Explicit method for Black-Scholes equation

The application of finite difference methods developed in the previous sections for the Black-Scholes equation is straightforward. For instance, we consider the application of the explicit method but the implicit methods would be similar. We consider the truncated region $\mathcal{R}_V^T$ defined in (4), for a sufficiently large value of S^* and in order to replace the terminal value problem associated with Black-Scholes equation by an initial value problem we perform a change of variables $U(S,t) := V(S, T-t)$ and consider the problem

$$\left\{ \begin{array}{cc} \frac{\partial U}{\partial t} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} - rU & \text{in } \mathcal{R}_V^T \\ U(S,0) = V(S,T) & S \in [0, S^*] \\ U(0,t) = V(0, T-t) & t \in [0,T] \\ U(S^*,t) = V(S^*, T-t) & t \in [0,T]. \end{array} \right. \tag{42}$$

Then, we define the grid of points of the form (S_i, t_j), where

$$S_i = h_S i,\ i = 0, 1, ..., N_S, \quad \text{where} \quad h_S = \frac{S^*}{N_S},$$

$$t_j = h_t j,\ j = 0, 1, ..., N_t, \quad \text{where} \quad h_t = \frac{T}{N_t}$$

and will use the notation $U_{i,j} = U(S_i, t_j)$.
We consider the finite difference approximations

$$\frac{\partial^2 U}{\partial S^2}(S_i, t_j) \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h_S^2},$$

$$\frac{\partial U}{\partial S}(S_i, t_j) \approx \frac{U_{i+1,j} - U_{i-1,j}}{2h_S}$$

and

$$\frac{\partial U}{\partial t}(S_i, t_j) \approx \frac{U_{i,j+1} - U_{i,j}}{h_t}.$$

Therefore, at a general point (S_i, t_j), we impose

$$\frac{U_{i,j+1} - U_{i,j}}{h_t} = \frac{\sigma^2}{2} S_i^2 \left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h_S^2} \right) + rS_i \left(\frac{U_{i+1,j} - U_{i-1,j}}{2h_S} \right) - rU_{i,j}$$

that can be written as

$$\begin{aligned} U_{i,j+1} &= U_{i,j} + \frac{\sigma^2}{2} S_i^2 h_t \left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h_S^2} \right) + rS_i h_t \left(\frac{U_{i+1,j} - U_{i-1,j}}{2h_S} \right) - rh_t U_{i,j} \\ &= U_{i-1,j} \left(\frac{\sigma^2}{2} h_t \frac{S_i^2}{h_S^2} - r \frac{h_t S_i}{2h_S} \right) + U_{i,j} \left(1 - \frac{\sigma^2}{2} h_t \frac{S_i^2}{h_S^2} - rh_t \right) + U_{i+1,j} \left(\frac{\sigma^2}{2} h_t \frac{S_i^2}{h_S^2} + r \frac{h_t S_i}{2h_S} \right) \\ &= U_{i-1,j} \frac{h_t}{2} \left(\sigma^2 i^2 - ri \right) + U_{i,j} \left(1 - \sigma^2 i^2 h_t - rh_t \right) + U_{i+1,j} \frac{h_t}{2} \left(\sigma^2 i^2 + ri \right) \end{aligned}$$

because $\frac{S_i}{h_S} = i$.

Figure 16 shows the plot of the solution in the case of an European call option, obtained with the parameters $r = 0.06$, $\sigma = 0.3$, $T = 1$ and $K = 10$. We took $S^* = 15$, $N_S = 50$ and $N_t = 10000$.

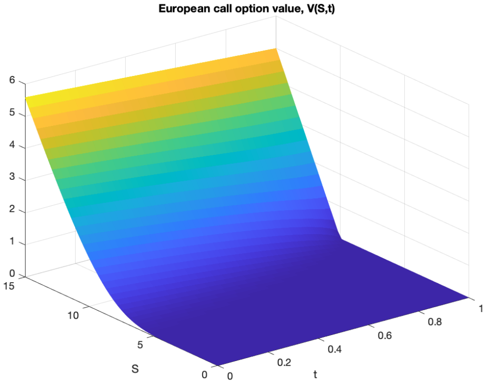

Figure: Plot of $V(S, t)$ in the case of an European call option, obtained with the parameters $r = 0.06$, $\sigma = 0.3$, $T = 1$ and $K = 10$. We took $S^* = 15$, $N_S = 50$ and $N_t = 10000$.

Figure 17 shows the plot of the solution for $t = 0$, $t = 0.5$ and $t = 1$, for which we have the payoff function.

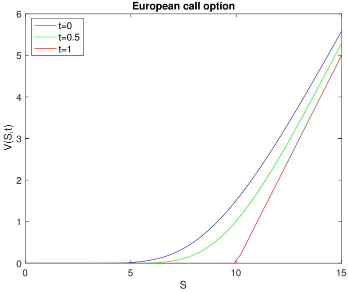

Figure: Plot of $V(S, t)$ for $t = 0$, $t = 0.5$ and $t = 1$, for which we have the payoff function.

The stability analysis would be similar to the analysis performed for the heat equation, but the calculations involved in the case of Black-Scholes equation are much more difficult.

Crank-Nicolson method for Black-Scholes equation

In the Crank-Nicolson method we define the scheme

$$\frac{U_{i,j+1}-U_{i,j}}{h_t} = \frac{1}{2}\left[\frac{\sigma^2}{2}S_i^2\left(\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{h_S^2}\right)+rS_i\left(\frac{U_{i+1,j}-U_{i-1,j}}{2h_S}\right)-rU_{i,j}\right]$$
$$+\frac{1}{2}\left[\frac{\sigma^2}{2}S_i^2\left(\frac{U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}}{h_S^2}\right)+rS_i\left(\frac{U_{i+1,j+1}-U_{i-1,j+1}}{2h_S}\right)-rU_{i,j+1}\right]$$

that can be written as

$$U_{i-1,j+1}\left(-\frac{\sigma^2}{4}h_t\frac{S_i^2}{h_S^2}+\frac{rh_t}{4}\frac{S_i}{h_S}\right)+U_{i,j+1}\left(1+\frac{\sigma^2}{2}h_t\frac{S_i^2}{h_S^2}+\frac{rh_t}{2}\right)+U_{i-1,j+1}\left(-\frac{\sigma^2}{4}h_t\frac{S_i^2}{h_S^2}-\frac{rh_t}{4}\frac{S_i}{h_S}\right)$$
$$=U_{i-1,j}\left(\frac{\sigma^2}{4}h_t\frac{S_i^2}{h_S^2}-\frac{rh_t}{4}\frac{S_i}{h_S}\right)+U_{i,j}\left(1-\frac{\sigma^2}{2}h_t\frac{S_i^2}{h_S^2}-\frac{rh_t}{2}\right)+U_{i-1,j}\left(\frac{\sigma^2}{4}h_t\frac{S_i^2}{h_S^2}+\frac{rh_t}{4}\frac{S_i}{h_S}\right)$$

that is

$$U_{i-1,j+1}\left(-\frac{\sigma^2}{4}h_t i^2+\frac{rh_t}{4}i\right)+U_{i,j+1}\left(1+\frac{\sigma^2}{2}h_t i^2+\frac{rh_t}{2}\right)+U_{i-1,j+1}\left(-\frac{\sigma^2}{4}h_t i^2-\frac{rh_t}{4}i\right)$$
$$=U_{i-1,j}\left(\frac{\sigma^2}{4}h_t i^2-\frac{rh_t}{4}i\right)+U_{i,j}\left(1-\frac{\sigma^2}{2}h_t i^2-\frac{rh_t}{2}\right)+U_{i-1,j}\left(\frac{\sigma^2}{4}h_t i^2+\frac{rh_t}{4}i\right)$$

and defining $a_i=-\frac{\sigma^2}{4}h_t i^2+\frac{rh_t}{4}i,\ b_i=1+\frac{\sigma^2}{2}h_t i^2+\frac{rh_t}{2}$, $c_i=-\frac{\sigma^2}{4}h_t i^2-\frac{rh_t}{4}i$ and $d_i=1-\frac{\sigma^2}{2}h_t i^2-\frac{rh_t}{2}$ we get

$$U_{i-1,j+1}\ a_i+U_{i,j+1}\ b_i+U_{i-1,j+1}\ c_i=-U_{i-1,j}\ a_i+U_{i,j}\ d_i-U_{i-1,j}\ c_i$$

and the values $U_{i+1,j+1}$, $U_{i,j+1}$ and $U_{i-1,j+1}$ are calculated by solving a linear system involving the values $U_{i+1,j}$, $U_{i,j}$ and $U_{i-1,j}$, which are assumed to be already calculated,

$$
\begin{bmatrix}
b_1 & c_1 & 0 & \dots & \dots & \dots & 0 \\
a_2 & b_2 & c_2 & 0 & \dots & \dots & 0 \\
0 & a_3 & b_3 & c_3 & 0 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & 0 & a_{N_S-3} & b_{N_S-3} & c_{N_S-3} & 0 \\
0 & \dots & \dots & 0 & a_{N_S-2} & b_{N_S-2} & c_{N_S-2} \\
0 & \dots & \dots & \dots & 0 & a_{N_S-1} & b_{N_S-1}
\end{bmatrix}
\begin{bmatrix}
U_{1,j+1} \\ U_{2,j+1} \\ U_{3,j+1} \\ \vdots \\ U_{N_S-3,j+1} \\ U_{N_S-2,j+1} \\ U_{N_S-1,j+1}
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
-a_1 & d_1 & -c_1 & 0 & \dots & \dots & \dots & \dots & 0 \\
0 & -a_2 & d_2 & -c_2 & 0 & \dots & \dots & \dots & 0 \\
0 & 0 & -a_3 & d_3 & -c_3 & \dots & \dots & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \vdots \\
0 & \dots & 0 & 0 & -a_{N_S-3} & d_{N_S-3} & -c_{N_S-3} & 0 & 0 \\
0 & \dots & \dots & \dots & 0 & -a_{N_S-2} & d_{N_S-2} & -c_{N_S-2} & 0 \\
0 & \dots & \dots & \dots & \dots & 0 & -a_{N_S-1} & d_{N_S-1} & -c_{N_S-1}
\end{bmatrix}
\begin{bmatrix}
U_{0,j} \\ U_{1,j} \\ U_{2,j} \\ \vdots \\ U_{N_S-2,j} \\ U_{N_S-1,j} \\ U_{N_S,j}
\end{bmatrix}
+
\begin{bmatrix}
-a_1 U_{0,j+1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -c_{N_s-1} U_{N_S,j+1}
\end{bmatrix}.
$$

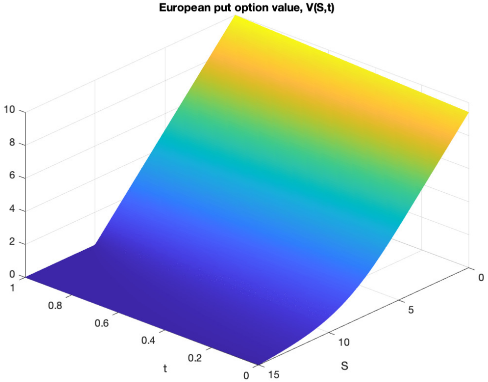

Figure: Plot of $V(S,t)$ in the case of an European put option, calculated by Crank-Nicolson method, obtained with the parameters $r = 0.06$, $\sigma = 0.3$, $T = 1$ and $K = 10$. We took $S^* = 15$, $N_S = 200$ and $N_t = 1000$.