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \vec{i}(6+1) - \vec{j}(8-2) + \vec{k}(-4-6)$$
$$= 7\vec{i} - 6\vec{j} - 12\vec{k}$$

Now,

$$(\vec{a}\times\vec{b})\cdot\vec{a} = (7\vec{i} - 6\vec{j} - 12\vec{k})\cdot(4\vec{i} + 3\vec{j} + \vec{k})$$
$$= 28 - 18 - 12$$
$$= -2$$

$(\vec{a}\times\vec{b})\cdot\vec{b} = (7\vec{i}-6\vec{j}-12\vec{k})\cdot(2\vec{i}-\vec{j}+2\vec{k})$

$= 14+6-24$

$= 20-24$

$= -4$

20. Find the unit vector normal to the vectors given by $\vec{a} = 3\vec{i}-2\vec{j}+4\vec{k}$ and $\vec{b} = \vec{i}+\vec{j}-2\vec{k}$

Sol^n

$\vec{a} = 3\vec{i}-2\vec{j}+4\vec{k}$

$\vec{b} = \vec{i}+\vec{j}-2\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix}$$

$= \vec{i}(4-4) - \vec{j}(-6-4) + \vec{k}(3+2)$

$= 10\vec{j}+5\vec{k}$

$|\vec{a}\times\vec{b}| = \sqrt{10^2+5^2} = \sqrt{125} = 5\sqrt{5}$

Let $\hat{n}$ be unit vector along $\vec{a}\times\vec{b}$ then

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{10\vec{j}+5\vec{k}}{5\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}(2\vec{j}+\vec{k})$$

Q1. Verify that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

Here,

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})$$

$$=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\cdot\vec{c}+(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}+(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}$$

$$=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}+(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}+(\vec{b}\cdot\vec{c})\vec{a}-(\vec{a}\cdot\vec{c})\vec{b}$$

$$=\vec{0}$$

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