

ENGINEERING PHYSICS



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Energy of a Simple Harmonic oscillator / SHM

let us take a particle having mass m is in SHM with displacement x then;

$$\text{Potential Energy } (U) = \int -F \cdot dx$$

$$= \int_0^x (-kx) \cdot dx$$

$$= -\frac{1}{2} kx^2$$

$$(U) = \frac{1}{2} k A^2 \sin^2(\omega t + \phi) = 0$$

And,

$$K.E = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \left(\frac{dx}{dt} \right)^2$$

$$= \frac{1}{2} m \omega^2 A^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} k A^2 \cos^2(\omega t + \phi) \quad [\because k = m \omega^2]$$

$$\text{Now; The total energy (TE)} = P.E + K.E$$

$$= \frac{1}{2} k A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} k A^2$$

Q. Show that average k.E is equal to average p.E is equal to ~~one~~ $\frac{1}{2}$ of the total energy of a SHM.

Ans:- We have.

$$k.E = \frac{1}{2} kA^2 \cos^2(\omega t + \phi)$$

Average k.E is,

$$\begin{aligned} \langle k.E \rangle &= \frac{1}{T} \int_0^T (k.E) dt \\ &= \frac{1}{T} \int_0^T \frac{1}{2} kA^2 \cos^2(\omega t + \phi) dt \end{aligned}$$

For simplicity we assume $\phi = 0$

$$\begin{aligned} \therefore \langle k.E \rangle &= \frac{1}{T} \int_0^T \frac{1}{2} kA^2 \cos^2(\omega t) dt \\ &= \frac{1}{2T} \int_0^T \frac{kA^2}{2} \cos 2\omega t + \frac{1}{2} dt \\ &= \frac{1}{4T} kA^2 \int_0^T 1 + \cos 2\omega t dt \\ &= \frac{1}{4T} kA^2 \left[t + \frac{\sin 2\omega t}{2\omega} \right]_0^T \\ &= \frac{1}{4T} kA^2 \left[T + \frac{\sin 2\omega T - \sin 0}{2\omega} \right] \\ &= \frac{1}{4T} kA^2 \left[T + \frac{\sin 2\pi}{2\pi} \right] \\ &= \frac{KA^2}{4\pi} \left[\frac{T + 1}{2} \right] \\ &= \frac{KA^2}{4\pi} \left[\frac{T}{2} + \frac{1}{2} \right] \\ &= \frac{KA^2}{4\pi} \times \frac{2}{2} \\ &\equiv \frac{KA^2}{4} \rightarrow ① \end{aligned}$$

$$\because \langle \cos^2 \theta \rangle = \langle \sin^2 \theta \rangle = \frac{1}{2}$$

Similarly Average P.E is

$$\begin{aligned}\langle P.E \rangle &= \frac{1}{T} \int_0^T (P.E) dt \\ &= \frac{1}{T} \int_0^T \frac{1}{2} kA^2 \sin^2(\omega t + \phi) dt\end{aligned}$$

For simplicity we assume $\phi = 0$.

$$\begin{aligned}\langle P.E \rangle &= \frac{1}{T} \int_0^T \frac{1}{2} (1 - \cos 2\omega t) dt \\ &= \frac{kA^2}{2T} \left\{ \int_0^T \frac{1}{2} dt - \int_0^T \frac{\cos 2\omega t}{2} dt \right\} \\ &= \frac{kA^2 \times T}{4T} \\ &= \frac{kA^2}{4} \quad \text{--- (i)}$$

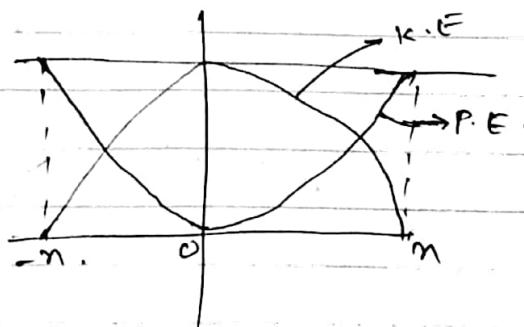
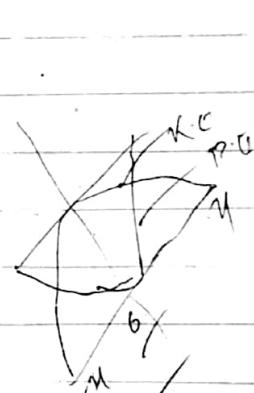
$$\langle K.E \rangle = \langle P.E \rangle = \frac{1}{2} \langle T.E \rangle$$

$$\text{or}, \quad \frac{kA^2}{4} = \frac{kA^2}{4} = \frac{1}{2} \times \frac{1}{2} kA^2$$

$$\therefore \frac{kA^2}{4} = \frac{kA^2}{4} = \frac{kA^2}{4} \quad \text{proved.}$$

Examples of SHM

① Simple Pendulum.



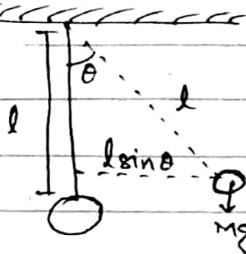
Simple Pendulum.

Let us take a bob having point mass 'm' is oscillating about a horizontal axis with angular displacement ' θ ' and moment of inertia about the point of suspension is $I = ml^2$ where;

l is the length of suspension.

Then the torque acting on the bob is:

$$\tau = I\alpha = I \frac{d^2\theta}{dt^2} \quad \text{--- (i)}$$



and restoring torque/force = force \times \perp^{er} distance
 $= -mgl \sin \theta \quad \text{--- (ii)}$

equating eqn (i) & (ii).

$$\Rightarrow I \frac{d^2\theta}{dt^2} = -mgl \sin \theta$$

For small angle $\sin \theta \approx \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{mgl}{I} \theta = 0$$

Comparing above eqn with general differential eqn of SHM,

$$\frac{mgl}{I} = \omega^2$$

$$\Rightarrow T = d\theta \sqrt{\frac{I}{mgl}} = d\theta \sqrt{\frac{mgl^2}{mgl}}$$

$$\therefore T = d\theta \sqrt{\frac{l}{g}}$$

d. Torsional Pendulum.

Let us take a torsional disc having moment of inertia 'I' is attached with metallic wire having length 'l' and modulus of rigidity of wire is 'M'. Then after the disc is rotated such that the wire is twisted and we calculate the time period. Let ' θ ' be the twisting angle then torque acting is

$$\tau \propto \theta$$

$$\text{or, } \tau = -c\theta \quad \text{--- (i)}$$

where $c = \frac{\pi M r^4}{8l}$ is the torsional constant. Also, r is the radius of metallic wire.

We have;

$$\tau = I\alpha = I \frac{d^2\theta}{dt^2} \quad \text{--- (ii)}$$

Equating eqⁱ & (ii) we get;

$$I \frac{d^2\theta}{dt^2} + c\theta = 0 \Rightarrow \frac{d^2\theta}{dt^2} + \frac{c\theta}{I} = 0$$

Comparing above equation with general diffⁿ eqⁿ of SHM we get,

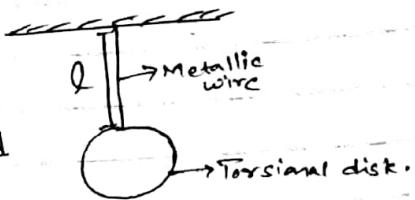
$$\omega^2 = \frac{c}{I}$$

$$\text{We know; } \omega = \frac{\alpha}{T}$$

$$\text{or, } \sqrt{\frac{c}{I}} = \frac{\alpha}{T}$$

$\therefore T = \alpha \sqrt{\frac{I}{c}}$ is the required time period for the torsional pendulum

This shows that time period of torsional pendulum doesn't depend on the accⁿ due to gravity.



Let us take a ring having moment of inertia 'I' is kept above the disc and the metallic wire is twisted then the time period is given by:

$$T_1 = 2\pi \sqrt{\frac{I+I_1}{C}}$$

→ Applications

(i) Calculation of modulus of rigidity of wire.

Now, the time period of torsional pendulum is

$$T = 2\pi \sqrt{\frac{I}{C}} \quad \text{when ring is kept above the torsional} \quad \textcircled{1}$$

Pendulum then the time period is given by

$$T_1 = 2\pi \sqrt{\frac{I+I_1}{C}} \quad \textcircled{1}$$

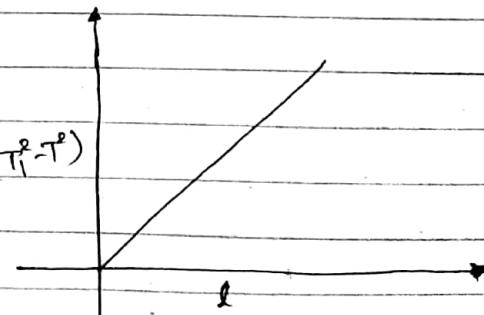
Comparing Squaring eqn ① & ② and subtracting ① from ② we get;

$$T_1^2 - T^2 = 4\pi^2 \left[\frac{I+I_1}{C} - \frac{I}{C} \right]$$

$$\Rightarrow T_1^2 - T^2 = 4\pi^2 \frac{I_1}{C}$$

$$\text{or, } T_1^2 - T^2 = \frac{4\pi^2 I_1}{\left(\frac{\pi M r^4}{8l} \right)}$$

$$\therefore \eta = \frac{8\pi I_1 l}{(T_1^2 - T^2) r^4} \text{ which is}$$



the modulus of rigidity of metallic wire at
Unit of η is dyne/cm² constant Temperature.

(ii) To find the Moment of inertia of non-uniform irregular rigid body.

We have the time period for torsional pendulum is

$$T = da \sqrt{\frac{I}{C}} \quad \text{--- (i)}$$

When ring is kept above the torsional pendulum time period is given by

$$T_1 = da \sqrt{\frac{I+I_1}{C}} \quad \text{--- (ii)}$$

Let us take a non-uniform rigid body having moment of Inertia ' I ' is kept on the disc and twisted such that the time period is

$$T_2 = da \sqrt{\frac{I+I_2}{C}} \quad \text{--- (iii)}$$

Squaring (i), (ii) & (iii) and divided by (ii) if (iii) by (i)

$$\Rightarrow \frac{T_1^2}{T^2} = \frac{I+I_1}{I} \quad \text{and}$$

$$\frac{T_2^2}{T^2} = \frac{I+I_2}{I}$$

$$\frac{T_1^2 - T_2^2}{T^2} = \frac{I_1}{I} \quad \text{--- (iv)}$$

$$\text{and } \frac{T_2^2 - T_1^2}{T^2} = \frac{I_2}{I} \quad \text{--- (v)}$$

Dividing eqn (iv) by (v)

$$\frac{\frac{T_1^2 - T^2}{T^2}}{\frac{T_2^2 - T^2}{T^2}} = \frac{I_1}{I_2}$$

$$\Rightarrow I_2 = I_1 \left(\frac{T_2^2 - T^2}{T_1^2 - T^2} \right) \quad \text{--- (vi)}$$

Eqn (vi) gives the moment of inertia of non-uniform rigid body.

(3) Physical Pendulum:-

A rigid body which is capable to oscillates about a horizontal axis is called physical pendulum or compound pendulum.

Let us take a rigid body which is capable to oscillates about a horizontal axis with moment of inertia I as shown in figure. 'S' is the point of suspension, 'G' is the point of CG. The distance betⁿ point of suspension and centre of gravity is called length of suspension (l). When it is oscillate at an angle ' θ ' Then the restoring torque act on the pendulum is

$$T = -mgl \sin\theta \quad \text{--- (i)}$$

We have;

$$T = I\alpha = \frac{Id^2\theta}{dt^2} \quad \text{--- (ii)}$$

Comparing Equating eqn (i) & (ii)

$$\frac{Id^2\theta}{dt^2} + mgl \sin\theta = 0$$

$$\therefore \frac{d^2\theta}{dt^2} + \frac{mgl}{I} \theta = 0 \quad (\text{For small angle } \sin\theta \approx \theta)$$

Comparing it with SHM differential equation we get;

$$\omega^2 = \frac{mgl}{I}$$

$$\text{We know, } \omega = \frac{\alpha}{T}$$

$$\therefore T = \frac{d\theta}{\sqrt{\frac{mgl}{I}}} = d\theta \sqrt{\frac{I}{mgl}}$$

Now, the moment of inertia of the compound pendulum is
 $I = mk^2 + ml^2$ where; k is the radius of gyration in which the time period of compound pendulum is minimum.

Then the time period is;

$$T = 2\pi \sqrt{\frac{mk^2 + ml^2}{mgl}}$$

$$\therefore T = 2\pi \sqrt{\frac{k^2 + l^2}{gl}}$$

$$\therefore T = 2\pi \sqrt{\frac{k^2 + l^2}{g}}$$

(4) Bar Pendulum:

It is the type of physical pendulum.

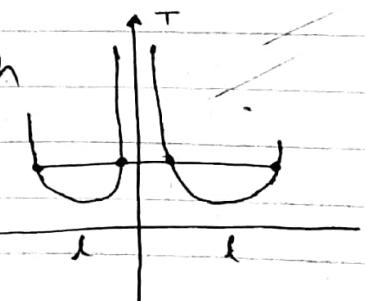
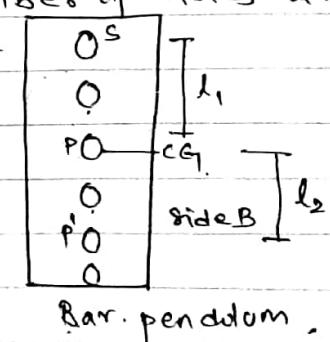
It is the metallic bar in which large number of holes are created which is at equidistance from each other.

Let S be the point of suspension, (P) be the point of centre of gravity. The distance between point of suspension and centre of gravity is called length of suspension. Let us take another point (P') which is at distance

$\frac{k^2}{l}$, from the centre of gravity is called length of oscillation. Then the time period of oscillation is:

$$T = 2\pi \sqrt{\frac{k^2/l + l^2}{g}}$$

where k is a radius of gyration i.e. the perpendicular distance from the centre of mass to axis of rotation.



Case I: Show that there are four collinear points in Bar pendulum.

Let us take a side A of bar pendulum in which the length of suspension is l_1 , then the time period is given by

$$T = 2\pi \sqrt{\frac{k^2}{l_1} + l_1}$$

$$\Rightarrow T^2 = 4\pi^2 \left(\frac{k^2 + l_1^2}{gl_1} \right)$$

$$\Rightarrow \frac{l_1^2 - gl_1 T^2}{4\pi^2} + k^2 = 0.$$

$$\therefore l_1 = \frac{gT^2}{4\pi^2} \pm \sqrt{\left(\frac{gT^2}{4\pi^2}\right)^2 - 4k^2}$$

It shows that there are two length in same time period for side A. Similarly for side B we can obtain that there are two length in same time period. From this relation we can say that there are four collinear points in Bar pendulum.

Case II: To find the radius of gyration,

let l_1 be the length of suspension and $l_2 = k^2/l_1$ is the length of oscillation. Then the time period of oscillation is

$$T = 2\pi \sqrt{\frac{k^2}{l_1} + l_1}$$

$$\text{Now; } \frac{k^2 + l_1}{l_1} = l_2 + l_1$$

$$\Rightarrow k^2 = l_1 l_2$$

$$\Rightarrow k = \sqrt{l_1 l_2}$$

Case III: To find the condition in which time period is maximum & minimum.

We have:

$$T = 2\pi \sqrt{\frac{k^2 + l_1}{l_1 g}}$$

The term $\frac{k^2 + l_1}{l_1}$ represent the length of the pendulum. The

time period is maximum when $\frac{k^2 + l_1}{l_1}$ is maximum. It is possible

only when $l_1 = \infty$ or 0 but practically infinite length of suspension is not possible such that the time period is maximum when $l_1 = 0$ i.e. when length of suspension is 0. It means length of suspension and CG coincide with each other.

The time period is minimum when $\frac{k^2 + l_1}{l_1}$ is minimum.

$$\frac{dT}{dl_1} = \frac{d(k^2/l_1 + l_1)}{dl_1} = 0$$

$$\text{or, } -\frac{k^2}{l_1^2} + 1 = 0$$

$$\Rightarrow k^2 = l_1^2$$

$$\Rightarrow k = l_1 \quad \text{--- (1)}$$

Again we have $k = \sqrt{l_1 l_2}$

$$\Rightarrow l_1 l_2 = l_1^2$$

$$\Rightarrow l_2 = l_1 \quad \text{--- (2)}$$

from eqn (1) it shows that the time period is minimum when length of suspension is equal to radius of gyration or from eqn (2) it shows that the time period is minimum when length of suspension is equal to length of oscillation.

Case IV: Show that point of suspension and point of oscillation are interchangeable.

Let l_1 be the length of suspension and l_2 be the length of oscillation for side A, then time period.

$$T_1 = 2\pi \sqrt{\frac{k^2 + l_1}{\frac{l_1}{g}}} \quad \text{--- (i)}$$

If we interchange the point of suspension and point of oscillation then the time period are

$$T_2 = 2\pi \sqrt{\frac{k^2 + l_2}{\frac{l_2}{g}}} \quad \text{--- (ii)}$$

We have;

$$k^2 = l_1 l_2$$

Putting this value in eqn (i) & (ii) we get;

$$T_1 = 2\pi \sqrt{\frac{l_1 l_2 + l_1}{\frac{l_1}{g}}} = 2\pi \sqrt{\frac{l_2 + l_1}{\frac{l_1}{g}}}$$

$$T_2 = 2\pi \sqrt{\frac{l_1 + l_2}{\frac{l_2}{g}}}$$

$$\therefore T_1 = T_2$$

Hence the point of suspension and point of oscillation can be interchanged.

Types of oscillation.

→ free oscillation:

The oscillation in which the total energy of amplitude remain constant is called free oscillation. It exists only in vacuum. In free oscillation resistive force does not exist in that medium.

Now, the k.E of oscillation having displacement x is

$$k.E = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2$$

Potential energy (P.E) = $\frac{1}{2} k x^2$

∴ Total energy = $\frac{1}{2} k x^2 + \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 = \text{constant}$.

Differentiating w.r.t time on both sides we get;

$$\frac{1}{2} k \cancel{m} \frac{d^2x}{dt^2} + \frac{1}{2} m \cancel{k} \frac{dx}{dt} \left(\frac{d^2x}{dt^2} \right) = 0$$

$$\Rightarrow kx + m \left(\frac{d^2x}{dt^2} \right) = 0$$

$$\Rightarrow m \frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega^2 = k/m$$

We have $\omega = \sqrt{k/m}$

∴ $f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$ is the reqd freq² of free oscillation.

Pix 9/

2) Damped oscillation/vibration.

The oscillation in which amplitude or energy decreases with increase in time due to internal and external resistance factor is called damped oscillation. The external force does not applied on the damped oscillation.

The resisting force is $-b \frac{dn}{dt}$ where b is a damping force constant and n is the displacement at any instant of time t . The SI unit of damping force constant (b) is kg/s.

Let us take a particle having mass (m) is in damping oscillation/vibration, n be the displacement at any instant of time, then

$$F = -kn - b \frac{dn}{dt}$$

$$\Rightarrow m \frac{d^2n}{dt^2} + kn + b \frac{dn}{dt} = 0$$

$$\Rightarrow \frac{d^2n}{dt^2} + \frac{b}{m} \frac{dn}{dt} + \frac{k}{m} n = 0 \quad \text{--- (1)}$$

Let $n = A_0 e^{rt}$ is the general solution of eqⁿ 1 that it should satisfy the eqⁿ 1

$$\Rightarrow \frac{d(A_0 e^{rt})}{dt^2} + \frac{b}{m} \frac{d(A_0 e^{rt})}{dt} + \frac{k}{m} A_0 e^{rt} = 0$$

$$\Rightarrow A_0 r^2 e^{rt} + \frac{b}{m} A_0 r e^{rt} + \frac{k}{m} A_0 e^{rt} = 0$$

$$\Rightarrow r^2 + \frac{b}{m} r + \frac{k}{m} = 0$$

$$\Rightarrow m r^2 + b r + k = 0 \quad \text{which is quadratic in } r, \text{ then -}$$

$$\therefore r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

$$= \frac{-b}{2m} \pm i \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Then $n = A_0 e^{-\frac{b}{2m}t}$

$$= A_0 e^{-\frac{b}{2m}t} \cdot e^{\pm(i\sqrt{k/m - b^2/4m^2})t}$$

$$= A_0 e^{-\frac{b}{2m}t} \cos \omega' t (\text{or } \sin \omega' t) \quad \text{(ii)}$$

where $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ is the angular freq² of damped

oscillation.

⇒ frequency of damped oscillation

$$f' = \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \left[\begin{array}{l} \text{where } k \text{ is force constant} \\ b \text{ is resistive force constant} \end{array} \right] \quad \text{(iii)}$$

Eqn (ii) shows that the amplitude of damped oscillation decreases exponentially with time, where A_0 is the initial amplitude.

Case I:

When $k/m = b^2/4m^2$ this condition is called critical damping in which the frequency of oscillation is suddenly die out.

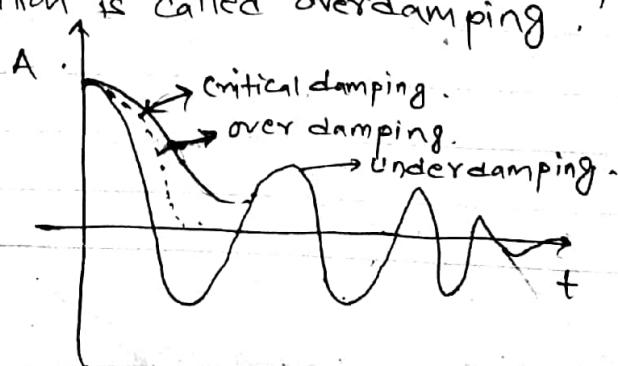
Case II:

When $k/m > b^2/4m^2$ this condition is called underdamping in which the oscillation is die out after a long time.

Case III:

When $k/m < b^2/4m^2$ then the ~~second~~ the frequency of damped oscillation is imaginary which condition is called overdamping.

anti-damping



The energy of damped oscillation is

$$E = \frac{1}{2} k A_0^2 e^{-b/m t}$$

Quality factor of damped oscillation.

Quality factor is the measurement of sharpness of resonance. In other forms;

$$\text{Quality factor (Q)} = \frac{\text{Energy stored}}{\text{Energy lost per cycle}}$$

$$= \frac{1}{2\pi} \frac{\text{Energy stored}}{\text{Power loss} \times \text{Time period}}$$

Now;

$$\text{Power loss} = -\frac{dE}{dt}$$

$$\text{We have, } E = \frac{1}{2} k A_0^2 e^{-b/m t}$$

$$\frac{dE}{dt} = \frac{1}{2} k A_0^2 \left(-\frac{b}{m} \right) e^{-b/m t}$$

$$\therefore \frac{dE}{dt} = -\frac{b}{m} E$$

$$\therefore \text{Power loss (P)} = -\frac{dE}{dt} = \frac{b}{m} E$$

$$\therefore Q = \frac{\frac{1}{2} k A_0^2}{\frac{b}{m} E T} = \frac{k A_0^2}{T b} = \frac{\omega m}{b}$$

which is the quality factor of damped oscillation shows that quality factor is inversely proportional to resistive force constant (b)

Forced / Driven oscillation

When external periodic force is applied to remove or reduce the damping nature of oscillation is called force or driven oscillation.

The general differential equation of forced / driven oscillation is

$$m \frac{d^2n}{dt^2} + b \frac{dn}{dt} + kn = F_0 \sin \omega' t$$

where b is resistive force constant. F_0 is applied force & $\omega' = 2\pi f'$ where f' is the applied frequency. n is the displacement of particle having mass m at any instant of time t .

$$\Rightarrow \frac{d^2n}{dt^2} + \frac{b}{m} \frac{dn}{dt} + \frac{k}{m} n = \frac{F_0}{m} \sin \omega' t \quad \text{--- (1)}$$

It is inhomogeneous equation such that let $n = A \sin(\omega t - \phi)$ is the general solution of eqn (1) where ~~ω is the natural freq of the body then~~

$$\frac{dn}{dt} = Aw \cos(\omega t - \phi)$$

$$\frac{d^2n}{dt^2} = -Aw^2 \sin(\omega t - \phi)$$

$$\Rightarrow -Aw^2 \sin(\omega t - \phi) + \frac{b}{m} Aw \cos(\omega t - \phi) + \frac{k}{m} A \sin(\omega t - \phi) = \frac{F_0}{m} \sin(\omega' t - \phi + \phi)$$

$$\Rightarrow \sin(\omega' t - \phi) \left[\frac{A k}{m} - Aw^2 \right] + \frac{b}{m} Aw \cos(\omega t - \phi) = \frac{F_0}{m} \left[\sin(\omega' t - \phi) \cos \phi + \cos(\omega' t - \phi) \sin \phi \right]$$

To hold this equation the coefficient of $\cos(\omega' t - \phi)$ & $\sin(\omega' t - \phi)$ should be equal

$$\Rightarrow \frac{A k}{m} - Aw^2 = \frac{F_0}{m} \cos \phi \quad \text{--- (ii)}$$

$$\frac{b}{m} Aw = \frac{F_0}{m} \sin \phi \quad \text{--- (iii)}$$

Squaring & adding ⑪ & ⑫ we get:

$$\frac{f_0^2}{m^2} (\cos^2 \phi + \sin^2 \phi) = \left[\frac{b^2}{m^2} \omega'^2 + \left(\frac{k}{m} - \omega'^2 \right)^2 \right] A^2$$

$$\Rightarrow A^2 = \frac{f_0^2}{m^2 \left[\frac{b^2}{m^2} \omega'^2 + \left(\frac{k}{m} - \omega'^2 \right)^2 \right]}$$

$$\Rightarrow A = \frac{f_0}{\sqrt{m^2 (\omega_0^2 - \omega^2)^2 + \frac{b^2}{m^2} \omega^2}} \quad \text{where } \omega_0^2 = k/m$$

which is the amplitude of forced oscillation. When $\omega_0 = \omega'$ then the amplitude is maximum which is the condition of resonance.

f_r is resonance frequency.

f_1 is lower cut off frequency. Amplitude

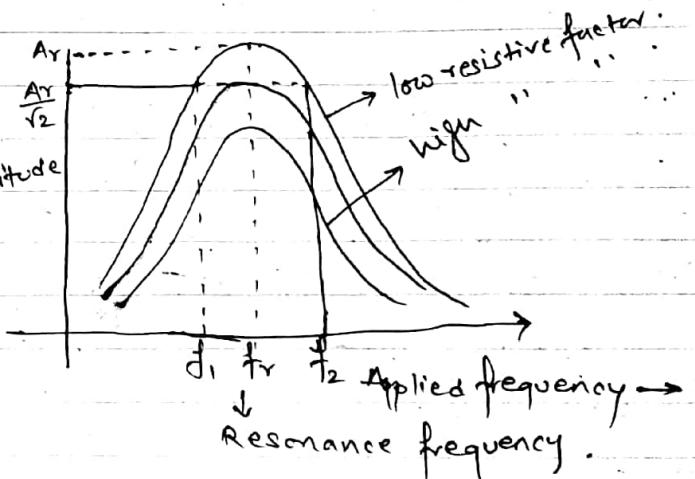
f_2 is higher cut off frequency.

Now,

The Quality factor of forced oscillation is:

$$Q = \frac{\omega_r f_r}{f_2 - f_1}$$

band width.



Electromagnetic oscillation.

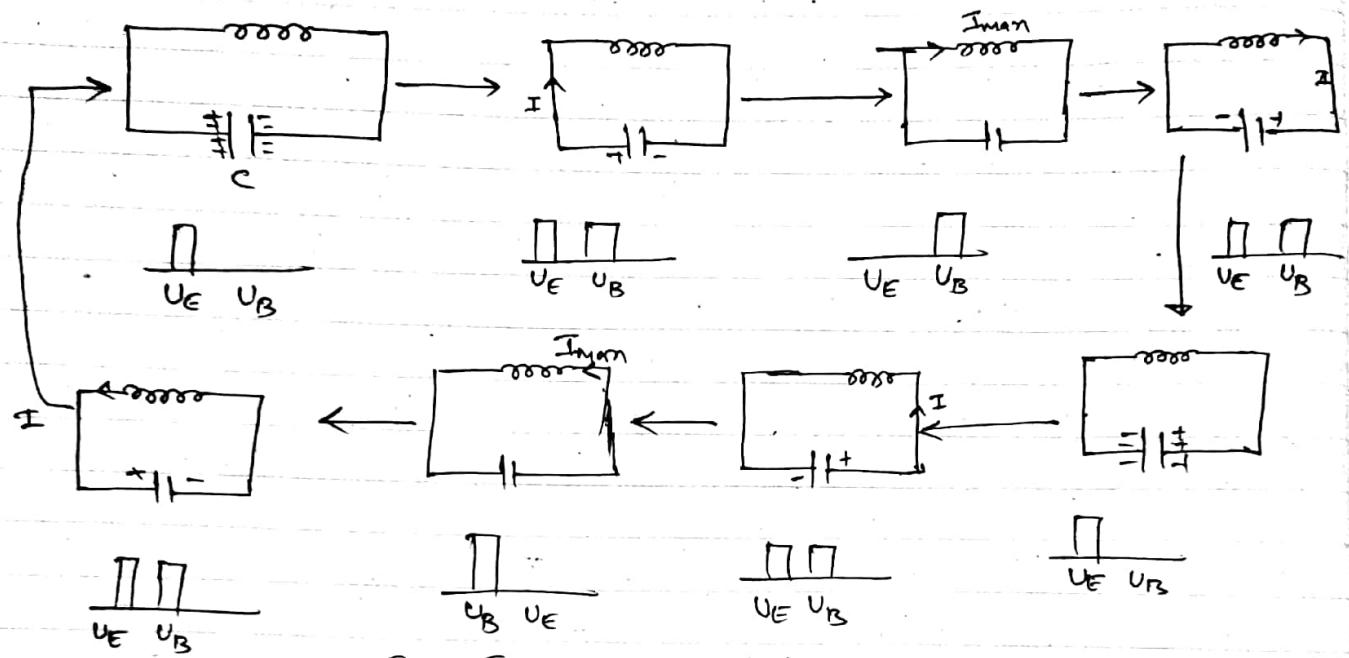


Fig: Free LC oscillation.

→ **Capacitor:-** It is a device which is used to stored the energy in the form of electrical energy. The energy stored in the capacitor is

$$U_E = \frac{1}{2} CV^2 = \frac{Q^2}{2C} \quad [\because Q = CV]$$

where Q is the charge of capacitor at any instant of time t .

The capacity to stored the energy of the capacitor is called capacitance of the capacitor. Unit of capacitance is farad (F)

→ Inductor:-

It is a device which is used to stored the energy in the form of magnetic energy and it is also used to control the current. The energy stored in the inductor having inductance L is

$$U_B = \frac{1}{2} L I^2$$

Unit of inductor henry (H).

→ Free LC oscillation.

Let us take a capacitor having capacitance 'C' is fully charged initially then it is connected in series with inductor having inductance (L) then the capacitor starts to discharge and the oscillation occurs as shown in figure.

It is assumed that there is no any resistive or energy loss factor. Now, the energy stored in the capacitor at any instant of time t is

$$U_E = \frac{Q^2}{2C}$$

and Energy stored in the inductor is $U_B = \frac{1}{2} L I^2$.

Since there is no any energy loss factor, total energy remains constant.

$$\therefore U_E + U_B = \text{constant}.$$

$$\frac{Q^2}{2C} + \frac{1}{2} L I^2 = \text{constant} \quad \text{--- (1)}$$

Diffr. (1) w.r.t time,

$$\frac{1}{2C} \cdot \frac{dQ}{dt} \cdot \frac{dQ}{dt} + \frac{1}{2} L \cdot \frac{dI}{dt} \cdot \frac{dI}{dt} = 0$$

$$\Rightarrow \frac{Q}{C} \frac{dQ}{dt} + \frac{LI}{2} \frac{dI}{dt} = 0$$

$$> \frac{Q}{C} \frac{dQ}{dt} + L \cdot \frac{dQ}{dt} \cdot \frac{d^2Q}{dt^2} = 0 \quad [\because I = \frac{dQ}{dt}]$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0 \quad \textcircled{1}$$

Let the general soln of eqn 11 be

$$Q = Q_0 \cos(\omega t + \phi)$$

$$\frac{dQ}{dt} = -\omega Q_0 \sin(\omega t + \phi)$$

$$\therefore \frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos(\omega t + \phi).$$

Then eqn 11 becomes,

$$-\omega^2 Q_0 \cos(\omega t + \phi) + \frac{Q_0 \cos(\omega t + \phi)}{LC} = 0$$

$$\text{or}, \quad -\omega^2 + \frac{1}{LC} = 0$$

$$\text{or}, \quad \omega^2 = \frac{1}{LC}$$

$$\text{or}, \quad f = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

$$\therefore f = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

$$\text{Now; } I_{\text{max}} = \left(\frac{dQ}{dt} \right)_{\text{max}} = \omega Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{CV}{\sqrt{LC}} \quad \text{--- } \textcircled{2}$$

Then energy stored in the capacitor.

$$U_E = \frac{1}{2} \frac{Q^2}{C} = \frac{Q_0^2 \cos^2(\omega t + \phi)}{2C}$$

$$\Rightarrow U_{E,\text{max}} = \frac{Q_0^2}{2C} \quad \text{--- } \textcircled{3}$$

$$\langle U_E \rangle = \frac{Q_0^2}{4C} \quad [\because \langle \cos^2(\omega t + \phi) \rangle = \frac{1}{2}]$$

Energy stored in the inductor.

$$U_B = \frac{1}{2} L I^2$$

$$= \frac{1}{2} L \left(\frac{dQ}{dt} \right)^2$$

$$= \frac{1}{2} L \cdot \left[-\omega Q_0 \sin^2(\omega t + \phi) \right]'$$

$$= \frac{1}{2} L \omega^2 Q_0^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} L \cdot \frac{1}{LC} Q_0^2 \sin^2(\omega t + \phi) \quad (\because \omega^2 = 1/LC).$$

$$= \frac{Q_0^2}{2C} \sin^2(\omega t + \phi)$$

$$\therefore U_{B,\text{max}} = \frac{Q_0^2}{2C} \quad \text{--- } \textcircled{V}$$

$$\Rightarrow \langle U_B \rangle = \frac{Q_0^2}{4C} \quad [\because \langle \sin^2(\omega t + \phi) \rangle = \frac{1}{2}]$$

Now, Total energy,

$$U = U_E + U_B$$

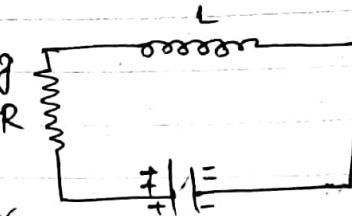
$$= \frac{Q_0^2}{4C} \cos^2(\omega t + \phi) + \frac{Q_0^2}{2C} \sin^2(\omega t + \phi)$$

$$\therefore U = \frac{Q_0^2}{2C} \quad \text{--- } \textcircled{W}$$

It shows that from above equation the maximum energy stored in the capacitor is i.e. maximum electric energy stored is equal to maximum energy stored in the inductor i.e. magnetic energy stored which is equal to the total energy. The average electric energy stored in capacitor is equal to average energy stored in the inductor which is equal to $\frac{1}{2}$ of the total energy.

Damped LCR oscillation:-

Let us take a capacitor having capacitance C is fully charged initially connected in series with resistor having resistance R and inductor having inductance L as shown in figure. Then the capacitor starts to discharge and oscillation occurs but due to the presence of resistance the energy is losted with increase in time such that it is called damped oscillation.



Now, the voltage dropped across R is IR

$$\text{and at capacitor } V = \frac{Q}{C}$$

$$\therefore L \frac{dI}{dt} + IR + \frac{Q}{C} = 0 \text{ at any instant of time } t;$$

$$\Rightarrow L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = 0$$

Let $Q = Q_0 e^{\lambda t}$ be the general soln of eqn ①, where λ is arbitrary constant which is to be determine

$$\Rightarrow Q_0 \lambda^2 e^{\lambda t} + \frac{R}{L} Q_0 \lambda e^{\lambda t} + \frac{Q_0 e^{\lambda t}}{LC} = 0$$

$$\Rightarrow \lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC} = 0$$

$$\text{or, } LC\lambda^2 + RC\lambda + 1 = 0$$

$$\therefore \lambda = \frac{-RC \pm \sqrt{R^2 C^2 - 4LC}}{2LC} = -\frac{R}{2L} \pm i \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

ENGINEERING PHYSICS



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Prepared By :
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2072 batch

$$\text{Then } Q = Q_0 e^{\pm \omega t}$$

$$= Q_0 e^{-R/2L} e^{\pm i\sqrt{1/C - R^2/4L^2}t}$$

$$= Q_0 e^{-R/2L} e^{\pm (i\sqrt{1/C - R^2/4L^2})t}$$

$$= Q_0 e^{-R/2L} \cos \omega' t \quad \text{--- (1)}$$

$$\text{where } \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\Rightarrow f' = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\Rightarrow T' = \frac{1}{f'}$$

Eqn (1) is the general solution of damped LCR oscillation.

Case I:

When $\sqrt{1/C} > R/2L$, the soln is oscillatory.

Case II: $\frac{1}{LC} = \frac{R^2}{4L^2}$ the soln is oscillatory but suddenly die-out.

Case III:

When $\sqrt{1/C} < R/2L$, the soln is non-oscillatory.

Q1. A ckt has $L = 1.2 \text{ mH}$, $C = 1.6 \mu\text{F}$, $R = 1.5 \Omega$

a) After what time t will the amplitude of the charge oscillation droped through one half of its initial value. and do how many periods of oscillation does this corresponds?

Q2. A $2\mu\text{F}$ capacitor is charged upto 100V then the battery is disconnected and the capacitor is connected to the 60mH with across LC oscillation. Calculate the maximum current, total energy and average magnetic energy stored in the inductor.

$$m = A \sin(\omega t - \phi)$$

(+) Solution

$$\text{Given } L = 1.2 \text{ mH} = 1.2 \times 10^{-3} \text{ H}$$

$$C = 1.6 \mu F = 1.6 \times 10^{-6} F$$

$$R = 1.5 \Omega$$

Now; We have;

$$Q = Q_0 e^{-\frac{Rt}{2L}}$$

By question

$$\frac{dQ}{dt} = Q_0 e^{-\frac{Rt}{2L}} \cdot \frac{-R}{2L}$$

$$\text{or, } \frac{1}{Q} = e^{-\frac{Rt}{2L}}$$

$$\text{or, } \ln(Q) = -\frac{Rt}{2L} \ln e$$

$$\text{or, } -0.693 = -\frac{1.5 \times t}{1.2 \times 10^{-3}} \times 1$$

$$\therefore t = 1.1088 \times 10^{-3} \text{ sec.}$$

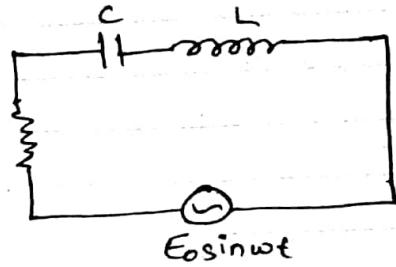
Now;

$$\begin{aligned} \text{We have; } f' &= \sqrt{\frac{1}{LC} - \frac{R^2}{Q_L^2}} \cdot \frac{1}{2\pi} \\ &= \sqrt{\frac{1}{1.2 \times 10^{-3} \times 1.6 \times 10^{-6}} - \frac{(1.5)^2}{9 \times (1.2 \times 10^{-3})^2}} \times \frac{1}{2\pi} \end{aligned}$$

H

Forced / driven LCR oscillation.

Let us take a resistance 'R'
inductance 'L' and capacitor having
Capacitance 'C' is connected in series.
with AC source having frequency 'ω'.



Then:

$$\frac{L dI}{dt} + IR + \frac{Q}{C} = E_0 \sin \omega t \quad (\text{where } \omega = 2\pi f, f \text{ is applied frequency})$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = \frac{E_0}{L} \sin \omega t \quad \text{--- (1)}$$

Also, $\frac{dQ}{dt} = I_0 \cos(\omega t - \phi)$ where $I_0 = Q_0 \omega$.

Let $Q = Q_0 \sin(\omega t - \phi)$ be the general solution of eqⁿ (1)

Then,

$$\Rightarrow -Q_0 \omega^2 \sin(\omega t - \phi) + \frac{R Q_0 \omega \cos(\omega t - \phi)}{L} + \frac{Q_0 \sin(\omega t - \phi)}{LC} = \frac{E_0 \sin \omega t}{L}$$

$$\Rightarrow -Q_0 \omega^2 \sin(\omega t - \phi) + \frac{R Q_0 \omega \cos(\omega t - \phi)}{L} + \frac{Q_0 \sin(\omega t - \phi)}{LC} = \frac{E_0 \sin(\omega t - \phi + \alpha)}{L}$$

$$\Rightarrow -Q_0 \omega^2 \sin(\omega t - \phi) + \frac{R Q_0 \omega \cos(\omega t - \phi)}{L} + \frac{Q_0 \sin(\omega t - \phi)}{LC} = \frac{E_0}{L} [\sin(\omega t - \phi) \cos \alpha + \cos(\omega t - \phi) \sin \alpha]$$

To hold this equation coefficient of $\cos(\omega t - \phi)$ and $\sin(\omega t - \phi)$ must be equal

$$-Q_0 \omega^2 + \frac{Q_0}{LC} = \frac{E_0 \cos \alpha}{L} \quad \text{--- (1)}$$

$$\frac{Q_0 R \omega}{L} = \frac{E_0 \sin \alpha}{L} \quad \text{--- (ii).}$$

Squaring eqⁿ (1) + (ii) & adding (1) + (ii) we get:

$$\left(\frac{Q_0}{LC} - Q_0 \omega^2 \right)^2 + \left(\frac{Q_0 R \omega}{L} \right)^2 = \frac{E_0^2}{L^2}$$

$$Q_0^2 \left(\left(\frac{1}{LC} - \omega^2 \right)^2 + \frac{\omega^2 R^2}{L^2} \right) = \frac{E_0^2}{L^2}$$

$$\Rightarrow Q_0 = \frac{E_0 / L}{\sqrt{\frac{R^2 \omega^2}{L^2} + \left(\frac{1}{LC} - \omega^2 \right)^2}}$$

$$\left(\frac{1}{\omega C} - \frac{1}{\omega} \right)^2$$

Now current amplitude;

$$\begin{aligned} I_0 &= Q_0 \omega \\ // \quad I_0 &= \frac{E_0 \cdot \omega}{L} = \frac{E_0 \cdot \omega}{\sqrt{\frac{R^2 \omega^2}{L^2} + \left(\frac{1}{LC} - \omega^2 \right)^2}} = \frac{E_0 \cdot \omega}{\sqrt{R^2 + \left(\frac{1}{LC} - \omega^2 \right)^2 / \frac{L^2}{\omega^2}}} \\ \Rightarrow I_0 &= \frac{E_0}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L \right)^2}} \\ // \quad I_0 &= \frac{E_0}{\sqrt{R^2 + (X_L - X_C)^2}} \end{aligned}$$

where $X_L = \omega L = 2\pi f L$ is inductive reactance

$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$ is capacitive reactance

$\therefore Z = \sqrt{R^2 + (X_L - X_C)^2}$ is impedance of LCR circuit.

When $X_L = X_C$ then the amplitude ie. current is maximum which is called the resonance current. In resonance the natural frequency of body coincide with the applied frequency.

At resonance

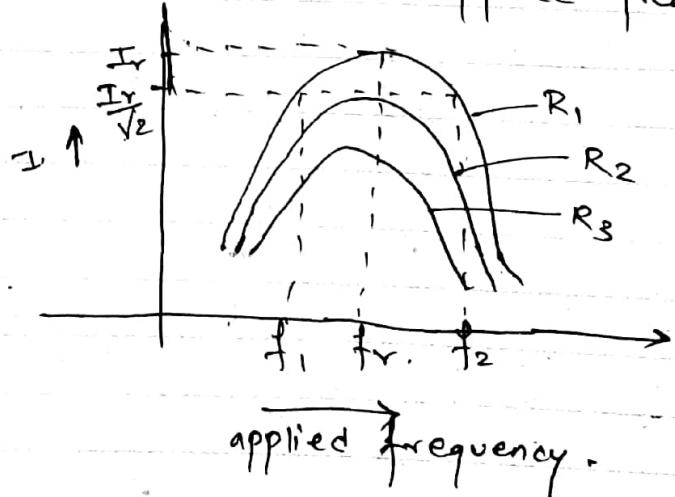
$$X_L = X_C$$

$$\Rightarrow \omega L = \frac{1}{\omega C}$$

$$\therefore f_{res} = \frac{1}{2\pi\sqrt{LC}} = f_r.$$

Quality factor.

$$Q = \frac{2\pi f_r}{f_2 - f_1}$$



applied frequency.

- ④ If $E_0 = 10V$, $L = 30\text{mH}$, $C = 1000\mu\text{F}$, $R = 100\Omega$. Calculate the voltage drop across inductance, resonance current and quality factor.
- Solution:

Given $E_0 = 10V$

$$L = 30\text{mH} = 30 \times 10^{-3}\text{H}$$

$$C = 1000\mu\text{F} = 1000 \times 10^{-6}\text{F}$$

$$R = 100\Omega$$

frequency of source = 50Hz .

$$I_0 = \frac{E_0}{\sqrt{R^2 + (X_L - X_C)^2}}$$

$$I_{0L} =$$

$$I_r = \frac{E_0}{R}$$

To find Voltage across inductance (V_L) = ?

Resonance current (I_0) = ?

Quality factor (Q) = ?

Now;

$$I_0 = \frac{E_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{10}{\sqrt{100^2 + (2\pi \times 50 \times 30 \times 10^{-3})^2 - \frac{1}{(2\pi \times 50 \times 10^{-3})^2}}}$$

$$\therefore I_0 = 0.0998A.$$

Now;

Voltage across inductance (V_L) = $I_0 X_L$

$$= 0.0998 \times 2\pi \times 50 \times 30 \times 10^{-3}$$

$$= 0.94V.$$

$$\text{Resonance current } (I_r) = \frac{E_0}{R} = \frac{10}{100} = 0.1A.$$

$$\text{Quality factor } (Q) = \frac{\omega_0 f_r}{f_2 - f_1}$$

$$\text{Resonance frequency } (f_r) = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{30 \times 10^{-3} \times 10^{-6}}} =$$

$$\therefore f_r = 29.05\text{Hz}.$$

$(f_1 - f_2)$ can't be determine directly.

② The amplitude of damped oscillation decreases by 3% in one cycle. What factor of the energy of oscillation is lost in each oscillation?

Solution;

$$\text{we have; } A = A_0 e^{-b/2m\tau}.$$

Since A is decreased by 3%.

$$\text{Also; } E = \frac{1}{2} k A^2$$

$$\therefore E \propto A^2$$

$$(e^n)^2$$

$$e^{2n}$$

$$67\%$$

$$A = 67\% A$$

$$E = \frac{1}{2} k A^2$$

$$A = A - 3\% \text{ of } A$$

$$= \frac{97A}{100}$$

$$\log_{10} \text{of loss} = [0.06\%]$$

50

- ③ A simple pendulum of length ~~10~~ cm and mass 70 gm is suspended in a car that is travelling with constant speed 60 m/s around a circle of radius 120 m. If the pendulum undergoes small oscillation in a radial direction about its equilibrium position. Calculate the time period & frequency of oscillation.

Solution:

$$\begin{aligned}g_{eff} &= \sqrt{\left(\frac{v^2}{r}\right)^2 + g^2} \\&= \sqrt{\left(\frac{60^2}{120}\right)^2 + 9.81^2}\end{aligned}$$

$$= 31.56$$

$$T = 2\pi \sqrt{\frac{l}{g_{eff}}} = 0.79 \text{ sec.}$$

Acoustic:

Reverberation:-

Due to the presence of reflecting wall, ceiling, floor & closed window and door in a hall the sound waves are reflected, such that there may be ~~inf~~ interference of sound waves which produced unclear in sound and even if the source of sound is stopped, the listener's also listen the sound which is due to reflection from the different part is called reverberation.

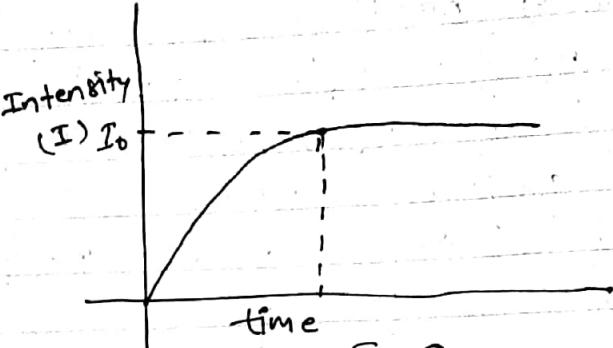


Fig ①

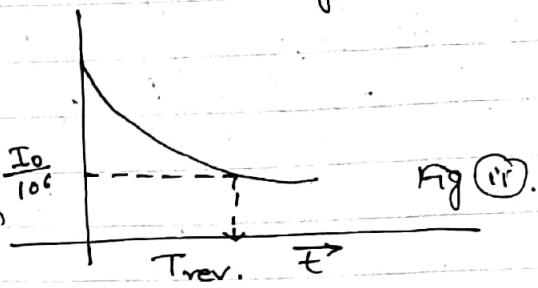


Fig ②

The time in which the listener's listen the sound after the source of sound is stopped is called reverberation time, in which Intensity falls to $I_0/10^6$ where I_0 is the intensity when the source is just stopped as shown in fig ②.

Absorption Co-efficient:-

Let S_1, S_2, \dots, S_n be the surface area of absorber having absorption co-efficient $\alpha_1, \alpha_2, \dots, \alpha_n$ then the average absorption coefficient is

$$\alpha_{av} = \frac{\sum_{i=1}^n \alpha_i S_i}{\sum_{i=1}^n S_i}$$

Sabine's Relation to find the reverberation time of a hall.

It is an empirical relation to find the reverberation time of a hall by using its parameter in terms of volume, surface area and absorption factor.

Let us take a hall having volume(V), Total surface area(S) and mean absorption coefficient of a hall is (α). Let I be the intensity of sound in the hall, δI be the decrease in intensity after the source is stopped at time δt then

$$\delta I = -\alpha n I \delta t$$

where n be the no. of reflections per second in that hall.

Now Mean free path / The distance travelled by the wave between two successive reflection is:

$$(\lambda_{\text{mean}}) = \frac{4V}{S}$$

and relaxation time / the time between two successive reflection of wave is

$$T = \frac{4V}{Sv} \text{ where } v \text{ be the velocity of sound wave in that hall}$$

$$\therefore n = \frac{1}{T} = \frac{Sv}{4V}$$

Putting this value in ①

$$\delta I = -\frac{Sv}{4V} \alpha I \delta t$$

$$\Rightarrow \frac{\delta I}{I} = -\frac{Sv}{4V} \alpha \delta t$$

$$\int_{I_0}^{I_t} \frac{\delta I}{I} = -\frac{Sv}{4V} \alpha \int_0^{T_{\text{rev}}} \delta t$$

$$\text{or, } \ln\left(\frac{I_t}{I_0}\right) = -\frac{Sv}{4V} \alpha T_{\text{rev}}$$

$$\text{Also; } I_t = I_0/10^6 \quad v = 350 \text{ m/s.}$$

$$\therefore T_{\text{rev}} = \frac{4V}{Sv\alpha} \ln\left(\frac{I_0}{10^6} \times \frac{1}{I_0}\right)$$

$$T_{\text{rev}} = \frac{-4V}{S \times 350 \cdot \alpha} \ln\left(\frac{1}{10^6}\right)$$

$$\therefore T_{\text{rev}} = \frac{1.57 \times 10^{-1} V}{S \alpha} = \frac{0.158 V}{S \alpha}$$

In FPS system

$$T_{rev} = \frac{0.05V}{\alpha S}$$

This is a reverberation time of a hall with volume (V), surface area (S) and absorption coefficient α .
Now,

Intensity level.

$$I_L = 10 \log_{10} \left(\frac{I}{I_0} \right) \text{ where } I_0 \text{ is reference intensity (standard intensity).}$$

Q. Calculate the reverberation time in a hall measuring $90 \times 10 \times 20$ ft. with the following parameter, 7500 sq.ft of plaster, $\alpha_1 = 0.03$, 900 sq. ft of glass $\alpha_2 = 0.025$, 6000 sq.ft of wood and floor. $\alpha_3 = 0.06$, audience of 500 person $\alpha_4 = 0.03$, $\alpha_5 = 1.0$

Solution,

$$\alpha_{av} = \frac{\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3}{S_1 + S_2 + S_3} = \frac{0.03 \times 7500 + 900 \times 0.025 + 6000 \times 0.06}{7500 + 900 + 6000} \\ = 0.0928$$

$$\alpha = \alpha_{av} + 6000 \times 0.03 + 500 \times 1.0 = 0.0928 + 0.03 + 9 = \frac{0.0928 \times 6000 + 0.03 \times 6000 + 500 \times 1.0 \times 6000}{6000} \approx 0.1809$$

$$T = \frac{0.05V}{\alpha S} = \frac{0.05 \times 90 \times 10 \times 20}{0.1809 \times 13900} \\ = 1.9 \times 10^{-5} \text{ sec.}$$

② Surface	Area	absorption coefficient
a Plastered wall	98	0.03
b Plastered ceiling	199	0.09
c Wooden door.	15	0.06
d Cushioned chairs	88	1.0
e Audience	150	4.70

Volume of a hall = 1500m³

Calculate Trev.

Solution:

$$\alpha_{av} = \frac{\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3}{S_1 + S_2 + S_3}$$

$$= 98 \times 0.03 + 199 \times 0.09 +$$

③ If a hall has dimension $30 \times 20 \times 50$ m. Calculate mean free path and relaxation time.

Solution:

$$\text{Given: Volume of a hall } (V) = 30 \times 20 \times 50$$

$$= 30000 \text{ m}^3$$

$$\text{Total Area of a hall } (S) = \cancel{30 \times 20 + 20 \times 50 + 30 \times 50} \\ = 2(30 \times 20 + 20 \times 50 + 30 \times 50) \\ = 6200 \text{ m}^2$$

Now:

$$\text{Mean free path } (\lambda_{\text{mean}}) = \frac{4V}{S} = \frac{4 \times 30000}{6200} = 19.35 \text{ m}$$

$$\text{Relaxation time } (T) = \frac{4V}{S \alpha} = \frac{4 \times 30000}{6200 \times 350} = 0.086 \text{ sec.}$$

④ If an empty hall has reverberation time 1.50 s. If 500 audience are present in the hall the reverberation time falls to 1.46 sec. Calculate the no. of person present in the hall if the reverberation time of that hall falls 1.90 sec.

Solution:

$$\text{Reverberation time for empty hall } (T'_{\text{rev}}) = 1.50 \text{ s}$$

$$\text{Reverberation time if } 500 \text{ person are present } (T''_{\text{rev}}) = 1.46 \text{ sec.}$$

No. of person ($n = ?$) if T'_{rev} falls to 1.90 sec.

$$1.50 = \frac{0.158V}{\alpha S}$$

$$1.46 = \frac{0.158V}{(\alpha + 500\alpha_1)S}$$

$$1.90 = \frac{0.158V}{(\alpha + n\alpha_1)S}$$

$$\frac{1.96}{1.50} = \frac{\alpha}{\alpha + 500\alpha}$$

$$1.027\alpha = \alpha + 500\alpha$$

$$\text{or, } 2.7 \times 10^{-2}\alpha = 500\alpha$$

$$\text{or, } 5.979 \times 10^{-5}\alpha = \alpha \quad \text{--- (i)}$$

$$\frac{1.90}{1.50} = \frac{\alpha}{\alpha + n\alpha}$$

$$1.071\alpha = \alpha + n\alpha$$

$$\text{or, } 1.071\alpha - \alpha = n\alpha$$

$$\text{or, } 7.19 \times 10^{-2}\alpha = n\alpha$$

$$\text{or, } 7.19 \times 10^{-2}\alpha = \alpha \quad \text{--- (ii)}$$

Then from (i) & (ii)

$$\frac{5.979 \times 10^{-5}\alpha \times n}{7.19 \times 10^{-2}\alpha} = 1$$

$$\text{or, } n = 1303.$$

Optical Fiber

It is a cylindrical device consists of core having refractive index (n_1) and cladding having refractive index (n_2) where $n_1 > n_2$. It works in the principle of total internal reflection.

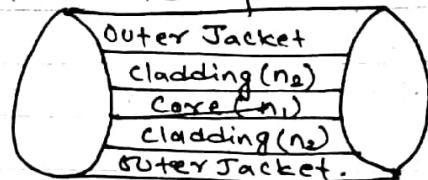


Fig: Optical fiber.

The glass material is used to construct the optical fiber. The total internal reflection occurs when angle of incidence is equal to the critical angle and angle of refraction is equal to 90° .

From Snell's law

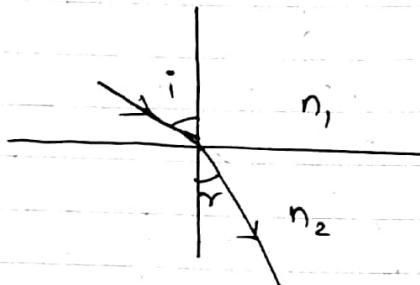
$$n_1 \sin i = n_2 \sin r$$

For total internal reflection

$$i = c \text{ & } r = 90^\circ$$

$$\therefore n_1 \sin c = n_2 \sin 90^\circ$$

$$\therefore \sin c = \frac{n_2}{n_1}$$



The internal reflection occurs when the wave travels from a denser medium to rarer medium.

Let us take an optical fiber in which the refractive index of core is n_1 and cladding is n_2 where $n_1 > n_2$. When

input signal is incident on the core with angle of incidence i and

angle of refraction on the core is θ as shown in figure. The signal is totally reflected from the cladding surface as shown in figure.

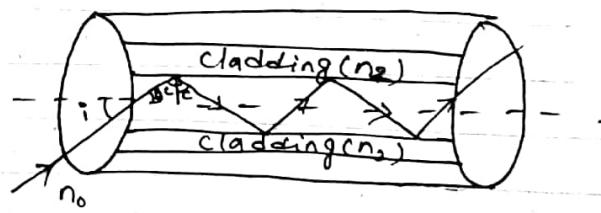


Fig: Light propagation in optical fibre.

Then from Snell's law.

$$n_0 \sin i = n_1 \sin \theta \quad [\text{where } i \text{ is called acceptance angle}]$$

$$\text{or, } \frac{n_0 \sin i}{n_1} = \sin(\theta - c) \quad [\because \theta = 90^\circ - c]$$

$$\Rightarrow \frac{n_0 \sin i}{n_1} = \cos c \quad \text{--- (i)}$$

And $n_1 \sin c = n_2 \sin 90^\circ$

$$\Rightarrow \sin c = \frac{n_2}{n_1} \quad \text{--- (ii)} \quad [\because \text{where } c \text{ is the critical angle between core \& cladding surface}]$$

Squaring eqn (i) & (ii) & add.

$$\sin^2 c + \cos^2 c = \frac{n_2^2}{n_1^2} + \frac{n_0^2 \sin^2 i}{n_1^2}$$

$$\Rightarrow \frac{n_0^2 \sin^2 i}{n_1^2} = 1 - \frac{n_2^2}{n_1^2}$$

$$\Rightarrow \sin i = \sqrt{\frac{n_1^2 - n_2^2}{n_0^2}}$$

For air $n_0 = 1$

$$\therefore \sin i = \sqrt{n_1^2 - n_2^2}$$

Now, Numerical aperture is the power of gathering of wave.

$$\therefore NA = \sqrt{n_1^2 - n_2^2} \quad \text{--- (iii)}$$

Now the fractional refractive index change:-

$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \quad \text{--- (iv).} \quad = \frac{n_1 - n_2}{n_1} \quad [\text{if } n_1 = n_2]$$

$$\Rightarrow NA = n \sqrt{2\Delta} \quad \text{--- (v).}$$

Normalized frequency or v-number

$$v = \frac{2\pi}{\lambda} \gamma(NA) \quad \text{where } \gamma \text{ radius of optical fiber, } \lambda \text{ is wavelength of signal.}$$

① The refractive index of core and cladding is 1.98 & 1.96 respectively. Calculate the acceptance angle, critical angle, Numerical aperture, fraction refractive index change. If the wavelength of signal is 600nm and diameter of optical fibre is 98μm. Calculate the V-number.

Solution:

Given, Refractive index of core (n_1) = 1.98

" " cladding (n_2) = 1.96.

$$a) \text{ Q} \sin i = \sqrt{n_1^2 - n_2^2}$$

$$\sin i = \sqrt{1.98^2 - 1.96^2}$$

$$\therefore i = 19.03^\circ$$

b) Critical angle = ?

$$\sin c = \frac{n_2}{n_1}$$

$$\therefore c = \sin^{-1} \left(\frac{1.96}{1.98} \right) = 80.56$$

$$c) \text{ Numerical Aperture (NA)} = \sqrt{n_1^2 - n_2^2} = 0.292$$

$$d) \text{ Fraction refractive index } (\Delta) = \frac{n_1^2 - n_2^2}{dn_1^2} = \frac{(0.292)^2}{2 \times 1.98^2} = 0.0139$$

e) V-number = ?

$$\lambda = 600\text{nm} = 600 \times 10^{-9}\text{m}$$

$$d = 98\mu\text{m} = 98 \times 10^{-6}\text{m}$$

$$\therefore V = \frac{\lambda}{d} \cdot \nu(\text{NA})$$

② An optical fiber has a numerical aperture of 0.22. The core has refractive index 1.60. Calculate the acceptance angle in water having refractive index 1.33. Also calculate critical angle, fractional refractive index.

Solution:

Given

$$\text{Numerical aperture (NA)} = 0.22.$$

$$\text{Refractive index of core (n}_1\text{)} = 1.60$$

$$\text{Refractive index of water (n)} = 1.33.$$

Now;

$$NA = \sqrt{n_1^2 - n_2^2}$$

$$\therefore (0.22)^2 = (1.60)^2 - n_2^2$$

$$\therefore n_2 = 1.589$$

Now;

$$\sin C = \frac{n_2}{n_1}$$

$$\therefore C = \sin^{-1} \left(\frac{1.589}{1.60} \right) = 80.93^\circ$$

$$\text{Now; } \frac{n}{n_1} \sin i = \cos C$$

$$\therefore \frac{1.33}{1.60} \sin(i) = \cos 80.93^\circ$$

$$\therefore i = 10.93^\circ$$

$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} = \frac{1.60^2 - (1.589)^2}{2 \times 1.60^2}$$

$$= 9.95 \times 10^{-3}.$$

Types of optical fiber.

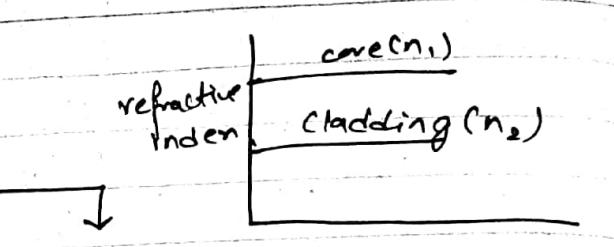
(1) Step index optical fiber.

Monomode

Step index

- Laser diode is used.

The refractive index of core and cladding is fixed and $n_1 > n_2$.

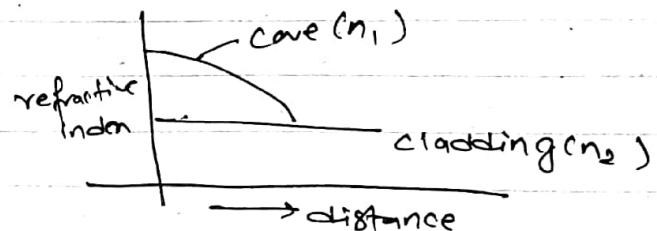


Multimode

Step index

- Light emitting diode is used.

(2) Graded index optical fiber.



Power loss in optical fiber

$$P_{\text{loss}} = \frac{1}{L} 10 \log_{10} \frac{P_{\text{in}}}{P_{\text{out}}}$$

where L is in km.

Wave Motion:-

The disturbance in medium produce ~~any~~ wave.

Generally, there are two types of wave.

a) Mechanical wave.

b) Non-Mechanical wave or electromagnetic wave.

The wave which carry the energy and amplitude remains constant is called progressive wave.

Let us consider a progressive wave, let a particle is in motion at O , then the displacement of particle is

$$y = A \sin \omega t$$

and the another particle is in motion at O' then the displacement of particle at O' is

$$y = A \sin (\omega t - \phi)$$

$$\text{where } \phi = \frac{2\pi n}{\lambda}$$

$$\text{Then, } y = A \sin \left(\omega t - \frac{2\pi n}{\lambda} \right)$$

$$= A \sin \left(\omega ft - \frac{2\pi n}{\lambda} \right)$$

$$= A \sin \frac{2\pi}{\lambda} (ft - n)$$

$$= A \sin \frac{2\pi}{\lambda} (vt - n) \quad \text{--- (i)}$$

where v is the velocity of wave.

Eqn (i) is the progressive wave equation moving along positive x -direction with wave velocity v .

Similarly the progressive wave equation moving along $-ve x$ axis with velocity v is

$$y = A \sin \frac{2\pi}{\lambda} (vt + n) \quad \text{--- (ii)}$$

General differential equation of wave:-
we have,

$$y = A \sin \frac{2\pi}{\lambda} (vt - n)$$

$$\frac{dy}{dt} = \cancel{\frac{2\pi}{\lambda}} \frac{d}{dt} A \cos \frac{2\pi}{\lambda} (vt - n)$$

$$\frac{d^2y}{dt^2} = \left(\frac{2\pi v}{\lambda}\right)^2 (-) A \sin \frac{2\pi}{\lambda} (vt - n) \quad \text{--- (i)}$$

and $\frac{dy}{dn} = (-) \frac{2\pi}{\lambda} A \cos \frac{2\pi}{\lambda} (vt - n)$

$$\frac{d^2y}{dn^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A \sin \frac{2\pi}{\lambda} (vt - n) \quad \text{--- (ii)}$$

from eqn (i) & (ii).

$$\boxed{\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dn^2}} \quad \text{--- (iii)}$$

Eqn (iii) is the general differential eqn of wave motion in one direction.

$$\nabla = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right)$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

$\nabla \rightarrow$ Nebula

$\nabla^2 \Rightarrow$ Laplacian.

In three dimension:

$$v^2 \nabla^2 y = \frac{d^2y}{dt^2}$$

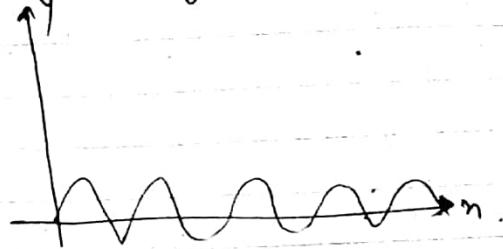
* Show that the velocity of particle is equal to the velocity of wave times the -ve slope of the curve in the string

Ans:-

We have;

$$y = A \sin(\omega t - \phi)$$

$$\text{Also; } y = A \sin \frac{\omega}{2} (vt - n)$$



where v is the wave velocity.

$$\text{Now; } v_p = \frac{dy}{dt} = \frac{d\pi}{2} \cdot v \cos \frac{\omega}{2} (vt - n) \quad \text{--- (i)}$$

$$\text{Also; } \frac{dy}{dn} = (-) \frac{d\pi}{2} A \cos \frac{\omega}{2} (vt - n)$$

$$\text{or, } -\frac{dy}{dn} = \frac{d\pi}{2} A \cos \frac{\omega}{2} (vt - n). \quad \text{--- (ii)}$$

$$\boxed{v_p = -\frac{d}{dn} \frac{dy}{dn}}$$

$$\therefore v_p = A \frac{d\pi}{2} v \cos \frac{\omega}{2} (vt - n)$$

$$\frac{d^2y}{dt^2} = a_p = (-) \left(\frac{d\pi}{2} v \right)^2 A \sin \frac{\omega}{2} (vt - n)$$

$$\therefore a_p = \frac{d^2y}{dt^2} = -\omega^2 y$$

Hence the particle of a wave vibrate in a SHM.

$\sqrt{2}Y$

Energy, Power and Intensity of progressive wave.

Now, the displacement of particle at any instant of time t.

$$y = A \sin \frac{2\pi}{\lambda} (vt - n)$$

Now, the potential energy

$$U = - \int_0^Y F dy$$

$$F = m \frac{dy^2}{dt^2} = m \left(\frac{2\pi}{\lambda} v \right)^2 (-) A \sin \frac{2\pi}{\lambda} (vt - n)$$

$$\Rightarrow F = -m \left(\frac{2\pi}{\lambda} v \right)^2 y$$

\therefore Potential energy per unit volume.

$$\begin{aligned} U_{\text{volume}} &= \frac{U}{V} = - \int_0^Y \frac{F}{V} dy \\ &= \left(\frac{2\pi}{\lambda} v \right)^2 \int_0^Y y dy \\ &= \left(\frac{2\pi}{\lambda} v \right)^2 \frac{y^2}{2} \end{aligned}$$

$$U_{\text{volume}} = \left(\frac{2\pi}{\lambda} v \right)^2 \frac{1}{2} \cdot A^2 \sin^2 \frac{2\pi}{\lambda} (vt - n) \quad \text{--- (i)}$$

Now, Kinetic energy per unit volume.

$$\begin{aligned} KE_{\text{volume}} &= \frac{K.E.}{V} = \frac{1}{2} m v^2 \\ &= \frac{1}{2} \int \left(\frac{dy}{dt} \right)^2 \\ &= \frac{1}{2} \int \left(\frac{2\pi}{\lambda} v A \cos \frac{2\pi}{\lambda} (vt - n) \right)^2 \\ &= \frac{1}{2} \int \left(\frac{2\pi}{\lambda} v \right)^2 A^2 \cos^2 \frac{2\pi}{\lambda} (vt - n) \quad \text{--- (ii)} \end{aligned}$$

Total energy per unit volume is

$$\begin{aligned} (\text{K.E})_{\text{volume}} &= (\text{P.E})_{\text{volume}} + (\text{P.E})_{\text{volume}} \\ &= \frac{g}{2} \left(\frac{\rho \pi v}{2} \right)^2 A^2 \cos^2 \frac{\omega}{2} (vt - n) + \frac{g}{2} \left(\frac{\rho \pi v}{2} \right)^2 A^2 \\ &\quad \sin^2 \frac{\omega}{2} (vt - n) \\ &= \frac{g}{2} \left(\frac{\rho \pi v}{2} \right)^2 A^2 \\ &= 2\pi^2 f A^2 f^2. \end{aligned}$$

Now, the total energy

$$\begin{aligned} E &= 2\pi^2 f A^2 f^2 \cdot V \text{ where } V \text{ is volume.} \\ &= 2\pi^2 f A^2 f^2 \cdot A \times l \text{ where } a \text{ is area.} \\ &= 2\pi^2 f A^2 f^2 \cdot a \times v \times t \end{aligned}$$

$$\text{Power} = \frac{E}{t} = P = 2\pi^2 f A^2 f^2 a v$$

$$\text{And intensity } I = P/a = 2\pi^2 f A^2 f^2 v.$$

$$\text{Intensity level } (\alpha) = 10 \log_{10} (I/I_0)$$

Q: A source of sound has 500 Hz frequency and amplitude 0.95 cm

Calculate the energy flow across a square of unit area per sec. The velocity of sound in air is 332 m/s. and density of air is 1.29 kg/m³.

Solution:

$$\text{Given; } f = 500 \text{ Hz}$$

$$A = 0.95 \text{ cm} = 0.95 \times 10^{-2} \text{ m.}$$

$$V = 332 \text{ m/s}$$

$$\rho = 1.29 \text{ kg/m}^3.$$

$$E = 2\pi^2 f A^2 f^2 \cdot a \times v \times t$$

$$\frac{E}{at} = 2\pi^2 \times 1.29 \times (0.95)^2 \times 10^{-4} \times (500)^2 \times 332$$

$$\therefore I = 4.28 \times 10^{-9} \text{ J/m}^2 \text{ s.}$$

Q. How much acoustic power enters the window of area 1.58 m^2 where the standard intensity level is 10^{-16} W/cm^2 . The window opens on a street where the street noise result in an intensity level at the window of 60 dB

Solution:

$$\text{Area } (A) = 1.58 \text{ m}^2$$

$$\text{Intensity level } (\alpha) = 10^{-16} \text{ W/cm}^2 = \frac{10^{-16}}{10^{-4}} \text{ W/m}^2 = \cancel{10^{-12} \text{ W/m}^2}$$

$$\text{We have } \alpha = \log_{10} \left(\frac{I}{I_0} \right)$$

$$\text{or, } 60 = 10 \log_{10} \left(\frac{I}{10^{-12}} \right)$$

$$\text{or, } I = 10^{-6} \text{ W/m}^2$$

Now,

$$\begin{aligned} P &= IXa \\ &= 10^{-16} \times 1.58 \\ &= 1.58 \times 10^{-16} \text{ W.} \end{aligned}$$

Velocity of transverse wave along a stretched string

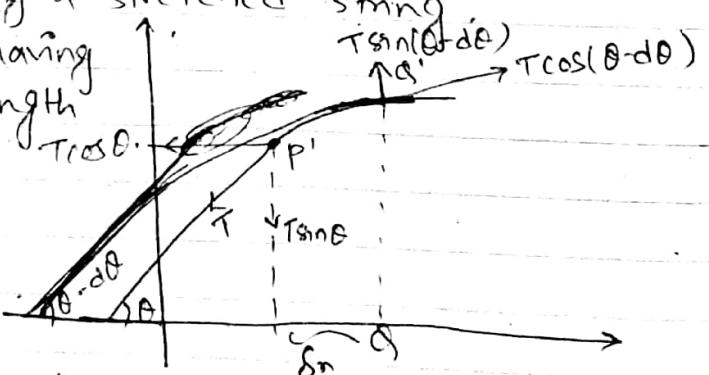
Let us take a string having linear mass density σ with length δn .

When tension T is applied on the string it starts to vibrate and generate the transverse wave as shown in

figure. Horizontal force acting on the component $P'Q'$ is

$$= T\cos(\theta - d\theta) - T\cos\theta$$

Since $d\theta$ is very small $\cos(\theta - d\theta) \approx \cos\theta$.



\therefore Net horizontal force = 0.

And

Net vertical force acting on the component $P'Q'$ is

$$T \sin \theta - T \sin(\theta - d\theta)$$

for small angle θ

$$\sin \theta \approx \tan \theta \text{ and } \sin(\theta - d\theta) \approx \tan(\theta - d\theta)$$

\therefore Net vertical force acting on the element = $T [\tan \theta - \tan(\theta - d\theta)]$

$$\tan \theta = \frac{dy}{dx} \approx \text{slope at } P'$$

$$\text{Now, slope at } Q' = \frac{dy}{dx} - \frac{d}{dx} \left(\frac{dy}{dx} \right) s_n$$

$$\begin{aligned} \therefore \text{Net vertical force} &= \left\{ \frac{dy}{dx} - \left[\frac{dy}{dx} - \frac{d^2y}{dx^2} \cdot s_n \right] \right\} T \\ &= T \frac{d^2y}{dx^2} s_n \quad \text{--- (i)} \end{aligned}$$

Now from Newton's law

$$F = ma = \sigma s_n \cdot \frac{d^2y}{dt^2} \quad \text{--- (ii)}$$

Equating (i) & (ii)

$$T \frac{d^2y}{dx^2} s_n = \sigma s_n \frac{d^2y}{dt^2}$$

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{T}{\sigma} \frac{d^2y}{dx^2}$$

Comparing it with $\frac{d^2y}{dt^2} = V^2 \frac{d^2y}{dx^2}$ we get

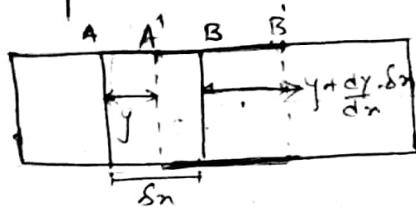
$$V = \sqrt{\frac{T}{\sigma}}$$

It shows that the velocity of transverse wave in stretched string depends on the applied tension and linear density of the material

\therefore

Velocity of longitudinal wave in **Isotropic rod**-

The medium whose physical properties do not change with direction i.e. same in all direction is called isotropic medium.



Anisotropic medium is the medium whose physical properties changes with direction.

Let us take a uniform rod having cross sectional area 'A' and density is 'f'. Let us take two arbitrary plane section A and B before passing the wave. When wave is passed in the rod the plane section shifts A to A' and B to B' as shown in fig.

The increase in length after passing the wave is

$A'B' - AB$

$$= f \frac{dy}{dn} \cdot \delta_n + f \frac{dy}{dn} \cdot \delta_n$$

$$= f_n \frac{dy}{dn}$$

$$\text{Now the strain} = \frac{\frac{dy}{dn} \cdot \delta_n}{\delta_n} = \frac{dy}{dn}$$

$$\text{Young's Modulus} (Y) = \frac{F/A}{dy/dn} = \alpha$$

$$\therefore F = \alpha \frac{dy}{dn}$$

Similarly, in section B' to B*

$$F + \frac{dF}{dn} \cdot \delta_n$$

$$\Rightarrow \alpha \frac{dy}{dn} + \frac{dF}{dn} \cdot \delta_n$$

$$\Rightarrow \alpha Y \frac{dy}{dx} + \alpha Y \frac{d^2y}{dx^2} \cdot \delta x.$$

The resultant force is $\alpha Y \frac{d^2y}{dx^2} \delta x$ — (i)

We have $F=ma$

$$= \alpha \delta x \cdot \delta \frac{d^2y}{dt^2} — (ii)$$

Equating (i) & (ii) we get;

$$\frac{d^2y}{dt^2} = \frac{Y}{\rho} \frac{d^2y}{dx^2}$$

$$\Rightarrow v = \sqrt{\frac{Y}{\rho}} \quad \text{where } Y \text{ is young's modulus of elasticity.}$$

For fluid or gases medium.

$$v = \sqrt{\frac{k}{\rho}} \quad \text{where } k \text{ is bulk modulus.}$$

Q. The speed of transverse wave on a string is 150m/s when the tension T is 105N. What value must the tension be changed to raised the wave speed of 190m/s.

Solution:

We have;

$$v = \sqrt{\frac{T}{\rho}}$$

$$150 = \sqrt{\frac{105}{\rho}} — (i)$$

For condition 2,

$$190 = \sqrt{\frac{T}{\rho}} — (ii)$$

Dividing eqn ① by ②.

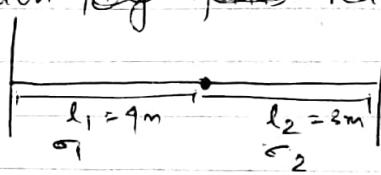
$$\frac{150}{190} = \sqrt{\frac{10.5}{T_2}}$$

$$\therefore T_2 = 168.46 \text{ N.}$$

$$\Delta T = 168.46 - 105$$

$$= 63.4667 \text{ N.}$$

Q. In figure two strings have been tight together with the knot and the ~~stressed~~ stretched between two rigid support. The spring has linear density, $\sigma_1 = 1.5 \times 10^{-9} \text{ kg/m}$ and $\sigma_2 = 2.9 \times 10^{-9} \text{ kg/m}$. $l_1 = 9 \text{ m}$ and $l_2 = 2 \text{ m}$. and 900 N tension is applied which reaches the first?



Solution;

$$\text{we have, } T = \sqrt{\frac{T}{\sigma_i}}$$

$$\propto, \frac{l_1}{t_1} = \sqrt{\frac{900}{1.5 \times 10^{-9}}}$$

$$\therefore t_1 = 4 \times \sqrt{\frac{1.5 \times 10^{-9}}{900}} = 0.00299 \text{ sec.}$$

$$\text{Now, } v_2 = \sqrt{\frac{T}{\sigma_2}}$$

$$\propto, t_2 = 3 \times \sqrt{\frac{2.9 \times 10^{-9}}{900}} = 0.00255 \text{ sec.}$$

Since the time taken by the string having length 4m reaches first.

Phase wave/Velocity.

Let us take a particle moving with velocity v then the displacement of particle can be written as

$$y = A \sin(\omega t - kn)$$

phase.

The phase is constant $\omega t - kn = \text{constant} \rightarrow \textcircled{1}$

Dif: ① w.r.t time,

$$\omega - k \frac{dn}{dt} = 0$$

$$\Rightarrow \frac{dn}{dt} = \frac{\omega}{k} = \frac{\omega f}{\lambda} = f \lambda$$

\therefore Phase velocity $V_{\text{phase}} = \frac{dn}{dt} = f \lambda$

Now in case of matter wave

We have;

$$mc^2 = hf$$

$$\Rightarrow f = \frac{mc^2}{h}$$

and $\lambda = \frac{h}{mv}$.

$$\therefore V_{\text{phase}} = \frac{mc^2}{h} \cdot \frac{h}{mv} = \frac{c^2}{v} \quad \text{--- ii}$$

v — velocity of particle

Eqn ii shows that the waves left behind the particle which is impossible such that matter wave is not a phase wave.

Group wave/wave packet:

When matter moves in certain velocity it generates two or more than two waves having slightly different wavelength such that there is interference between waves due to the constructive interference the amplitude is maximum which is taken as matter wave and it is called group wave or wave packet.

Let us take the displacement of particle at any instant time with frequency f and wave no. k then

$$y_1 = a \sin(\omega t - kn)$$

Again take another wave having frequency $(f + \delta f)$ and wave no. $(k + \delta k)$ then

$$y_2 = a \sin((\omega t + \delta \omega)t - (k + \delta k)n)$$

∴ Resultant wave becomes;

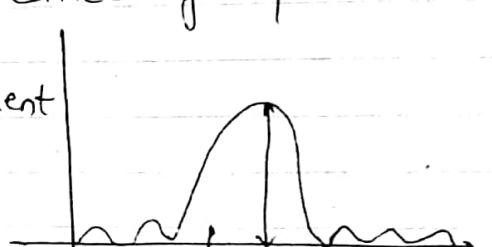
$$y = y_1 + y_2 \\ = a [\sin(\omega t - kn) + \sin((\omega + \delta \omega)t - (k + \delta k)n)]$$

$$= a \frac{1}{2} \left[\cancel{\sin}(\omega t - kn) + \cancel{\sin}((\omega + \delta \omega)t - (k + \delta k)n) \right]$$

$$= a \frac{1}{2} \cos \left(\cancel{\omega t - kn} - \frac{1}{2} \cancel{[(\omega + \delta \omega)t - (k + \delta k)n]} \right)$$

$$\text{or, } y = \frac{1}{2} a \sin \left(\frac{\omega t - kn + \omega t + t\delta \omega - k\delta n - \delta kn}{2} \right) \left(\frac{\omega t - kn - \omega t - \delta \omega t + k\delta n + \delta kn}{2} \right)$$

$$\text{or, } y = \frac{1}{2} a \sin \left(\frac{\omega \cancel{t} + t\delta \omega - \delta kn}{2} \right) \left(\frac{\delta kn - \delta \omega t}{2} \right)$$



matter wave
(wave packet + group wave)

$$\text{or, } y = A \sin(\omega t - kn) \cos\left(\frac{\delta k \cdot n - \delta \omega \cdot t}{2}\right) \quad \left[\begin{array}{l} \therefore \delta \omega + \delta \omega = 2\omega \\ \delta k + \delta k = 2k \end{array} \right]$$

$$\therefore y = A \sin(\omega t - kn) \cos\left(\frac{\delta k \cdot n - \delta \omega \cdot t}{2}\right) \quad \left[\because A = 2A \right]$$

The resultant amplitude to be maximum

$$\text{if } \delta k \cdot n - \delta \omega \cdot t = 0$$

$$\Rightarrow \frac{n}{t} = \frac{\delta \omega}{\delta k}$$

$$\Rightarrow \text{group velocity } v_g = \frac{d\omega}{dk} \quad \text{Taking limit } \delta k \rightarrow 0.$$

* Relationship between particle velocity and group velocity.

\because we have,

$$\text{group velocity } v_g = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{dE}{dP}$$

$$\text{we have } E = \frac{P^2}{2m}$$

$$\Rightarrow \frac{dE}{dP} = \frac{P}{m} = \frac{mv}{m} = v$$

$$\Rightarrow v_g = v \quad \text{where } v \text{ is particle velocity}$$

It shows that in matter waves the group velocity is equal to particle velocity.

* Relationship between group velocity and phase velocity.

$$\text{We have, } v_g = \frac{d\omega}{dk} = \frac{d\omega}{dx} \cdot \frac{dx}{dk}$$

Again we have

$$\omega = dx/dt = \frac{dx}{dt} V_{\text{phase}}$$

$$\Rightarrow \frac{d\omega}{d\lambda} = V_{\text{phase}} \left(-\frac{d\eta}{\lambda^2} \right) + \frac{d\eta}{\lambda} \frac{dV_{\text{phase}}}{d\lambda}$$

And

$$k = \frac{d\eta}{\lambda}$$

$$\frac{dk}{d\lambda} = -\frac{d\eta}{\lambda^2}$$

Now;

$$\begin{aligned} V_g &= \frac{d\omega}{d\lambda} \cdot \frac{d\lambda}{dk} = \frac{V_{\text{phase}} \left(-\frac{d\eta}{\lambda^2} \right) + \frac{d\eta}{\lambda} \frac{dV_{\text{phase}}}{d\lambda}}{-\frac{d\eta}{\lambda^2}} \\ &= \frac{\frac{d\eta}{\lambda} \left(-\frac{1}{\lambda} V_{\text{phase}} + \frac{dV_{\text{phase}}}{d\lambda} \right)}{-\frac{d\eta}{\lambda^2}} \end{aligned}$$

$$V_g = \left(V_{\text{phase}} - \frac{dV_{\text{phase}}}{d\lambda} \right)$$

It shows that the group velocity is less than the phase velocity in dispersive medium. In non dispersive medium;

$$\frac{dV_{\text{phase}}}{d\lambda} = 0$$

$$\therefore V_g = V_{\text{phase}}$$

$$h = 6.62 \times 10^{-34} \text{ J.S}$$

$$\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34} \text{ J.S.}$$

Photon & Matter Wave

The quanta of electro magnetic wave or light wave is called photon. The minimum packet of energy is called quanta.

The quanta of sound wave is called phonon.

Matter Wave (De-Broglie wave).

When any object or particle is in motion they generates the wave which is called matter wave. Let us take a photon; the energy of the photon is given by plank's quantum theory is

$$E = hf = \hbar\omega \quad \text{--- (i)}$$

Now from photoelectric effect, the photon acts as a particle, so the energy of the photon from Einstein's energy mass relation is $E = mc^2$ --- (ii).

Equating eqⁿ (i) and (ii) we get;

$$\cancel{mc^2} = \cancel{hf} = mc^2$$

$$\cancel{h \cdot c^2} = \cancel{mc^2}$$

$$\therefore \lambda = \frac{h}{mc} \quad \text{--- (iii)}$$

If a particle having mass m moving with velocity v, then

$$\lambda = \frac{h}{mv}$$

① If you are moving with velocity 20 km per hour - Calculate the de-broglie wavelength.

Solution;

$$\text{Mass (m)} = 5.5 \text{ kg}$$

$$v = 20 \text{ km/hr} = \frac{20 \times 10^3}{3600} = 5.556$$

Now,

$$\lambda = \frac{h}{mv} = \frac{6.62 \times 10^{-34}}{5.5 \times 5.556} = 2.16 \times 10^{-36} \text{ m}$$

② Calculate the de-broglie's wave length of an electron if 1 keV ~~concrete~~ is supplied on electron.

Solution;

$$\text{We know, } \frac{1}{2}mv^2 = eV$$

} if
v > 10⁷ or 10⁸
m is charge

$$\text{or, } v = \sqrt{\frac{2eV}{m}}$$

$$\therefore \lambda = \frac{h}{\sqrt{2meV}} = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^19}}$$

$$\therefore v = \sqrt{\frac{2 \times 10 \times 10^3 \times 1.6 \times 10^{19}}{9.1 \times 10^{-31}}} = 5.93 \times 10^7 \text{ m/s.}$$

Now,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{9.1 \times 10^{-31}}{\sqrt{1 - \frac{(5.93 \times 10^7)^2}{(3 \times 10^8)^2}}} = 9.28 \times 10^{-31} \text{ kg}$$

$$\therefore \lambda = \frac{h}{mv} = \frac{6.62 \times 10^{-34}}{9.28 \times 10^{-31} \times 5.93 \times 10^7} = 1.20 \times 10^{-11} \text{ m.}$$

Potential Barrier / Tunneling effect.

Let us take a particle having energy E is incident on a potential barrier having height V_0 where $E < V_0$.

Classically the particle totally reflected back from $n=0$. But in quantum mechanics there

is some probability of transition of a particle from the barrier having width l . This effect is called tunnelling effect.

For example: α -decay, β -decay and tunneling diode.

In ① region the SWE can be written as:

$$-\frac{\hbar^2}{dm} \frac{\partial^2 \psi_I(m)}{\partial m^2} + 0 = E \psi_I(m)$$

$$\Rightarrow \frac{\partial^2 \psi_I(m)}{\partial m^2} + \frac{dm \cdot E}{\hbar^2} \psi_I(m) = 0$$

$$\text{let } \alpha^2 = \frac{dm \cdot E}{\hbar^2}$$

$$\text{Then; } \frac{\partial^2 \psi_I(m)}{\partial m^2} + \alpha^2 \psi_I(m) = 0 \quad \text{--- ①.}$$

The general soln of eqn ① is

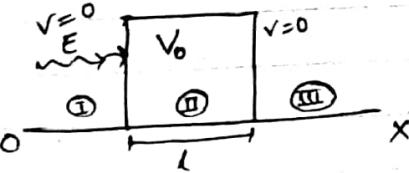
$$\psi_I(m) = A e^{im} + B e^{-im} \quad \text{--- ②}$$

The first term $A e^{im}$ is the amplitude of incident wave and the second term $B e^{-im}$ is the amplitude of wave reflected from $n=0$.

In ② region the SWE can be written as

$$-\frac{\hbar^2}{dm} \frac{\partial^2 \psi_{II}(m)}{\partial m^2} + V_0(m) \psi_{II}(m) = E \psi_{II}(m)$$

$$\Rightarrow \frac{\partial^2 \psi_{II}(m)}{\partial m^2} - \frac{dm(V_0 - E)}{\hbar^2} \psi_{II}(m) = 0 \quad \text{--- ③}$$



In Nut

The general solution of ~~Eqn (ii)~~ let $\beta^2 = \frac{2m(V_0 - E)}{\hbar^2}$

then $\frac{\partial^2 \Psi_{II}(n)}{\partial n^2} - \beta^2 \Psi_{II}(n) = 0$

The general solution is

$$\Psi_{II}(n) = Ce^{\beta n} + De^{-\beta n} \quad \text{(ii)}$$

The first term $Ce^{\beta n}$ gives the amplitude of incident wave on $n=L$ and the second term $De^{-\beta n}$ gives the amplitude of reflected wave from $n=L$.

Now, the SWE for region (III) is

$$\frac{\partial^2 \Psi_{III}(n)}{\partial n^2} + \frac{2mE}{\hbar^2} \Psi_{III}(n) = 0$$

Let $\alpha^2 = \frac{2mE}{\hbar^2}$

$$\Rightarrow \Psi_{III}(n) = Fe^{i\alpha n} + Ge^{-i\alpha n}$$

Since there is no any barrier in third region then there is no possibility of reflection of wave such that $Ge^{-i\alpha n} = 0$

Hence, $\Psi_{III}(n) = Fe^{i\alpha n} \quad \text{(iv)}$

Using boundary condition we get transmission-coefficient.

$$T = \left| \frac{F}{A} \right|^2 = \frac{16E(V_0 - E)}{V_0^2} e^{-2\beta L}$$

where; $\beta = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$

L = width of potential barrier.

E = Energy of particle.

V_0 = barrier potential/height.

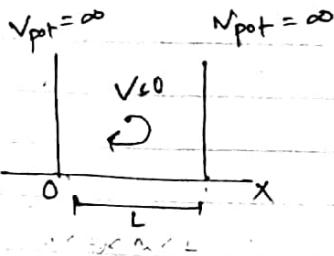
m = mass of incident particle.

Imp

Particle in a box / Infinitely deep potential well

Let us take a particle having mass m is confined in an infinitely deep potential well such that the particle moves in to and fro motion entire the well as shown in figure. The potential energy is zero inside the well.

$V=0; 0 < n < L$
 $V=\infty \quad n \leq 0 \text{ or } n \geq L$



Now, we have Schrodinger time independent wave equation is

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(n)}{\partial n^2} + V_{\text{pot}} \cdot \psi(n) = E \psi(n)$$

Since $V_{\text{pot}}=0$, inside the well, then it becomes:

$$\frac{\partial^2 \psi(n)}{\partial n^2} + \frac{2mE}{\hbar^2} \psi(n) = 0$$

$$\text{Let } k^2 = \frac{2mE}{\hbar^2}$$

$$\text{Then, } \frac{\partial^2 \psi(n)}{\partial n^2} + k^2 \psi(n) = 0 \quad \text{--- (1)}$$

The general soln of eqn (1) is

$$\psi(n) = A \sin kn + B \cos kn$$

$$\text{Now, } \psi(n) = 0 \Big|_{n=0}$$

$$0 = A \cdot 0 + B$$

$$\therefore B = 0$$

$$\text{Again, } \psi(n) = 0 \Big|_{n=L}$$

$$0 = A \sin kL + B \cos kL$$

$$\Rightarrow A \sin kL = 0 \quad (\because A \neq 0)$$

$$\Rightarrow \sin kL = 0$$

$$\Rightarrow \sin kn = 0$$

$$\Rightarrow k = \frac{n\pi}{L}$$

We have $k^2 = \frac{\partial m E}{\partial x^2}$

$$\therefore \frac{\partial m E}{\partial x^2} = \frac{n^2 \pi^2}{L^2}$$

$$\Rightarrow E_n = \frac{n^2 \pi^2 \hbar^2}{8m L^2} \quad (\text{energy eigen value})$$

where $n=1, 2, 3$

which gives the energy Eigen value of a particle confined in a infinitely deep potential wave along x -axis having width L . It shows that energy of particle is quantized. contd.

- ① An electron is confined to an infinite ~~height~~ box of size 0.1nm . Calculate the ground state energy of the electron. How this energy can be put to the third energy state.

Solution;

$$L = 0.1\text{nm} = 0.1 \times 10^{-9}\text{m.}$$

For ground state, $n = 1$

~~$E_1 = \frac{1^2 \times \pi^2 \times \hbar^2}{2m L^2} =$~~

For third state, $n = 3$

$$E_3 = \frac{n^2 \pi^2 \hbar^2}{8m L^2} = \frac{3^2 \times \pi^2 \times 6.69 \times 10^{-39}}{8 \times 9.1 \times 10^{-31} \times (0.1 \times 10^{-9})^2} = 1.03 \times 10^{18}\text{J.}$$

① If a football having mass 500gm is confined in a Dashrath Rangasala having width 1km. Calculate the velocity of the football.

Solution

$$\text{Mass}(m) = 500 \times 10^{-3} \text{ kg.}$$

$$n = 1 \text{ (for ground state)}$$

$$L = 1 \text{ km} = 1000 \text{ m}$$

$$\therefore E_n = \frac{n^2 \pi^2 \hbar^2}{dm L^2} = \frac{1^2 \pi^2 \times (1.05 \times 10^{-39})^2}{dm \times 500 \times 10^{-3} \times (1000)^2} = \cancel{1.08 \times 10^{-73}} \text{ J.}$$

Now;

$$E_n = \frac{1}{2} mv^2$$

$$\therefore \cancel{1.08 \times 10^{-73}} \quad 1.08 \times 10^{-73} = \frac{1}{2} \times 500 \times 10^{-3} \times v^2$$

$$\therefore v = 6.57 \times 10^{-37} \text{ m/s.}$$

contd....

The wavefunction of the particle inside the infinitely deep potential well is

$$\Psi(n) = A \sin kn$$

To find the normalization constant A;

$$\int_0^L \Psi^*(n) \Psi(n) dn = 1 \quad \int_0^L$$

$$\therefore \int_0^L 1^2 \sin^2 kn dn = 1$$

$$\therefore A^2 \left[\int_0^L \frac{1 - \cos 2kn}{2} dn \right] = 1$$

$$\therefore A^2 \left[\frac{1 \times L}{2} - \frac{1}{2} \int_0^L \cos 2kn dn \right] = 1$$

$$\text{or, } A^2 \left[\frac{L}{\alpha} - \frac{1}{2} \left[\frac{\sin \alpha kn}{\alpha k} \right]_0^L \right] = 1$$

$$\text{or, } A^2 \left[\frac{L}{\alpha} - \frac{1}{2} (\sin \alpha k) \right] = 1$$

$$A^2 \left[\frac{L}{\alpha} - \frac{1}{2} \left(\frac{\sin \alpha kn}{\alpha k} \right)_0^L \right] = 1$$

$$\text{or, } A^2 \left[\frac{L}{\alpha} - \frac{1}{2} \times \frac{\sin \alpha n\pi}{\alpha} \right] = 1$$

$$\text{or, } A^2 \left[\frac{L}{\alpha} - \frac{1}{2} \times \frac{\sin \alpha n\pi}{\alpha} \right] = 1$$

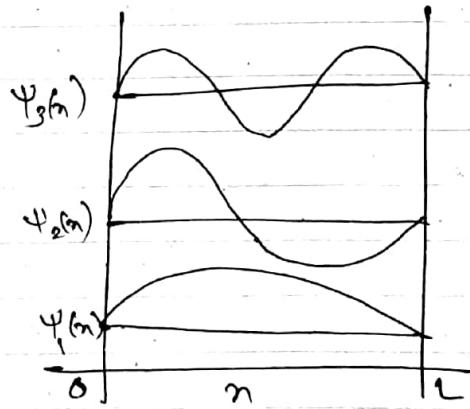
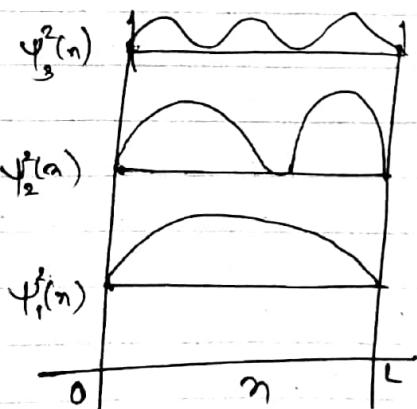
$$\text{or, } A^2 \left[\frac{L}{\alpha} - \frac{1}{2} \times \frac{1}{2} \left(\sin \alpha n\pi - \sin 0 \right) \right] = 1$$

$$\text{or, } A^2 \left[\frac{L}{\alpha} - \frac{1}{2} \times \frac{1}{2} \left(\sin \alpha n\pi \right) \right] = 1$$

$$\text{or, } A^2 = \frac{1}{4} \quad \text{since } n=1, 2, 3, \dots$$

$$\therefore A = \sqrt{\frac{1}{4}}$$

$$\therefore \Psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$$



Let R be the reflection co-efficient, then

$$R+T=1$$

$$\Rightarrow R=1-T$$

- ① If an electron beam having energy 5eV is incident on a potential barrier having height 6eV where the width of the potential barrier is 15 \AA . Calculate the reflection coeff and transmission coefficient.

Solution:

Given; Energy of electron (E) = 5eV = $5 \times 1.6 \times 10^{-19} \text{ J}$

Energy Height or barrier potential (V_0) = 6eV. = $6 \times 1.6 \times 10^{-19} \text{ J}$

Width of potential barrier (l) = $15\text{ \AA} = 15 \times 10^{-10} \text{ m}$.

Now,

$$\beta = \sqrt{\frac{2m(V_0-E)}{\pi^2}} = \sqrt{\frac{2 \times 9.1 \times 10^{-31} \times (6 \times 1.6 \times 10^{-19} - 5 \times 1.6 \times 10^{-19})}{(1.05 \times 10^{-34})^2}}$$
$$= 5.26 \times 10^{-8}$$

Now; Transmission coeff (T) = $\frac{16E(V_0-E)}{V_0^2} e^{-2\beta l}$

$$= \frac{16 \times 5 \times (1.6 \times 10^{-19}) (6 \times 1.6 \times 10^{-19} - 5 \times 1.6 \times 10^{-19})}{(6 \times 1.6 \times 10^{-19})^2} \times e^{-\frac{(2 \times 5.26 \times 10^{-8})}{(15 \times 10^{-10})}}$$

$$= 8.82 \times 10^{-9}$$

$$=$$

$$\therefore \text{Reflection coefficient (R)} = 1 - T$$
$$= 1 - 8.82 \times 10^{-9}$$
$$= 9.99 \times 10^{-1}$$

Heisenberg Uncertainty principle.

Any two ~~two~~ canonically conjugate variable cannot be measure accurately and simultaneously which is called heisenberg Uncertainty principle. We have group velocity;

$$V_g = \frac{d\omega}{dk} = \hbar \frac{d\omega}{dp} = V_{\text{particle}} = \frac{dn}{dt}$$

for small increments

$$\frac{\Delta n}{\Delta t} = \hbar \frac{\Delta \omega}{\Delta p}$$

Now,

$$\begin{aligned}\Delta \omega, \Delta t &\gg 1 \\ \Rightarrow \Delta n, \Delta p &\gg \hbar\end{aligned}$$

Direct Current.

The rate of flow of charge is called current.

Dc.

- The polarity of Dc is constant.
- a) frequency remains constant.

AC.

- The polarity of AC changes with time.
- 2) frequency changes -

Ohm's law

"At a constant physical condition like temp, pressure etc the current through the conductor is directly proportional to the potential difference applied across the end of the conductor".

Mathematically

$$\Rightarrow V \propto I$$

$$V = IR \text{ where } R \text{ is a proportionality}$$

constant called resistance at constant temp.

If a p.d V is applied on the conductor then the current flow in the conductor is given by

$$V = IR$$

Let us take a conductor having length l and cross-sectional area A . then the current flowing in the conductor I .

$$I = \frac{q}{t} = \frac{Ne}{t} = \frac{n e A l}{t} = V n e A \text{ where } n \text{ is charge carrier density.}$$

n is no. of charge carriers.

$$\text{Current Density } (J) = \frac{I}{A}$$

$$= \cancel{\text{something}} \Rightarrow I = \int J \cdot dA$$

$$\therefore n = \frac{\rho N_A}{M} \text{ where}$$

M = Molar mass

ρ = Density of material

N_A = Avogadro's number

Resistance and Resistivity.

Resistance in conductor is due to the collision of electrons. Let us take a conductor having length ' l '. If the p.d V is applied at the two ends of the conductor then:

$$E = \frac{V}{l}$$

The modified Ohm's law is

$$J = \sigma E \text{ where } \sigma \text{ is conductivity.}$$

$$\therefore \frac{J}{\sigma} = \frac{V}{l}$$

$$\Rightarrow V = \frac{lJ}{\sigma} = \frac{lI}{A\sigma} = IR$$

$$\Rightarrow R = \frac{l}{A\sigma} = \frac{\ell l}{A} \text{ where } \ell \text{ is resistivity of the material.}$$

$$\Rightarrow R = \frac{\ell l^2}{A \times l}$$

$$\Rightarrow [R \propto l^2]$$

- ① Calculate the current flowing between the R and $R/2$ where R is radius of wire. If current density is ~~10^9~~ A/m^2 & diameter of conductor is 75mm.

Solution.

$$J = 10^9 A/m^2$$

$$d = 75\text{mm} = 37.5\text{mm} = 37.5 \times 10^{-3}\text{m}$$

$$\therefore R = 37.5 \times 10^{-3}\text{m}$$

 ~~$R/2 = 18.75 \times 10^{-3}\text{m}$~~

$$A = \pi r^2$$

$$\Delta A = 2\pi r dr$$

$$I = \int_{r_2}^r J \cdot 2\pi r dr$$

$$= \sigma \times J \int_{r_2}^r r dr$$

$$= \frac{\sigma \times 10^{-9}}{2} \left[r^2 \right]_r^{r_2}$$

$$= 3.19 \times 10^{-9} (r^2 - (r_2)^2)$$

$$= 3.19 \times 10^{-9} \left((37.5 \times 10^3)^2 - \left(\frac{37.5 \times 10^3}{2}\right)^2 \right)$$

$$= 3.31 \times 10^{-7} A$$

② If a wire having length l is enlonged to $5l$. calculate the % change in resistance.

Sol:-

we know, $R \propto l^2$

$$\frac{R_1}{R_2} = \left(\frac{l_1}{5l_1}\right)^2 \quad \frac{R_1}{R_2} = \frac{1}{25}$$

$$\therefore R_2 = 25 R_1$$

Now,

$$\Delta R = \frac{25R_1 - R_1}{R_1} \times 100\%$$

$$= 2400\%$$

Show that $J = \sigma E$

Soln.

$$J = \frac{I}{A} = \frac{V}{RA} = \frac{V}{\cancel{Sl} \cdot A} = \frac{IE}{\cancel{A}} = \sigma E \quad (\because E = \frac{V}{l}),$$

Microscopic view of Ohm's law.

When an electric field E is supplied on a conductor then the force experienced by electron is

$$F = eE = \frac{eJ}{\sigma} \quad \text{--- (i)}$$

We have from Newton's law

$$F = ma$$

$$= m \cdot v_d \rightarrow \text{drift velocity.} \quad \text{--- (ii)}$$

where m is mass of electron

τ = Relaxation time / mean free time.

Equating eqn (i) & (ii)

$$\frac{eJ}{\sigma} = \frac{mv_d}{\tau}$$

$$\text{or, } \sigma = \frac{ne^2}{m} \left[\because J = nev_d \right]$$

is conductivity of the electron.

We have;

$$\frac{1}{2} m v_d^2 = \frac{3}{2} k_B T$$

The mean free path $\lambda = v_d \times \tau$.

And Mobility (μ) = $\left| \frac{v_d}{E} \right|$

Q. A current of $1.2 \times 10^{-8} A$ exist in a copper wire having diameter 3mm. Calculate the current density, electrical conductivity, mobility of electrons, charge carrier density, mean free path, ($\beta = 9 \text{ gm/sec}$, $M = 63.5$) and each Cu atom has 1 free electron.
 Solution: (Mean free time = 10^{-8} sec)

$$I = 1.2 \times 10^{-8} A.$$

$$d = 3 \text{ mm} \Rightarrow r = 1.5 \times 10^{-3} = 0.0015 \text{ m.}$$

a) $J = \frac{I}{A} = \frac{1.2 \times 10^{-8}}{\pi \times (0.0015)^2} \approx 1.69 \times 10^3 \text{ A/m}^2$

b) $\sigma = \frac{J N_A}{M} = \frac{9 \times 10^{-3} \times 6.02 \times 10^{23}}{63.5 \times 10^{-6} \times 10^{-3}} \approx 8.5 \times 10^{18}$

c) $\sigma = \frac{n e^2 \tau}{m_e} = \frac{8.5 \times 10^{18} \times (1.6 \times 10^{-19})^2 \times 10^{-8}}{9.1 \times 10^{-31}}$

Semiconductor:-

The material whose conductivity are between conductor and ~~insulator~~, at low temp., the semiconductor act as insulator and in at high temp it acts as conductor. Mainly there are two type of semiconductor.

1) Intrinsic or pure semiconductor.

The Silicon, Germanium are the intrinsic semiconductor.

2) Extrinsic or impure semiconductor.:-

When pure semiconductor is dopped by pentavalent or trivalent atom like Phosphorus & Boron then there conductivity is increases such type of semiconductor are called extrinsic or impure semiconductor.

a) N-type semiconductor;

When pentavalent atom is ~~dopped~~ ^{dopped} into silicon or Germanium then electron acts as majority charge carriers which are called N-type Semiconductor.

b) P-type Semiconductor.

When trivalent atom is dopped into Si or Ge then holes acts as majority charge carriers which are called P-type Semiconductor.

Conductivity of Semi-conductor.

$$\sigma = n_i e (M_e + M_h)$$

where M_h = mobility of hole

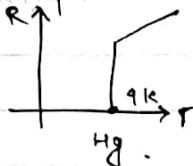
M_e = mobility of electron.

$$n_i = \sqrt{np}$$

SuperConductor.

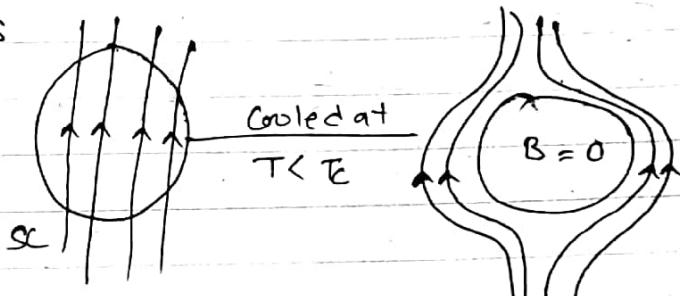
When some metal or alloy cooled at very low temperature i.e. below the critical temp^r then there resistivity or resistance suddenly drops to zero. This phenomenon is called super conductivity and the material which show this phenomenon is called Super conductor.

for eg. Mercury



Meissner's effect.

When Superconductors are placed in magnetic field and cooled at very low temperature i.e. below the critical temperature then the magnetic field lines are ejected from the superconductor which shows the diamagnetic property of the superconductor which is called Meissner's effect



Let H be the applied magnetic field on the superconductor and M be the magnetization then

Total magnetic field;

$$B = M_0(H+M)$$

Since $B=0$

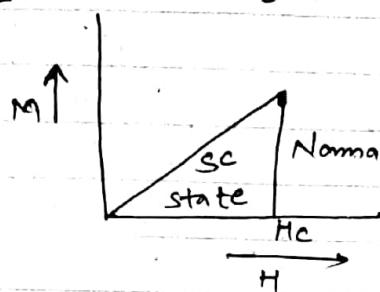
$$\text{Then } H+M=0$$

\therefore Magnetic susceptibility.

$$\text{If } \chi = \frac{M}{H} = -1 \text{ shows diamagnetic property.}$$

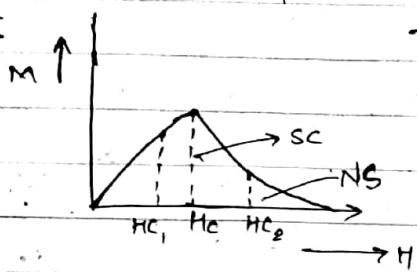
Types of Super Conductor

(1) Type I.



The super conductor which exactly follow the Meissner's effect and there exist only one vertex state, is called type I Superconductor.

(2) Type II



The superconductor which does not follow the Meissner's effect and have two or more than two vertex state is called type II superconductor.

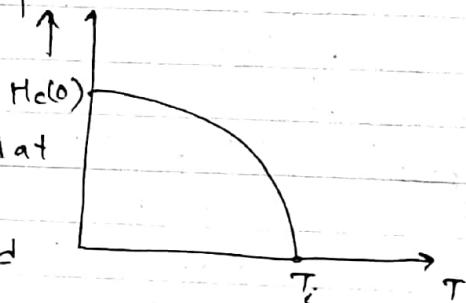
Critical Magnetic field and critical temp^r.

$$H_c(T) = H_c(0) \left[1 - \left(\frac{T}{T_c} \right)^2 \right]$$

where $H_c(0)$ = critical magnetic field at 0K

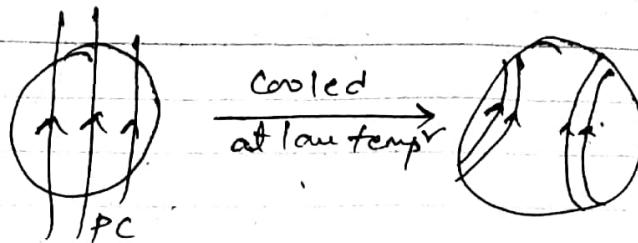
$H_c(T)$ = critical magnetic field at temp^r T

T_c = critical temp^r at zero magnetic field.



Perfect Conductor.

The conductor which has zero resistance at any temp^r. It is an idealized concept. It is used to connect the mathematical relation. When a perfect conductor is placed in magnetic field and cooled at very low temp^r then the magnetic field lines are freezed inside it.



Induced magnetic field / Modified Ampere's law / Displacement current
from Faraday law of induction, we have

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi_B}{\partial t} \quad \text{where } \Phi_B = BA = \oint \vec{B} \cdot d\vec{A} \text{ is magnetic flux.}$$

Similarly;

Maxwell suppose that the change in electric flux produced magnetic field, why not.

Then,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t}$$

Where $\epsilon_0 \frac{\partial \Phi_E}{\partial t}$ is called displacement current

where $\Phi_E = EA = \oint \vec{E} \cdot d\vec{A}$ Electric flux.

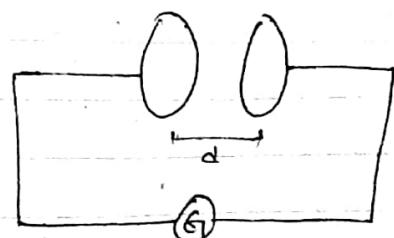
Then Modified Ampere's law becomes:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (I + I_d) \quad \text{where } I_d = \epsilon_0 \frac{\partial \Phi_E}{\partial t}$$

→ What is displacement current? Show that the displacement current is equal to the conduction current in a circularly parallel plate capacitor.

Ans: Displacement current is a fictitious current. The displacement current is given by

$$I_d = \epsilon_0 \frac{\partial \Phi_E}{\partial t} \quad \text{where } \Phi_E = EA \\ = \oint \vec{E} \cdot d\vec{l}$$



Ans.

Let us take two circularly parallel plate capacitor having area (A) and separated by a finite distance ' d '. When they are connected with the varying electric field then the charge on the capacitor at any instant of time ' t ' can be written as

$$q = CV \text{ where } C \text{ is the capacitance of the capacitor.}$$

Then current is given by

$$\begin{aligned} I &= \frac{dq}{dt} = C \frac{dV}{dt} = \frac{\epsilon_0 A}{d} \frac{dV}{dt} \\ &= \epsilon_0 A \frac{d(V/d)}{dt} \end{aligned}$$

$$= \epsilon_0 A \frac{dE}{dt}$$

$$= \epsilon_0 \frac{d\Phi_E}{dt} \text{ where } \Phi_E = EA$$

$$\boxed{I = I_d}$$

Q. Calculate the displacement current betn the capacitor plates of area $1.5 \times 10^{-2} \text{ m}^2$ and rate of electric field change is $1.5 \times 10^{12} \text{ V/ms}$ and also calculate induced magnetic field?

Solution.

$$\text{Given } A = 1.5 \times 10^{-2} \text{ m}^2$$

$$\frac{dE}{dt} = 1.5 \times 10^{12} \text{ V/ms}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m.}$$

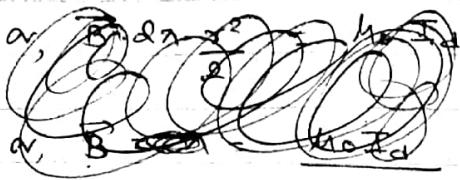
$$\text{Now } I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

$$= \epsilon_0 A \frac{dE}{dt}$$

$$= 8.85 \times 10^{-12} \times 1.5 \times 10^{-2} \times 1.5 \times 10^{12}$$

$$= 0.199 \text{ A.}$$

$$\text{Now, } \oint \vec{B} d\vec{l} = \mu_0 E_0 \frac{\partial \Phi_E}{\partial t} = \mu_0 I_d$$



$$\begin{aligned} B &= \frac{\mu_0 I_d}{2\pi r} = \frac{4\pi \times 10^{-7} \times 0.199}{2\pi \times \sqrt{1.5 \times 10^{-2}}} \\ &= \cancel{0.015} \\ &= 5.75 \times 10^{-7} \text{ T.} \end{aligned}$$

$$I_d = \frac{E_0 C_d}{2\pi r}$$

$$C_d = \frac{\pi r^2}{2\pi r}$$

Q. A variable field of $10^2 \text{ V/m}^2 \text{s}$ is applied to a parallel plate capacitor with circular plate of diameter 10cm. Calculate the displacement current density, current and induced magnetic field.

Solution:

$$\frac{dE}{dt} = 10^2 \text{ V/m}^2 \text{s}$$

$$\text{diameter } (d) = 10\text{ cm} = 5\text{ cm} = 5 \times 10^{-2} \text{ m} = 0.05 \text{ m}$$

$$I_d = ?$$

$$J_d = ?$$

$$B = ?$$

$$\begin{aligned} \text{Now, } I_d &= E_0 A \frac{dE}{dt} = 8.85 \times 10^{-12} \times \pi \times (0.05)^2 \times 10^2 \\ &= 6.9 \times 10^{-2} \text{ A.} \end{aligned}$$

$$J_D = \frac{I_D}{A} = \frac{6.9 \times 10^{-2}}{\pi \times (0.05)^2} \approx 8.785 \text{ A/m}^2.$$

$$B = \frac{\mu_0 I_D}{4\pi r} = 2.76 \times 10^{-7} T.$$

Electromagnetic wave.

1) Maxwell Equations in integral form

The basic form law of electricity and magnetism in which the Ampere's law is modified by Maxwell are called Maxwell equation.

$$\text{(i)} \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} = \oint \frac{f}{\epsilon_0} dv$$

which is the gauss law of electricity.
It states that the total electric flux inside a closed surface is equal to the charge enclosed by the surface.

$$\text{(ii)} \oint \vec{B} \cdot d\vec{A} = 0$$

which is the gauss law of magnetism.

It states that the total magnetic flux inside a closed surface is equal to zero i.e there is no existence of magnetic monopole/ consisting of dipole.

$$\text{(iii)} \oint \vec{E} \cdot d\vec{l} = - \frac{\partial \Phi_B}{\partial t} = - \frac{\partial}{\partial t} \oint_S \vec{B} \cdot d\vec{A}$$

which is the Faraday's law of induction.

It states that the change in magnetic flux produces induced emf.

$$\text{(iv)} \oint_S \vec{B} \cdot d\vec{A} = \mu_0 (I + I_d) \Rightarrow \\ = \mu_0 \left(\oint_S \vec{P} \cdot d\vec{A} + \epsilon_0 \frac{\partial}{\partial t} \oint_S \vec{E} \cdot d\vec{A} \right)$$

which is the modification of Ampere's law by Maxwell.

It states that the change in electric flux also produces the magnetic field.

Conversion of Maxwell equation into differential form

(i) We have

$$\oint_S \vec{P} \cdot d\vec{A} = \frac{q}{\epsilon_0} = \oint_V \frac{\delta dv}{\epsilon_0}$$

Using Gauss divergence theorem,

$$\oint_S \vec{P} \cdot d\vec{A} = \oint_V (\nabla \cdot \vec{P}) dv$$

$$\therefore \oint_V (\nabla \cdot \vec{P}) dv = \int_V \frac{\delta dv}{\epsilon_0}$$

$$\Rightarrow \nabla \cdot \vec{P} = \frac{\delta}{\epsilon_0} \quad \text{since } D = \epsilon_0 \vec{P}$$

is the required Maxwell 1st equation in differential form.

Note:

$$\nabla \times \nabla \Phi = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

(ii) We have

$$\oint_S \vec{B} \cdot d\vec{A} = 0$$

$$\oint_L \vec{F} \cdot d\vec{l} = \oint_V (\nabla \cdot \vec{F}) dV$$

↳ Gauss divergence theorem

$$\text{Using Gauss div. Theorem } \oint_S \vec{B} \cdot d\vec{A} = \oint_V (\nabla \cdot \vec{B}) dV \\ \oint_V (\nabla \cdot \vec{B}) dV = 0$$

$$\oint_L \vec{F} \cdot d\vec{l} = \oint_S (\nabla \times \vec{F}) \cdot d\vec{A}$$

↳ Stoke's theorem

$$\Rightarrow \nabla \cdot \vec{B} = 0 - (i)$$

Maxwell 2nd eqn in differential forms.

(iii) We have

$$\oint_L \vec{E} \cdot d\vec{l} = - \frac{\partial \Phi_B}{\partial t} = \oint_S - \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Now

$$\oint_L \vec{E} \cdot d\vec{l} = \oint_S (\nabla \times \vec{E}) \cdot d\vec{A} \text{ Using Stoke's theorem}$$

$$\therefore \oint_S (\nabla \times \vec{E}) \cdot d\vec{A} = \oint_S - \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

$$\Rightarrow \nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \text{ is Maxwell third equation in diff. form}$$

(iv) We have

$$\oint_S \vec{B} \cdot d\vec{l} = \mu_0 (I + I_\alpha) \\ = \mu_0 \oint_S \vec{J} \cdot d\vec{A} + \mu_0 E_0 \oint_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

Using Stoke's theorem

$$\oint_S \vec{B} \cdot d\vec{l} = \oint_S (\nabla \times \vec{B}) \cdot d\vec{A}$$

$$\therefore \oint_S (\nabla \times \vec{B}) \cdot d\vec{A} = \mu_0 \oint_S \left(\vec{J} + E_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

$$\epsilon_0 \vec{E} = \vec{D}$$
$$B = \mu_0 H$$

$\Rightarrow \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$ is Maxwell 4th eqn in diff. form.

since $B = \mu_0 H$

$$\nabla \times \vec{H} = \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \quad \vec{D} = \epsilon_0 \vec{E} \text{ is displacement vector.}$$

→ Write the Maxwell equation in differential form and then convert it into integral form.

Q. Show that charge conservation theorem. $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

Sol:-

We have from Maxwell 4th eqn is

$$\nabla \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}) \quad \text{--- (1)}$$

Taking divergence on both side of eqn (1).

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \left[\nabla \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) \right]$$

$$\therefore 0 = \mu_0 \left[\nabla \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) \right]$$

$$\therefore \nabla \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) = 0$$

$$\therefore \nabla \cdot \vec{J} + \epsilon_0 \frac{\partial}{\partial t} (\rho/\epsilon_0) = 0 \quad (\because \nabla \cdot \vec{E} = \rho/\epsilon_0)$$

$$\therefore \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\therefore \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \text{ proved.}$$

~~Ans~~

Maxwell equation's in free space and plane wave or em wave in free space

In free space the charge density 'ρ' and the current density (J) is zero. Such that Maxwell equation becomes:-

$$\nabla \cdot \vec{E} = 0 \quad \text{--- (i)}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{--- (ii)}$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad \text{--- (iii).}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{--- (iv).}$$

To find the e.m./plane wave equations. Taking curl on both sides of eqn (iii).

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$

$$\Rightarrow \overset{\circ}{(\nabla \cdot \vec{E})} \nabla - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \mu_0 \epsilon_0 \left(\frac{\partial \vec{E}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{--- (v)}$$

Similarly; In terms of magnetic field vector \vec{B} :

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \left\{ \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right\}$$

$$\Rightarrow \overset{\circ}{(\nabla \cdot \vec{B})} \nabla - \vec{B} \nabla^2 = \mu_0 \epsilon_0 \cancel{\nabla \times} \frac{\partial (\nabla \times \vec{E})}{\partial t}$$

$$\Rightarrow \cancel{\nabla \times} - \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{--- (vi)}$$

Comparing eqn (v) & (vi). with general diff. wave equation.

$$V^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\therefore V = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3 \times 10^8 \text{ m/s}$$

It shows that the em wave travels with speed of light (c) in free space.

The general soln of (v) & (vi) is

$$E = E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$B = B_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

In one dimension.

$$E = E_0 e^{ikn-wt}$$

$$B = B_0 e^{ikn-wt}$$

If the wave is sinusoidal

$$E = E_0 \sin(kn-wt)$$

$$B = B_0 \sin(kn-wt)$$

Q. Show that $E/B = E_0/B_0 = c$.

We have Maxwell 3rd eqn is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

In one dimension it becomes; $\frac{\partial \vec{E}}{\partial n} = -\frac{\partial \vec{B}}{\partial t}$

Now we have $E = E_0 e^{ikn-wt}$

$$B = B_0 e^{ikn-wt}$$

or, $E_0 e^{ikn-wt} \cdot iK = (-) B_0 e^{ikn-wt} \cdot \omega i$

or, $\frac{E_0}{B_0} = -\frac{\omega}{K} = \frac{\omega f}{\lambda f} = \lambda f = c$

$$\therefore \frac{E_0}{B_0} = c$$

Q. Show that magnetic energy density is equal to electric energy density.

Solution.

We have, Electric energy density (U_E) = $\frac{1}{2} \epsilon_0 E^2$ — (i)

Magnetic energy density (U_B) = $\frac{B^2}{2\mu_0}$ — (ii).

Dividing eqn ① by ④

$$\frac{U_E}{U_B} = \frac{\frac{1}{\alpha} E_0 E^2}{\frac{B^2}{\alpha \mu_0}} = \frac{\frac{1}{\alpha} E_0 E^2 \times \alpha \mu_0}{B^2} = \frac{E_0}{B^2}$$

$$= \frac{U_B}{U_E} = \frac{B^2}{\alpha \mu_0} \times \frac{\alpha}{E_0 E^2}$$

$$= \frac{(B/E)^2}{(\sqrt{\mu_0 E_0})^2}$$

$$= \frac{(Vc)^2 \times C^2}{1}$$

Maxwell equations in Non-conducting/dielectric isotropic medium

In dielectric or non conducting medium charge density ' ρ ' and current density ' J ' is zero and permeability

$$\mu = \mu_r \mu_0 \text{ and permittivity } \epsilon = \epsilon_r \epsilon_0$$

Then the Maxwell equation becomes

$$\nabla \cdot \vec{E} = 0 \quad \text{--- ①}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{--- ②}$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad \text{--- ③}$$

$$\nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{--- ④}$$

To find the e.m/plane wave equation. Taking curl on both sides of eqn ③

$$\nabla \times (\nabla \times \vec{E}) = - \frac{\partial}{\partial t} (\nabla \times \vec{B})$$

$$\Rightarrow (\nabla \cdot \vec{E}) \nabla - \nabla^2 \vec{E} = - \frac{\partial}{\partial t} \mu \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{--- ⑤}$$

Similarly in terms of magnetic field vector \vec{B} .

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \mu_0 E \frac{\partial \vec{E}}{\partial t}$$

$$\therefore \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \quad \rightarrow \textcircled{i}$$

Comparing eqn \textcircled{i} & \textcircled{v} with general diff wave eqn

$$\begin{aligned} \text{Velocity of em wave is } v &= \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{\mu_0 \rho_0 \cdot \epsilon_r \epsilon_0}} \\ &= \frac{c}{\sqrt{\rho_0 \epsilon_r}} \end{aligned}$$

Maxwell eqn in conducting medium.

In conducting medium let the ~~conductivity~~ ^{conductivity} be σ and charge density ρ is zero.

Then the Maxwell eqn becomes

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \left[\sigma \vec{E} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right].$$

Then to find cm/plane wave equation.

$$\nabla \times (\nabla \times \vec{E}) = - \frac{\partial}{\partial t} (\nabla \times \vec{B})$$

$$(\nabla \cdot \vec{E}) \nabla - \nabla^2 \vec{E} = - \frac{\partial}{\partial t} \mu_0 \left[\sigma \vec{E} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

$$- \nabla^2 \vec{E} = - \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\therefore \nabla^2 \vec{E} = \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.$$

$$\nabla \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

$$= \mu_0 \vec{J}$$

Similarly in terms of magnetic field.

$$\nabla^2 \vec{B} = \mu_0 \frac{\partial \vec{B}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Maxwell eqn in slowly varying field.

In slowly varying field electric field \vec{E} and magnetic field \vec{B} slowly changes with time so $\frac{\partial \vec{E}}{\partial t} = 0$ and $\frac{\partial \vec{B}}{\partial t} = 0$.

$$\nabla \cdot \vec{E} = \rho / \epsilon_0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Then)

$$\nabla \times (\nabla \times \vec{E}) = - \frac{\partial}{\partial t} (\nabla \times \vec{B})$$

$$\Rightarrow (\nabla \cdot \vec{E}) \nabla - \nabla^2 \vec{E} = - \frac{\partial}{\partial t} \mu_0 \vec{J}$$

$$\Rightarrow \frac{\rho}{\epsilon_0} \nabla - \nabla^2 \vec{E} = - \frac{\partial}{\partial t} \mu_0 \vec{J}$$

$$\Rightarrow \frac{\rho}{\epsilon_0} \nabla = \nabla^2 \vec{E} + \mu_0 \frac{\partial \vec{J}}{\partial t}$$

Poynting vector.

When an object is placed in the path of electromagnetic wave then the em wave transfers the energy on the object in the form of electric and magnetic energy. The rate of transfer of energy per unit area is called poynting vector. It can be written as

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{4\pi}$$

Let us take an object having area A and thickness dn is placed in a path of em wave then at any instant of time t the energy transfer on the object can be written as

$$dU = dU_B + dU_E$$

$$= (U_B + U_E)Adn \quad \text{where } U_E = \text{electric energy density}, \\ U_B = \text{magnetic energy density}$$

Dividing both sides by dt ,

$$\frac{dU}{dt} = (U_E + U_B) \cdot A \frac{dn}{dt}$$

$$\frac{dU}{Adt} = (U_E + U_B) \cdot c$$

$$\text{or, } S = \left(\frac{1}{2} E_0 E^2 + \frac{B^2}{2\mu_0} \right) c$$

$$\text{or } S = \left(\frac{\mu_0 E_0 E^2 + B^2}{2\mu_0} \right) c$$

$$\text{or } S = \left(\frac{E^2}{c^2} + \frac{B^2}{2\mu_0} \right) c$$

$$\text{or, } S = \frac{E^2 + B^2 c^2}{2\mu_0 c}$$

We have

~~$E/B = c$~~

$$\text{Then } S = \frac{\partial E^2}{\partial \mu_0 c} = \frac{EB}{\mu_0}$$

In vector form; $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$. Since E and B are 90° to each other

The electromagnetic wave is sinusoidal such that we have-

$$E = E_0 \sin(kn - \omega t)$$

$$B = B_0 \sin(kn - \omega t)$$

$$\therefore S = \frac{E_0 B_0}{2\mu_0} \sin^2(kn - \omega t)$$

average

The average pointing vector or average energy intensity is

$$\langle S \rangle = I_{av} = \frac{E_0 B_0}{2\mu_0} \left[\because \langle \sin^2(kn - \omega t) \rangle = \frac{1}{2} \right]$$

$$\langle S \rangle = \frac{E_0}{\sqrt{2}} \cdot \frac{B_0}{\sqrt{2}} \cdot \frac{1}{\mu_0}$$

$$= \frac{E_{rms} B_{rms}}{\mu_0}$$

$$\text{Also; } \langle S \rangle = \frac{E_0^2}{2\mu_0 c} \left[\because E/B_0 = c \right]$$

$$I_{av} = \frac{c B_0 B_0}{2\mu_0} \\ = \frac{B_0^2 c}{2\mu_0}$$

$$= \frac{E_{rms}^2}{\mu_0 c} \\ = \frac{B_{rms}^2}{\mu_0} \cdot c$$

$$\therefore I_{av} = \langle S \rangle = \frac{P}{A} \quad \text{where } P = \text{Power} \\ A = \text{Area}$$

The maximum electric field from an isotropic point source of power is dV/m . What is the maximum value of magnetic field, average energy and power of source?

Solution.

Given: Electric field (E) = dV/m

$$I_{av} = \langle S \rangle = \frac{E_0^2}{2\mu_0 c} = \frac{\text{d}^2}{\text{d}x^2 \pi \times 10^{-7} \times 3 \times 10^8} = 0.0853 \text{ W/m}^2$$

✓

$$I_{av}(s) = \frac{P}{A}$$

$$\text{Also, } B_0 = \frac{E_0}{c} = \frac{0.0053}{3 \times 10^8} = 6.667 \times 10^{-9} \text{ T.}$$

$$\therefore 0.0053 = \frac{P}{4\pi \times 10^2}$$

$$\therefore P = 6.660 \text{ W.}$$

Q. Calculate the maximum electric and magnetic field if solar constant
 $I_s = 1.9 \text{ kW/m}^2$

Solution;

$$I_{av} = \frac{1.9 \text{ kW}}{\text{m}^2} = 1.9 \times 10^3 \text{ W/m}^2.$$

$$\therefore I_{av} = \frac{E_0^2}{240 \text{ C.}}$$

$$1.9 \times 10^3 = \frac{E_0^2}{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8}$$

$$\therefore E_0 = \sqrt{1.06 \times 10^6} = 1.03 \times 10^3 \text{ V}$$

$$I_{av} = \frac{B_0^2 C}{2 M_0}$$

$$B_0^2 = \frac{1.9 \times 10^3 \times 2 \times 4\pi \times 10^{-7}}{3 \times 10^8}$$

$$\therefore B_0 = 3.92 \times 10^{-6} \text{ T.}$$

$$1 \text{ mile} = 1852 \text{ m.}$$

Q. What is the average magnitude of pointing vector s_{mile} from the radio ~~transmitter~~ broad casting isotropically with average power 250 kW?

Sol:-

$$\text{Av. Power } \langle P \rangle = 250 \text{ kW} = 250 \times 10^3 \text{ W.} \quad r = \text{smile} = 5 \times 1852 = 9.26 \times 10^3 \text{ m}$$

$$\begin{aligned} I_{\text{av}} &= \langle s \rangle = \frac{P}{A} = \frac{250 \times 10^3}{4\pi \times (9.26 \times 10^3)^2} \\ &= 2.38 \times 10^{-9} \text{ W/m}^2. \end{aligned}$$

Radiation pressure.

Q. If intensity of direct sunlight at a point on the surface of earth is 0.85 kW/m^2 . Calculate the momentum density and radiation pressure (rate of flow of momentum).

Sol:-

$$\begin{aligned} \text{Given; } I &= 0.85 \text{ kW/m}^2 = 850 \text{ W/m}^2. & \frac{\Delta P}{V} &= \frac{\Delta E}{C \cdot A \cdot l} \\ \text{Radiation pressure (Pr)} &= \frac{I}{c} & &= \frac{\Delta E}{\Delta t \cdot C \cdot A l} \\ &= \frac{850}{3 \times 10^8} & &= \frac{850}{\Delta t} \\ &= 2.83 \times 10^{-6} \text{ Pa.} & & \end{aligned}$$

$$\begin{aligned} \text{Momentum density} &= \frac{I}{c^2} & &= \frac{P}{C A C} \\ &= \frac{850}{(3 \times 10^8)^2} & &= \frac{I}{c^2} \\ &= 9.44 \times 10^{-15} \text{ Pa/m}^3. & & \end{aligned}$$

$$\frac{\text{W/m}^2}{(\text{m/s})^2} \times \frac{\text{m}^2}{\text{m}^2}$$

Q. A laser beam having power 10mW and wavelength 680nm. The laser beam is focused until its diameter is equal to the ~~d~~ on a sphere. If the sphere has density $5 \times 10^9 \text{ kg/m}^3$. What is the beam intensity at the sphere, the radiation pressure on the sphere, corresponding force and magnitude of acceleration.

801-7

$$I = \frac{P}{A} = \frac{10 \times 10^{-3}}{\pi \times (680 \times 10^{-9})^2}$$

$$= 6.88 \times 10^9 \text{ W/m}^2$$

$$\text{Radiation pressure (Pr)} = I/c$$

$$= \frac{6.88 \times 10^9}{3.0 \times 10^8}$$

$$= 2.29 \times 10$$

$$I = \frac{P}{A} = \frac{P_0}{4\pi r^2}$$

$$Pr = \frac{I}{c}$$

radiation
press

~~E = mc²~~

Electromagnetism).

Magnetic field intensity (B) = $\mu_0(H+M)$ and its unit is Tesla.

Magnetic flux:-

The total number of imaginary lines of magnetic field in a surface is called magnetic flux. Let B be the magnetic field intensity then the magnetic flux

$$\Phi_B = BA = \oint \vec{B} \cdot d\vec{A}$$
 in any closed surface.

In a closed surface the total no. of incoming magnetic field lines is equal to the total number of outgoing magnetic field lines such that

$\oint \vec{B} \cdot d\vec{A} = 0$ which is called gauss law of magnetism It states that, "there is no existence of magnetic monopole".

Magnetic field and force:-

Let us consider a charged particle having charge q is in dynamic motion. When magnetic field ' B ' is applied on a charged particle moving with velocity ' v ' then the magnetic force experienced by charged particle is

$$F_B \propto B$$

$$F_B \propto q$$

$$F_B \propto v$$

$$F_B \propto \sin \theta$$

Then, $F_B = Bqv \sin \theta$
 $= q(\vec{v} \times \vec{B})$

$$F_e = q\vec{E}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$
 which is a lorentz force

Magnetic force on a current carrying conductor.

Consider a conductor having length l and area A . Let N be the total number of electron flowing in a conductor and I be the current flowing in the conductor. Then the charge is given by

$$q = Ne = nAxe$$

Then the magnetic force is given by

$$F_B = Bqv \sin \theta$$

$$= BnAxe v \sin \theta$$

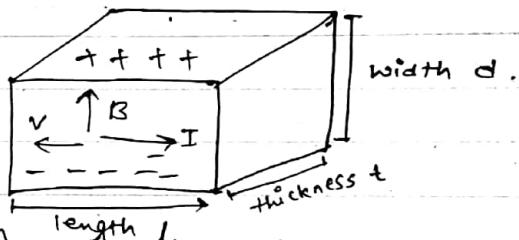
$$= IBl \sin \theta \quad [\because I = veA]$$

$$\therefore \vec{F}_B = I(\vec{l} \times \vec{B})$$

$$\therefore d\vec{F}_B = I(d\vec{l} \times \vec{B}) \quad (\text{for elementary length } dl)$$

Hall effect.

Let us take a conducting strip having length ' l ', width ' w ' and thickness ' t ' as shown in figure. The current I is flowing along lengthwise in x -direction. When



magnetic field is applied along z direction that is \perp to thickness. then a voltage is developed across width or along y -direction which is called Hall potential. It is called Hall effect in which Hall potential is \perp to both ' B ' and ' I ' at steady state.

The magnetic force experienced by a charge particle is equal to electric force experienced by charge particle.

$$\text{i.e. } eE_H = Bev$$

where E_H is Hall field.

$$\text{ie. } e \frac{E_H}{d} = Be \Rightarrow$$

$$\Rightarrow V_H = Bd$$

$$\text{or, } V_H = B \cdot \frac{I}{neA} d$$

$$\text{or, } V_H = \frac{BId}{ne \times d}$$

$$\therefore V_H = \frac{BI}{ne}$$

* Hall coefficient.

$$R_H = \frac{E_H}{J_n B} = \frac{BV}{J_n B} = \frac{BV}{nevB} = \frac{1}{ne}$$

* Hall Mobility

$$M_H = \left| \frac{v}{E} \right| = \frac{J}{nevE} = \frac{\sigma}{ne} = R_H \sigma$$



* Hall angle.

$$\begin{aligned} \theta &= \frac{J_y}{J_x} = \frac{\sigma E_y}{J_n} = \frac{\sigma E_H}{J_n} \\ &= \sigma \frac{E_H}{J_n B} \\ &= \sigma R_H B \end{aligned}$$

Hall Resistance

$$\text{We have, } V_H = \frac{BI}{ne}$$

$$\Rightarrow V_H = IR \text{ where } R = \frac{B}{ne}$$

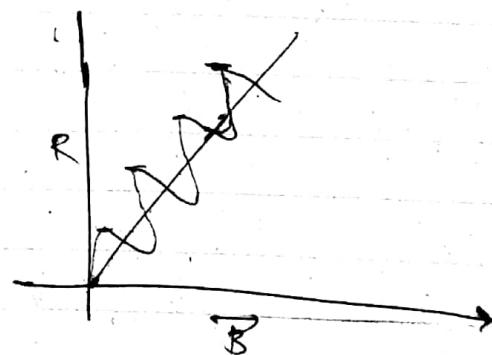


Fig: Hall resistance quantized.

$$n - \text{charge carrier density} = \frac{\text{Total charge}}{\text{Volume}}$$

Q: In a Hall experiment a current of 3A sent lengthwise through a conductor of ~~1cm~~ 1cm width, 4cm length and 1mm thick produces a transverse hall voltage of 10mV. When a magnetic field of 1.5T is passed perpendicular through the thickness of conductor. Calculate drift velocity, charge carries density, hall field, hall coefficient and hall resistance.

Solution:

$$\text{Given, } I = 3\text{A}, \quad l = 4\text{cm} = 4 \times 10^{-2}\text{m}$$

$$d = 1\text{cm} = 1 \times 10^{-2}\text{m}$$

$$t = 1\text{mm} = 1 \times 10^{-3}\text{m}$$

$$V_H = 10\text{mV} = 10 \times 10^{-6}\text{V}$$

$$B = 1.5\text{T}$$

$$\text{Now, } V_H = Bd$$

$$\therefore 10 \times 10^{-6} = 1.5 \times v \times 1 \times 10^{-2}$$

$$\therefore v = 6.66 \times 10^{-7} \text{ m/s.}$$

Now;

$$E_H = Bv = 1.5 \times 6.66 \times 10^{-7} \approx 10^{-9} \text{ V.}$$

$$\text{Hall coeff (R}_H) = \frac{1}{ne} = \frac{1}{\left(\frac{BI}{V_{H\text{eff}}}\right) \cdot e} = \frac{10 \times 10^{-6} \times 1 \times 10^{-2}}{1.5 \times 3} = 2.22 \times 10^{-12}.$$

$$\text{Hall resistance (R)} = \frac{B}{ne} = \frac{1.5}{2.81 \times 10^{30} \times 1.6 \times 10^{-19} \times 1 \times 10^{-6}} = 3.33 \times 10^6$$

$$n = \frac{BI}{V_{H\text{eff}}} = \frac{1.5 \times 3}{10 \times 10^{-6} \times e \times 1 \times 10^{-6}} = 2.81 \times 10^{30} \text{ /m}^3.$$

(Q) A copper strip having 4.5cm width, 1.5mm thick is placed in a magnetic field of αT per to the plane of the stripe. If a current of $1.5A$ is set up into the stripe. Calculate the hall potential difference across a stripe.

Given

$$d = 4.5\text{cm} = 4.5 \times 10^{-2}\text{m}$$

$$t = 1.5\text{mm} = 1.5 \times 10^{-3}\text{m}$$

$$B = \alpha T$$

$$I = 1.5A$$

$$\text{Soln. } V_H = ?$$

$$eE_H = BeJ$$

$$\therefore \frac{V_H}{d} = B \cdot e J$$

$$\therefore V_H = Bd$$

$$V_H = \frac{BI}{(n\epsilon)}$$

$$A = \pi r^2$$

$$= \pi \times \left(\frac{1.5 \times 10^{-3}}{2}\right)^2$$

$$(I \neq v \epsilon n A)$$

27.

Maximum KE of particle is

$$(K.E)_{\max} = \frac{1}{2} m V_{\max}^2$$

$$= \frac{1}{2} m \cdot \frac{B^2 q^2}{m^2} r_{\max}^2$$

$$= \frac{B^2 q^2 r_{\max}^2}{2m}$$

$$= \frac{B^2 q^2}{4\pi^2 m^2} \cdot 2q^2 m \cdot r_{\max}^2$$

$$\therefore K.E_{\max} = \frac{q^2 m f^2 r_{\max}^2}{8\pi^2}$$

We have:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Due to the constant magnetic field and has limitation on size of Dees we use synchrotron to obtain high energy particles in $f-f_0$ and m be the mass then to obtain the resonance form is constant in synchrotron.

Suppose a cyclotron is operated at an frequency of 15MHz of ~~Dee~~ Dee of radius 55cm. What is the resulting KE of nuclear? What is the magnetic field needed for the deuterium? ($m_D = 3.34 \times 10^{-27} \text{ kg}$).

(Q.12)

$$\text{Frequency } f = 15 \text{ MHz} = 15 \times 10^6 \text{ Hz}$$

$$\text{Radius } (r) = 55 \text{ cm} = 55 \times 10^{-2} \text{ m}$$

$$\text{Now, } K.E_{\max} = \frac{q^2 m f^2 r_{\max}^2}{8\pi^2}$$

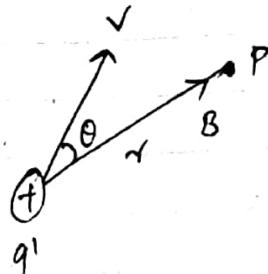
$$= \frac{2q^2}{8\pi^2} \times 3.34 \times 10^{-27} \times (15 \times 10^6)^2 \times (55 \times 10^{-2})^2$$
$$= 9.47 \times 10^{-19} \text{ J}$$

Biot-Savart Law.

$$B = \frac{qv \sin \theta}{r^2}$$

$$B = \frac{qv \sin \theta \cdot r}{r^3}$$

$$\therefore \vec{B} = \frac{q(v \times \vec{r})}{r^3}$$



Let us take a charged particle having charge q moving with velocity v then the magnetic field due to the charged particle at point P which is at distance ' r ' from charged particle is given by

$$B \propto q$$

$$B \propto v$$

$$B \propto \frac{1}{r^2}$$

$$B \propto \sin \theta$$

$$\therefore B = \frac{qv \sin \theta}{r^2} \frac{\mu_0}{4\pi}$$

Magnetic field due to current carrying element.

Let us take a conductor carrying current I , let dl be the ~~segment~~ segment of the conductor then the charge on the segment is

$$= Nq = nA dl \cdot q$$

where A is the cross sectional area.

Then the magnetic field due to the current element dl is

$$\vec{dB} = \frac{\mu_0}{4\pi} \frac{nA dl q v \sin \theta}{r^2}$$

$$\vec{dB} = \frac{\mu_0}{4\pi} \frac{I dl \sin \theta}{r^2}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

$$\vec{B} = \int \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

Application of Biot-Savart law.

(1) Magnetic field due to a conductor having length l .

Let us consider a conductor having

length l carrying current I as shown in figure. Let ' $d\vec{l}$ ' be the small length element. Then the magnetic field due to small length element is

$$dB = \frac{\mu_0}{4\pi} \frac{Id\vec{l} \sin \theta}{r^2}$$

$$\sin \theta = \sin(\pi - \theta) = \frac{R}{\sqrt{R^2 + r^2}}$$

Then,

$$\begin{aligned} \text{Total magnetic field } B &= 2 \int_0^{l/2} \frac{\mu_0}{4\pi} I d\vec{l} \cdot \frac{R}{\sqrt{R^2 + r^2}} \cdot \frac{1}{r^2} \\ &= 2 \int_0^{l/2} \frac{\mu_0}{4\pi} I \frac{d\vec{l} \cdot R}{(R^2 + r^2)^{3/2}} \\ &= 2 \frac{\mu_0}{4\pi} I R \int_0^{l/2} \frac{dn}{(R^2 + r^2)^{3/2}} \\ &= \frac{2\mu_0 I R}{4\pi} \left[\int_0^{l/2} \frac{dn}{(R^2 + r^2)^{3/2}} \right] \end{aligned}$$

PUT $r = R \tan \alpha$

$$dn = R \sec^2 \alpha d\alpha$$

AS $\alpha \rightarrow 0$ $\alpha \rightarrow 0$
 $\alpha \rightarrow l/2$ $\alpha \rightarrow \infty$

Put $n = R \tan \theta$

$$dn = R \sec^2 \theta d\theta$$

$$\overline{B}^2 = \frac{\mu_0 I}{2\pi} R \int_0^{l/2} (R^2 + n^2)^{-3/2} dn$$

$$\int (R^2 + n^2)^{-3/2} dn$$

$$= \int \frac{1}{(R^2 + n^2)^{3/2}} dn$$

$$= \int \frac{1}{(R^2 + R^2 \tan^2 \theta)^{3/2}} dn$$

$$= \int \frac{1}{R^3 \sec^3 \theta} dn$$

$$= \frac{1}{R^3} \int \cos^3 \theta \cdot R \sec^2 \theta d\theta$$

$$= \frac{1}{R^2} \int \cos \theta d\theta$$

$$= \frac{1}{R^2} \sin \theta$$

$$= \frac{\sin \theta}{R^2}$$

$$= \frac{1}{R^2} \left[\frac{R}{\sqrt{R^2 + n^2}} \right]_{0}^{l/2}$$

$$= \frac{1}{R} \left[\frac{1}{\sqrt{R^2 + (l/2)^2}} - \frac{1}{\sqrt{R^2}} \right]$$

$$= \frac{1}{R} \left[\frac{1}{\sqrt{R^2 + l^2}} - \frac{1}{R} \right]$$

$$= \frac{1}{R \sqrt{R^2 + l^2}} - \frac{1}{R^2}$$

At the centre of the coil $m=0$ such that magnetic field at centre. $B = \frac{\mu_0 I}{2R}$. If there are N coils having same radius

then the M.F due to N coils is

$$B_N = \frac{\cancel{N} \cancel{I}}{\cancel{2R}} \frac{\mu_0 N I R^2}{2(m^2 + R^2)^{3/2}}$$

Magnetic dipole moment

$$M_d = NIA$$

$$\therefore B_M = \frac{\mu_0 N I \pi R^2}{2(\pi^2 + R^2)^{3/2}}$$

$$\therefore B_M = \frac{\mu_0 M_d}{2\pi(m^2 + R^2)^{3/2}}$$

(3) Magnetic field due to circular arc.

Let us consider a circular arc having radius 'R'. Let 'dl' be the small length element. Then magnetic field due to small length element is

$$dB = \frac{\mu_0 I dl}{4\pi R^2}$$

Total magnetic field

$$\begin{aligned} B &= \int dB = \frac{\mu_0 I}{4\pi R^2} \int_0^{\pi/2} dl \\ &= \frac{\mu_0 I}{4\pi R^2} \int_0^{\pi/2} R d\theta \quad (\because dl = R d\theta) \\ &= \frac{\mu_0 I}{4\pi R} \cdot \frac{\pi}{2} \\ B &= \frac{\mu_0 I}{8R} \end{aligned}$$

Ampere's Law.

The magnetic field produced by steady current flowing in a closed loop which is symmetrical in nature is given by

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Application.

→ Calculate the magnetic field inside and outside a current carrying wire.

Let us take a wire having radius 'a' carrying current I then the magnetic field outside the wire ($r > a$). We consider a Amperian loop at a distance of r from the centre. Then

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \text{ for } (r > a)$$

$$a B \cdot 2\pi r = \mu_0 I$$

$$\therefore B = \frac{\mu_0 I}{2\pi r}$$

Case II : Inside $r < a$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I'$. [I' is the current enclosed by Amperian loop in fig (ii)]

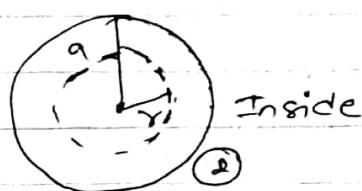
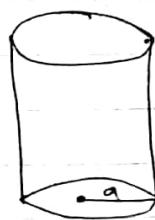
Now, we have the current density J is constant everywhere.

$$J = \frac{I}{A} = \frac{I'}{A'}$$

$$\Rightarrow \frac{I}{\pi a^2} = \frac{I'}{\pi r^2}$$

$$\therefore I' = \frac{r^2}{a^2} I$$

$$\therefore \pi r B = \mu_0 \frac{r^2}{a^2} I \quad \therefore B_{\text{inside}} = \frac{\mu_0 I}{2\pi a^2} r$$



$$\begin{aligned} I &= \frac{I}{a} \\ \frac{I}{\pi a^2} &= \frac{I'}{\pi r^2} \end{aligned}$$

$$I' = \frac{r^2}{a^2} I$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I'$$

$$\frac{B}{B} \cdot 2\pi r = \mu_0 I'$$

$$\frac{B}{B} = \frac{\mu_0 I}{2\pi a^2 r}$$

$$\therefore E_{ind} = -\frac{\partial \Phi_B}{\partial t}$$

Induced electric field.

Let us take a conducting loop having charge 'q' which is moving along the direction of electric field then the charged particle of the loops experience the electric force ~~force~~ is

$$\vec{F} = q\vec{E}$$

From Faraday's law of induction we have:

$$E_{ind} = -\frac{\partial \Phi_B}{\partial t}$$

The work done by the electric force is

$$W = F \cdot d\vec{r}$$

$$W = qE \cdot d\vec{r} \quad \text{--- (i)}$$

Workdone by induced emf is

$$W = qE_{ind} \quad \text{--- (ii)}$$

$$\therefore E_{in} = E \cdot d\vec{r} \quad (\text{on equating (i) & (ii)})$$

$$-\frac{\partial \Phi_B}{\partial t} = \oint \vec{E} \cdot d\vec{l}$$

Using Stokes theorem

$$\oint \vec{E} \cdot d\vec{l} = \oint (\nabla \times \vec{E}) \cdot d\vec{A}$$

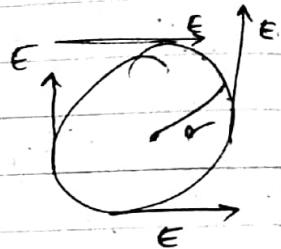
$$\text{and } \Phi_B = \oint \vec{B} \cdot d\vec{A}$$

$$\text{Now, } -\frac{\partial \Phi_B}{\partial t} = \oint (\nabla \times \vec{E}) \cdot d\vec{A}$$

$$\text{or } -\frac{\partial \oint \vec{B} \cdot d\vec{A}}{\partial t} = \oint (\nabla \times \vec{E}) \cdot d\vec{A}$$

$$\text{or } -\frac{\partial \vec{B} \cdot d\vec{A}}{\partial t} = \nabla \times \vec{E}$$

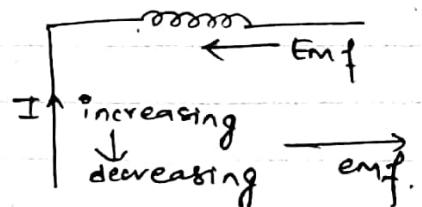
$$\therefore \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$



which shows that the variation in magnetic field on conducting loop gives the induced electric field.

Lenz's law.

The induced current flows always in such direction as to oppose the change causing it.



// Induction and Energy transformation.

Let us take a conductor having length 'l' is move on a magnetic field ~~B~~ \vec{B} then the magnetic force experience by the charged particle is

$$\vec{F} = q(\vec{v} \times \vec{B}) \quad \textcircled{1}$$

The magnetic force is ~~per~~ to both velocity and \vec{B} . Such that the electrons of the conductor accumulated at lower surface leaving positive charge on upper surface which set up the potential difference between charged particle then the electric force experience by charged particle is

$$\vec{F} = q\vec{E} \quad \textcircled{2}$$

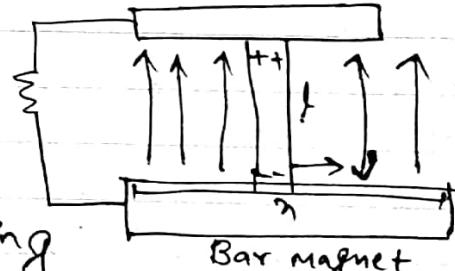
Taking magnitude and equating $\textcircled{1} + \textcircled{2}$

$$E = VB$$

Now the potential difference

$$V = El$$

$$\text{or, } V = BBl \quad \textcircled{3}$$



-R

and from ~~opposite~~ induced emf

$$E_{\text{ind}} = -\frac{d\phi_B}{dt} \quad \text{where } \phi_B = BA = Bl^2$$

$$\therefore E_{\text{ind}} = -Bl \frac{dv}{dt} = -Blv \quad \text{--- (iv)}$$

from (iii) & (iv) we get the mechanical power is equal to the electrical power.

$$P = \frac{B^2 l^2 v^2}{R}$$

i.e Mechanical Energy \rightarrow Electrical energy \rightarrow Thermal energy

Self Inductance

An induced emf appears in a coil if we change the current in the coil is called self induction.

According to Henry; the total magnetic flux on the coil is directly proportional to the current flowing in the coil. Let 'N' be the number of turns on the coil then

$$N\phi_B \propto I$$

$$\textcircled{1} \rightarrow N\phi_B = LI \quad \text{where } L \text{ is the self inductance}$$

Differentiating $\textcircled{1}$

$$N \frac{d\phi_B}{dt} = L \frac{dI}{dt}$$

$$\text{we have, } E_{\text{ind}} = -\frac{d\phi_B}{dt} = -L \frac{dI}{dt} \quad \text{--- (v)}$$

Mutual Inductance.

The phenomenon of producing an induced emf in a circuit due to variation of current in a neighbouring circuit is called Mutual Inductance.

Let us take a two current loop ① & ② the current I_1 is flowing on loop ① as shown in figure. Let B_1 be the magnetic field produced at loop ② then the magnetic flux on loop ② due to loop ① is

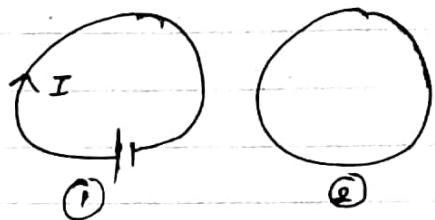
$$\Phi_{21} = \oint \vec{B}_1 \cdot d\vec{s}_2$$

$$\Phi_{21} = M_{21} I_1$$

Now

$$-\frac{\partial \Phi_{21}}{\partial t} = -M_{21} \frac{\partial I_1}{\partial t}$$

$$\therefore E_{21} = -M_{21} \frac{dI_1}{dt}$$



Similarly if current I_2 is flowing on loop ② then the induced emf on loop ① can be written as

$$E_{12} = -M_{12} \frac{\partial I_2}{\partial t}$$

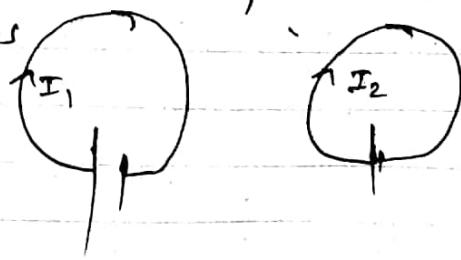
Relationship b/w Mutual inductance and Self inductance

Let us take a two current loops in which current I_1 is flowing in loop ① and I_2 is flowing in loop ② having self inductance of loop ① is L_1 and loop ② is L_2 .

Then let N_1 be the no. of turns on loop ① and N_2 be the no. of turns on loop ②

$$N_1 \Phi_1 = L_1 I_1$$

$$N_2 \Phi_2 = L_2 I_2$$



$$\text{and } N_2 \phi_{21} = M_{21} I_1$$

$$N_1 \phi_{12} = M_{12} I_2$$

$$\frac{N_1 N_2 \phi_{21} \phi_{12}}{I_1 I_2} = M_{12} M_{21}$$

Since $\phi_{21} = \phi_1$ and $\phi_{12} = \phi_2$

$$\text{With } M_{12} = M_{21} = M$$

$$\text{Then } M^2 = \frac{L I_1 L I_2}{I_1 I_2}$$

$$\therefore M = \sqrt{L C_2}$$

~~Q~~ Calculate the self inductance of solenoid and toroid.

Now, the magnetic field on the axis of solenoid having length l , number of turns 'N' and current flowing on solenoid is I the magnetic field B is

$$B = \frac{\mu_0 N I}{l}$$

Let A be the cross sectional area of solenoid then

$$\Phi_B = B A = \frac{\mu_0 N I A}{l}$$

$$\text{Total flux} = \Phi_{B_{\text{total}}} = N \Phi_B$$

$$\Phi_{B_{\text{total}}} = \frac{\mu_0 N^2 I A}{l}$$

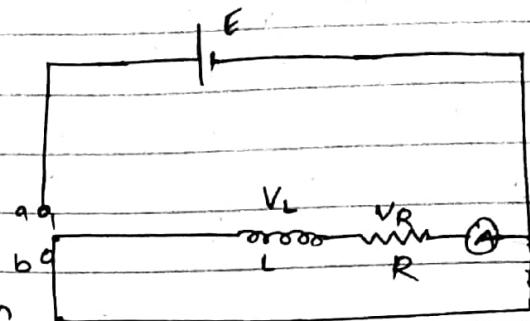
$$E_{\text{ind}} = -\frac{\partial \Phi_B}{\partial t} = -L \frac{dI}{dt}$$

$$\text{or } -\frac{\partial}{\partial t} \left(\frac{\mu_0 N^2 I A}{l} \right) = -L \frac{dI}{dt}$$

$$\text{or } \frac{\mu_0 N^2 A}{l} \frac{-\partial I}{\partial t} = -L \frac{dI}{dt}$$

Series L-R circuit.

Let us take a battery having emf (E), inductor having inductance ' L ', Resistor having resistor R are connected in series with ammeter A as shown in figure.



Growing Current:

When terminal a is connected then the current flow through the inductor and resistor, then at instant of time ' t ' using Kirchoff's law we can write

$$E = V_L + V_R$$

$$\text{or, } \frac{I_0 R}{L} = L \frac{dI}{dt} + IR \quad L \frac{dI}{dt} = I_0 \frac{R}{L} - IR$$

$$\text{or, } \frac{dI}{dt} = (I_0 - I) \frac{R}{L}$$

$$(I_0 - I) \frac{R}{L}$$

$$\text{or, } \frac{dI}{dt} = -(I - I_0) \frac{R}{L}$$

$$\text{or, } \int_{0}^{I} \frac{dI}{(I - I_0)} = -\frac{R}{L} \int_{0}^{t} dt$$

$$\text{or, } \ln(I - I_0) \Big|_0^I = -\frac{R}{L} t \Big|_0^t$$

$$\therefore \ln(I - I_0) - \ln(-I_0) = -\frac{R}{L} t$$

$$\text{or, } \ln \left(\frac{I - I_0}{-I_0} \right) = -\frac{R}{L} t$$

$$\text{or, } \frac{I - I_0}{-I_0} = e^{-\frac{R}{L} t}$$

$$\text{or, } I = I_0 (1 - e^{-\frac{R}{L} t}) \quad \text{which is the growing current in LR circuit}$$

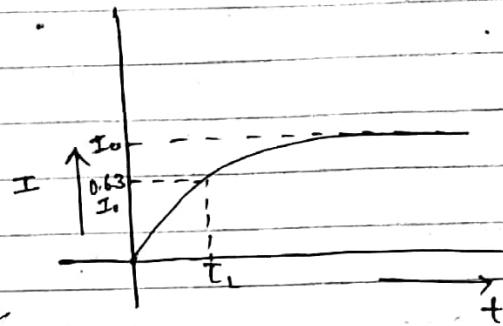
Let $T_L = \frac{L}{R}$ is the inductive time constant.

$$\text{At } t = T_L$$

$$I = 0.63 I_0$$

When the current reaches to 63% of final or equilibrium current then the inductive time constant

$$T_L = t$$



Decay current in LR circuit.

When terminal b is connected and a is open then the current starts to decay. Using Kirchoff's law;

$$V_L + V_R = 0$$

$$\text{or, } \frac{L dI}{dt} + IR = 0$$

$$\text{or, } \frac{dI}{I} = -\frac{R}{L} dt$$

$$\text{or, } \int_{I_0}^I \frac{dI}{I} = -\frac{R}{L} \int_0^t dt$$

$$\text{or, } \ln I \Big|_{I_0}^I = -\frac{R}{L} t$$

$$\text{or, } \ln \left(\frac{I}{I_0} \right) = -\frac{R}{L} t$$

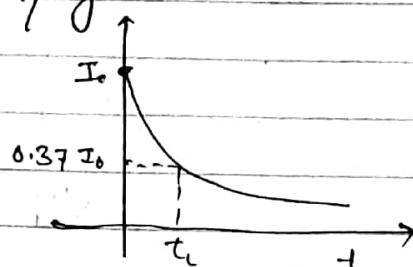
$$\therefore I = I_0 e^{-\frac{R}{L} t}$$

$$\therefore I = I_0 e^{-t/T_L}$$

At $t = T_1$

$$\therefore I = 0.37 I_0$$

The time required to reach the current of 37% of initial current is called inductive time constant in case of decaying current.



- ① A solenoid having an inductance of 0.6 mH is connected in series with 1.2 k Ω resistor. If a 12V battery is connected across the pair. How long will it take for the current to reach 80% of equilibrium value? What is the current through the resistor at time $t = T_1$.

Solution,

$$\text{Here; } L = 0.6 \text{ mH} = 0.6 \times 10^{-3} \text{ H}$$

$$R = 1.2 \text{ k}\Omega = 1.2 \times 10^3 \Omega$$

$$E = 12 \text{ V.}$$

We know

$$I = I_0 (1 - e^{-R_L t})$$

$$\therefore 80\% \text{ of } I_0 = I_0 (1 - e^{-R_L t})$$

$$\therefore \frac{4}{5} I_0 = I_0 (1 - e^{-R_L t})$$

$$\text{or, } \frac{4}{5} = 1 - e^{-R_L t}$$

$$\text{or, } e^{-R_L t} = \frac{1}{5}$$

$$\therefore -R_L t = \ln(\frac{1}{5}) \quad \therefore t = \frac{0.6 \times 10^{-3} \ln(\frac{1}{5})}{1.2 \times 10^3} = 8.9 \times 10^{-5} \text{ sec}$$

$$I = I_0 (1 - e^{-R_L t})$$

At $t = T_L$

~~VV~~

Energy stored in a magnetic field.

Let us take an inductor having inductance L is connected to a generator which terminal voltage can be varied when the current is increasing. Induced emf is

$$E_{\text{ind}} = -L \frac{dI}{dt} \quad \text{appears across the inductor. The}$$

generator must do work to push the charges through the inductor against induced emf. The rate at which work done against the emf is:

$$P = IE$$

Now Work done

$$W = U_B = \int_0^I P dt = \int_0^I I L \frac{dI}{dt} dt = \frac{1}{2} L I^2$$

which is the energy stored in inductor having inductance

Let us take a solenoid having length 'l' and area 'A' with self inductance 'L' then the magnetic energy density is given by

$$\frac{U_B}{\text{Volume}} = \frac{1/2 LI^2}{Axl}$$

We have the self inductance of solenoid.

$$L = \frac{\mu_0 N^2 A}{l}$$

$$\begin{aligned} \text{Then } \frac{U_B}{\sqrt{}} &= \frac{1/2 \times \frac{\mu_0 N^2 A}{l} \times I^2 \times \frac{1}{Axl}}{\sqrt{}} \\ &= \frac{\mu_0 I^2 N^2 A}{2 Al l^2} \\ &= \frac{\mu_0 N^2 I^2}{2 l^2} \end{aligned}$$

Now, the magnetic field of solenoid is $B = \frac{\mu_0 NI}{l}$

$$\therefore \frac{U_B}{\text{Volume}} \rightarrow \frac{B^2}{2 \mu_0}$$