MATHEMATIX CLUB

NISER Mathematics Club

SRS 39

Exploring the Hardy-Littlewood Maximal Function and the Hilbert Transform

Sandipan Samanta

Overview

1. Almost Everywhere Convergence

2. Hardy-Littlewood Maximal Function

3. Marcinkiewicz Interpolation Theorem

4. Dyadic Maximal Function

5. Calderón-Zygmund Decomposition

6. Back to Maximal Function

7. Hilbert Transform

A brief description of measure and Lp spaces

Definition (Measure)

Let X be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of X. A **measure** on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ that satisfies the following properties:

1. **Non-negativity:** $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.
2. **Null empty set:** $\mu(\emptyset) = 0$.
3. **Countable additivity (σ-additivity):** For any countable collection of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Definition (Lebesgue Integration)

Let $f : X \to \mathbb{R}$ be a measurable function on a measure space $(X, \mathcal{A}, \mu)$. The **Lebesgue integral** of f over X is defined as follows:

1. **For a non-negative simple function** $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $a_i \geq 0$ and $A_i \in \mathcal{A}$,

$$\int_X \varphi \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

2. **For a non-negative measurable function** f,

$$\int_X f \, d\mu = \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq f, \, \varphi \text{ is simple} \right\}.$$

3. **For an integrable function** f, decompose $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Then,

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

Definition (Lp and weak Lp space)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $0 < p < \infty$.

1. L^p **Space:**

   The space $L^p(X)$ consists of all measurable functions $f : X \to \mathbb{R}$ (or $\mathbb{C}$) such that

   $$\|f\|_{L^p} = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} < \infty.$$

   This quantity $\|f\|_{L^p}$ is called the L^p-norm of f.

2. **Weak** L^p **Space (denoted** $L^{p,\infty}$**):**

   The weak L^p space consists of all measurable functions $f : X \to \mathbb{R}$ (or $\mathbb{C}$) for which there exists a constant $C \geq 0$ such that

   $$\mu\left(\{x \in X : |f(x)| > \lambda\}\right) \leq \frac{C}{\lambda^p} \quad \text{for all } \lambda > 0.$$

   The smallest such constant C is called the weak L^p-quasi-norm of f and is denoted by $\|f\|_{L^{p,\infty}}$.

Definition (Strong and weak (p,q) operators)

Let T be a linear operator acting on functions in a measure space $(X, \mathcal{A}, \mu)$.

1. **Strong** (p, q) **Operator:**
   The operator T is said to be a **strong** (p, q) **operator** if there exists a constant $C > 0$ such that for all $f \in L^p(X)$,
   $$\|Tf\|_{L^q} \leq C\|f\|_{L^p}.$$
2. **Weak** (p, q) **Operator:**
   The operator T is said to be a **weak** (p, q) **operator** if there exists a constant $C > 0$ such that for all $f \in L^p(X)$ and all $\lambda > 0$,
   $$\mu\left(\{x \in X : |Tf(x)| > \lambda\}\right) \leq \frac{C\|f\|_{L^p}^q}{\lambda^q}.$$

Almost Everywhere Convergence

Almost Everywhere Convergence

In this section we'll establish the relationship between weak (p, q) inequalities and almost everywhere convergence.

Theorem

Let $\{T_t\}$ be a family of linear operators on $L^p(X, \mu)$ and define the maximal operator associated with this family $T^*f(x) = \sup_t |T_t f(x)|$. If T^* is weak (p, q) then the set

$$\left\{ f \in L^p(X, \mu) : \lim_{t \to t_0} T_t f(x) = f(x) \text{ a.e.} \right\}$$

is closed in $L^p(X, \mu)$.

Proof.

Let $\{f_n\}$ be a sequence of functions which converges to f in $L^p(X,\mu)$ norm such that $\lim_{t\to t_0} T_t f(x) = f(x)$ a.e. Then, for any $\lambda > 0$,

$$
\begin{aligned}
&\mu(\{x \in X : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\}) \\
&\quad \le \mu(\{x \in X : \limsup_{t\to t_0} |T_t(f - f_n)(x) - (f - f_n)(x)| > \lambda\}) \\
&\quad \le \mu(\{x \in X : \limsup_{t\to t_0} |T_t(f - f_n)(x)| > \lambda/2\}) + \mu(\{x \in X : |(f - f_n)(x)| > \lambda\}) \\
&\quad \le \mu(\{x \in X : T^*(f - f_n)(x) > \lambda/2\}) + \mu(\{x \in X : |(f - f_n)(x)| > \lambda\}) \\
&\quad \le \left(\frac{2C}{\lambda}\|f - f_n\|_p\right)^q + \left(\frac{2}{\lambda}\|f - f_n\|_p\right)^p
\end{aligned}
$$

Proof cont.

As this is true for all $n \in \mathbb{N}$ and $\|f - f_n\|_p \to 0$ as $n \to 0$, we get

$$\mu(\{x \in X : \limsup_{t \to t_0} |T_t f(x) - f(x)| > \lambda\}) = 0, \quad \forall \lambda > 0.$$

Therefore,

$$\begin{aligned} &\mu(\{x \in X : \limsup_{t \to t_0} |T_t f(x) - f(x)| > 0\}) \\ &\leq \sum_{k=1}^{\infty} \mu(\{x \in X : \limsup_{t \to t_0} |T_t f(x) - f(x)| > \frac{1}{k}\}) \\ &= 0 \end{aligned}$$

Thus $T_t f(x)$ converges to $f(x)$ almost everywhere. $\square$

Similarly, if we consider

$$\mu(\{x \in X : \limsup_{t \to t_0} T_t f(x) - \liminf_{t \to t_0} T_t f(x) > \lambda\})$$

we can prove that the set

$$\left\{ f \in L^p(X, \mu) : \lim_{t \to t_0} T_t f(x) \text{ exists a.e.} \right\}$$

is closed. This comes from the fact that

$$\limsup_{t \to t_0} T_t f(x) - \liminf_{t \to t_0} T_t f(x) \leq 2T^* f(x)$$

Hardy-Littlewood Maximal Function

Hardy-Littlewood Maximal Function

What is this function?

Definition (Hardy-Littewood Maximal Function)

For a locally integrable function f on $\mathbb{R}^d$, it is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy$$

where $B_r = B(0, r)$

Cubic Maximal Function

Definition (Cubic Maximal Function)

For a locally integrable function f on $\mathbb{R}^d$, it is defined by

$$M'f(x) = \sup_{r>0} \frac{1}{(2r)^d} \int_{Q_r} |f(x-y)|dy$$

where $Q_r = [-r, r]^d$

Note

For any $d \in \mathbb{N}$, there exist constants c_d and C_d, depending only on d, such that

$$c_d M'f(x) \leq Mf(x) \leq C_d M'f(x), \quad \forall x \in \mathbb{R}^d$$

Hardy-Littlewood Maximal Inequality

Theorem (Hardy-Littlewood Maximal Inequality)

The operator M is weak $(1,1)$ and strong (p,p), for $1 < p \leq \infty$, i.e.,

$$\|Mf\|_{1,\infty} \lesssim_d \|f\|_1 \quad \text{and} \quad \|Mf\|_p \lesssim_d \|f\|_p$$

Remark

By the previous note, the same result holds for M' as well.

From the definition it is clear that $\|Mf\|_\infty \leq \|f\|_\infty$.

We will prove the rest using Marcinkiewicz interpolation theorem and Dyadic Maximal Function

Marcinkiewicz Interpolation Theorem

Distribution Function

Definition

Let (X, μ) be a measure space and let $f : X \to \mathbb{C}$ be a measurable function. We call the the function $a_f : (0, \infty) \to [0, \infty]$, given by

$$a_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}),$$

the distribution function of f (associated with μ).

Lemma

Let $\phi : [0, \infty) \to [0, \infty)$ be a differentiable and increasing function such that $\phi(0) = 0$. Then

$$\int_X \phi(|f(x)|)d\mu = \int_0^\infty \phi'(\lambda)a_f(\lambda)d\lambda$$

If, in particular, $\phi(\lambda) = \lambda^p$ then

$$\|f\|_p^p = p\int_0^\infty \lambda^{p-1}a_f(\lambda)d\lambda.$$

Proof of the lemma.

$$
\begin{aligned}
\int_X \phi(|f(x)|)d\mu &= \int_X \int_0^{|f(x)|} \phi'(\lambda)d\lambda d\mu \\
&= \int_X \int_0^{\infty} \phi'(\lambda)\chi_{\{\lambda\in(0,\infty):\lambda<|f(x)|\}}d\lambda d\mu \\
&= \int_0^{\infty} \int_X \phi'(\lambda)\chi_{\{x\in X:|f(x)|>\lambda\}}d\mu d\lambda \\
&= \int_0^{\infty} \phi'(\lambda)\mu(\{x \in X : |f(x)| > \lambda\})d\lambda \\
&= \int_0^{\infty} \phi'(\lambda)a_f(\lambda)d\lambda
\end{aligned}
$$

□ □

The Marcienkiewicz Interpolation Theorem

Theorem (Marcinkiewicz Interpolation)

Let (X, μ) *and* (Y, ν) *be measure spaces,* $1 \leq p_0 < p1 \leq \infty$*, and T be a sublinear operator from* $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ *to the space of measurable functions on* Y *that is weak* (p_0, p_0) *and weak* (p_1, p_1)*. Then T is strong* (p, p) *for* $p_0 < p < p_1$*.*

Proof

Given $f \in L^p$, for each λ decompose f as $f_0 + f_1$, where $f_0 = f\chi_{\{x:|f(x)|>c\lambda\}}$ and $f_1 = f\chi_{\{x:|f(x)|\le c\lambda\}}$. The constant c will be fixed below such that $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$.

Then due to sublinearity we have $|Tf(x)| \le |Tf_0(x)| + |Tf_1(x)|$.

This implies $a_{Tf}(\lambda) \le a_{Tf_0}(\lambda/2) + a_{Tf_1}(\lambda/2)$.

Case 1: $p_1 = \infty$.

Choose $c = 1/(2A_1)$, where A_1 is such that $\|Tg\|_\infty \le A_1\|g\|_\infty$.

Then $a_{Tf_1}(\lambda/2) = 0$. Also, by the weak (p_0, p_0) inequality,

$$a_{Tf_0}(\lambda/2) \le \left(\frac{2A_0}{\lambda}\|f_0\|_{p_0}\right)^{p_0}$$

Proof Cont...

Hence, by the previous lemma

$$\begin{aligned}
\|Tf\|_p^p \leq & p\int_0^\infty \lambda^{p-1-p_0}(2A_0)^{p_0}\int_{\{x:|f(x)|>c\lambda\}}|f(x)|^{p_0}\,d\mu d\lambda \\
= & p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{|f(x)|/c}\lambda^{p-1-p_0}\,d\lambda d\mu \\
= & \frac{p}{p-p_0}(2A_0)^{p_0}(2A_1)^{p-p_0}\|f\|_p^p
\end{aligned}$$

Proof Cont...

Case 2: $p_1 < \infty$.

In this case we get the pair of inequalities

$$a_{Tf_i}(\lambda/2) \leq \left(\frac{2A_i}{\lambda} \|f_i\|_{p_i} \right)^{p_i}, \quad i = 0, 1.$$

Choose c such that $(2A_0)^{p_0} = (2A_1)^{p_1}$, then arguing similarly we get

$$\|Tf\|_p^p \leq p2^p \left(\frac{1}{p - p_0} + \frac{1}{p_1 - p} \right) A_0^{p_0 \frac{p_1 - p}{p_1 - p_0}} A_1^{p_1 \frac{p_0 - p}{p_0 - p_1}} \|f\|_p^p$$

$\square$

In general, we can write the strong (p, p) norm inequality in this theorem precisely as

$$\|Tf\|_p \leq 2p^{1/p} \left(\frac{1}{p - p_0} + \frac{1}{p_1 - p} \right)^{1/p} A_0^{1-\theta} A_1^{\theta} \|f\|_p,$$

where

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}, \quad 0 < \theta < 1$$

Dyadic Maximal Function

Dyadic Cubes

Definition (Dyadic Cubes)

In $\mathbb{R}^d$, we define D_k to be the family of cubes, open to the right, whose vertices are adjacent points of the lattice $(2^{-k}\mathbb{Z})^d$, i.e.,

$$D_k = \left\{ 2^{-k} \left([0,1)^d + m \right) : m \in \mathbb{Z}^d \right\}.$$

The cubes in $D = \bigcup_k D_k$ are called dyadic cubes.

Some properties of dyadic cubes

- For each $x \in \mathbb{R}^d$ there is a unique cube in each family D_k which contains it.
- Any two dyadic cubes are either disjoint or one is wholly contained in the other.
- A dyadic cube in D_k is contained in a unique cube of each family D_j, $j < k$, and contains 2^d dyadic cubes of D_{k+1}

Given a function $f \in L^1_{loc}(\mathbb{R}^d)$, define

$$E_k f(x) = \sum_{Q \in D_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x).$$

$E_k f$ is the conditional expectation of f with respect to the $\sigma-$ algebra geterated by D_k. It also satisfies

$$\int_D E_k f = \int_D f.$$

Dyadic Maximal Function

Definition (Dyadic Maximal Function)

For a locally integrable function on $\mathbb{R}^d$, the dyadic maximal function is defined by

$$M_d f(x) = \sup_k |E_k f(x)|$$

Theorem (Dyadic Maximal Inequality)

The dyadic maximal function is weak $(1,1)$*, i.e.,*

$$\sup_\lambda \lambda |\{x \in \mathbb{R}^d : M_d f(x) > \lambda\}| \lesssim_d \|f\|_1.$$

Proof of dyadic maximal inequality

Fix $f \in L^1$. If f is real, it can be decomposed into positive and negative parts, and if it is complex then it can be decomposed into real and imaginary parts, thus we may assume f is non-negative.

Note that since $f \in L^1$, $E_k f(x) \to 0$ as $k \to 0$.

So for any $\lambda > 0$,

$$\{x \in \mathbb{R}^d : M_d f(x) > \lambda\} = \bigcup_k \Omega_k$$

where

$$\Omega_k = \{x \in \mathbb{R}^d : E_k f(x) > \lambda \text{ and } E_j f(x) \leq \lambda \text{ if } j < k\}$$

By the construction of the sets Ω_k they are disjoint, and each one of them can be written as the union of cubes in D_k.

Proof Cont..

Hence,

$$
\begin{aligned}
|\{x \in \mathbb{R}^d : M_d f(x) > \lambda\}| &= \sum_k |\Omega_k| = \sum_k \int_{\Omega_k} \chi_{\Omega_k} \\
&\leq \sum_k \frac{1}{\lambda} \int_{\Omega_k} E_k f = \frac{1}{\lambda} \sum_k \int_{\Omega_k} f = \frac{1}{\lambda} \int_{\bigcup_k \Omega_k} f \\
&\leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f = \frac{1}{\lambda} \|f\|_1
\end{aligned}
$$

Since this is true for all $\lambda > 0$, we get

$$
\|M_d f(x)\|_{1,\infty} = \sup_\lambda \lambda |\{x \in \mathbb{R}^d : M_d f(x) > \lambda\}| \lesssim_d \|f\|_1
$$

$\square$

Lemma (Discrete analog of an approximation of the identity)

If $f \in L^1_{loc}$*, then* $\lim_{k\to\infty} E_k f(x) = f(x)$ *a.e.*

Proof.

It is clearly true if f is continuous, and so by the theorem in the section almost everywhere convergence, it holds for $f \in L^1$ (because continuous functions are dense in L^1).

Now, if $f \in L^1_{loc}$ then $f\chi_Q \in L^1$ for every $Q \in D_0$. Hence, the result holds for almost every $x \in Q$, and so for almost every $x \in \mathbb{R}^d$. $\square$

Calderón-Zygmund Decomposition

Calderón Zygmund Covering

The previous proof that dyadic maximal function is weak $(1, 1)$ uses decomposition of $\mathbb{R}^d$ which is extremely useful. We state it precisely as follows.

Lemma (Calderón-Zygmund Covering)

Given a non-negative integrable function f and given a positive number λ, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that

1. $f(x) \leq \lambda$ *for almost every* $x \notin \bigcup_j Q_j$
2. $\left|\bigcup_j Q_j\right| \leq \frac{1}{\lambda}\|f\|_1$
3. $\lambda < \frac{1}{|Q_j|}\int_{Q_j} f \leq 2^d \lambda$

Proof.

We have $E_k f(x) = \sum_{Q \in D_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x)$

and $\Omega_k = \{x \in \mathbb{R}^d : E_k f(x) > \lambda \text{ and } E_j f(x) \leq \lambda \text{ if } j < k\}$.

Then $\bigcup_k \Omega_k = \Omega = \bigcup_j Q_j$ for some family $\{Q_j\}$ of disjoint dyadic cubes.

1. By the previous lemma, for almost every $x \notin \bigcup_j Q_j$, $f(x) = \lim_{k \to 0} E_k f(x) \leq \lambda$.
2. This part is exactly the weak $(1,1)$ inequality of dyadic maximal function.
3. The first inequality comes from the definition of the sets Ω_k. Now, for each Q_j consider $\tilde{Q}_j$ to be the dyadic cube containing Q_j whose sides are twice as long. Then $\tilde{Q}_j$ is not in the family $\{Q_j\}$. Therefore,

$$\frac{1}{|Q_j|} \int_{Q_j} f \leq \frac{|\tilde{Q}_j|}{|Q_j|} \frac{1}{|\tilde{Q}_j|} \int_{\tilde{Q}_j} f \leq 2^d \lambda$$

$\square$

Calderón Zygmund Decomposition

Now, we will decompose these kind of functions into "good" and "bad" parts.

Theorem (Calderón Zygmund Decomposition)

If f is a non-negative integrable function and λ is a positive number, then f can be written as sum of a "good" function g and a "bad" function b, $f = g + b$ such that

1. $g(x) \leq \lambda$ *for almost every* $x \notin \bigcup_j Q_j$ *and* $g(x) \leq 2^d \lambda$ *for* $x \in \bigcup_j Q_j$
2. $b(x) = 0$ *for every* $x \notin \bigcup_j Q_j$ *and* $\frac{1}{|Q_j|} \int_{Q_j} b = 0$

Proof.

Using the covering lemma for f at height λ, we have the sequence $\{Q_j\}$ of disjoint dyadic cubes. Now, consider $\Omega = \bigcup_j Q_j$ and define

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega \\ \dfrac{1}{|Q_j|}\displaystyle\int_{|Q_j|} f & \text{if } x \in Q_j \end{cases}$$

and

$$b(x) = \sum_j b_j(x) \text{ where } b_j(x) = \left(f(x) - \frac{1}{|Q_j|}\int_{|Q_j|} f \right) \chi_{Q_j}(x).$$

The rest comes directly from the covering lemma. □

What we have shown so far?

Quick recap

- If T^* is weak (p,q) then the set $\left\{ f \in L^p(X,\mu) : \lim_{t\to t_0} T_t f(x) = f(x) \text{ a.e.} \right\}$ is closed in $L^p(X,\mu)$.
- The Hardy-Littlewood Maximal Function is weak (∞,∞).
- Marcinkiewicz Interpolation Theorem.
- The Dyadic Maximal Function is weak $(1,1)$.
- Calderón Zygmund Covering lemma and Decomposition theorem.

Back to Maximal Function

Lemma

If f is a non-negative integrable function then

$$|\{x \in \mathbb{R}^d : M'f(x) > 4^d\lambda\}| \leq 2^d|\{x \in \mathbb{R}^d : M_d f(x) > \lambda\}|$$

Proof.

Using the Calderón Zygmund decomposition, we have $\{x \in \mathbb{R}^d : M_d f(x) > \lambda\} = \bigcup_j Q_j$.
Let $2Q_j$ represent the cube with the same center as Q_j with twice the side length. Then it is sufficient to prove that

$$\{x \in \mathbb{R}^d : M'f(x) > 4^d\lambda\} \subset \bigcup_j 2Q_j.$$

Proof.

Let $x \notin \bigcup_j 2Q_j$ and Q be any cube centered at x. If l denotes the side length of Q, then there exists $k \in \mathbb{Z}$ such that $2^{k-1} \leq l < 2^k$. Then Q intersects $m(\leq 2^d)$ dyadic cubes in D_k, call them $R_1, R_2, \ldots, R_m$. If $R_i \subset Q_j$ for some i and j, then being the center of Q, $x \in 2R_i \subset 2Q_j$, which raises a contradiction. Hence,

$$\frac{1}{|Q|}\int_Q f = \frac{1}{|Q|}\sum_{i=1}^{m}\int_{Q\cap R_i} f \leq \sum_{i=1}^{m}\frac{|R_i|}{|Q|}\int_{R_i} f \leq 2^d m\lambda \leq 4^d\lambda.$$

This implies $x \notin \{x \in \mathbb{R}^d : M_d f(x) > \lambda\}$. $\square$

Proof of Hardy-Littlewood Maximal Inequality.

Using the previous lemma and by the weak $(1,1)$ inequality for M_d, we get

$$|\{x \in \mathbb{R}^d : M'f(x) > \lambda\}| \leq 2^d |\{x \in \mathbb{R}^d : M_d f(x) > 4^{-d}\lambda\}| \leq \frac{8^d}{\lambda}\|f\|_1.$$

That is, M', and hence M is weak $(1,1)$. Also, we know M is weak (∞,∞). So, by Marcinkiewicz Interpolation Theorem, M is strong (p,p) for $1 < p \leq \infty$. $\square$

Lebesgue Differentiation Theorem

Theorem (Lebesque Differentiation Theorem)

If $f \in L^1_{loc}(\mathbb{R}^d)$ *then*

$$\lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r} f(x-y)dy = f(x) \text{ a.e.}$$

This is just a corollary of the weak $(1,1)$ maximal inequality and the theorem shown in almost everywhere convergence.

Hilbert Transform

Schwartz Space and Tempered Distributions

Definition (Schwartz Space)

The Schwartz space is the space of functions

$$S(\mathbb{R}^d, \mathbb{C}) = \{f \in C^\infty(\mathbb{R}^d, \mathbb{C}) : p_{\alpha,\beta}(f) < \infty \ \forall \alpha, \beta \in \mathbb{N}^d\}$$

where $p_{\alpha,\beta} = \sup_x |x^\alpha D^\beta f(x)|$.

Definition (Tempered Distributions)

Tempered Distributions S' is the space of all bounded linear functionals on the Schwartz space S.

Remark

A linear map T from S to C is in S' if $\lim_{k\to\infty} T(\phi_k) = 0$ whenever $\lim_{k\to\infty} \phi_k = 0$ in S.

The Principle Value of 1/x

Definition

We define a tempered distribution called the principal value of $1/x$, abbreviated as p.v.$\frac{1}{x}$, by

$$\text{p.v.}\frac{1}{x}(\phi) = \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{\phi(x)}{x}dx, \quad \phi \in S(\mathbb{R}).$$

To see that this defines a tempered distribution, we can write it as

$$\text{p.v.}\frac{1}{x}(\phi) = \lim_{\epsilon\to 0}\int_{\epsilon<|x|<1} \frac{\phi(x)-\phi(0)}{x}dx + \int_{|x|>1}\frac{\phi(x)}{x}dx.$$

Thus,

$$\left|\text{p.v.}\frac{1}{x}(\phi)\right| \le \lim_{\epsilon\to 0}\int_{\epsilon<|x|<1} \left\|\phi'\right\|_\infty dx + \int_{|x|>1}\frac{\|x\phi\|_\infty}{x}dx \le C\left(\left\|\phi'\right\|_\infty + \|x\phi\|_\infty\right).$$

Definition (Conjugate Poisson kernel)

$Q_t(x) = \frac{1}{\pi}\frac{x}{t^2+x^2}$ is called the conjugate Poisson kernel. It satisfies
$\widehat{Q_t}(\xi) = -i\,\mathrm{sgn}(\xi)e^{-2\pi t|\xi|}$.

Proposition

In S', $\lim_{t\to 0} Q_t = \text{p.v. } \frac{1}{x}$.

As a result of this proposition we get that

$$\lim_{t\to 0} Q_t * f(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{f(x-y)}{y}dy,$$

and by the continuity of Fourier transform we get

$$\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right)^{\wedge}(\xi) = -i\,\mathrm{sgn}(\xi).$$

Hilbert Transform

Definition (Hilbert Transform)

For a function $f \in S$, we define its Hilbert transform by any one of the following equivalent expressions:

$$Hf = \lim_{t \to 0} Q_t * f(x)$$
$$Hf = \frac{1}{\pi} \text{p.v.} \frac{1}{x} * f$$
$$(Hf)\widehat{\ }(\xi) = -i\,\text{sgn}(\xi)\widehat{f}(\xi).$$

The third expression lets us define the Hilbert transform of function in $L^2(\mathbb{R})$ and it satisfies

$$\|Hf\|_2 = \|f\|_2, \quad H(Hf) = -f, \quad \int Hf \cdot g = -\int f \cdot Hg.$$

Theorem (Kolmogorov)

H is weak $(1,1)$:

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

Proof.

Fix $\lambda > 0$ and f non-negative, then apply Calderón Zygmund decomposition to get a sequence of disjoint intervals $\{I_j\}$ and $f = g + b$ where $g(x) \leq 2\lambda$ a.e., $b = \sum_j b_j$, and b_j is supported on I_j and has zero integral.

Since $Hf = Hg + Hb$,

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq |\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| + |\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}|.$$

Proof cont.

$$
\begin{aligned}
|\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| &\leq \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |Hg(x)|^2 dx \\
&= \frac{4}{\lambda^2} \int_{\mathbb{R}} g(x)^2 dx \\
&\leq \frac{8}{\lambda} \int_{\mathbb{R}} g(x) dx \\
&= \frac{8}{\lambda} \|f\|_1
\end{aligned}
$$

Now, let $2I_j$ be the interval with same center as I_j and twice the length. Let $\Omega^* = \bigcup_j 2I_j$ then $|\Omega^*| \leq 2|\Omega| \leq \dfrac{2}{\lambda}\|f\|_1$.

Proof cont.

$$
\begin{aligned}
|\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}| &\leq |\Omega^*| + |\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}| \\
&\leq \frac{2}{\lambda}\|f\|_1 + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)|dx
\end{aligned}
$$

So, it is sufficient to prove that

$$
\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)|dx \leq C\|f\|_1.
$$

The proof of this omitted because it just contains a lot of calculation.

Thus, we have H is weak $(1,1)$.

Theorem (M. Riesz)

H is strong (p, p), $1 < p < \infty$:

$$\|Hf\|_p \le C_p \|f\|_p.$$

Proof.

Since H is weak $(1, 1)$ and strong $(2, 2)$, by the Marcinkiewicz interpolation theorem we have the strong (p, p) inequality for $1 < p < 2$.

For $p > 2$, let $q \in \mathbb{R}$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

Proof.

$$
\begin{aligned}
\|Hf\|_p &= \sup\left\{\left|\int_{\mathbb{R}} Hf \cdot g\right| : \|g\|_q \leq 1\right\} \\
&= \sup\left\{\left|\int_{\mathbb{R}} f \cdot Hg\right| : \|g\|_q \leq 1\right\} \\
&\leq \|f\|_p \sup\{\|Hg\|_q : \|g\|_q \leq 1\} \\
&\leq C_q \|f\|_p.
\end{aligned}
$$

□

Thank You

References

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