Topology

What is the difference between a coffee mug and a doughnut?
Ask a Topologist !!!

Krishanu Dan
School of Mathematical Sciences
NISER, Bhubaneswar

Acknowledgment

Do I really need to say the obvious? I guess so. Here it goes:

No originality was claimed and no copyright infringement was intended.

This is a class note and should be treated as such. The main reference for this note is "Topology" by James R. Munkres. It was assumed that the students are familiar with the concepts of metric spaces. Readers are strongly encouraged to consult the book (or any other book of their preferences) as much as possible. This note is not a replacement, rather a companion. For any mistake and/or typo, please email to krishanu@niser.ac.in

The purpose of this note is purely academic.

Krishanu Dan

Contents

1. Topological Spaces

Definition 1.1. Let X be a set and let $\mathscr{P}(X)$ be the set of subsets of X. A *topology* τ on X is a subset of $\mathscr{P}(X)$ satisfying the followings:

- $\emptyset \in \tau$ and $X \in \tau$;
- if $\{U_\alpha : \alpha \in \Lambda\}$ is a collection of elements of τ, then $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$;
- if $\{U_i : 1 \leq i \leq n\}$ is a collection of elements of τ, then $\bigcap_{i=1}^{n} U_i \in \tau$.

The set X together with the topology τ is called a *topological space*, denoted by (X, τ). The elements of τ are called *open subsets* of X.

Example 1.2. *(1.)* Let X be a set and let $\tau = \{\emptyset, X\} \subseteq \mathscr{P}(X)$. Then τ defines a topology on X, called *trivial topology* or *indiscrete topology* on X.

(2.) Let X be a set and let $\tau = \mathscr{P}(X)$. Then τ defines a topology on X, called *discrete topology* on X, and the topological space (X, τ) is called *discrete space*.

(3.) Let (X, d) be a metric space. Let $r > 0$ be a real number. For any $x \in X$, consider the open ball $B(x, r) := \{y \in X : d(x, y) < r\}$. Let τ be the collection of subsets of X defined by the following property:

$$U \in \tau \iff \text{for every } x \in U, B(x, r_x) \subseteq U \text{ for some real number } r_x > 0.$$

Then τ is a topology on X. Thus the open sets in the metric space (X, d) and the open sets in the topological space (X, τ) are the same. We say that *the topology τ is generated by the metric d*. Whenever we consider a metric space (X, d), unless specified explicitly, we will always assume that X is equipped with the topology generated by the metric d.

(4.) Let $X = \mathbb{R}^n$ and let d be the Euclidean metric on $\mathbb{R}^n$, induced by the Euclidean norm: $||(a_1, \cdots, a_n)|| = (a_1^2 + \cdots + a_n^2)^{1/2}$. The topology τ on $\mathbb{R}^n$ induced by Euclidean norm will be called *Euclidean topology* and $\mathbb{R}^n$ equipped with Euclidean topology will be called *Euclidean space*. It is also known as the *standard topology* on $\mathbb{R}^n$ or even the *usual topology* on $\mathbb{R}^n$. Unless mentioned otherwise explicitly, we will assume $\mathbb{R}^n$ is equipped with the Euclidean topology.

(5.) Let X be a non-empty set and let $\tau \subseteq \mathscr{P}(X)$ be defined by

$$U \in \tau \iff \text{either } U = \emptyset \text{ or } X \setminus U \text{ is a finite set.}$$

Let's check the three defining properties of a topology:

- $\emptyset \in \tau$ and $X \in \tau$.
- Let $\{U_\alpha : \alpha \in \Lambda\}$ be a collection of elements of τ. Then
$$X \setminus \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcap_{\alpha\in\Lambda} (X \setminus U_\alpha) \implies \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau.$$
- Let $\{U_i : 1 \leq i \leq n\}$ be a collection of elements of τ. Since
$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i) \implies \bigcap_{i=1}^{n} U_i \in \tau.$$

The topology τ on X is called the *co-finite topology* or *finite-complement topology*.

(6.) Let X be an infinite set and let τ be the collection of subsets of X defined by the following property:

$$U \in \tau \iff \text{either } U = \emptyset \text{ or } X \setminus U \text{ is a countable set.}$$

Arguing similar to the co-finite topology we can show that τ is a topology on X. The topology τ on X is called the *co-countable topology*.

(7.) Let X be an infinite set and let $a \in X$. Define

$$\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subseteq X : a \in V \text{ and } X \setminus V \text{ is finite}\} \subseteq \mathscr{P}(X).$$

We claim that τ_a is a topology on X.

(i) Clearly $\emptyset \in \tau_a$ and $X \in \tau_a$.

(ii) Let $\{V_i : 1 \leq i \leq n\}$ be a collection of elements of τ_a.

- Suppose $a \in V_i, \forall i = 1, \cdots, n$. Then $a \in \bigcap_{i=1}^n V_i$. In this case

$$X \setminus \left(\bigcap_{i=1}^{n} V_i \right) = \bigcup_{i=1}^{n} (X \setminus V_i) \implies \bigcap_{i=1}^{n} V_i \in \tau_a.$$

- Suppose there is $i \in \{1, \cdots, n\}$ such that $a \notin V_i$, Then $\bigcap_{i=1}^n V_i \subseteq X \setminus \{a\}$ and hence $\bigcap_{i=1}^n V_i \in \tau_a$.

(iii) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of elements of τ_a.

- Suppose there is $\beta \in \Lambda$ such that $a \in V_\beta$. Then $a \in \bigcup_{\alpha \in \Lambda} V_\alpha$ and

$$X \setminus \left(\bigcup_{\alpha \in \Lambda} V_\alpha \right) = \bigcap_{\alpha \in \Lambda} (X \setminus V_\alpha) \subseteq X \setminus V_\beta \implies \bigcup_{\alpha \in \Lambda} V_\alpha \in \tau_a.$$

- Suppose $a \notin V_\alpha, \forall \alpha \in \Lambda$. Then $\bigcup_{\alpha \in \Lambda} V_\alpha \subseteq X \setminus \{a\}$ so that $\bigcup_{\alpha \in \Lambda} V_\alpha \in \tau_a$.

Hence τ_a is a topology on X.

Exercise 1.3. (1.) Let $X = \mathbb{N}$ and define $\tau_1, \tau_2 \subseteq \mathscr{P}(\mathbb{N})$ as follows:

$$\tau_1 := \{\emptyset, \mathbb{N}, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \cdots\};$$
$$\tau_2 := \{\emptyset, \mathbb{N}, \mathbb{N} \setminus \{1\}, \mathbb{N} \setminus \{1,2\}, \mathbb{N} \setminus \{1,2,3\}, \mathbb{N} \setminus \{1,2,3,4\}, \cdots\}.$$

Check if τ_1, τ_2 are topologies on $\mathbb{N}$.

(2.) Let $X = \{a, b, c, d, e, f\}$. In each of the following cases, check if the given subset of $\mathscr{P}(X)$ defines a topology on X:

- $\tau_1 = \{\emptyset, X, \{b\}, \{a,c\}, \{a,c,e\}, \{a,b,c\}, \{a,b,c,e\}\}$.
- $\tau_2 = \{\emptyset, X, \{a\}, \{a,c\}, \{a,c,d\}, \{a,b,c\}, \{a,b,c,d\}\}$.
- $\tau_3 = \{\emptyset, X, \{e\}, \{f\}, \{e,f\}, \{b,e,f\}, \{c,e,f\}, \{d,e,f\}\}$.

(3.) Let X be an infinite set and let $a, b \in X$ be distinct points. Define

$$\tau = \mathscr{P}(X \setminus \{a,b\}) \bigcup \{V \subseteq X : a \in V \text{ or } b \in V \text{ and } X \setminus V \text{ is finite}\}.$$

Is τ a topology on X?

(4.) Let $X = \mathbb{R}$, fix $a \in \mathbb{R}$ and define subsets $\tau_1, \tau_2 \subseteq \mathscr{P}(\mathbb{R})$ as follows:

$$\tau_1 := \{V \subseteq \mathbb{R} : V = \emptyset \text{ or } a \in V\} \text{ and } \tau_2 := \{V \subseteq \mathbb{R} : V = \mathbb{R} \text{ or } a \notin V\}.$$

Check if τ_1, τ_2 are topologies on $\mathbb{R}$.

Definition 1.4. Let (X, τ) be a topological space. A set $V \subseteq X$ is called *closed* if $X \setminus V$ is open, i.e. $X \setminus V \in \tau$. A set $A \subseteq X$ which is both closed and open subset of X is called a *clopen set*.

Exercise 1.5. Let (X, τ) be a topological space. Prove that

- the empty set $\emptyset$ and X are closed;
- union of finitely many closed subsets of X is closed;
- intersection of arbitrary collection of closed subsets of X is closed.

In any topological space (X, τ), the empty set $\emptyset$ and the whole space X are always clopen sets. If τ is the discrete topology on X, then every subset of X is a clopen set. On the other hand, a clopen set other than $\emptyset$ and X may not exists. For example, let X be an infinite set equipped with co-finite topology. Since non-empty open subsets are infinite and proper closed subsets are finite, no non-empty proper subset of X can be both open and closed.

Definition 1.6. Let (X, τ) be a topological space.

- By a *neighbourhood* of $x \in X$ we mean a set $N_x \subseteq X$ such that there is an open set $V \subseteq X$ satisfying $x \in V \subseteq N_x$. The collection $\mathscr{N}_x$ of all neighbourhoods of x is called the *neighbourhood system* at x.
- A point $x \in X$ is said to be a *limit point* of a set $A \subseteq X$ if every neighbourhood N_x of x contains a point of A other than x, i.e. $A \bigcap (N_x \setminus \{x\}) \neq \emptyset$.

Notice that, in the definition of neighbourhood of a point $x \in X$, we have not assumed that N_x is an open subset of X. If in addition, a neighbourhood N_x of x is itself an open subset of X, we say that N_x is an *open neighbourhood* of x.

Example 1.7. *(1.)* Let $X = \{a, b, c, d, e\}$ and let $\tau \subset \mathscr{P}(X)$ be a topology on X defined by

$$\tau = \{\, \emptyset,\ X,\ \{a\},\ \{c, d\},\ \{a, c, d\},\ \{b, c, d, e\}\,\}$$

Let $A = \{a, b, c\}$. Here A is not open, but it is a neighbourhood of the point a. In this case, a and c are not limit points of A (e.g. $N_a = \{a\}$ and $N_c = \{c, d\}$). On the other hand, b, d, e are all limit points of A.

(2.) Let X be an infinite set, τ be the co-finite topology on X and let $A \subseteq X$ be an infinite set. Let $x \in X$ and let $N_x \subseteq X$ be a neighbourhood of x. Since $X \setminus N_x$ is a finite set, $A \cap (N_x \setminus \{x\}) \neq \emptyset$. Hence every point of X is a limit point of A.

(3.) Let X be a non-empty set and let τ be the discrete topology on X. Let $\emptyset \neq A \subsetneq X$ and $x \notin X \setminus A$. Then x can not be a limit point of A since $\{x\}$ is a neighbourhood of x. Moreover, every non-empty subset of X is a neighbourhood of each of the points it contained.

Lemma 1.8. *Let (X, τ) be a topological space and let $Y \subseteq X$. Then*

(i) Y is open if and only if Y contains a neighbourhood of each of its points.

(ii) Y is closed if and only if Y contains all its limit points.

Proof. (i) Suppose Y is open and let $y \in Y$. In this case we can choose $N_y = Y$ and we are done. For the converse, for each $y \in Y$, choose a neighbourhood $N_y \in \mathscr{N}_y$ such that $N_y \subseteq Y$. Moreover, for each $y \in Y$, let $V_y \subseteq X$ be an open set such that $y \in V_y \subseteq N_y$. Then we can write $Y = \bigcup_{y \in Y} V_y$ and hence $Y \in \tau$.

(ii) Suppose Y is closed. Since $X \setminus Y$ is open, it an open neighbourhood of every $x \in X \setminus Y$. Hence no $x \in X \setminus Y$ can be a limit point of Y.

Now assume Y contains all of its limit points. Let $x \in X \setminus Y$. Since x is not a limit point of Y, there is $N_x \in \mathscr{N}_x$ such that $(N_x \setminus \{x\}) \bigcap Y = \emptyset$ so that $N_x \subseteq X \setminus Y$. Hence $X \setminus Y$ is open by (i). □

Definition 1.9. Let (X, τ) be a topological space and let $A \subseteq X$.

- The *closure* of A, denoted by $\overline{A}$ or $\mathrm{Cl}_X(A)$, is the intersection of all the closed subsets of X containing A.
- The *interior* of A, denoted by $\mathrm{Int}(A)$, is the union of all the open subsets of X contained in A.
- The *boundary* of A, denoted by ∂A, is the set of all points $x \in X$ such that every neighbourhood of x intersects both A and $X \setminus A$.

Let (X,τ) be a topological space and let $A \subseteq X$. Then $\overline{A}$ is a closed subset of X and $\mathrm{Int}(A)$ is an open subset of X. Moreover, $\overline{A}$ is the smallest closed subset of X containing A and $\mathrm{Int}(A)$ is the largest open subset of X contained in A.

Lemma 1.10. *Let (X,τ) be a topological space, $x \in X$ and let $A \subseteq X$. Then*

(i) $A \in \mathscr{N}_x$ if and only if $x \in Int(A)$.

(ii) $x \in \overline{A}$ if and only if $A \bigcap N \neq \emptyset$, for every $N \in \mathscr{N}_x$.

Proof. (i) Let $A \in \mathscr{N}_x$. Then there is an open set $V \subseteq X$ such that $x \in V \subseteq A$. Hence $x \in V \subseteq \mathrm{Int}(A)$. Conversely, assume that $x \in \mathrm{Int}(A)$. Since $\mathrm{Int}(A)$ is open, $A \subseteq \mathscr{N}_x$.

(ii) Let $A \bigcap N \neq \emptyset$, for all $N \in \mathscr{N}_x$. If $x \in A$, then $x \in \overline{A}$. Now assume $x \notin A$. Then x is a limit point of A. Let $E \subseteq X$ be a closed set satisfying $A \subseteq E$. Then $E \bigcap (N \setminus \{x\}) \neq \emptyset$. By Lemma 1.8, $x \in E$. Hence $x \in \overline{A}$.

Let $x \in \overline{A}$. If $x \in A$, then $A \bigcap N \neq \emptyset$, for all $N \in \mathscr{N}_x$. So assume $x \notin A$. Suppose for some $N \in \mathscr{N}_x$, we have $A \bigcap N = \emptyset$. Then $X \setminus \mathrm{Int}(N)$ is a closed subset of X and $A \subseteq X \setminus \mathrm{Int}(N)$. Thus $\overline{A} \subseteq X \setminus \mathrm{Int}(N)$, a contradiction since $x \notin X \setminus \mathrm{Int}(N)$. $\square$

Exercise 1.11. *(1.)* Let X be an infinite set (resp. uncountable set), τ be the co-finite topology (resp. co-countable topology) on X and let $A \subseteq X$. Prove that

- if A is finite (resp. countable), then $\overline{A} = A$;
- if A is infinite (resp. uncountable), then $\overline{A} = X$.
- if A is neither open nor closed, then $\partial A = X$.

(2.) Let (X,τ) be a topological space and let $A, B \subseteq X$. Prove that

- $\overline{\overline{A}} = \overline{A}$ and if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$;
- $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$ and $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$;
- Given an example where $\overline{A \bigcap B} \neq \overline{A} \bigcap \overline{B}$.

(3.) Let (X,τ) be a topological space and let $A, B \subseteq X$. Prove that

- $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$ and if $A \subseteq B$, then $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$;
- $\mathrm{Int}(A \bigcup B) \supseteq \mathrm{Int}(A) \bigcup \mathrm{Int}(B)$ and $\mathrm{Int}(A \bigcap B) = \mathrm{Int}(A) \bigcap \mathrm{Int}(B)$;
- Given an example where $\mathrm{Int}(A \bigcup B) \neq \mathrm{Int}(A) \bigcup \mathrm{Int}(B)$.

(4.) Let (X,τ) be a topological space and let $A \subseteq X$. Prove that

- $x \in \partial A$ if and only if $x \notin \mathrm{Int}(A)$ and $x \notin \mathrm{Int}(X \setminus A)$;
- $\partial A = \partial(X \setminus A)$
- $\partial A = \overline{A} \bigcap \overline{X \setminus A} = \overline{A} \setminus \mathrm{Int}(A)$;
- $\overline{A} = A \bigcup \partial A$ and $\mathrm{Int}(A) = A \setminus \partial A$.

Let (X,τ) be a topological space and let $Y \subseteq X$. Set

$$\tau_Y := \{V \cap Y : V \in \tau\} \subseteq \mathscr{P}(Y).$$

We claim that τ_Y defines a topology on Y:

- We have $\emptyset \in \tau_Y$ and $Y = Y \bigcap X \in \tau_Y$;
- Let $\{V_\alpha \bigcap Y : \alpha \in \Lambda \text{ and } V_\alpha \in \tau \text{ for each } \alpha \in \Lambda\}$ be a family of elements of τ_Y. Then $\bigcup_{\alpha\in\Lambda} (V_\alpha \bigcap Y) = (\bigcup_{\alpha\in\Lambda} V_\alpha) \bigcap Y \in \tau_Y$ since $\bigcup_{\alpha\in\Lambda} V_\alpha \in \tau$.
- Let $\{V_i \bigcap Y : V_i \in \tau, 1 \leq i \leq n\}$ be a family of elements of τ_Y. Since $\bigcap_{i=1}^n V_i \in \tau$, we have $\bigcap_{i=1}^n (V_i \bigcap Y) = (\bigcap_{i=1}^n V_i) \bigcap Y \in \tau_Y$.

Hence τ_Y is a topology on Y. We say τ_Y on Y is *induced from the topology* τ on X.

Definition 1.12. (X,τ) be a topological space, $Y \subseteq X$ and let τ_Y be the topology on Y as defined above. The topology τ_Y is called the *subspace topology* on Y and the topological space (Y, τ_Y) is called a *subspace* of (X,τ).

The open subsets of Y are the subsets of Y that can be written as an intersection of an open subset of X with Y. Notice that, this intersection need not be an open subset of X. For example, let $X = \mathbb{R}$ be equipped with the Euclidean topology and let $Y = [0,1]$. Then $[0,1/2) = Y \cap (-1,1/2)$ is open in Y, but not in $\mathbb{R}$.

Lemma 1.13. *Let (X,τ) be a topological space, (Y,τ_Y) be a subspace of (X,τ) and let $A \subseteq Y$.*

(i) A is closed in (Y,τ_Y) if and only if A can be written as $A = Y \bigcap E$ for some closed set E in (X,τ).

(ii) A point $a \in Y$ is a limit point of A in (Y,τ_Y) if and only if it is a limit point of A in (X,τ).

(iii) The closure of A in (Y,τ_Y) is precisely the intersection of Y and the closure of A in (X,τ).

Proof. (i) Follows directly from definition and left as an exercise.

(ii) Suppose $a \in Y$ is a limit point of A in (Y,τ_Y). Then for any neighbourhood V_a of a in (Y,τ_Y), $A \bigcap (V_a \setminus \{a\}) \neq \emptyset$. Let W_a be a neighbourhood of a in (X,τ). Then $W_a \cap Y$ is a neighbourhood of a in (Y,τ_Y) and hence

$$A \bigcap (W_a \setminus \{a\}) \;=\; A \bigcap \Big((W_a \cap Y) \setminus \{a\} \Big) \neq \emptyset\,.$$

Thus a is a limit point of A in (X,τ).

Conversely, assume that a is a limit point of A in (X,τ). Let V_a be a neighbourhood of a in (Y,τ_Y). Choose a neighbourhood W_a of a in (X,τ) so that $V_a = W_a \bigcap Y$. Then

$$A \bigcap (V_a \setminus \{a\}) \;=\; A \bigcap \Big((W_a \cap Y) \setminus \{a\} \Big) \;=\; A \bigcap (W_a \setminus \{a\}) \bigcap Y \neq \emptyset.$$

Hence $a \in Y$ is a limit point of A in (Y,τ_Y).

(iii) Let $\mathrm{Cl}_X(A)$ and $\mathrm{Cl}_Y(A)$ denote the closures of A in (X,τ) and (Y,τ_Y), respectively. By definition

$$\mathrm{Cl}_Y(A) \;=\; \bigcap \{E \subseteq Y \,:\, A \subseteq E \;\text{ and }\; E \;\text{ is closed in }\; (Y,\tau_Y)\}$$

and

$$\begin{aligned}\mathrm{Cl}_X(A) \bigcap Y \;&=\; \Big(\bigcap \{F \subseteq X \,:\, A \subseteq F \;\text{ and }\; F \;\text{ is closed in }\; (X,\tau)\} \Big) \bigcap Y \\ &=\; \{F \cap Y \,:\, A \subseteq F \subseteq X \;\text{ and }\; F \;\text{ is closed in }\; (X,\tau)\}\end{aligned}$$

By (i), we have $\mathrm{Cl}_Y(A) \;=\; \mathrm{Cl}_X(A) \bigcap Y$. □

Let (X,τ) be a topological space, (Y,τ_Y) be a subspace of (X,τ) and let $A \subseteq Y$.

- Suppose Y is open in X. Then A is open in Y if and only if A is open in X.
- Suppose Y is closed in X. Then A is closed in Y if and only if A is closed in X.

Exercise 1.14. *(1.)* Give example of a topological space (X,τ) and a set $Y \subseteq X$ containing at least two elements such that τ_Y is the indiscrete topology on Y but τ is not the indiscrete topology on X.

(2.) Give example of a topological space (X,τ) and an infinite set $Y \subseteq X$ such that τ_Y is the discrete topology on Y but τ is not the discrete topology on X.

Definition 1.15. Let (X, τ) be a topological space. A set $D \subseteq X$ is called *dense* in X if $\overline{D} = X$.

In the Euclidean space $\mathbb{R}$, both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense. If X is an infinite set equipped with co-finite topology τ, then any infinite subset of X is dense in X.

Lemma 1.16. *Let (X, τ) be a topological space and $D \subseteq X$. Then D is dense in X if and only if $D \bigcap V \neq \emptyset$, for every open set $\emptyset \neq V \subseteq X$.*

Proof. Suppose D is dense in X and let $\emptyset \neq V \subseteq X$ be an open set. Choose $x \in V$. Then V is a neighbourhood of x and hence $D \bigcap V \neq \emptyset$ by Lemma 1.10.

Conversely, assume that $D \bigcap V \neq \emptyset$, for every open set $\emptyset \neq V \subseteq X$. Let $y \in X$. Then for every $N \in \mathscr{N}_y$, $D \bigcap N \neq \emptyset$. So by Lemma 1.10, $y \in \overline{D}$. Hence $\overline{D} = X$. $\square$

In the above Lemma, we can not replace "open set" by "closed set". For example, let $\mathbb{R}$ be the Euclidean space, $D = \mathbb{R} \setminus \mathbb{Q}$ and $E = \mathbb{Z}$. Then D is dense in $\mathbb{R}$ and E is a closed subset of $\mathbb{R}$ but $D \bigcap E = \emptyset$.

Exercise 1.17. *(1.)* Let (X, τ) be a topological space and let $\emptyset \neq D \subseteq X$. Then the following conditions are equivalent:

(i) D is dense in X;

(ii) for any closed set $E \subseteq X$, if $D \subseteq E$, then $E = X$;

(iii) $\text{Int}(X \setminus D) = \emptyset$.

(2.) Let (X, τ) be a non-empty topological space and let $D \subseteq X$ be a dense set. Prove that $\overline{V \bigcap D} = \overline{V}$, for every open set $\emptyset \neq V \subseteq X$.

Lemma 1.18. *Let X be a non-empty set. Define a function $\phi : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ satisfying*

(a) $\phi(\emptyset) = \emptyset$;

(b) $A \subseteq \phi(A)$, for every $A \in \mathscr{P}(X)$;

(c) $\phi \circ \phi(A) = \phi(A)$, for every $A \in \mathscr{P}(X)$;

(d) $\phi(A \bigcup B) = \phi(A) \bigcup \phi(B)$, for every $A, B \in \mathscr{P}(X)$.

Then $\tau = \{X \setminus \phi(A) : A \in \mathscr{P}(X)\}$ is a topology on X and in the topological space (X, τ) we have $\phi(A) = \overline{A}$, for every $A \in \mathscr{P}(X)$.

Proof. Left as an exercise. $\square$

Let X be an infinite set and fix $a_0 \in X$. Define $\phi : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ by $\phi(\emptyset) = \emptyset$ and $\phi(A) = A \bigcup \{a_0\}$, for $\emptyset \neq A \in \mathscr{P}(X)$. It's easy to see that ϕ satisfies the hypothesis of Lemma 1.18. Let τ be the topology on X as in Lemma 1.18. Then $\{a_0\}$ is closed in X and for any $a_0 \neq x \in X$, $\{x\}$ is open in X, but not closed.

Lemma 1.19. *Let X be a non-empty set. Define a function $\psi : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ satisfying*

(a) $\psi(X) = X$;

(b) $\psi(A) \subseteq A$, for every $A \in \mathscr{P}(X)$;

(c) $\psi \circ \psi(A) = \psi(A)$, for every $A \in \mathscr{P}(X)$;

(d) $\psi(A \bigcap B) = \psi(A) \bigcap \psi(B)$, for every $A, B \in \mathscr{P}(X)$.

Then $\tau = \{\psi(A) : A \in \mathscr{P}(X)\}$ is a topology on X and in the topological space (X, τ) we have $\psi(A) = Int(A)$, for every $A \in \mathscr{P}(X)$.

Proof. Left as an exercise. $\square$

Let X be an infinite set and let $\emptyset \neq Y \subseteq X$ be such that $X \setminus Y$ contains at least two points. Define $\psi : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ by $\psi(X) = X$ and $\psi(A) = A \bigcap Y$, for $X \neq A \in \mathscr{P}(X)$. It's easy to see that ψ satisfies the hypothesis of Lemma 1.19. Let τ be the topology on X as in Lemma 1.19. The open subsets of X are X and all subsets of Y.

Exercise 1.20. *(1.)* Let X be a non-empty finite set and let τ be the co-finite topology on X. Prove that $\tau = \mathscr{P}(X)$.

(2.) Let X be a non-empty countable set and let τ be the co-countable topology on X. Prove that $\tau = \mathscr{P}(X)$.

(3.) Let (X, τ) be a non-empty topological space. Prove that τ is the discrete topology on X if and only if $\{x\} \in \tau, \forall x \in X$.

(4.) Let X be an infinite set and let $a \in X$. Define

$$\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subseteq X : a \in V \text{ and } X \setminus V \text{ is finite}\}.$$

Let $A \subseteq X$ be a non-empty set. Find $\overline{A}$ and $\mathrm{Int}(A)$ in (X, τ_a). (*Hint:* Consider two separate cases: A is finite and A is infinite.)

(5.) Given $n \in \mathbb{N}$, let $D_n \subseteq \mathbb{N}$ be the set of all divisors of n. Set

$$\tau := \{\emptyset, X\} \bigcup \{D_n : n \in \mathbb{N}\} \subseteq \mathscr{P}(\mathbb{N}).$$

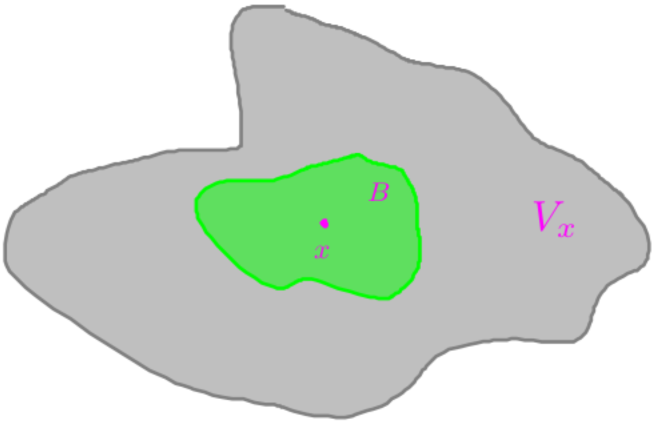
Is τ a topology on $\mathbb{N}$?

(6.) Let $X = \mathbb{R}$ and let $\tau \subseteq \mathscr{P}(\mathbb{R})$ be defined by

$$\tau := \{V \subseteq \mathbb{R} : V = \emptyset \text{ or if } x \in V, \text{ then there is } a_x \in \mathbb{R} \text{ such that } x \in (a_x, \infty) \subseteq V\}.$$

Check if τ is a topology on $\mathbb{R}$.

Definition 2.1. Let (X,τ) be a topological space. A set $\mathscr{B} \subseteq \mathscr{P}(X)$ is called a *base for the topology* τ if $\mathscr{B} \subseteq \tau$ as subsets of $\mathscr{P}(X)$ and given any $x \in X$ and any neighbourhood $V_x \in \mathscr{N}_x$, there is $B \in \mathscr{B}$ such that $x \in B \subseteq V_x$. The elements of $\mathscr{B}$ are called *basic open subsets* of (X,τ).



Example 2.2. *(1.)* Let (X,d) be a metric space. Then the set

$$\mathscr{B} := \{B(x,r) : x \in X,\, r \in \mathbb{R},\, r > 0\}$$

is a basis for the topology induced by the metric d.

(2.) Let $X = \{a,b,c,d,e\}$ and let

$$\tau = \{\emptyset,\ X,\ \{a\},\ \{a,b\},\ \{a,b,c\},\ \{a,d,e\}\,\}\,.$$

Then (X,τ) is a topological space. Set

$$\mathscr{B}_1 := \{\,\{a\},\ \{a,b\},\ \{a,b,c\},\ \{a,d,e\}\,\} \subseteq \mathscr{P}(X)$$
$$\mathscr{B}_2 := \{\,\{a\},\ \{a,b,c\},\ \{a,d,e\}\,\} \subseteq \mathscr{P}(X).$$

Then $\mathscr{B}_1$ is a base for τ. But $\mathscr{B}_2$ is not a basis of τ. For example, there is no element $B \in \mathscr{B}_2$ satisfying $b \in B \subseteq \{a,b\}$.

(3.) Let $X = \{a,b,c,d,e\}$ and let

$$\tau = \{\emptyset,\ X,\ \{a\},\ \{a,b\},\ \{a,b,c\},\ \{a,b,d\},\ \{a,b,c,d\}\,\}\,.$$

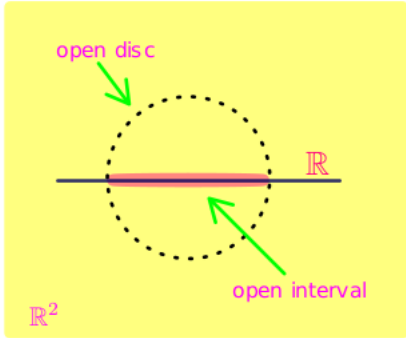
Then (X,τ) is a topological space. Set

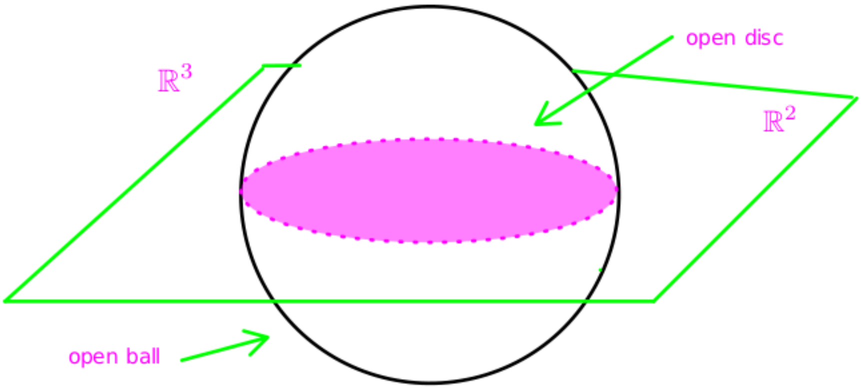
$$\mathscr{B}_1 := \{\,\{a\},\ \{a,b\},\ \{a,b,c\},\ \{a,b,d\}\,\} \subseteq \mathscr{P}(X)$$
$$\mathscr{B}_2 := \{\,\{a\},\ \{a,b\},\ \{a,b,c\},\ \{a,b,d\},\ X\,\} \subseteq \mathscr{P}(X).$$

Then $\mathscr{B}_2$ is a base for the topology τ, but $\mathscr{B}_1$ is not a base for the topology τ since there is no $B \in \mathscr{B}_1$ satisfying $e \in B \subseteq X$.

Exercise 2.3. *(1.)* Let (X,τ) be a topological space and let $\mathscr{B}$ be a base for τ. Prove that any subset $\mathscr{U} \subseteq \mathscr{P}(X)$ satisfying $\mathscr{B} \subseteq \mathscr{U} \subseteq \tau$ is also a base for τ.

(2.) Let (X,τ) be a topological space, $\mathscr{B}$ be a base for τ and let $Y \subseteq X$. Set $\mathscr{B}_Y = \{B \cap Y : B \in \mathscr{B}\}$. Prove that $\mathscr{B}_Y$ is a base for the subspace topology on Y.





Lemma 2.4. *Let (X, τ) be a topological space and let $\mathscr{B} \subseteq \tau$. Then $\mathscr{B}$ is a base for τ if and only if every member of τ can be written as a union of members of $\mathscr{B}$.*

Proof. Let $\mathscr{B}$ be a base for τ and let $V \in \tau$. For any $x \in V$, there is a basic open set $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq V$. Then $V = \bigcup_{x \in V} B_x$. The empty set can be written as an empty union of members of $\mathscr{B}$.

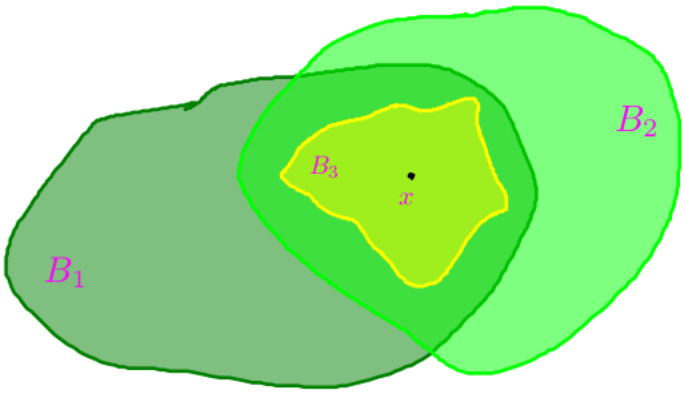
Conversely assume that every member of τ can be written as a union of members of $\mathscr{B}$. Let $x \in X$. Choose an open set $V \subseteq X$ such that $x \in V$. By the given condition, there is $B \in \mathscr{B}$ such that $x \in B \subseteq V$. Hence $\mathscr{B}$ is a basis for τ. $\square$

In Example 2.2(*2.*), the open set $\{a, b\}$ can not be written as a union of elements of $\mathscr{B}_2$. Similarly, in Example 2.2(*3.*), X can not be written as a union of elements of $\mathscr{B}_1$.

Proposition 2.5. *Let X be a non-empty set and let $\mathscr{B} \subseteq \mathscr{P}(X)$. Then the following conditions are equivalent:*

(i) *$\mathscr{B}$ is a base for some topology on X;*

(ii) *$\mathscr{B}$ satisfies the following two properties:*
- $X = \bigcup_{B \in \mathscr{B}} B$;
- *for any $B_1, B_2 \in \mathscr{B}$ having non-empty intersection and any $x \in B_1 \cap B_2$, there is $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*



Proof. $(i) \implies (ii)$ Suppose $\mathscr{B}$ is a base for a topology τ on X. Then $\mathscr{B} \subseteq \tau$. Now the result follows from Lemma 2.4.

$(ii) \implies (i)$ Assume $\mathscr{B}$ satisfies the given condition. Set

$$\tau := \left\{ \bigcup \mathscr{A} \,:\, \mathscr{A} \subseteq \mathscr{B} \right\} \subseteq \mathscr{P}(X)$$

i.e. τ is the set of all subsets of X that can be written as a union of some elements of $\mathscr{B}$. Then $\emptyset, X \in \tau$. Let's check the remaining two defining conditions of a topology:

- Let $\{V_\alpha : \alpha \in \Lambda\}$ be a collection of elements of τ. For each $\alpha \in \Lambda$, choose a subset $\mathscr{A}_\alpha \subseteq \mathscr{B}$ such that $V_\alpha = \bigcup \mathscr{A}_\alpha$. Set $\mathscr{A} = \bigcup_{\alpha\in\Lambda} \mathscr{A}_\alpha \subseteq \mathscr{B}$. Then $\bigcup_{\alpha\in\Lambda} V_\alpha = \bigcup \mathscr{A}$ and hence $\bigcup_{\alpha\in\Lambda} V_\alpha \in \tau$.
- Let $V_1, V_2 \in \tau$ and let $x \in V_1 \cap V_2$. Choose $B_1, B_2 \in \mathscr{B}$ such that $x \in B_1 \subseteq V_1$ and $x \in B_2 \subseteq V_2$. Then there is $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq B_1 \cap B_2 \subseteq V_1 \cap V_2$. Set $\mathscr{A}_x := \{B_x \in \mathscr{B} : x \in V_1 \cap V_2\}$. Then $V_1 \cap V_2 = \bigcup \mathscr{A}_x$.

Hence τ is a topology on X and $\mathscr{B}$ is a base for τ. $\square$

Let X be a non-empty set and let $\mathscr{B} \subseteq \mathscr{P}(X)$ satisfies

- $X = \bigcup_{B\in\mathscr{B}} B$;
- for any $B_1, B_2 \in \mathscr{B}$ having non-empty intersection and any $x \in B_1 \cap B_2$, there is $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

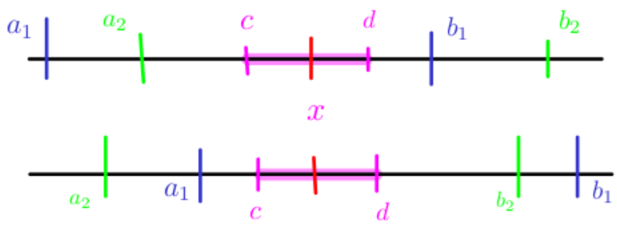
Then the topology

$$\tau := \left\{ \cup \mathscr{A} \,:\, \mathscr{A} \subseteq \mathscr{B} \right\} \subseteq \mathscr{P}(X)$$

on X is said to be *generated by* $\mathscr{B}$.

Exercise 2.6. Let (X, τ) be a topological space and let $\mathscr{B}$ be a base for τ. Let τ' be the topology on X generated by $\mathscr{B}$. Prove that $\tau = \tau'$.

Example 2.7. *(1.)* Let $X = \mathbb{R}$ and consider the set $\mathscr{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\} \subseteq \mathscr{P}(X)$. Clearly $\mathbb{R} = \bigcup_{B\in\mathscr{B}} B$. Let $B_1 = [a_1, b_1), B_2 = [a_2, b_2) \in \mathscr{B}$ and let $x \in B_1 \cap B_2$. Then there are $c, d \in \mathbb{R}$ with $c < x < d$ such that $x \in [c, d) \subseteq B_1 \cap B_2$.



Thus $\mathscr{B}$ satisfies the properties of Proposition 2.5 and hence generates a topology on $\mathbb{R}$, known as the *lower limit topology* on $\mathbb{R}$. The set of real numbers $\mathbb{R}$ equipped with the lower limit topology will be denoted by $\mathbb{R}_\ell$, called *Sorgenfrey line*.

(2.) Let $X = \mathbb{R}$ and consider the set $\mathscr{B} = \{(a, b] : a, b \in \mathbb{R}, a < b\} \subseteq \mathscr{P}(X)$. Arguing as above, we get that $\mathscr{B}$ generates a topology on $\mathbb{R}$, called the *upper limit topology* on $\mathbb{R}$.

(3.) Let $X = \{a, b, c, d\}$ and let

$$\mathscr{B} := \{\{a\}, \{b\}, \{d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\} \subseteq \mathscr{P}(X).$$

Notice that, there is no $V \in \mathscr{B}$ satisfying $c \in V \subseteq \{a, b, c\} \bigcap \{a, c, d\}$. Thus $\mathscr{B}$ is not a base for a topology on X.

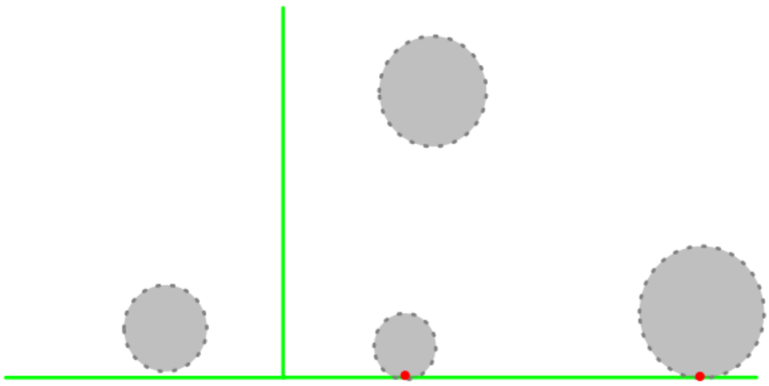
Exercise 2.8. *(1.)* Let $K = \{1/n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ and set

$$\mathscr{B}_K := \{(a, b) : a, b \in \mathbb{R},\ a < b\} \bigcup \{(a, b) \setminus K : a, b \in \mathbb{R},\ a < b\} \subseteq \mathscr{P}(\mathbb{R}).$$

Prove that $\mathscr{B}_K$ is a base for some topology on $\mathbb{R}$. The topology τ_K on $\mathbb{R}$ generated by the base $\mathscr{B}_K$ is known as *K-topology*.

(2.) Let $X = \{x \in \mathbb{R} : x > 0\} \bigcup \{\infty\} \bigcup \{-\infty\}$ and let $\mathscr{B} \subseteq \mathscr{P}(X)$ be the collection consists of intervals (a, b) with $a, b \in \mathbb{R}, 0 < a < b$ together with the sets of the form $(0, s) \cup \{\infty\}, s \in \mathbb{R}, s > 0$ and $(0, t) \cup \{-\infty\}, t \in \mathbb{R}, t > 0$. Prove that $\mathscr{B}$ is a base for some topology on X.

(3.) Let $X = \{(a, b) \in \mathbb{R}^2 : b \geq 0\}$ and let $\mathscr{B} \subseteq \mathscr{P}(X)$ be the collection consists of



- all open discs of the form $B\big((a, b); r\big)$ with $0 < r \leq b$;
- all sets of the form $B\big((a, b); r\big) \bigcup \{(a, 0)\}$ where $r = b > 0$.

Prove that $\mathscr{B}$ is a base for some topology on X.

Definition 2.9. Let X be a set and let τ_1, τ_2 be two topologies on X.

- We say τ_2 is *larger* (resp. *strictly larger*) than τ_1 and τ_1 is *smaller* (resp. *strictly smaller*) than τ_2 if $\tau_1 \subseteq \tau_2$ (resp. $\tau_1 \subsetneq \tau_2$) as subsets of $\mathscr{P}(X)$.
- We say that τ_1 and τ_2 are *comparable* if $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$ as subsets of $\mathscr{P}(X)$.

The set $\mathscr{P}(\mathscr{P}(X))$ is a partially ordered set with respect to inclusion: for $\mathscr{C}, \mathscr{D}$ in $\mathscr{P}(\mathscr{P}(X))$, $\mathscr{C} \leq \mathscr{D}$ if and only if $\mathscr{C} \subseteq \mathscr{D} \subseteq \mathscr{P}(X)$. Two topologies on X are comparable if they are comparable as elements of the partially ordered set $\mathscr{P}(\mathscr{P}(X))$. Similarly, for two comparable topologies on X, one is larger (resp. smaller) if it is larger (resp. smaller) as elements of the partially ordered set $\mathscr{P}(\mathscr{P}(X))$.

Example 2.10. *(1.)* Let X be a set and let τ_0, τ_1 be the indiscrete and discrete topologies on X, respectively. If τ is any topology on X, we always have $\tau_0 \subseteq \tau \subseteq \tau_1 = \mathscr{P}(X)$. In particular, the indiscrete topology is the smallest topology and the discrete topology is the largest topology on a set.

(2.) Let $X = \{a, b, c\}$ and consider the topologies on X:

$$\tau_1 = \{\emptyset, X, \{a\}\}; \;\; \tau_2 = \{\emptyset, X, \{b\}\}; \;\; \tau_3 = \{\emptyset, X, \{a\}, \{a, b\}\}.$$

Then τ_1 and τ_2 are not comparable, but $\tau_1 \subsetneq \tau_3$ and $\tau_2 \subsetneq \tau_3$.

(3.) Let X be an infinite set and let τ_1, τ_2 be the co-finite and co-countable topologies on X, respectively. Then $\tau_1 \subseteq \tau_2$.

(4.) Let $X = \mathbb{R}$ and let τ_1, τ_2 and τ_3 be the Euclidean topology, co-finite topology and co-countable topology on $\mathbb{R}$, respectively.

- $\tau_2 \subseteq \tau_1$: Suppose $U \in \tau_2$ and $\emptyset \neq U \subsetneq X$. Then $U = \mathbb{R} \setminus \{a_1, \cdots, a_n\}$, for some $a_1, \cdots, a_n \in \mathbb{R}$. Renaming, if necessary, we may assume that $a_1 < \cdots < a_n$. Then $U = (-\infty, a_1) \bigcup (a_1, a_2) \bigcup \cdots \bigcup (a_{n-1}, a_n) \bigcup (a_n, \infty) \in \tau_1$.
- τ_1 and τ_3 are not comparable: $\mathbb{R} \setminus \mathbb{Q} \in \tau_3$ but $\mathbb{R} \setminus \mathbb{Q} \notin \tau_1$; $(0,1) \in \tau_1$ but $(0,1) \notin \tau_3$.

Proposition 2.11. *Let X be a non-empty set, τ_1, τ_2 be two topologies on X and let $\mathscr{B}_i$ be a base for τ_i, $i = 1, 2$. Then the following conditions are equivalent:*

(i) τ_1 is smaller than τ_2, i.e. $\tau_1 \subseteq \tau_2$;

(ii) for any $V \in \mathscr{B}_1$ and any $x \in V$, there is $B \in \mathscr{B}_2$ such that $x \in B \subseteq V$.

Proof. $(i) \implies (ii)$ Assume $\tau_1 \subseteq \tau_2$. Let $V \in \mathscr{B}_1$ and $x \in V$. Then $V \in \tau_2$ and hence there is $B \in \mathscr{B}_2$ such that $x \in B \subseteq V$.

$(ii) \implies (i)$ Assume the condition holds. Let $V \in \tau_1$. For any $x \in V$, there is $U_x \in \mathscr{B}_1$ such that $x \in U_x \subseteq V$. By the given condition, there is $B_x \in \mathscr{B}_2$ such that $x \in B_x \subseteq U_x \subseteq V$. Hence $V = \bigcup_{x \in V} B_x$ and consequently, $V \in \tau_2$. □

Let X be a non-empty set and let τ_1, τ_2 be two comparable topologies on X. Another notation is used in literature in place of "larger" and "smaller": if $\tau_1 \subseteq \tau_2$ then τ_2 is said to be *finer than* τ_1 and τ_1 is said to be *coarser than* τ_2. Motivation behind this terminology comes from Proposition 2.11: if $\tau_1 \subseteq \tau_2$, then τ_2 has "finer" open sets than τ_1.

Example 2.12. *(1.)* Let $X = \mathbb{R}$ and let τ, τ_ℓ be the Euclidean topology and the lower limit topology on $\mathbb{R}$, respectively. Then $\mathscr{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is a base for τ and $\mathscr{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is a base for τ_ℓ. Let $(a, b) \in \mathscr{B}$. Then we can write $(a, b) = \bigcup_{n \geq N_0} [a + \frac{1}{n}, b)$, for some $N_0 \in \mathbb{N}$. By Proposition 2.11, $\tau \subseteq \tau_\ell$. On the other hand, there is no open interval (a, b) satisfying $0 \in (a, b) \subseteq [0, 1)$. Thus $\tau \subsetneq \tau_\ell$, i.e. τ_ℓ is strictly larger than τ or τ is strictly smaller than τ_ℓ.

(2.) Let $X = \mathbb{R}$ and consider the sets

$$\mathscr{B}_1 := \{(a, b) : a, b \in \mathbb{R}, a < b\} \text{ and } \mathscr{B}_2 := \{(a, b) : a, b \in \mathbb{Q}, a < b\}.$$

Notice that $\mathscr{B}_1$ is a base for the Euclidean topology on $\mathbb{R}$ and $\mathscr{B}_2$ is a base for some topology τ on $\mathbb{R}$. We claim that these two topologies are the same:

- Let $(a, b) \in \mathscr{B}_1$ and $x \in (a, b)$. There are $c, d \in \mathbb{Q}$ satisfying $a < c < x < d < b$. Then $x \in (c, d) \subseteq (a, b)$ and $(c, d) \in \mathscr{B}_2$. Thus τ is larger than the Euclidean topology.
- On the other hand, $\mathscr{B}_2 \subseteq \mathscr{B}_1$. Thus τ is smaller than the Euclidean topology.

($3.$) Let $X = \mathbb{R}$ and consider the sets

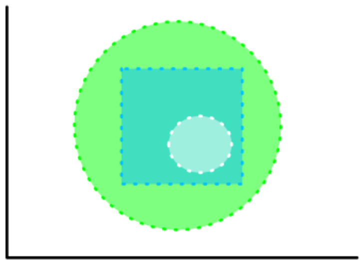
$$\mathscr{B}_1 := \{[a, b) : a, b \in \mathbb{R},\ a < b\} \text{ and } \mathscr{B}_2 := \{[a, b) : a, b \in \mathbb{Q},\ a < b\}.$$

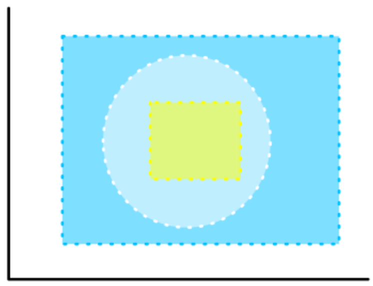
Then $\mathscr{B}_1$ is a base for the lower limit topology τ_ℓ on $\mathbb{R}$ and $\mathscr{B}_2$ is a base for some topology τ on $\mathbb{R}$. Since $\mathscr{B}_2 \subseteq \mathscr{B}_1$, we have $\tau \subseteq \tau_\ell$. Since there is no $V \in \mathscr{B}_2$ satisfying $\sqrt{2} \in V \subseteq [\sqrt{2}, \sqrt{3})$, $\tau \neq \tau_\ell$. Thus τ_ℓ is strictly larger that τ.

Exercise 2.13. ($1.$) Let $X = \mathbb{R}^2$ and define

$$\mathscr{B} := \left\{(a, b) \times (c, d) \subseteq \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \ :\ a, b, c, d \in \mathbb{R},\ a < b,\ c < d\right\}.$$

Prove that $\mathscr{B}$ is a base for the Euclidean topology on $\mathbb{R}^2$.





($2.$) Let X be a non-empty set and let d_1, d_2 be two metrics on X and assume there are real numbers $\alpha, \beta > 0$ such that $\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y)$, for every $(x, y) \in X \times X$. Let τ_i be the topology on X generated by the metric d_i, $i = 1, 2$. Prove that $\tau_1 = \tau_2$.

($3.$) Let (X, d) be a metric space and define

$$d_1 : X \times X \longrightarrow \mathbb{R},\ d_1(x, y) = \min\{1, d(x, y)\}$$

$$d_2 : X \times X \longrightarrow \mathbb{R},\ d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Prove that

(i) both d_1 and d_2 are metrics on X;

(ii) the metrics d, d_1 and d_2 generates the same topology on X.

Let X be a non-empty set and let $\{\tau_\alpha : \alpha \in \Lambda\}$ be a family of topologies on X. Considering each τ_α as a subset of X, we can define their intersection $\tau := \bigcap_{\alpha \in \Lambda} \tau_\alpha \subseteq \mathscr{P}(X)$.

- By construction, $\emptyset \in \tau$ and $X \in \tau$.
- Let $\{V_i : i \in I\}$ be a collection of elements of τ. Then

  $$V_i \in \tau_\alpha, \text{ for every } i \in I \text{ and } \alpha \in \Lambda \implies \bigcup_{i \in I} V_i \in \tau_\alpha, \forall \alpha \in \Lambda$$

  so that $\bigcup_{i \in I} V_i \in \tau$.
- Let $\{V_i : 1 \leq i \leq n\}$ be a collection of elements of τ. Then

  $$V_i \in \tau_\alpha, \text{ for every } i \in \{1, \cdots, n\} \text{ and } \alpha \in \Lambda \implies \bigcap_{i=1}^{n} V_i \in \tau_\alpha, \forall \alpha \in \Lambda$$

  so that $\bigcap_{i=1}^{n} V_i \in \tau$.

Thus τ is a topology on X. Moreover, τ is the *unique largest topology* on X such that $\tau \subsetneq \tau_\alpha$, for every $\alpha \in \Lambda$.

Let $X = \{a, b, c\}$ and consider the two topologies on X given by

$$\tau_1 := \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\} \text{ and } \tau_2 := \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\}$$

Then

$$\tau_1 \bigcup \tau_2 := \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\} \subsetneq \mathscr{P}(X).$$

Notice that $\tau_1 \bigcup \tau_2$ is not a topology on X as $\{b\} \bigcup \{c\} \notin \tau_1 \bigcup \tau_2$. Thus, in general, union of two topologies need not be a topology.

Let X be a non-empty set and let τ_1, τ_2 be two topologies on X. Let τ' be the discrete topology on X. Then $\tau_1 \subseteq \tau'$ and $\tau_2 \subseteq \tau'$ as subsets of $\mathscr{P}(X)$. Set

$$\Sigma := \{\mu \subseteq \mathscr{P}(X) : \mu \text{ is a topology on } X \text{ and } \tau_1 \subseteq \mu,\, \tau_2 \subseteq \mu\}.$$

Since $\tau' \in \Sigma$, $\Sigma \neq \emptyset$. Define a relation $\preceq$ on Σ as follows: $\mu_1 \preceq \mu_2$ if and only if $\mu_2 \subseteq \mu_1$ as subsets of $\mathscr{P}(X)$. It's easy to see that $\preceq$ is a partial order relation on Σ. Let's consider a chain $\mu_1 \preceq \mu_2 \preceq \mu_2 \preceq \cdots$ of elements of Σ. Then $\mu_1 \supseteq \mu_2 \supseteq \mu_2 \supseteq \cdots$ as subsets of $\mathscr{P}(X)$. Set $\mu := \bigcap_{i \geq 1} \mu_i \subseteq \mathscr{P}(X)$. Then μ is a topology on X. By construction, $\tau_1 \subseteq \mu$ and $\tau_2 \subseteq \mu$ so that $\mu \in \Sigma$. By Zorn's Lemma, Σ has a maximal element, say τ. Then τ is the *unique smallest topology* on X such that $\tau_1 \subseteq \tau$ and $\tau_2 \subseteq \tau$.

Exercise 2.14. *(1.)* Let X be a set with cardinality one or two. Prove that union of any two topologies on X is again a topology on X.

(2.) Let $X = \{a, b, c, d\}$ and consider the two topologies on X given by

$$\tau_1 := \{\emptyset, X, \{a\}, \{a,b,c\}\} \text{ and } \tau_2 := \{\emptyset, X, \{b\}, \{a,b,d\}\}$$

Find the unique smallest topology τ on X such that $\tau_1 \subseteq \tau$ and $\tau_2 \subseteq \tau$.

(3.) Let $X = \mathbb{R}$ and let τ_ℓ and τ_u be the lower limit topology and upper limit topology on X, respectively. Find the unique smallest topology τ on X such that $\tau_\ell \subseteq \tau$ and $\tau_u \subseteq \tau$.

(4.) Let X be a non-empty set and let $\{\tau_\alpha : \alpha \in \Lambda\}$ be a collection of topologies on X. Does there exists a *unique smallest topology* $\tau \subseteq \mathscr{P}(X)$ on X such that $\tau_\alpha \subseteq \tau, \forall \alpha \in \Lambda$?

Let $\mathscr{S} := \{A_\alpha : \alpha \in \Lambda\}$ be a non-empty collection of sets and let $X := \bigcup_{\alpha \in \Lambda} A_\alpha$. Set

$$\mathscr{B} := \{A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \subseteq X : \alpha_1, \cdots, \alpha_n \in \Lambda,\, n \in \mathbb{N}\}$$

i.e. $\mathscr{B}$ is the set of all subsets of X that can be written as a finite intersection of members of $\mathscr{S}$. We claim that $\mathscr{B}$ is a base for some topology on X:

- Since $X = \bigcup_{\alpha \in \Lambda} A_\alpha$, we have $X = \bigcup_{B \in \mathscr{B}} B$.
- Let $V, V' \in \mathscr{B}$. Then there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ and $\beta_1, \cdots, \beta_m \in \Lambda$ such that $V = \bigcap_{t=1}^n A_{\alpha_t}$ and $V' = \bigcap_{t=1}^m A_{\beta_t}$. Thus $V \bigcap V' \in \mathscr{B}$.

By Proposition 2.5, $\mathscr{B}$ is a base for a topology on X.

Definition 2.15. Let (X, τ) be a topological space and let $\emptyset \neq \mathscr{S} \subseteq \mathscr{P}(X)$. We say $\mathscr{S}$ is a *sub-base for the topology* τ if the collection of all subsets of X that can be written as a finite intersection of members of $\mathscr{S}$ forms a base $\mathscr{B}$ for τ. We say the base $\mathscr{B}$ and the topology τ is *generated* by the sub-base $\mathscr{S}$.

Example 2.16. *(1.)* Let $X = \mathbb{R}$ and let

$$\mathscr{S} := \{(-\infty, s) : s \in \mathbb{R}\} \bigcup \{(t, \infty) : t \in \mathbb{R}\} \subseteq \mathscr{P}(\mathbb{R}).$$

Let $a, b \in \mathbb{R}$ with $a < b$. Then we can write $(a, b) = (-\infty, b) \bigcap (a, \infty)$. Thus $\mathscr{S}$ generates the Euclidean topology on $\mathbb{R}$.

(2.) Let $X = \mathbb{R}$ and let

$$\mathscr{S} := \{(-\infty, s) : s \in \mathbb{R}\} \bigcup \{[t, \infty) : t \in \mathbb{R}\} \subseteq \mathscr{P}(\mathbb{R}).$$

Let $a, b \in \mathbb{R}$ with $a < b$. Then we can write $[a, b) = (-\infty, b) \bigcap [a, \infty)$. Thus $\mathscr{S}$ generates the lower limit topology on $\mathbb{R}$.

Exercise 2.17. (1.) Let $X = \{a, b, c, d\}$ and let $\mathscr{S} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, d\}\}$. Write down the topology generated by $\mathscr{S}$ on X explicitly.

(2.) Let $X = \{a, b, c, d, e\}$ and let $\mathscr{S} = \{\{a\}, \{a, b, c\}, \{b, c, d\}, \{c, e\}\}$. Write down the topology generated by $\mathscr{S}$ on X explicitly.

(3.) Let $X = \{a, b, c, d, e, f\}$ and let $\mathscr{S} = \{\{a, b\}, \{b, c\}, \{c, d\}, \{c, d, f\}\}$. Write down the topology generated by $\mathscr{S}$ on X explicitly.

(4.) Let X be an infinite set. Show that the collection $\mathscr{S} = \{X \setminus \{x\} : x \in X\}$ forms a sub-base for the co-finite topology on X.

Example 2.18. (1.) Let $(X, \preceq)$ be a totally ordered set. For $a, b \in X$ with $a \prec b$, define

$$\begin{aligned} \text{open interval } (a, b) &:= \{x \in X : a \prec x \prec b\} \\ \text{closed interval } [a, b] &:= \{x \in X : a \preceq x \preceq b\} \\ \text{half-open interval } [a, b) &:= \{x \in X : a \preceq x \prec b\} \\ \text{half-open interval } (a, b] &:= \{x \in X : a \prec x \preceq b\} \end{aligned}$$

Let $\mathscr{B} \subseteq \mathscr{P}(X)$ be the collection of subsets of X of the form

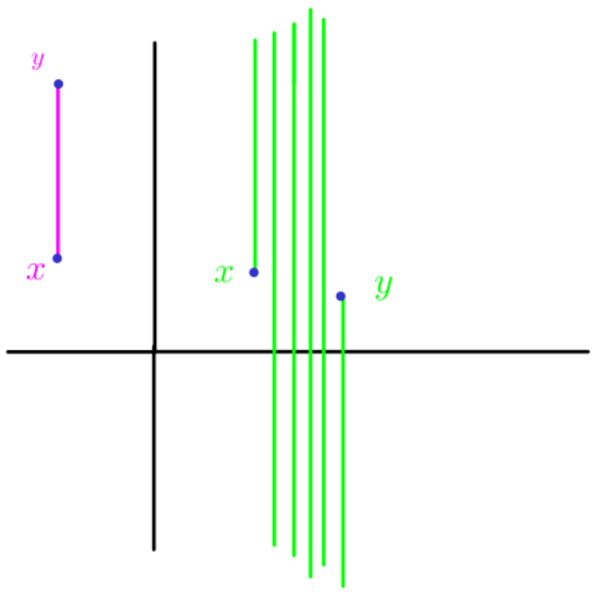
- open intervals (a, b) in X;
- half-open intervals $[a_0, b)$ in X where $a_0 \in X$ is the smallest element, if exists;
- half-open intervals $(a, b_0]$ in X where $b_0 \in X$ is the largest element, if exists.

Then its easy to see that $\mathscr{B}$ satisfies the properties of Proposition 2.5. Thus $\mathscr{B}$ generates a topology on $(X, \preceq)$, called the *order topology*.

(2.) On the set $\mathbb{R}^2$, define a relation

$$x = (x_1, x_2) \prec y = (y_1, y_2) \iff x_1 < y_1 \text{ or } x_1 = y_1 \text{ and } x_2 < y_2.$$

Then $\prec$ is a total order relation on $\mathbb{R}^2$, known as *lexicographic ordering* or *dictionary ordering*. For $x, y \in \mathbb{R}^2$ with $x \prec y$, the open interval will look like



Since $(\mathbb{R}^2, \prec)$ has no smallest and no largest elements, $\mathscr{B} := \{(x, y) : x, y \in \mathbb{R}^2,\ x \prec y\}$ is a base for the order topology on $(\mathbb{R}^2, \prec)$.

Exercise 2.19. *(1.)* Let X be the set of all real valued continuous functions on the closed bounded interval $[0, 1]$. For every $f \in X$ and every real number $\varepsilon > 0$, set

$$M(f;\varepsilon) := \left\{ g \in X \ : \ \int_0^1 |f - g| \ < \varepsilon \right\}$$

and

$$B(f;\varepsilon) := \{g \in X \ : \ \sup\{|g(x) - f(x)| : x \in [0, 1]\} \ < \varepsilon\}.$$

- Prove that the collections
$$\mathscr{M} = \{M(f;\varepsilon) : f \in X, \varepsilon > 0\} \ \text{ and } \ \mathscr{B} = \{B(f;\varepsilon) : f \in X, \varepsilon > 0\}$$
form bases for some topologies on X.
- Compare the topologies, if possible, generated by $\mathscr{M}$ and $\mathscr{B}$.

(2.) Let $M(n, \mathbb{R})$ be the set of all $n \times n$ matrices with real entries. For $A = (a_{ij}) \in M(n, \mathbb{R})$ and real $r > 0$, set

$$B(A;r) := \{B = (b_{ij}) \in M(n, \mathbb{R}) \ : \ |a_{ij} - b_{ij}| < r, \forall\, i, j = 1, \cdots, n\}.$$

Prove that $\{B(A;r) : A \in M(n, \mathbb{R}), r \in \mathbb{R}, r > 0\}$ is a base for some topology on $M(n, \mathbb{R})$.

(3.) Let X be a non-empty set and let $\{\tau_i : i \in I\}$ be a collection of topologies on X. Set $\mathscr{S} := \bigcup_{i \in I} \tau_i \subseteq \mathscr{P}(X)$. Prove that the topology τ on X generated by the subbase $\mathscr{S}$ is the unique smallest topology on X satisfying $\tau \subseteq \tau_i, \forall i \in I$.

(4.) Let $X = \mathbb{R}$ and let τ, τ_ℓ, τ_K be the Euclidean topology, lower limit topology and K-topology on $\mathbb{R}$, respectively. Compare these topologies.

Definition 2.20. Let (X, τ) be a topological space and let $x \in X$. By a *local base* at x, we mean a collection $\mathscr{B}_x$ of neighbourhoods of x satisfying the following: given any $N \in \mathscr{N}_x$, there is $B \in \mathscr{B}_x$ such that $B \subseteq N$.

Example 2.21. *(1.)* Let (X, τ) be a non-empty topological space.

- Suppose τ is the discrete topology on X. Then for any $x \in X$, $\mathscr{B}_x = \{\{x\}\}$ is a local base at x.
- Suppose τ is the indiscrete topology on X. Then for any $x \in X$, $\mathscr{B}_x = \{X\}$ is the only possible local base at x.

(2.) Let (Z, d) is a metric space. Then for any $z \in Z$,

$$\mathscr{B}_z := \{B(z;r) \ : \ r \in \mathbb{R},\ r > 0\} \ \text{ and } \ \mathscr{U}_z := \{B(z;\ 1/n) \ : \ n \in \mathbb{N}\}$$

are both local bases at z.

Exercise 2.22. *(1.)* Let (X, τ) be a topological space and let $\mathscr{B}$ be a base for the topology τ and let $x \in X$. Prove that $\mathscr{B}_x := \{B \in \mathscr{B} : x \in B\}$ is a local base at x.

(2.) Let (X, τ) be a non-empty topological space. For each $x \in X$, let $\mathscr{B}_x$ be a local base at x. Prove that

(i) $E \subseteq X$ is closed if and only if for each $x \in X \setminus E$, there is $B \in \mathscr{B}_x$ satisfying $B \bigcap E = \emptyset$;

(ii) for any $A \subseteq X$, $\overline{A} = \{x \in X \ : \ B \bigcap A \neq \emptyset, \forall B \in \mathscr{B}_x\}$;

(iii) for any $A \subseteq X$, $\operatorname{Int}(A) = \{x \in X \ : \ B \subseteq A$ for some $B \in \mathscr{B}_x\}$;

(iv) for any $A \subseteq X$, $\partial A = \{x \in X \ : \ B \bigcap A \neq \emptyset \neq B \bigcap (X \setminus A), \forall B \in \mathscr{B}_x\}$.

3. Continuous Function

From now on we will write "X be a topological space" instead of "(X, τ) be a topological space", unless explicit mention of the topology on X is needed.

Definition 3.1. Let X and Y be two topological spaces. A function $f : X \longrightarrow Y$ is said to be *continuous* if for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.

Exercise 3.2. Let X, Y be topological spaces, $f : X \longrightarrow Y$ be a function and let $\mathscr{B}$ and $\mathscr{S}$ be a base and a sub-base for the topology on Y, respectively. Prove that

- f is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathscr{B}$;
- f is continuous if and only if $f^{-1}(S)$ is open in X for every $S \in \mathscr{S}$.

Example 3.3. *(1.)* Let (X, d_1) and (Y, d_2) be metric spaces and let τ_1, τ_2 be the topologies, on X and Y, generated by d_1 and d_2, respectively. Then a map $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ between topological spaces is continuous if and only if it is continuous as a map $f : (X, d_1) \longrightarrow (Y, d_2)$ between metric spaces.

(2.) Let $f : X \longrightarrow Y$ be a function between two non-empty sets. If X is equipped with the discrete topology, then for any topology on Y, f is continuous. On the other hand, if Y is equipped with the indiscrete topology, then for any topology on X, f is continuous.

(3.) Let $X = \mathbb{R}$ and let τ, τ_ℓ be the Euclidean topology and the lower limit topology on X, respectively. Define

$$f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \tau_\ell),\ x \mapsto x \ \text{ and } \ g : (\mathbb{R}, \tau_\ell) \longrightarrow (\mathbb{R}, \tau),\ x \mapsto x\,.$$

Since $[0, 1) = f^{-1}[0, 1)$ is not open in $(\mathbb{R}, \tau)$, by Exercise 3.2, f is not continuous. On the other hand, we can write $g^{-1}(a, b) = (a, b) = \bigcup_{n \geq N_0} [a + 1/n,\, b)$, for some $N_0 \in \mathbb{N}$. Hence by Exercise 3.2, g is continuous.

(4.) Let X be an uncountable set and let $a, b \in X$ with $a \neq b$. Consider the two topologies on X defined by

$$\begin{aligned} \tau_a &= \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subseteq X \,:\, a \in V \text{ and } X \setminus V \text{ is finite}\} \\ \tau_b &= \mathscr{P}(X \setminus \{b\}) \bigcup \{V \subseteq X \,:\, b \in V \text{ and } X \setminus V \text{ is finite}\}\,. \end{aligned}$$

Define $f : (X, \tau_a) \longrightarrow (X, \tau_b)$ by $f(x) = x$ for $x \neq a, b$; $f(a) = b$ and $f(b) = a$. Notice that the set

$$\mathscr{B} := \{\{x\} \,:\, x \in X \text{ and } x \neq b\} \bigcup \{V \subseteq X \,:\, b \in V \text{ and } X \setminus V \text{ is finite}\}$$

is a base for the topology τ_b. For any $x \in X \setminus \{a, b\}$, we have $f^{-1}(x) = \{x\}$ and $f^{-1}(a) = \{b\}$. Thus $f^{-1}(\{x\})$ is open in (X, τ_a), for any $x \in X \setminus \{b\}$. Let $V \in \tau_b$ be such that $b \in V$. Then $X \setminus V$ is finite and $a \in f^{-1}V$. We consider two cases:

- Let $a \in V$. Then $f^{-1}(V \setminus \{a, b\}) = V \setminus \{a, b\}$ and hence $X \setminus f^{-1}(V)$ is finite. Consequently, $f^{-1}(V) \in \tau_a$.
- Let $a \notin V$. Then $f^{-1}(V \setminus \{b\}) = V \setminus \{b\}$ and $f^{-1}(b) = \{a\}$ so that $X \setminus f^{-1}(V)$ is finite. Consequently, $f^{-1}(V) \in \tau_a$.

Hence f is a continuous function.

Exercise 3.4. *(1.)* Let $f : X \longrightarrow Y$ be a function between two topological spaces X and Y. Prove that f is continuous if and only if $f^{-1}(E) \subseteq X$ is closed for every closed set $E \subseteq Y$.

(2.) Let X be a topological space and let $A \subseteq X$ be a subspace. Prove that the natural inclusion map $\iota_A : A \longrightarrow X, a \mapsto a$ is continuous.

(3.) Let X, Y, Z be topological spaces and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two continuous functions. Prove that $g \circ f : X \longrightarrow Z$ is continuous.

(4.) Let $f : X \longrightarrow Y$ be a function between two topological spaces. Let $A \subseteq X$ and $B \subseteq Y$ be two subspaces such that $f(A) \subseteq B$. Prove that the restriction map $f|_A : A \longrightarrow B$ is continuous.

(5.) Let X, Y be topological spaces and $f : X \longrightarrow Y$ be a function. Suppose X can be written as $X = \bigcup_{\alpha \in \Lambda} V_\alpha$ where each V_α is open in X. Assume the restriction map $f|_{V_\alpha} : V_\alpha \longrightarrow Y$ is continuous, for each $\alpha \in \Lambda$, when V_α is equipped with the subspace topology. Prove that $f : X \longrightarrow Y$ is continuous.

Proposition 3.5. *Let X and Y be two topological spaces and let $f : X \longrightarrow Y$ be a function. Then the following conditions are equivalent:*

(i) f is continuous;

(ii) for every set $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$;

(iii) for every set $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Proof. $(i) \implies (ii)$ Assume f is continuous and let $A \subseteq X$. Let $x \in \overline{A}$ and let $V_{f(x)} \subseteq Y$ be a neighbourhood of $f(x)$. Then

$$f^{-1}V_{f(x)} \in \mathscr{N}_x \implies A \bigcap f^{-1}V_{f(x)} \neq \emptyset \implies f(A) \bigcap V_{f(x)} \neq \emptyset.$$

Since this is true for every neighbourhood of $f(x)$, $f(x) \in \overline{f(A)}$.

$(ii) \implies (iii)$ Suppose $f(\overline{A}) \subseteq \overline{f(A)}$ holds for every $A \subseteq X$. Let $B \subseteq Y$. Then

$$f(\overline{f^{-1}B}) \subseteq \overline{f(f^{-1}B)} = \overline{B} \implies \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}).$$

$(iii) \implies (i)$ Suppose $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ holds for every $B \subseteq Y$. Let $E \subseteq Y$ be a closed subset. Then $f^{-1}E \subseteq \overline{f^{-1}(E)} \subseteq f^{-1}(\overline{E}) = f^{-1}E$ so that $f^{-1}E$ is a closed set in X. Hence f is continuous by the above Exercise. $\square$

Exercise 3.6. Let X, Y be topological spaces and suppose we can write $X = \bigcup_{\alpha \in \Lambda} V_\alpha$ where each V_α is open in X. For each $\alpha \in \Lambda$, let $f_\alpha : V_\alpha \longrightarrow Y$ be a continuous map where V_α is equipped with subspace topology. If $f_\alpha|_{V_\alpha \cap V_\beta} = f_\beta|_{V_\alpha \cap V_\beta}$ holds for every $\alpha, \beta \in \Lambda$, then prove that there is a unique continuous map $f : X \longrightarrow Y$ such that $f|_{V_\alpha} = f_\alpha, \forall \alpha \in \Lambda$.

Lemma 3.7. *Let X, Y be two topological spaces and assume $X = A_1 \bigcup A_2$ for some closed subsets $A_1, A_2 \subseteq X$. Let Y be a topological space and let $f_1 : A_1 \longrightarrow Y$ and $f_2 : A_2 \longrightarrow Y$ be two continuous functions such that $f_1(x) = f_2(x), \forall x \in A_1 \cap A_2$. Then the map*

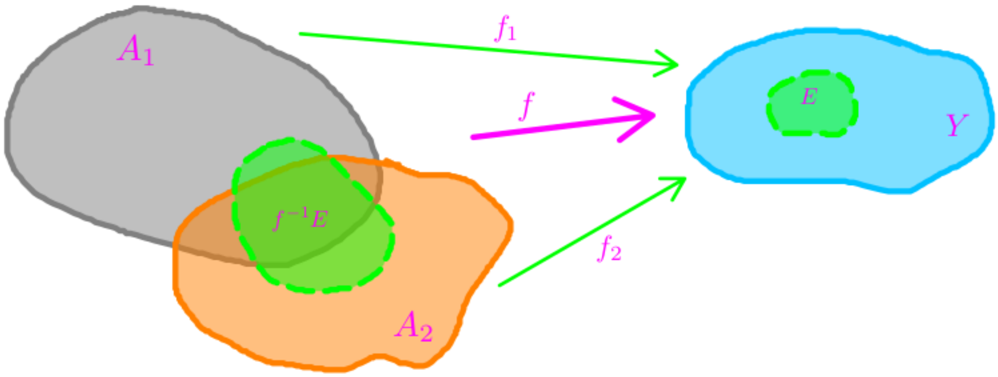
$$f : X \longrightarrow Y, \ \ f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1 \\ f_2(x) & \text{if } x \in A_2 \end{cases}$$

is continuous.

Proof. Let $E \subseteq Y$ be a closed set. We can write

$$f^{-1}E = \left(A_1 \bigcap f^{-1}E\right) \bigcup \left(A_2 \bigcap f^{-1}E\right) = f_1^{-1}E \bigcup f_2^{-1}E$$

Since both f_1, f_2 are continuous and $A_1, A_2 \subseteq X$ are closed, $f_1^{-1}E, f_2^{-1}E$ are closed subsets of X. Thus the set $f^{-1}E$ is closed in X. Hence f is continuous. $\square$



Corollary 3.8. *Let X, Y be two topological spaces and suppose we can write $X = \bigcup_{i=1}^{n} A_i$ where each A_i is a non-empty closed subset of X. For each $i = 1, \cdots, n$, let $f_i : A_i \longrightarrow Y$ be a continuous map satisfying $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}, \forall\, i, j \in \{1, \cdots, n\}$. Define $f : X \longrightarrow Y$ by $f(x) = f_i(x)$, if $x \in A_i$. Then f is continuous.*

Let $X = \{1/n : n \in \mathbb{N}\} \bigcup \{0\} \subseteq \mathbb{R}$ be equipped with subspace topology. Set $A_0 = \{0\}$ and $A_n = \{1/n\}$ for $n \in \mathbb{N}$. Then $X = \bigcup_{n \geq 0} A_n$ and each A_n is a closed subset of X. Define

$$f_0 : A_0 \longrightarrow \mathbb{R},\;\; f_0(0) = 1 \;\text{ and }\; f_n : A_n \longrightarrow \mathbb{R},\;\; f_n(1/n) = 0,\; \forall\, n \in \mathbb{N}.$$

Suppose $f : X \longrightarrow \mathbb{R}$ be a function such that $f|_{A_n} = f_n$, for every integer $n \geq 0$. Since $f^{-1}(1/2, 3/2) = \{0\}$, f is not continuous. Thus Corollary 3.8 can not be extended for infinite collection of closed subsets.

Definition 3.9. Let $f : X \longrightarrow Y$ be a map between two topological spaces X and Y. We say f is

- an *open map* if for every open set $U \subseteq X$, $f(U)$ is open in Y.
- a *closed map* if for every closed set $E \subseteq X$, $f(E)$ is closed in Y.
- a *homeomorphism* if f is a continuous map and there is a continuous map $g : Y \longrightarrow X$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$.

If $f : X \longrightarrow Y$ is a homeomorphism, we say X and Y are *homeomorphic*.

Example 3.10. *(1.)* Let τ, τ_ℓ be the Euclidean topology and lower limit topology on $\mathbb{R}$, respectively. Define $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \tau_\ell)$ by $f(x) = x$. Then f is not continuous. But f is an open map. Since any non-empty open set in $(\mathbb{R}, \tau)$ can be written as a union of countably many disjoint open intervals, it is sufficient to show that the image of every open interval in open in $(\mathbb{R}, \tau_\ell)$ and this is true.

(2.) Define $g : (\mathbb{R}, \tau_\ell) \longrightarrow (\mathbb{R}, \tau)$ by $g(x) = x$ where τ, τ_ℓ are as above. Then g is bijective and continuous. The inverse of g is nothing but the map f above and it is not continuous. Thus a bijective continuous map need be a homeomorphism.

(3.) Define $f : \mathbb{R} \longrightarrow \mathbb{Z}$ by $f(x) = \lfloor x \rfloor$ where $\mathbb{R}$ is equipped with Euclidean topology and $\mathbb{Z}$ is equipped with discrete topology. Then f is not continuous as $f^{-1}\{0\} = [0, 1)$. On the other hand, f is both open and closed map since every subset of $\mathbb{Z}$ is both open and closed.

Exercise 3.11. *(1.)* Let X be a non-empty set and let τ_1, τ_2 be two topologies on X. Let $f : (X, \tau_1) \longrightarrow (X, \tau_2)$ be a function defined by $f(x) = x$. Prove that

- f is continuous if and only if $\tau_2 \subseteq \tau_1$.

- f is an open map if and only if $\tau_1 \subseteq \tau_2$.

(2.) Let $X = (-\infty, 0] \bigcup (1, \infty) \subseteq \mathbb{R}$ be a subspace of the Euclidean space $\mathbb{R}$. Define

$$f : \mathbb{R} \longrightarrow X,\ \ f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$$

Prove that f is both open and closed map, but not continuous.

Proposition 3.12. *Let $f : X \longrightarrow Y$ be a bijective continuous map between two topological spaces X and Y. Then the following conditions are equivalent:*

(i) f is homeomorphism;

(ii) f is an open map;

(iii) f is a closed map.

Proof. Left as an exercise. □

Example 3.13. Let $X = \mathbb{N}$ and $Y = \{1/n : n \in \mathbb{N}\}$ be equipped with the subspace topology of $\mathbb{R}$. Define

$$f : X \longrightarrow Y,\ \ n \mapsto \frac{1}{n}.$$

Since both the topological spaces X and Y are discrete, f is a bijective, continuous open map. Hence f is a homeomorphism.

Exercise 3.14. *(1.)* Let $\mathbb{R}$ be equipped with Euclidean topology and assume the subsets of $\mathbb{R}$ are equipped with the subspace topology induced from that of $\mathbb{R}$.

(i) Prove that any two non-empty bounded open intervals are homeomorphic.

(ii) Prove that any two non-empty bounded closed intervals are homeomorphic.

(iii) Prove that, for any two real numbers $a, b \in \mathbb{R}$, $(-\infty, a]$ and $[b, \infty)$ are homeomorphic.

(iv) Prove that the map $f : (-1, 1) \longrightarrow \mathbb{R}, f(x) = \frac{1}{1-|x|}$ is a homeomorphism.

(2.) Let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a homeomorphism between two topological spaces. Justify the statement: If any of the topology is the discrete topology (resp. indiscrete topology), then the other must be the same.

(3.) Let $f : X \longrightarrow Y$ be a bijective function between two topological spaces. Prove that f is a homeomorphism if and only if both f and f^{-1} are open maps.

Definition 3.15. *(1.)* Let X, Y be two topological spaces and let $f : X \longrightarrow Y$ be an injective continuous map. If $f : X \longrightarrow f(X)$ is a homeomorphism, we say f is a *topological embedding* (or simply, an *embedding*) of X into Y.

(2.) Given two topological spaces X and Y, we say X can be *topologically embedded* (or simply, *embedded*) into Y if there is an embedding $f : X \longrightarrow Y$.

Suppose a topological space X can be embedded into another topological space Y via an embedding $f : X \longrightarrow Y$. Then identifying X with its homeomorphic image $f(X) \subseteq Y$, we may consider X as a subspace of Y.

Exercise 3.16. Set $X = \{1/n : n \in \mathbb{N}\}$ and $Y = \{1/n : n \in \mathbb{Z}, n \neq 0\}$. Check if the following pair of subspaces of $\mathbb{R}$ are homeomorphic:

$$X \text{ and } Y; \quad X \cup \{0\} \text{ and } Y; \quad X \text{ and } Y \cup \{0\}; \quad X \cup \{0\} \text{ and } Y \cup \{0\}.$$

If homeomorphic, write down an explicit homeomorphism.

4. Product Topology

Let $f : X \longrightarrow Y$ be a function between two non-empty sets. Suppose Y is equipped with a topology τ_Y. We want to know what topology we need on X that makes f a continuous function. Such a topology always exists, for example, we can choose the discrete topology on X. Consider the set

$$\Sigma := \{\tau \subseteq \mathscr{P}(X) \ : \ \tau \ \text{ is a topology on } \ X \ \text{ and} \\ f : (X, \tau) \longrightarrow (Y, \tau_Y) \ \text{ is continuous}\}.$$

Set $\tau_X := \bigcap_{\tau \in \Sigma} \tau$. Then τ_X is a topology on X such that $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is continuous. Moreover, if τ is another topology on X such that $f : (X, \tau) \longrightarrow (Y, \tau_Y)$ is continuous, then $\tau_X \subseteq \tau$. In other words, τ is the *unique smallest topology* on X for which f is continuous.

Since f is continuous, $f^{-1}V \in \tau_X$ for every $V \in \tau_Y$. Also finite intersections of these sets must in τ_X. Set

$$\mathscr{B} := \{f^{-1}V_1 \cap \cdots \cap f^{-1}V_n \subseteq X \ : \ V_1, \cdots, V_n \in \tau_Y, n \in \mathbb{N}\}.$$

It's easy to see that $\mathscr{B}$ is a base for some topology, say τ_f, on X. Clearly, $\tau_f \subseteq \tau_X$. On the other hand, $f : (X, \tau_f) \longrightarrow (Y, \tau_Y)$ is continuous. Hence $\tau_X = \tau_f$. In particular, the set $\mathscr{S} = \{f^{-1}V : V \in \tau_Y\}$ is a sub-base for the topology τ_X.

As a concrete example, let (Y, τ_Y) be a topological space, $\emptyset \neq X \subseteq Y$ and consider the natural inclusion map $\iota_X : X \longrightarrow Y$. Applying the above construction we see that the subspace topology τ_X on X (induced from τ_Y) is same as the unique smallest topology on X that makes the natural inclusion ι_X a continuous function.

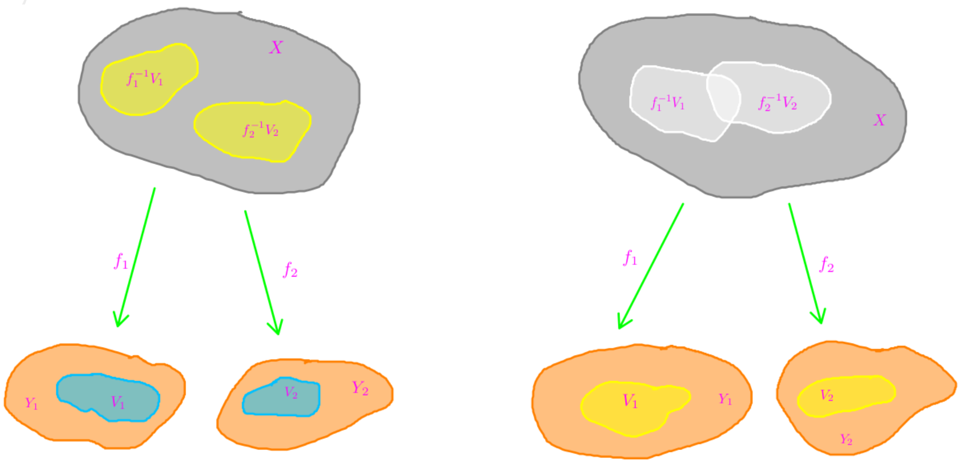
Let X be a non-empty set, $\{Y_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces and for each $\alpha \in \Lambda$, let $f_\alpha : X \longrightarrow Y_\alpha$ be a function. Set

$$\Sigma := \{\tau \subseteq \mathscr{P}(X) \ : \ \tau \ \text{ is a topology on } \ X \ \text{ and} \\ f_\alpha : (X, \tau) \longrightarrow Y_\alpha \ \text{ is continuous for every } \ \alpha \in \Lambda\}.$$

Arguing as above, we see that $\tau_X = \bigcap_{\tau \in \Sigma} \tau \subseteq \mathscr{P}(X)$ is the *unique smallest topology* on X such that $f_\alpha : (X, \tau_X) \longrightarrow Y_\alpha$ is continuous for every $\alpha \in \Lambda$. Moreover, the set

$$\mathscr{S} := \bigcup_{\alpha \in \Lambda} \{f_\alpha^{-1}V \subseteq X \ : \ V \subseteq Y_\alpha \ \text{ is open}\} \subseteq \mathscr{P}(X)$$

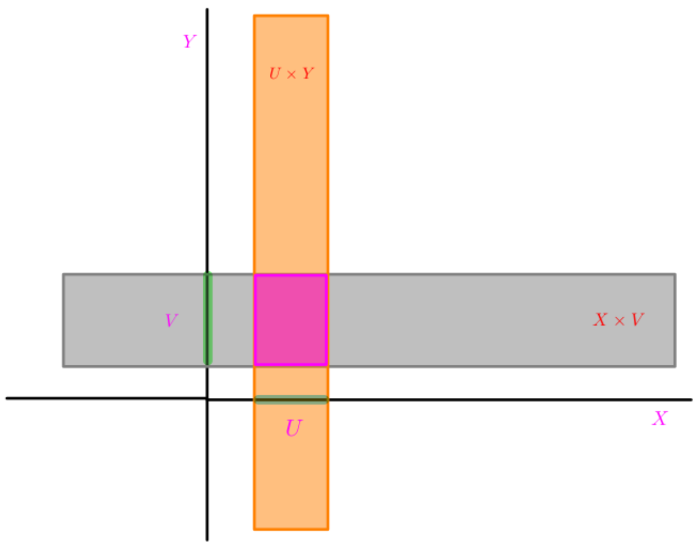
forms a sub-base for the topology τ_X on X.



Example 4.1. (*1.*) Let X, Y be two topological spaces and let $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ be two projections. Let τ be the unique smallest topology on $X \times Y$ such that both the projection maps p_1 and p_2 are continuous. The topology τ is called the *product topology* on $X \times Y$ and the set $X \times Y$ equipped with the product topology is called the *product space* of X and Y. The set

$$\mathscr{S} := \{X \times V \,:\, V \subseteq Y \text{ is open}\} \bigcup \{U \times Y \,:\, U \subseteq X \text{ is open}\}$$

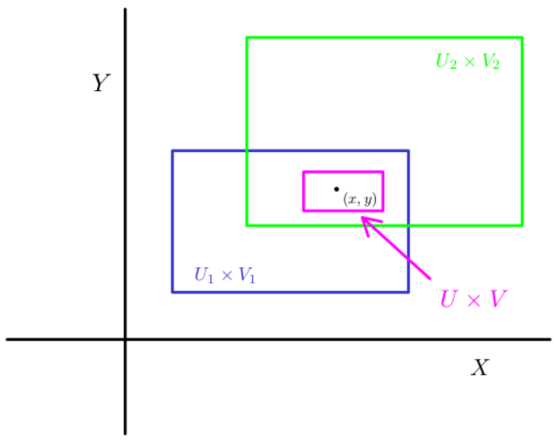
is a sub-base for the product topology on $X \times Y$.



Thus the set

$$\mathscr{B} := \{U \times V \,:\, U \subseteq X \text{ is open and } V \subseteq Y \text{ is open}\}$$

is a base for the product topology $X \times Y$.



(*2.*) Let $X_1, \cdots, X_n$ be topological spaces and let $p_j : \prod_{i=1}^{n} X_i \longrightarrow X_j$ be the j-th coordinate projection. The unique smallest topology on $\prod_{i=1}^{n} X_i$ for which each p_j is continuous is called the *product topology* on $\prod_{i=1}^{n} X_i$ and the set $\prod_{i=1}^{n} X_i$ equipped with the product topology is called the *product space* of $X_i, 1 \leq i \leq n$. The set

$$\mathscr{S} := \{p_j^{-1} V \,:\, V \subseteq X_j \text{ is open}, \;\; 1 \leq j \leq n\}$$

is a subbase for the product topology on $\prod_{i=1}^{n} X_i$ and the set

$$\mathscr{B} := \{V_1 \times \cdots \times V_n \,:\, V_i \subseteq X_i \text{ is open}, \;\; 1 \leq i \leq n\}$$

is a base for the product topology $\prod_{i=1}^{n} X_i$.

(3.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Recall, the Cartesian product of the X_α's is defined by

$$\prod_{\alpha\in\Lambda} X_\alpha = \left\{ x : \Lambda \longrightarrow \bigcup_{\alpha\in\Lambda} X_\alpha \ : \ x_\alpha := x(\alpha) \in X_\alpha, \forall \alpha \in \Lambda \right\}.$$

If all the X_α's are non-empty, then by Axiom of Choice, $\prod_{\alpha\in\Lambda} X_\alpha$ is non-empty. Let $\beta \in \Lambda$ and define $p_\beta : \prod_{\alpha\in\Lambda} X_\alpha \longrightarrow X_\beta$ by $p_\beta(x) = x_\beta$. The map p_β is called the β-th coordinate projection. Let $\alpha_1, \cdots, \alpha_n \in \Lambda$ and for each $k \in \{1, \cdots, n\}$, let V_{α_k} be a subset of X_{α_k}. Then, we can write

$$\begin{aligned} p_{\alpha_k}^{-1}(V_{\alpha_k}) &= \prod_{\alpha\in\Lambda} W_\alpha \quad \text{where} \quad W_\alpha = \begin{cases} V_{\alpha_k} & \text{if } \alpha = \alpha_k \\ X_\alpha & \text{if } \alpha \neq \alpha_k \end{cases} \\ &= \left\{ x \in \prod_{\alpha\in\Lambda} X_\alpha \ : \ x_{\alpha_k} \in V_{\alpha_k} \right\} \end{aligned}$$

and

$$\begin{aligned} \bigcap_{k=1}^{n} p_{\alpha_k}^{-1}(V_{\alpha_k}) &= \prod_{\alpha\in\Lambda} W_\alpha \quad \text{where} \quad W_\alpha = \begin{cases} V_{\alpha_k} & \text{if } \alpha = \alpha_1, \cdots, \alpha_n \\ X_\alpha & \text{otherwise} \end{cases} \\ &= \left\{ x \in \prod_{\alpha\in\Lambda} X_\alpha \ : \ x_{\alpha_k} \in V_{\alpha_k}, \forall k = 1, \cdots, n \right\}. \end{aligned}$$

Let τ be the unique smallest topology on $\prod_{\alpha\in\Lambda} X_\alpha$ such that each p_α is continuous. We call this topology the *product topology* on $\prod_{\alpha\in\Lambda} X_\alpha$ and the set $\prod_{\alpha\in\Lambda} X_\alpha$ equipped with the product topology is called the *product space* of the family $\{X_\alpha : \alpha \in \Lambda\}$. Notice that the set

$$\mathscr{S} := \bigcup_{\alpha\in\Lambda} \{p_\alpha^{-1} V : V \subseteq X_\alpha \text{ is open}\}$$

is a sub-base for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ and the set

$$\mathscr{B} := \left\{ \bigcap_{k=1}^{n} p_{\alpha_k}^{-1}(V_{\alpha_k}) \ : \ \text{each } V_{\alpha_k} \subseteq X_{\alpha_k} \text{ is open and } \{\alpha_1, \cdots, \alpha_n\} \subseteq \Lambda, n \in \mathbb{N} \right\}$$

is a base for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

Exercise 4.2. Prove that the product topology on $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n-times) is same as the Euclidean topology on $\mathbb{R}^n$.

Notation: Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Unless explicitly specified otherwise, we will assume that the Cartesian product $\prod_{\alpha\in\Lambda} X_\alpha$ is equipped with the product topology as defined above. Moreover, the sub-base $\mathscr{S}$ and the base $\mathscr{B}$ for the product topology as defined above will be called *canonical sub-base* and *canonical base* for the product topology, respectively.

Proposition 4.3. *Let X be a non-empty set, $\{Y_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces and for each $\alpha \in \Lambda$, let $f_\alpha : X \longrightarrow Y_\alpha$ be a function. Let τ_X be the unique smallest topology on X such that $f_\alpha : X \longrightarrow Y_\alpha$ is continuous for every $\alpha \in \Lambda$. Let Z be a topological space and let $g : Z \longrightarrow X$ be a function. Then g is continuous if and only if $f_\alpha \circ g : Z \longrightarrow Y_\alpha$ is continuous for every $\alpha \in \Lambda$.*

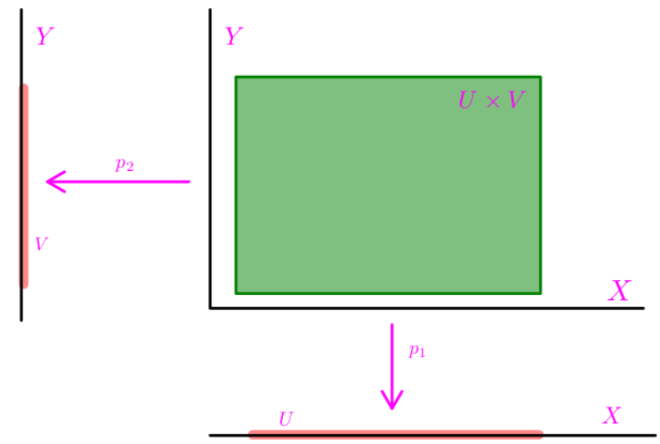
$$
\begin{tikzcd}
 & & Y_\alpha \\
Z \arrow[r, "g"] \arrow[urr, bend left, "f_\alpha \circ g"] \arrow[drr, bend right, "f_\beta \circ g"'] & X \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & \\
 & & Y_\beta
\end{tikzcd}
$$

Proof. If g is continuous, then so is $f_\alpha \circ g, \forall \alpha \in \Lambda$. Conversely, assume that $f_\alpha \circ g$ is continuous for every $\alpha \in \Lambda$. It is sufficient to show that $g^{-1}(S) \subseteq Z$ is open for every $S \in \mathscr{S} = \bigcup_{\alpha\in\Lambda} \{f_\alpha^{-1} V \subseteq X : V \subseteq Y_\alpha \text{ is open}\}$. Let $\alpha \in \Lambda$ and let $V \subseteq Y_\alpha$ be an open set. By the given condition, $g^{-1}(f_\alpha^{-1} V) = (f_\alpha \circ g)^{-1}(V)$ is open in Z. Hence g is continuous. $\square$

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of non-empty sets, Z be a non-empty set and for each $\alpha \in \Lambda$, let $f_\alpha : Z \longrightarrow X_\alpha$ be a function. Then there is a induced function $f : Z \longrightarrow \prod_{\alpha\in\Lambda} X_\alpha$ such that $p_\alpha \circ f = f_\alpha$, for each projection map $p_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \longrightarrow X_\alpha$. The map f is defined as follows: for each $a \in Z$, $f(a) : \Lambda \longrightarrow \bigcup_{\alpha\in\Lambda} X_\alpha$ is a function defined by $f(a)(\alpha) = f_\alpha(a)$.

Corollary 4.4. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces, Z be a topological space and for each $\alpha \in \Lambda$, let $f_\alpha : Z \longrightarrow X_\alpha$ be a function. Let $f : Z \longrightarrow \prod_{\alpha\in\Lambda} X_\alpha$ be the induced function to the product space. Then f is continuous if and only if each f_α is continuous.*

Lemma 4.5. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Then for any $\beta \in \Lambda$, the β-th coordinate projection $p_\beta : \prod_{\alpha\in\Lambda} X_\alpha \longrightarrow X_\beta$ is an open map.*



Proof. Since the image of an arbitrary union of sets is the union of their images, it is sufficient to prove that image of every canonical basic open set is open. Let $V = \prod_{\alpha\in\Lambda} V_\alpha$ be a canonical basic open set. Then $p_\beta(V) = V_\beta \subseteq X_\beta$ is open. $\square$

Exercise 4.6. Give examples of topological spaces X and Y such that the projection map $X \times Y \longrightarrow X$ is not a closed map.

Corollary 4.7. *Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be two collection of topological spaces, index by the same set Λ. For each $\alpha \in \Lambda$, let $f_\alpha : Y_\alpha \longrightarrow X_\alpha$ be a function and let $\prod_{\alpha\in\Lambda} f_\alpha : \prod_{\alpha\in\Lambda} Y_\alpha \longrightarrow \prod_{\alpha\in\Lambda} X_\alpha$ be the induced function. Then $\prod_{\alpha\in\Lambda} f_\alpha$ is continuous if and only if each f_α is continuous.*

Proof. For $\beta \in \Lambda$, let's denote the β-th coordinate projections by $p_\beta : \prod_{\alpha\in\Lambda} X_\alpha \longrightarrow X_\beta$ and $q_\beta : \prod_{\alpha\in\Lambda} Y_\alpha \longrightarrow Y_\beta$. We have a commutative diagram:

$$\begin{tikzcd}
\prod_{\alpha\in\Lambda} Y_\alpha \arrow[r, "\prod_{\alpha\in\Lambda} f_\alpha"] \arrow[d, "q_\beta"'] & \prod_{\alpha\in\Lambda} X_\alpha \arrow[d, "p_\beta"] \\
Y_\beta \arrow[r, "f_\beta"] & X_\beta
\end{tikzcd}$$

Suppose each f_α is continuous. Let $y \in \prod_{\alpha\in\Lambda} Y_\alpha$. Then for each $\beta \in \Lambda$, $f_\beta(q_\beta(y)) \in X_\beta$. Since $f_\beta \circ q_\beta : \prod_{\alpha\in\Lambda} Y_\alpha \longrightarrow X_\beta$ is continuous for every $\beta \in \Lambda$, by Corollary 4.4, $\prod_{\alpha\in\Lambda} f_\alpha$ is continuous.

Now assume that $\prod_{\alpha\in\Lambda} f_\alpha$ is continuous and let $V \subseteq X_\beta$ be an open set. Then

$$q_\beta^{-1} f_\beta^{-1} V = \Big(\prod_{\alpha\in\Lambda} f_\alpha\Big)^{-1} p_\beta^{-1} V \subseteq \prod_{\alpha\in\Lambda} Y_\alpha$$

is open. By Lemma 4.5, $f_\beta^{-1} V = q_\beta(q_\beta^{-1} f_\beta^{-1} V) \subseteq Y_\beta$ is open. Hence f_β is continuous. $\square$

Exercise 4.8. *(1.)* Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces and let $\phi : \Lambda \longrightarrow \Lambda$ be a bijection. Prove that the product spaces $\prod_{\alpha\in\Lambda} X_\alpha$ and $\prod_{\alpha\in\Lambda} X_{\phi(\alpha)}$ are homeomorphic.

(2.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces and let $\Lambda = \bigcup_{t\in\Omega} \Lambda_t$ be a partition of Λ. Prove that the product spaces $\prod_{\alpha\in\Lambda} X_\alpha$ and $\prod_{t\in\Omega}\left(\prod_{\alpha\in\Lambda_t} X_\alpha\right)$ are homeomorphic (associativity of product spaces).

(3.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces and for each $\alpha \in \Lambda$, let Y_α be a subspace of X_α. Prove that the product topology on $\prod_{\alpha\in\Lambda} Y_\alpha$ is same as the subspace topology on $\prod_{\alpha\in\Lambda} Y_\alpha$ induced by the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

(4.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of non-empty topological spaces. Fix $\beta \in \Lambda$ and for each $\alpha \in \Lambda, \alpha \neq \beta$, choose $x_\alpha \in X_\alpha$. Set $A := \prod_{\alpha\in\Lambda} A_\alpha \subseteq \prod_{\alpha\in\Lambda} X_\alpha$ where $A_\beta = X_\beta$ and for for each $\alpha \in \Lambda, \alpha \neq \beta$, $A_\alpha = \{x_\alpha\}$. Prove that A (with subspace topology induced from the product space) is homeomorphic to X_β.

(5.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. For each $\alpha \in \Lambda$, let $\mathscr{B}_\alpha$ be a base for the topology on X_α. Prove that the set $\mathscr{S} = \{p_\alpha^{-1} B : B \in \mathscr{B}_\alpha, \alpha \in \Lambda\}$ is a sub-base for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

Lemma 4.9. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces and for every $\alpha \in \Lambda$, let $A_\alpha \subseteq X_\alpha$. Then*

$$\prod_{\alpha\in\Lambda} \overline{A_\alpha} = \overline{\prod_{\alpha\in\Lambda} A_\alpha}$$

In particular, product of closed subsets of X_α is closed in the product space.

Proof. Let $x \in \prod_{\alpha\in\Lambda} \overline{A_\alpha}$. Then for each $\alpha \in \Lambda, x_\alpha \in \overline{A_\alpha}$. Fix $\beta \in \Lambda$. Let $V_\beta \subseteq X_\beta$ be a neighbourhood of x_β and define $N_x = \prod_{\alpha\in\Lambda} W_\alpha$ where $W_\beta = V_\beta$ and $W_\alpha = X_\alpha$ for $\alpha \neq \beta$. Then N_x is a neighbourhood of x. Since $V_\beta \bigcap A_\beta \neq \emptyset$, $N_x \bigcap \left(\prod_{\alpha\in\Lambda} A_\alpha\right) \neq \emptyset$. Thus x is in the closure of $\prod_{\alpha\in\Lambda} A_\alpha$.

Now let $y \in \overline{\prod_{\alpha\in\Lambda} A_\alpha}$ and let $p_\beta^{-1}(V_\beta)$ be a canonical basic open open set containing y. Then $\left(\prod_{\alpha\in\Lambda} A_\alpha\right) \bigcap p_\beta^{-1}(V_\beta) \neq \emptyset$. In particular, we have $A_\beta \bigcap V_\beta \neq \emptyset$. Fixing $\beta \in \Lambda$ and varying the the open set $V_\beta \subseteq X_\beta$ subject to the condition $y_\beta \in V_\beta$, we see that A_β intersects every neighbourhood of y_β. Hence $y_\beta \in \overline{A_\beta}$. Now varying $\beta \in \Lambda$, we get $y \in \prod_{\alpha\in\Lambda} \overline{A_\alpha}$. $\square$

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces and for each $\alpha \in \Lambda$, let $\emptyset \neq V_\alpha \subsetneq X_\alpha$ be an open set. Set $V := \prod_{\alpha\in\Lambda} V_\alpha$ and assume that Λ is an infinite set. Let B be a canonical basic open subset of the product space $\prod_{\alpha\in\Lambda} X_\alpha$. We can write $B = \prod_{\alpha\in\Lambda} W_\alpha$ where $W_\alpha = X_\alpha$ for all but finitely many $\alpha \in \Lambda$. Thus $B \not\subset V$. Hence V is not an open set. So the product of open subsets of X_α may not be open in the product space.

Let Λ be an infinite set. For each $\alpha \in \Lambda$, let X_α be the discrete topological space $\{0,1\}$. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be the product space. Set $V = \prod_{\alpha\in\Lambda} V_\alpha$ where $V_\alpha = \{0\}, \forall \alpha \in \Lambda$. By above, V is not an open set in the product space. Thus, infinite product of discrete spaces is not a discrete space.

Corollary 4.10. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces and for every $\alpha \in \Lambda$, let $D_\alpha \subseteq X_\alpha$ be a dense set. Then $\prod_{\alpha\in\Lambda} D_\alpha$ is dense in the product space $\prod_{\alpha\in\Lambda} X_\alpha$.*

Exercise 4.11. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of non-empty topological spaces and let $x_0 \in \prod_{\alpha\in\Lambda} X_\alpha$. Consider the set

$$D := \left\{ x \in \prod_{\alpha\in\Lambda} X_\alpha \;:\; \{\alpha \in \Lambda : x(\alpha) \neq x_0(\alpha)\} \text{ is a finite set} \right\}$$

Prove that D is dense in $\prod_{\alpha\in\Lambda} X_\alpha$.

Definition 4.12. A topological space X is said to be a *T_2-space* or a *Hausdorff space* if for each pair of distinct points $x, y \in X$, there is a neighbourhood V_x of x and a neighbourhood V_y of y such that $V_x \bigcap V_y = \emptyset$.

Exercise 4.13. Let (X, τ) be a non-empty topological space and let $\mathscr{B}$ be a base for the topology τ. Then the following conditions are equivalent:

- X is a Hausdorff space;
- given any two distinct points $x, y \in X$, there are basic open sets $B_x, B_y \in \mathscr{B}$ such that $x \in B_x, y \in B_y$ and $B_x \bigcap B_y = \emptyset$.

Example 4.14. *(1.)* Let (X, d) be a metric space and let τ be the topology on X generated by the metric d. Then (X, τ) is a Hausdorff space. In particular, the Euclidean space $\mathbb{R}^n$ is a Hausdorff space.

(2.) Let X be an infinite set and $a \in X$. Define a topology on X as follows:

$$\tau_a := \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subset X : a \in V \text{ and } X \setminus V \text{ is finite}\}.$$

Let $x, y \in X$ with $x \neq y$. We have to consider two cases:

- Suppose $x, y \in X \setminus \{a\}$. Then we can choose $V_x = \{x\}$ and $V_y = \{y\}$.
- Suppose $x = a$. In this case, we can choose $V_x = X \setminus \{y\}$ and $V_y = \{y\}$.

Thus (X, τ_a) is a topological space.

(3.) Let X be an infinite (resp. uncountable) set and let τ be the co-finite (resp. co-countable) topology on X. Let $x, y \in X$ be such that $x \neq y$. Let $V_x \subseteq X$ and $V_y \subseteq X$ be two open sets containing x and y, respectively. Since X is infinite (resp. uncountable), $V_x \bigcap V_y \neq \emptyset$. Thus (X, τ) is not a Hausdorff topological space.

(4.) Let $\mathbb{R}_\ell$ be the lower limit topological space and let $x, y \in \mathbb{R}_\ell$ with $x < y$. Choose real numbers a, b, c, d such that $a < x < b < c < y < d$. Set $V_x = [a, b)$ and $V_y = [c, d)$. Then $x \in V_x, y \in V_y$ and $V_x \bigcap V_y = \emptyset$. Thus $\mathbb{R}_\ell$ is a Hausdorff space.

(5.) Let $X = \{x \in \mathbb{R} : x > 0\} \bigcup \{\infty\} \bigcup \{-\infty\}$ and let $\mathscr{B} \subseteq \mathscr{P}(X)$ be the collection consists of intervals (a, b) with $a, b \in \mathbb{R}, 0 < a < b$ together with the sets of the form $(0, s) \cup \{\infty\}, s \in \mathbb{R}, s > 0$ and $(0, t) \cup \{-\infty\}, t \in \mathbb{R}, t > 0$. Let τ be the topology on X generated by $\mathscr{B}$. Notice that, given any $B, B' \in \mathscr{B}$ with $\infty \in B$ and $-\infty \in B'$, we have $(0, a) \subseteq B \bigcap B'$, for some real $a > 0$. Thus (X, τ) is not a Hausdorff space.

Lemma 4.15. *Let X be a non-empty Hausdorff topological space. Then*

(i) any non-empty subspace of X is Hausdorff;

(ii) any finite subset of X is closed in X.

Proof. (i) Let $\emptyset \neq Y \subseteq X$ be a subspace and let $y_1, y_2 \in Y$ be two distinct points. Then there are open sets V_1, V_2 in X such that $y_1 \in V_1, y_2 \in V_2$ and $V_1 \bigcap V_2 = \emptyset$. The sets $V_1 \bigcap Y$ and $V_2 \bigcap Y$ are non-empty open subsets of Y. Hence Y is Hausdorff.

(ii) It's sufficient to show that any single point subset of X is closed in X. Let $a \in X$. Then for any $x \in X$ with $x \neq a$, there is an open set V_x in X such that $a \in X \setminus V_x$. Hence $\{a\} = \bigcap_{x \in X \setminus \{x\}} (X \setminus V_x)$ is closed in X. □

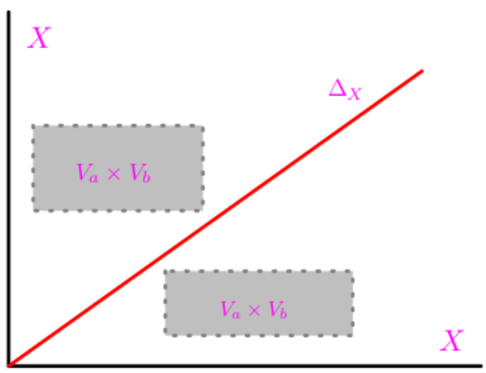
Proposition 4.16. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. Then the product space $X = \prod_{\alpha \in \Lambda} X_\alpha$ is Hausdorff if and only if each X_α is Hausdorff.*

Proof. Suppose each X_α is Hausdorff. Let $a, b \in X$ be two distinct points. Then for some $\beta \in \Lambda$, $a_\beta \neq b_\beta$ in X_β. Since X_β is Hausdorff, there are open sets V_1, V_2 in X_β such that $a_\beta \in V_1$, $b_\beta \in V_2$ and $V_1 \bigcap V_2 = \emptyset$. Set $V_a = p_\beta^{-1} V_1$ and $V_b = p_\beta^{-1} V_2$. Then $a \in V_a$, $b \in V_b$ are open sets in X and $V_a \bigcap V_b = \emptyset$. Hence X is Hausdorff.

Assume the product space X is Hausdorff. Fix $\beta \in \Lambda$ and let $x, y \in X_\beta$ be two distinct points. Let $a, b \in X$ be such that $a_\beta = x, b_\beta = y_\beta$ and $a_\alpha = b_\alpha$ for every $\alpha \neq \beta$. Then $a \neq b$. Since X is Hausdorff, there are open sets V_a, V_b in X such that $a \in V_a, b \in V_b$ and $V_a \bigcap V_b = \emptyset$. By our choice of a and b, the open subsets $p_\beta(V_a) \subseteq X_\beta$ and $p_\beta(V_b) \subseteq X_\beta$ are disjoint. Hence X_β is Hausdorff. □

Proposition 4.17. *A non-empty topological space X is Hausdorff if and only if the diagonal $\Delta_X := \{(x, x) : x \in X\}$ is closed in the product space $X \times X$.*

Proof. Let X be a Hausdorff space and let $(a, b) \in X \times X$ with $a \neq b$. Choose neighbourhoods V_a of a and V_b of b such that $V_a \bigcap V_b = \emptyset$. Then the open set $V_a \times V_b \subseteq X \times X$ is disjoint from Δ_X. Since $(a, b) \in (X \times X) \setminus \Delta_X$ is arbitrary, $(X \times X) \setminus \Delta_X$ is an open subset of $X \times X$.



Conversely assume that Δ_X is closed in $X \times X$. Let $a, b \in X$ with $a \neq b$. Then there is a canonical basic open set $U \times V \subseteq X \times X$ such that $(a, b) \in U \times V$ and $(U \times V) \bigcap \Delta_X = \emptyset$. Hence $U \bigcap V = \emptyset$. $\square$

Corollary 4.18. *Let $f, g : X \longrightarrow Y$ be two continuous maps between two topological spaces and assume Y is a Hausdorff space. Then*

(i) the set $\{x \in X : f(x) = g(x)\}$ is closed in X;

(ii) if D is a dense subset of X and $f|_D = g|_D$, then $f = g$;

(iii) the graph $\Gamma_f = \{(x, f(x)) : x \in X\}$ of f is closed in $X \times Y$.

Proof. (i) Define $\phi : X \longrightarrow Y \times Y$ by $\phi(x) = \big(f(x), g(x)\big)$. Then, by Corollary 4.4, ϕ is a continuous map. Thus $\{x \in X : f(x) = g(x)\} = \phi^{-1}\Delta_Y$ is closed by Proposition 4.17.

(ii) By (i) we have $X = \overline{D} \subseteq \{x \in X : f(x) = g(x)\}$. Hence $f = g$.

(iii) Define $\psi : X \times Y \longrightarrow Y \times Y$ by $\psi(x, y) = \big(f(x), y\big)$. Then, by Corollary 4.4, ψ is continuous and $\psi^{-1}\Delta_Y = \Gamma_f$. Hence Γ_f is closed in $X \times Y$ by Proposition 4.17. $\square$

Exercise 4.19. *(1.)* Let $f : X \longrightarrow Y$ be a continuous open surjective map between two topological spaces X and Y. Prove that the set $\{(x_1, x_2) : f(x_1) = f(x_2)\} \subseteq X \times X$ is closed in $X \times X$ if and only if Y is a Hausdorff space.

(2.) Let $f : X \longrightarrow Y$ be an open continuous surjective map between topological spaces X and Y and assume that X is a Hausdorff space. Give an example to show that Y need not be a Hausdorff space.

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces and consider the set

$$\mathscr{U} := \Big\{ \prod_{\alpha \in \Lambda} V_\alpha \ : \ V_\alpha \subseteq X_\alpha \ \text{is open, for every} \ \alpha \in \Lambda \Big\}.$$

The $\mathscr{B}$ is a base for some topology on the Cartesian product $\prod_{\alpha \in \Lambda} X_\alpha$. We call this topology the *box topology*. Every canonical basic open set in the product topology is an element of $\mathscr{U}$. Thus the product topology is smaller than the box topology on the Cartesian product $\prod_{\alpha \in \Lambda} X_\alpha$. If Λ is a finite set, then the topologies are same. When Λ is infinite, product topology is strictly smaller than the box topology.

Example 4.20. For each $n \in \mathbb{N}$, let X_n be the Euclidean space $\mathbb{R}$ and set

$$\mathbb{R}^\omega := \prod_{n \in \mathbb{N}} X_n$$

the countable product of $\mathbb{R}$ and let's equip $\mathbb{R}^\omega$ with box topology. Define

$$f : \mathbb{R} \longrightarrow \mathbb{R}^\omega, \quad f(x) = (x, x, x, \cdots)$$

where $\mathbb{R}$ is equipped with Euclidean topology. Consider the open subset $V \subseteq \mathbb{R}^\omega$ given by

$$V = (-1, 1) \times \Big(-\frac{1}{2}, \frac{1}{2}\Big) \times \Big(-\frac{1}{3}, \frac{1}{3}\Big) \times \cdots$$

Let $\varepsilon > 0$ be a real number. Choose $m \in \mathbb{N}$ such that $(-1/m, 1/m) \subsetneq (-\varepsilon, \varepsilon)$. Hence, there is no neighbourhood N_0 of $0 \in f^{-1}V$ such that $f(N_0) \subseteq V$. In particular, f is not continuous. If $p_n : \mathbb{R}^\omega \longrightarrow \mathbb{R}$ be the n-th coordinate projection, then $p_n \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is the identity map and hence continuous. Thus the box topology violets the conclusion of Corollary 4.4.

Exercise 4.21. *(1.)* Let $f : X \longrightarrow Y$ be a continuous map between two topological spaces X and Y. Let $\Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$ be the graph of f. Prove that X is homeomorphic to Γ_f, where Γ_f is equipped with subspace topology of $X \times Y$.

(2.) Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be two family of non-empty topological spaces. Suppose for each $\alpha \in \Lambda$, there is an open map $f_\alpha : X_\alpha \longrightarrow Y_\alpha$. If all but finitely many f_α's are surjective, prove that the induced map $\prod_{\alpha\in\Lambda} f_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \longrightarrow \prod_{\alpha\in\Lambda} Y_\alpha$ is an open map.

(3.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. For each $\alpha \in \Lambda$, let A_α be a subset of X_α. If $\prod_{\alpha\in\Lambda} A_\alpha$ is dense in $\prod_{\alpha\in\Lambda} X_\alpha$, then prove that A_α is dense in X_α, for each $\alpha \in \Lambda$.

(4.) Let X, Y, Z be three non-empty topological spaces. Check if the following statement is true or not: If $X \times Y$ is homeomorphic to $X \times Z$, then Y is homeomorphic to Z.

"If a city is compact, it can be guarded by a finite number of arbitrarily short-sighted policemen." – H. Weyl

Definition 5.1. Let X be a topological space.

- A *cover* of X is a collection $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ of subsets of X such that $X = \bigcup_{\alpha\in\Lambda} V_\alpha$.
- Let $\mathscr{U}$ be a cover of X. A subset $\mathscr{A}$ of $\mathscr{U}$ is called a *subcover* of X if $X = \bigcup_{A\in\mathscr{A}} A$.
- A cover $\mathscr{U}$ is called an *open cover* of X if each members of $\mathscr{U}$ are open subsets of X.
- We say X is *compact* if every open cover of X has a finite subcover, i.e. given any open cover $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ of X, there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $X = \bigcup_{i=1}^n V_{\alpha_i}$.

Example 5.2. *(1.)* Let X be an infinite set and let τ be the co-finite topology on X. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. We may further assume that $V_\alpha \neq \emptyset, \forall \alpha \in \Lambda$. Fix $\beta \in \Lambda$ and write $X \setminus V_\beta = \{x_1, \cdots, x_n\}$. For each $i = 1, \cdots, n$, choose $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $x_i \in V_{\alpha_i}$. Then $X = V_\beta \bigcup V_{\alpha_1} \bigcup \cdots \bigcup V_{\alpha_n}$. Hence X is compact.

(2.) Let X be an infinite set and let τ be the discrete topology on X. Then X is not compact as the open cover $\mathscr{U} = \{\{x\} : x \in X\}$ has no finite subcover.

(3.) The Euclidean space $\mathbb{R}$ is not compact: the open cover $\{(-n, n) : n \in \mathbb{N}\}$ has no finite subcover. The lower limit topological space $\mathbb{R}_\ell$ is not compact: the open cover $\{[-n, n) : n \in \mathbb{N}\}$ has no finite subcover.

(4.) Let X be an infinite set, $a \in X$ and define a topology on X by

$$\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subseteq X : a \in V \text{ and } X \setminus V \text{ is finite}\}.$$

Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Choose $\beta \in \Lambda$ such that $a \in V_\beta$. Since $X \setminus V_\beta$ is a finite set, arguing as in the case of co-finite topology, we see that (X, τ) is compact.

Exercise 5.3. *(1.)* Let X be an uncountable set equipped with the co-countable topology on X. Is X compact?

(2.) On $X = \mathbb{R}$, consider the two topologies:

$$\tau := \{V \subseteq X : V = X \text{ or } 0 \notin V\} \text{ and } \tau' := \{V \subseteq X : V = \emptyset \text{ or } 0 \in V\}.$$

Check if the topological spaces (X, τ) and (X, τ') are compact.

(3.) Let X be a non-empty topological space and let $\mathscr{B}$ be a base for the topology on X. Prove that X is compact if and only if every open cover of X consisting of elements of $\mathscr{B}$ has a finite subcover.

Let X be a topological space and let $Y \subseteq X$. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be a family of open sets of X. If $Y \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$, we say $\mathscr{U}$ is a *cover of Y by open subsets of X*. We say *Y is compact in X* or *Y is a compact subset of X* if every cover of Y by open subsets of X has a finite subcover.

Lemma 5.4. *Let X be a topological space and let $Y \subseteq X$. Then Y is compact in X if and only if Y is a compact topological space as a subspace of X.*

Proof. Let Y be compact in X and let $\mathscr{A} = \{A_\alpha : \alpha \in \Lambda\}$ be an open cover of Y in the subspace topology. Then for each $\alpha \in \Lambda$, there is an open set V_α in X such that $A_\alpha = V_\alpha \bigcap Y$. The family $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ is a cover of Y by open subsets of X. Thus there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $Y \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ and hence $Y = \bigcup_{i=1}^n (V_{\alpha_i} \bigcap Y) = \bigcup_{i=1}^n A_{\alpha_i}$, i.e. the open cover $\mathscr{A}$ has a finite subcover.

Conversely assume Y is a compact topological space as a subspace of X. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of Y by open subsets of X. Write $A_\alpha = V_\alpha \bigcap Y, \alpha \in \Lambda$. Then $\mathscr{A} = \{A_\alpha : \alpha \in \Lambda\}$ is an open cover of Y in subspace topology. By the given condition, there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $Y \subseteq \bigcup_{i=1}^n A_{\alpha_i} \subseteq \bigcup_{i=1}^n V_{\alpha_i}$. Hence Y is compact in X. $\square$

Corollary 5.5. *Let X be a topological space and let $Z \subseteq Y \subseteq X$ be equipped with subspace topology. Then Z is compact in Y if and only if Z is compact in X.*

In view of the above Lemma, we will use the terms "Y is compact in X, "Y is a compact subset of X" and "Y is a compact subspace of X" interchangeably. When there is no ambiguity about the topology on X, we will also say that "Y is compact", omitting X. But it should be understood that the topology on Y is induced from that of X.

Example 5.6. *(1.)* Let $X = \{1/n : n \in \mathbb{N}\}$ be equipped with the subspace of topology of the Euclidean space $\mathbb{R}$. Let $\mathscr{U} = \{V_n : n \in \mathbb{N}\}$ be a cover of X where for each $n \in \mathbb{N}$,

$$V_n := \left(\frac{1}{n} - \delta_n, \frac{1}{n} + \delta_n\right) \text{ and } \delta_n := \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Then $V_n \bigcap V_m = \emptyset$, for every $n, m \in \mathbb{N}$ with $n \neq m$. Thus $\mathscr{U}$ has no finite subcover and hence X is not a compact subset of the Euclidean space $\mathbb{R}$.

(2.) Let $X = \{1/n : n \in \mathbb{N}\} \bigcup \{0\}$ be equipped with the subspace topology of the Euclidean space $\mathbb{R}$. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be a cover of X by open subsets of $\mathbb{R}$. Let $\beta \in \Lambda$ be such that $0 \in V_\beta$. By Archimedean property, there is $m \in \mathbb{N}$ such that $1/n \in V_\beta, \forall n \geq m+1$. Choose $V_{\alpha_r} \in \mathscr{U}, 1 \leq r \leq m$, such that $1/r \in V_{\alpha_r}$. Then $X \subseteq V_\beta \bigcup \left(\bigcup_{r=1}^m V_{\alpha_r}\right)$. Hence X is a compact subset of the Euclidean space $\mathbb{R}$.

(3.) Let τ_K be the K-topology on $\mathbb{R}$ and let $Y = [0,1]$. For each $n \in \mathbb{N}$, choose $t_n \in \mathbb{R}$ satisfying $\frac{1}{n+1} < t_{n+1} < \frac{1}{n}$ and $t_1 = 3/2$. Then

$$\mathscr{U} := \{(t_{n+1}, t_n) : n \in \mathbb{N}\} \bigcup \{(-2, 2) \setminus K\}$$

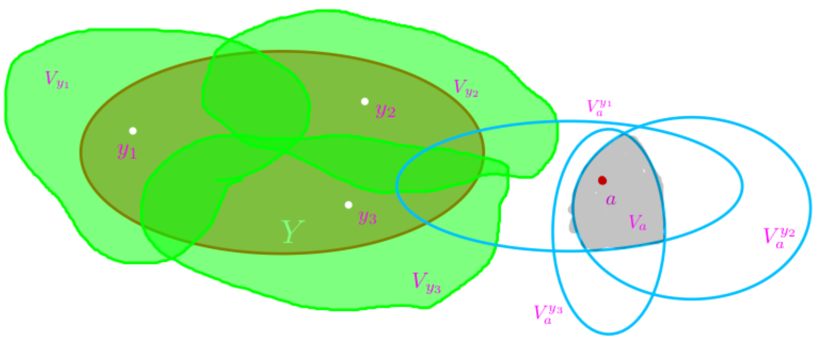
is an open cover of Y having no finite subcover. Hence Y is not a compact in $(\mathbb{R}, \tau_K)$.

Lemma 5.7. *Let X be a topological space and let $Y \subseteq X$.*

(i) If X is compact and Y is closed, then Y is compact in X.

(ii) If X is Hausdorff and Y is compact, then Y is a closed subset of X.

Proof. (i) Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be a cover of Y by open subsets of X. Then $\mathscr{U} \bigcup \{X \setminus Y\}$ is an open cover of X. Since X is compact, there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $X = \left(\bigcup_{i=1}^n V_{\alpha_i}\right) \bigcup (X \setminus Y)$. Hence $Y \subseteq \bigcup_{i=1}^n V_{\alpha_i}$.



(ii) Let $a \in X \setminus Y$. Then for every $y \in Y$. Then there are open sets V_y, V_a^y in X such that $y \in V_y$, $a \in V_a^y$ and $V_y \bigcap V_a^y = \emptyset$. The family $\{V_y : y \in Y\}$ is a cover Y by open subsets of X. Since Y is compact, there are $y_1, \cdots, y_n \in Y$ such that $Y \subseteq V_Y := \bigcup_{i=1}^n V_{y_i}$. Set $V_a = \bigcap_{i=1} V_a^{y_i}$. Then $a \in V_a$ is an open set in X and $V_a \bigcap V_Y = \emptyset$. Thus a is not a limit point of Y. Hence Y is closed. $\square$

Corollary 5.8. *Let X be a compact Hausdorff topological space, $Y \subsetneq X$ be a closed set and $a \in X \setminus Y$. Then there are open sets V_Y, V_a in X such that $Y \subseteq V_Y, a \in V_a$ and $V_Y \bigcap V_a = \emptyset$.*

An arbitrary subspace of a compact topological space need not be compact. For example, let X be an infinite set, $a \in X$ and let the topology on X is given by

$$\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subseteq X : a \in V \text{ and } X \setminus V \text{ is finite}\}.$$

Then (X, τ_a) is compact. Now let $Y = X \setminus \{a\}$. Since X is infinite and the subspace topology on Y is the discrete topology, Y is not compact.

Let X be a compact topological space and let $Y \subseteq X$ be a compact subset. If X is not Hausdorff, Y need not be closed. For example, let X be an uncountable set equipped with co-finite topology. Then X is compact and any subset of X is compact. But no proper infinite subset of X is closed.

Lemma 5.9. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces. If $Z \subseteq X$ is compact in X, then $f(Z)$ is compact in Y.*

Proof. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f(Z)$ by open subsets of Y. Set $f^{-1}\mathscr{U} := \{f^{-1}V_\alpha : \alpha \in \Lambda\}$. Then $f^{-1}\mathscr{U}$ is a cover of Z by open subsets of X. Since Z is compact in X, there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $Z \subseteq \bigcup_{i=1}^n f^{-1}V_{\alpha_i}$. Then $f(Z) \subseteq \bigcup_{i=1}^n V_{\alpha_i}$. Thus $f(Z)$ is compact in Y. $\square$

Corollary 5.10. *Let $f : X \longrightarrow Y$ be a bijective continuous map between topological spaces. Assume X is compact and Y is Hausdorff. Then f is a homeomorphism.*

Proof. It is sufficient to show that f is a closed map. Let $Z \subseteq X$ be a closed subset. Since X is compact, by Lemma 5.7 and Lemma 5.9, $f(Z)$ is compact in Y. Since Y is Hausdorff, by Lemma 5.7, $f(Z)$ is closed in Y. $\square$

Exercise 5.11. Prove that the two subspaces

$$X := \{1/n : n \in \mathbb{N}\} \bigcup \{0\} \quad \text{and} \quad Y := \{1/m : m \in \mathbb{Z}, m \neq 0\} \bigcup \{0\}$$

of the Euclidean space $\mathbb{R}$ are homeomorphic.

Let $\mathscr{A}$ be a non-empty collection of non-empty sets. We say $\mathscr{A}$ has *finite intersection property* (FIP, in short) if given any finite subfamily $\{A_1, \cdots, A_n\}$ of $\mathscr{A}$, we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

Exercise 5.12. In each of the following cases, check if the given collection of subsets of $\mathbb{R}$ satisfies finite intersection property:

$$\mathscr{U}_1 := \{(n, n^2) : n \in \mathbb{N}\}; \ \mathscr{U}_2 := \{(0, n) : n \in \mathbb{N}\}; \ \mathscr{U}_3 := \{(-\infty, n] : n \in \mathbb{N}\}.$$

Proposition 5.13. *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(i) X is compact;

(ii) every non-empty family of closed subsets of X having finite intersection property has a non-empty intersection;

(iii) every non-empty family of closed subsets of X having empty intersection contains a finite subfamily having empty intersection.

Proof. $(i) \implies (ii)$ Let X be compact, $\mathscr{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a non-empty family of closed subsets of X having finite intersection property. Suppose on the contrary that $\bigcap_{\alpha\in\Lambda} A_\alpha = \emptyset$. Then $\mathscr{U} = \{X \setminus A_\alpha : \alpha \in \Lambda\}$ is an open cover of X. Since X is compact, we can write $X = \bigcup_{i=1}^{n} (X \setminus A_{\alpha_i})$, for some $\alpha_1, \cdots, \alpha_n \in \Lambda$. But then $\bigcap_{i=1}^{n} A_{\alpha_i} = \emptyset$, a contradiction.

$(ii) \iff (iii)$ This is just a restatement.

$(iii) \implies (i)$ Suppose the condition holds and let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then $\mathscr{A} = \{X \setminus V_\alpha : \alpha \in \Lambda\}$ is a non-empty family of closed subsets of X. Moreover, $\bigcap_{\alpha\in\Lambda} (X \setminus V_\alpha) = \emptyset$. Thus there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $\bigcap_{i=1}^{n} (X \setminus V_{\alpha_i}) = \emptyset$. But then $\bigcup_{i=1}^{n} V_{\alpha_i} = X$. Hence X is compact. □

Let X be a non-empty set and let $\mathscr{U} \subseteq \mathscr{P}(X)$ be a family of subsets of X.

- We say $\mathscr{U}$ is an *adequate family* if $\bigcup_{V\in\mathscr{U}} V = X$; otherwise $\mathscr{U}$ is said to be an *inadequate family*.
- $\mathscr{U}$ is said to be a *finitely adequate family* if a finite subfamily of $\mathscr{U}$ cover X; otherwise $\mathscr{U}$ is said to be a *finitely inadequate family*.

For a topological space X, the following conditions are equivalent:

- X is compact;
- every adequate family of open sets in X is finitely adequate;
- every finitely inadequate family of open sets in X is inadequate.

Lemma 5.14. *Let X be a topological space and let $\mathscr{U}$ be a finitely inadequate family of open subsets of X. Then there is a maximal (with respect to set inclusion) finitely inadequate family $\mathscr{A}$ of open subsets of X such that $\mathscr{U} \subseteq \mathscr{A}$.*

Proof. Let Σ be the collection of all finitely inadequate family of open subsets of X containing $\mathscr{U}$. Then Σ is a partially ordered set with respect to set inclusion. Let Γ be a chain in Σ and set $\mathscr{D} = \bigcup\{\mathscr{C} : \mathscr{C} \in \Gamma\}$. Let $V_1, \cdots, V_n \in \mathscr{D}$ and choose $\mathscr{C}_1, \cdots, \mathscr{C}_n \in \Gamma$ such that $V_i \in \mathscr{C}_i, 1 \le i \le n$. Since Γ is a chain, there is $\mathscr{C}^* \in \Gamma$ such that $\mathscr{C}_i \subseteq \mathscr{C}^*, \forall i = 1, \cdots, n$. Then $V_i \in \mathscr{C}^*, \forall i = 1, \cdots, n$. Since $\mathscr{C}^*$ is a finitely inadequate family, $X \neq \bigcup_{i=1}^{n} V_i$. Thus $\mathscr{D}$ is finitely inadequate and $\mathscr{D}$ is an upper bound of Γ in Σ. By Zorn's Lemma, Σ has a maximal element, say $\mathscr{A}$. Then $\mathscr{A}$ is a maximal finitely inadequate family of open subsets of X such that $\mathscr{U} \subseteq \mathscr{A}$. □

Theorem 5.15. (*Alexander Sub-base Theorem*) *Let X be a topological space and let $\mathscr{S}$ be a sub-base for the topology on X. Then X is compact if and only if every open cover of X consisting of elements of $\mathscr{S}$ has a finite subcover.*

Proof. Assume that every open cover of X consisting of elements of $\mathscr{S}$ has a finite subcover. Let $\mathscr{C}$ be a finitely inadequate family of open sets in X. By Lemma 5.14, there is a maximal finitely inadequate family $\mathscr{U}$ of open sets in X such that $\mathscr{C} \subseteq \mathscr{U}$. We will show that $\mathscr{U}$ is an inadequate family. Then $\mathscr{C}$ will also be an inadequate family. Let's make two crucial observations:

- Let $V \subseteq X$ be an open set and $V \notin \mathscr{U}$. By the maximality of $\mathscr{U}$, there are $A_1, \cdots, A_n \in \mathscr{U}$ such that $X = V \bigcup (\bigcup_{i=1}^{n} A_i)$. Hence no open subset of X that contains V can be a member of $\mathscr{U}$.
- Let $V' \subseteq X$ be an open set such that $V' \notin \mathscr{U}$. Choose $B_1, \cdots, B_m \in \mathscr{U}$ such that $X = V' \bigcup (\bigcup_{i=1}^{m} B_i)$. Then
$$\left(V \bigcap V'\right) \bigcup A_1 \bigcup \cdots \bigcup A_n \bigcup B_1 \bigcup \cdots \bigcup B_m = X$$
so that $V \bigcap V' \notin \mathscr{U}$.

Let $\mathscr{D} = \{V_i : 1 \le i \le r\}$ be a finite family of open sets in X and let $W \subseteq X$ be an open set. Assume $V_i \notin \mathscr{U}$, for every $i = 1, \cdots, n$. Then

- if $V_i \subseteq W$, for some $i = 1, \cdots r$, $W \notin \mathscr{U}$;
- if W contains intersection of some members of $\mathscr{D}$, $W \notin \mathscr{U}$.

Consequently, if a member of $\mathscr{U}$ contains a finite intersection $G_1 \bigcap \cdots \bigcap G_t$ of open subsets of X, then some G_j must be a member of $\mathscr{U}$.

Let $x \in \bigcup_{V \in \mathscr{U}} V$ and choose $A_0 \in \mathscr{U}$ such that $x \in A_0$. Since $\mathscr{S}$ is a bus-base for the topology on X, there are $V_1, \cdots, V_n \in \mathscr{U}$ such that $x \in \bigcap_{i=1}^n V_i \subseteq A_0$. Then by the above observation, $V_j \subseteq A_0$, for some $j \in \{1, \cdots, n\}$. In particular, $\bigcup \mathscr{U} = \bigcup (\mathscr{S} \bigcap \mathscr{U})$. Since $\mathscr{U}$ is a finite inadequate family of open sets in X, the same is true for $\mathscr{S} \bigcap \mathscr{U}$. On the other hand, by our assumption, $\mathscr{S} \bigcap \mathscr{U}$ must be an inadequate family of open sets in X. Hence $\mathscr{U}$ is an inadequate family of open sets in X.

The converse follows from definition. $\square$

Let X be a topological, $\mathscr{B}$ be a base for the topology on X and let $\mathscr{S}$ be a sub-base for the topology on X. Then the following conditions are equivalent:

- X is compact;
- any finitely inadequate family of open sets in X is inadequate;
- any finitely inadequate family consisting of elements of $\mathscr{B}$ is inadequate;
- any finitely inadequate family consisting of elements of $\mathscr{S}$ is inadequate.

Theorem 5.16. (*Tychonoff*) *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of non-empty topological spaces. Then the product space $X = \prod_{\alpha \in \Lambda} X_\alpha$ is compact if and only if each X_α is compact.*

Proof. Suppose the product space X is compact. Let $p_\alpha : X \longrightarrow X_\alpha$ be the α-th coordinate projection. Since $X_\alpha = p_\alpha(X)$, by Lemma 5.9, X_α is compact.

Now assume each X_α is compact. Let $\mathscr{S}$ be the canonical sub-base for the product topology on X and let $\mathscr{U} = \{V : V \in \mathscr{S}\}$ be a finite inadequate family. For each $\alpha \in \Lambda$, set

$$\mathscr{U}_\alpha := \{V : V \subseteq X_\alpha \text{ is open and } p_\alpha^{-1} V \in \mathscr{U}\}.$$

Since $\mathscr{U}$ is a finite inadequate family, there is $\beta \in \Lambda$ such that $\mathscr{U}_\beta$ is a finite inadequate family of open sets on X_β. Since X_β is compact, $\bigcup \mathscr{U}_\beta \neq X_\beta$. Choose $a_\beta \in X_\beta \setminus (\bigcup \mathscr{U}_\beta)$ and define $x : \Lambda \to \bigcup_{\alpha \in \Lambda} X_\alpha$ such that $x(\beta) = x_\beta$. Then $x \in X \setminus (\bigcup \mathscr{U})$. Thus $\mathscr{U}$ is an inadequate family. By above, X is compact. $\square$

Proposition 5.17. *In the Euclidean space $\mathbb{R}$, any closed and bounded interval is a compact.*

Proof. Let $I = [a, b]$ be a closed and bounded interval. Let $\mathscr{U}$ be a cover of I consisting of open intervals in $\mathbb{R}$. Set

$$Y := \{x \in [a, b] : [a, x] \text{ can be covered by a finite subfamily of } \mathscr{U}\}.$$

Since $a \in Y$, it is non-empty. Let $x_0 \in Y$ and let $V_{x_0} \in \mathscr{U}$ be such that $x_0 \in V_{x_0}$. Then for any $y \in V_{x_0}$ with $x_0 < y \le b$, $[x_0, y] \subseteq V_{x_0}$. By our choice, $[a, x_0]$ can be covered by a finite subfamily $\mathscr{U}_{x_0}$ of $\mathscr{U}$. Thus $\mathscr{U}_{x_0} \bigcup \{V_{x_0}\}$ is also a finite subcover of $[a, y]$. In other words, if $Y \cap V \neq \emptyset$ for some $V \in \mathscr{U}$, $V \bigcap I \subseteq Y$. Let $c = \sup Y$. Then $a < c \le b$. If $c < b$, choose $c \in V_c \in \mathscr{U}$. Since V_c is an open interval in $\mathbb{R}$ and $c < b$, there are $s, t \in V_c \bigcap I$ such that $a < s < c < t < b$. But then $t \in Y$, a contradiction. Hence $c = b$. $\square$

Corollary 5.18. *Let $I_1, \cdots, I_n \subseteq \mathbb{R}$ be closed bounded intervals. Then $I_1 \times \cdots \times I_n$ is a compact subset of the Euclidean space $\mathbb{R}^n$.*

Theorem 5.19. (*Heine - Borel*) *A subset of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Let $Y \subseteq \mathbb{R}^n$ be a compact subset. Then by Lemma 5.7, Y is closed. Let $\mathscr{U} = \{B(0,n) : n \in \mathbb{N}\}$ be an open cover of Y. Since Y is compact, there are positive integers $n_1 < \cdots < n_r$ such that $Y \subseteq \bigcup_{i=1}^r B(0,n_i) = B(0,n_r)$. Hence Y is bounded.

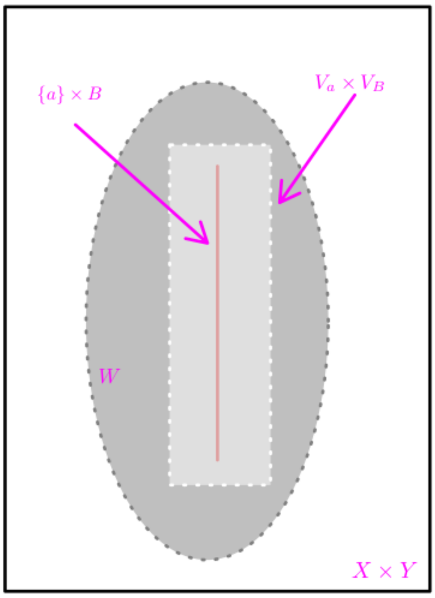
Now assume Y is a closed and bounded subset of $\mathbb{R}^n$. Then there are closed and bounded intervals $I_1, \cdots, I_n \subseteq \mathbb{R}^n$ such that $Y \subseteq \prod_{j=1}^n I_j$. Now the result follows from Corollary 5.18 and Lemma 5.7. $\square$

Corollary 5.20. (*Extreme Value Theorem*) *Let X be a non-empty compact topological space and $f : X \longrightarrow \mathbb{R}$ be a continuous map. Then there are $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$, for every $x \in X$.*

Proof. Since X is compact, $f(X)$ is a compact subset of $\mathbb{R}$. By Heine - Borel theorem, $f(X)$ is closed and bounded. Let $M = \sup\{f(x) : x \in X\}$ and $m = \inf\{f(x) : x \in X\}$. Since $f(X)$ is closed, we must have $M, m \in f(X)$. $\square$

Exercise 5.21. Let X be a subspace of the Euclidean space $\mathbb{R}$ and assume that every continuous map $f : X \longrightarrow \mathbb{R}$ is bounded. Is X a compact subspace of $\mathbb{R}$?

Lemma 5.22. (*Tube Lemma*) *Let X, Y be two non-empty topological spaces and let $B \subseteq Y$ be a compact subspace. Let $a \in X$ and let $W \subseteq X \times Y$ be an open set containing $\{a\} \times B$. Then there are open sets $V_a \subseteq X$ and $V_B \subseteq Y$ such that $\{a\} \times B \subseteq V_a \times V_B \subseteq W$.*



Proof. Since $W \subseteq X \times Y$ is an open set containing $\{a\} \times B$, for each $y \in B$, there is a canonical basic open set $U_y \times V_y$ such that $(a, y) \in U_y \times V_y \subseteq W$. Thus $\{V_y : y \in B\}$ is an open cover of B. Since B is compact, there are $y_1, \cdots, y_r \in B$ such that $B \subseteq \bigcup_{j=1}^r V_{y_j}$. Set $V_a := \bigcap_{j=1}^r U_{y_j}$ and $V_B := \bigcup_{j=1}^r V_{y_j}$. Then $V_a \subseteq X$ is an open set, $V_B \subseteq Y$ is an open set and $\{a\} \times B \subseteq V_a \times V_B \subseteq W$. $\square$

Exercise 5.23. Let $X = Y = \mathbb{R}$ be equipped with the Euclidean topology and set

$$W := \left\{ (x, y) \in \mathbb{R}^2 \; : \; |\, x \,| \; < \; \frac{1}{1 + y^2} \right\}.$$

Prove that

(i) W is an open subset of $\mathbb{R}^2$ containing $\{0\} \times \mathbb{R}$ (the y-axis);

(ii) there is no open neighbourhood V of 0 satisfying $\{0\} \times \mathbb{R} \subseteq V \times \mathbb{R} \subseteq W$.

(This exercise shows the conclusion of Lemma 5.22 need not be true if one does not assume B is a compact subset of Y.)

Corollary 5.24. *If X and Y are compact topological spaces, then so is $X \times Y$.*

Proof. Let $\mathscr{U} = \{A_\alpha : \alpha \in \Lambda\}$ be an open cover of $X \times Y$. Since Y is compact, for each $x \in X$, there is $r_x \in \mathbb{N}$ $\alpha_{x,1}, \cdots, \alpha_{x,r_x} \in \Lambda$ such that $\{x\} \times Y \subseteq \bigcup_{j=1}^{r_x} A_{\alpha_{x,j}}$. By Lemma 5.22, there is an open neighbourhood $V_x \subseteq X$ of x such that $\{x\} \times Y \subseteq V_x \times Y \subseteq \bigcup_{j=1}^{r_x} A_{\alpha_{x,j}}$. Then the collection $\{V_x : x \in X\}$ is an open cover of X. Since X is compact, there are $x_1, \cdots, x_n \in X$ such that $X = \bigcup_{j=1}^{n} V_{x_j}$. Then the finite subfamily

$$\{A_{x_i, j} \subseteq X \times Y \; : \; 1 \leq j \leq r_{x_i} \text{ and } 1 \leq i \leq n\}$$

of $\mathscr{U}$ is an open cover of $X \times Y$. Hence $X \times Y$ is compact. $\square$

Proposition 5.25. (*Kuratowski*) *Let X be a non-empty topological space. Then X is compact if and only if for any topological space Y, the projection map $p : X \times Y \longrightarrow Y$ is a closed map.*

Proof. Suppose X is compact Let $E \subseteq X \times Y$ be a closed subset and let $y \in Y \setminus p(E)$. Then $X \times \{y\}$ is contained in the open set $(X \times Y) \setminus E$. By Lemma 5.22, there is an open neighbourhood $V_y \subseteq Y$ of y such that $X \times \{y\} \subseteq X \times V_y \subseteq (X \times Y) \setminus E$. Thus $y \in V_y \subseteq Y \setminus p(E)$ and hence y can not be a limit point of E. Consequently, $p(E)$ is a closed subset of Y.

Now assume that for any topological space Y, the projection map $p : X \times Y \longrightarrow Y$ is a closed map. Let $\mathscr{U}$ be an open cover of X. Choose $\infty \notin X$ and set $Y = X \bigcup \{\infty\}$. Let Y be equipped with the topology generated by the sub-base

$$\mathscr{S} := \{\{x\} : x \in X\} \bigcup \{Y \setminus V : V \in \mathscr{U}\} \subseteq \mathscr{P}(Y).$$

Let $Z = \{(x, x) : x \in X\} \subseteq X \times Y$. Let $p : X \times Y \longrightarrow Y$ be the projection map. Then $p(\mathrm{Cl}_{X \times Y}(Z)) = \mathrm{Cl}_Y(p(Z)) = \mathrm{Cl}_Y(X)$. Given $a \in X$, choose $V_a \in \mathscr{U}$ such that $a \in V_a$. Then $Z \bigcap (V_a \times (Y \setminus V_a)) = \emptyset$. In particular, $(a, \infty) \in X \times Y$ is not a limit point of Z. Thus $\mathrm{Cl}_Y(X) = X$ so that $\{\infty\}$ is an open subset of Y and $\{\infty\}$ can be written as a finite intersection of elements of $\mathscr{S}$. Choose $V_1, \cdots, V_n \in \mathscr{U}$ such that $\bigcap_{i=1}^{n} (Y \setminus V_i) = \{\infty\}$. Then $X = \bigcup_{i=1}^{n} V_i$. Hence X is compact. $\square$

Corollary 5.26. *Let X, Y be two non-empty topological space and let $f : X \longrightarrow Y$ be a function. If Y is compact and the graph Γ_f of f is closed in $X \times Y$, f is a continuous map.*

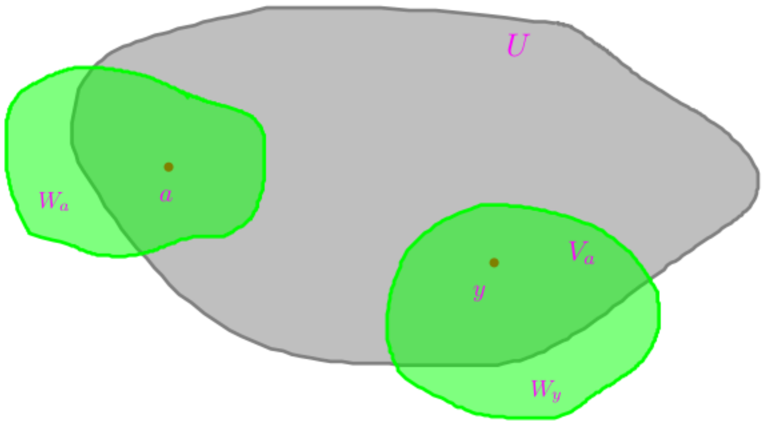
Proof. Let $E \subseteq Y$ be a closed subset. Then $X \times E \subseteq X \times Y$ is a closed subset and hence so is $A = (X \times E) \bigcap \Gamma_f$. Let $p : X \times Y \longrightarrow X$ be the first coordinate projection. By Proposition 5.25, $p(A)$ is a closed subset of X. Notice that

$$A \; = \; (X \times E) \bigcap \Gamma_f \; = \; \{(x, y) \in X \times Y \; : \; y = f(x) \in E\}$$

so that $p(A) = f^{-1}E$. Hence f is continuous. $\square$

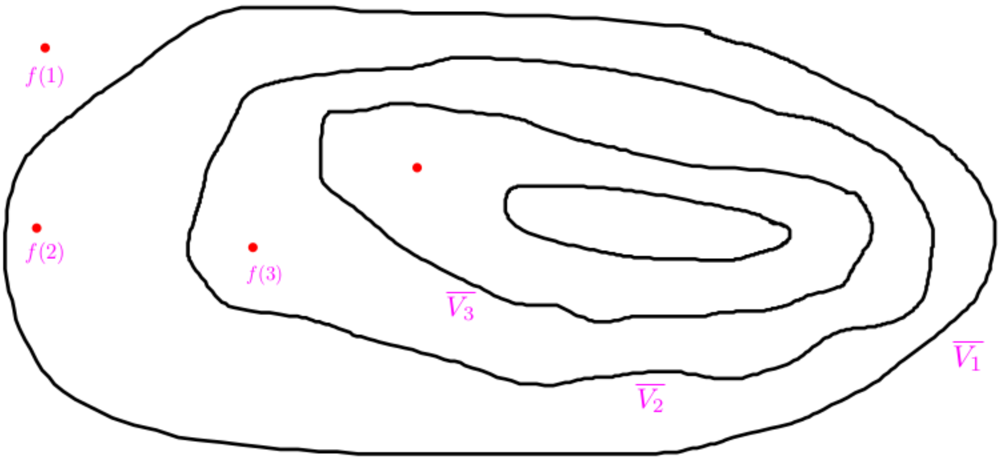
Proposition 5.27. *Let X be a non-empty compact Hausdorff space with no isolated points. Then X is uncountable.*

Proof. Let $U \subseteq X$ be a non-empty set and let $a \in X$. Since X has no isolated point, there is $y \in U$ such that $a \neq y$. Sine X is Hausdorff, there are open sets $W_a, W_y \subseteq X$ such that $a \in W_a, y \in W_y$ and $W_a \bigcap W_y = \emptyset$. Set $V_a = W_y \bigcap U$. Then $a \notin \overline{V_a}$.



Let $f : \mathbb{N} \longrightarrow X$ be any function. We construct a sequence of open subsets of X as follows:

- For $U = X$, there is a non-empty open set $V_1 \subseteq X$ such that $f(1) \notin \overline{V_1}$.
- For $U = V_1$, there is an non-empty open set $V_2 \subseteq V_1$ such that $f(2) \notin \overline{V_2}$.
- For $U = V_2$, there is an non-empty open set $V_3 \subseteq V_1$ such that $f(3) \notin \overline{V_3}$ and so on.



Thus we have a decreasing chain of non-empty closed subsets $\overline{V_1} \supseteq \overline{V_2} \supseteq \overline{V_3} \supseteq \cdots$ of X. Since X is compact, by Proposition 5.13, $\bigcap_{n\in\mathbb{N}} \overline{V_n} \neq \emptyset$. Since $f(n) \neq \overline{V_n}, \forall n \in \mathbb{N}$, $f(\mathbb{N}) \bigcap (\bigcap_{n\in\mathbb{N}} \overline{V_n}) = \emptyset$ so that f is not a surjective map. Hence X is uncountable. $\square$

Exercise 5.28. *(1.)* Let $\mathbb{R}_\ell$ be the lower limit topological space. Check if the interval $I = [0, 1]$ is a compact subset of $\mathbb{R}_\ell$.

(2.) Let X be a compact topological space and let A be an infinite subset of X. Prove that the set of limit points of A is non-empty.

(3.) Let X be a non-empty topological space and let $Y_1, \cdots, Y_n$ be a finite collection of compact subspaces of X. Prove that $\bigcup_{i=1}^{n} Y_i$ is a compact subspace of X. Give an example to show that infinite union of compact subspaces need not be compact.

(4.) Let X be a non-empty Hausdorff space and let $\{Y_\alpha : \alpha \in \Lambda\}$ be a collection of compact subsets of X. Prove that $\bigcap_{\alpha\in\Lambda} Y_\alpha$ is a compact subset of X.

(5.) Give an example of a topological space X and two compact subspaces Y_1, Y_2 of X such that $Y_1 \bigcap Y_2$ is not compact.

(6.) Let X be a non-empty compact Hausdorff space and let $\mathscr{A}$ be a collection of continuous functions $X \longrightarrow \mathbb{R}$ satisfying the following:

- if $f, g \in \mathscr{A}$, then $fg \in \mathscr{A}$;
- given any $x \in Y$, there is an open neighbourhood $V_x \subseteq X$ of x and some $f \in \mathscr{A}$ such that $f(t) = 0, \forall t \in V_x$.

Prove that $\mathscr{A}$ contains the zero function $\mathbb{O} : X \longrightarrow \mathbb{R}, \mathbb{O}(t) = 0, \forall t \in X$.

(7.) Let $n \in \mathbb{N}$ and $J = [-1, 1] \subset \mathbb{R}$.

- Define $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ by $\phi(0) = 0$ and
$$\phi(x) = \frac{\max\{|x_1|, \cdots, |x_n|\}}{||x||} \text{ for } x = (x_1, \cdots, x_n) \neq (0, \cdots, 0)$$
Prove that ϕ is a continuous function.
- Show that ϕ induces a homeomorphism $f : J^n \longrightarrow \overline{B(0;1)}$.

(8.) Let (X, d) be a compact metric space. Prove that X is compact if and only if every continuous map $f : X \longrightarrow \mathbb{R}$ is bounded.

6. Quotient Topology

Let $f : X \longrightarrow Y$ be a map between two non-empty sets and assume that X is equipped with a topology τ_X. If τ is a topology on Y for which f is continuous, then for any $U \in \tau$, $f^{-1}(U) \in \tau_X$. Consider the set

$$\tau_Y := \{V \subseteq Y : f^{-1}V \subseteq X \text{ is open in } X\} \subseteq \mathscr{P}(Y).$$

We claim that τ_Y is a topology on X:

- By definition $\emptyset, Y \in \tau_Y$.
- Let $\{V_\alpha : \alpha \in \Lambda\}$ be a collection of elements of τ_Y. Then
$$f^{-1}\Big(\bigcup_{\alpha\in\Lambda} V_\alpha\Big) = \bigcup_{\alpha\in\Lambda} f^{-1}V_\alpha \in \tau_X \implies \bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_Y.$$
- Let $V_1, \cdots, V_n \in \tau_Y$. Then $f^{-1}\big(\bigcap_{i=1}^n V_i\big) = \bigcap_{i=1}^n f^{-1}V_i \in \tau_X$. Hence $\bigcap_{i=1}^n V_i \in \tau_Y$.

Thus τ_Y is a topology on Y and $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is continuous. Let τ be another topology on Y for which $f : (X, \tau_X) \longrightarrow (Y, \tau)$ is continuous. Let $V \in \tau$. Then $f^{-1}V \in \tau_X$ and hence $V \in \tau_Y$. Thus τ_Y is the *unique largest topology* on Y for which the map f is continuous.

Lemma 6.1. *Let $\emptyset \neq X$ be a topological space, Y be a non-empty set and let $f : X \longrightarrow Y$ be a map. Let's equip Y with the unique largest topology on Y for which f is continuous. Let Z be a topological space and let $g : Y \longrightarrow Z$ be a function. Then g is continuous if and only if $g \circ f : X \longrightarrow Z$ is continuous.*

Proof. Assume $g \circ f : X \longrightarrow Z$ is continuous. Let $U \subseteq Z$ be an open set and $V = g^{-1}U \subseteq Y$. Then $f^{-1}V = f^{-1}g^{-1}U \subseteq X$ is open and hence $V \subseteq Y$ is open. Thus g is continuous. The converse is trivial. □

Exercise 6.2. Let $f : X \longrightarrow Y$ be a function from a topological space X to a set Y. Let τ be the unique largest topology on Y for which f is continuous. Then $E \subseteq Y$ is closed in Y if and only if $f^{-1}E \subseteq X$ is closed in X.

Definition 6.3. Let X and Y be two topological spaces and let $q : X \longrightarrow Y$ be a surjective map (of sets). We say q is a *quotient map* if it satisfies the following property: $V \subseteq Y$ is open if and only if $q^{-1}V \subseteq X$ is open.

Let $q : X \longrightarrow Y$ be a quotient map between two topological spaces X and Y. Notice that, by definition, q is continuous. Moreover, by Exercise 6.2, the followings are equivalent:

- q is a quotient map;
- $E \subseteq Y$ is closed if and only if $q^{-1}E \subseteq X$ is closed.

Exercise 6.4. Let X, Y be two topological spaces and let $q : X \longrightarrow Y$ be a surjective continuous map.

(i) If q is an open map or a closed map, then prove that q is a quotient map.
(ii) If X is compact and Y is Hausdorff, then prove that q is a quotient map.

Let X be a topological space and let $\mathscr{R}$ be an equivalence relation on the set X. Let $X/\mathscr{R}$ be the set of all equivalence classes of elements of X. Then we have a canonical surjective map (of sets) $q : X \longrightarrow X/\mathscr{R}$ defined by $q(x) = [x]$ where $[x]$ is the equivalent class containing x. Let τ be the unique largest topology on $X/\mathscr{R}$ for which the map q is continuous. The topology τ in $X/\mathscr{R}$ is called *quotient topology* and the topological space $X/\mathscr{R}$ is called the *quotient space* of X by the equivalence relation $\mathscr{R}$. By construction, the canonical surjective map $q : X \longrightarrow X/\mathscr{R}$ is a quotient map.

Let $q : X \longrightarrow Y$ be a quotient map between two topological spaces. Define a relation $\mathscr{R}$ on X as follows: $x\,\mathscr{R}\,y$ if and only if $q(x) = q(y)$. Then $\mathscr{R}$ is an equivalence relation on X. Let $X/\mathscr{R}$ be the quotient space of X by the equivalence relation $\mathscr{R}$ and let $q' : X \longrightarrow X/\mathscr{R}$ be the canonical quotient map. Let $f : X/\mathscr{R} \longrightarrow Y$ be the induced set theoretic map that makes the following diagram commutative:

$$\begin{tikzcd}
X \arrow[rr, "q"] \arrow[dr, "q'"'] & & Y \\
& X/\mathscr{R} \arrow[ur, dashed, "f"'] &
\end{tikzcd}$$

By Lemma 6.1, f is continuous. By construction, f is bijective. Now let V be an open subset of $X/\mathscr{R}$. Then

$$q^{-1}(f(V)) = (f \circ q')^{-1}(f(V)) = q'^{-1}f^{-1}(f(V)) = q'^{-1}V \subseteq X \text{ is open}$$

(the third equality holds since f is bijective). Since $q : X \longrightarrow Y$ is a quotient map, $f(V)$ is open in Y. Thus f is an open map and hence $f : X/\mathscr{R} \longrightarrow Y$ is a homeomorphism.

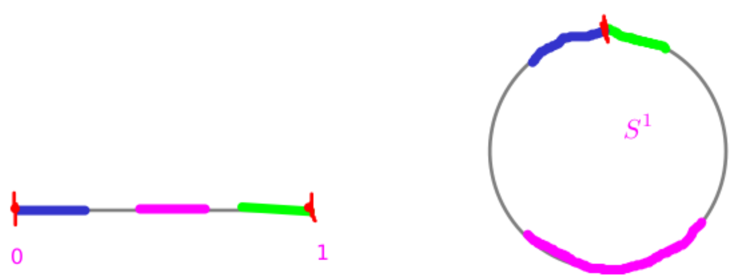
Proposition 6.5. *Let X, Y be two non-empty topological spaces and let $q : X \longrightarrow Y$ be a continuous surjective map. The following conditions are equivalent:*

- *q is a quotient map;*
- *Y is a quotient space of X by some equivalence relation on X;*
- *a set $E \subseteq Y$ is closed in Y if and only if $q^{-1}E \subseteq X$ is closed in X.*

Example 6.6. *(1.)* Let $X = [0,1] \subset \mathbb{R}$ be equipped with subspace topology and define a relation $\sim$ on X as follows: $x \sim x$ if $x \in (0,1)$ and $0 \sim 1$. Then $\sim$ is an equivalence relation on X. Define

$$g : [0,1] \longrightarrow \mathbb{S}^1, \quad g(t) = \exp(2\pi i t).$$

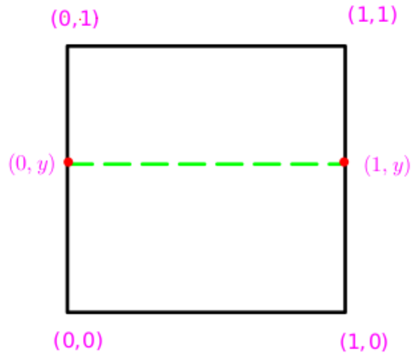
Then g surjective continuous map. Since X is compact and $\mathbb{S}^1$ is Hausdorff, by Exercise 6.4, g is a quotient map. Notice that, for any $(x,y) \in \mathbb{S}^{-1} \setminus \{(0,1)\}$, $g^{-1}(x,y)$ is a single point set where as $g^{-1}(0,1) = \{0,1\}$. Thus the quotient space $X/\sim$ is homeomorphic to $\mathbb{S}^1$.

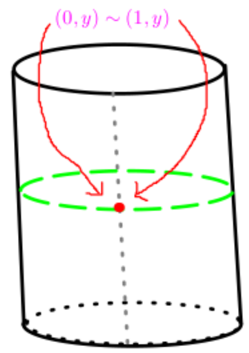


(2.) Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ be equipped with subspace topology and define a relation $\sim$ on X as follows: $(x,y) \sim (x,y)$ if $x \in (0,1)$ and $(0,y) \sim (1,y)$. Then $\sim$ is an equivalence relation on X. Define

$$g : X \longrightarrow \mathbb{S}^1 \times [0,1], \quad g(s,t) = \big(\exp(2\pi i s), t\big).$$

Then g is continuous surjective map. By Exercise 6.4, g is a quotient map. Notice that the fibers of g are precisely the equivalence classes of $\sim$. Thus the quotient space $X/\sim$ is homeomorphic to the cylinder $\mathbb{S}^1 \times [0,1]$.

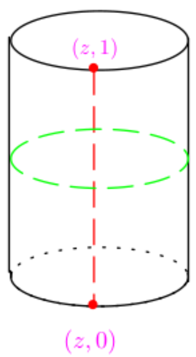


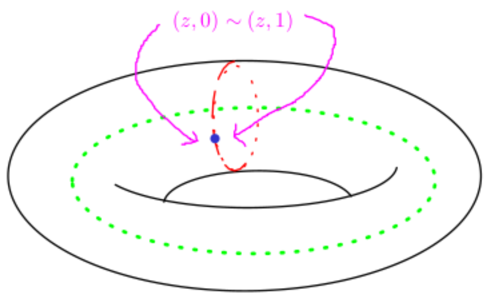


(3.) Let $X = \mathbb{S}^1 \times [0,1] \subseteq \mathbb{R}^3$ be equipped with the subspace topology and define a relation $\sim$ on X as follows: $(z,t) \sim (z,t)$ if $t \in (0,1)$ and $(z,0) \sim (z,1)$. Then $\sim$ is an equivalence relation on X. The map

$$g : \mathbb{S}^1 \times [0,1] \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1, \ \ (z,t) \mapsto \big(z, \exp(2\pi i t)\big)$$

is a continuous surjection and hence by Exercise 6.4, g is a quotient map. Since the fibers of g are precisely the equivalence classes of $\sim$, the quotient space $X/\sim$ is homeomorphic to the cylinder $\mathbb{S}^1 \times \mathbb{S}^1$.





(4.) Define a map

$$g : \mathbb{R} \longrightarrow \mathbb{S}^1, \ \ t \mapsto \exp(2\pi i t) \ = \ \big(\cos(2\pi t), \sin(2\pi t)\big)$$

Then g is a surjective continuous map. We claim that f is an open map. It is sufficient to show that image of an open interval is open. Let $(a,b) \subseteq \mathbb{R}$ be a non-empty open interval. Since $g(t+1) = \exp(2\pi i t + 2\pi i) = \exp(2\pi i t) = g(t)$, for every $t \in \mathbb{R}$, we have $g\big((a,b)\big) = \mathbb{S}^1$ whenever $b - a > 1$. Now assume $b - a \leq 1$. Let $s_0 \in (a,b)$ be such that $g(s_0) = \big(\cos(2\pi s_0), \sin(2\pi s_0)\big) \in g\big((a,b)\big)$. Set $E = [a, a+1] \setminus (a,b)$. By Heine-Borel theorem, E is a compact subset of $\mathbb{R}$. Since $\mathbb{S}^1$ is Hausdorff and $g(E)$ is a compact subset of $\mathbb{S}^1$, $g(E)$ is a closed subset of $\mathbb{S}^1$. Then

$$\mathbb{S}^1 \ = \ g([a, a+1]) \ = \ g(E) \bigcup g\big((a,b)\big) \ \text{ and } \ g(E) \bigcap g\big((a,b)\big) \ = \ \emptyset.$$

Thus $g\big((a,b)\big) = \mathbb{S}^1 \setminus g(E)$ is an open set. Hence g is an open map. By Exercise 6.4, g is a quotient map. Since the fibers of g are $\mathbb{Z}$, the quotient space is usually denoted by $\mathbb{R}/\mathbb{Z}$.

Lemma 6.7. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces, Y be non-empty set and let $f_\alpha : X_\alpha \longrightarrow Y$ be a function, for each $\alpha \in \Lambda$. Set*

$$\tau_Y := \{V \subseteq Y : f_\alpha^{-1}V \subseteq X_\alpha \text{ is open in } X_\alpha, \forall \alpha \in \Lambda\} \subseteq \mathscr{P}(Y).$$

(i) The set $\tau_Y \subseteq \mathscr{P}(Y)$ is a topology on Y.

(ii) The topology τ_Y is the unique largest topology on Y for which each $f_\alpha : X_\alpha \longrightarrow Y$ is continuous.

(iii) Let Z be a topological space and let $g : Y \longrightarrow X$ be a function. Then g is continuous if and only if $g \circ f_\alpha : X_\alpha \longrightarrow Z$ continuous, for every $\alpha \in \Lambda$.

Proof. Left as an exercise. □

Example 6.8. *(1.)* Let X, Y be two topological spaces. Recall, a disjoint union of X and Y is the set

$$X \coprod Y := \{(x,0) : x \in X\} \bigcup \{(y,1) : y \in Y\} = (X \times \{0\}) \bigcup (Y \times \{1\})$$

We have two natural injective map between sets

$$\iota_X : X \longrightarrow X \coprod Y,\ x \mapsto (x,0) \quad \text{and} \quad \iota_Y : Y \longrightarrow X \coprod Y,\ y \mapsto (y,1).$$

Let τ be the unique largest topology on $X \coprod Y$ for which both the maps ι_X and ι_Y are continuous. Notice that the sets $X \times \{0\}$ and $Y \times \{1\}$ are both open in $X \coprod Y$. Thus the maps $\iota_X : X \longrightarrow X \times \{0\}$ and $\iota_Y : Y \longrightarrow Y \times \{1\}$ are homeomorphism.

(2.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. The disjoint union of the sets X_α is defined by

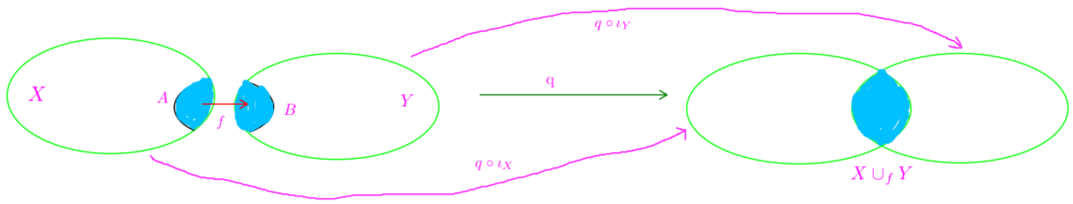
$$\coprod_{\alpha \in \Lambda} X_\alpha := \bigcup_{\alpha \in \Lambda} (X_\alpha \times \{\alpha\}).$$

For each $\alpha \in \Lambda$, the natural map $\iota_\alpha : X_\alpha \longrightarrow \coprod_{\alpha \in \Lambda} X_\alpha$, $x \mapsto (x, \alpha)$ is injective. Let τ be the unique largest topology on $\coprod_{\alpha \in \Lambda} X_\alpha$ such that each ι_α is continuous. For each $\alpha \in \Lambda$, $X_\alpha \times \{\alpha\}$ is open in $\coprod_{\alpha \in \Lambda} X_\alpha$ and the map $\iota_\alpha : X_\alpha \longrightarrow X_\alpha \times \{\alpha\}$ is a homeomorphism.

Let X and Y be two topological spaces, $\emptyset \neq A \subsetneq X$ and $\emptyset \neq B \subseteq Y$ be open subspaces and let $f : A \longrightarrow B$ be a homeomorphism. On the topological space $X \coprod Y$, define a relation $\sim$ as follows:

- $(x,0) \sim (x,0)$ and $(y,1) \sim (y,1)$ if $x \notin A$ and $y \notin B$;
- $(x,0) \sim (f(x),1)$ if $x \in A$.

Then $\sim$ is an equivalence relation on $X \coprod Y$. Let $X \bigcup_f Y$ be the quotient space of $X \coprod Y$ by the relation $\sim$ and let $q : X \coprod Y \longrightarrow X \bigcup_f Y$ be the quotient map.

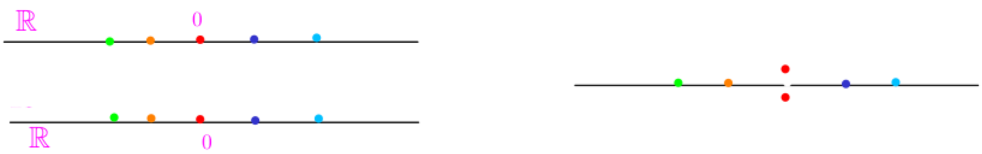


The maps $q \circ \iota_X : X \longrightarrow X \bigcup_f Y$ and $q \circ \iota_Y : Y \longrightarrow X \bigcup_f Y$ are continuous and injective. By construction, $T := q \circ \iota_X(A) = q \circ \iota_Y(B) \subsetneq X \bigcup_f Y$. Since $q^{-1}T = (A \times \{0\}) \bigcup (B \times \{1\})$, T is an open subset of $X \bigcup_f Y$. Let W be an open subset of X. Since $q \circ \iota_X$ is injective, $q^{-1}(q \circ \iota_X(W)) = W \times \{0\}$. Thus $q \circ \iota_X(W)$ is open in $X \bigcup_f Y$ and hence $q \circ \iota_X$ is an open map. Similarly, $q \circ \iota_Y$ is an open map. Thus X and Y are homeomorphic to open subsets of

$X \bigcup_f Y$. Identifying X and Y with these open subsets of $X \bigcup_f Y$, we may consider X and Y as open subsets of $X \bigcup_f Y$. We say the topological space $X \bigcup_f Y$ is obtained from X and Y by *gluing* or *identifying* the subspaces A and B via f.

In the above situation, if $A \subseteq X$ and $B \subseteq Y$ are closed subsets, then arguing as above, we can show that the maps $q \circ \iota_X : X \longrightarrow X \bigcup_f Y$ and $q \circ \iota_Y : Y \longrightarrow X \bigcup_f Y$ are injective, continuous and closed maps. Thus X and Y are homeomorphic to closed subsets of $X \bigcup_f Y$. As above, identifying X and Y with their homeomorphic images, we may assume that both X and Y are closed subsets of $X \bigcup_f Y$.

Example 6.9. *(1.)* Let $X = Y = \mathbb{R}$, $U = V = \mathbb{R} \setminus \{0\}$ and $f : U \longrightarrow V$ be the identity map. Then the quotient space $X \bigcup_f Y$, that obtained as above, is called the *real line with origin double.*



(2.) Let $X = Y = \mathbb{R}^2$, $U = V = \mathbb{R}^2 \setminus \{(0,0)\}$ and $f : U \longrightarrow V$ be the identity map. Then the quotient space Z, that obtained as above, is called the *real plane with origin double.*

Exercise 6.10. *(1.)* Let X, Y be two non-empty topological spaces. Prove that the topological space $X \coprod Y$ is compact if and only if both X and Y are compact.

(2.) Let X, Y be two non-empty topological spaces. Prove that the topological space $X \coprod Y$ is Hausdorff if and only if both X and Y are Hausdorff topological spaces.

Let $X = Y = \mathbb{R}$, $U = V = \mathbb{R} \setminus \{0\}$ and $f : U \longrightarrow V$ be the identity map. By the above exercise, $X \coprod Y$ is a Hausdorff topological space. Let $X \bigcup_f Y$ be the quotient space of $X \coprod Y$ obtained by identifying U and V via the map f. Let $q : X \coprod Y \longrightarrow X \bigcup_f Y$ be the quotient map. For simplicity, let O_1 (resp. O_2) the image of the origin of $X = \mathbb{R}$ (resp. of $Y = \mathbb{R}$) in $X \bigcup_f Y$. By construction, $O_1 \neq O_2$. Let W_1 (resp. W_2) be an open set in $X \bigcup_f Y$ containing O_1 (resp. O_2). Then there is a real number $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq (q \circ \iota_X)^{-1} W_1$ in X and $(-\varepsilon, \varepsilon) \subseteq (q \circ \iota_Y)^{-1} W_2$ in Y. Thus

$$q \circ \iota_X\big((-\varepsilon, \varepsilon) \setminus \{0\}\big) = q \circ \iota_Y\big((-\varepsilon, \varepsilon) \setminus \{0\}\big) \subseteq W_1 \bigcap W_2 \implies W_1 \bigcap W_2 \neq \emptyset.$$

Hence $X \bigcup_f Y$ is not a Hausdorff space. This example shows that quotient space of a Hausdorff topological space need not be Hausdorff.

Definition 7.1. Let X be a topological space.

- We say X is *connected* if it can not be written as a union of two disjoint non-empty open subsets of X. Otherwise we say X is *disconnected*.
- A set $Y \subseteq X$ is said to be a *connected subset* of X if Y is connected as a subspace of X; otherwise we say Y is a *disconnected subset* of X.

Example 7.2. *(1.)* The empty set and the single point spaces are connected. A set X with at least two points and equipped with discrete topology is not connected: For any point $a \in X$, the sets $\{a\}$ and $X \setminus \{a\}$ are disjoint open subsets of X and their union is X.

(2.) Let X be an infinite set equipped with co-finite topology. Then X is connected: Let V_1, V_2 be two non-empty open subset of X such that $X = V_1 \bigcup V_2$. Since both $X \setminus V_1$ and $X \setminus V_2$ are finite sets, $V_1 \bigcap V_2 \neq \emptyset$.

(3.) Let $\mathbb{R}_\ell$ be the lower limit topological space. Then $\mathbb{R}_\ell$ is not connected: we can write $\mathbb{R}_\ell = (-\infty, 0) \bigcup [0, \infty)$.

(4.) Consider $\mathbb{Q}$ as a subspace of the Euclidean space $\mathbb{R}$. Then $\mathbb{Q}$ is not connected: we can write $\mathbb{Q} = ((-\infty, \sqrt{2}) \bigcap \mathbb{Q}) \cup ((\sqrt{2}, \infty) \bigcap \mathbb{Q})$.

Exercise 7.3. *(1.)* Let X and Y be two homeomorphic topological spaces. If one of them is connected, prove that the other one is also connected.

(2.) Let X be topological space. Prove that the following conditions are equivalent:

(i) X is disconnected;

(ii) there are non-empty disjoint closed sets $E, F \subseteq X$ such that $X = E \bigcup F$;

(iii) there is a non-empty proper clopen subset of X.

Lemma 7.4. *Let X be a topological space and $\emptyset \neq Y \subseteq X$. Then the following conditions are equivalent:*

(i) Y is a connected subset of X;

(ii) there do not exist open sets V_1, V_2 in X satisfying

$$Y \subseteq V_1 \bigcup V_2, \quad Y \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2, \quad Y \bigcap V_1 \bigcap V_2 = \emptyset.$$

Proof. $(i) \implies (ii)$ Suppose Y is a connected subset of X and assume there are non-empty open sets V_1, V_2 in X satisfying the condition of (ii). Set $A_i = Y \bigcap V_i, i = 1, 2$. Then A_1, A_2 are non-empty disjoint open sets in Y such that $Y = A_1 \bigcup A_2$ and hence Y is a disconnected subset of X, a contradiction.

$(ii) \implies (i)$ Suppose the condition holds and assume Y is a disconnected subset of X. Then there are non-empty disjoint open sets A_1, A_2 in Y such that $Y = A_1 \bigcup A_2$. Since A_1, A_2 are open in Y, there are open sets V_1, V_2 in X such that $A_1 = V_1 \bigcap Y$ and $A_2 = V_2 \bigcap Y$. Now its easy to check that $Y \subseteq V_1 \bigcup V_2$, $Y \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2$ and $Y \bigcap V_1 \bigcap V_2 = \emptyset$, a contradiction. $\square$

Corollary 7.5. *Let X be a topological space and let $Z \subseteq Y \subseteq X$ be equipped with subspace topology. Then Z is a connected subset of Y if and only if Z is a connected subset of X.*

In other words, connectedness of a subset of a topological space does not depend on the subspace topology. In view of this, we will often write "$A \subseteq X$ is connected (resp. disconnected)" instead of "A is a connected (resp. disconnected) subset of X".

Lemma 7.6. *Let $f : X \longrightarrow Y$ be a continuous map between two topological spaces and let $A \subseteq X$ be a connected subset. Then $f(A)$ is a connected subset of Y.*

Proof. Suppose $f(A) \subseteq Y$ is disconnected. By Lemma 7.4, there are non-empty open sets V_1, V_2 in Y such that

$$f(A) \subseteq V_1 \bigcup V_2, \;\; f(A) \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2, \;\; f(A) \bigcap V_1 \bigcap V_2 = \emptyset.$$

Set $W_i = f^{-1} V_i, i = 1, 2$. Then W_1, W_2 are non-empty open sets in X satisfying

$$A \subseteq W_1 \bigcup W_2, \;\; A \bigcap W_1 \neq \emptyset \neq A \bigcap W_2, \;\; A \bigcap W_1 \bigcap W_2 = \emptyset.$$

This contradicts A is connected, by Lemma 7.4. Hence $f(A)$ is connected. $\square$

Corollary 7.7. *Let X be a non-empty topological space and assume the set $\{0, 1\}$ is equipped with the discrete topology. Then X is connected if and only if any continuous function $f : X \longrightarrow \{0, 1\}$ is constant.*

Proof. Suppose X is connected. Then by Lemma 7.6, $f(X)$ is connected. The non-empty connected subsets of $\{0, 1\}$ are $\{0\}$ and $\{1\}$. Hence either $f(X) = \{0\}$ or $f(X) = \{1\}$.

Suppose the condition holds. If X is not connected, there are non-empty disjoint closed sets E, F in X such that $X = E \bigcup F$. Define $g : E \longrightarrow \{0, 1\}$ by $g(x) = 0, \forall x \in E$ and $h : F \longrightarrow \{0, 1\}$ by $h(x) = 1, \forall x \in F$. Then both g and h are continuous maps and the map $f : X \longrightarrow \{0, 1\}$ define by $f(x) = 0$, if $x \in E$ and $f(x) = 1$, if $x \in F$ is continuous, a contradiction. $\square$

Lemma 7.8. *A set $Y \subseteq \mathbb{R}$ is connected if and only if Y is an interval.*

Proof. To avoid triviality, we will assume X contains at least two points. Suppose Y is connected. If Y is not an interval, there are $a, b, c \in \mathbb{R}$ such that $a < b < c$, $a, c \in Y$ but $b \notin Y$. Set $V_1 = (-\infty, b)$ and $V_2 = (b, \infty)$. Then V_1, V_2 are non-empty open subsets of $\mathbb{R}$ satisfying $Y \subseteq V_1 \bigcup V_2$, $Y \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2$ and $Y \bigcap V_1 \bigcap V_2 = \emptyset$, a contradiction by Lemma 7.4. Hence Y is an interval.

Conversely assume that Y is an interval. If Y is not connected, then there are non-empty open subsets V_1, V_2 of $\mathbb{R}$ satisfying $Y \subseteq V_1 \bigcup V_2$, $Y \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2$ and $Y \bigcap V_1 \bigcap V_2 = \emptyset$. Let $a \in Y \bigcap V_1$ and $b \in Y \bigcap V_2$. Since Y is an interval, $[a, b] \subseteq Y \subseteq V_1 \bigcup V_2$. Let $c = \sup \{[a, b] \bigcap V_1\}$. Then $c \in \overline{[a, b] \bigcap V_1}$ and hence $c \notin V_2$. Since $a < c \leq b$ and $c \notin V_2$, $a < c < b$ and $c \in V_1$. Since V_1 is open, there is a real number $\delta > 0$ such that $(c - 2\delta, c + 2\delta) \subseteq V_1$ and $c + 2\delta < b$. Then $c + \delta \in [a, b] \bigcap V_1$, a contradiction. $\square$

Let $X = \mathbb{R}_\ell$ be the lower limit topological space and let $Y \subseteq X$ be a set containing at least two points. Choose $a, b, c \in X$ such that $a < b < c$ and $a, c \in Y$ (we are not claiming that $b \in Y$). Set $V_1 = (-\infty, b)$ and $V_2 = [b, \infty)$. Then both V_1, V_2 are open sets in X satisfying

$$Y \subseteq V_1 \bigcup V_2, \;\; Y \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2, \;\; Y \bigcap V_1 \bigcap V_2 = \emptyset.$$

Thus Y is disconnected. In particular, only the empty set and the single point sets are connected subsets of X. Thus the above Lemma does not hold for $\mathbb{R}_\ell$. This example also shows that on the same (non-empty) set, connected sets may change drastically if we change the topology on the set (even comparable topologies).

Corollary 7.9. *A real valued continuous function $f : X \longrightarrow \mathbb{R}$ from a non-empty connected space X satisfies the Intermediate Value Property: Let $a, b \in X$ and assume $f(a) < f(b)$. Then for any $r \in (f(a), f(b))$, $f(c) = r$ for some $c \in X$.*

Proof. By Lemma 7.8, $f(X) \subseteq \mathbb{R}$ is an interval. Thus $[f(a), f(b)] \subseteq f(X)$. Hence there is $c \in X$ such that $f(c) = r$. $\square$

Exercise 7.10. *(1.)* Given example of a function $f : X \longrightarrow \mathbb{R}$, for some non-empty connected topological space X, that satisfies Intermediate Value Property but not continuous.

(2.) Given example of a non-empty disconnected topological space X and a continuous function $f : X \longrightarrow \mathbb{R}$ that satisfies Intermediate Value Property.

Lemma 7.11. *Let X be a topological space, Y be a connected subset of X and let $Y \subseteq Z \subseteq \overline{Y}$. Then Z is connected. In particular, $\overline{Y}$ is connected.*

Proof. Suppose Z is not connected. Let V_1, V_2 be two open sets in X satisfying $Z \subseteq V_1 \bigcup V_2$, $Z \bigcap V_1 \neq \emptyset \neq Z \bigcap V_2$ and $Z \bigcap V_1 \bigcap V_2 = \emptyset$. Since $Y \subseteq Z \subseteq \overline{Y}$ and $Z \bigcap V_1 \neq \emptyset \neq Z \bigcap V_2$, we also have $Y \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2$. Then by Lemma 7.4, Y is disconnected, a contradiction. □

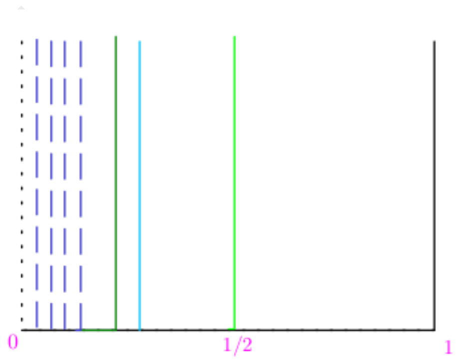
Lemma 7.12. *Let X be a topological space and let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of connected subsets of X such that $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected.*

Proof. Let $\{0, 1\}$ be the two point discrete space and let $f : \bigcup_{\alpha \in \Lambda} A_\alpha \longrightarrow \{0, 1\}$ be a continuous map. Then for each $\alpha \in \Lambda$, $f|_{A_\alpha}$ is constant by Corollary 7.7. Fix $\beta \in \Lambda$. Since $A_\alpha \bigcap A_\beta \neq \emptyset, \forall \alpha \in \Lambda$, $f(A_\alpha) = f(A_\beta), \forall \alpha \in \Lambda$. Hence f is constant. By Corollary 7.7, $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected. □

Corollary 7.13. *Let X be a topological space, $\{A_\alpha : \alpha \in \Lambda\}$ be a family of connected subsets of X and let $B \subseteq X$ be a connected subset such that $B \bigcap A_\alpha \neq \emptyset$. Then $B \bigcup \left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)$ is compact.*

Proof. For each $\alpha \in \Lambda$, set $Y_\alpha = B \bigcup A_\alpha$. Then $\bigcup_{\alpha \in \Lambda} Y_\alpha = B \bigcup \left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)$. By Lemma 7.12, each Y_α is connected. Moreover, $\emptyset \neq B \subseteq \bigcap_{\alpha \in \Lambda} Y_\alpha$. Again using Lemma 7.12, we get $\bigcup_{\alpha \in \Lambda} Y_\alpha$ is connected. □

For each $n \in \mathbb{N}$, set $A_n := \{(1/n, y) : 0 \leq y \leq 1\} \subseteq \mathbb{R}^2$. Then each A_n is a connected subset of $\mathbb{R}^2$. Let $L := \{(x, 0) : 0 \leq x \leq 1\} \subseteq \mathbb{R}^2$. Then L is also a connected subset of $\mathbb{R}^2$ and $L \bigcap A_n \neq \emptyset, \forall n \in \mathbb{N}$. By Corollary 7.13, $Y := L \bigcup \left(\bigcup_{n \in \mathbb{N}} A_n\right)$ is a connected subset of $\mathbb{R}^2$. Thus the closure $\overline{Y} = \{(x, y) : x = 0 \text{ or } x = 1/n, n \in \mathbb{N} \text{ and } 0 \leq y \leq 1\} \bigcup L$ is a connected subset of $\mathbb{R}^2$. The subspace $\overline{Y}$ is known as *Comb space*.



Lemma 7.14. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty connected topological spaces and let $x, y \in X := \prod_{\alpha \in \Lambda} X_\alpha$ be such that x and y differs in only finitely many coordinates. Then there is a connected set $A \subseteq X$ such that $x, y \in A$.*

Proof. To avoid triviality, we assume that $x \neq y$. We will use induction on the number of coordinates in which x and y differs. Suppose x and y differs in only one coordinate, say $\beta \in \Lambda$. Define $A = \prod_{\alpha\in\Lambda} A_\alpha \subseteq X$ by $A_\beta = X_\beta$ and for $\alpha \neq \beta$, $A_\alpha = \{x_\alpha\} = \{y_\alpha\}$. Then $x, y \in A$. Since A is homeomorphic to X_β, it is connected.

Suppose the result is true if x and y differs in $n \geq 1$ coordinates. Now assume x and y differs in $n+1$ coordinates, say $\beta_1, \cdots, \beta_{n+1} \in \Lambda$. Define

$$z : \Lambda \longrightarrow \bigcup_{\alpha\in\Lambda} X_\alpha, \quad z_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in \Lambda \setminus \{\beta_{n+1}\} \\ y_{\beta_{n+1}} & \text{if } \alpha = \beta_{n+1} \end{cases}$$

Then $z \in X$ has the following property: x and z differs in only one coordinate and y and z differs in n coordinates. By Induction hypothesis, there is are connected sets $A_1, A_2 \subseteq X$ such that $x, z \in A_1$ and $y, z \in A_2$. Since $A_1 \bigcap A_2 \neq \emptyset$, by Lemma 7.12, $A_1 \bigcup A_2$ is connected subset of X containing both x and y. $\square$

Proposition 7.15. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. Then the product space $\prod_{\alpha\in\Lambda} X_\alpha$ is connected if and only if each X_α is connected.*

Proof. Let $p_\beta : \prod_{\alpha\in\Lambda} X_\alpha \longrightarrow X_\beta$ be the β-th coordinate projection. If $\prod_{\alpha\in\Lambda} X_\alpha$ is connected, then by Lemma 7.6, $X_\beta = p_\beta\left(\prod_{\alpha\in\Lambda} X_\alpha\right)$ is connected, for each $\beta \in \Lambda$.

Conversely assume that each X_α is connected. Fix $y^0 \in X$ and set

$$D := \left\{ x \in \prod_{\alpha\in\Lambda} X_\alpha \ : \ \{\alpha \in \Lambda : x_\alpha \neq y^0_\alpha\} \text{ is finite} \right\}.$$

Then D is dense in $\prod_{\alpha\in\Lambda} X_\alpha$. Let

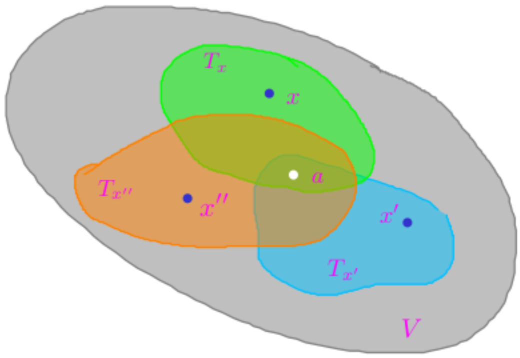
$$A := \left\{ B \subseteq \prod_{\alpha\in\Lambda} X_\alpha \ : \ y^0 \in B \text{ and } B \text{ is connected} \right\}.$$

By Lemma 7.12, A is a connected subset of $\prod_{\alpha\in\Lambda} X_\alpha$. Let $x \in D$ and assume $x \neq y^0$. Then x and y^0 differs in only finitely many coordinates. By Lemma 7.14, there is a connected set $C \subseteq \prod_{\alpha\in\Lambda} X_\alpha$ such that $x, y^0 \in C$. Thus $x \in C \subseteq A$ and consequently, $D \subseteq A$. Now by Lemma 7.11, $\overline{A} = \overline{D} = \prod_{\alpha\in\Lambda} X_\alpha$ is connected. $\square$

Let X be a non-empty topological space. Define a relation $\sim$ on X as follows: $x \sim y$ if and only if there is a connected set $A \subseteq X$ such that $x, y \in A$. We claim that $\sim$ is an equivalence relation on X:

- Clearly $x \sim x$ holds for every $x \in X$. If for $x, y \in X$ we have $x \sim y$, then $y \sim x$. Thus $\sim$ is both reflexive and symmetric.
- Let $x, y, z \in X$ be such that $x \sim y$ and $y \sim z$ holds. Let $A_1, A_2 \subseteq X$ be connected sets such that $x, y \in A_1$ and $y, z \in A_2$. Since $A_1 \bigcap A_2 \neq \emptyset$, by Lemma 7.12, $A_1 \bigcup A_2$ is a connected subset of X containing both x and z. Hence $x \sim z$. Consequently $\sim$ is transitive.

Let $\emptyset \neq V \subseteq X$ be an equivalence class of $\sim$. Fix $a \in V$ and let $x \in V$. Then there is a connected set $T_x \subseteq X$ such that $a, x \in T_x$. Now for any $y \in T_x$, we have $a \sim y$ and hence $y \in V$. Thus $V = \bigcup_{x\in V} T_x$. Since $\bigcap_{x\in V} T_x \neq \emptyset$, by Lemma 7.12, V is connected. By Lemma 7.11, $\overline{V}$ is connected. Thus for any $z \in \overline{V}$, $a \sim z$. Hence we must have $V = \overline{V}$. In other words, the equivalence classes of $\sim$ are closed and connected subsets of X. If Z is connected subset of X containing a, then we must have $Z \subseteq V = \overline{V}$. Thus the equivalence classes of $\sim$ are precisely the maximal (with respect to set inclusion) connected subsets of X.



Definition 7.16. The maximal (with respect to set inclusion) connected subsets of a non-empty topological space X are called *connected components* of X.

Lemma 7.17. *Let X be a non-empty topological space. Then the connected components of X are closed in X.*

If X is connected, then X is the only connected component of X. Recall, a non-empty open subset G of the Euclidean space $\mathbb{R}$ can be written as a union of countably many disjoint open intervals. These open intervals are the connected components of the subspace G. Let $X = \mathbb{R}_\ell$ be the lower limit topological space. Then single point sets are the connected components of X.

Exercise 7.18. *(1.)* Let X be a non-empty topological space and let $a \in X$. Set

$$T_a := \bigcup \{V \subseteq X \,:\, a \in V \text{ and } V \text{ is connecetd}\}\,.$$

Prove that T_a is a connected component of X containing a.

(2.) Let $f : X \longrightarrow Y$ be a homeomorphism between two non-empty topological spaces. Prove that f induces a one-one correspondence between the set of all connected components of X and the set of all connected components of Y such that if a connected component $T \subseteq X$ corresponds to the connected component $S \subseteq Y$, then $f(T) \subseteq S$ and the restriction map $f : T \longrightarrow S$ is a homeomorphism.

Example 7.19. *(1.)* Let $X = (0,1) \subseteq \mathbb{R}$ and $Y = [0,1] \subseteq \mathbb{R}$ be equipped with subspace topologies. Suppose there is a homeomorphism $f : X \longrightarrow Y$. Let $c = f^{-1}(0) \in X$. Then the restriction map $f : X \setminus \{c\} \longrightarrow Y \setminus \{0\}$ is also a homeomorphism. Notice that $X \setminus \{c\}$ is disconnected but $Y \setminus \{0\}$ is connected. Hence no such homeomorphism $f : X \longrightarrow Y$ exists. By a similar argument, we can show that $(0,1) \subseteq \mathbb{R}$ is not homeomorphic to $(0,1] \subseteq \mathbb{R}$. Since any two non-empty bounded closed (resp. open, half-open) intervals are homeomorphic in subspace topology of $\mathbb{R}$, we conclude that a bounded open interval can not be homeomorphic to a bounded closed interval or a bounded half-open interval (as subspaces of $\mathbb{R}$).

(2.) [2, Chapter V, Section 3, Example 6] Consider two subspaces of $\mathbb{R}$:

$$\begin{aligned} X &:= \bigcup_{n \geq 0} \Big((3n, 3n+1) \bigcup \{3n+2\} \Big) \\ &:= (0,1) \bigcup \{2\} \bigcup (3,4) \bigcup \{5\} \bigcup \cdots \bigcup (3n, 3n+1) \bigcup \{3n+2\} \bigcup \cdots \end{aligned}$$

and

$$
\begin{aligned}
Y &:= (0,1] \bigcup \Big[\bigcup_{n\geq 1} \Big((3n, 3n+1) \bigcup \{3n+2\} \Big) \Big] \\
&:= (0,1] \bigcup (3,4) \bigcup \{5\} \bigcup \cdots \bigcup (3n, 3n+1) \bigcup \{3n+2\} \bigcup \cdots
\end{aligned}
$$

Since the connected component $(0,1]$ of Y is not homeomorphic any of the connected components of X, X and Y are not homeomorphic. The map

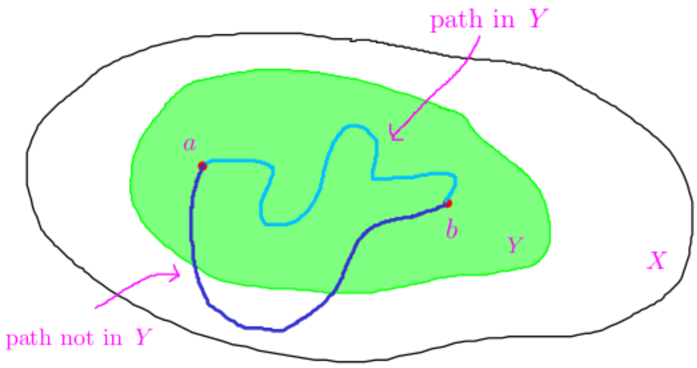
$$
f : X \longrightarrow Y, \;\; f(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}
$$

is a continuous bijection. Define

$$
g : Y \longrightarrow X, \;\; g(x) = \begin{cases} x/2 & \text{if } x \in (0,1] \\ (x-2)/2 & \text{if } x \in (3,4) \\ x-3 & \text{otherwise} \end{cases}
$$

Then g is also continuous bijection. Thus there are non-empty topological spaces X and Y and two continuous bijection $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ but X and Y are not homeomorphic.

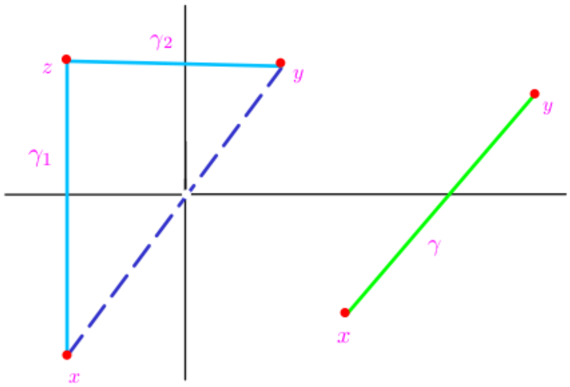
Definition 7.20. Let X be a non-empty topological space and let $\emptyset \neq Y \subseteq X$. We say Y is *path connected* if given any two point $a, b \in Y$, there is a continuous map $\gamma : [0,1] \longrightarrow Y$ such that $\gamma(0) = a$ and $\gamma(1) = b$. The continuous map γ and its image in X are both called a *path* in Y joining the points a and b.



Example 7.21. *(1.)* For any $n \in \mathbb{N}$, the Euclidean space $\mathbb{R}^n$ is path connected: for any two points $x, y \in \mathbb{R}^n$, define $\gamma : [0,1] \longrightarrow \mathbb{R}^n$ by $\gamma(t) = (1-t)x + ty$. Then γ is a path in $\mathbb{R}^n$.

(2.) Let $n \geq 2$ and consider the subspace $X = \mathbb{R}^n \setminus \{0\}$ of the Euclidean space $\mathbb{R}^n$. Let $x, y \in X$. If the origin does not lie on the line segment in $\mathbb{R}^n$ joining x and y, then the path γ as defined above is path in X joining x and y. Now assume the origin lies on the line segment in $\mathbb{R}^n$ joining x and y. Choose a point $0 \neq z \in \mathbb{R}^n$ that does not lie on the line in $\mathbb{R}^n$ joining x and y. Define

$$
\gamma_1, \gamma_2 : [0,1] \longrightarrow X, \;\; \gamma_1(t) = (1-t)x + tz \;\text{ and }\; \gamma_1(s) = (1-s)z + sy\,.
$$



Then γ_1 (resp. γ_2) is a path in X joining x and z (resp. joining z and y). Now define

$$\gamma : [0,1] \longrightarrow X, \;\; \gamma(t) \; = \; \begin{cases} \gamma_1(2t) & \text{if } \; x \in [0,1/2] \\ \gamma_2(2t-1) & \text{if } \; x \in [1/2,1] \end{cases}$$

Then γ is a path in X joining x and y. Hence X is path connected.

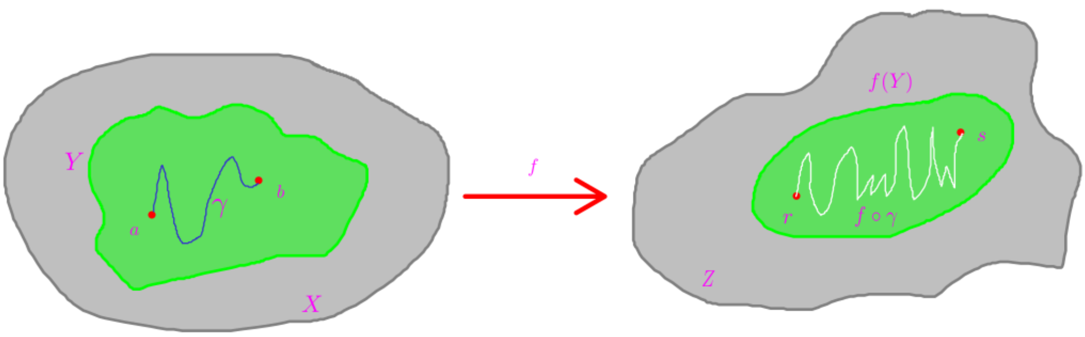
Exercise 7.22. Let X be a non-empty topological space. Prove that X is path connected if and only if there is $a \in X$ such that every $x \in X$ can be joined to a via a path in X.

Lemma 7.23. *Let $\emptyset \neq Y$ be a path connected subset of a topological space X.*

(i) Then Y is connected.

(ii) If Z is a non-empty topological space and $f : X \longrightarrow Z$ is a continuous map, then $f(Y)$ is a path connected subset of Z.

Proof. (i) Suppose Y is not connected. Then there are open sets V_1, V_2 in X such that $Y \subseteq V_1 \bigcap V_2$, $Y \bigcap V_1 \neq \emptyset \neq Y \bigcap V_2$ and $Y \bigcap V_1 \bigcap V_2 = \emptyset$. Let $a \in Y \bigcap V_1$ and $b \in Y \bigcap V_2$. Then there is a continuous map $\gamma : [0,1] \longrightarrow Y$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Let $W_1 = \gamma^{-1}(Y \bigcap V_1)$ and $W_2 = \gamma^{-1}(Y \bigcap V_2)$. Then W_1, W_2 are non-empty disjoint open sets in $[0,1]$ and we can write $[0,1] = W_1 \bigcup W_2$, a contradiction since $[0,1]$ is connected. Hence Y is connected.



(ii) Let $r, s \in f(Y)$ and choose $a, b \in Y$ such that $f(a) = r$ and $f(s) = s$. Since Y is path connected, there is a continuous map $\gamma : [0,1] \longrightarrow Y$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Then $f \circ \gamma : [0,1] \longrightarrow f(Y)$ is a continuous functions satisfying $f \circ \gamma(0) = r$ and $f \circ \gamma(1) = s$. Hence $f(Y)$ is path connected. $\square$

For $n \in \mathbb{N}$, consider the map

$$f : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{S}^n, \ \ x \mapsto \frac{x}{||x||}.$$

Then f is continuous and surjective. Since $\mathbb{R}^{n+1} \setminus \{0\}$ is path connected, by Lemma 7.23, the n-sphere $\mathbb{S}^n$ is path connected and connected.

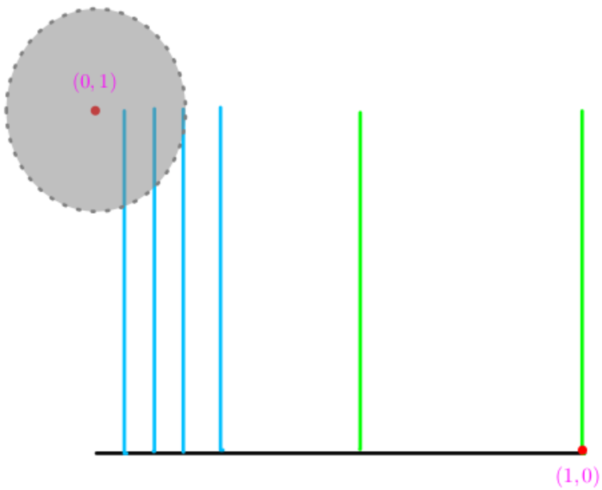
Example 7.24. *(1.)* Consider the following subspaces of $\mathbb{R}^2$:

- for each $n \in \mathbb{N}$, $A_n := \{(1/n, y) \subseteq \mathbb{R}^2 \, : \, 0 \leq y \leq 1\}$ and $L := [0,1] \times \{0\}$;
- $Y := L \bigcup \big(\bigcup_{n \in \mathbb{N}} A_n \big)$.

By Exercise 7.22 and Lemma 7.23, Y is both a connected and a path connected subset of $\mathbb{R}^2$. Let $X = Y \bigcup \{(0,1)\}$. Since $Y \subseteq X \subseteq \overline{Y}$, X is connected. Let $\gamma : [0,1] \longrightarrow X$ be a continuous map such that $\gamma(0) = (0,1)$ and $\gamma(1) = (1,0)$. Let $t_0 = \sup \{t \in (0,1) : \gamma(t) = (0,1)\}$. Since $\gamma^{-1}\{(0,1)\}$ is closed, $\gamma(t_0) = (0,1)$. Choose a real number $\delta > 0$ such that

$$\gamma[t_0, t_0 + \delta) \ \subseteq \ B\big((0,1); 1/4\big) \bigcap X \ \implies \ \gamma(t_0, t_1] \ \subseteq \ \bigcup_{n \in \mathbb{N}} A_n, \ \text{ for some } \ t_1 \in (t_0, t_0 + \delta).$$

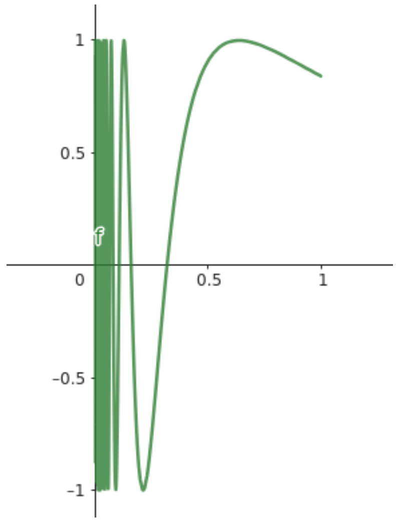
Let $p_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the first coordinate projection. Then the image of the continuous map $p_1 \circ \gamma : (t_0, t_1] \longrightarrow \mathbb{R}$ is contained in the set $\{1/n : n \in \mathbb{N}\}$. Since $\{1/n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is a discrete subspace, we must have $p_1 \circ \gamma(t_0, t_1] = \{1/m\}$, for some $m \in \mathbb{N}$. Then the image of the restriction map $p_1 \circ \gamma : [0, t_1] \longrightarrow \mathbb{R}$ is a two points subspace of $\mathbb{R}$, a contradiction. Hence no such path γ exists. Thus X is not path connected.



(2.) Consider the subspace $Y = \{(x, \sin(1/x)) \ : \ x \in (0,1]\} \subseteq \mathbb{R}^2$. Since Y is the continuous image of the map $f : (0,1] \longrightarrow \mathbb{R}^2, t \mapsto (t, \sin(1/t))$, Y is connected. Thus $\overline{Y}$ is also connected subset of $\mathbb{R}^2$. We claim that $\overline{Y} = Y \bigcup \big(\{0\} \times [-1,1]\big)$. First notice that, if $(x,y) \in \mathbb{R}^2$ with $|y| > 1$ or $x < 0$ or $x > 1$, then $(x,y) \notin \overline{Y}$. Thus any $(x,y) \in \overline{Y} \setminus Y$ must satisfy $|y| \leq 1$ and $0 \leq x \leq 1$. Define a map

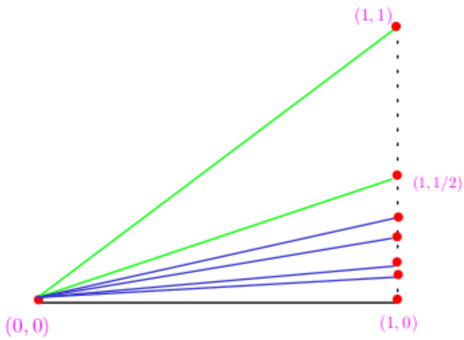
$$g : (0,1] \times [-1,1] \longrightarrow \mathbb{R}, \ \ g(s,t) \ = \ t - \sin(1/s) \, .$$

Then g is continuous and $g^{-1}\{0\} = Y$. Moreover Y is closed in $(0,1] \times [-1,1]$ and hence $\overline{Y} \subseteq [0,1] \times [-1,1]$. Thus $\overline{Y} \setminus Y \subseteq \{0\} \times [-1,1]$. Let $(0, y_0) \in \{0\} \times [-1,1]$, $\varepsilon > 0$ be a real number and let B_ε denotes the open ball centered at $(0, y_0)$ of radius ε. Choose $n \in \mathbb{N}$ such that $1/n\pi < \varepsilon$. Then $(x, y_0) \in B_\varepsilon$, for all $x \in [-1/n\pi, 1/n\pi]$. Since the map $\sigma : [1/(n+2)\pi, 1/n\pi] \longrightarrow [-1,1], s \mapsto \sin(1/s)$ is surjective, there is $x_0 \in [1/(n\pi+2), 1/n\pi]$ such that $\sigma(x_0) = y_0$. Hence $(x_0, y_0) \in Y \bigcap (B_\varepsilon \setminus \{0\})$. Since this is true for every real number $\varepsilon > 0$, $(0, y_0) \in \overline{Y}$. Consequently, $\overline{Y} = Y \bigcup \big(\{0\} \times [-1,1]\big)$. The subspace $\overline{Y}$ is called a *topologist's sine curve.*



Suppose there is a continuous map $\gamma : [0,1] \longrightarrow \overline{Y}$ such that $\gamma(0) = (0,0)$ and $\gamma(1) = (1/\pi, 0)$. Let $t_0 = \sup\{t \in [0,1] : \gamma(t) = (0,0)\}$. Then $\gamma(t_0) = (0,0)$. Let $p_1, p_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the first and second coordinate projections, respectively. Since $p_2 \circ \gamma$ is continuous on $[0,1]$, there is a real number $\delta > 0$ such that $|p_2 \circ \gamma(t)| < 1$ for every $t \in [t_0, t_0 + \delta)$. Since $t_0 < 1$, there is $t_1 \in (t_0, t_0 + \delta)$ such that $\ell := p_1 \circ \gamma(t_1) > 0$. Now for all large $n \in \mathbb{N}$, $2/n\pi \in [0, \ell]$. Let $t', t'' \in (t_0, t_1)$ be such that $p_1 \circ \gamma(t') = 2/(4m+1)\pi$ and $p_1 \circ \gamma(t'') = 2/(4m-1)\pi$, for some $m \in \mathbb{N}$. Then $p_2 \circ \gamma(t') = 1$ and $p_2 \circ \gamma(t'') = -1$ since $p_2 \circ \gamma(t) = \sin(1/p_1 \circ \gamma(t))$, if $p_1 \circ \gamma(t) \neq 0$. Thus $p_2 \circ \gamma : [t_0, t_1] \longrightarrow [-1, 1]$ is a surjective map. This contradicts $t_1 \in (t_0, t_0 + \delta)$. Hence no such continuous map γ exists and $\overline{Y}$ is not path connected. This example also shows that closure of a path connected space need not be path connected.

Exercise 7.25. *(1.)* For each $n \in \mathbb{N}$, let $A_n \subseteq \mathbb{R}^2$ be the line segment joining the points $(0,0)$ and $(1, 1/n)$ and let $Y = \bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{R}^2$.



(i) Prove that Y is path connected.

(ii) Prove that $\overline{Y} = Y \bigcup \{(x, 0) : 0 \leq x \leq 1\}$.

(iii) Let $X = Y \bigcup \{(1,0)\}$. Prove that X is not path connected.

(2.) Let $X = \mathbb{R}^2 \setminus \{(0,0)\} \subseteq \mathbb{R}^2$ be equipped with subspace topology. For each $n \in \mathbb{N}$, set $A_n = \{(1/n, y) : 0 \leq y \leq 1\}$ and let $L = (0,1] \times \{0\}$. Set $Y = \left(\bigcup_{n \in \mathbb{N}} A_n\right) \bigcup L \subseteq X$.

(i) Prove that $\overline{Y} = Y \bigcup (\{0\} \times (0,1])$.

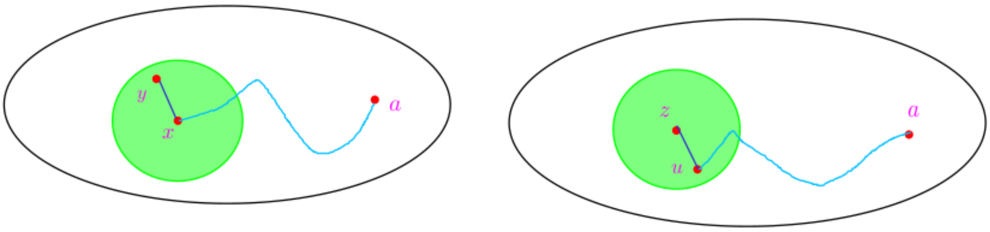
(ii) Prove that given any $(0, y) \in \{0\} \times (0,1]$, there is no continuous map $\gamma : [0,1] \longrightarrow \overline{Y}$ such that $\gamma(0) = (0, y)$ and $\gamma(1) = (1, 0)$.

This exercise shows that closure of a path connected set need not be path connected.

Proposition 7.26. *A non-empty open connected subset of the Euclidean space $\mathbb{R}^n$ is path connected.*

Proof. Let $V \subseteq \mathbb{R}^n$ be a non-empty open connected set. Fix $a \in V$ and set

$$W := \{x \in V \,:\, x \text{ and } a \text{ can be joinned by a path in } V\} \text{ and } E := V \setminus W.$$



Since $a \in W$, $W \neq \emptyset$. Let $x \in W$. Then there is $\delta \in \mathbb{R}_{>0}$ such that $B(x;\delta) \subseteq V$. Let $y \in B(x;\delta)$. Then y and a can be joined by a path in V: consider the line segment joining y and x in the open ball $B(x;\delta)$ together with a path in V joining x and a. Thus $B(x;\delta) \subseteq V$. Hence W is an open set in V.

Let $z \in E$. Then there is $\delta' \in \mathbb{R}_{>0}$ such that $B(z;\delta') \subseteq V$. Let $u \in B(z;\delta')$. Suppose u and a can be joined by a path in V. Adjoining this path with the line segment joining u and z in $B(z;\delta')$, we get a path in V joining z and a. This contradicts that $z \in E$. Hence $B(z;\delta') \subseteq E$ and E is an open set in V.

Thus $V = W \bigcup E$, both W and E are open sets in V and $W \bigcap E = \emptyset$. Since V is connected and $W \neq \emptyset$, $E = \emptyset$. Thus $W = V$ and hence V is path connected. $\square$

Proposition 7.27. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. Then the product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ is path connected if and only if each X_α is path connected.*

Proof. For $\alpha \in \Lambda$, let $p_\alpha : X \longrightarrow X_\alpha$ be the α-th coordinate projection. If X is path connected, by Lemma 7.23, $X_\alpha = p_\alpha(X)$ is path connected.

Conversely assume that each X_α is path connected. Let $x, y \in X$. Then for each $\alpha \in \Lambda$, there is a continuous map $\gamma_\alpha : [0,1] \longrightarrow X_\alpha$ such that $\gamma_\alpha(0) = x_\alpha$ and $\gamma_\alpha(1) = y_\alpha$. Define $\gamma : [0,1] \longrightarrow X$ by $\gamma(t)(\alpha) = \gamma_\alpha(t)$. Then γ is continuous, $\gamma(0) = x$ and $\gamma(1) = y$. Hence X is path connected. $\square$

Let X be a non-empty topological space and let $a \in X$. Set

$$T_a := \{x \in X \,:\, x \text{ and } a \text{ can be joined by a path in } X\}.$$

Then T_a is a path connected subset of X containing a. Moreover, T_a is a maximal (with respect to set inclusion) path connected subset of X containing a. Let $Y \subseteq X$ be a path connected subset such that $T_a \bigcap Y \neq \emptyset$. Let $x \in T_a \bigcap Y$ and $y \in Y$. Then there is a path in X joining a and x and a path in X joining x and y. Combining, we get a path in X joining a and y. Hence $Y \subseteq T_a$. Thus the collection $\{T_a : a \in X\}$ forms a partition of X.

Definition 7.28. Let X be a non-empty topological space. The maximal (with respect to set inclusion) path connected subsets of X are called *path connected components* of X.

Let X be a non-empty topological space and let T be a path connected component of X. Since T is connected, T is contained in some connected component of X. Thus path connected components of X induces a partition of connected components of X.

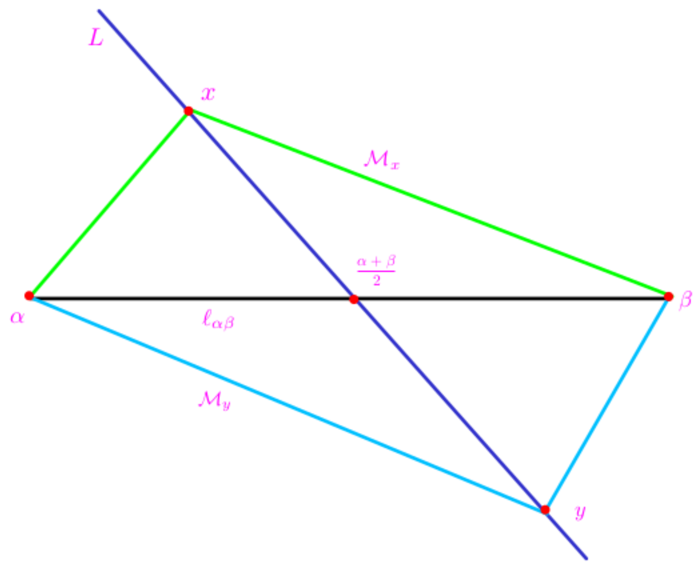
In general, path connected components and connected components are not same. For example, the topologist's sine curve is connected so it has only one connected component.

But it has two path connected components: $\{(t, \sin(1/t)) : t \in (0,1]\}$ and $\{0\} \times [-1,1]$. In particular, a path connected component need not be closed.

Exercise 7.29. Let X be a non-empty topological space and let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of path connected subsets of X such that $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \emptyset$. Is $\bigcup_{\alpha\in\Lambda} A_\alpha$ path connected?

Proposition 7.30. *Let $n \geq 2$ be an integer and let $A \subset \mathbb{R}^n$ be a countable subset. Then $\mathbb{R}^n \setminus A$ is a path connected (and hence a connected) subset of $\mathbb{R}^n$.*

Proof. Let $\alpha, \beta \in \mathbb{R}^n \setminus A$ and $\alpha \neq \beta$. Let L be a line in $\mathbb{R}^2$ that intersects the line segment $\ell_{\alpha\beta}$ joining α and β at exactly the point $(\alpha+\beta)/2$.



For any $x \in L \setminus \{(\alpha+\beta)/2\}$, set

$$\psi_x : [0,1] \longrightarrow \mathbb{R}^2, \;\; \psi_x(t) = \begin{cases} (1-2t)\alpha + 2tx & \text{if } t \in [0,1/2] \\ (2-2t)x + (2t-1)\beta & \text{if } t \in [1/2,1] \end{cases}$$

Let $\mathscr{M}_x := \mathrm{Im}(\psi_x)$. Then $\mathscr{M}_x \subseteq \mathbb{R}^2$ is a path joining α and β. Note that $\ell_{\alpha\beta} \bigcap \mathscr{M}_x = \{\alpha, \beta\}$, for every $x \in L \setminus \{(\alpha+\beta)/2\}$ and for any $x, y \in L \setminus \{(\alpha+\beta)/2\}$ with $x \neq y$,

$$(A \cap \mathscr{M}_x) \cap (A \cap \mathscr{M}_y) \;=\; A \cap \mathscr{M}_x \cap \mathscr{M}_y \;=\; \emptyset.$$

We claim that there is $x \in L \setminus \{(\alpha+\beta)/2\}$ such that $\mathscr{M}_x \bigcap A = \emptyset$. If not, then for each $x \in L \setminus \{(\alpha+\beta)/2\}$, choose $a_x \in \mathscr{M}_x \bigcap A$. Thus the map

$$L \setminus \{(\alpha+\beta)/2\} \longrightarrow A, \;\; x \mapsto a_x$$

is injective, a contradiction since A is countable. So there is $x \in L \setminus \{(\alpha+\beta)/2\}$ such that $\mathscr{M}_x \bigcap A = \emptyset$, i.e. $\mathscr{M}_x$ is a path in $\mathbb{R}^n \setminus A$ joining α and β. Thus $\mathbb{R}^n \setminus A$ is a path connected (and hence a connected) subset of $\mathbb{R}^n$. $\square$

Exercise 7.31. Let $n \geq 2$ be an integer and let $f = \sum_{i=1}^n c_i x_i \in \mathbb{R}[x_1, \cdots, x_n]$ be a non-zero polynomial. Let $H = \{(a_1, \cdots, a_n) : f(a_1, \cdots, a_n) = 0\}$. Prove that H is a path connected subspace of $\mathbb{R}^n$. Does the result hold if we choose f to be a non-constant polynomial of degree at least two?

Proposition 7.32. *Let $n \geq 2$ be an integer and let $\mathscr{M} \subseteq \mathbb{R}^n$ be an $\mathbb{R}$-linear vector subspace with $\dim_{\mathbb{R}} \mathscr{M} \leq n-2$. Then $\mathbb{R}^n \setminus \mathscr{M}$ is a path connected subset of $\mathbb{R}^n$.*

Proof. If $n = 2$, then $\mathscr{M} = \{(0,0)\}$. In this case, $\mathbb{R}^n \setminus \mathscr{M}$ is path connected. So assume $n \geq 3$. Let $\{v_1, \cdots, v_{n-2}\}$ be an $\mathbb{R}$-basis of $\mathscr{M}$. Extend this to an $\mathbb{R}$-basis $\{v_1, \cdots, v_{n-2}, v_{n-1}, v_n\}$ of $\mathbb{R}^n$. Let $\alpha, \beta \in \mathbb{R}^n \setminus \mathscr{M}$. Then for some $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$, we have $\alpha = \sum_{i=1}^n a_i v_i$ and $\beta = \sum_{i=1}^n b_i v_i$. Since $\alpha, \beta \notin \mathscr{M}$, we have $(a_{n-1}, a_n), (b_{n-1}, b_n) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ be a path such that $\gamma(0) = (a_{n-1}, a_n)$ and $\gamma(1) = (b_{n-1}, b_n)$. Since $n - 2 \geq 1$, there is a path $\mu : [0,1] \longrightarrow \mathbb{R}^{n-2}$ such that $\mu(0) = (a_1, \cdots, a_{n-2})$ and $\mu(1) = (b_1, \cdots, b_{n-2})$. Set $\gamma_i = p_i \circ \gamma$ and $\mu_j = q_j \circ \mu$ where $p_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (resp. $q_j : \mathbb{R}^{n-2} \longrightarrow \mathbb{R}$) be the i-th coordinate (resp. j-th coordinate) projection, $i \in \{1,2\}$ (resp. $j \in \{1, \cdots, n-2\}$). Define

$$\delta : [0,1] \longrightarrow \mathbb{R}^n, \quad \delta(t) = \sum_{j=1}^{n-2} \mu_j(t) v_j + \gamma_1(t) v_{n-1} + \gamma_2(t) v_n.$$

Then δ is a continuous map with $\delta(0) = \alpha$ and $\delta(1) = \beta$. Since γ is a path in $\mathbb{R}^2 \setminus \{(0,0)\}$, δ is a path in $\mathbb{R}^n \setminus \mathscr{M}$. Thus $\mathbb{R}^n \setminus \mathscr{M}$ is a path connected subset of $\mathbb{R}^n$. $\square$

Definition 7.33. Let X be a non-empty topological space and let $a \in X$. We say X is *locally connected at the point a* if for any neighbourhood W of a, there is an open connected neighbourhood V_a of a such that $V_a \subseteq W$. We say X is *locally connected* if it is locally connected at every point.

Example 7.34. *(1.)* The Euclidean space $\mathbb{R}^n$ is locally connected as the open balls $B(x;r)$ are connected.

(2.) Any non-empty discrete space is locally connected as the single point sets are connected open sets. By choosing a set with at least two points, we get an example of a locally connected topological space that is not connected.

(3.) Any infinite set (resp. uncountable set) with co-finite topology (resp. co-countable topology) is locally connected.

Lemma 7.35. *A non-empty topological space X is locally connected if and only if the connected components of each open subset are open.*

Proof. Suppose X is locally connected. Let $\emptyset \neq G$ be an open subset of X and let A be a connected component of G. Let $a \in A$. Then there is an open connected neighbourhood V_a of a such that $V_a \subseteq G$. Then $A \bigcup V_a$ is a connected subset of G and hence $A = A \bigcup V_a$. Thus $a \in V_a \subseteq A$ so that $A = \bigcup_{a \in A} V_a$ is open.

Conversely assume the connected components of each open subset are open. Let $x \in X$ and let W be a neighbourhood of x. Let T_x be the connected component of $\mathrm{Int}(W)$ containing x. Then T_x is open and $x \in T_x \subseteq W$. Hence X is locally connected at x. $\square$

Continuous image of a locally connected space need not be locally connected. For example, consider the subspaces $X = \mathbb{N} \bigcup \{0\}$ and $Y = \{1/n : n \in \mathbb{N}\} \bigcup \{0\}$ of $\mathbb{R}$. Define $f : X \longrightarrow Y$ by $f(0) = 0$ and $f(n) = 1/n$, for $n \in \mathbb{N}$. Then f is a continuous bijection. X is locally connected since it is a discrete space. We claim that Y is not locally connected. Notice that for any $n \in \mathbb{N}$, $\{1/n\}$ is a connected open subset of Y. Let V be a neighbourhood of 0. Then for all but finitely many $n \in \mathbb{N}$, $\{1/n\} \subseteq V$. Thus there is no open connected neighbourhood T_0 of 0. satisfying $0 \in T_0 \subseteq V$.

Exercise 7.36. Let X be a locally connected topological space and let $f : X \longrightarrow Y$ be a continuous open surjection. Prove that Y is locally connected.

Lemma 7.37. *Let X be a locally connected topological space and let $f : X \longrightarrow Y$ be a continuous closed surjection. Then Y is locally connected.*

Proof. Let $\emptyset \neq W \subseteq Y$ be an open set and let $A \subseteq W$ be a connected component of W. Let $x_0 \in f^{-1}A$ and V_{x_0} be an open connected neighbourhood of x_0 such that $V_{x_0} \subseteq f^{-1}W$. Then $f(x_0) \in f(V_{x_0})$ is a connected subset of W and since A is the connected component of W containing $f(x_0)$, we have $f(x_0) \in f(V_{x_0}) \subseteq A$. Thus $x_0 \in V_{x_0} \subseteq f^{-1}A$ and hence $f^{-1}A = \bigcup_{x_0 \in f^{-1}A} V_{x_0}$ is open in X. Since f is closed, $f(X \setminus f^{-1}A) = Y \setminus A$ is closed in Y and hence A is open in Y. Now the conclusion follows from Lemma 7.35. $\square$

Proposition 7.38. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. Then the product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ is locally connected if and only if each X_α is locally connected and all but finitely many $\alpha \in \Lambda$, X_α is connected.*

Proof. For each $\alpha \in \Lambda$, let $p_\alpha : X \longrightarrow X_\alpha$ be the α-th coordinate projection. Suppose X is locally connected and let $A \subseteq X$ be a connected component of X. By Lemma 7.35, A is open. Let $B \subseteq A$ be a canonical basic open set in X. Then $p_\alpha(B) = X_\alpha$, except possibly for finitely many $\alpha \in \Lambda$. Hence $p_\alpha(A) = X_\alpha$, except possibly for finitely many $\alpha \in \Lambda$. Since A is connected, so is $p_\alpha(A)$. Let $a \in X_\beta$ and let $a \in V \subseteq X_\beta$ be an open set. Then $p_\beta^{-1}V$ is open in X. Choose $x \in X$ such that $x_\beta = a$. Since X is locally connected, there is an open connected set $W_x \subseteq X$ such that $x \in W_x \subseteq p_\beta^{-1}V$. Since p_β is an open map, $a = p_\beta(x) \in p_\beta(W_x) \subseteq V$ and $p_\beta(W_x) \subseteq X_\beta$ is connected. Hence X_β is locally connected.

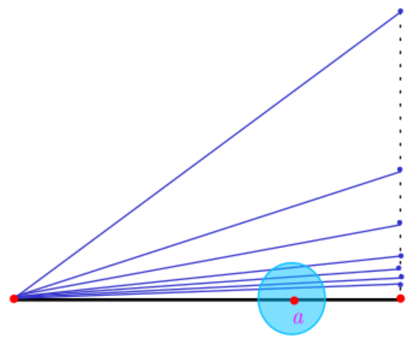
Conversely assume that each X_α is locally connected and let $\Omega \subseteq \Lambda$ be a finite set such that X_α is connected for every $\alpha \in \Lambda \setminus \Omega$. Let $x \in X$ and $\bigcap_{i=1}^n p_{\alpha_i}^{-1} W_{\alpha_i}$ be a canonical basic open set containing x. Then for each $i = 1, \cdots, n$, W_{α_i} is a neighbourhood of x_{α_i}. For each $i = 1, \cdots, n$, choose an open connected set $V_{\alpha_i} \subseteq X_{\alpha_i}$ such that $x_{\alpha_i} \in V_{\alpha_i} \subseteq W_{\alpha_i}$. Let $\Omega \setminus \{\alpha_1, \cdots, \alpha_n\} = \{\beta_1, \cdots, \beta_m\}$. For each $j = 1, \cdots, m$, choose a connected neighbourhood $V_{\beta_j} \subseteq X_{\beta_j}$ of x_{β_j}. Then $\left(\bigcap_{i=1}^n p_{\alpha_i}^{-1} V_{\alpha_i}\right) \bigcap \left(\bigcap_{j=1}^m p_{\beta_j}^{-1} V_{\beta_j}\right) \subseteq \bigcap_{i=1}^n p_{\alpha_i}^{-1} W_{\alpha_i}$ is an open connected neighbourhood of x. Hence X is locally connected. $\square$

Lemma 7.39. *Let X be a non-empty locally connected topological space and let $\sim$ be an equivalence relation on X. Then the quotient space $X/\sim$ is locally connected.*

Proof. Let $q : X \longrightarrow X/\sim$ be the quotient map and let $V \subseteq X/\sim$ be a non-empty open set. By Lemma 7.35, it is sufficient to show that the connected components of V are open. Let A be a connected component of V and let $x \in q^{-1}A$. Let W be the connected component of $q^{-1}V$ containing x. By Lemma 7.35, W is an open subset of X. Since A is a connected component of V, we have $q(x) \in q(W) \subseteq A$. Thus $x \in W \subseteq q^{-1}A$. Consequently, $q^{-1}A$ is an open subset of X. Hence A is open in $X/\sim$. $\square$

Definition 7.40. Let X be a non-empty topological space and let $a \in X$. We say X is *locally path connected at a* if given any neighbourhood W of a, there is an open path connected neighbourhood V_a of a such that $V_a \subseteq W$. We say X is *locally path connected* if it is locally path connected at every point.

Example 7.41. *(1.)* For each $n \in \mathbb{N}$, let A_n be the line segment in $\mathbb{R}^2$ joining the points $(0, 0)$ and $(1, 1/n)$ and set $A_0 = \{(x, 0) : 0 \leq x \leq 1\}$. Then the subspace $Y = \bigcup_{n\geq 0} A_n \subseteq \mathbb{R}^2$ is path connected. Let $a \in A_0$ be a point distinct from the origin and let $B(a; r)$ be an open ball in $\mathbb{R}^2$ not containing the origin. Then $B(a; r) \bigcap Y$ is a neighbourhood of a in Y.



Notice that, for all large $n \in \mathbb{N}$, $T_n := A_n \bigcap B(a;r)$ is a path connected component of $B(a;r) \bigcap Y$ and $T_n \bigcap T_m = \emptyset$ for $m \neq n$. Thus $B(a;r) \bigcap Y$ is not connected. Hence Y is not locally path connected. Moreover, this example also shows that a path connected (or connected) topological space need not be locally connected.

Exercise 7.42. Let X be a locally path connected topological space and let $f : X \longrightarrow Y$ be a continuous closed or open surjection. Then Y is locally path connected connected.

Continuous image of a locally path connected space need not be locally path connected. For example, let $\Gamma \subseteq \mathbb{R}^2$ be the graph of the continuous map $(0,1] \longrightarrow \mathbb{R}, t \mapsto \sin(1/t)$. Set $X = \Gamma \bigcup \{(-2024,0)\}$ and $Y = \Gamma \bigcup \{(0,0)\}$. Define $f : X \longrightarrow Y$ by $f(x) = x, \forall x \in \Gamma$ and $f(-2024,0) = (0,0)$. Then f is a continuous bijection. Notice that X is locally path connected, but Y is not.

Lemma 7.43. *A non-empty topological space X is locally path connected if and only if the path connected components of each open subset are open.*

Proof. Suppose X is locally path connected and let $\emptyset \neq W \subseteq X$ be an open set. Let $T \subseteq W$ be a path connected component and let $a \in T$. Choose a path connected open set V_a such that $a \in V_a \subseteq W$. Then $T \bigcup V_a$ is a path connected subset of W and hence $V_a \subseteq T$. Thus $T = \bigcup_{a \in T} V_a$ is open. The converse follows from definition. $\square$

Lemma 7.44. *Let X be a locally path connected topological space. Then every path connected component of X is clopen. In particular, every path connected component is also a connected component of X.*

Proof. Let $\{T_\alpha : \alpha \in \Lambda\}$ be the collection of all path connected components of X. By Lemma 7.43, each T_α is open. Fix $\beta \in \Lambda$. Then $T_\beta = X \setminus \left(\bigcup_{\alpha \neq \beta} T_\alpha\right)$ is closed. Let A be a connected component of X containing T_β. Since T_β is a clopen set, $T_\beta = A$. $\square$

Corollary 7.45. *Let X be a connected and locally path connected topological space. Then X is path connected.*

Proposition 7.46. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. Then the product space $X = \prod_{\alpha \in \Lambda} X_\alpha$ is locally path connected if and only if each X_α is locally path connected and all but finitely many $\alpha \in \Lambda$, X_α are path connected.*

Proof. Similar to that of Proposition 7.38. $\square$

Exercise 7.47. Let X be a non-empty locally path connected topological space and let $\sim$ be an equivalence relation on X. Prove that quotient space $X/\sim$ is locally path connected.

Let (X, τ) be a topological space and let $\emptyset \neq Y \subseteq X$. We say Y is *irreducible* if it can not be written as $Y = Y_1 \bigcup Y_2$ of two proper subsets of Y, each of which is closed in Y. Otherwise, we say Y is *reducible*. The empty set $\emptyset$ is not considered to be irreducible.

Lemma 7.48. *Let (X, τ) be a topological space and let $\emptyset \neq Y \subseteq X$. Prove that*

(i) X is irreducible if and only if every non-empty open subset of X is dense;

(ii) if X is irreducible, then every non-empty open subset of X is irreducible;

(iii) if Y is irreducible, then so is $\overline{Y}$.

Proof. Left as an exercise. □

Let $X = \mathbb{N}$ and define a topology τ on X by

$$\tau := \{\emptyset,\ X,\ \{1\},\ \{1,2\},\ \{1,2,3\},\ \{1,2,3,4\},\ \cdots\}$$

Since every non-empty open subset of X is dense in X, by Lemma 7.48, X is irreducible. On the other hand, a Hausdorff topological space with more than one point can not be irreducible. Irreducibility of a topological space is much more stronger condition than connectedness. Any irreducible topological space is connected, but the converse is not true. For example, the Euclidean space $\mathbb{R}$ is connected but not irreducible.

Exercise 7.49. *(1.)* Let $f : X \longrightarrow Y$ be a continuous map between two non-empty topological spaces. If $Z \subseteq X$ is irreducible, prove that $f(Z)$ is irreducible in Y.

(2.) Let X be a non-empty topological space and let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X such that $V_\alpha \bigcap V_\beta \neq \emptyset$, for every $\alpha, \beta \in \Lambda$. If each V_α is irreducible, prove that X is irreducible.

(3.) Let X be a topological space and let $\emptyset \neq Y \subseteq X$. Suppose $\overline{Y}$ is irreducible. Is Y irreducible?

(4.) Let X be a non-empty topological space and let $\{A_n : n \in \mathbb{N}\}$ be a family of connected subsets of X such that $A_n \bigcap A_{n+1} \neq \emptyset, \forall n \in \mathbb{N}$. Prove that $\bigcup_{n\in\mathbb{N}} A_n$ is connected.

(5.) Let $n \geq 2$ be an integer and let $A \subseteq \mathbb{R}^n$ be a non-empty connected subset. Are $\mathrm{Int}(A)$ and ∂A connected subset of $\mathbb{R}^n$? If $B \subseteq \mathbb{R}^n$ is a non-empty subset of $\mathbb{R}^n$ such that $\mathrm{Int}(B)$ and ∂B are connected subset of $\mathbb{R}^n$, does this imply that B is connected?

(6.) Let $f : X \longrightarrow Y$ be a homeomorphism between two non-empty topological spaces. For $x \in X$ (resp. $y \in Y$), let $\xi(x, X)$ (resp. $\eta(y, Y)$) be the path connected component of X (resp. of Y) containing x (resp. containing y). Let $\pi_0(X)$ (resp. $\pi_0(Y)$) be the set of all path connected components of X (resp. of Y). Prove that the map $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$ define by $\xi(x, X) \mapsto \eta(f(x), Y)$ is a bijection.

(7.) Let X be a non-empty topological space. A point $x \in X$ is called a *cut point* if X is connected but $X \setminus \{x\}$ is disconnected. Let $f : X \longrightarrow Y$ be a homeomorphism. Prove that f induces a bijection between the set of all cut points of X and the set of all cut points of Y. Conclude that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n, n \geq 2$.

(8.) For each $n \in \mathbb{N}$, set $A_n := \{(1/n,\ y) \in \mathbb{R}^2\ :\ -n \leq y \leq n\}$. Prove that $\mathbb{R}^2 \setminus \left(\bigcup_{n\in\mathbb{N}} A_n\right)$ is a connected subset of $\mathbb{R}^2$.

(9.) Let X be a topological space and let $\{V_n : n \in \mathbb{N}\}$ be a collection of connected subsets of X satisfying $V_n \supsetneq V_{n+1}, \forall n \in \mathbb{N}$. Is $\bigcap_{n\in\mathbb{N}} V_n$ connected?

8. Countability Axioms

Definition 8.1. Let (X, τ) be a topological space. We say X is *second countable* if it has a countable base for τ and X is *separable* if it has a countable dense subset.

Example 8.2. *(1.)* Let $\emptyset \neq X$ be a countable infinite set and let τ be the co-finite topology on X. Let A be an infinite subset of X. Then for any non-empty open set $V \subseteq X$, $A \bigcap V \neq \emptyset$. Thus A is dense in X and X is separable. Notice that the set

$$\mathscr{B} = \{X \setminus \{x_1, \cdots, x_n\} : x_1, \cdots, x_n \in X, \ n \in \mathbb{N}\}$$

is a base for τ. Since X is countable, collection of all finite subsets of X is countable. Thus $\mathscr{B}$ is countable and hence X is a second countable space.

(2.) The Euclidean space $\mathbb{R}^n$ is both second countable and separable:

- The set $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$;
- The set

  $$\mathscr{B} := \{B(x; 1/r) : x \in \mathbb{Q}^n, \ r \in \mathbb{N}\}$$

  is a countable base of the Euclidean topology on $\mathbb{R}^n$.

Exercise 8.3. *(1.)* Let $\{X_i : 1 \leq i \leq n\}$ be a finite collection of non-empty topological spaces. Prove that

(i) if each X_i is second countable, then so is $\prod_{i=1}^n X_i$.
(ii) if each X_i is separable, then so is $\prod_{i=1}^n X_i$.

(2.) Prove that a subspace of a second countable space is second countable.

On $X = \mathbb{R}$, consider the topology

$$\tau := \{V \subseteq \mathbb{R} : V = \emptyset \text{ or } 0 \in V\} \subseteq \mathscr{P}(X).$$

Since $\{0\}$ is a dense open subset of X, X is separable. Notice that, the subspace topology on $Y = \mathbb{R} \setminus \{0\}$ is same as the discrete topology on Y. Since Y is uncountable, Y not separable. Thus a subspace of a separable topological space need not be separable.

Lemma 8.4. *A second countable topological space is separable.*

Proof. Let X be a second countable topological space and let $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base for the topology on X. We may assume that $X \neq \emptyset$ and $B_k \neq \emptyset, \forall k \in \mathbb{N}$. Set $D = \{a_n \in B_n : n \in \mathbb{N}\}$. Then D is a countable set. Let $\emptyset \neq V \subseteq X$ be an open set. Choose $B_\ell \in \mathscr{B}$ such that $B_\ell \subseteq V$. Then $a_\ell \in V \cap D$. Hence D is a countable dense subset of X. $\square$

Example 8.5. *(1.)* Let X be an uncountable set equipped with co-finite topology. Since any infinite set $A \subseteq X$ is dense in X, X is separable. Suppose X is second countable and let $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base. Then for any $a \in X$

$$\bigcap_{a \in V \in \mathscr{B}} V = \{a\} \implies X \setminus \{a\} = \bigcup_{a \in V \in \mathscr{B}} (X \setminus V)$$

a contradiction, since $X \setminus V$ is finite and $\mathscr{B}$ is countable.

(2.) Let $X = \mathbb{R}_\ell$, the lower limit topological space. It's easy to see that $\mathbb{Q}$ is dense in X. Let $\mathscr{B}$ be a base for the topology on X. For any $x \in X$, there is $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq [x, x+1)$. Notice that, for $x, y \in X$ with $x \neq y$, $B_x \neq B_y$ since $x = \inf B_x$ and $y = \inf B_y$. Thus the map $X \longrightarrow \mathscr{B}, x \mapsto B_x$ is injective. Hence $\mathscr{B}$ is not countable. In other words, every base for the topology on X is uncountable. Hence X is not second countable.

Lemma 8.6. *Let X, Y be two non-empty topological spaces and let $f : X \longrightarrow Y$ be a surjective continuous map.*

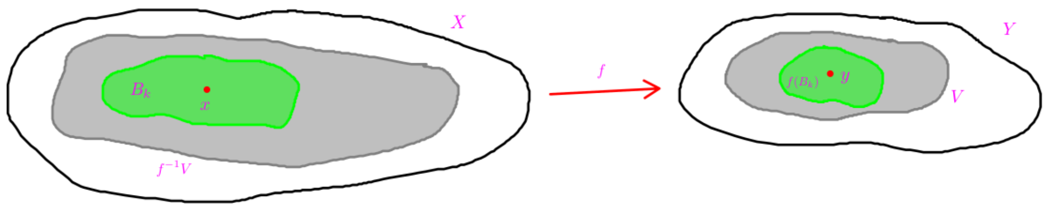
(i) If X is separable and the so is Y.

(ii) If X is second countable and f is an open map, then Y is second countable.

Proof. (i) Assume X is separable and let D be a countable dense subset of X. Since f is continuous, we have

$$f(\overline{D}) \subseteq \overline{f(D)} \implies f(X) \subseteq \overline{f(D)} \implies Y = \overline{f(D)}$$

since f is surjective. Hence Y is separable.



(ii) Suppose X is second countable and f is an open map. Let $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base for the topology of X. Then each $f(B_n)$ is an open subset of Y. Let $\emptyset \neq V$ be an open set in Y and let $y \in V$. Choose $x \in f^{-1}(y)$ and $B_k \in \mathscr{B}$ such that $x \in B_k \subseteq f^{-1}V$ so that $y \in f(B_k) \subseteq V$. Since f is surjective, $f(\mathscr{B}) := \{f(B_n) : n \in \mathbb{N}\}$ is a base for the topology on Y. $\square$

Let $f : X \longrightarrow Y$ be a surjective continuous map between two topological spaces X and Y and let X is second countable. Then Y need not be second countable. For example let $X = \mathbb{R}$ be equipped with Euclidean topology and let $Y = \mathbb{R}$ be equipped with co-finite topology. Then the map $f : X \longrightarrow Y, f(x) = x$ is a continuous surjection. Here X is second countable, but Y is not.

Proposition 8.7. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty second countable topological spaces. Then the product space $\prod_{\alpha\in\Lambda} X_\alpha$ is second countable if and only if all but a countable number of the topological spaces X_α is equipped with indiscrete topology.*

Proof. Let $\Omega \subseteq \Lambda$ be a countable set such that for each $\alpha \in \Lambda \setminus \Omega$, the topological space X_α is equipped with indiscrete topology. For each $\alpha \in \Lambda$, let $\mathscr{B}_\alpha$ be a countable base for the topology on X_α. Then $\mathscr{B}_\alpha = \{X_\alpha\}$, for every $\alpha \in \Lambda \setminus \Omega$. Notice that the set $\mathscr{S} := \{p_\alpha^{-1}B : B \in \mathscr{B}_\alpha, \alpha \in \Lambda\}$ is a sub-base for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ where $p_\beta : \prod_{\alpha\in\Lambda} X_\alpha \longrightarrow X_\beta$ is the β-th coordinate projection map. Since $\mathscr{B}_\alpha = \{X_\alpha\}, \forall \alpha \in \Lambda \setminus \Omega$ and Ω is countable, for all but countably many $p_\alpha^{-1}B \in \mathscr{S}$ equals to the whole space. Since the collection of all finite intersections of elements of $\mathscr{S}$ forms a base for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$, the product space $\prod_{\alpha\in\Lambda} X_\alpha$ is second countable.

Conversely assume that the product space $\prod_{\alpha\in\Lambda} X_\alpha$ is second countable and let $\mathscr{B}$ be a countable basis for product topology on $\prod_{\alpha\in\Lambda} X_\alpha$. Assume there is an uncountable set $\Omega \subseteq \Lambda$ such that for each $\alpha \in \Omega$, X_α has a base containing a non-empty proper open subset of X_α. Since every $B \in \mathscr{B}$ contains a canonical basic open set, for all but finitely many $\alpha \in \Lambda$, we have $p_\alpha(B) = X_\alpha$. Since $\mathscr{B}$ is countable and Ω is uncountable, there is $\beta \in \Omega$ such that $p_\beta(B) = X_\beta, \forall B \in \mathscr{B}$. Let $\emptyset \neq V \subsetneq X_\beta$ be a basic open set (such an open set exists by our assumption). Then there is no $B \in \mathscr{B}$ such that $B \subseteq p_\beta^{-1}(V)$, a contradiction. Hence at most countably many X_α is equipped with other than indiscrete topology. $\square$

Definition 8.8. A topological space X is said to be *Lindelöf* if every open cover of X has a countable subcover.

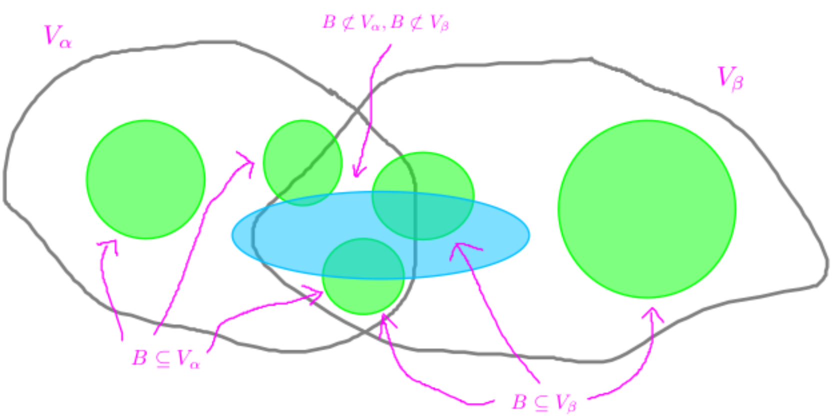
Any compact topological space is Lindelöf, by definition, but the converse is not true. For example, the Euclidean space $\mathbb{R}^n$ is Lindelöf (by the next Lemma), but not compact.

Lemma 8.9. *(Lindeöf) A second countable topological space is Lindelöf.*

Proof. Let (X, τ) be a second countable topological space and let $\mathscr{B}$ be a countable base for the topology τ. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then the family

$$\mathscr{A} := \{B \in \mathscr{B} : B \subseteq V_\alpha, \text{ for some } \alpha \in \Lambda\}$$

is a countable open cover X. Write $\mathscr{A} = \{B_{r_k} : k \in \mathbb{N}\}$ and for each $k \in \mathbb{N}$, choose $\alpha_k \in \Lambda$ such that $B_{r_k} \subseteq V_{\alpha_k}$. Then $\{V_{r_k} : k \in \mathbb{N}\}$ is a countable subcover $\mathscr{U}$. Hence X is Lindelöf. □



Example 8.10. *(1.)* Let $\mathbb{R}_\ell$ be the real line equipped with lower limit topology. We have seen before that $\mathbb{R}_\ell$ is not second countable. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $\mathbb{R}_\ell$. Set

$$\mathscr{A} := \{[a,b) : [a,b) \in V_\alpha, \text{ for some } \alpha \in \Lambda\} \text{ and } Y := \bigcup \{(a,b) : [a,b) \in \mathscr{A}\}.$$

Since the set $\{(a,b) : [a,b) \in \mathscr{A}\}$ forms an open cover of Y with respect to the Euclidean topology on $\mathbb{R}$, by Lemma 8.9, for each $n \in \mathbb{N}$ there is $[a_n, b_n) \in \mathscr{A}$ such that $Y = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$. For each $n \in \mathbb{N}$, choose $\alpha_n \in \Lambda$ such that $[a_n, b_n) \subseteq V_{\alpha_n}$. Then $Y \subseteq \bigcup_{n \in \mathbb{N}} V_{\alpha_n}$. Let $x \in \mathbb{R} \setminus Y$. Then x_0 is the left end point of some half open interval of the form $[a,b) \in \mathscr{A}$. Let's denote such an interval by $[s_0, t_0) \in \mathscr{A}$ where $x_0 = s_0$. Let $x_1 \in \mathbb{R} \setminus Y$ be such that $x_0 < x_1$ and let $[s_1, t_1) \in \mathscr{A}$ be such that $x_1 = s_1$. Notice that $t_0 \le s_1 = x_1$. Otherwise, $x_1 \in (s_0, t_0) \subseteq Y$, a contradiction. Choose rational numbers q_0, q_1 such that $s_0 < q_0 < t_0$ and $s_1 < q_1 < t_1$. Then $q_0 < q_1$. Hence the map $\mathbb{R} \setminus Y \longrightarrow \mathbb{Q}, x \mapsto q_x$ is one-one. In particular, $\mathbb{R} \setminus Y$ is countable. Let's write $\mathbb{R} \setminus Y = \{z_1, z_2, z_3, \cdots\}$ and for each $m \in \mathbb{N}$, choose $\beta_m \in \Lambda$ such that $z_m \in V_{\beta_m}$. Then

$$\mathbb{R} \setminus Y \subseteq \bigcup_{m \in \mathbb{N}} V_{\beta_m} \implies \mathbb{R} = \Big(\bigcup_{n \in \mathbb{N}} V_{\alpha_n}\Big) \bigcup \Big(\bigcup_{m \in \mathbb{N}} V_{\beta_m}\Big).$$

Thus $\{V_{\alpha_i}, V_{\beta_j} : i, j \in \mathbb{N}\}$ is a countable subcover of $\mathscr{U}$ and hence $\mathbb{R}_\ell$ is Lindelöf.

(2.) Let X be an uncountable set and let $a \in X$. Equip X with the topology $\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subset X : a \in V \text{ and } X \setminus V \text{ is finite}\}$. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover X. Choose $\alpha_0 \in \Lambda$ such that $a \in V_{\alpha_0}$. Then there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $X \setminus V_{\alpha_0} \subseteq \bigcup_{j=1}^n V_{\alpha_j}$. Hence $\{V_{\alpha_j} : j = 0, 1, \cdots, n\}$ is a finite cover of X. Hence X is Lindelöf. Now let $Y = X \setminus \{a\}$. The Y is a discrete subspace of X. Since Y is uncountable, it is not Lindelöf. Thus a subspace of a Lindelöf space need not be Lindelöf.

Lemma 8.11. *A closed subspace of a Lindelöf space is Lindelöf.*

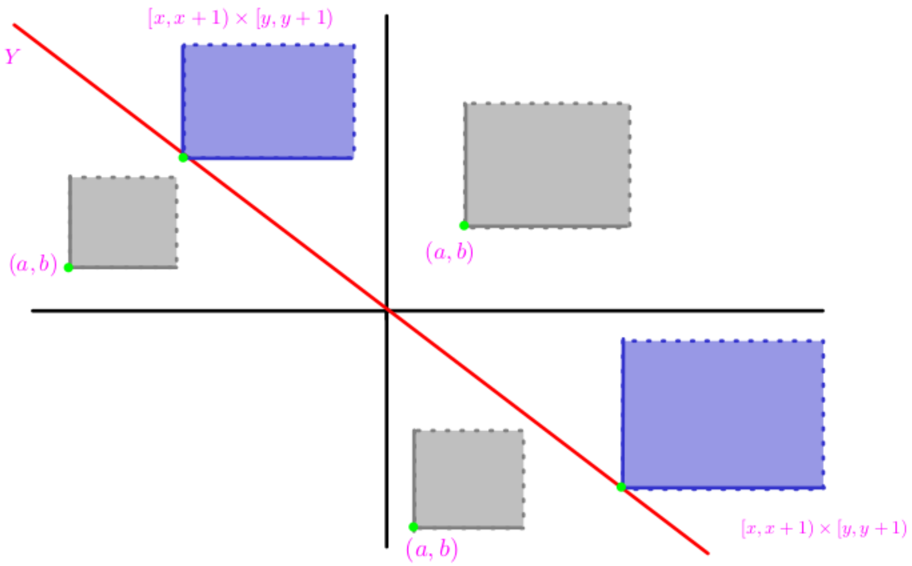
Proof. Let X be a non-empty Lindelöf topological space, Y be a closed non-empty subspace of X and let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be a cover of Y by open subsets of X. Since Y is closed in X, the collection $\mathscr{U} \bigcup (X \setminus Y)$ is an open cover of X. Since X is Lindelöf, there is a countable set $\{\alpha_n \in \Lambda : n \in \mathbb{N}\}$ such that $X = \left(\bigcup_{n\in\mathbb{N}} V_{\alpha_n}\right) \bigcup (X \setminus Y)$. Hence $\{V_{\alpha_n} : n \in \mathbb{N}\}$ is a countable subcover of Y. $\square$

Example 8.12. Let $\mathbb{R}_\ell$ be the real line equipped with lower limit topology. Set

$$Y := \{(x, y) \in \mathbb{R}_\ell \times \mathbb{R}_\ell : x + y = 0\}$$

Let $(a, b) \in X \setminus Y$. Then either $a + b > 0$ or $a + b < 0$. Consider the two cases:

- If $a + b > 0$, then $([a, a + 1) \times [b, b + 1)) \bigcap Y = \emptyset$.
- If $a + b > 0$, then $([a, a + t) \times [b, b + t)) \bigcap Y = \emptyset$ where $t = -(a + b)/3$.



Hence Y is a closed subset of X. For any $(x, y) \in Y, Y \bigcap ([x, x + 1) \times [y, y + 1)) = \{(x, y)\}$. Thus Y is a discrete closed subspace of X. Since Y is uncountable, it is not Lindelöf. By Lemma 8.11, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelöf.

Exercise 8.13. Let X, Y be two topological spaces and let $f : X \longrightarrow Y$ be a continuous surjective map. If X is Lindelöf, prove that so is Y.

Definition 8.14. Let X be a topological space. We say X is *first countable* if every point of X has a countable local base.

Example 8.15. *(1.)* Let $\mathbb{R}$ be the Euclidean space and let $x \in \mathbb{R}$. Then the collection $\mathscr{B}_x = \{(x - 1/n, x + 1/n) : n \in \mathbb{N}\}$ is a countable local base at x. Since this is true for every $x \in \mathbb{R}$, $\mathbb{R}$ is first countable.

(2.) Let $\mathbb{R}_\ell$ be the real line equipped with lower limit topology. Then for any $x \in \mathbb{R}_\ell$, $B_x = \{[x, x + 1/n) : n \in \mathbb{N}\}$ is a countable local base at x. Hence $\mathbb{R}_\ell$ is first countable.

(3.) Let X be an uncountable set equipped with co-finite topology. Let $a \in X$ and $\mathscr{U}_a = \{V_n : n \in \mathbb{N}\}$ be a countable local base at a. Then for any $x \in X \setminus \{a\}$, there is

$V_j \in \mathscr{U}_a$ such that $a \in V_j \subseteq X \setminus \{x\}$. Hence $X \setminus \{a\} = \bigcup_{n\in\mathbb{N}} (X \setminus V_n)$, a contradiction since X is uncountable. Thus a does not have a countable local base and this is true for every $a \in X$. Hence X is not first countable.

(4.) Let X be an uncountable set and let $a \in X$. Let us equip X with the topology $\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subset X : a \in V \text{ and } X \setminus V \text{ is finite}\}$. For any $x \in X \setminus \{a\}$, $B_x = \{x\}$ is a countable local base. Let $B_a = \{V_n : n \in \mathbb{N}\}$ be a countable local base. Arguing as above, we see that $X \setminus \{a\} = \bigcup_{n\in\mathbb{N}} (X \setminus V_n)$, a contradiction. Hence X is not first countable.

Exercise 8.16. *(1.)* Give example of a topological space X such that it contains exactly n points $x_1, \cdots, x_n \in X$ at which there is no countable local base, but every other point of X has a countable local base.

(2.) Prove that

(a) a second countable topological space is first countable.

(b) a subspace of a first countable topological space is first countable.

(c) an open subspace of a separable topological space is separable.

(3.) Let $f : X \longrightarrow Y$ be a continuous surjective map between two topological spaces X and Y and assume X is first countable. Give an example to show that Y need not be first countable.

(4.) Let $f : X \longrightarrow Y$ be an open continuous surjective map between two topological spaces X and Y and assume X is first countable . Prove that Y is first countable.

Example 8.17. Let $X = \mathbb{R}$ be the Euclidean space. Define a relation $\sim$ in X by $x \sim y$ if and only if $x, y \in \mathbb{Z}$. Then $\sim$ is an equivalence relation on X. Let $Y := X/\!\sim$ be the quotient space and let $q : X \longrightarrow Y$ be a quotient map. Let $[0] \in Y$ denotes the equivalence class of $0 \in \mathbb{R}$ and let $\{V_n \subseteq Y : n \in \mathbb{Z}\}$ be a local base at $[0]$. Then each $q^{-1}V_n$ is an open set in $\mathbb{R}$ containing $\mathbb{Z}$. For each $n, m \in \mathbb{Z}$, choose $r_{m,n} \in (0, 1/2)$ such that $\bigcup_{m\in\mathbb{Z}} (m - r_{m,n},\, m + r_{m,n}) \subseteq q^{-1}V_n$. Now for each $m \in \mathbb{Z}$, set $\delta_m := r_{m,m}/2$ and $W := \bigcup_{m\in\mathbb{Z}} (m - \delta_m,\, m + \delta_m)$. Then W is an open set in $\mathbb{R}$ containing $\mathbb{Z}$. Since $q^{-1}(q(W)) = W$, $q(W) \subseteq Y$ is an open set containing $[0]$. Fix $n \in \mathbb{N}$ and choose $x \in (n - r_{n,n},\, n + r_{n,n}) \setminus (n - \delta_n,\, n + \delta_n)$. Then $q(x) \in V_n \setminus q(W)$ so that $V_n \not\subset q(W)$, for every $n \in \mathbb{N}$, a contradiction. Thus quotient of a first (resp. second) countable topological space need not be first (resp. second) countable.

Proposition 8.18. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces such that each X_α is first countable. Then the product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ is first countable if and only if all but a countable number of the topological spaces X_α is equipped with indiscrete topology.*

Proof. Let $\Omega \subseteq \Lambda$ be a countable set such that for each $\alpha \in \Lambda \setminus \Omega$, the topological space X_α is equipped with trivial topology. Let $x \in X$ and let $\mathscr{U}_\alpha$ be a countable local base at $x_\alpha := x(\alpha) \in X_\alpha$. Then $\mathscr{U}_\alpha = \{X_\alpha\}$ for every $\alpha \in \Lambda \setminus \Omega$. For $\beta \in \Lambda$, let $p_\beta : X \longrightarrow X_\beta$ be the β-th coordinate projection. Then the family $\mathscr{S} := \{p_\alpha^{-1}V : V \in \mathscr{U}_\alpha,\, \alpha \in \Lambda\}$ is countable and hence the family $\mathscr{U}_x$ obtained from $\mathscr{S}$ by taking finite intersections of elements of $\mathscr{S}$ is also countable. Notice that $\mathscr{U}_x$ is a local base at x. Hence X is first countable.

Conversely assume that the product space X is first countable and $x \in X$. Let $\mathscr{U}$ be a countable local base at x. Assume there is an uncountable set $\Omega \subseteq \Lambda$ such that for each $\alpha \in \Omega$, x_α has a neighbourhood which is a proper open subset of X_α. Since every $V \in \mathscr{U}$ contains a canonical basic open set, for all but finitely many $\alpha \in \Lambda$, we have $p_\alpha(V) = X_\alpha$. Since $\mathscr{U}$ is countable and Ω is uncountable, there is $\beta \in \Omega$ such that $p_\beta(V) = X_\beta, \forall V \in \mathscr{U}$. Let $x_\beta \in V_\beta \subsetneq X_\beta$ be a neighbourhood of x_β (such an open set exists by our assumption). Then

$x \in p_\beta^{-1} V_\beta \subseteq X$ is a neighbourhood of x. But there is no $V \in \mathscr{U}$ such that $x \in V \subseteq p_\beta^{-1} V_\beta$, a contradiction. Hence no such uncountable set $\Omega \subseteq \Lambda$ exists. $\square$

Example 8.19. *(1.)* Let X be an uncountable set and let $a \in X$. Equip X with the topology $\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subset X : a \in V \text{ and } X \setminus V \text{ is finite}\}$. Then X is Lindelöf, not separable, not first countable, not second countable.

(2.) Let X be an uncountable set equipped with co-finite topology. Then X is Lindelöf, seprable, but not first countable.

(3.) Let Y be an uncountable set and let Z be an infinite set with $Y \bigcap Z = \emptyset$. On the set $X = Y \bigcup Z$, define a topology as follows:

$$\tau := \{V \subseteq X : V = \emptyset \text{ or } Z \setminus V \text{ is finite}\}.$$

Since any infinite subset of Z is dense in X, X is separable. Consider the open cover of X given by $\mathscr{U} = \{Z, \{y\} \bigcup Z : y \in Y\}$. Then $\mathscr{U}$ is an uncountable family and it has not countable subcover. Hence X is not Lindelöf.

Let X be a non-empty set. A *sequence* in X is a function $\mathbb{N} \longrightarrow X, n \mapsto x_n$. It will be denoted by (x_n) or by $(x_n : n \in \mathbb{N})$.

Definition 8.20. Let X be a topological space and let (x_n) be a sequence in X. We say that the sequence (x_n) *converges to* $x \in X$, if for any neighbourhood V_x of x, $x_n \in V_x$ for all but finitely many $n \in \mathbb{N}$. If the sequence (x_n) converges to x, we say x is a *limit* of the sequence (x_n).

Example 8.21. *(1.)* Let $X = \mathbb{N}$ be equipped with co-finite topology and consider the sequence (x_n) where $x_n = n, n \in \mathbb{N}$. Fix $a \in \mathbb{N}$ and let V_a be a neighbourhood of a. Since $\mathbb{N} \setminus V_a$ is finite, all finitely many $n \in \mathbb{N}$, $x_n \in V_a$. Thus (x_n) converges to a. This is true for every $a \in \mathbb{N}$. This example shows that unlike the case of metric spaces, limit of a sequence need not be unique.

(2.) Let X be a non-empty set with at least two points. Let τ_1, τ_2 be the discrete topology and indiscrete topology on X, respectively. Then the convergent sequences of (X, τ_1) are eventually constant sequences and any sequence in (X, τ_2) is convergent and every point of X is a limit point of every sequence.

Lemma 8.22. *Let X be a non-empty Hausdorff topological space. Then every convergent sequence in X has a unique limit.*

Proof. Let (x_n) be a convergent sequence in X with a limit $x \in X$. Let $y \in X \setminus \{x\}$. Then there are two neighbourhoods V_x and V_y of x and y, respectively, such that $V_x \bigcap V_y = \emptyset$. By definition, for all but finitely many $n \in \mathbb{N}$, $x_n \in V_x$. Hence y can not be a limit of (x_n). Thus limit of (x_n) is unique. $\square$

Let $X = \mathbb{R}$ be equipped with co-countable topology and let (x_n) be a convergent sequence in X with a limit $x \in X$. Set $T = \{x_n : x_n \neq x\}$. Then T is countable and hence $X \setminus T$ is a neighbourhood of x. Hence we must have $x_n = x$, for all but finitely many $n \in \mathbb{N}$. In other words, the convergent sequences in X are the eventually constant sequences and hence every convergent sequence in X has a unique limit. But X is not Hausdorff.

Lemma 8.23. *Let X be a non-empty first countable topological space. Then X is Hausdorff if and only if every convergent sequence in X has a unique limit.*

Proof. If X is Hausdorff, then by Lemma 8.22, every convergent sequence in X has a unique limit. Conversely, assume that every convergent sequence in X has a unique limit. Suppose X is not Hausdorff. Then there are two distinct points $a, b \in X$ such that every open neighbourhood of a intersects every open neighbourhood of b. Let $\mathscr{B}_a = \{V_n : n \in \mathbb{N}\}$ and $\mathscr{B}_b = \{W_n : n \in \mathbb{N}\}$ be two local base at a and b, respectively. We may further assume that $V_{n+1} \subseteq V_n$ and $W_{n+1} \subseteq W_n$, for every $n \in \mathbb{N}$. By our assumption, $V_n \bigcap W_n \neq \emptyset, \forall n \in \mathbb{N}$. Choose $x_n \in V_n \bigcap W_n$. Then (x_n) is a sequence in X that converges to both a and b, a contradiction. Hence X is a Hausdorff topological space. $\square$

Lemma 8.24. *Let X, Y be two non-empty topological spaces and let $f : X \longrightarrow Y$ be a continuous map. If (x_n) is a sequence in X converges to $x \in X$, then the sequence $(f(x_n))$ converges to $f(x)$ in Y.*

Proof. Left as an exercise. $\square$

Let $X = \mathbb{R}$ be equipped with co-countable topology, $Y = \mathbb{R}$ be the Euclidean space and let $f : X \longrightarrow Y$ be defined by $f(x) = x$. Then f is not continuous. As we have seen before, if (x_n) is a convergent sequence in X, then it is eventually constant. Hence $(f(x_n))$ is a convergent sequence in Y.

Lemma 8.25. *Let $f : X \longrightarrow Y$ be a function between two non-empty topological spaces and assume X is first countable. Then the following conditions are equivalent:*

(i) f is continuous;

(ii) for every convergent sequence (x_n) in X converging to $x \in X$, the sequence $(f(x_n))$ converges to $f(x)$ in Y.

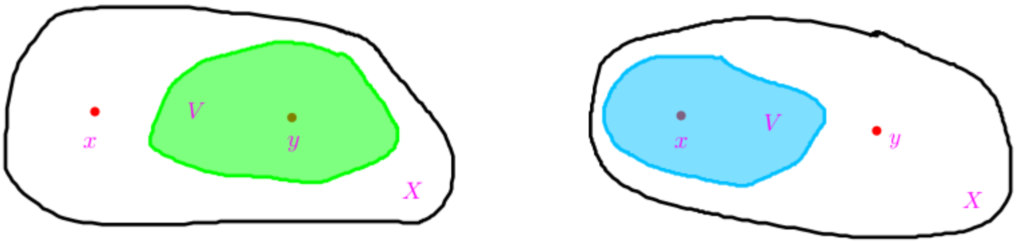
Proof. $(ii) \implies (i)$ Let $\emptyset \neq A \subseteq X$ and let $x_0 \in \overline{A}$. Let $\mathscr{B}_{x_0} = \{V_n : n \in \mathbb{N}\}$ be a local base of x_0. For each $n \in \mathbb{N}$, choose $a_n \in V_n \bigcap A$. Then (a_n) is a sequence in A converging to x_0. By the given condition, the sequence $(f(a_n))$ converges to $f(x_0)$ in Y. Then for any neighbourhood V of $f(x_0)$, we have $f(A) \bigcap V \neq \emptyset$. Thus $f(x_0) \in \overline{f(A)}$. Since $x_0 \in \overline{A}$ is arbitrary, $f(\overline{A}) \subseteq \overline{f(A)}$. Hence f is continuous. $(i) \implies (ii)$ is trivial. $\square$

Exercise 8.26. *(1.)* Let X be a compact metric space. Prove that X is second countable, separable and Lindelöf. (<u>Hint:</u> For each $n \in \mathbb{N}$, the open cover $\mathscr{D}_n = \{B(x; 1/n) : x \in X\}$ of X has a finite subcover, say $\mathscr{A}_n$. Set $\mathscr{B} = \bigcup_{n \in \mathbb{N}} \mathscr{A}_n$. Then $\mathscr{B}$ is base for the topology on X.)

(2.) Let X be a second countable, separable and Lindelöf space. Give an example to show that X need not be compact.

9. Separation Axioms

Definition 9.1. A topological space X is said to be a *T_0-space* if for each pair of distinct points of X, there is a neighbourhood of one point which does not contain the other point.



Example 9.2. *(1.)* Let $X = \mathbb{N}$ and $\tau = \{\emptyset, \mathbb{N}, \{1\}, \{1,2\}, \{1,2,3\}, \cdots\}$. For $x, y \in \mathbb{N}, x \neq y$, we have

- if $x < y$, then $x \in V_x = \{1, \cdots, x\}$ and $y \notin V_x$;
- if $x > y$, then $y \in V_y = \{1, \cdots, y\}$ and $x \notin V_y$.

(2.) Let (X, d) be a metric space and let τ be the topology on X generated by the metric d. Then (X, τ) is a T_0-space. In particular, the Euclidean space $\mathbb{R}^n$ is a T_0 space.

(3.) Let $X = \{a, b, c\}$ and let $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then (X, τ_1) is not a T_0-space but (X, τ_2) is a T_0-space.

Exercise 9.3. *(1.)* Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of T_0-spaces. Prove that the product space $\prod_{\alpha \in \Lambda} X_\alpha$ is also a T_0-space.

(2.) On the set $X = \mathbb{R}$, consider the topology

$$\tau := \{V \subseteq \mathbb{R} : V = \emptyset \text{ or if } x \in V, \text{ then there is } a_x \in \mathbb{R} \text{ such that } x \in (a_x, \infty) \subseteq V\}.$$

Prove that (X, τ) be a T_0 topological space.

(3.) A *pseudometric* on a non-empty set X is function $\rho : X \times X \longrightarrow \mathbb{R}$ satisfying

- $\rho(x, y) \geq 0, \forall x, y \in X$ and $\rho(x, x) = 0, \forall x \in X$;
- $\rho(x, y) = \rho(y, x), \forall x, y \in X$;
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y), \forall x, y, z \in X$.

A non-empty set X together with a pseudometric ρ on it is called a pseudometric space.

(i) Let A be an infinite set, X be the set of all functions $A \longrightarrow \mathbb{R}$ and let $a_0 \in A$. For $f, g \in X$, set $\rho(f, g) = |f(a_0) - g(a_0)|$. Prove that ρ is a pseudometric on X.

(ii) Let $\mathscr{R}[0, 1]$ be the set of all real valued Riemann integrable functions on $[0, 1]$. For $f, g \in \mathscr{R}[0, 1]$, set $\rho(f, g) = \int_0^1 |f - g|$. Prove that ρ is a pseudometric on $\mathscr{R}[0, 1]$.

(iii) Let (X, ρ) be a pseudometric space. For any $x \in X$ and $r \in \mathbb{R}, r > 0$, define the open ball centered at x of radius r to be $B(x; r) := \{y \in X : \rho(x, y) < r\}$. Show that the set $\mathscr{B} := \{B(x; r) : x \in X, r \in \mathbb{R}, r > 0\}$ forms a base for some topology τ on X. We say τ is generated by the pseudometric ρ.

(iv) Let (X, ρ) be a pseudometric space and let τ be the topology on X generated by the pseudometric ρ. Prove that ρ is a metric if and only if (X, τ) is a T_0-space.

Let X be a non-empty topological space. For $x \in X$, let $\mathscr{N}_x^0$ be the set of all open neighbourhoods of x. Define a relation on X as follows: $x \sim y$ if and only if $\mathscr{N}_x^0 = \mathscr{N}_y^0$.

Then $\sim$ is an equivalence relation on X. Let quotient space $X/\sim$ is also called *Kolmogorov quotient* of X. Let $q : X \longrightarrow X/\sim$ be the quotient map.

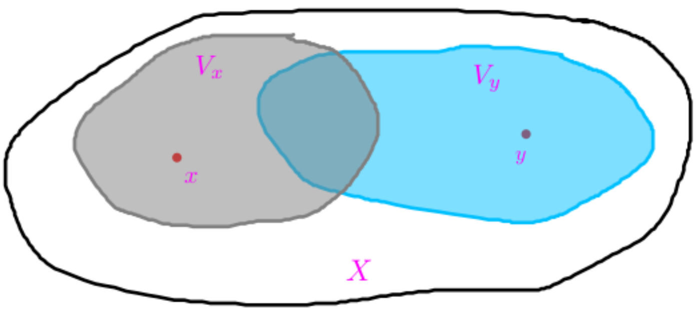
- Let $\emptyset \neq V \subseteq X$ be an open set. Then

$$\begin{aligned} q^{-1}(q(V)) &= \{x \in X \,:\, q(x) \in q(V)\} = \{x \in X \,:\, q(x) = q(y) \text{ for some } y \in V\} \\ &= \{x \in X \,:\, x \sim y \text{ for some } y \in V\} = V. \end{aligned}$$

  Thus $q^{-1}(q(V)) \subseteq X$ is open and $q(V)$ is open in $X/\sim$. Hence q is an open map.
- Let $x, y \in X$ be such that $q(x) \neq q(y)$. Then $x \not\sim y$ and hence there is an open set $V \subseteq X$, $x \in V$ but $y \notin V$. Then $q(x) \in q(V)$ and $q(y) \notin q(V)$. Since q is an open map, $X/\sim$ is a T_0-space.

Notice that, if X is a T_0-space, then the quotient map $q : X \longrightarrow X/\sim$ is injective and hence a homeomorphism.

Definition 9.4. A topological space X is said to be a *T_1-space* if for each pair of distinct points $x, y \in X$, there is a neighbourhood V_x of x and a neighbourhood V_y of y such that $x \notin V_y$ and $y \notin V_x$.



Any T_1 topological space is a T_0-space, by definition, but the converse is not true: see Example 9.2 (*1.*) and Exercise 9.3 (*2.*). Let (X, d) be a metric space and let τ be the topology on X generated by the metric d. Then (X, τ) is a T_1-space. In particular, the Euclidean space $\mathbb{R}^n$ is a T_1-space.

Lemma 9.5. *Let X be a topological space. The X is a T_1-space if and only if $\{x\}$ is closed in X, for every $x \in X$.*

Proof. Let X be a T_1-space and let $x \in X$. Then for any $y \in X \setminus \{x\}$, there is a neighbourhood V_y of y such that $x \notin V_y$. Thus $\bigcap_{y \in X \setminus \{x\}} (X \setminus V_y) = \{x\}$ and hence $\{x\}$ is closed in X.

Conversely assume that $\{x\}$ is closed in X, for every $x \in X$. Let $a, b \in X$ be two distinct point. Set $V_a = X \setminus \{b\}$ and $V_b = X \setminus \{a\}$. Then the neighbourhoods $a \in V_a$ and $b \in V_b$ satisfies the required condition. □

Corollary 9.6. *Let X be a T_1 topological space and let $A \subseteq X$. A point $x \in X$ is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.*

Proof. Let x be a limit point of X. Suppose there is an open neighbourhood V_x of x such that $E = (A \bigcap V_x)$ is a finite set. Since X is a T_1 topological space, by the above Lemma, E is a closed subset of X. Then $W = V_x \setminus (E \setminus \{x\})$ is an open neighbourhood of x and $W \bigcap (A \setminus \{x\}) = \emptyset$, a contradiction since x is a limit point of A. Hence every neighbourhood of x contains infinitely many points of A. The converse is trivial. □

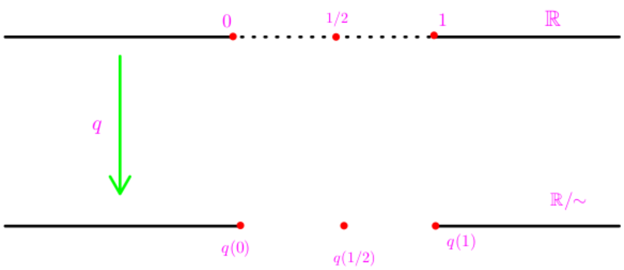
Continuous image of a T_1-space need not be T_1. For example, let X be a set with at least two elements. Let τ be the discrete topology on X and let μ be the indiscrete topology on X. Then the map $f : (X, \tau) \longrightarrow (X, \mu)$ is continuous. Here (X, τ) is a T_1-space, but (X, μ) is not a T_1-space.

Exercise 9.7. *(1.)* Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of non-empty topological spaces. Prove that the product space $\prod_{\alpha\in\Lambda} X_\alpha$ is a T_1-space if and only if every X_α is a T_1-space.

(2.) Prove that a finite T_1-space is discrete.

(3.) Let X be a T_1-space and let $\emptyset \neq Y \subseteq X$ be a connected set. If $|Y| > 1$, then prove that Y is an infinite set.

On the Euclidean space $\mathbb{R}$, define a relation $\sim$ as follows: $x \sim 1/2$ if $x \in (0, 1)$ and $y \sim y$ if $y \notin (0, 1)$. Then $\sim$ is an equivalence relation on $\mathbb{R}$. Let $\mathbb{R}/\!\sim$ be the quotient space and $q : \mathbb{R} \longrightarrow \mathbb{R}/\!\sim$ be the quotient map. Let $q(0) \in V \subseteq \mathbb{R}/\!\sim$ be an open set. Since $q^{-1}V$ is an open set in $\mathbb{R}$ containing 0, we must have $q^{-1}V \bigcap (0, 1) \neq \emptyset$. Thus $q(1/2) \in V$. In other words, every open neighbourhood of $q(0)$ also contain $q(1/2)$. Hence $\mathbb{R}/\!\sim$ is not a T_1-space.



Exercise 9.8. Let X be a non-empty topological space and let $\sim$ be an equivalence relation on X. For $x \in X$, let $[x] \subseteq X$ be the equivalence class of x. Prove that the quotient space $X/\!\sim$ is a T_1-space if and only if $[x]$ is a closed subset of X, for every $x \in X$.

Recall, a topological space X is said to be T_2 or a Hausdorff space if for each pair of distinct points $x, y \in X$, there are neighbourhoods V_x of x and V_y of y such that $V_x \bigcap V_y = \emptyset$. By definition, a Hausdorff space if T_1, but the converse is not true. For example, consider the set $X = \mathbb{R}$ equipped with co-finite topology. Then X is a T_1-space (by Lemma 9.5), but it is not Hausdorff.

Proposition 9.9. *Let X be a non-empty topological space and let $\sim$ be an equivalence relation on X. Set $\Gamma(\sim) := \{(x, x') \in X \times X : x \sim x'\}$. Let $X/\!\sim$ be the quotient space and $q : X \longrightarrow X/\!\sim$ be the quotient map.*

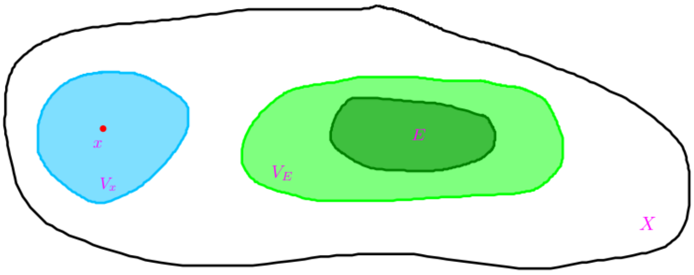
(i) If $X/\!\sim$ is Hausdorff, then $\Gamma(\sim)$ is a closed subset of $X \times X$.

(ii) If $\Gamma(\sim) \subseteq X \times X$ is closed and q is an open map, then $X/\!\sim$ is Hausdorff.

Proof. (i) Suppose $X/\!\sim$ is Hausdorff and let $x, y \in X$ be such that $x \not\sim y$. Then $q(x) \neq q(y)$ and hence there are open sets $V, W \subseteq X/\!\sim$ such that $q(x) \in V$, $q(y) \in W$ and $V \bigcap V = \emptyset$. Thus $(x, y) \in q^{-1}V \times q^{-1}W \subseteq (X \times X) \setminus \Gamma(\sim)$. Hence $\Gamma(\sim) \subseteq X \times X$ is closed.

(ii) Now assume $\Gamma(\sim) \subseteq X \times X$ is closed and q is an open map. Let $x, y \in X$ be such that $q(x) \neq q(y)$. Then $(x, y) \in (X \times X) \setminus \Gamma(\sim)$. Hence there are open sets $T, S \subseteq X$ such that $(x, y) \in T \times S \subseteq (X \times X) \setminus \Gamma(\sim)$. Since q is an open map, $q(T), q(S)$ are open sets in $X/\!\sim$. Moreover, $q(T) \bigcap q(S) = \emptyset$. Hence $X/\!\sim$ is Hausdorff. $\square$

Definition 9.10. A topological space X is said to be *regular* if given any closed set $E \subseteq X$ and point $x \in X \setminus E$, there are open sets V_E and V_x in X such that $E \subseteq V_E$, $x \in V_x$ and $V_E \bigcap V_x = \emptyset$. A regular T_1-topological space is called a *T_3-space*.



Example 9.11. *(1.)* Let X be an infinite set, $a \in X$ and define

$$\tau_a := \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subset X \,:\, a \in V \text{ and } X \setminus V \text{ is finite}\}.$$

Then (X, τ_a) is a T_3 space.

(2.) Any non-empty set equipped with discrete topology is a regular space. Any compact Hausdorff space is regular.

(3.) Let $\mathbb{R}$ be the Euclidean space, $E \subseteq \mathbb{R}$ is closed and let $a \in \mathbb{R} \setminus E$. Since $\mathbb{R} \setminus E$ is open, there is a real number $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subseteq \mathbb{R} \setminus E$ and hence $E \subseteq \mathbb{R} \setminus [a-\varepsilon/2, a+\varepsilon/2]$. By setting $V_E = \mathbb{R} \setminus [a-\varepsilon/2, a+\varepsilon/2]$ and $V_a = (a-\varepsilon/2, a+\varepsilon/2)$, we see that $V_E \bigcap V_a = \emptyset$. Hence $\mathbb{R}$ is a T_3 space.

(4.) Let $X = \{a, b, c, d\}$ and consider the topology $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then for any closed set $E \subsetneq X$ and any $x \in X \setminus E$, there are open sets V_E and V_x in X such that $E \subseteq V_E$, $x \in V_x$ and $V_E \bigcap V_x = \emptyset$. Thus X is a regular topological space. But X is not a T_1-space and hence not a Hausdorff space.

(5.) Let $X = \mathbb{R}$, $K = \{1/n \,:\, n \in \mathbb{N}\}$ and set

$$\mathscr{B} := \{(a, b) : a, b \in \mathbb{R},\ a < b\} \bigcup \{(a, b) \setminus K \,:\, a, b \in \mathbb{R},\ a < b\}$$

Let τ be the topology of $\mathbb{R}$ generated by the base $\mathscr{B}$. Note that $(\mathbb{R}, \tau)$ is Hausdorff. Set $E = K$ and $a = 0$. Clearly K is closed in $(\mathbb{R}, \tau)$. Suppose there are disjoint open subsets V_E and V_a of X such that $E \subseteq V_E$ and $a \in V_a$. Let B be a basic open set satisfying $0 \in B \subseteq V_a$. Then $B = (a, b) - K$, for some $a, b \in \mathbb{R}$ with $a < b$. Choose $m \in \mathbb{N}$ such that $1/m \in (a, b)$. Let B' be a sub-basic open set satisfying $1/m \in B' \subseteq V_E$. Then $B' = (c, d)$, for some $c, d \in \mathbb{R}$ with $c < d$. Then for any $x \in \mathbb{R}$ with $x < 1/m$ and $x > \max\{c, 1/(m+1)\}$, we have $x \in V_E \cap V_a$, a contradiction. Hence $(\mathbb{R}, \tau)$ is not regular. Thus a Hausdorff topological space need not be regular.

Lemma 9.12. *A T_3-space is Hausdorff.*

Proof. Let X be a T_3 space and let $x, y \in X$ be distinct points. Since X is T_1, both $\{x\}$ and $\{y\}$ are closed subsets of X. Since X is regular, there are open sets V_x, V_y in X such that $x \in V_x$, $y \in V_y$ and $V_x \bigcap V_y = \emptyset$. □

Proposition 9.13. *For a topological space $X \neq \emptyset$, the following conditions are equivalent:*

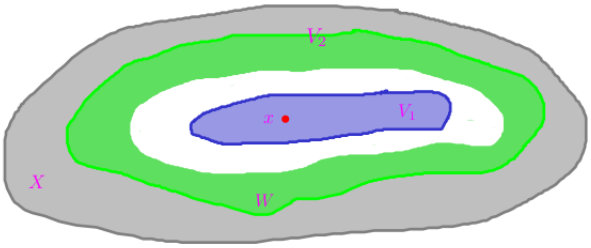
(i) X is regular;

(ii) for every $x \in X$ and every open neighbourhood W of x, there is an open set $V_x \subseteq X$ such that $x \in V_x \subseteq \overline{V_x} \subseteq W$;

(iii) for every $x \in X$ and every closed set $E \subseteq X$ with $x \notin E$, there is an open set $V_x \subseteq X$ such that $x \in V_x$ and $\overline{V_x} \bigcap E = \emptyset$.

Proof. $(i) \Longrightarrow (ii)$ Let X be regular, $x \in X$ and let W be an open neighbourhood of x. Then there are open sets V_1, V_2 such that $x \in V_1$, $X \setminus W \subseteq V_2$ and $V_1 \bigcap V_2 = \emptyset$. Thus $V_1 \subseteq X \setminus V_2$ and hence $\overline{V_1} \subseteq X \setminus V_2$. Now we have

$$\overline{V_1} \bigcap (X \setminus W) \subseteq \overline{V_1} \bigcap V_2 = \emptyset \implies \overline{V_1} \subseteq W.$$



$(ii) \Longrightarrow (iii)$ Apply the condition for $W = X \setminus E$.

$(iii) \Longrightarrow (i)$ Let $E \subseteq X$ be a closed set and let $x \notin E$. By the given condition, there is an open set $V_x \subseteq X$ such that $x \in V_x$ and $\overline{V_x} \bigcap E = \emptyset$. By setting $V_E = X \setminus \overline{V_x}$, we get V_E is open in X, $E \subseteq V_E$ and $V_E \bigcap V_x = \emptyset$. $\square$

Proposition 9.14. *(1.) Subspace of a regular topological space (resp. a T_3-space) is regular (resp. a T_3-space).*

(2.) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty of topological spaces. Then the product space $X = \prod_{\alpha \in \Lambda} X_\alpha$ is regular (resp. a T_3-space) if and only if each X_α is regular (resp. a T_3-space).

Proof. (*1.*) Left as an exercise.

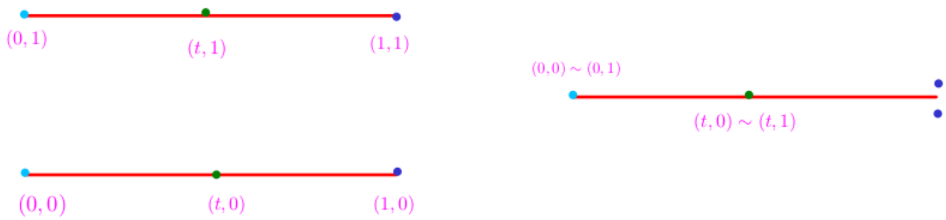
(*2.*) Suppose each X_α is regular. Let $x \in X$ and let $U \subseteq X$ be an open neighbourhood of x. Let $\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(W_{\alpha_i})$ be a canonical basic open set such that $x \in \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(W_{\alpha_i}) \subseteq U$. For each $i = 1, \cdots, n$, choose an open set $V_{\alpha_i} \subseteq X_{\alpha_i}$ such that $x_{\alpha_i} \in V_{\alpha_i} \subseteq \overline{V_{\alpha_i}} \subseteq W_{\alpha_i}$ (Proposition 9.13). Set $V := \bigcap_{i=1}^{n} p_{\alpha_i}^{-1} V_{\alpha_i} \subseteq X$. Then V is an open set in X satisfying

$$x \in V \subseteq \overline{V} \subseteq \bigcap_{i=1}^{n} p_{\alpha_i}^{-1} \overline{V_{\alpha_i}} \subseteq \bigcap_{i=1}^{n} p_{\alpha_i}^{-1} W_{\alpha_i} \subseteq U.$$

By Proposition 9.13, X is regular. The converse follows from (*1.*).

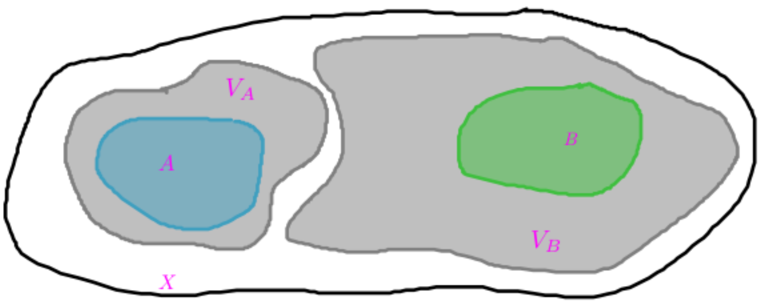
Since product topological spaces is a T_1-space if and only if each topological space is a T_1-space, the corresponding result of T_3-spaces holds. $\square$

Example 9.15. On the subspace $X = [0,1] \times \{0,1\} \subseteq \mathbb{R}^2$, define a relation $\sim$ as follows: $(t,0) \sim (t,1)$ if and only if $t \in [0,1)$. Then $\sim$ is an equivalence relation on X. Let $X/\sim$ be the quotient space and let $q : X \longrightarrow X/\sim$ be quotient map.



For any $t \in [0,1)$, $q^{-1}(q(t,0)) = \{(t,0),(t,1)\} = q^{-1}(q(t,1))$. On the other hand, $q^{-1}(q(1,0)) = \{(1,0)\}, q^{-1}(q(1,1)) = \{(1,1)\}$. Thus $X/\sim$ is a T_1-space. It's easy to check that the for any two open sets $V, W \subseteq X/\sim$ satisfying $q(1,0) \in V$ and $q(1,1) \in W$, must have a non-empty intersection. In particular, $X/\sim$ is not Hausdorff. On the other hand, X is a T_3-space, by Proposition 9.14. Thus quotient of a T_3-space need not be a T_3-space. In fact, the quotient map q is an open map. This example also shows that the image of a T_3-space under a continuous surjective open map need not be a T_3-space.

Definition 9.16. A topological space X is said to be *normal* if for any two disjoint closed sets $A, B \subseteq X$, there are two disjoint open sets $V_A, V_B \subseteq X$ such that $A \subseteq V_A$ and $B \subseteq V_B$. A normal T_1-space is called a *T_4-space*.



Example 9.17. *(1.)* Let (X, d) be a metric space and let τ be the topology on X generated by the metric X. Let $E, F \subseteq X$ be two disjoint closed subsets. For each $x \in E$ (resp. $y \in F$), choose a real number $\varepsilon_x > 0$ (resp. $\delta_y > 0$) such that $B(x;\varepsilon_x) \cap F = \emptyset$ (resp. $E \cap B(y;\delta_y) = \emptyset$). Set

$$V_E := \bigcup_{x \in E} B(x;\varepsilon_x/2) \text{ and } V_F := \bigcup_{y \in F} B(y;\delta_y/2)\,.$$

Then $V_E, V_F \subseteq X$ are open sets containing E and F, respectively. If $z \in V_E \bigcap V_F$, then for some $x \in E$ and $y \in F$, we have $z \in B(x;\varepsilon_x/2) \bigcap B(y;\delta_y/2)$. Then $d(x,y) < \max\{\varepsilon_x, \delta_y\}$, a contradiction by our choice of ε_x and δ_y. Hence V_E and V_F are disjoint. Since X is Hausdorff, it is a T_4-space.

(2.) Let $\mathbb{R}_\ell$ be the lower limit topological space and let $E, F \subseteq \mathbb{R}_\ell$ be two disjoint closed subsets. For each $x \in E$ (resp. $y \in F$), choose a basic open set $[x, \varepsilon_x)$ (resp. $[y, \delta_y)$) such that $[x, \varepsilon_x) \cap F = \emptyset$ (resp. $E \cap [y, \delta_y) = \emptyset$). Set

$$V_E := \bigcup_{x \in E} [x, \varepsilon_x) \text{ and } V_F := \bigcup_{y \in F} [y, \delta_y)\,.$$

Then $V_E, V_F \subseteq \mathbb{R}_\ell$ are open sets containing E and F, respectively. If $z \in V_E \bigcap V_F$, then for some $x \in E$ and $y \in F$, we have $z \in [x, \varepsilon_x) \bigcap [y, \delta_y)$, a contradiction. Hence V_E and V_F are disjoint. Since $\mathbb{R}_\ell$ is Hausdorff, it is a T_4-space.

Lemma 9.18. *A T_4-space is T_3.*

Proof. Let X be a T_4-space, $E \subseteq X$ be a closed set and $x \in X \setminus E$. Since X is T_1, $\{x\}$ is closed in X. Thus there are open sets V_x and V_E such that $x \in V_x$, $E \subseteq V_E$ and $V_x \bigcap V_E = \emptyset$. Hence X is regular. Since X is T_1, it is T_3. □

Let $X = \{a, b, c, d\}$ and let τ be a topology on X defined by

$$\tau = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}\,.$$

Thus the closed subsets of X are

$$X,\ \emptyset,\ \{b,c,d\},\ \{c,d\},\ \{b,d\},\ \{d\}$$

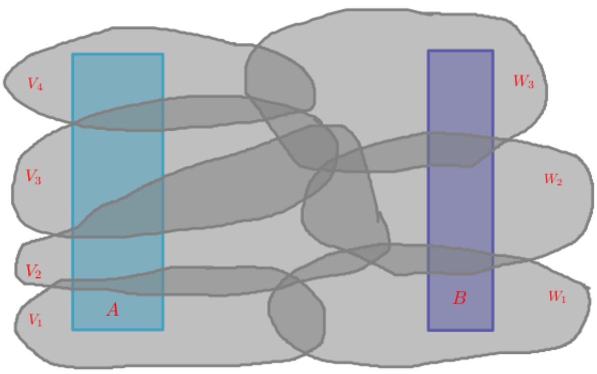
Now it's easy to see that (X,τ) is a normal space which is not regular. Let $Y=\{a,b,c\}\subseteq X$ and let τ_Y be the subspace topology on Y induced from X. Notice that $\{b\}$ and $\{c\}$ are closed subsets of Y, but there are no open sets V_b, V_c in Y satisfying $b\in V_b, c\in V_c$ and $V_b\bigcap V_c=\emptyset$. Thus a subspace of normal space need not be normal. Product of normal space need not be normal: in fact, $\mathbb{R}_\ell\times\mathbb{R}_\ell$ is not normal (see [4, Chapter 4, Section 31]).

Lemma 9.19. *A regular Lindelöf space is normal.*

Proof. Let X be a regular Lindelöf space and let A,B be two disjoint non-empty closed subsets of X. Since X is regular, by Proposition 9.13, for each $a\in A$, there is a neighbourhood V_a of a such that $\overline{V_a}\bigcap B=\emptyset$. Set

$$\mathscr{U}_A := \{V : V\subseteq X \text{ is open and } \overline{V}\cap B=\emptyset\}$$
$$\mathscr{U}_B := \{W : W\subseteq X \text{ is open and } \overline{W}\cap A=\emptyset\}.$$

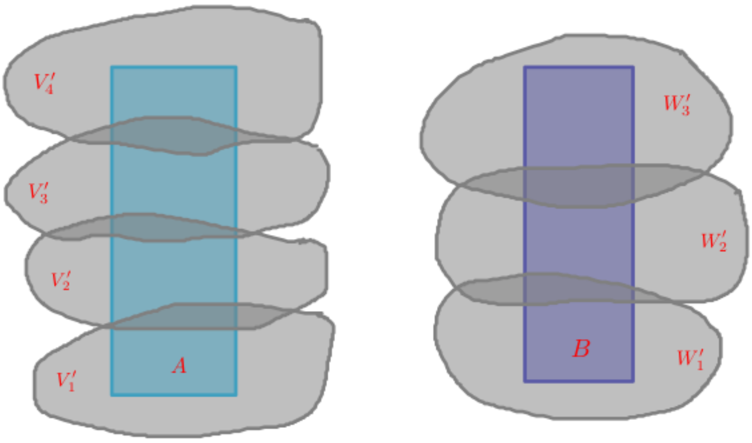
Then $\mathscr{U}_A$ and $\mathscr{U}_B$ are open covers of A and B, respectively. Since A,B are closed subsets of X, both are Lindelöf. Thus both $\mathscr{U}_A$ and $\mathscr{U}_B$ has countable subcover $\{V_n : n\in\mathbb{N}\}$ and $\{W_n : n\in\mathbb{N}\}$, respectively.



For each $n\in\mathbb{N}$, set

$$V_n' := V_n \setminus \left(\cup_{i=1}^n \overline{W_n}\right) \text{ and } W_n' := W_n \setminus \left(\cup_{i=1}^n \overline{V_n}\right).$$

For $m,n\in\mathbb{N}$ with $m\le n$, we have $V_n'\bigcap W_m=\emptyset$ so that $V_n'\bigcap W_m'=\emptyset$. Interchanging the roles of V_n' and W_m', we get $V_n'\bigcap W_m'=\emptyset$, for every $m,n\in\mathbb{N}$.



Since $A \bigcap W_m = \emptyset$ and $B \bigcap V_n = \emptyset$, for all $m, n \in \mathbb{N}$, we have

$$A \subseteq V_A := \bigcup_{n\geq 1} V_n', \quad B \subseteq V_B := \bigcup_{n\geq 1} W_n' \quad \text{and} \quad V_A \bigcap V_B = \emptyset\,.$$

Hence X is normal. □

Exercise 9.20. *(1.)* Prove that a compact Hausdorff space is a T_4-space.

(2.) For a topological space $X \neq \emptyset$, prove that the following conditions are equivalent:

(i) X is normal;

(ii) for any closed set $A \subseteq X$ and any open set $W \subseteq X$ containing A, there is an open set $V_A \subseteq X$ such that $A \subseteq V_A \subseteq \overline{V_A} \subseteq W$;

(iii) for any two disjoint closed sets $A, B \subseteq X$, there is an open set $V \subseteq X$ such that $A \subseteq V$ and $\overline{V} \bigcap B = \emptyset$.

(3.) Prove that a closed subspace of a normal space is normal.

(4.) Give an example to show that quotient of a T_4-space need not be a T_4-space.

Theorem 9.21. *(Urysohn Lemma) Let X be a normal topological space and let A, B be two disjoint non-empty closed subsets of X. Then there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Lemma 9.22. *Let X be a non-empty set, D be a dense subset of positive real numbers and assume for each $t \in D$, we have assigned a set $E_t \subseteq X$ satisfying*

- *if $t < s$, then $E_t \subseteq E_s$;*
- *$X = \bigcup\{E_t : t \in D\}$.*

Define a map $f : X \longrightarrow \mathbb{R}$ by $f(x) = \inf\{t \in D : x \in E_t\}$. Then, for each $s \in \mathbb{R}$

(i) $\{x : f(x) < s\} = \bigcup\{E_t : t \in D$ and $t < s\}$; and

(ii) $\{x : f(x) \leq s\} = \bigcap\{E_t : t \in D$ and $t > s\}$.

Proof. [3, Chapter 4, 2 Lemma] *(i)* We have

$$f(x) = \inf\{t \in D : x \in E_t\} < s \iff x \in E_t \text{ and } t < s \text{ for some } t \in D.$$

Thus we get the equality.

(ii) Let $x \in \bigcap\{E_t : t \in D$ and $t > s\}$. Then $f(x) = \inf\{t \in D : x \in E_t\} \leq s$. Notice that

$$f(x) = \inf\{t \in D : x \in E_t\} \leq s \implies \text{for any } r \in \mathbb{R} \text{ and } s < r \text{ there is } t \in D \text{ such that } x \in E_t \text{ and } t < r.$$

In particular, if we choose $r \in D$, then $x \in E_t \subseteq E_r$. Since this is true for any $r \in D$ with $r > s$, we have $x \in \bigcap\{E_t : t \in D$ and $t > s\}$. □

Lemma 9.23. *Let X be a non-empty topological space, D be a dense subset of positive real numbers and assume for each $t \in D$, we have assigned an open set $E_t \subseteq X$ satisfying*

- *if $t < s$, then $\overline{E_t} \subseteq E_s$;*
- *$X = \bigcup\{E_t : t \in D\}$.*

Then the map $f : X \longrightarrow \mathbb{R}$ defined by $f(x) = \inf\{t \in D : x \in E_t\}$ is continuous.

Proof. [3, Chapter 4, 3 Lemma] Since $\mathscr{S} := \{(-\infty, a) : a \in \mathbb{R}\} \bigcup \{(b, \infty) : b \in \mathbb{R}\}$ is a sub-base for the Euclidean topology on $\mathbb{R}$, it is sufficient to show that, for each $s \in \mathbb{R}$, the set $\{x \in X : f(x) < s\}$ is open in X and the set $\{x \in X : f(x) \leq s\}$ is closed in X. By

Lemma 9.22, $\{x \in X : f(x) < s\}$ is open in X, being a union of open subsets of X. For each $t \in D$ with $t > s$, choose $r \in D$ such that $t > r > s$. By the given condition, $\overline{E_r} \subseteq E_t$. Thus

$$\bigcap \{E_t : t \in D \text{ and } t > s\} = \bigcap \{\overline{E_t} : t \in D \text{ and } t > s\}$$

and the right hand side is a closed subset of X, being an intersection of closed subsets of X. Hence f is continuous. $\square$

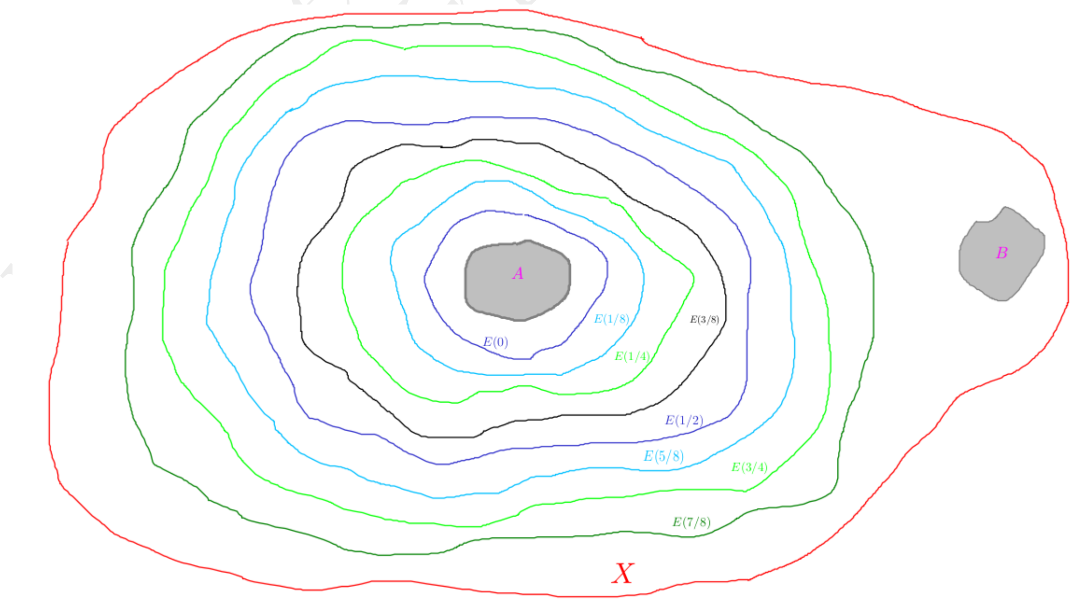
Let $A = \{m \cdot 2^{-n} : m, n \in \mathbb{Z}, n > 0\} \subseteq \mathbb{R}$ be the set of *dyadic rational numbers*. Let $r < s$ be two real numbers. By Archimedean property, there is $n \in \mathbb{N}$ such that $2^{-n} < s - r$. Now choose $m \in \mathbb{Z}$ such that $m - 1 \leq r \cdot 2^n < m$. Then

$$r < m \cdot 2^{-n} \text{ and } m \leq r \cdot 2^n + 1 < s \cdot 2^n \implies r < m \cdot 2^{-n} < s.$$

Hence A is dense in $\mathbb{R}$.

Proof. (of Urysohn Lemma) Let D be the set of all positive dyadic rational numbers. For each $n \in \mathbb{N}$, set $D_n = \{m \cdot 2^{-n} : m = 1, \cdots, 2^n\}$. Then $D_n \subseteq D_{n+1}, \forall n \in \mathbb{N}$. Set $E(t) = X$ for $t \in D$ and $t > 1$, $E(1) = X \setminus B$ and $E(0)$ be an open set containing A such that $\overline{E(0)} \bigcap B = \emptyset$ (such an open set exists since X is normal), i.e. $\overline{E(0)} \subseteq E(1)$. We will use induction on n to construct open set $E(t) \subseteq X$, for each $t \in D$, satisfying the hypothesis of Lemma 9.23:

- $n = 1$: There is an open set $E(1/2) \subseteq X$ such that
$$\overline{E(0)} \subseteq E(1/2) \subseteq \overline{E(1/2)} \subseteq E(1).$$
- $n = 2$: There are open sets $E(1/4), E(3/4) \subseteq X$ such that
$$\overline{E(0)} \subseteq E(1/4) \subseteq \overline{E(1/4)} \subseteq E(1/2)$$
$$\overline{E(1/2)} \subseteq E(3/4) \subseteq \overline{E(3/4)} \subseteq E(1).$$
- $n = 3$: There are open sets $E(1/8), E(3/8), E(5/8), E(7/8) \subseteq X$ such that
$$\overline{E(0)} \subseteq E(1/8) \subseteq \overline{E(1/8)} \subseteq E(1/4) \subseteq \overline{E(1/4)} \subseteq E(3/8) \subseteq \overline{E(3/8)} \subseteq E(1/2)$$
$$\overline{E(1/2)} \subseteq E(5/8) \subseteq \overline{E(5/8)} \subseteq E(3/4) \subseteq \overline{E(3/4)} \subseteq E(7/8) \subseteq \overline{E(7/8)} \subseteq E(1).$$



- Assume for some $n \geq 2$ and every $m = 0, 1, \cdots, 2^n$, the open subsets $E(m/2^n) \subseteq X$ satisfying $t < t'$ in D_n implies $\overline{E(t)} \subseteq E(t')$ is defined.

- Since X is normal, for $m = 1, 3, 5, \cdots, 2^{n+1} - 1$, there is an open set $E(m/2^{n+1}) \subseteq X$ such that

$$\overline{E((m-1)/2^{n+1})} \subseteq E(m/2^{n+1}) \subseteq \overline{E(m/2^{n+1})} \subseteq E((m+1)/2^{n+1}).$$

Thus, for each $t \in D$, there is an open set $E(t) \subseteq X$ satisfying the hypothesis of Lemma 9.23. Hence the map

$$f : X \longrightarrow \mathbb{R}, \quad f(x) := \inf \{t \in D : x \in E(t)\}$$

is continuous. Since $A \subseteq E(t), \forall t \in D$, $f(A) = \{0\}$. Since $E(1) = X \setminus B$ and $E(t) = X$ for $t \in D$ and $t > 1$, $f(B) = \{1\}$. Moreover, by Lemma 9.22, we have $\{x \in X : f(x) \leq 1\} = \bigcap \{E(t) : t \in D \text{ and } t > 1\} = X$. Hence $f(X) \subseteq [0, 1]$. $\square$

Corollary 9.24. *Let X be a normal space, A, B be two non-empty disjoint closed subsets of X and let $\alpha < \beta$ be two real numbers. Then there is a continuous function $f : X \longrightarrow [\alpha, \beta]$ such that $f(A) = \{\alpha\}$ and $f(B) = \{\beta\}$.*

Proof. Left as an exercise. $\square$

Urysohn Lemma does not assert that $A = f^{-1}(0)$ or $B = f^{-1}(1)$. It only asserts that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$. In general, these could be strict inclusions.

Lemma 9.25. *Let X be a T_4-space and let A, B be two non-empty disjoint closed subsets X. Then there is a continuous map $f : X \longrightarrow [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$ if and only if both A and B are G_δ-sets.*

Proof. See [2, Chapter VII, Corollary 4.2] $\square$

Theorem 9.26. (*Tietze Extension Theorem*) *Let X be a normal topological space, $A \subseteq X$ be a closed subset and let $f : A \longrightarrow \mathbb{R}$ be a bounded continuous function. Then f can be extended to a bounded continuous function $g : X \longrightarrow \mathbb{R}$. Moreover, we may choose g in such a way that g is bounded by the bounds of f.*

Proof. See [4, Chapter 4, Section 35]. $\square$

Exercise 9.27. Let X be a non-empty topological space satisfying the the following: for every closed set $\emptyset \neq E \subseteq X$ and every continuous function $f : E \longrightarrow [0, 1] \subseteq \mathbb{R}$, there is a continuous function $\widehat{f} : X \longrightarrow [0, 1]$ such that $\widehat{f}|_E = f$. Prove that X is normal.

Let X be a non-empty topological space and let $f : X \longrightarrow \mathbb{R}$ be a continuous function. The *support of f* is defined as

$$\mathrm{Supp}(f) := \overline{\{x \in X : f(x) \neq 0\}} \subseteq X.$$

Definition 9.28. Let X be a non-empty topological space and let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X such that for any $x \in X$ the set $\{\alpha \in \Lambda : x \in V_\alpha\}$ is a finite set. A *partition of unity subordinate to $\mathscr{U}$* is a family of continuous functions $\{\psi_\alpha : X \longrightarrow \mathbb{R} : \alpha \in \Lambda\}$ satisfying the followings:

- $0 \leq \psi_\alpha(x) \leq 1$, for every $\alpha \in \Lambda$ and $x \in X$;
- $\mathrm{Supp}(\psi_\alpha) \subseteq V_\alpha$, for every $\alpha \in \Lambda$;
- $\sum_{\alpha \in \Lambda} \psi_\alpha(x) = 1$, for every $x \in X$.

By the given condition, for any $x \in X$, there are only finitely many $\alpha \in \Lambda$ such that $x \in V_\alpha$. Since $\mathrm{Supp}(\psi_\alpha) \subseteq V_\alpha$, for every $\alpha \in \Lambda$, for any $x \in X$, there are only finitely many $\alpha \in \Lambda$ such that $\psi_\alpha(x) \neq 0$. Hence, for every $x \in X$, the sum $\sum_{\alpha \in \Lambda} \psi_\alpha(x)$ can be thought of as a finite sum.

Lemma 9.29. (*Existence*) *Let X be a normal topological space and let $\mathscr{U} = \{V_1, \cdots, V_n\}$ be an open cover of X. Then there is a partition of unity subordinate to $\mathscr{U}$.*

Proof. Set $A_1 := X \setminus \bigcup_{i=2}^n V_i$. Then A_1 is a closed subset of X and $A_1 \subseteq V_1$. Since X is normal, there is an open set $W_1 \subseteq X$ such that $A \subseteq W_1 \subseteq \overline{W_1} \subseteq V_1$. Then $\{W_1, V_2, \cdots, V_n\}$ is an open cover of X. Let $2 \leq k \leq n-1$ and assume that for each $i \in \{1, \cdots, k\}$, there is an open set $W_i \subseteq X$ such that $\overline{W_i} \subseteq V_i$ and $\{W_1, \cdots, W_k, V_{k+1}, \cdots, V_n\}$ is an open cover of X. Set $A_{k+1} = X \setminus [(\bigcup_{i=1}^k W_i) \bigcup (\bigcup_{j=k+2}^n V_j)]$. Then A_{k+1} is a closed subset of X and $A_{k+1} \subseteq V_{k+1}$. Thus there is an open set $W_{k+1} \subseteq X$ such that $A \subseteq W_{k+1} \subseteq \overline{W_{k+1}} \subseteq V_{k+1}$. Moreover, $\{W_1, \cdots, W_{k+1}, V_{k+2}, \cdots, V_n\}$ is an open cover of X. By Induction, we get an open cover $\mathscr{W} = \{W_1, \cdots, W_n\}$ of X such that $\overline{W_i} \subseteq V_i, \forall i = 1, \cdots, n$. Using the same argument, we get an open cover $\mathscr{T} = \{T_1, \cdots, T_n\}$ of X such that $\overline{T_i} \subseteq W_i, \forall i = 1, \cdots, n$.

For each $i \in \{1, \cdots, n\}$, $\overline{T_i}$ and $X \setminus W_i$ are two disjoint closed subsets of X. Since X is normal, by Urysohn Lemma, there is a continuous function $\phi_i : X \longrightarrow [0,1]$ such that $\phi_i(\overline{T_i}) = \{1\}$ and $\phi_i(X \setminus W_i) = \{0\}$. Then $\mathrm{Supp}(\phi_i) \subseteq \overline{W_i} \subseteq V_i$. Define

$$\psi : X \longrightarrow \mathbb{R}, \text{ by } \psi(x) := \sum_{i=1}^n \phi_i(x).$$

Then ψ is a continuous function. For each $\ell \in \{1, \cdots, n\}$, define

$$\psi_\ell : X \longrightarrow [0,1], \text{ by } \psi_\ell(x) := \frac{\phi_\ell(x)}{\psi(x)}.$$

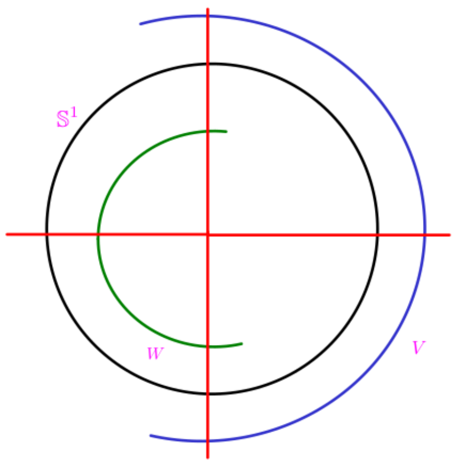
Since $X = \bigcup_{i=1}^n W_i$, for any $x \in X$, $\psi(x) > 0$. Thus ψ_ℓ is a well-defined continuous map. Now it's easy to see that $\{\psi_1, \cdots \psi_n\}$ is a partition of unity subordinate of $\mathscr{U}$. $\square$

Definition 9.30. A non-empty second countable, Hausdorff topological space X is said to be an *n-manifold* if every $x \in X$ has an open neighbourhood V_x such that V_x is homeomorphic to an open ball in $\mathbb{R}^n$ or equivalently, homeomorphic to $\mathbb{R}^n$.

Example 9.31. (*1.*) The Euclidean space $\mathbb{R}^n$ is an n-manifold by definition.

(*2.*) The circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is 1-manifold. Let $\delta \in (0,1)$ and set

$$V := \{(x,y) \in \mathbb{S}^1 : x > -\delta\} \text{ and } W := \{(x,y) \in \mathbb{S}^1 : x < \delta\}.$$



Notice that both V and W are open subsets of X and both homeomorphic to $\mathbb{R}$. Since $X = V \bigcup W$, we get X is a 1-manifold.

Exercise 9.32. *(1.)* Prove that any non-empty open subset of $\mathbb{R}^n$ is an n-manifold.

(2.) Let X be an n-manifold. Prove that any non-empty open subset of X is an n-manifold.

Theorem 9.33. *If $m \neq n$, then a non-empty topological space can not be simultaneously an m-manifold and an n-manifold.*

Proof. Beyond the scope of this note. □

Proposition 9.34. *Let X be a compact n-manifold. Then X can be topologically embedded in $\mathbb{R}^k$, for some $k \in \mathbb{N}$.*

Proof. For each $x \in X$, let V_x be an open neighbourhood of x such that V_x is homeomorphic to $\mathbb{R}^n$. Then $\{V_x : x \in X\}$ is an open cover of X. Since X is compact, it has a finite subcover. By renaming, we may assume that $\mathscr{U} = \{V_1, \cdots, V_t\}$ be such an open cover of X. For each $i \in \{1, \cdots, t\}$, fix a homeomorphism $f_i : V_i \longrightarrow \mathbb{R}^n$. Since X is compact Hausdorff topological space, it is normal. Let $\{\psi_1, \cdots, \psi_t\}$ be a partition of unity subordinate of $\mathscr{U}$. For each $i \in \{1, \cdots, t\}$, define

$$g_i : X \longrightarrow \mathbb{R}^m, \;\; g_i(x) := \begin{cases} \psi_i(x) \cdot f_i(x) & \text{if } x \in V_i \\ 0 & \text{if } x \in X \setminus \mathrm{Supp}(\psi_i) \end{cases}$$

Since $\mathrm{Supp}(\psi_i) \subseteq V_i$, g_i is a well-defined continuous function. Define

$$\begin{aligned} \phi : X &\longrightarrow \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{t\text{-times}} \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{t\text{-times}} \\ x &\mapsto (\psi_1(x), \cdots, \psi_t(x), g_1(x), \cdots, g_t(x)) \end{aligned}$$

Then ϕ is a continuous map. Let $x, y \in X$ be such that $\phi(x) = \phi(y)$. Choose $i \in \{1, \cdots, t\}$ such that $\psi_i(x) > 0$. Then $\psi_i(y) > 0$. Thus $x, y \in \mathrm{Supp}(\psi_i) \subseteq V_i$ and

$$\psi_i(x) \cdot f_i(x) = \psi_i(y) \cdot f_i(y) \implies f_i(x) = f_i(y) \implies x = y$$

since f_i is injective. Thus $\phi : X \longrightarrow \phi(X)$ is a bijective continuous map. Since X is compact and $\phi(X)$ is Hausdorff, ϕ is a homeomorphism. □

Definition 9.35. A topological space X is said to be *completely regular* if for any closed set $A \subsetneq X$ and any $x \in X \setminus A$, there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1, \forall y \in A$. A completely regular T_1-space is called a *Tychonoff space*, denoted by $T_{3\frac{1}{2}}$ or by T_{3a}.

A normal (resp. T_4) topological space is completely regular (resp. Tychonoff) and a complete regular space (rep. Tychonoff space) is regular (resp. T_3). But the converses are not true. The examples are hard to construct, [2, Chpater VII, Section 7].

Exercise 9.36. Every subspace of a completely regular space (resp. Tychonoff space) is completely regular (resp. Tychonoff)

Proposition 9.37. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. Then the product space $\prod_{\alpha \in \Lambda} X_\alpha$ is a completely regular space (resp. a Tychonoff space) if and only if each X_α is a completely regular space (resp. a Tychonoff space).*

Proof. It is sufficient to prove the case of completely regular spaces. Set $X = \prod_{\alpha \in \Lambda} X_\alpha$ and for each $\alpha \in \Lambda$, let $p_\alpha : X \longrightarrow X_\alpha$ be the α-th coordinate function.

Assume that each X_α is a completely regular space. Let $E \subsetneq X$ be a closed set and $x \in X \setminus E$. Choose $\alpha_1, \cdots, \alpha_n \in \Lambda$ and open sets $V_{\alpha_i} \subseteq X_{\alpha_i}, 1 \leq i \leq n$, such that

$x \in \bigcap_{i=1}^{n} p_{\alpha_i}^{-1} V_{\alpha_i} \subseteq X \setminus E$. By the given condition, for each $i = 1, \cdots, n$, there is a continuous map $f_{\alpha_i} : X_{\alpha_i} \longrightarrow [0,1]$ such that $f_{\alpha_i}(x_{\alpha_i}) = 0$ and $f_{\alpha_i}(X_{\alpha_i} \setminus V_{\alpha_i}) = \{1\}$. Define

$$f : X \longrightarrow \mathbb{R}, \quad f(y) \;=\; \max\{f_{\alpha_i}(y_{\alpha_i}) \,:\, 1 \le i \le n\}.$$

Then f is the maximum of the continuous functions $f_{\alpha_i} \circ p_{\alpha_i} : X \longrightarrow X_{\alpha_i}$, $1 \le i \le n$ and hence continuous. Moreover, $f(X) \subseteq [0,1]$.

$$\begin{tikzcd}
\prod_{\alpha\in\Lambda} X_\alpha \arrow[rr, "p_{\alpha_i}"] \arrow[dr, "f_{\alpha_i}\circ p_{\alpha_i}"'] & & X_{\alpha_i} \arrow[dl, "f_{\alpha_i}"] \\
& {[0,1]} &
\end{tikzcd}$$

Then $f(x) = 0$. Let $y \in E$. Choose $k \in \{1, \cdots, n\}$ such that $y_{\alpha_k} \notin V_{\alpha_k}$. Then $f_{\alpha_k}(y_{\alpha_k}) = 1$. Hence $f(E) = \{1\}$. Thus X is completely regular.

Now assume that X is completely regular. Since each X_α is homeomorphic to a subspace of X, by the above Exercise, X_α is completely regular. $\square$

Let X be a topological space and let $\mathscr{F}$ be a family of functions such that each $f \in \mathscr{F}$ is a function from X to a topological space Y_f (for different f's, Y_f's need not be distinct). For each $f \in \mathscr{F}$, let $p_f : \prod_{f\in\mathscr{F}} Y_f \longrightarrow Y_f$ the f-th coordinate projection. Define

$$\mathscr{E} := \mathscr{E}_{\mathscr{F}} : X \longrightarrow \prod_{f\in\mathscr{F}} Y_f \;\text{ such that }\; p_f(\mathscr{E}_{\mathscr{F}}(x)) \;=\; f(x)\,.$$

$$\begin{tikzcd}
X \arrow[rr, "\mathscr{E}"] \arrow[dr, "f"'] & & \prod_{f\in\mathscr{F}} Y_f \arrow[dl, "p_f"] \\
& Y_f &
\end{tikzcd}$$

The map $\mathscr{E}$ is called *evaluation map*. We say $\mathscr{F}$ *distinguishes points* in X if for any two distinct points $x, y \in X$, there is $f \in \mathscr{F}$ such that $f(x) \neq f(y)$ in Y_f. And we say $\mathscr{F}$ *distinguishes points and closed sets* in X if for any closed set $A \subseteq X$ and any $x \in X \setminus A$, there is $f \in \mathscr{F}$ such that $f(x) \notin \overline{f(A)} \subseteq Y_f$.

Lemma 9.38. (*Embedding Lemma*) *Let X be a topological space and let $\mathscr{F}$ be a family of functions such that each $f \in \mathscr{F}$ is a function from X to a topological space Y_f. Then*

(i) the evaluation map $\mathscr{E}$ is continuous if and only if the function $f : X \longrightarrow Y_f$ is continuous, for every $f \in \mathscr{F}$.

(ii) $\mathscr{E}$ is one-one if and only if $\mathscr{F}$ distinguishes points in X.

(iii) if $\mathscr{F}$ distinguishes points and closed sets in X, $\mathscr{E} : X \longrightarrow \mathscr{E}(X) \subseteq \prod_{f\in\mathscr{F}} Y_f$ is an open map.

Proof. (i) Follows from the properties of product spaces.

(ii) Suppose $\mathscr{E}$ is one-one. Let $x, x' \in X$ be two distinct points. Then there is $f \in \mathscr{F}$ such that $f(x) = p_f(\mathscr{E}(x)) \neq p_f(\mathscr{E}(x')) = f(x')$ in Y_f. Hence $\mathscr{F}$ distinguishes points in X. The converse is trivial.

(iii) Suppose $\mathscr{F}$ distinguishes points and closed sets in X. Let $\emptyset \neq V \subseteq X$ be an open set and let $y_0 \in \mathscr{E}(V)$. Choose $x_0 \in V$ such that $\mathscr{E}(x_0) = y_0$. Since $x_0 \notin X \setminus V$, by the given condition, there is $f \in \mathscr{F}$ such that

$$p_f(\mathscr{E}(x_0)) = f(x_0) \notin \overline{f(X \setminus V)} \subseteq Y_f \implies \mathscr{E}(x_0) \notin p_f^{-1}\big(\overline{f(X \setminus V)}\big) \subseteq \prod_{f\in\mathscr{F}} Y_f.$$

Set $W' := \left(\prod_{f\in\mathscr{F}} Y_f\right) \setminus p_f^{-1}\left(\overline{f(X\setminus V)}\right)$. Then W' is an open set in $\prod_{f\in\mathscr{F}} Y_f$ and thus $W := W' \bigcap \mathscr{E}(X)$ is an open set in $\mathscr{E}(X)$ containing y_0. Let $y \in W$. Choose $x \in X$ such that $\mathscr{E}(x) = y$. Then

$$\mathscr{E}(x) \notin p_f^{-1}\left(\overline{f(X\setminus V)}\right) \implies f(x) = p_f(\mathscr{E}(x)) \notin \overline{f(X\setminus V)}.$$

Thus $x \in V, y = f(x) \in \mathscr{E}(V)$ and $W \subseteq \mathscr{E}(V)$. In other words, $\mathscr{E}(V)$ contains a neighbourhood of each of its point. Hence $\mathscr{E} : X \longrightarrow \mathscr{E}(X)$ is an open map. $\square$

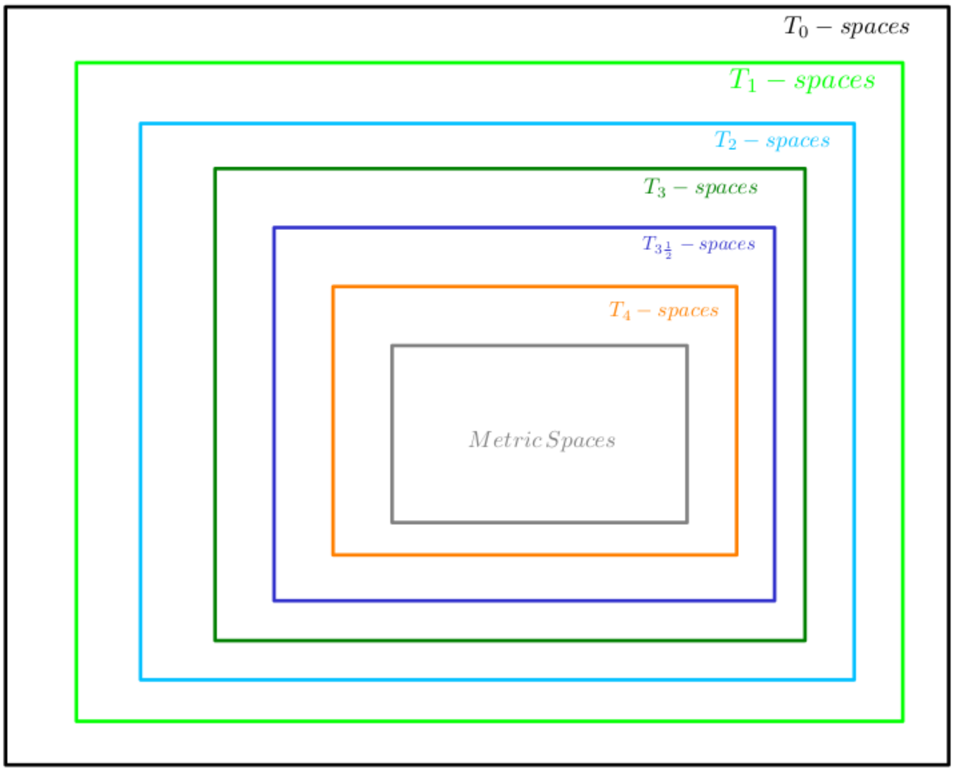
Let $I = [0,1] \subseteq \mathbb{R}$ be the closed unit interval. By a *cube* we mean a Cartesian product of non-empty family of closed unit interval I. By Tychonoff Theorem, a cube is a compact Hausdorff space and hence a T_4-space.

Theorem 9.39. (*Tychonoff Embedding Theorem*) *Let X be a non-empty topological space. Then X is a Tychonoff space if and only if X is homeomorphic to a subspace of a cube.*

Proof. Let $\mathscr{F}$ be the family of continuous maps $X \longrightarrow I$. Since X is a Tychonoff space, $\mathscr{F}$ distinguishes points and closed sets in X and $\mathscr{F}$ distinguishes points in X (single point sets are closed in X). Denote $\mathscr{I} := \prod_{f\in\mathscr{F}} Y_f$ where $Y_f = I, \forall f \in \mathscr{F}$. Then $\mathscr{I}$ is a cube. By Embedding Lemma (Lemma 9.38), the evaluation map $\mathscr{E} : X \longrightarrow \mathscr{I}$ is a homeomorphism onto a subspace of the cube $\mathscr{I}$.

Let X be a topological space such that X is homeomorphic to a subspace Y of a cube $\mathscr{I}$. Since $\mathscr{I}$ is a Tychonoff space, so is Y. Hence X is a Tychonoff space. $\square$

Exercise 9.40. Let X be a connected completely regular space. If $|X| > 1$, the prove that every non-empty open subset of X is uncountable.



Definition 10.1. Let X be a topological space.

- We say X is *locally compact at $x \in X$* if there is an open neighbourhood V_x of x such that its closure $\overline{V_x}$ is compact in X.
- We say X is *locally compact* if it is locally compact at every point.

Example 10.2. *(1.)* Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. For each $j = 1, \cdots, n$, choose a bounded open interval $V_j \subseteq \mathbb{R}$ containing x_j. Then $x \in \prod_{j=1}^n V_j \subseteq \mathbb{R}^n$ is a neighbourhood of x and $\overline{\prod_{j=1}^n V_j} = \prod_{j=1}^n \overline{V_j}$ is compact. Thus the Euclidean space $\mathbb{R}^n$ is locally compact.

(2.) Let X be an uncountable set, $a \in X$ and consider the topology on X:

$$\tau_a = \mathscr{P}(X \setminus \{a\}) \bigcup \{V \subseteq X \,:\, a \in V \text{ and } X \setminus V \text{ is finite}\}.$$

For any $x \in X \setminus \{a\}$, the set $\{x\}$ is a clopen set. Hence X is locally compact at x. Let V be an open set containing a. Then $\overline{V} = V$ and hence X is locally compact at a. Thus X is locally compact.

(3.) Let X be a compact topological space. Then for any $x \in X$ and any neighbourhood V of X, $\overline{V}$ is compact. Hence X is locally compact. On the other hand, a locally compact topological space need not be compact. For example, let Y be an infinite set equipped with discrete topology. Then Y is locally compact, but not compact.

(4.) Let $X = \mathbb{N}$ and define a topology on X as follows:

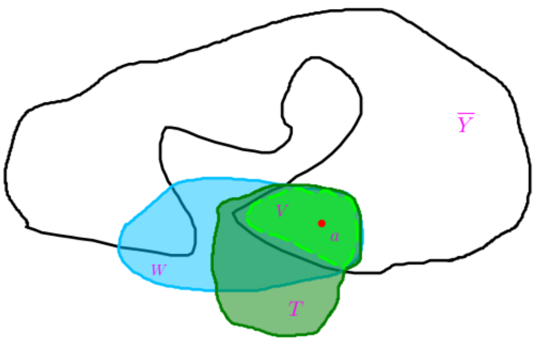
$$\tau := \{\emptyset, \mathbb{N}, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \cdots\}.$$

Let $n \in \mathbb{N}$. Then $V_n = \{1, 2, \cdots, n\}$ is a neighbourhood of n and we have $\overline{V_n} = X$. Moreover, any neighbourhood W of n must contain V_n and for any such neighbourhood we have $\overline{W} = X$. Since X is not compact, X is not locally compact at n. This is true for every $n \in \mathbb{N}$.

Lemma 10.3. *Let X be a non-empty Hausdorff space. Then X is locally compact at $a \in X$ if and only if given any open neighbourhood W of a, there is an open neighbourhood V of a such that $a \in V \subseteq \overline{V} \subseteq W$ and $\overline{V}$ is compact in X.*

Proof. Suppose X is locally compact at a and let Y be an open neighbourhood of a such that $\overline{Y}$ is compact. Let W be an open neighbourhood of a. Then $\overline{Y} \bigcap W$ is an open neighbourhood of a in $\overline{Y}$. Since $\overline{Y}$ is compact and Hausdorff, it is regular. Thus there is an open set Z in $\overline{Y}$ such that $a \in Z \subseteq \mathrm{Cl}_{\overline{Y}}(Z) \subseteq \overline{Y} \bigcap W$. Let $T \subseteq X$ be an open set such that $Z = \overline{Y} \bigcap T$. Set $V = T \bigcap Y$. Then

$$a \in V \subseteq \overline{V} \subseteq \overline{T} \bigcap \overline{Y} = \mathrm{Cl}_{\overline{Y}}(\overline{Y} \cap T) = \mathrm{Cl}_{\overline{Y}}(Z) \subseteq \overline{Y} \bigcap W \subseteq W.$$



Moreover, $\overline{V}$ is compact being a closed subspace of the compact space $\overline{Y}$. $\square$

Corollary 10.4. *Let X be a Hausdorff locally compact topological space. Then any open subset of X is locally compact in subspace topology.*

Proof. Let $Y \subseteq X$ be an open set and let $y \in Y$. Since Y is a neighbourhood of y, by Lemma 10.3, there is an open set V in X such that $x \in V \subseteq \overline{V} \subseteq Y$ and $\overline{V}$ is compact in X. Notice that $Cl_Y(V) = \overline{V} \subseteq Y$. Thus Y is locally compact at y. Since $y \in Y$ is arbitrary, Y is locally compact in subspace topology. □

Arbitrary subspaces of a Hausdorff locally compact topological space need not be locally compact. For example, $\mathbb{Q}$ is not locally compact as subspace of the Euclidean space $\mathbb{R}$: Let $x \in \mathbb{Q}$ and $V \subseteq \mathbb{Q}$ be an open set containing x such that $\mathrm{Cl}_{\mathbb{Q}}(V)$ is compact in $\mathbb{Q}$. Then $\mathrm{Cl}_{\mathbb{Q}}(V)$ is compact in $\mathbb{R}$ and hence a closed and bounded subset of $\mathbb{R}$, by Heine-Borel theorem. Let $W \subseteq \mathbb{R}$ be an open set such that $V = \mathbb{Q} \bigcap W$. Then any irrational point in W is a limit point of V in $\mathbb{R}$. This contradicts that $\mathrm{Cl}_{\mathbb{Q}}(V)$ is closed in $\mathbb{R}$.

Exercise 10.5. *(1.)* Is $\mathbb{R}_\ell$ locally compact?

(2.) Prove that a closed subset of a locally compact topological space is locally compact in subspace topology.

Proposition 10.6. *Let X be Hausdorff locally compact topological space and let $Y \subseteq X$. Then Y is locally compact in subspace topology if and only if there is a closed set $E \subseteq X$ and an open set $V \subseteq X$ such that $Y = E \bigcap V$.*

Proof. Suppose Y can be written as $Y = E \bigcap V$ for some closed set $E \subseteq X$ and open set $V \subseteq X$. Then by Corollary 10.4 and the above Exercise, Y is locally compact in subspace topology.

Conversely assume that Y is locally compact in subspace topology. Let $y \in Y$ and let V_y be an open set in Y containing y such that $\mathrm{Cl}_Y(V_y)$ is compact in Y. Choose an open set W_y in X such that $V_y = W_y \bigcap Y$. Then

$$\overline{W_y \cap Y} \cap Y = \overline{V_y} \cap Y = \mathrm{Cl}_Y(V_y) \implies \overline{W_y \cap Y} \cap Y \text{ is closed in } X$$

since $\mathrm{Cl}_Y(V_y)$ is compact and X is Hausdorff. Thus

$$\overline{W_y \cap Y} \subseteq \overline{W_y \cap Y} \cap Y \subseteq Y \implies \overline{Y} \cap W_y \subseteq \overline{W_y \cap Y} \subseteq Y.$$

Now $\overline{Y} \bigcap W_y$ is an open set of $\overline{Y}$ containing y and contained in Y. Hence Y is open in $\overline{Y}$. Thus there is an open set G in X such that $Y = G \bigcap \overline{Y}$. □

Corollary 10.7. *A dense subset of a compact Hausdorff space is locally compact if and only if it is open.*

Let X be a topological space. A subset of X is called *locally closed* if it can be written as a intersection of a closed subset and an open subset of X. By the above Proposition, the locally compact subspaces of a Hausdorff locally compact space are precisely the locally closed subspaces.

Let (X, τ) be a non-empty topological space which is not locally compact. Let τ' be a the discrete topology on X and consider the map $f : (X, \tau') \longrightarrow (X, \tau)$ defined by $f(x) = x$. Then f is continuous, (X, τ) is locally compact but $\mathrm{Im}(f)$ is not locally compact. In other words, continuous image of a locally compact space need not be locally compact.

Exercise 10.8. Let $f : X \longrightarrow Y$ be an open surjective continuous map between two topological spaces. If X is locally compact, prove that Y is locally compact.

Proposition 10.9. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. The product space $X := \prod_{\alpha\in\Lambda} X_\alpha$ is locally compact if and only if each X_α is locally compact and all but finitely many X_α are compact.*

Proof. For each $\alpha \in \Lambda$, let $p_\alpha : X \longrightarrow X_\alpha$ be the α-th coordinate projection. Suppose X is locally compact. Since each p_α is open continuous surjection, by the above Exercise, X_α is locally compact. Let $x \in X$ and let $x \in V \subseteq X$ be an open set such that $\overline{V}$ is compact. Choose $\alpha_1, \cdots, \alpha_n \in \Lambda$ and open sets $V_{\alpha_i} \subseteq X_{\alpha_i}$ such that $x \in \bigcap_{i=1}^n p_{\alpha_i}^{-1} V_{\alpha_i} \subseteq V$. Then for any $\beta \in \Lambda$ with $\beta \neq \alpha$, we have $p_\beta(\overline{V}) = X_\beta$. Hence each such X_β is compact.

Now assume each X_α is locally compact and there are $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that for any $\beta \in \Lambda$ with $\beta \neq \alpha_i, 1 \leq i \leq n$, X_β is compact. Let $x \in X$. For each $i = 1, \cdots, n$, choose an open set $V_{\alpha_i} \subseteq X_{\alpha_i}$ containing x_{α_i} such that $\overline{V_{\alpha_i}}$ is compact. Set $V = \bigcap_{i=1}^n p_{\alpha_i}^{-1} V_{\alpha_i} \subseteq X$. Then V is an open neighbourhood of x and by Tychonoff theorem, $\overline{V}$ is compact. Thus X is locally compact. □

Let X be a topological space and let ∞ be a point not in X. Consider the set $X_\infty = X \bigcup \{\infty\}$ and a set $\tau_\infty \subseteq \mathscr{P}(X_\infty)$ defined as follows:

- for $\infty \notin V$, $V \in \tau_\infty$ if and only if V is open in X;
- for $\infty \in V$, $V \in \tau_\infty$ if and only if $X_\infty \setminus V \subseteq X$ is closed and compact.

We claim that τ_∞ is a topology on X_∞:

(i) The empty set $\emptyset$ and X_∞ is in τ_∞.

(ii) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a collection of members of τ_∞. Based on whether $\infty \in V_\alpha$ or not, we consider two cases:
- Suppose $\infty \notin V_\alpha, \forall \alpha \in \Lambda$. Since each V_α is open in X, $\bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_\infty$.
- Suppose $\infty \in V_\alpha$ for some $\alpha \in \Lambda$. Since $\bigcap_{\alpha\in\Lambda} (X_\infty \setminus V_\alpha)$ is closed and compact in X, we have $\bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_\infty$.

(iii) Let $\{V_i : 1 \leq i \leq n\}$ be a finite collection of members of τ_∞. Based on whether $\infty \in V_i$ or not, we consider three cases:
- Suppose $\infty \notin V_i, \forall\, i$. Since each V_i is open in X, $\bigcap_{i=1}^n V_\alpha \in \tau_\infty$.
- Suppose $\infty \in V_i, \forall\, i$. Since $\bigcup_{i=1}^n (X \setminus V_i)$ is closed and compact in X, $\bigcap_{i=1}^n V_i \in \tau_\infty$.
- Suppose $\infty \in V_i$, for some $j \in \{1, \cdots, n\}$, but not all. Renaming, if necessary, we may assume that $\infty \in V_i, 1 \leq i \leq k$ and $\infty \notin V_j, k+1 \leq j \leq n$, for some $1 \leq k \leq n-1$. Since $(\bigcap_{i=1}^k V_i) \bigcap X = X \setminus (\bigcup_{i=1}^k (X_\infty \setminus V_i))$ and $X_\infty \setminus V_i$ is closed in X for each $i \in \{1, \cdots, k\}$, $(\bigcap_{i=1}^k V_i) \bigcap X$ is open in X. Hence $\bigcap_{i=1}^n V_i$ is open in X so that $\bigcap_{i=1}^n V_i \in \tau_\infty$.

Thus (X_∞, τ_∞) is a topological space. Notice that X is an open subset of X_∞ and the subspace topology on X induced from X_∞ is same as the original topology on X.

Lemma 10.10. *Let X be a non-empty topological space and let X_∞ be the topological space defined as above. Then X_∞ is compact.*

Proof. Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X_∞. Choose $\beta \in \Lambda$ such that $\infty \in V_\beta$. Notice that $\mathscr{U}_X = \{V_\alpha \bigcap X : \alpha \in \Lambda, \alpha \neq \beta\}$ is an open cover of $X_\infty \setminus V_\beta$. Since $X_\infty \setminus V_\beta$ is compact in X, there is $\alpha_1, \cdots, \alpha_n \in \Lambda$ such that $X_\infty \setminus V_\beta \subseteq \bigcup_{i=1}^n (V_{\alpha_i} \bigcap X)$. Thus $\{V_{\alpha_1}, \cdots, V_{\alpha_n}, V_\beta\}$ is a finite subcover of $\mathscr{U}$. Hence X_∞ is compact. □

Let X be a non-compact topological space. The compact topological X_∞ space defined as above is called the (Alexandroff) *one-point compactification* of X.

Lemma 10.11. *Let X be a non-compact topological space and let X_∞ be the one-point compactification of X. Then X is dense in X_∞.*

Proof. Let $\emptyset \neq V \subseteq X_\infty$ be an open set. If $\infty \notin V$, then $V \subseteq X$. If $\infty \in V$, then $X \bigcap V \neq \emptyset$ as X is not compact. Hence X is dense in X_∞. □

Proposition 10.12. *Let X be a non-compact topological space. Then then one-point compactification X_∞ of X is Hausdorff if and only if X is Hausdorff and locally compact.*

Proof. Suppose X_∞ is a Hausdorff space. Then X is Hausdorff. Let $x \in X$. Choose two open sets V_1, V_2 in X_∞ such that $x \in V_1, \infty \in V_2$ and $V_1 \bigcap V_2 = \emptyset$. Then $V_1 \subseteq X_\infty \setminus V_2$ and hence $\overline{V_1} \subseteq X_\infty \setminus V_2$ is a compact subset of X. Thus X is locally compact at x. Since this is true for every $x \in X$, X is locally compact.

Now assume X is a Hausdorff locally compact space. Let $a, b \in X_\infty$ be two distinct points. If $a, b \in X$, then there are open sets V_1, V_2 in X such that $a \in V_1, b \in V_2$ and $V_1 \bigcap V_2 = \emptyset$. Now assume $b = \infty$. Let $V \subseteq X$ be an open set such that $a \in V$ and $\mathrm{Cl}_X(V)$ is compact in X. Set $W = X_\infty \setminus \mathrm{Cl}_X(V)$. Then W is an open subset of X_∞ containing ∞. By construction, $V \bigcap W = \emptyset$. Hence X_∞ is Hausdorff. □

Corollary 10.13. *A locally compact Hausdorff space is Tychonoff.*

Proof. If X is compact, then there is nothing to prove. We assume now that X is non-compact. Let X_∞ be the one-point compactification of X. Then X_∞ is a compact Hausdorff space and hence a Tychonoff space. Hence X is a Tychonoff space. □

Let $X = \mathbb{Q}$ be equipped with subspace topology from $\mathbb{R}$. Since $\mathbb{Q}$ is not locally compact, X_∞ is not Hausdorff. Since X_∞ is compact, it is locally compact. Moreover, $\mathbb{Q}$ is an open subset of X_∞. This gives an example of a locally compact non-Hausdorff space which contains an open set which is not locally compact (see Corollary 10.4).

Lemma 10.14. *Let X be a Hausdorff locally compact non-compact space and let X_∞ be the one-point compactification of X. Let Y be a compact Hausdorff space and $f : X \longrightarrow Y$ be a homeomorphism onto a subspace of Y such that $Y \setminus f(X)$ is a single point set. Then f can be extended to a homeomorphism $g : X_\infty \longrightarrow Y$.*

Proof. Let $Y \setminus f(X) = \{y_0\}$. Since Y is Hausdorff, $f(X)$ is open in Y. Define

$$g : X_\infty \longrightarrow Y, \;\; g(x) = \begin{cases} f(x) & \text{if } x \in X \\ y_0 & \text{if } x = \infty \end{cases}$$

Then g is a bijection. Let $E \subseteq Y$ be a closed set. Suppose $y_0 \notin E$. Then E is a closed subset of $f(X)$. Thus $g^{-1}E = f^{-1}E$ is closed in X. Since Y is a compact Hausdorff space, E is also compact. Hence $f^{-1}E$ is a compact subset of X. Thus $g^{-1}E$ is a closed subset of X_∞. Now suppose $y_0 \in E$ and let $E' = E \setminus \{y_0\}$. Then E' is a closed subset of $f(X)$. Now $X_\infty \setminus g^{-1}E = X \setminus f^{-1}E'$ is open in X and hence open in X_∞. Thus $g^{-1}E$ is closed in X_∞. Hence g is continuous.

Let V be an open subset of X_∞. If $\infty \notin V$, then V is an open subset of X and hence $g(V) = f(V)$ is an open subset of Y. Now assume $\infty \in V$. Then $X_\infty \setminus V$ is a closed and compact subset of X. Thus $g(X_\infty \setminus V) = f(X_\infty \setminus V)$ is a compact subset of $f(X)$. Since Y is Hausdorff, $Y \setminus g(V) = g(X_\infty \setminus V)$ is closed in Y so that $g(V)$ is open in Y. Hence g is an open map. □

Example 10.15. *(1.)* Let $X = (0, 1]$ be subspace of $\mathbb{R}$. Then X_∞ is homeomorphic to the closed interval $[0, 1]$.

(2.) Let $X = \mathbb{N}$ be equipped with the subspace topology of $\mathbb{R}$. Then X_∞ is homeomorphic to the subspace $\{1/n : n \in \mathbb{N}\} \bigcup \{0\}$ of $\mathbb{R}$.

(3.) Let $n \in \mathbb{N}$ and $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ be the n-sphere. Via stereographic projection, $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n \setminus \{\text{north pole}\}$. Hence the one-point compactification of $\mathbb{R}^n$ is $\mathbb{S}^n$.

Exercise 10.16. *(1.)* Let X, Y be Hausdorff locally compact non-compact topological spaces and let X_∞ and $Y_{\infty'}$ be the one-point compactification of X and Y, respectively. Prove that any homeomorphism $f : X \longrightarrow Y$ can be extended to a homeomorphism $\widehat{f} : X_\infty \longrightarrow Y_{\infty'}$.

(2.) Let X be a Hausdorff locally compact non-compact topological space and let X_∞ be the one-point compactification of X. Let Y be a Hausdorff locally compact topological space. Fix $y_0 \in Y$. Suppose $f : X \longrightarrow Y$ be a continuous map satisfying the following: for any closed subset $E \subseteq Y$ with $y_0 \neq E$, $f^{-1}E$ is a compact subset of X. Prove that f can be extended uniquely to a continuous map $\widehat{f} : X_\infty \longrightarrow Y$.

Definition 10.17. Let X be a non-empty topological space. A *compactification* of X is a pair (K_X, h_X) where K_X is a compact topological space and $h_X : X \longrightarrow K_X$ is an embedding such that $h_X(X)$ is dense in K_X. If in addition, K_X is a Hausdorff space, we say (K_X, h_X) is a *Hausdorff compactification* of X.

If X is non-compact space, then an one-point compactification of X is a compactification of X. Moreover, if X is Hausdorff, locally compact and non-compact, then the one-point compactification of X is a Hausdorff compactification of X. By Corollary 10.13, any such space is a Tychonoff space. So we ask the following question: *does every Tychonoff space admit a Hausdorff compactification ?*

Let X be a non-empty Tychonoff space and let $I := [0,1] \subseteq \mathbb{R}$ be the closed unit interval. Let $\mathscr{F}$ be the family of all continuous functions $X \longrightarrow I$. For each $f \in \mathscr{F}$, let $p_f : \mathscr{I} := \prod_{f\in\mathscr{F}} I_f \longrightarrow I_f$ the f-th coordinate projection where $I_f = I, \forall f \in \mathscr{F}$. Recall, the evaluation map is defined as

$$\mathscr{E} := \mathscr{E}_{\mathscr{F}} : X \longrightarrow \mathscr{I} \quad \text{such that} \quad p_f(\mathscr{E}_{\mathscr{F}}(x)) \;=\; f(x)\,.$$

Since X is Tychonoff, by Tychonoff embedding theorem (Theorem 9.39), $\mathscr{E}$ is an embedding.

Definition 10.18. The *Stone-Čech compactification* of a Tychonoff space is a pair $(\beta(X), \mathscr{E})$ where $\beta(X)$ is the closure of $\mathscr{E}(X)$ in the cube $\mathscr{I}$.

The Stone-Čech compactification of a Tychonoff space is a Hausdorff compactification: since the cube $\mathscr{I}$ is Hausdorff and compact and $\beta(X)$ is a closed subset of $\mathscr{I}$, it is also Hausdorff and compact; and $\mathscr{E}(X)$ is dense in $\beta(X)$. By identifying X with its homeomorphic image $\mathscr{E}(X)$, we may assume that X is a subspace of $\beta(X)$. ‘

Proposition 10.19. *(Extension Property) Let X be a Tychonoff space, $(\beta(X), \mathscr{E})$ be the Stone-Čech compactification of X and let K be a compact Hausdorff space. Then any continuous map $\sigma : X \longrightarrow K$ can be extended to a continuous map $\phi : \beta(X) \longrightarrow K$:*

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\mathscr{E}"'] & & K \\
& \beta(X) \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}$$

Proof. Since K is compact Hausdorff space, it is a Tychonoff space. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be the family of all continuous functions $X \longrightarrow I$ (resp. $K \longrightarrow I$). For each $f \in \mathscr{F}$ and $g \in \mathscr{G}$, set $I_f = I_g = I$ and $\mathscr{I} := \prod_{f\in\mathscr{F}} I_f$, $\mathscr{I}' := \prod_{g\in\mathscr{G}} I_g$. For each $f \in \mathscr{F}$ and $g \in \mathscr{G}$, let $p_f : \mathscr{I} \longrightarrow I_f$ and $p_g : \mathscr{I}' \longrightarrow I_g$ be the f-th and g-th coordinate projections, respectively. Moreover, let $\mathscr{E}' : K \longrightarrow \mathscr{I}'$ be the evaluation map. Define $\psi : \mathscr{I} \longrightarrow \mathscr{I}'$ as follows: for $t := (t_f)_{f\in\mathscr{F}} \in \mathscr{F}$, define $\psi(t) \in \mathscr{I}'$ such that $p'_g(\psi(t)) = t_{g\circ\sigma} = p_{g\circ\sigma}(t)$, for every $g \in \mathscr{G}$. Then ψ is continuous and we have a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\mathscr{E}"] \arrow[d, "\sigma"'] & \mathscr{E}(X) \arrow[r, hook, "\text{inc}"] \arrow[d, dashed] & \mathscr{I} \arrow[d, "\psi"] \\
K \arrow[r, "\mathscr{E}'"] & \mathscr{E}'(K) \arrow[r, hook, "\text{inc}"] & \mathscr{I}'
\end{tikzcd}$$

Let $x \in X$ and $g \in \mathscr{G}$. Then

$$p'_g(\psi(\mathscr{E}(x))) \;=\; p_{g\circ\sigma}(\mathscr{E}(x)) \;=\; g\circ\sigma(x) \;=\; p'_g(\mathscr{E}'(\sigma(x))) \;\implies\; \psi(\mathscr{E}(x)) = \mathscr{E}'(\sigma(x)).$$

Thus $\psi(\mathscr{E}(X)) \subseteq \mathscr{E}(K)$. Moreover,

$$\overline{\mathscr{E}(X)} \;=\; \beta(X) \;\implies\; \psi(\,\overline{\mathscr{E}(X)}\,) \subseteq \overline{\psi(\mathscr{E}(X))} \;\implies\; \overline{\psi(\mathscr{E}(X))} \;=\; \psi(\beta(X)).$$

Since $\mathscr{I}'$ is Hausdorff and $\mathscr{E}'(K)$ is compact, it is closed. Since $\psi(\mathscr{E}(X)) \subseteq \mathscr{E}'(K)$, we have $\psi(\beta(X)) \subseteq \mathscr{E}'(K)$. Set $\phi := \mathscr{E}'^{-1} \circ \psi|_{\beta(X)} : \beta(X) \longrightarrow K$:

$$\begin{tikzcd}
\beta(X) \arrow[r, "\psi|_{\beta(X)}"] & \mathscr{E}'(X) \arrow[r, "\mathscr{E}'^{-1}"] & K
\end{tikzcd}$$

Then ϕ is continuous. for any $x \in X$, we have

$$\phi(\mathscr{E}(x)) \;=\; \mathscr{E}'^{-1} \circ \psi(\mathscr{E}(x)) \;=\; \mathscr{E}'^{-1}(\mathscr{E}'(\sigma(x))) \;=\; \sigma(x)\,.$$

Thus $\phi \circ \mathscr{E} = \sigma$. $\square$

The one-point compactification of a Hausdorff locally compact space does not satisfy Extension Property in general. The closed unit interval $[0,1]$ is the one-point compactification of $(0,1]$. The continuous map $\sigma : (0,1] \longrightarrow [-1,1]$ defined by $\sigma(x) = \sin(1/x)$ has no continuous extension to $[0,1]$.

The Stone-Čech compactification of a Tychonoff space can be characterized by the Extension Property, see [8, Chapter 6, Section 19].

Theorem 10.20. (*Whitehead*) *Let Y, Z be two non-empty topological space and $q : Y \longrightarrow Z$ be a quotient map. If X is a locally compact Hausdorff space, then $1 \times q : X \times Y \longrightarrow X \times Z$ is a quotient map.*

Proof. Let $U \subseteq X \times Z$ be a non-empty such that $(1 \times q)^{-1}(U)$ is an open subset of $X \times Y$. Let $(x_0, z_0) \in U$ and choose $y_0 \in Y$ such that $q(y_0) = z_0$. Then $(x_0, y_0) \in (1\times q)^{-1}(U)$. Since X is locally compact and Hausdorff, there is an open neighbourhood $V \subseteq X$ of x_0 such that $\overline{V}$ is compact and $\overline{V} \times \{y_0\} \subseteq (1 \times q)^{-1}(U)$. Notice that, for any $y \in Y$, we have

$$\overline{V} \times \{y\} \;\subseteq\; (1\times q)^{-1}(U) \;\implies\; \overline{V} \times q^{-1}(q(y)) \;\subseteq\; (1\times q)^{-1}(U).$$

In particular, $\overline{V} \times q^{-1}(z_0) \;\subseteq\; (1\times q)^{-1}(U)$. Set $W = \{z \in Z : \overline{V} \times q^{-1}(z) \;\subseteq\; (1\times q)^{-1}(U)\}$. Then $(x_0, z_0) \subseteq V \times W \;\subseteq\; U$. Let $\pi : \overline{V} \times Y \longrightarrow Y$ be the projection map. Since $\overline{V}$ is compact, π is a closed map. Thus the image of the set $E := (\overline{V} \times Y) \setminus (1 \times q)^{-1}(U)$ is a closed set in Y. In particular,

$$q^{-1}W \;=\; \{y \in Y \;:\; \overline{V} \times \{y\} \;\subseteq\; (1\times q)^{-1}(U)\}$$

is open in Y. Since q is quotient map, W is an open subset of Z. Thus $V \times W$ is an open neighbourhood of (x_0, z_0) contained in W. Hence W is an open subset of $X \times Z$. $\square$

Corollary 10.21. *Let X, Y, Z, W be non-empty topological spaces and let $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$ be two quotient maps. Suppose X and W are locally compact Hausdorff spaces. Then $f \times g : X \times Y \longrightarrow Z \times W$ is a quotient map.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
X \times Y \arrow[rr, "f \times g"] \arrow[dr, "1_X \times g"'] & & Z \times W \\
& X \times W \arrow[ur, "f \times 1_W"'] &
\end{tikzcd}$$

Since X (resp. W) is a locally compact Hausdorff space, $1_X \times g$ (resp. $f \times 1_W$) is a quotient map. It remains to show that composition of two quotient maps is again a quotient map. We leave that as an exercise. $\square$

In general, product of two quotient maps need not be a quotient map. For an example, see [4, Chapter 3, Section 22], [6, Chapter 6, Example 6.1.14].

Definition 11.1. A topological space (X, τ) is said to be *metrizable* if there is a metric d on X such that the topology τ is generated by d.

We do not claim that for a metrizable space X, the topology on X is generated by a unique metric on X. In fact, this is far from being true in general.

Definition 11.2. Let d_1, d_2 be two metrics on a non-empty set X. We say d_1 and d_2 are *topologically equivalent* if they generate the same topology on X.

Exercise 11.3. *(1.)* Let (X, d) be a metric space. Define

$$d_1 : X \times X \longrightarrow \mathbb{R}, \quad d_1(x,y) \;=\; \min\{1, d(x,y)\}$$
$$d_2 : X \times X \longrightarrow \mathbb{R}, \quad d_2(x,y) \;=\; \frac{d(x,y)}{1 + d(x,y)}$$

Prove that

(i) both d_1 and d_2 are metrics on X;
(ii) the metrics d, d_1 and d_2 are topologically equivalent.

(2.) Let X be a metrizable space and let $Y \subseteq X$ be a subspace. Prove that Y is a metrizable space.

(3.) Let $X_1, X_2, \cdots, X_n$ be metrizable topological spaces. Prove that their product $\prod_{i=1}^n X_i$ is a metrizable topological space.

Proposition 11.4. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of non-empty topological spaces. Then the product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ is metrizable if and only if each X_α is metrizable and all but countably many X_α are single point sets.*

Proof. Suppose X is metrizable. Since each X_α can be identified as a subspace of X, each X_α is metrizable. Since any metrizable space of first countable, X and each X_α is first countable. Hence all but countably many X_α are equipped with indiscrete topology and hence must be a single point set.

Now assume each X_α is metrizable and $\Omega \subseteq \Lambda$ be a countable set such that X_α is a single point set, for each $\alpha \in \Lambda \setminus \Omega$. To check if X is metrizable, it is sufficient to check if $\prod_{\beta\in\Omega} X_\beta$ is metrizable. In other words, we may consider only the product of countably many metric spaces. In view of the above exercise, we will consider countably infinitely many metric spaces.

Let $\{X_n : n \in \mathbb{N}\}$ be a collection of metric spaces. By the above exercise, we may assume, for each $n \in \mathbb{N}$, the metric d_n on X_n satisfies $d_n(a,b) \le 1$, for all $a, b \in X_n$. Let $X = \prod_{n\ge1} X_n$ and define

$$d : X \times X \longrightarrow \mathbb{R}, \quad d(x,y) \;=\; \sum_{n\ge1} \frac{1}{2^n} d_n(x_n, y_n)$$

where $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in X$. It's easy to see that d is a metric on X. Let τ and τ' be the product topology on X and the topology on X generated by d, respectively. For each $n \in \mathbb{N}$, let $p_n : X \longrightarrow X_n$ be the n-th coordinate projection.

Let $x = (x_n)_{n\in\mathbb{N}} \in X$ and let $r > 0$ be a real number. Choose $m \in \mathbb{N}$ such that $\sum_{k=m+1}^\infty 1/2^k = 1/2^m < r$. For each $i = 1, \cdots, m$, set $V_i = B(x_i; r_i/2) \subseteq X_i$. Then $x \in \bigcap_{i=1}^m p_i^{-1} V_i$ and for any $y = (y_n)_{n\in\mathbb{N}} \in \bigcap_{i=1}^m p_i^{-1} V_i$, we have

$$d(x,y) \;\le\; \sum_{k=1}^m \frac{1}{2^k} d_k(x_k, y_k) + \sum_{k=m+1}^\infty \frac{1}{2^k} \;<\; \frac{r}{2} + \frac{r}{2} \;=\; r \;\implies\; y \in B(x; r)$$

so that $\bigcap_{i=1}^m p_i^{-1}V_i \subseteq B(x;r)$. Thus $\tau' \subseteq \tau$.

Now let $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in X$ and assume $d(x,y) < \varepsilon$, for some real number $\varepsilon > 0$. Then $d_n(x_n, y_n) < \varepsilon\, 2^n$, for every $n \in \mathbb{N}$. Let $p_n^{-1}W \subseteq X$ be a canonical sub-basic open set in X. Let $x \in p_n^{-1}W$ and $x_n := p_n(x) \in W \subseteq X_n$. Since W is open in X_n, there is a real number $r > 0$ such that $B(x_n; r) \subseteq W$. Then $B(x; r/2^{-n}) \subseteq p_n^{-1}W$. Thus, for each $n \in \mathbb{N}$, the projection map $p_n : X \longrightarrow X_n$ is continuous whenever X is equipped with the topology τ'. By definition of product topology, we have $\tau \subseteq \tau'$. Hence $\tau = \tau'$. □

For each $n \in \mathbb{N}$, $X_n = \mathbb{R}$. Then $\mathbb{R}^{\aleph_0} := \prod_{n=1} X_n$ is a metrizable topological space, by the above Proposition. Notice that $\mathbb{R}^{\aleph_0}$ is a separable, both first and second countable metrizable topological space.

For each $n \in \mathbb{N}$, $X_n = I = [0,1] \subseteq \mathbb{R}$, the closed unit interval equipped with subspace topology. By the above Proposition, $I^{\aleph_0} := \prod_{n=1} X_n$ is a metrizable topological space, known as *Hilbert cube*. By Proposition 11.4, the topology on $I^{\aleph_0}$ is generated by the metric

$$d((x_n),\, (y_n)) \;:=\; \sum_{j=1}^{\infty} \frac{1}{2^j}\,|x_j - y_j|.$$

Let $\ell_2 := \ell_2(\mathbb{R})$ be the set of all real sequences $x = (x_1, x_2, \cdots)$ satisfying $\sum_{i=1}^{\infty} x_i^2 < \infty$. Define

$$d : \ell_2 \times \ell_2 \longrightarrow \mathbb{R}, \quad d(x,y) \;=\; \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{1/2}.$$

Then d is a metric on ℓ_2. Moreover, ℓ_2 is a separable complete metric space and hence a separable Hilbert space. Let $\mathbf{0} = (0,0,\cdots)$ be the zero sequence in ℓ_2 and let $\varepsilon > 0$ be a real number. Define a sequence (x_n) in ℓ_2 as follows: for $x_n = (x_{n,1},\, x_{n,2}, \cdots)$, $x_{n,k} = 0$, if $k \neq n$ and $x_{n,n} = \varepsilon$:

$$x_1 \;=\; (\varepsilon, 0, 0, 0, \cdots);\;\; x_2 \;=\; (0, \varepsilon, 0, 0, \cdots);\;\; x_3 \;=\; (0, 0, \varepsilon, 0, \cdots);\;\; \cdots$$

Then (x_n) is a sequence in $\overline{B}(\mathbf{0};\varepsilon) = \{x \in \ell_2 : d(x,\mathbf{0}) \le \varepsilon\}$ of distinct points. Since $d(x_m, x_n) = \sqrt{2}\varepsilon$, for every $m \neq n$, this sequence does not have any convergent subsequence. Thus $\overline{B}(\mathbf{0};\varepsilon)$ is not compact. Let V be an open neighbourhood of $\mathbf{0}$ such that $\overline{V}$ is compact. Choose a real number $\varepsilon > 0$ such that $B(\mathbf{0};\varepsilon) \subseteq V$. Then $\overline{B}(\mathbf{0};\varepsilon) \subseteq \overline{V}$ is compact, a contradiction. Hence ℓ_2 is not locally compact at $\mathbf{0}$ (in fact, at any of its points).

Exercise 11.5. Define a metric d in $\mathbb{R}$ by $d(a,b) = \min\{1, |a-b|\}$, for $a, b \in \mathbb{R}$. For each $n \in \mathbb{N}$, let $X_n = \mathbb{R}$ be equipped with the metric $d_n = d$. Define

$$D : \mathbb{R}^{\aleph_0} \times \mathbb{R}^{\aleph_0} \longrightarrow \mathbb{R}, \quad D(x,y) = \sup\left\{\frac{1}{n}\, d_n(x_n, y_n) : n \in \mathbb{N}\right\}$$

where $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\aleph_0}$.

(i) Prove that D is a metric on $\mathbb{R}^{\aleph_0}$.

(ii) Is the topology generated by D same as the product topology on $\mathbb{R}^{\aleph_0}$?

Theorem 11.6. (*Urysohn Metrization Theorem*) *Let X be a T_1-topological space. Then the followings are equivalent:*

(i) X is regular and second countable;

(ii) X is separable and metrizable;

(iii) X can be embedded as a subspace of the Hilbert cube $I^{\aleph_0}$.

Proof. *(iii)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(i)* left as an exercise.

$(i) \implies (iii)$ Since X is second countable, it is a Lindelöf space. Moreover, since X is regular, it is normal. Let $\mathscr{B}$ be a countable base for the topology on X. Set

$$\mathscr{U} := \{(V,W) : V,W \in \mathscr{B} \text{ and } \overline{V} \subseteq W\}.$$

Then $\mathscr{U}$ is a countable set. Since X is normal, for any $(V,W) \in \mathscr{U}$, there is continuous map $f_{VW} : X \longrightarrow I$ such that $f_{VW}(\overline{V}) = 0$ and $f_{VW}(X \setminus W) = 1$. Let

$$\mathscr{F} := \{f_{VW} : X \longrightarrow I : (V,W) \in \mathscr{U}\}.$$

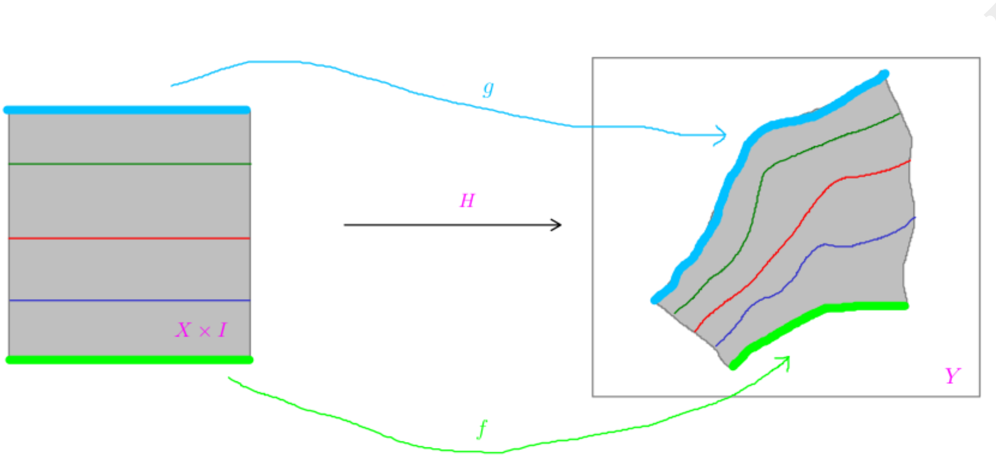
Then $\mathscr{F}$ is countable. For each $g \in \mathscr{F}$, set $I_g = I$. Clearly $\mathscr{F}$ separates points in X and points and closed sets in X. Thus, by Tychonoff Embedding Theorem, the evaluation map $\mathscr{E} : X \longrightarrow \prod_{g \in \mathscr{F}} I_g \subseteq I^{\aleph_0}$ is an embedding. $\square$

Corollary 11.7. *Let X be a compact metric space, Y be a Hausdorff topological space and let $f : X \longrightarrow Y$ be a continuous map. Then $f(X)$ is metrizable.*

Proof. Replacing Y by $f(X)$, if necessary, we may assume that f is surjective. Then Y is a compact Hausdorff space and hence regular. By Urysohn Metrization Theorem, we need to show that Y is second countable. Since X is a compact metric space, it is second countable. Let $\mathscr{B}$ be a countable base for the topology on X and let $\mathscr{U}$ be the collection of open subsets of X that can be written as a finite union of basic open sets from $\mathscr{B}$. Then $\mathscr{U}$ is a countable collection. Set $\mathscr{A} = \{Y \setminus f(X \setminus V) : V \in \mathscr{U}\}$. Since X is compact and Y is Hausdorff, f is a closed map. Thus $\mathscr{A}$ is a countable collection of open subsets of Y. Let $y \in Y$ and let $W \subseteq Y$ be a neighbourhood of y. Then $f^{-1}W \subseteq X$ is a neighbourhood of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact, being a closed subset of a compact space, there are $B_1, \cdots, B_m \in \mathscr{B}$ such that $f^{-1}y \in \bigcup_{i=1}^m B_i \subseteq f^{-1}W$. Set $V = \bigcup_{i=1}^m B_i$. Then $y \in Y \setminus f(X \setminus V) \subseteq W$. Thus $\mathscr{A}$ is a base for the topology on Y. $\square$

Notation: The closed bounded interval $[0, 1]$ equipped with the subspace topology of the Euclidean space $\mathbb{R}$ will be denoted by I.

Definition 12.1. Let X, Y be two non-empty topological spaces and let $f, g : X \longrightarrow Y$ be two continuous maps. We say f is *homotopic* to g, denoted by $f \simeq g$, if there a continuous map $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for every $x \in X$. In this case, H is called a *homotopy* between f and g.



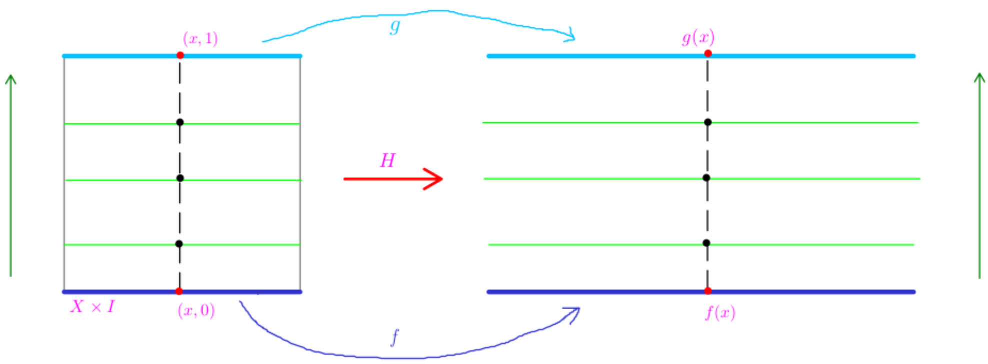
Let X, Y be two non-empty topological spaces and let $f, g : X \longrightarrow Y$ be two continuous maps. Suppose f and g are homotopic and let $H : X \times I \longrightarrow Y$ be a homotopy between f and g. Then for each $t \in [0, 1]$, $H_t : X \longrightarrow Y, x \mapsto H(x, t)$ is a continuous map with $H_0 = f$ and $H_1 = g$. Thus we can think of H as a family of continuous maps from X to Y, parametrized by I, starting with f and ending with g.

Intuitively, we can think the homotopy H between f and g as a "continuous deformation" of f to g that occurs during a unit length of time. At the start of the process i.e. at $t = 0$, we have $H_0 = f$, at the end i.e. at $t = 1$, we have $H_1 = g$ and for $t \in (0, 1)$, H_t is the in-between stage at the t-th moment.

Example 12.2. *(1.)* Let X be a non-empty topological space and let $f, g : X \longrightarrow \mathbb{R}^n$ be two continuous maps. Define

$$H : X \times I \longrightarrow \mathbb{R}^n, \ \ (x, t) \mapsto (1 - t)f(x) + tg(x)$$

Then H is a continuous map with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Hence $f \simeq g$. In particular, any continuous map $X \longrightarrow \mathbb{R}^n$ is homotopic to any constant map $X \longrightarrow \mathbb{R}^n$.



(2.) Let $Y \subseteq \mathbb{R}^n$ be a convex subset. Let X be a non-empty topological space and let $f, g \in X$ be two continuous maps. Then for any $x \in X$, the line segment ℓ_x joining $f(x)$ and $g(x)$ in $\mathbb{R}^n$ lies in Y. Define

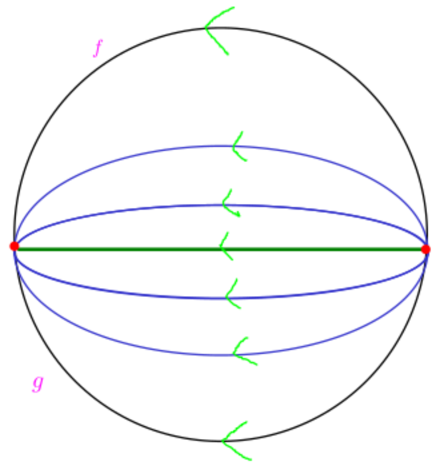
$$H : X \times I \longrightarrow \mathbb{R}^n, \ \ (x,t) \mapsto (1-t)f(x) + tg(x)$$

Then H is a continuous map with $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Hence $f \simeq g$. This type of homotopy is called *linear homotopy* or *straight-line homotopy.* For a fixed $x \in X$, the points $\{H(x,t) : t \in [0,1]\}$ moves from $H(x,0) = f(x)$ to $H(x,1) = g(x)$ via the the line segment joining $f(x)$ and $g(x)$.

(3.) Define $f, g : I \longrightarrow \mathbb{R}^2$ by $f(x) = \big(\cos(\pi x), \sin(\pi x)\big)$ and $g(x) = \big(\cos(\pi x), -\sin(\pi x)\big)$. Define

$$H : I \times I \longrightarrow \mathbb{R}^2, \ \ (x,t) \mapsto \big(\cos(\pi x), \, (1-2t)\sin(\pi x)\big)$$

Then H is continuous function with $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Hence $f \simeq g$.



(4.) Let X be a non-empty topological space and define $f, g : X \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ by $f(x) = (1,0)$ and $g(x) = (-1,0)$. Fix a path $\gamma : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ such that $\gamma(0) = (1,0)$ and $\gamma(1) = (-1,0)$. Define

$$H : X \times I \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}, \ \ (x,t) \mapsto \gamma(t).$$

Then H is a homotopy between f and g. Notice that, the line segment joining $(1,0)$ and $(-1,0)$ in $\mathbb{R}^2$ does not contained in $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus there us no straight-line homotopy between f and g.

(5.) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ be two constant maps defined by $f(x) = 1$ and $g(x) = -1$, for every $x \in \mathbb{R}$. Let $H : \mathbb{R} \times I \longrightarrow \mathbb{R} \setminus \{0\}$ be a continuous map. Since $\mathbb{R} \times I$ is connected, image of H must contained in one of the connected components of $\mathbb{R} \setminus \{0\}$. Thus H can not be a homotopy between f and g. Hence f and g are not homotopic.

(6.) Let X, Y be a two non-empty topological spaces and let $f, g : X \longrightarrow Y$ be two continuous map. Suppose $H : X \times I \longrightarrow Y$ be a homotopy between f and g. Define

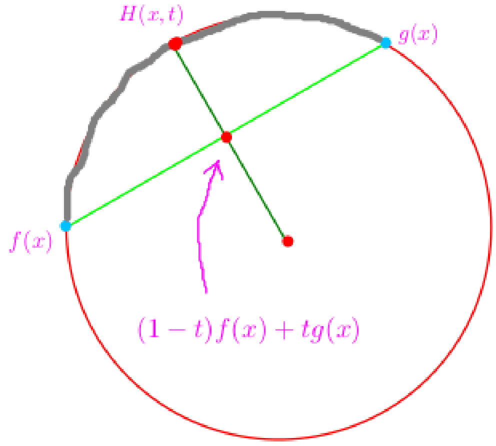
$$H' : X \times I \longrightarrow Y, \ \ H'(x,t) := H(x, t^2).$$

Then H' is also a homotopy between f and g, i.e. a homotopy between f and g is not unique.

(7.) Let X be a non-empty topological space and let $f, g : X \longrightarrow \mathbb{S}^n$ be two continuous map such that $f(x) \neq -g(x)$, for every $x \in X$ (i.e. $f(x)$ and $g(x)$ are not antipodal points). Define

$$H : X \times I \longrightarrow \mathbb{S}^n, \ \ (x,t) \mapsto \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}$$

where $||\cdot||$ is the Euclidean norm of $\mathbb{R}^{n+1}$. Since $f(x) \neq -g(x, \forall x \in X$, $(1-t)f(x)+tg(x) \neq 0$ in $\mathbb{R}^{n+1}$, for every $t \in I$. Thus the above map is well-defined. Clearly, H is continuous. Moreover, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Thus $f \simeq g$.



Exercise 12.3. Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a continuous map.

(i) Suppose f has no fixed point, i.e. $f(x) \neq x, \forall x \in \mathbb{S}^n$. Prove that f is homotopic to the *antipodal map* $g : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ defined by $g(x) = -x$.

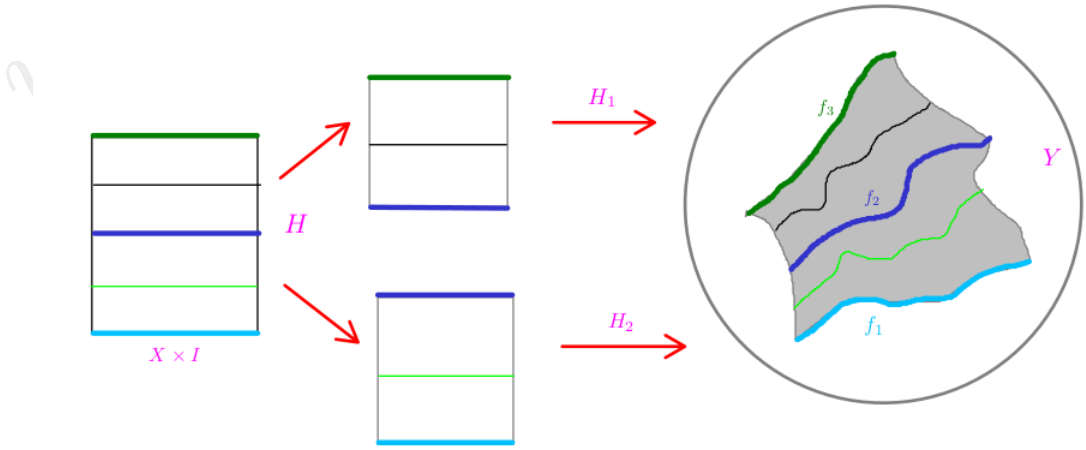
(ii) Suppose $f(x) \neq -x, \forall x \in \mathbb{S}^n$. Prove that f is homotopic to the identity map $Id_{\mathbb{S}^n}$.

Lemma 12.4. *Let X, Y be two non-empty topological spaces and let $C(X,Y)$ be the set of all continuous maps from X to Y. Define a relation $\mathscr{R}$ on $C(X,Y)$ as follows: $f\mathscr{R}g$ if and only if $f \simeq g$. Then $\mathscr{R}$ is an equivalence relation on $C(X,Y)$.*

Proof. We leave the proof of reflexivity and symmetry as exercise. We will prove the transitivity. Suppose $f_1, f_2, f_3 : X \longrightarrow Y$ be continuous maps such that $f_1 \simeq f_2$ and $f_2 \simeq f_3$. Let $H_1 : X \times I \longrightarrow Y$ (resp. $H_2 : X \times I \longrightarrow Y$) be a homotopy between f_1 and f_2 (resp. between f_2 and f_3). Define

$$H : X \times I \longrightarrow Y, \ H(x,t) := \begin{cases} H_1(x, 2t) & \text{if } t \in [0,\ 1/2] \\ H_2(x, 2t-1) & \text{if } t \in [1/2,\ 1] \end{cases}$$

Then H is a homotopy between f_1 and f_3. $\square$



Lemma 12.5. *Let X, Y be two non-empty topological spaces and let $f, g : X \longrightarrow Y$ be two continuous maps. Suppose f is homotopic to g.*

- *Let Z be a non-empty topological space and let $\phi : Z \longrightarrow X$ be a continuous map. Then continuous the maps $f \circ \phi$, $g \circ \phi : Z \longrightarrow Y$ are homotopic.*
- *Let W be a non-empty topological space and let $\psi : Y \longrightarrow W$ be a continuous map. Then the continuous maps $\psi \circ f$, $\psi \circ g : X \longrightarrow W$ are homotopic.*

$$\begin{tikzcd} Z \arrow[r, "\phi"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "\psi"] & W \end{tikzcd}$$

Proof. Let $H : X \times I \longrightarrow Y$ be a homotopy between f and g. Define

$$H' : Z \times I \longrightarrow Y, \;\; H'(z, t) := H(\phi(z), t).$$

Then H' is a continuous map with $H'(z, 0) = H(\phi(z), 0) = f \circ \phi(z)$ and $H'(z, 1) = H(\phi(z), 1) = g \circ \phi(z)$. Thus $f \circ \phi \simeq g \circ \phi$. The rest is similar. $\square$

Corollary 12.6. *Let X, Y, Z be non-empty topological spaces and let $f_1, g_1 : X \longrightarrow Y$ and $f_2, g_2 : Y \longrightarrow Z$ be continuous maps. Suppose $f_1 \simeq g_1$ and $f_2 \simeq g_2$. Then the continuous maps $f_2 \circ f_1$, $g_2 \circ g_1 : X \longrightarrow Z$ are homotopic.*

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] & Y \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] & Z \end{tikzcd}$$

Proof. By repeated use of Lemma 12.5, we get $f_2 \circ f_1 \simeq f_2 \circ g_1 \simeq g_2 \circ g_1$. $\square$

Definition 12.7. Let X, Y be a non-empty topological spaces and $f : X \longrightarrow Y$ be a continuous map.

- We say f is *null-homotopic* if f is homotopic to a constant map.
- We say X is *contractible* if the identity map $1_X : X \longrightarrow X$ is null-homotopic

Exercise 12.8. Prove that

(i) the Euclidean space $\mathbb{R}^n$ is contractible.
(ii) a contractible space is path connected.

Corollary 12.9. *Let X be a non-empty topological space. Then X is contractible if and only if given any non-empty topological space Z and any two continuous function $f, g : Z \longrightarrow X$, f is homotopic to g.*

Proof. Suppose X is a contractible space. Let Z be a non-empty topological space and let $f, g : Z \longrightarrow X$ be two continuous functions. Suppose the identity map 1_X is homotopic to the constant map $c_{x_0} : X \longrightarrow X$ defined by $c_{x_0}(x) = x_0, \forall x \in X$, for some $x_0 \in X$. By Lemma 12.5, $1_X \circ f = f : Z \longrightarrow X$ is homotopic to the constant map $c_{x_0} \circ f : Z \longrightarrow X$. Similarly, $g = 1_X \circ g$ is homotopic to the constant map $c_{x_0} \circ g$. Since $c_{x_0} \circ f = c_{x_0} \circ g$, by Lemma 12.4, f is homotopic to g. The converse is trivial. $\square$

Definition 12.10. Let X and Y be two non-empty topological spaces.

- A continuous map $f : X \longrightarrow Y$ is said to be a *homotopy equivalence* if there is a continuous map $g : Y \longrightarrow X$ such that $g \circ f \simeq 1_X$ and $f \circ g \cong 1_Y$. In this case we say g is a *homotopy inverse* of f.
- If there is a homotopy equivalence from X to Y, we say that X is *homotopy equivalent* to Y or that X and Y have the *same homotopy type*.

If two non-empty topological spaces are homeomorphic, then they have the same homotopy type. But the converse is not true (see the Example below).

Example 12.11. Consider the two subspaces of $\mathbb{R}^2$:

$$X := \{re^{2\pi\sqrt{-1}\,\theta} : r \in [1,2],\, \theta \in [0,1]\} \text{ and } Y := \{e^{2\pi\sqrt{-1}\,\theta} : \theta \in [0,1]\} = \mathbb{S}^1.$$

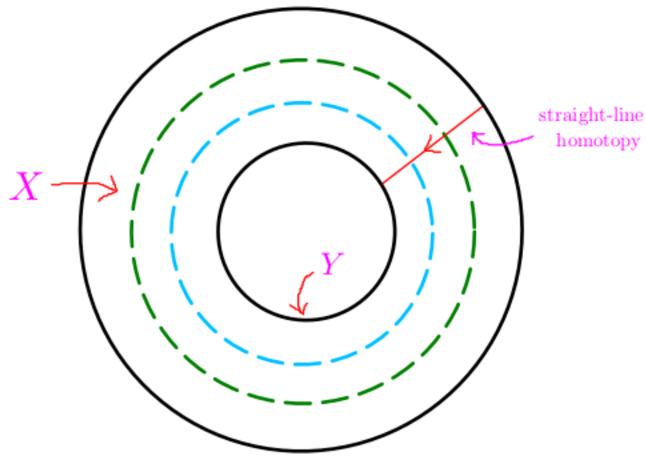
Define $\phi : X \longrightarrow Y$ by $re^{2\pi\sqrt{-1}\,\theta} \mapsto e^{2\pi\sqrt{-1}\,\theta}$ and let $\psi : Y \longrightarrow X$ be the natural inclusion map. Then both ϕ and ψ are continuous maps and $\phi \circ \psi = 1_Y$. Now define

$$H : X \times I \longrightarrow Y, \quad H(re^{2\pi\sqrt{-1}\,\theta}, t) := (1-t)re^{2\pi\sqrt{-1}\,\theta} + te^{2\pi\sqrt{-1}\,\theta}.$$

Then H is a continuous map such that

$$H(re^{2\pi\sqrt{-1}\,\theta}, 0) = re^{2\pi\sqrt{-1}\,\theta} \text{ and } H(re^{2\pi\sqrt{-1}\,\theta}, 1) = e^{2\pi\sqrt{-1}\,\theta} = \psi \circ \phi(re^{2\pi\sqrt{-1}\,\theta}).$$

Thus H is a homotopy between 1_X and $\psi \circ \phi$. Hence ϕ is a homotopy equivalence between X and Y so that X and Y have the same homotopy type.



Lemma 12.12. *Being "same homotopy type" is an equivalence relation on any collection of non-empty topological spaces.*

Proof. We will prove only the transitivity, reflexivity and symmetry are left as an exercise. Let X, Y, Z be three non-empty topological spaces. Assume X and Y have the same homotopy type and Y and Z have the same homotopy type. Let $f : X \longrightarrow Y$ (resp. $\phi : Y \longrightarrow Z$) be a homotopy equivalence with homotopy inverse $g : Y \longrightarrow X$ (resp. $\psi : Z \longrightarrow Y$). Then $g \circ f \simeq 1_X, f \circ g \simeq 1_Y$ and $\psi \circ \phi \simeq 1_Y$, $\phi \circ \psi \simeq 1_Z$.

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[r, bend left, "\phi"] & Z \arrow[l, bend left, "\psi"]
\end{tikzcd}$$

By Lemma 12.5, we have

$$(g \circ \psi) \circ (\phi \circ f) \;=\; g \circ (\psi \circ \phi) \circ f \;\simeq\; g \circ 1_Y \circ f \;=\; g \circ f \;\simeq\; 1_X$$
$$(\phi \circ f) \circ (g \circ \psi) \;=\; \phi \circ (f \circ g) \circ \psi \;\simeq\; \phi \circ 1_Y \circ \psi \;=\; \phi \circ \psi \;\simeq\; 1_Z.$$

Then $\phi \circ f : X \longrightarrow Z$ is a homotopy equivalence with homotopy inverse $g \circ \psi : Z \longrightarrow X$. $\square$

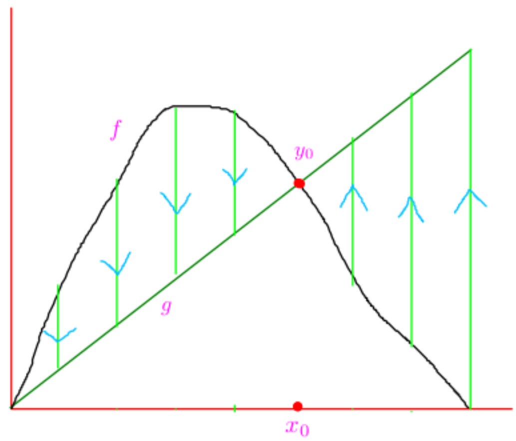
Exercise 12.13. *(1.)* Prove that two contractible spaces have the same homotopy type.

(2.) Let X, Y be two non-empty topological spaces.

(i) If Y is contractible then prove that any two continuous maps $X \longrightarrow Y$ are homotopic.

(ii) If both X and Y are contractible, prove that any continuous map $X \longrightarrow Y$ is a homotopy equivalence.

By a *pointed topological space* we mean an ordered pair (X, x_0) where X is a non-empty topological space and a *marked point* $x_0 \in X$. A continuous map between two pointed topological spaces (X, x_0) and (Y, y_0) is a continuous map $f : X \longrightarrow Y$ such that $f(x_0) = y_0$.



Let $f, g : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map between pointed topological spaces. We say f is *homotopic* to g if there are is a continuous map $H : X \times I \longrightarrow Y$ such that

- $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for every $x \in X$;
- $H(x_0, t) = y_0$, for every $t \in [0, 1]$.

Exercise 12.14. *(1.)* Let (X, x_0) and (Y, y_0) be two pointed topological spaces and let $\mathscr{L}$ be the set of all continuous maps $(X, x_0) \longrightarrow (Y, y_0)$. Define a relation $\mathscr{R}$ on L as follows: $f \mathscr{R} g$ if and only if f and g are homotopic (in the sense of above definition). Is $\mathscr{R}$ an equivalence relation on $\mathscr{L}$?

(2.) Let $f, g : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map between pointed topological spaces. Suppose f is homotopic to g (in the sense of above definition).

- Let (Z, z_0) be a pointed topological space and let $\phi : (Z, z_0) \longrightarrow (X, x_0)$ be a continuous map. Prove that $f \circ \phi$ and $g \circ \phi$ are homotopic (in the sense of above definition).
- Let (W, w_0) be a pointed topological space and let $\psi : (Y, y_0) \longrightarrow (W, w_0)$ be a continuous map. Prove that $\psi \circ f$ are $\psi \circ g$ are homotopic (in the sense of above definition).

(3.) Let X, Y be two non-empty topological spaces, $\emptyset \neq A \subseteq X$ and let $f, g : X \longrightarrow Y$ be two continuous maps such that $f(a) = g(a)$, $\forall a \in A$. We say *f is homotopic to g relative to A*, denoted by $f \simeq g$ rel A, if there there exists a homotopy $H : X \times I \longrightarrow Y$ between f and g such that $H(a, t) = f(a) = g(a)$, $\forall a \in A$ and $t \in I$.

(i) Let $\mathscr{K}$ be the set of all continuous maps $X \longrightarrow Y$ such that $f|_A = g|_A$, $\forall f, g \in \mathscr{K}$. Define a relation $\mathscr{R}$ on $\mathscr{K}$ as follows: $f \mathscr{R} g$ if and only if $f \simeq g$ rel A. Prove that $\mathscr{R}$ is an equivalence relation on $\mathscr{K}$.

(ii) Let Z be a non-empty topological space and let $\emptyset \neq B \subseteq Y$. Let $f_1, g_1 : X \longrightarrow Y$ and let $f_2, g_2 : Y \longrightarrow Z$ be continuous maps. If $f_1 \simeq g_1$ rel A, $f_2 \simeq g_2$ rel B and $f_1(A) \subseteq B$, then prove that $f_2 \circ f_1 \simeq g_2 \circ g_1$ rel A.

Proposition 12.15. *(i) If $n \in \mathbb{N}$ is an odd integer, then the antipodal map $g : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is homotopic to the identity map $Id_{\mathbb{S}^n} : \mathbb{S}^n \longrightarrow \mathbb{S}^n$.*

(ii) If $n \in \mathbb{N}$ is an even integer, then the antipodal map $g : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is not homotopic to the identity map $Id_{\mathbb{S}^n} : \mathbb{S}^n \longrightarrow \mathbb{S}^n$.

Proof. (i) Let $n = 2m - 1$. Considering $\mathbb{S}^n \subseteq \mathbb{R}^{2m}$, we denote the an element of $\mathbb{S}^n$ by $(x_1, y_1, x_2, y_2, \cdots, x_m, y_m)$. For $j = 1, \cdots, m$, consider the complex number $z_j = x_j + \sqrt{-1} y_j$. Then $\sum_{j=1}^m |z_j|^2 = 1$. In other words, every element of $\mathbb{S}^n$ can be written as $(z_1, \cdots, z_m)$ where $z_j = x_j + \sqrt{-1} y_j \in \mathbb{C}$ with $\sum_{j=1}^m |z_j|^2 = 1$. For every $u \in \mathbb{C}$ with $|u| = 1$ and every $z = (z_1, \cdots, z_m) \in \mathbb{S}^n$. define $u \cdot z = (uz_1, \cdots, uz_m)$. Then $u \cdot z \in \mathbb{S}^n$. Define

$$H : \mathbb{S}^n \times I \longrightarrow \mathbb{S}^n, \;\; H(z,t) := e^{t\pi\sqrt{-1}} \cdot z.$$

Then H is a continuous map with $H(z, 0) = z$ and $H(z, 1) = -z$. Thus H is homotopy between the identity map $Id_{\mathbb{S}^n}$ and the antipodal map g.

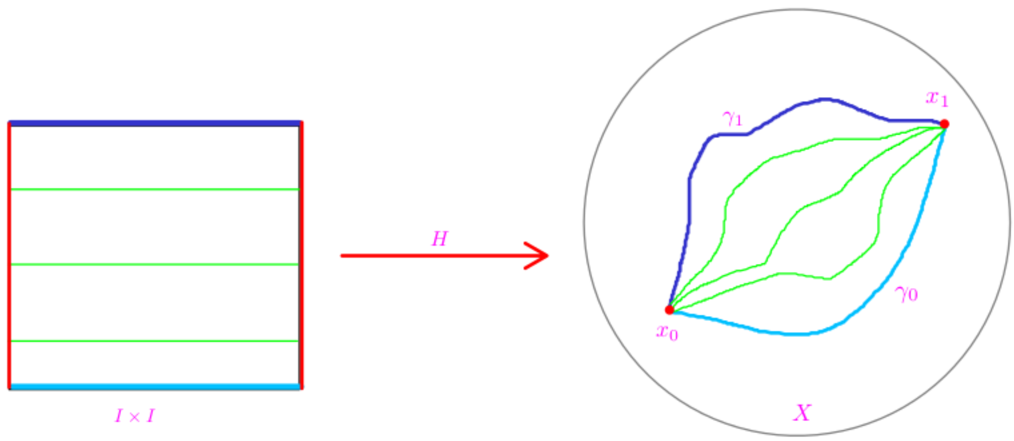
(ii) Beyond the scope of this note. $\square$

Let X be a non-empty topological space and let $x_0, x_1 \in X$. Recall, a path from x_0 to x_1 in X is a continuous map $\gamma : I \longrightarrow X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We say x_0 is the *initial point* and x_1 is the *final point* or *terminal point* of the path γ. When $x_0 = x_1$, we say that γ is a *loop* in X and the point x_0 is called the *base point* of the loop γ. Notice that, by definition, any constant path $I \longrightarrow X$ is a loop. We define the *opposite path* of γ to be a path in X from x_1 to x_0 by $\overline{\gamma} : I \longrightarrow X$, $t \mapsto \gamma(1-t)$.

Definition 13.1. Let X be a non-empty topological space and let $\gamma_0, \gamma_1 : I \longrightarrow X$ be two paths in X with $\gamma_0(0) = \gamma_1(0) = x_0 \in X$ and $\gamma_0(1) = \gamma_1(1) = x_1 \in X$. We say γ_0 and γ_1 are *path homotopic*, denoted by $\gamma_0 \simeq \gamma_1$, if there is a continuous map $H : I \times I \longrightarrow X$ such that

- $H(s, 0) = \gamma_0(x)$ and $H(s, 1) = \gamma_1(x)$, for every $s \in I$;
- $H(0, t) = x_0$ and $H(1, t) = x_1$, for every $t \in [0, 1]$.

In this case, H is said to be *path homotopy* between γ_0 and γ_1.



Unless explicitly mentioned otherwise, a homotopy between two paths $I \longrightarrow X$ with same initial and terminal points will always mean a path homotopy.

Lemma 13.2. *Let X be a non-empty topological space and let $x_0, x_1 \in X$. Let Σ be the set of all paths in X with initial point x_0 and terminal point x_1. Define a relation $\mathscr{R}$ on Σ as follows: $\gamma \mathscr{R} \gamma'$ if and only if $\gamma \simeq \gamma'$. Then $\mathscr{R}$ is an equivalence relation in Σ.*

Proof. Similar to that of Lemma 12.4 and left as an exercise. $\square$

Definition 13.3. Let X be a non-empty topological space and let $x_0, x_1, x_2 \in X$. Suppose $\gamma_0 : I \longrightarrow X$ (resp $\gamma_1 : I \longrightarrow X$) be a path in X from x_0 to x_1 (resp. from x_1 to x_2). We define the *product* of the paths γ_0 and γ_1 to be a path $\gamma_0 * \gamma_1 : I \longrightarrow X$ in X from x_0 to x_2 as follows:

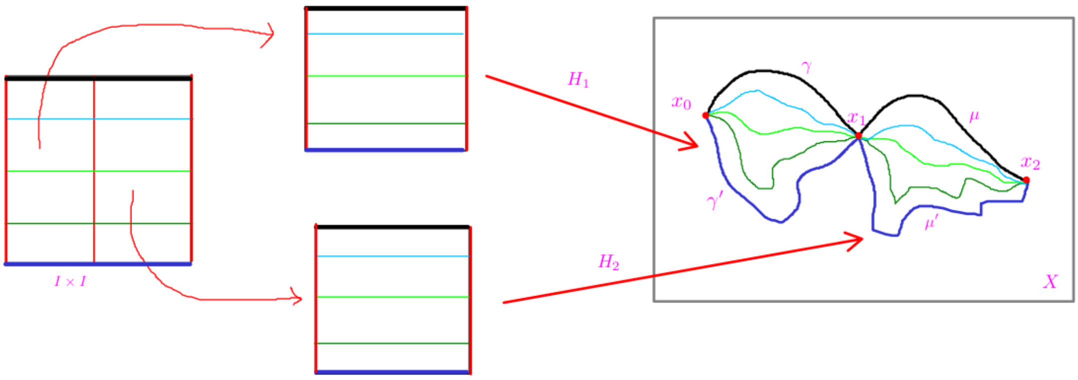
$$\gamma_0 * \gamma_1(s) := \begin{cases} \gamma_0(2s) & \text{if } s \in [0, 1/2] \\ \gamma_1(2s-1) & \text{if } s \in [1/2, 1] \end{cases}$$

Let X be a non-empty topological space and let $x_0, x_1, x_2 \in X$. Let Σ_0 (resp. Σ_1) be the set of all paths in X from x_0 to x_1 (resp. from x_1 to x_2). For $\gamma \in \Sigma_0$ and $\mu \in \Sigma_1$, set

$$[\gamma] := \{\gamma' \in \Sigma_0 : \gamma \simeq \gamma'\} \text{ and } [\mu] := \{\mu' \in \Sigma_1 : \mu \simeq \mu'\}.$$

Let $\gamma' \in [\gamma]$ and $\mu' \in [\mu]$. Let $H_1 : I \times I \longrightarrow X$ (resp. $H_2 : I \times I \longrightarrow X$) be a path homotopy between γ and γ' (resp. between μ and μ'). Define

$$H : I \times I \longrightarrow X, \;\; H(s,t) := \begin{cases} H_1(2s,t) & \text{if } s \in [0,\, 1/2] \\ H_2(2s-1,t) & \text{if } s \in [1, 2,\, 1] \end{cases}$$



Since for every $t \in I$, $H_1(1,t) = x_1 = H_2(0,t)$, by pasting lemma, H is a continuous map. Moreover,

$$H(s,0) = \begin{cases} \gamma(2s) & \text{if } s \in [0,\, 1/2] \\ \mu(2s-1) & \text{if } s \in [1, 2,\, 1] \end{cases} = \gamma * \mu(s)$$

$$H(s,1) = \begin{cases} \gamma'(2s) & \text{if } s \in [0,\, 1/2] \\ \mu'(2s-1) & \text{if } s \in [1, 2,\, 1] \end{cases} = \gamma' * \mu'(s)$$

and $H(0,t) = x_0, H(1,t) = x_2, \forall t \in I$. Thus H is a path homotopy between $\gamma * \mu$ and $\gamma' * \mu'$.

Lemma 13.4. *Let X be a topological space and let $x_0, x_1, x_2 \in X$. Let $\gamma, \gamma' : I \longrightarrow X$ (resp. $\mu, \mu' : I \longrightarrow X$) be two paths from x_0 to x_1 (resp. from x_1 to x_2). If $\gamma \simeq \gamma'$ and $\mu \simeq \mu'$, then $\gamma * \mu \simeq \gamma' * \mu'$.*

Let X be non-empty topological space and let $\gamma : I \longrightarrow X$ be a path in X. Recall, a *reparametrization* of the path γ is a path $\gamma \circ \sigma : I \longrightarrow X$ where $\sigma : I \longrightarrow I$ is a continuous function such that $\sigma(0) = 0$ and $\sigma(1) = 1$.

Lemma 13.5. *Let X be a topological space, $x_0, x_1 \in X$ and let $\gamma : I \longrightarrow X$ be a path from x_0 to x_1. If $\gamma \circ \sigma : I \longrightarrow X$ is a reparametrization of the γ, then $\gamma \circ \sigma \simeq \gamma$.*

Proof. For each $t \in I$, define

$$\psi_t(s) : I \longrightarrow I, \;\; s \mapsto (1-t)\sigma(s) + ts.$$

Then ψ_t is a continuous function, for every $t \in I$. Now define

$$H : I \times I \longrightarrow X, \;\; (s,t) \mapsto \gamma(\psi_t(s)).$$

Then H is a continuous function. Moreover,

- $H(s,0) = \gamma \circ \sigma(s)$ and $H(s,1) = \gamma(s)$, for every $s \in I$;
- $H(0,t) = \gamma((1-t)\sigma(0)) = \gamma(0) = x_0$ and $H(1,t) = \gamma(1) = x_1$, for every $s \in I$.

Thus H is a path homotopy between $\gamma \circ \sigma$ and γ. □

Let X be a non-empty topological space and fix a point $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of all homotopic classes of loops in X with base point x_0, i.e.

$$\pi_1(X, x_0) := \{[\gamma] \,:\, \gamma : I \longrightarrow X \text{ is a path with } \gamma(0) = \gamma(1) = x_0\}.$$

Define

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0), \quad ([\gamma], [\mu]) \mapsto [\gamma * \mu].$$

By Lemma 13.4 and Lemma 13.5, the above operation is well-defined and independent of the reparametrization of loops.

Proposition 13.6. *Let X be a non-empty topological space and fix a point $x_0 \in X$. Then $\pi_1(X, x_0)$ forms a groups with respect to the the binary operation*

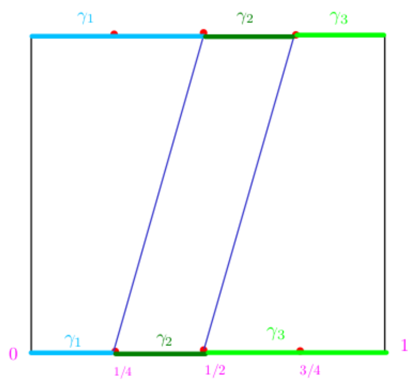
$$\pi_1(X, x_0) \times \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0), \quad [\gamma] * [\mu] := [\gamma * \mu].$$

Proof. Associativity: Let $\gamma_1, \gamma_2, \gamma_3 : I \longrightarrow X$ be three loops with base point x_0. It is sufficient to show that $(\gamma_1 * \gamma_2) * \gamma_3 \simeq \gamma_1 * (\gamma_2 * \gamma_3)$. Note that

$$(\gamma_1 * \gamma_2) * \gamma_3(t) = \begin{cases} \gamma_1(4t) & \text{if } t \in [0, 1/4] \\ \gamma_2(4t-1) & \text{if } t \in [1/4, 1/2] \\ \gamma_3(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

and

$$\gamma_1 * (\gamma_2 * \gamma_3)(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, 1/2] \\ \gamma_2(4t-2) & \text{if } t \in [1/2, 3/4] \\ \gamma_3(4t-3) & \text{if } t \in [3/4, 1] \end{cases}$$



Define

$$\sigma : I \longrightarrow I, \quad \sigma(t) := \begin{cases} t/2 & \text{if } t \in [0, 1/2] \\ t - 1/4 & \text{if } t \in [1/2, 3/4] \\ 2t - 1 & \text{if } t \in [3/4, 1] \end{cases}$$

Then $(\gamma_1 * \gamma_2) * \gamma_3 \circ \sigma = \gamma_1 * (\gamma_2 * \gamma_3)$ so that $\gamma_1 * (\gamma_2 * \gamma_3)$ is a reparametrization of $(\gamma_1 * \gamma_2) * \gamma_3$. Thus by Lemma 13.5, $(\gamma_1 * \gamma_2) * \gamma_3 \simeq \gamma_1 * (\gamma_2 * \gamma_3)$.

Existence of Identity: Let $\gamma : I \longrightarrow X$ be a loop with base point x_0 and let $c_{x_0} : I \longrightarrow X$ be the constant map $c_{x_0}(t) = x_0, \forall t \in I$. Define

$$\sigma : I \longrightarrow I, \quad \sigma(t) := \begin{cases} 2t & \text{if } t \in [0, 1/2] \\ 1 & \text{if } t \in [1/2, 1] \end{cases}$$

Then $(\gamma * c_{x_0}) \circ \sigma = \gamma$ so that $\gamma * c_{x_0}$ is a reparametrization of γ. Thus by Lemma 13.5, $\gamma * c_{x_0} \simeq \gamma$. Define

$$\tau : I \longrightarrow I, \quad \tau(t) := \begin{cases} 0 & \text{if } t \in [0, 1/2] \\ 2t - 1 & \text{if } t \in [1/2, 1] \end{cases}$$

Then $(c_{x_0} * \gamma) \circ \tau = \gamma$ so that $c_{x_0} * \gamma$ is a reparametrization of γ. Thus by Lemma 13.5, $c_{x_0} * \gamma \simeq \gamma$. Thus for any loop $\gamma : I \longrightarrow X$ with base point x_0, we have

$$[\gamma] * [c_{x_0}] = [\gamma * c_{x_0}] = [\gamma] = [c_{x_0} * \gamma] = [c_{x_0}] * [\gamma].$$

Existence of Inverse: Let $\gamma : I \longrightarrow X$ be a loop with base point x_0. Let $\overline{\gamma} : I \longrightarrow X$ be the opposite loop of γ with base point x_0. For each $t \in I$, consider the paths

$$\phi_t : I \longrightarrow X, \ \ \phi_t(s) := \begin{cases} \gamma(s) & \text{if } \ s \in [0,\, 1-t] \\ \overline{\gamma}(t) & \text{if } \ s \in [1-t,\, 1] \end{cases}$$

and

$$\psi_t : I \longrightarrow X, \ \ \psi_t(s) := \phi_t(1-s).$$

Now define

$$H : I \times I \longrightarrow X, \ \ H(s,t) := \phi_t(s) * \psi_t(s).$$

Then H is a continuous map with

$$H(s,0) = \begin{cases} \phi_0(2s) & \text{if } \ s \in [0,\, 1/2] \\ \psi_0(2s-1) & \text{if } \ s \in [1/2,\, 1] \end{cases} = \begin{cases} \gamma(2s) & \text{if } \ s \in [0,\, 1/2] \\ \overline{\gamma}(2s-1) & \text{if } \ s \in [1/2,\, 1] \end{cases} = \gamma * \overline{\gamma}(s)$$

and

$$H(s,1) = \begin{cases} \phi_1(2s) & \text{if } \ s \in [0,\, 1/2] \\ \psi_1(2s-1) & \text{if } \ s \in [1/2,\, 1] \end{cases} = \begin{cases} \gamma(1) & \text{if } \ s \in [0,\, 1/2] \\ \overline{\gamma}(1) & \text{if } \ s \in [1/2,\, 1] \end{cases} = c_{x_0}$$

Thus H is a path homotopy between the loops $\gamma * \overline{\gamma}$ and c_{x_0} with base point x_0 and hence $[\gamma] * [\overline{\gamma}] = [\gamma * \overline{\gamma}] = [c_{x_0}]$. Similarly, one can show that $[\overline{\gamma}] * [\gamma] = [c_{x_0}]$. $\square$

Definition 13.7. Let X be a non-empty topological space, $x_0 \in X$ and let $\pi_1(X, x_0)$ be the set of all homotopic classes of loops in X with base point x_0. The group $(\pi_1(X, x_0), *)$ as defined in Proposition 13.6 is called the *fundamental group of X with base point x_0*.

Example 13.8. Let $X = \mathbb{R}$ be the Euclidean space. Define

$$H : \mathbb{R} \times I \longrightarrow \mathbb{R}, \ \ H(s,t) := (1-t)x.$$

Then H is a continuous map satisfying
- $H(x,0) = x$ and $H(x,1) = 0$, for every $x \in \mathbb{R}$;
- $H(0,t) = 0$, for every $t \in I$.

Let $\gamma : I \longrightarrow \mathbb{R}$ be a loop with base point 0 and let $c_0 : I \longrightarrow \mathbb{R}$ be the constant map defined by $c_0(t) = 0, \forall t \in I$. Consider the continuous map

$$H \circ (\gamma \times Id_I) \ : \ I \times I \xrightarrow{\ \gamma \times Id_I\ } \mathbb{R} \times I \xrightarrow{\ \ H\ \ } \mathbb{R}$$

Then
- $H \circ (\gamma \times Id_I)(s,0) = \gamma(s)$ and $H \circ (\gamma \times Id_I)(s,1) = c_0(s)$, for every $s \in I$;
- $H \circ (\gamma \times Id_I)(0,t) = 0 = H \circ (\gamma \times Id_I)(0,t)$, for every $t \in I$.

Thus $H \circ (\gamma \times Id_I)$ is a path homotopy between γ and c_0. Hence $\pi_1(\mathbb{R}, 0)$ is trivial.

Exercise 13.9. *(1.)* Let $X = \mathbb{D}^2 := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disc in the plane. Prove that $\pi_1(\mathbb{D}^2, 1)$ is trivial.

(2.) Let X be a non-empty topological space, $x_0, x_1 \in X$ and let $\gamma : I \longrightarrow X$ be a path from x_0 to x_1. Let $\overline{\gamma} : I \longrightarrow X$ be the opposite path of γ. Prove that $c_{x_0} \simeq \gamma * \overline{\gamma}, c_{x_1} \simeq \overline{\gamma} * \gamma$, $c_{x_0} * \gamma \simeq \gamma$ and $\gamma * c_{x_1} \simeq \gamma$.

(3.) Let Let X be a non-empty topological space, $x_0, x_1 \in X$ and let $\gamma, \mu : I \longrightarrow X$ be two paths from x_0 to x_1. Let $\overline{\gamma}, \overline{\mu} : I \longrightarrow X$ be the opposite paths of γ and μ, respectively. If $\gamma \simeq \mu$, then prove that $\gamma * \overline{\mu} \simeq c_{x_0}$ and $\overline{\gamma} * \mu \simeq c_{x_1}$.

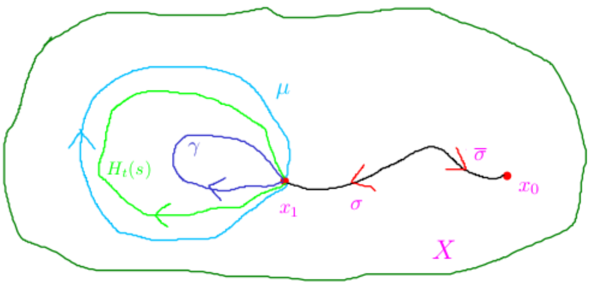
Let X be a non-empty topological space, $x_0, x_1 \in X$ and let $\sigma : I \longrightarrow X$ be a path from x_0 to x_1. For a loop $\gamma : I \longrightarrow X$ with base point x_1, define

$$\widehat{\gamma} : I \longrightarrow X, \ \ \widehat{\gamma}(t) \ := \ (\sigma * \gamma) * \overline{\sigma}(t).$$

where $\overline{\sigma}$ be the opposite path of σ. Then $\widehat{\gamma}(0) = \sigma(0) = x_0$ and $\widehat{\gamma}(1) = \overline{\sigma}(1) = x_0$. Thus $\widehat{\gamma}$ is a loop in X with base point x_0. Let $\mu : I \longrightarrow X$ be a loop with base point x_1 and let $F : I \times I \longrightarrow X$ be a path homotopy between γ and μ. For each $t \in I$, define $H_t : I \longrightarrow X$ by $H_t(s) := F(s,t)$. Then $(\sigma * H_t) * \overline{\sigma} : I \longrightarrow X$ is a loop with base point x_0, for each $t \in I$. Now define

$$H : I \times I \longrightarrow X, \ \ H(s,t) \ := \ (\sigma * H_t) * \overline{\sigma}(s).$$

It's easy to check that H is a path homotopy between $(\sigma * \gamma) * \overline{\sigma}$ and $(\sigma * \mu) * \overline{\sigma}$.



Thus we have a well-defined map

$$\Psi_\sigma : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0), \ \ [\gamma] \ \mapsto \ [(\sigma * \gamma) * \overline{\sigma}] \, .$$

Lemma 13.10. *Let X be a non-empty topological space, $x_0, x_1 \in X$ and let $\sigma : I \longrightarrow X$ be a path from x_0 to x_1. Then the map*

$$\Psi_\sigma : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0), \ \ [\gamma] \ \mapsto \ [(\sigma * \gamma) * \overline{\sigma}] \, .$$

is a group isomorphism.

Proof. Let $c_{x_0}, c_{x_1} : I \longrightarrow X$ be the constant paths $c_{x_0}(t) = x_0, \forall t \in I$ and $c_{x_1}(t) = x_1, \forall t \in I$. Then we have $\overline{\sigma} * \sigma \ \simeq \ c_{x_1}$ and $\sigma * \overline{\sigma} \ \simeq \ c_{x_0}$. Let γ, μ be two loops in X with base point x_1. Since

$$\sigma * (\gamma * \mu) * \overline{\sigma} \ \simeq \ \sigma * \gamma * c_{x_1} * \mu * \overline{\sigma} \ \simeq \ \sigma * \gamma * (\overline{\sigma} * \sigma) * \mu * \overline{\sigma} \ \simeq \ (\sigma * \gamma * \overline{\sigma}) * (\sigma * \mu * \overline{\sigma}) \, ,$$

Ψ_σ is a group homomorphism. Now define

$$\Phi_\sigma : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1), \ \ [\mu] \ \mapsto \ [(\overline{\sigma} * \mu) * \sigma].$$

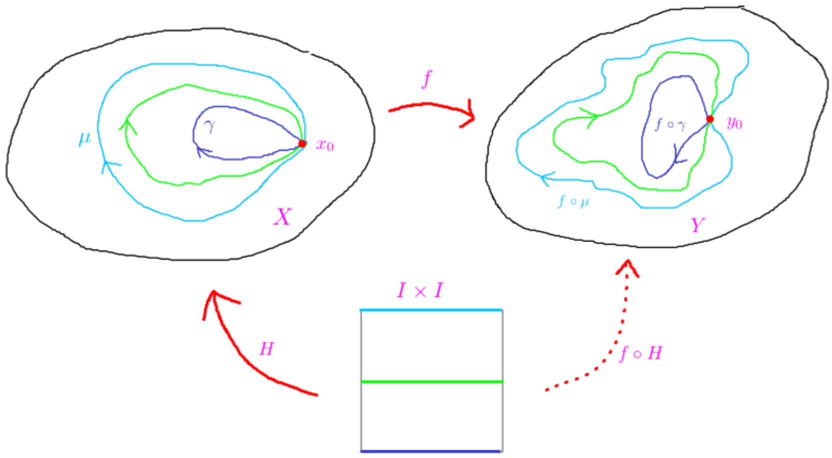
As above, Φ_σ is a well-defined group homomorphism. Let γ be a loop in X with base point x_1 and μ be a loop in X with base point x_0. Then

$$\begin{aligned} \overline{\sigma} * (\sigma * \gamma * \overline{\sigma}) * \sigma \ &\simeq \ (\overline{\sigma} * \sigma) * \gamma * (\overline{\sigma} * \sigma) \ \simeq \ c_{x_1} * \gamma * c_{x_1} \ \simeq \ \gamma \\ \sigma * (\overline{\sigma} * \mu * \sigma) * \overline{\sigma} \ &\simeq \ (\sigma * \overline{\sigma}) * \mu * (\sigma * \overline{\sigma}) \ \simeq \ c_{x_0} * \mu * c_{x_0} \ \simeq \ \mu \end{aligned}$$

so that $\Phi_\sigma \circ \Psi_\sigma([\gamma]) = [\gamma]$ and $\Psi_\sigma \circ \Phi_\sigma([\mu]) = [\mu]$. Hence Ψ_σ and Φ_σ are isomorphisms. $\square$

Corollary 13.11. *Let X be a non-empty path connected space. Then for any two points $x_0, x_1 \in X$, the fundamental groups $\pi_1(X, x_0)$ and $\pi(X, x_1)$ are isomorphic.*

Let X, Y be two non-empty topological spaces and let $f : X \longrightarrow Y$ be a continuous map. Fix $x_0 \in X$ and let $y_0 := f(x_0) \in Y$. Given a loop $\gamma : I \longrightarrow X$ with base point x_0, the map $f \circ \gamma : I \longrightarrow Y$ is a loop with base point y_0. Let $\mu : I \longrightarrow X$ be a loop with base point x_0 and let $H : I \times I \longrightarrow X$ be a path homotopy between γ and μ. It's easy to see that $f \circ H : I \times I \longrightarrow Y$ be a path homotopy between $f \circ \gamma$ and $f \circ \mu$.



Thus we have a well-defined map
$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \;\; [\gamma] \mapsto [f \circ \gamma].$$

Lemma 13.12. *Let X, Y be two non-empty topological spaces and let $f : X \longrightarrow Y$ be a continuous map. Fix $x_0 \in X$ and let $y_0 := f(x_0) \in Y$. Then the map*
$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \;\; [\gamma] \mapsto [f \circ \gamma]$$
is a group homomorphism.

Proof. Let γ, μ be two loops in X with base point x_0. Then
$$f_*([\gamma] * [\mu]) = [f \circ (\gamma * \mu)] = [(f \circ \gamma) * (f \circ \mu)] = [f \circ \gamma] * [f \circ \mu] = f_*([\gamma]) * f_*([\mu]).$$
Hence f_* is a group homomorphism. $\square$

Exercise 13.13. *(1.)* Let $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$ be continuous maps between pointed topological spaces. Prove that the following diagram commutes:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[rr, "f_*"] \arrow[dr, "(g \circ f)_*"'] & & \pi_1(Y, y_0) \arrow[dl, "g_*"] \\ & \pi_1(Z, z_0) & \end{tikzcd}$$

Conclude that if f is a homeomorphism, then f_* is a group isomorphism.

(2.) Let X be a non-empty topological space, $x_0 \in X$ and let $A \subseteq X$ be the path connected component of X containing x_0. Let $f : A \longrightarrow X$ be the natural inclusion map. Prove that the induced map $f_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ is a group isomorphism.

Proposition 13.14. *Let $f, g : X \longrightarrow Y$ be two continuous maps between two non-empty topological spaces. Let $x_0 \in X$ and set $y_0 := f(x_0) \in Y$ and $y_1 := g(x_0) \in Y$. Suppose $f \simeq g$. Then there is a path $\sigma : I \longrightarrow Y$ from y_0 to y_1 such that the induced group isomorphism $\Psi_\sigma : \pi_1(Y, y_1) \longrightarrow \pi_1(Y, y_0)$ defined by $[\gamma] \mapsto [(\sigma * \gamma) * \overline{\sigma}]$ makes the following the diagram commutative:*

$$
\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr, "f_*"] \arrow[dr, "g_*"'] & & \pi_1(Y, y_0) \\
& \pi_1(Y, y_1) \arrow[ur, "\Phi_\sigma"'] &
\end{tikzcd}
$$

Proof. Let $H' : X \times I \longrightarrow Y$ be a homotopy between f and g. Define

$$\sigma : I \longrightarrow Y, \;\; \sigma(t) := H'(x_0, t).$$

Then σ is a path in Y from y_0 to y_1. By Lemma 13.10, $\Psi_\sigma : \pi_1(Y, y_1) \longrightarrow \pi_1(Y, y_0)$ is a group isomorphism. Let $\gamma : I \longrightarrow X$ be a loop with base point x_0. Then $f \circ \gamma$ (resp. $g \circ \gamma$) is a loop in Y with base point y_0 (resp. with base point y_1). Define

$$H : I \times I \longrightarrow Y, \;\; H(s,t) := \begin{cases} \sigma(4s) & \text{if } s \in [0,\, t/4] \\ H'\Big(\gamma\Big(\dfrac{4s-t}{4-3t}\Big),\, t\Big) & \text{if } s \in [t/4,\, 1 - t/2] \\ \overline{\sigma}(2s-1) & \text{if } s \in [1 - t/2,\, 1] \end{cases}$$

Then H is a continuous map with

- $H(s, 0) = H'(\gamma(s), 0) = f \circ \gamma(s)$ and $H(s, 1) = (\sigma * (g \circ \gamma)) * \overline{\sigma}(s), \;\; \forall s \in I$;
- $H(0, t) = \sigma(0) = y_0$ and $H(1, t) = \overline{\sigma}(1) = y_0, \;\; \forall t \in I$.

Thus $f \circ \gamma \simeq (\sigma * (g \circ \gamma)) * \overline{\sigma}$ so that $f_*([\gamma]) = \Psi_\sigma \circ g_*([\gamma]) \in \pi_1(Y, y_0)$. $\square$

Corollary 13.15. *Let $f, g : (X, x_0) \longrightarrow (Y, y_0)$ be two continuous maps between pointed topological spaces. Suppose $f \simeq g$. Then there is a loop η in Y with base point y_0 such that $f_*([\gamma]) = [\eta] * g_*([\gamma]) * [\eta]^{-1} \in \pi_1(Y, y_0)$, for every $[\gamma] \in \pi_1(X, x_0)$, i.e. f_* and g_* are conjugate by an element of $\pi_1(Y, y_0)$.*

Proof. Let $H : X \times I \longrightarrow Y$ be a homotopy between f and g. Define $\eta : I \longrightarrow Y$ by $\eta(t) := H(x_0, t)$. Then η is a loop in Y with base point y_0. Since $[\overline{\eta}] = [\eta]^{-1} \in \pi_1(Y, y_0)$, the result follows from Proposition 13.14. $\square$

Exercise 13.16. Let X, Y be two non-empty topological spaces and let $f : X \longrightarrow Y$ be a homotopy equivalence. Let $x_0 \in X$ and set $y_0 := f(x_0) \in Y$. Prove that the induced group homomorphism $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism.

Definition 13.17. Let X be a non-empty topological space. We say X is *simply connected* if X is path connected and $\pi_1(X, x_0)$ is the trivial group for some (and hence any) $x_0 \in X$.

Corollary 13.18. *A contractible space is simply connected.*

Proof. Let X be a contractible topological space. Then it is path connected. Let $x_0 \in X$ and $c_{x_0} : I \longrightarrow X$ be the constant map $c_{x_0}(t) = x_0, \forall t \in I$ such that $1_X \simeq c_{x_0}$. By Corollary 13.15, the induced group homomorphisms $(1_X)_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$ and $(c_{x_0})_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$ are conjugate by an element of $\pi_1(X, x_0)$. Since $(c_{x_0})_*$ is the trivial group homomorphism and $Id_{\pi_1(X, x_0)} = (1_X)_*$, the image of $Id_{\pi_1(X, x_0)}$ is the trivial subgroup of $\pi_1(X, x_0)$. Hence $\pi_1(X, x_0)$ is the trivial group. $\square$

Exercise 13.19. *(1.)* Let $f : X \longrightarrow Y$ be a homeomorphism between non-empty topological spaces. If one of them is simply connected, prove that the other one is also simply connected.

(2.) Let X, Y be two non-empty topological spaces and let $(x_0, y_0) \in X \times Y$. Prove that $\pi_1(X \times Y,\, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

We will consider $\mathbb{S}^1$ as a subspace of $\mathbb{C}$: $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Let $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the exponential map $t \mapsto e^{2t\pi\sqrt{-1}}$. Then p is a surjective continuous group homomorphism. Moreover, the restriction map $p : [0, 1) \longrightarrow \mathbb{S}^1$ is a bijection. Since $p^{-1}(1) = \mathbb{Z}$, for any $t \in \mathbb{R}$, we have $p^{-1}(p(t)) = \{t + n : n \in \mathbb{Z}\}$. In particular, for any $z \in \mathbb{S}^1$ and any $t \in \mathbb{R}$ with $p(t) = z$, we have $p^{-1}(z) = \{t + n : n \in \mathbb{Z}\}$.

Lemma 14.1. *The exponential map $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ is an open map.*

Proof. Let $\emptyset \neq V \subseteq \mathbb{R}$ be an open set. Then $p^{-1}(p(V)) = \bigcup_{n\in\mathbb{Z}} (n + V)$ is an open set in $\mathbb{R}$. Set $E := \mathbb{S}^1 \setminus p(V)$. Then $p^{-1}E \subseteq \mathbb{R}$ is closed. Since for any $s \in \mathbb{R}$, there is $t \in [0, 1]$ such that $p(s) = p(t)$, we have $E = p(p^{-1}E) = p(p^{-1}E \bigcap [0, 1])$. Since $p^{-1}E \bigcap [0, 1] \subseteq \mathbb{R}$ is compact, so is $E \subseteq \mathbb{S}^1$. Hence E is closed in $\mathbb{S}^1$. $\square$

A non-empty set $X \subseteq \mathbb{R}^n$ is said to a *star shaped with respect to some $x_0 \in X$* if for any $x \in X$, the line segment joining x and x_0 in $\mathbb{R}^n$ lies in X.

Theorem 14.2. *Let $X \subseteq \mathbb{R}^n$ be a compact subset and assume that X is star shaped with respect to the origin. Let $f : X \longrightarrow \mathbb{S}^1$ be a continuous map such that $f(0) = 1$. Then for $m \in \mathbb{Z}$, there is a unique continuous map $\widetilde{f} : X \longrightarrow \mathbb{R}$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(0) = m$ where $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the exponential map.*

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & \mathbb{S}^1 \end{tikzcd}$$

Proof. Since X is compact, f is uniformly continuous. Thus there is a real number $\delta > 0$ such that $|f(x) - f(y)| < 2$ whenever $x, y \in X$ with $||x - y|| < \delta$. Since X is bounded, there is $N \in \mathbb{N}$ such that $X \subseteq B(0; N\delta)$. Let $x \in X$. Then, for any $j \in \{1, \cdots, N\}$,

$$\left|\left|\frac{j}{N}x - \frac{j-1}{N}x\right|\right| < \delta \implies \left|f\left(\frac{j}{N}x\right) - f\left(\frac{j-1}{N}x\right)\right| < 2$$

so that we have a continuous map

$$g_j : X \longrightarrow \mathbb{S}^1 \setminus \{-1\}, \quad g_j(x) := f\left(\frac{j}{N}x\right) \Big/ f\left(\frac{j-1}{N}x\right).$$

Notice that, $g_j(0) = 1, \forall j = 1, \cdots, N$ and f is the point-wise product of $g_1, \cdots, g_n$. Let $q : (-1/2, 1/2) \longrightarrow \mathbb{S}^1 \setminus \{-1\}$ be the restriction of p. Then q is a homeomorphism. Let $\phi : \prod_{j=1}^N \mathbb{S}^1 \setminus \{-1\} \longrightarrow \prod_{j=1}^N \mathbb{R}$ be the continuous map induced by the composition

$$\phi : \prod_{j=1}^N \mathbb{S}^1 \setminus \{-1\} \xrightarrow{\prod_{j=1}^N q^{-1}} \prod_{j=1}^N (-1/2, 1/2) \xhookrightarrow{\text{inc}} \prod_{j=1}^N \mathbb{R}$$

Fix $m \in \mathbb{Z}$ and define $\widetilde{f} : X \longrightarrow \mathbb{R}$ by the composition

$$\widetilde{f} : X \xrightarrow{\prod_{j=1}^N g_j} \prod_{j=1}^N (\mathbb{S}^1 \setminus \{-1\}) \xrightarrow{\phi} \prod_{j=1}^N \mathbb{R} \xrightarrow{\mathscr{A}} \mathbb{R}$$

where $\mathscr{A} : \prod_{j=1}^{N} \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $(a_1, \cdots, a_N) \mapsto m + \sum_{j=1}^{N} a_j$. For any $x \in X$,

$$\widetilde{f}(x) = m + \sum_{j=1}^{N} q^{-1} g_j(x) \implies p \circ \widetilde{f}(x) = p(m) \cdot \prod_{j=1}^{N} p(q^{-1} g_j(x)) = \prod_{j=1}^{N} g_j(x) = f(x)$$

and $\widetilde{f}(0) = m + \sum_{j=1}^{N} q^{-1} g_j(0) = m$.

Now assume $\widetilde{h} : X \longrightarrow \mathbb{R}$ be a continuous map such that $p \circ \widetilde{h} = f$ and $\widetilde{h}(0) = m$. Define $\psi : X \longrightarrow \mathbb{R}$ by $\psi(x) = \widetilde{f}(x) - \widetilde{h}(x)$. Then $p \circ \psi(x) = 1, \forall x \in X$. Since $p^{-1}(1) = \mathbb{Z}$, we have $\psi(x) \in \mathbb{Z}, \forall x \in X$. Since X is connected and $\psi(0) = 0$, $\psi \equiv 0$. Hence $\widetilde{f} = \widetilde{h}$. $\square$

Corollary 14.3. (*i*) (*Path lifting Property*) *Let $\gamma : I \longrightarrow \mathbb{S}^1$ be a path with $\gamma(0) = 1$. Then there is a unique path $\widetilde{\gamma} : I \longrightarrow \mathbb{R}$ such that $p \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = 0$.*

(*ii*) (*Homotopy lifting Property*) *Let $H : I \times I \longrightarrow \mathbb{S}^1$ be a continuous map with $H(0,0) = 1$. Then there is a unique continuous map $\widetilde{H} : I \times I \longrightarrow \mathbb{R}$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}(0,0) = 0$.*

Definition 14.4. Let $\gamma : I \longrightarrow \mathbb{S}^1$ be a loop with base point 1. Let $\widetilde{\gamma} : I \longrightarrow \mathbb{R}$ be the unique lifting of γ with $\widetilde{\gamma}(0) = 0$. We define the *degree* of γ to be the integer $\widetilde{\gamma}(1)$.

Let $\gamma, \mu : I \longrightarrow \mathbb{S}^1$ be two loops with base points 1. Let $\widetilde{\gamma}, \widetilde{\mu} : I \longrightarrow \mathbb{R}$ be the unique lifting of γ and μ, respectively, with $\widetilde{\gamma}(0) = 0 = \widetilde{\mu}(0)$. Now assume γ and μ are path homotopic and let $H : I \times I \longrightarrow \mathbb{S}^1$ be a path homotopy between γ and μ. Let $\widetilde{H} : I \times I \longrightarrow \mathbb{R}$ be the unique lifting of H with $\widetilde{H}(0,0) = 0$. Then

- $p \circ \widetilde{H}(s,0) = H(s,0) = \gamma(s)$, $p \circ \widetilde{H}(s,1) = H(s,1) = \mu(s), \forall s \in I$;
- $p \circ \widetilde{H}(0,t) = \gamma(0) = \mu(0) = 1$, $p \circ \widetilde{H}(1,t) = \gamma(1) = \mu(1) = 1, \forall t \in I$.

Since the map $I \longrightarrow \mathbb{S}^1$, $t \mapsto \widetilde{H}(0,t)$ is continuous and $p \circ \widetilde{H}(0,t) = 1, \forall t \in \mathbb{Z}$, $\widetilde{H}(0,t)$ is a fixed integer, for every $t \in I$. Similarly, $\widetilde{H}(1,t)$ is a fixed integer, for every $t \in I$. Since $\widetilde{H}(0,0) = 0$, we have $\widetilde{H}(0,t) = 0, \forall t \in I$. By the uniqueness of path lifting property, we have $\widetilde{\gamma}(s) = \widetilde{H}(s,0)$ and $\widetilde{\mu}(s) = \widetilde{H}(s,1), s \in I$. Thus $\widetilde{\gamma}(1) = \widetilde{H}(1,0) = \widetilde{H}(1,1) = \widetilde{\mu}(1) \in \mathbb{Z}$. So we have a well defined map

$$\deg : \pi_1(\mathbb{S}^1, 1) \longrightarrow \mathbb{Z}, \ [\gamma] \mapsto \deg(\gamma).$$

Lemma 14.5. *Let $\gamma, \mu : I \longrightarrow \mathbb{S}^1$ be two loops with base point 1. If $\deg(f) = \deg(g)$, then f and g are path homotopic.*

Proof. Let $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the exponential map and let $\widetilde{\gamma} : I \longrightarrow \mathbb{R}$ and $\widetilde{\mu} : I \longrightarrow \mathbb{R}$ be the unique lifting of γ and μ, respectively, with $\widetilde{\gamma}(0) = 0 = \widetilde{\mu}(0)$ (by Corollary 14.3). Since $\deg(f) = \deg(g)$, we have $\widetilde{\gamma}(1) = \widetilde{\mu}(1)$. Now define

$$H : I \times I \longrightarrow \mathbb{R}, \ H(s,t) := (1-t)\widetilde{\gamma}(s) + t\widetilde{\mu}(s).$$

Then H is a path homotopy between $\widetilde{\gamma}$ and $\widetilde{\mu}$. Thus $p \circ H : I \times I \longrightarrow \mathbb{S}^1$ is a path homotopy between $\gamma = p \circ \widetilde{\gamma}$ and $\mu = p \circ \widetilde{\mu}$. $\square$

Theorem 14.6. *The map $\deg : \pi_1(\mathbb{S}^1, 1) \longrightarrow \mathbb{Z}$, $[\gamma] \mapsto \deg(\gamma)$ is a group isomorphism.*

Proof. Let $n \in \mathbb{Z}$. Consider the path $\sigma : I \longrightarrow \mathbb{R}$, $s \mapsto ns$. Then $p \circ \sigma : I \longrightarrow \mathbb{S}^1$ is a loop with base point 1. Then the map deg is surjective. By Lemma 14.5, deg is a bijection.

Let $\gamma, \mu : I \longrightarrow \mathbb{S}^1$ be two loops with base point 1 and let $\widetilde{\gamma} : I \longrightarrow \mathbb{R}$ and $\widetilde{\mu} : I \longrightarrow \mathbb{R}$ be the unique lifting of γ and μ, respectively, with $\widetilde{\gamma}(0) = 0 = \widetilde{\mu}(0)$. Define a path

$$\sigma : I \longrightarrow \mathbb{R}, \ \sigma(t) := \begin{cases} \widetilde{\gamma}(2t) & \text{if } t \in [0, 1/2] \\ \widetilde{\gamma}(1) + \widetilde{\mu}(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

Then we have

$$p \circ \sigma : I \longrightarrow \mathbb{S}^1, \quad p \circ \sigma(t) := \begin{cases} \gamma(2t) & \text{if } t \in [0,\, 1/2] \\ \mu(2t-1) & \text{if } t \in [1/2,\, 1] \end{cases} = (\gamma * \mu)(t).$$

Thus $\sigma : I \longrightarrow \mathbb{R}$ is the unique lifting of $\gamma * \mu$ with $\sigma(0) = 0$. Hence

$$\deg(\gamma * \mu) = \sigma(1) = \widetilde{\gamma}(1) + \widetilde{\mu}(1) = \deg(\gamma) + \deg(\mu)$$

so that the map deg is a group homomorphism. $\square$

Lemma 14.7. *Let (X, x_0) be a pointed topological space and let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Assume $\mathscr{U}$ satisfies the following conditions:*

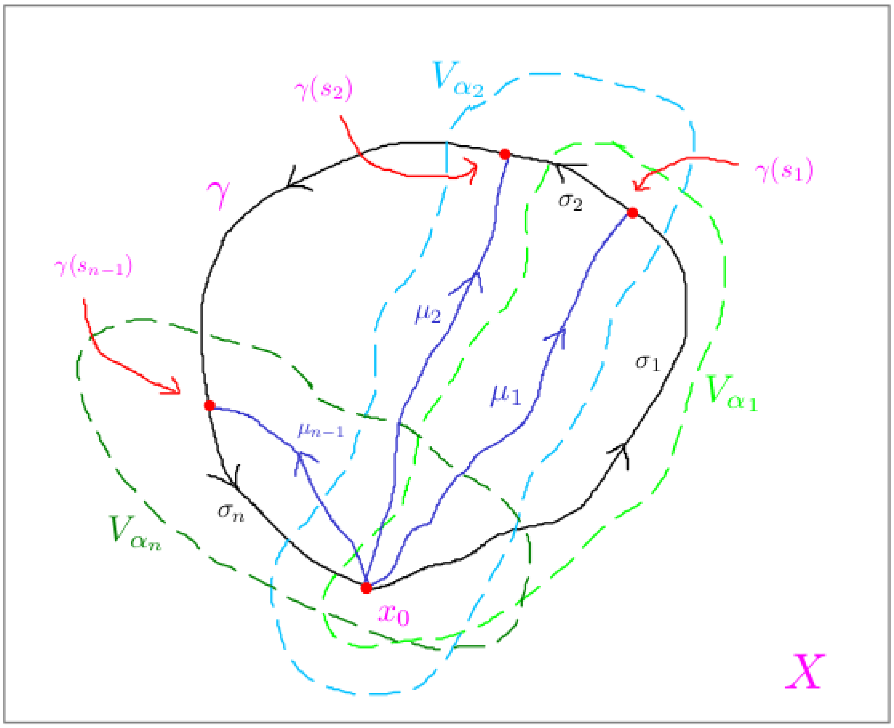
- *each V_α is path connected;*
- *$x_0 \in V_\alpha$, for every $\alpha \in \Lambda$;*
- *$V_\alpha \bigcap V_\beta$ is path connected, for every $\alpha, \beta \in \Lambda$.*

Let $\gamma : I \longrightarrow X$ be a loop with base point x_0. Then there are $\alpha_1, \cdots, \alpha_n \in \Lambda$, for some $n \in \mathbb{N}$, and loops $\gamma_j : I \longrightarrow V_{\alpha_j}$ with base point x_0 such that $[\gamma] = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n] \in \pi_1(X, x_0)$.

Proof. For each $t \in I$, there is a real number $\delta_t > 0$ such that $\gamma\big([t - \delta_t, t + \delta_t] \bigcap I\big) \subseteq V_{\alpha_t}$, for some $\alpha_t \in \Lambda$. Since I is compact and $\{(t - \delta_t, t + \delta_t) \bigcap I : t \in I\}$ is an open cover of I, there are finitely many points $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $\gamma([s_{j-1}, s_j]) \subseteq V_{\alpha_j}$, for some $\alpha_j \in \Lambda$. For each $j \in \{1, \cdots, n\}$, define

$$\sigma_j : I \longrightarrow V_{\alpha_j}, \quad \sigma_j(t) := \gamma((1-t)s_{j-1} + ts_j)$$

Notice that σ_j is noting but the potion of the path γ from $\gamma(s_{j-1})$ to $\gamma(s_j)$. Let $\mu_j : I \longrightarrow V_{\alpha_j}$ be a path from x_0 to $\gamma(s_j) = \sigma_j(1)$ and $\overline{\mu_j} : I \longrightarrow V_{\alpha_j}$ be the opposite path of μ_j, for each $j = 1, \cdots, n-1$ (such a path exists since V_{α_j} is path connected).



For each $j \in \{1, \cdots, n\}$, define

$$\gamma_j : I \longrightarrow X, \;\; \gamma_j \;:=\; \begin{cases} \sigma_1 * \overline{\mu_1} & \text{if } \; j = 1 \\ (\mu_{j-1} * \sigma_j) * \overline{\mu_j} & \text{if } \; 2 \leq j \leq n-1 \\ \mu_{n-1} * \sigma_n & \text{if } \; j = n \end{cases}$$

Then γ_j is a loop in V_{α_j} with base point x_0, for every $j \in \{1, \cdots, n\}$. Moreover,

$$\begin{aligned} \gamma_1 * \gamma_2 * \cdots \gamma_n \;&\simeq\; (\sigma_1 * \overline{\mu_1}) * (\mu_1 * \sigma_2 * \overline{\mu_2}) * \cdots * (\mu_{n-1} * \sigma_n) \\ &\simeq\; \sigma_1 * \sigma_2 * \cdots * \sigma_n \;\simeq\; \gamma. \end{aligned}$$

Thus $[\gamma] = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n] \in \pi_1(X, x_0)$. $\square$

Theorem 14.8. *For $n \geq 2$, the sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is simply connected.*

Proof. Since $\mathbb{S}^n$ is path connected, we only need to show that any loop in $\mathbb{S}^n$ with a some base point $x_0 \in \mathbb{S}^n$ is null homotopic. Let $A = (0, \cdots, 0, 1) \in \mathbb{S}^n$ and $B := -A \in \mathbb{S}^n$. Consider the open subsets of $\mathbb{S}^n$: $V := \mathbb{S}^n \setminus \{A\}$ and $W := \mathbb{S}^n \setminus \{B\}$. Then $\mathbb{S}^n = V \bigcup W$. Notice that the maps

$$\begin{aligned} \phi : V \longrightarrow \mathbb{R}^n, \;\; (x_0, \cdots, x_n) &\mapsto \frac{1}{1 - x_n}(x_0, \cdots, x_{n-1}) \\ \psi : W \longrightarrow V, \;\; (x_0, \cdots, x_n) &\mapsto (x_0, \cdots, x_{n-1}, -x_n) \end{aligned}$$

are homeomorphisms. Thus both V and W are simply connected. Since $n \geq 2$ and we have a homeomorphism $\phi : V \cap W \longrightarrow \mathbb{R}^n \setminus \{0\}$, $V \bigcap W$ is path connected. Fix $x_0 \in V \bigcap W$ and let $\gamma : I \longrightarrow \mathbb{S}^n$ be a loop with base point x_0. Then there is a loop $\gamma_1 : I \longrightarrow V$ with base point x_0 and a loop $\gamma_2 : I \longrightarrow W$ with base point x_0 such that $[\gamma] = [\gamma_1] * [\gamma_2] \in \pi_1(\mathbb{S}^n, x_0)$ (Lemma 14.7). Since both V and W are simply connected, γ_1 and γ_2 are homotopic to the constant path $c_{x_0} : I \longrightarrow \mathbb{S}^n, c_{x_0}(t) = x_0, \forall t \in I$. Hence γ is null homotopic. $\square$

Exercise 14.9. Let X be a non-empty topological space and let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover X. Suppose $\mathscr{U}$ satisfies the following conditions:

- each V_α is simply connected;
- $x_0 \in V_\alpha$, for every $\alpha \in \Lambda$;
- $V_\alpha \bigcap V_\beta$ is path connected, for every $\alpha, \beta \in \Lambda$.

Prove that X is simply connected.

Definition 15.1. Let X be a non-empty topological space and let $\emptyset \neq A \subseteq X$. We say A is a *retract* of X if there is a continuous map $r : X \longrightarrow A$ such that $r|_A : A \longrightarrow A$ is the identity map. In this case, the r is said to be a *retraction* of X onto A.

Example 15.2. *(1.)* Let X be a non-empty topological space and let $A = \{a\} \subseteq X$. Then A is a retract of X and the constant map $c_a : X \longrightarrow A$, $c_a(x) = a, \forall x \in X$ is a retraction of X onto A.

(2.) For any $n \in \mathbb{N}$, the map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{S}^n$, $x \mapsto x/||x||$ is a retraction of $\mathbb{R}^{n+1} \setminus \{0\}$ onto $\mathbb{S}^n$. Thus $\mathbb{S}^n$ is a retract of $\mathbb{R}^{n+1} \setminus \{0\}$.

(3.) Let $X = \{re^{2\pi\sqrt{-1}\,\theta} : r \in [1,2],\, \theta \in [0,1]\} \subseteq \mathbb{R}^2$ be an annulus. For $j = 1, 2$, set $A_j = \{je^{2\pi\sqrt{-1}\,\theta} : \theta \in [0,1]\}$. Define

$$r_1 : X \longrightarrow A_1,\ \ re^{2\pi\sqrt{-1}\,\theta} \mapsto e^{2\pi\sqrt{-1}\,\theta}$$
$$r_2 : X \longrightarrow A_2,\ \ re^{2\pi\sqrt{-1}\,\theta} \mapsto 2e^{2\pi\sqrt{-1}\,\theta}$$

Then r_j is a retraction of X onto A_j. Since X is connected and $A_1 \bigcup A_2$ is disconnected, there is no retraction of X to $A_1 \bigcup A_2$.

Exercise 15.3. Let X, Y be two non-empty topological spaces. Prove that both X and Y are retracts of $X \times Y$. Conclude that a circle is a retract of torus.

Proposition 15.4. *Let X be a non-empty topological space and let $\emptyset \neq A \subseteq X$ be a retract of X. Then for any $a_0 \in A$, the natural inclusion $\iota : A \longrightarrow X$ induces an injective group homomorphism $\iota_* : \pi_1(A, a_0) \longrightarrow \pi_1(X, x_0)$.*

Proof. Let $r : X \longrightarrow A$ be a retraction of X onto A and let $a_0 \in A$. Then $r \circ \iota = Id_A$ and we have a commutative diagram:

```tikzcd
\pi_1(A, a_0) \arrow[r, "\iota_*"] \arrow[dr, "(Id_A)_*"'] & \pi_1(X, a_0) \arrow[d, "r_*"] \\
 & \pi_1(A, a_0)
```

Since $Id_{\pi_1(A,a_0)} = (Id_A)_* : \pi_1(A, a_0) \longrightarrow \pi_1(A, a_0)$, ι_* is injective. □

Corollary 15.5. (*No-retraction Theorem*) *There is no retraction from the disc $\mathbb{D}^2$ onto $\mathbb{S}^1$.*

Proof. If such a retraction exists, then there is an injective group homomorphism

$$\mathbb{Z} \cong \pi_1(\mathbb{S}^1, 1) \longrightarrow \pi_1(\mathbb{D}^2, 1) = \{[Id_{\mathbb{D}^2}]\}.$$

Thus no such retraction exists. □

Definition 15.6. Let X be a non-empty topological space and let $\emptyset \neq A \subseteq X$. A *deformation retraction* of X onto A is a continuous map $H : X \times I \longrightarrow X$ such that

- $H(x, 0) = x$, $\forall x \in X$ and $H(x, 1) \in A$, $\forall x \in X$;
- $H(a, t) = a$, $\forall a \in A$ and $t \in I$.

If such a deformation retraction of X onto A exists, we say A is a *deformation retract* of X.

Let X be a non-empty topological space, $\emptyset \neq A \subseteq X$ and assume A is a deformation retract of X. Suppose $H : X \times I \longrightarrow X$ be a deformation retraction of X onto A. Define $r : X \longrightarrow A$ by $r(x) := H(x,1)$. Then r is a retraction of X onto A. Thus if A is a deformation retract of X, then A is a retract of X. But the converse need not be true. For example, if there is a deformation retraction of $\mathbb{S}^1$ to $\{1\} \subseteq \mathbb{S}^1$, then the identity map $Id_{\mathbb{S}^1}$ will be null homotopic. In this case, $\mathbb{S}^1$ is contractible, a contradiction since $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$. Thus $\{1\}$ is not a deformation retract of $\mathbb{S}^1$. On the other hand, $\{1\}$ is a retract of $\mathbb{S}^1$.

Example 15.7. *(1.)* Let $X = \{re^{\sqrt{-1}\theta} : 0 < r \leq 1, 0 \leq \theta < 2\pi\} \subseteq \mathbb{R}^2$ be the punctured closed disc in the plane. For each $t \in I$, set

$$f_t : X \longrightarrow X, \;\; re^{\sqrt{-1}\theta} \mapsto (t + (1-t)r)e^{\sqrt{-1}\theta}.$$

Then f_t is a continuous map. Now define

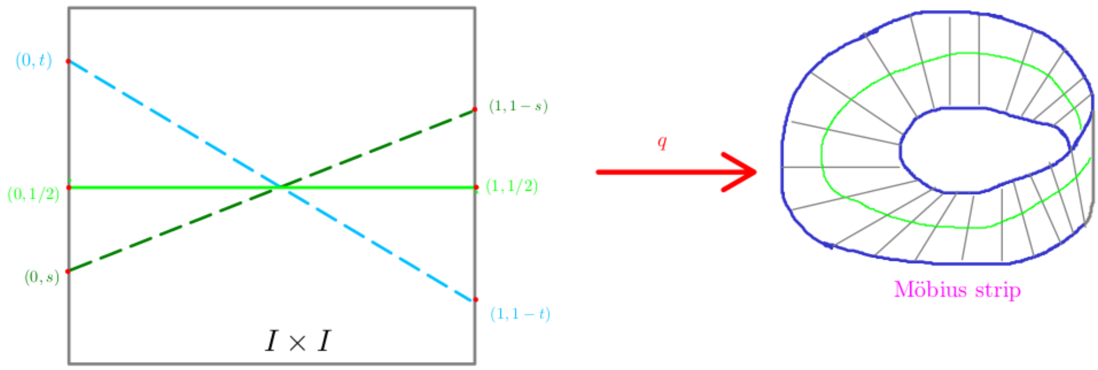
$$H : X \times I \longrightarrow \mathbb{S}^1, \;\; H(re^{\sqrt{-1}\theta}, t) := f_t(re^{\sqrt{-1}\theta}).$$

Then H is a continuous map such that

- $H(x,0) = x$, $\forall x \in X$ and $H(x,1) \in \mathbb{S}^1$, $\forall x \in X$;
- $H(a,t) = a$, $\forall a \in \mathbb{S}^1$ and $t \in I$.

Thus H is a deformation retraction of X onto $\mathbb{S}^1$.

(2.) Define a relation $\sim$ on $I \times I$ as follows: $(0,t) \sim (1, 1-t)$. Then $\sim$ is an equivalence relation $I \times I$. The quotient space $\mathscr{M} := I \times I/\sim$ is called the *Möbius strip*.



Let $q : I \times I \longrightarrow \mathscr{M}$ be the quotient map. Let $A = [0,1] \times \{1/2\} \subseteq I \times I$. Notice that, $q(A) \subseteq \mathscr{M}$ is homeomorphic to $\mathbb{S}^1$. Identifying $q(A)$ and $\mathbb{S}^1$, we consider $\mathbb{S}^1 \subseteq \mathscr{M}$, called the *center circle* of $\mathscr{M}$. Define

$$F : (I \times I) \times I \longrightarrow I \times I, \;\; F((x,y),t) := (x, \, t/2 + (1-t)y).$$

Then F is a continuous map satisfying

$$F((0,y),t) = (0, \, t/2 + (1-t)y) \;\text{ and }\; F((1, 1-y), t) = (1, \, 1 - [t/2 + (1-t)y])$$

for every $y \in I$ and $t \in I$. Thus we have a continuous map $H : \mathscr{M} \times I \longrightarrow \mathscr{M}$ making the following diagram commutative:

$$\begin{tikzcd}
(I \times I) \times I \arrow[r, "F"] \arrow[d, "q \times Id"'] & I \times I \arrow[d, "q"] \\
\mathscr{M} \times I \arrow[r, "H"] & \mathscr{M}
\end{tikzcd}$$

Now we have

- $H\big(q(x,y),0\big) \;=\; q(x,y),\ \forall (x,y) \in I \times I$;
- $H\big(q(x,y),1\big) \;=\; q(x,1/2) \in q(A),\ \forall (x,y) \in I \times I$;
- $H\big(q(x,1/2),t\big) \;=\; q(x,1/2) \in q(A),\ \forall (x,1/2) \in A$ and $t \in I$.

Thus H is a deformation retraction of $\mathscr{M}$ onto $q(A)$.

(3.) Consider the two subspaces of $\mathbb{R}^2$:

$$X \;:=\; \{re^{2\pi\sqrt{-1}\,\theta} \;:\; r \in [1,2],\, \theta \in [0,1]\} \;\text{ and }\; Y \;:=\; \{e^{2\pi\sqrt{-1}\,\theta} \;:\; \theta \in [0,1]\} \;=\; \mathbb{S}^1.$$

Define

$$H : X \times I \longrightarrow X, \quad H\big(re^{2\pi\sqrt{-1}\,\theta}, t\big) \;:=\; (1-t)re^{2\pi\sqrt{-1}\,\theta} + te^{2\pi\sqrt{-1}\,\theta}.$$

Then H is a continuous map such that

- $H(x,0) = x,\, \forall x \in X$ and $H(x,1) \in Y,\, \forall x \in X$;
- $H(x,t) = x,\, \forall x \in Y$ and $t \in I$.

Thus H is a deformation retract of X onto Y.

Proposition 15.8. *Let X be a non-empty topological space and let $\emptyset \neq A \subseteq X$ be a deformation retract of X. Then the natural inclusion map $\iota : A \longrightarrow X$ induces a group isomorphism $\pi_1(A,a_0) \longrightarrow \pi_1(X,a_0)$, for every $a_0 \in A$.*

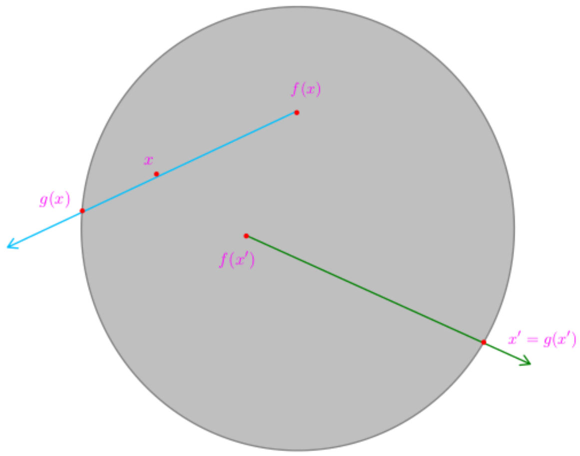
Proof. Let $H : X \times I \longrightarrow X$ be a deformation retraction of X onto A. Set

$$f : X \longrightarrow X,\;\; x \mapsto H(x,0) \;\text{ and }\; g : X \longrightarrow A,\;\; x \mapsto H(x,1)\,.$$

Then $f_0 = Id_X$ and $g \circ \iota = Id_A$. Moreover, H is a homotopy between Id_X and $\iota \circ g : X \longrightarrow X$. Hence g is a homotopy equivalence from X to A with a homotopy inverse $\iota : A \longrightarrow X$. Thus $\iota_* : \pi_1(A,a_0) \longrightarrow \pi_1(X,a_0)$ is a group isomorphism, for every $a_0 \in A$. $\square$

Since $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is a deformation retract of $X \;:=\; \{re^{2\pi\sqrt{-1}\,\theta} \;:\; r \in [1,2],\, \theta \in [0,1]\}$, by the above Proposition, we have $\pi_1(X,1) \cong \pi_1(\mathbb{S}^1,1) \cong \mathbb{Z}$. On the other hand, $\pi_1(\mathbb{D}^2,1)$ is trivial. Thus the annular region X is not homeomorphic (in fact, not homotopic) to the closed unit disc $\mathbb{D}^2$.

Theorem 15.9. (*Brouwer's Fixed Point Theorem*) *Every continuous map $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ has a fixed point.*



Proof. Let $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ be a continuous map which does not have a fixed point. Let $x \in \mathbb{D}^2$ and consider the half-line $\ell_x = \{tx + (1-t)f(x) : t \in \mathbb{R}, t \geq 0\}$ starting at $f(x)$ that passes through the point x. Let $t_x \geq 0$ be the unique[1] real number such that $t_x x + (1-t_x)f(x) \in \partial\mathbb{D}^2 = \mathbb{S}^1$. This is the point of intersection of ℓ_x with the boundary of $\mathbb{D}^2$. Define $g : \mathbb{D}^2 \longrightarrow \mathbb{S}^1$ by $g(x) := t_x x + (1-t_x)f(x)$. Then g is a continuous map and $g(x) = x$, for every $x \in \partial\mathbb{D}^2$. Thus g is a retraction of $\mathbb{D}^2$ onto $\mathbb{S}^1$, a contradiction to Corollary 15.5. Thus f has a fixed point. □

Exercise 15.10. *(1.)* Let X be a non-empty topological spaces and let $\emptyset \neq A \subseteq B \subseteq X$. If A is retract (resp. deformation retract) of B and B is a retract (resp. deformation retract) of X, then prove that A is a retract (resp. deformation retract) of X.

(2.) Let X, Y be two non-empty topological spaces and let $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$. If A is a retract (resp. deformation retract) of X and B is a retract (resp. deformation retract) of Y, then prove that $A \times B$ is a retract (resp. deformation retract) of $X \times Y$.

(3.) Prove that the circle $\mathbb{S}^1 \times \{0\}$ is a deformation retract of the cylinder $\mathbb{S}^1 \times I$. (Write down an explicit deformation retraction from $\mathbb{S}^1 \times I$ onto $\mathbb{S}^1 \times \{0\}$.)

[1] We can write down the formula for t_x as follows:

$$t_x = \frac{||f(x)||^2 - \langle x, f(x)\rangle + \sqrt{||x - f(x)||^2 - ||x||^2||f(x)||^2 + \langle x, f(x)\rangle^2}}{||x - f(x)||^2}$$

Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a continuous map and $q : I \longrightarrow \mathbb{S}^1$ be the quotient map $t \mapsto e^{2\pi t\sqrt{-1}}$. Define $f' : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ by $f'(z) := f(z)/f(1), z \in \mathbb{S}^1$. Then $f' \circ q : I \longrightarrow \mathbb{S}^1$ is a loop with base point 1. Notice that, given f, the map f' and the loop $f' \circ q$ are uniquely determined. We define the *degree* of f, denoted by $\deg(f)$, to be degree of the loop $f' \circ q$.

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d, "p : t \mapsto e^{2\pi t\sqrt{-1}}"] \\
I \arrow[r, "q"] \arrow[urr, dashed] & \mathbb{S}^1 \arrow[r, "f'"] & \mathbb{S}^1
\end{tikzcd}$$

Let $n \in \mathbb{N}$, consider the continuous map $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, z \mapsto z^n$. Then $f' = f$ and $f \circ q : I \longrightarrow \mathbb{S}^1$ is a loop with base point 1. Define $\widetilde{f} : I \longrightarrow \mathbb{R}$ by $\widetilde{f}(s) = ns$. Then $\widetilde{f}$ is the unique lifting of $f \circ q$ to $\mathbb{R}$ with $\widetilde{f}(0) = 0$. Then $\deg(f) = \deg(f \circ q) = \widetilde{f}(1) = n$.

Proposition 16.1. *Two continuous functions $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ are homotopic if and only if* $\deg(f) = \deg(g)$

Proof. Let $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be two continuous maps and let $q : I \longrightarrow \mathbb{S}^1$, $t \mapsto e^{2\pi t\sqrt{-1}}$ be the quotient map. Define $f', g' : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ by $f'(z) = f(z)/f(1)$ and $g'(z) = g(z)/g(1)$. Then both $f' \circ q$, $g' \circ q : I \longrightarrow \mathbb{S}^1$ are loops with base point 1.

Let $H : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1$ be a homotopy between f and g. Define

$$H' : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1, \ \ H'(z,t) \ := \ H(z,t)/H(1,t)\,.$$

Then H' is a continuous map such that

$$\begin{aligned} H'(z,0) \ &= \ H(z,0)/H(1,0) \ = \ f(z)/f(1) \ = \ f'(z) \\ H'(z,1) \ &= \ H(z,1)/H(1,1) \ = \ g(z)/g(1) \ = \ g'(z)\,. \end{aligned}$$

and $H'(1,t) = 1 = f'(1) = g'(1)$, $\forall t \in I$. Thus f' and g' are homotopic relative to $\{1\}$. Hence the loops $f' \circ q$, $g' \circ q : I \longrightarrow \mathbb{S}^1$ with base point 1 are homotopic. Thus

$$\deg(f) \ = \ \deg(f' \circ q) \ = \ \deg(g' \circ q) \ = \ \deg(g).$$

Now Suppose $\deg(f) = \deg(g)$. Then by Lemma 14.5, the loops $f' \circ q$ and $g' \circ q$ are homotopic. Let $F : I \times I \longrightarrow \mathbb{S}^1$ be a homotopy between $f' \circ q$ and $g' \circ q$. Then for any $t \in I$, $f' \circ q(0,t) = f' \circ q(1,t)$ and $g' \circ q(0,t) = g' \circ q(1,t)$. Thus F induces a continuous map $H : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1$ such the following diagram commutes:

$$\begin{tikzcd}
I \times I \arrow[r, "F"] \arrow[d, "q \times Id"'] & \mathbb{S}^1 \arrow[d, "Id"] \\
\mathbb{S}^1 \times I \arrow[r, "H"] & \mathbb{S}^1
\end{tikzcd}$$

Notice that H is a path homotopy between f' and g'. To show that f and g are homotopic, by Lemma 12.4, it is sufficient to show that f is homotopic to f'. For this, let $\theta_0 \in I$ be such that $f(1) = e^{2\pi\theta_0\sqrt{-1}}$ and define

$$H' : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1, \ \ H'(z,t) \ := \ e^{-2\pi t\theta_0\sqrt{-1}} f(z)\,.$$

Then H' is a continuous map with $H'(z,0) = f(z)$ and $H'(z,1) = f'(z)$, $\forall z \in \mathbb{S}^1$. Thus H' is a homotopy between f and f'. $\square$

Theorem 16.2. (*Fundamental Theorem of Algebra*) *Any non-constant polynomial with complex coefficients has a complex root.*

Proof. Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with complex co-efficient of degree $n \geq 1$. We may assume $a_n = 1$. Notice that, $f(z)$ has a root in $\mathbb{C}$ if and only if for every real $b \in \mathbb{R}, b > 0$, the polynomial $z^n + (a_{n-1}/b)z^{n-1} + \cdots + (a_1/b^{n-1})z + a_0/b^n$ has a root in $\mathbb{C}$. By choosing b sufficiently large, we may also assume $\sum_{j=0}^{n-1} |a_j/b^{n-j}| < 1$. Thus it is sufficient to assume that the co-efficient of $f(z)$ satisfies the condition $\sum_{j=0}^{n-1} |a_j| < 1$. Suppose $f(z) \neq 0, \forall z \in \mathbb{C}$. Thus f induces a continuous map $f : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$. Define

$$\Phi : \mathbb{S}^1 \times I \longrightarrow \mathbb{C} \setminus \{0\}, \;\; \Phi(z,t) := f((1-t)z).$$

Then Φ is a homotopy between $f|_{\mathbb{S}^1}$ and the constant map $c_{a_0} : \mathbb{S}^1 \longrightarrow \mathbb{C} \setminus \{0\}$ defined by $c_{a_0}(z) = a_0, \forall z \in \mathbb{S}^1$. For any $t \in I$ and $z \in \mathbb{S}^1$, we have

$$z^n + \sum_{j=0}^{n-1}(1-t)a_j z^j = 0 \implies 1 = \left|\sum_{j=0}^{n-1}(1-t)a_j z^j\right| \leq \sum_{j=0}^{n-1}(1-t)|a_j| \leq \sum_{j=0}^{n-1}|a_j| < 1$$

a contradiction. Thus we have a continuous map

$$H : \mathbb{S}^1 \times I \longrightarrow \mathbb{C} \setminus \{0\}, \;\; H(z,t) := z^n + \sum_{j=0}^{n-1}(1-t)a_j z^j.$$

Consider the continuous map $\Psi : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{S}^1, \; z \mapsto z/||z||$. Then the continuous map $\Psi \circ H : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1$ is a homotopy between $\Psi \circ f|_{\mathbb{S}^1}$ and the loop $\sigma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \; z \mapsto z^n$ with base point 1. On the other hand, $\Psi \circ \Phi : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1$ is a homotopy between $\Psi \circ f|_{\mathbb{S}^1}$ and the constant loop with base point 1. Hence σ is null homotopic, a contradiction since $\deg(\sigma) = n \geq 1$. Thus f must have a complex root. $\square$

Exercise 16.3. Let $f : \mathbb{S}^1 \longrightarrow \mathbb{R}$ be a continuous map. Prove that there is $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$.

Theorem 16.4. (*Borsuk-Ulam*) *Let $f : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$ be a continuous map. Then there is $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$.*

Proof. Suppose no such point $x \in \mathbb{S}^2$ exists. Then we have a continuous map

$$g : \mathbb{S}^2 \longrightarrow \mathbb{S}^1 \subseteq \mathbb{R}^2, \;\; g(x) := \frac{f(x) - f(-x)}{||f(x) - f(-x)||}.$$

Let $\gamma : I \longrightarrow \mathbb{S}^2$ is a map defined by $\gamma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$, i.e. γ is the equator of the sphere $\mathbb{S}^2$. Then $\sigma' := g \circ \gamma : I \longrightarrow \mathbb{S}^1$ is a path. Since $g(x) = -g(-x), \forall x \in \mathbb{S}^2$, we have $\sigma'(t + 1/2) = -\sigma'(t), \forall t \in [0, 1/2]$. Define $\sigma : I \longrightarrow \mathbb{S}^1$ by $\sigma(t) = \sigma'(t)/\sigma'(0)$. Then σ is a loop with base point $\sigma(0) = 1 = \sigma(1)$ and $\sigma(t + 1/2) = -\sigma(t), \forall t \in [0, 1/2]$. Let $\widetilde{\sigma} : I \longrightarrow \mathbb{R}$ be the unique lifting of σ to $\mathbb{R}$ such that $\widetilde{\sigma}(0) = 0$ (Corollary 14.3). Then, for any $t \in [0, 1/2]$

$$p \circ \widetilde{\sigma}(t + 1/2) = \sigma(t + 1/2) = -\sigma(t) = -p \circ \widetilde{\sigma}(t) \implies e^{2\pi\left(\widetilde{\sigma}(t+1/2) - \widetilde{\sigma}(t)\right)\sqrt{-1}} = -1.$$

Thus, for each $t \in [0, 1/2]$, $\widetilde{\sigma}(t + 1/2) = \widetilde{\sigma}(t) + m(t)/2$, for some odd integer $m(t)$. Since the map $[0, 1/2] \longrightarrow \mathbb{Z}, t \mapsto m(t)$ is continuous, $m(t) = m$, for some fixed odd integer m, for every $t \in [0, 1/2]$. Thus $\widetilde{\sigma}(1) = \widetilde{\sigma}(1/2) + m/2 = \widetilde{\sigma}(0) + m/2 + m/2 = m$ so that $\deg(\sigma) = m \neq 0$. Thus σ is not null-homotopic and hence σ' is not null-homotopic. On the other hand, γ is null-homotopic since $\mathbb{S}^2$ is simply connected and hence $\sigma' = g \circ \gamma$ is null-homotopic, a contradiction. Thus there $x \in \mathbb{S}^2$ such that $f(x) = -f(-x)$. $\square$

References

[1] Armstrong, M.A.; *Basic Topology*. Springer, UTM.

[2] Dugundji, James; *Topology*. Allyn and Bacon, 1996.

[3] Kelly, John L.; *General Topology*. Springer, GTM.

[4] Munkres, James R.; *Topology*. Second Edition, Prentice Hall India.

[5] Simmons, G.F.; *Introduction to Topology and Modern Analysis*. McGraw Hill Education.

[6] Sing, Tej Bahadur; *Introduction to Topology*. Springer.

[7] Viro, Ya. et al; *Elementary Topology: Problem Textbook*. American Mathematical Society.

[8] Willard, Stephen; *General Topology*. Dover Books on Mathematics .