Representation of Heisenberg group

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The aim here is to find all the irreducible representations (upto isomorphism) of the Heisenberg group over modulo p.

1 Heisenberg group

The Heisenberg group over modulo prime p is

$$H_3(\mathbb{Z}/p\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}.$$

Though this can be defined over any commutative ring with identity, here we will only study over $\mathbb{Z}/p\mathbb{Z}$, and denote the group by H.

1.0.1 Notation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \coloneqq \begin{pmatrix} 1 & b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

1.0.2 Properties

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a + a' + bc' \\ b + b' \\ c + c' \end{pmatrix} \text{ and } \begin{pmatrix} a \\ b \\ c \end{pmatrix}^{-1} = \begin{pmatrix} bc - a \\ -b \\ -c \end{pmatrix}$$

1.1 Commutator subgroup

For any $\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix} \in H$,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}^{-1} \begin{pmatrix} d \\ e \\ f \end{pmatrix}^{-1} = \begin{pmatrix} a + d + bf \\ b + e \\ c + f \end{pmatrix} \begin{pmatrix} bc - a + ef - d + bf \\ -b - e \\ -c - f \end{pmatrix} = \begin{pmatrix} bf - ce \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the commutator subgroup of H is

$$H' = \left\{ \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{Z}/p\mathbb{Z} \right\} \cong \mathbb{Z}/p\mathbb{Z}.$$

The proof for the isomorphism of H' with $\mathbb{Z}/p\mathbb{Z}$ is left to the reader as an exercise.

Note that

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha + a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$$

which can be use to prove that the center of the Heisenberg group, $Z(H) = H'$.

From this we also get that

$$H/H' = \left\{ \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} H' \,\middle|\, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}$$

and the map $\varphi : H/H' \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ given by

$$\varphi \left(\begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \right) = (b, c)$$

is an isomorphism.

Remark: As the order of Heisenberg group is p^3 and the dimension of an irreducible representation divides the order of the group, any irreducible representation of H has dimensions $1, p, p^2$ or p^3. Since the order of the group is equal to the sum of the squares of all its irreducible representations, p^2 and p^3 are not possible. Not only that, since H is not abelian, it is easy to conlude that there are p^2 1-dimensional representations and $(p-1)$ p-dimensional representations.

1.2 1-dimensional representations

All the 1 dimensional representations of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ are of the form $\tilde{\rho}_{j,k} : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times$, $j, k \in \mathbb{Z}/p\mathbb{Z}$, given by

$$\tilde{\rho}_{j,k}((m, n)) = \xi^{jm+kn} \text{ , where } \xi = e^{\frac{2i\pi}{p}}.$$

So, using φ and the canonical surjective homomorphism from H to H/H', we get that all the 1 dimensional representations of H are $\rho_{j,k} : H \to \mathbb{C}^\times$ given by

$$\rho_{j,k} \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = \xi^{jb+kc}.$$

[This uses the fact that $\mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathrm{Hom}(G/G', \mathbb{C}^\times) \cong G/G'$.]

1.3 p-dimensional representations

For this, we will try to find induced representation from 1-dimensional representaions of a subgroup of index p. Consider the following subgroup of H,

$$K = \left\{ \begin{pmatrix} a \\ b \\ b \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z}/p\mathbb{Z} \right\}$$

and the map $\phi : K \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ given by

$$\phi \left(\begin{pmatrix} a \\ b \\ b \end{pmatrix} \right) = \left(a - \frac{b(b+1)}{2}, b \right)$$

forms a group isomorphism.

$$\begin{aligned}
\phi\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\begin{pmatrix}c\\d\\d\end{pmatrix}\right) =&\phi\left(\begin{pmatrix}a+c+bd\\b+d\\b+d\end{pmatrix}\right)\\
=&\left(a+c+bd-\frac{(b+d)(b+d+1)}{2}, b+d\right)\\
=&\left(a+c-\frac{b(b+1)}{2}-\frac{d(d+1)}{2}, b+d\right)\\
=&\phi\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\right)+\phi\left(\begin{pmatrix}c\\d\\d\end{pmatrix}\right)
\end{aligned}$$

Since K is of index p (i.e., the smallest prime dividing the order of the group H), K is normal, and $K^{(s)} = sKs^{-1} \cap K = K$ for any $s \in H$.

And for any $\begin{pmatrix}a\\b\\c\end{pmatrix} \in H$, $\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\c-b\end{pmatrix}\begin{pmatrix}a\\b\\b\end{pmatrix}$. Thus, $S = \left\{\begin{pmatrix}0\\0\\\beta\end{pmatrix} \middle| \beta \in \mathbb{Z}/p\mathbb{Z}\right\}$ is a set of left coset representatives of K in H. For simplicity in writing, denote $s_\beta = \begin{pmatrix}0\\0\\\beta\end{pmatrix}$.

Now, any irreducible representation of K is of the form $\tilde{\theta}_{j,k} := \tilde{\rho}_{j,k} \circ \phi$ which can be explicitly computed to be

$$\tilde{\theta}_{j,k}\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\right) = \xi^{j(a-\frac{b(b+1)}{2})+kb}$$

and

$$\tilde{\theta}^{\beta}_{j,k}\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\right) = \xi^{j(a+b\beta-\frac{b(b+1)}{2})+kb}, \text{ where } \tilde{\theta}^{\beta}_{j,k} := (\tilde{\theta}_{j,k})^{(s_\beta)}.$$

Since $\tilde{\theta}_{j,k}$ is irreducible, by Mackey's irreducibility criteria, $\mathrm{Ind}_K^H(\tilde{\theta}_{j,k})$ is irreducible if and only if for each $\beta \in \mathbb{Z}/p\mathbb{Z}, \beta \neq 0$, $\tilde{\theta}^{\beta}_{j,k}$ and $\tilde{\theta}_{j,k}$ are disjoint, i.e.,

$$\left\langle \chi_{\tilde{\theta}^{\beta}_{j,k}}, \chi_{\tilde{\theta}_{j,k}} \right\rangle = 0.$$

For $j = 0$, and any $k \in \mathbb{Z}/p\mathbb{Z}$

$$\begin{aligned}
\left\langle \chi_{\tilde{\theta}^{\beta}_{j,k}}, \chi_{\tilde{\theta}_{j,k}} \right\rangle =&\frac{1}{p^2}\sum_{0\le a,b,\le p-1} \chi_{\tilde{\theta}^{\beta}_{j,k}}\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\right)\overline{\chi_{\tilde{\theta}_{j,k}}\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\right)}\\
=&\frac{1}{p^2}\sum_{0\le a,b,\le p-1} \xi^{kb}\overline{\xi^{kb}}\\
=&\frac{1}{p^2}\sum_{0\le a,b,\le p-1} 1\\
=&1
\end{aligned}$$

Thus, $\mathrm{Ind}_K^H(\tilde{\theta}_{0,k})$ is reducible. Next, we note that for any $j, k, l \in \mathbb{Z}/p\mathbb{Z}$, $j \neq 0$, choose $\beta = j^{-1}(l-k)$, then

$$\tilde{\theta}^{\beta}_{j,k}\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\right) = \xi^{j(a+b\beta-\frac{b(b+1)}{2})+kb} = \xi^{j(a-\frac{b(b+1)}{2})+lb} = \tilde{\theta}_{j,l}\left(\begin{pmatrix}a\\b\\b\end{pmatrix}\right), \quad \forall \begin{pmatrix}a\\b\\b\end{pmatrix} \in K,$$

i.e., orbit $\left(\tilde{\theta}_{j,k}\right)$ = orbit $\left(\tilde{\theta}_{j,k}\right)$ for any $j, k, l \in \mathbb{Z}/p\mathbb{Z}$, $j \neq 0$. So, they induce the same representation (upto equivalence) of H. So, it is sufficient to check the irreducibility of induced representations of $\tilde{\theta}_{j,0}$, where $1 \leq j \leq p-1$. Denote $\theta_j = \tilde{\theta}_{j,0}$.

Now, for any $1 \leq \beta \leq p-1$,

$$\begin{aligned}
\left\langle \chi_{\theta_j^\beta}, \chi_{\theta_j} \right\rangle =& \frac{1}{p^2} \sum_{0 \leq a,b, \leq p-1} \chi_{\theta_j^\beta}\left(\begin{pmatrix} a \\ b \\ b \end{pmatrix}\right) \overline{\chi_{\theta_j}\left(\begin{pmatrix} a \\ b \\ b \end{pmatrix}\right)} \\
=& \frac{1}{p^2} \sum_{0 \leq a,b, \leq p-1} \xi^{j\left(a + b\beta - \frac{b(b+1)}{2}\right)} \overline{\xi^{j\left(a - \frac{b(b+1)}{2}\right)}} \\
=& \frac{1}{p^2} \sum_{0 \leq a,b, \leq p-1} \xi^{jb\beta} \\
=& \frac{p}{p^2} \sum_{b=0}^{p-1} \xi^{jb\beta} \\
=& \frac{1}{p} \frac{1 - \xi^{jp\beta}}{1 - \xi^{j\beta}} \\
=& 0
\end{aligned}$$

Also, it is easy to check that orbit $(\theta_{j_1}) \neq$ orbit (θ_{j_2}) if $j_1 \neq j_2$, which implies the corresponding induced representations are not equivalent. Thus, all the p-dimensional representations of H are of the form $\rho_j' = \mathrm{Ind}_K^H(\theta_j)$. These can be explicitly computed as $\rho_j' : H \to \mathrm{GL}_p(\mathbb{C})$ given by

$$\left[\rho_j'\left(\begin{pmatrix} a \\ b \\ b \end{pmatrix}\right)\right]_{m,n} = \begin{cases} \xi^{j\left(a + bn - \frac{b(b+1)}{2}\right)} & \text{if } b - c = n - m \\ 0 & \text{otherwise.} \end{cases}$$