Method of lines for the Black-Scholes equation

Another possibility for solving the Black-Scholes equation is to apply the method of lines. The method of lines separates the problem of space discretization from the problem of evolution in time and we discretize in space only, keeping the time continuous. We will consider the initial value problem (41) and write the PDE of the problem as

$$\frac{\partial U}{\partial t}(S,t) = G(S,t), \quad \text{where } G(S,t) := \frac{\sigma^2}{2}S^2\frac{\partial^2 U}{\partial S^2}(S,t) + rS\frac{\partial U}{\partial S}(S,t) - rU(S,t).$$

$$\text{Now, we define } S_i = h_S i,\ i = 0, 1, ..., N_S, \quad \text{where} \quad h_S = \frac{S^*}{N_S}$$

and apply the finite difference formulas to approximate the partial derivatives in space arising in the definition of G at a certain point S_i for a fixed time. Negleting the $O(h_x^2)$ error term, we have

$$\begin{aligned}
&G(S_i,t) \approx \\
&\approx \frac{\sigma^2}{2}S_i^2\left(\frac{U(S_{i+1},t) - 2U(S_i,t) + U(S_{i-1},t)}{h_S^2}\right) + rS_i\left(\frac{U(S_{i+1},t) - U(S_{i-1},t)}{2h_S}\right) - rU(S_i,t) \\
&= U(S_{i-1},t)\left(\frac{\sigma^2}{2}\frac{S_i^2}{h_S^2} - \frac{rS_i}{2h_S}\right) + U(S_i,t)\left(-\sigma^2\frac{S_i^2}{h_S^2} - r\right) + U(S_{i+1},t)\left(\frac{\sigma^2}{2}\frac{S_i^2}{h_S^2} + \frac{rS_i}{2h_S}\right) \\
&= U(S_{i-1},t)\underbrace{\left(\frac{\sigma^2}{2}i^2 - \frac{ri}{2}\right)}_{:=\alpha_i} + U(S_i,t)\underbrace{\left(-\sigma^2 i^2 - r\right)}_{:=\beta_i} + U(S_{i+1},t)\underbrace{\left(\frac{\sigma^2}{2}i^2 + \frac{ri}{2}\right)}_{:=\gamma_i}.
\end{aligned}$$

Now we define

$$\mathbf{W}(t) = [U(S_1, t), U(S_2, t), ..., U(S_{N_S-1}, t)]^T$$

and the equations

$$\frac{\partial U}{\partial t}(S_i, t) = G(S_i, t), \; i = 1, ...N_S - 1$$

can be written as

$$\mathbf{W}'(t) = [G(S_1, t), G(S_2, t), ..., G(S_{N_S-1}, t)]^T \tag{45}$$

and we have

$$
\begin{bmatrix} G(S_1,t) \\ G(S_2,t) \\ G(S_3,t) \\ \vdots \\ G(S_{N_S-3},t) \\ G(S_{N_S-2},t) \\ G(S_{N_S-1},t) \end{bmatrix} = \underbrace{\begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & \dots & \dots & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \dots & \dots & 0 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_{N_S-3} & \beta_{N_S-3} & \gamma_{N_S-3} & 0 \\ 0 & \dots & \dots & 0 & \alpha_{N_S-2} & \beta_{N_S-2} & \gamma_{N_S-2} \\ 0 & \dots & \dots & \dots & 0 & \alpha_{N_S-1} & \beta_{N_S-1} \end{bmatrix}}_{:=\mathbf{A}^{ML}} \underbrace{\begin{bmatrix} U(S_1,t) \\ U(S_2,t) \\ U(S_3,t) \\ \vdots \\ U(S_{N_S-3},t) \\ U(S_{N_S-2},t) \\ U(S_{N_S-1},t) \end{bmatrix}}_{\mathbf{W}(t)} + \underbrace{\begin{bmatrix} \alpha_1 U(S_0,t) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \gamma_{N_S-1} U(S_{N_S},t) \end{bmatrix}}_{:=\mathbf{b}^{ML}}.
$$

Therefore, (45) can be written as an ODE system

$$
\mathbf{W}'(t) = \mathbf{A}^{ML} \cdot \mathbf{W}(t) + \mathbf{b}^{ML} \tag{46}
$$

which can be solved by any numerical method for ODEs, for instance, Runge-Kutta methods.

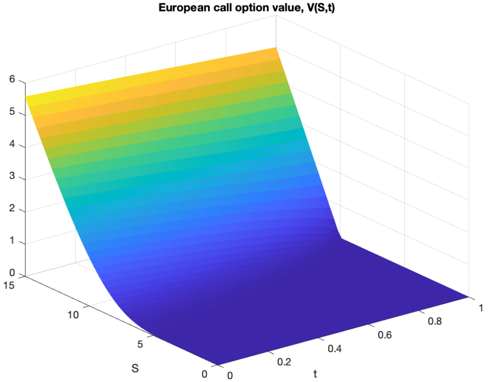

Figure: Plot of $V(S, t)$ obtained by the method of lines using fourth order Runge-Kutta method in the case of a European call option, obtained with the parameters $r = 0.06$, $\sigma = 0.3$, $T = 1$ and $K = 10$. We took $S^* = 15$, $N_S = 50$