Homework 1

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MA5250 Computational Fluid Dynamics

Q1:

Substitute $u=u_0+\varepsilon\tilde{u}$, $\rho=\rho_0+\varepsilon\tilde{\rho}$, $e=e_0+\varepsilon\tilde{e}$, $u_0=0$, $e_0=\frac{5}{2}$, $\gamma=\frac{7}{5}$, $\rho_0=1$ *into equations system(ignoring* $O(\varepsilon^2)$ *),we have:*

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0\Rightarrow\frac{\partial}{\partial t}(\rho_0+\varepsilon\tilde{\rho})+\frac{\partial}{\partial x}[(\rho_0+\varepsilon\tilde{\rho})(u_0+\varepsilon\tilde{u})]=0\\
\Rightarrow&\frac{\partial\tilde{\rho}}{\partial t}+\frac{\partial\tilde{u}}{\partial x}=0
\end{aligned}$$

$$p=(\gamma-1)\left(e_0+\varepsilon\tilde{e}-\frac{1}{2}(\rho_0+\varepsilon\tilde{\rho})(u_0+\varepsilon\tilde{u})^2\right)=(\gamma-1)(e_0+\varepsilon\tilde{e})$$

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+p)=0\Rightarrow\frac{\partial}{\partial t}(\rho_0\tilde{u}+\tilde{\rho}u_0)+\frac{\partial}{\partial x}((\rho_0+\varepsilon\tilde{\rho})(u_0+\varepsilon\tilde{u})^2+(\gamma-1)(e_0+\varepsilon\tilde{e}))\\
\Rightarrow&\frac{\partial\tilde{u}}{\partial t}+(\gamma-1)\frac{\partial\tilde{e}}{\partial x}=0
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial e}{\partial t}+\frac{\partial}{\partial x}(u(e+p))=0\Rightarrow\frac{\partial}{\partial t}(e_0+\varepsilon\tilde{e})+\frac{\partial}{\partial x}[(u_0+\varepsilon\tilde{u})(e_0+\varepsilon\tilde{e}+(\gamma-1)(e_0+\varepsilon\tilde{e}))=0\\
\Rightarrow&\frac{\partial\tilde{e}}{\partial t}+\gamma e_0\frac{\partial\tilde{u}}{\partial x}=0
\end{aligned}$$

So the system transform to:

$$\begin{aligned}
\frac{\partial\tilde{\rho}}{\partial t}+\frac{\partial\tilde{u}}{\partial x}&=0\\
\frac{\partial\tilde{u}}{\partial t}+(\gamma-1)\frac{\partial\tilde{e}}{\partial x}&=0\\
\frac{\partial\tilde{e}}{\partial t}+\gamma e_0\frac{\partial\tilde{u}}{\partial x}&=0
\end{aligned}$$

Let vector $\mathbf{q}=(\tilde{\rho},\tilde{u},\tilde{e})^T$. *Define matrix A as:*

$$A=\begin{pmatrix}0&1&0\\0&0&\gamma-1\\0&\gamma e_0&0\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&\frac{2}{5}\\0&\frac{7}{2}&0\end{pmatrix},\lambda=\pm\frac{\sqrt{35}}{5},0$$

$$U=\begin{pmatrix}1&\frac{2}{7}&\frac{2}{7}\\0&-\frac{2\sqrt{35}}{35}&\frac{2\sqrt{35}}{35}\\0&1&1\end{pmatrix},\Lambda=\begin{pmatrix}0&0&0\\0&-\frac{\sqrt{35}}{5}&0\\0&0&\frac{\sqrt{35}}{5}\end{pmatrix},U^{-1}=\begin{pmatrix}1&0&-\frac{2}{7}\\0&-\frac{\sqrt{35}}{4}&\frac{1}{2}\\0&\frac{\sqrt{35}}{4}&\frac{1}{2}\end{pmatrix}$$

Then the linear system can be expressed as:

$$\frac{\partial\mathbf{q}}{\partial t}+A\cdot\nabla\mathbf{q}=0$$

Let $w=U^{-1}\mathbf{q}$, *so we have:*$\frac{\partial w}{\partial t}+\Lambda\frac{\partial w}{\partial x}=0$. *The initial condition is:*

$$w(x,0)=\begin{pmatrix}\frac{5}{7}\\\frac{1}{2}-\frac{\sqrt{35}}{4}\\\frac{1}{2}+\frac{\sqrt{35}}{4}\end{pmatrix},x<0\,;w(x,0)=\begin{pmatrix}-\frac{2}{7}\\\frac{1}{2}+\frac{\sqrt{35}}{4}\\\frac{1}{2}-\frac{\sqrt{35}}{4}\end{pmatrix},x>0$$

The solution to this system is:

$$w^1(x,t)=\begin{cases}\frac{5}{7} & x<0\\ -\frac{2}{7} & x>0\end{cases}$$

$$w^2(x,t)=\begin{cases}\left(\frac{1}{2}-\frac{\sqrt{35}}{4}\right) & x<\frac{\sqrt{35}}{5}t\\ \left(\frac{1}{2}+\frac{\sqrt{35}}{4}\right) & x>\frac{\sqrt{35}}{5}t\end{cases}$$

$$w^3(x,t)=\begin{cases}\left(\frac{1}{2}+\frac{\sqrt{35}}{4}\right) & x<-\frac{\sqrt{35}}{5}t\\ \left(\frac{1}{2}-\frac{\sqrt{35}}{4}\right) & x>-\frac{\sqrt{35}}{5}t\end{cases}$$

Therefore, the solution to the initial equations system is:$\mathbf{q}=Uw$

Q2

(a) *Divide domain* Ω *into:*$\Omega=\bigcup_{i=1}^{N}\Omega_i, \mathrm{int}(\Omega_i)\cap\mathrm{int}(\Omega_j)=\varnothing, i\neq j$. *Then discretize with* Q_i

$$Q_i=\frac{1}{|\Omega_i|}\int_{\Omega_i}q(x,t)dx=\frac{1}{|\Delta x|}\int_{x_{i-1/2}}^{x_{i+1/2}}q(x,t)dx$$

Apply Fourier transform, we have:

$$\begin{aligned}Q_i^n&=\frac{1}{|\Delta x|}\int_{x_{i-1/2}}^{x_{i+1/2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{q}_h^n(k)e^{ikx}dkdx\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\sin(k\Delta x/2)}{k\Delta x/2}\hat{q}_h^n(k)e^{ikx_i}dk\end{aligned}$$

Then apply to the scheme, we got:

$$\begin{aligned}Q_i^{n+1}&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\sin(k\Delta x/2)}{k\Delta x/2}\hat{q}_h^n(k)\left(\frac{e^{ikx_{i+1}}+e^{ikx_{i-1}}}{2}-\frac{a\Delta t}{2\Delta x}(e^{ikx_{i+1}}-e^{ikx_{i-1}})\right)dk\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\sin(k\Delta x/2)}{k\Delta x/2}\hat{q}_h^n(k)e^{ikx_i}\left(\frac{e^{ik\Delta x}+e^{-ik\Delta x}}{2}-\frac{a\Delta t}{2\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})\right)dk\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\sin(k\Delta x/2)}{k\Delta x/2}\hat{q}_h^n(k)e^{ikx_i}\left(\left(1-\frac{a\Delta t}{\Delta x}\right)\cos(k\Delta x)\right)dk\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\sin(k\Delta x/2)}{k\Delta x/2}\hat{q}_h^0(k)e^{ikx_i}\left(\left(1-\frac{a\Delta t}{\Delta x}\right)\cos(k\Delta x)\right)^n dk\end{aligned}$$

Therefore for $\nu=|a|\frac{\Delta t}{\Delta x}<1$,$\left(\left(1-\frac{a\Delta t}{\Delta x}\right)\cos(k\Delta x)\right)^n$ *decays exponentially. It is a stable scheme*

(b)

$$Q_i^{n+1}=\frac{Q_{i-1}^n+Q_{i+1}^n}{2}-\frac{a\Delta t}{2\Delta x}(Q_{i+1}^n-Q_{i-1}^n)$$

To carry out the error analysis for above scheme, we need to define the exact solution for the equation. Let $Q_i^{n+1}=\mathcal{N}(Q_i^n)$, *and* $q(x,t_n)$ *is the exact solution at* t_n, *then:*

$$\tau=\left\|\frac{\mathcal{N}(Q^n)-q(x,t_{n+1})}{\Delta t}\right\|$$

For scheme, we have:

$$
\begin{aligned}
\frac{\mathcal{N}(Q^n)-q(x,t_{n+1})}{\Delta t}=&\frac{1}{2\Delta x\Delta t}\left(\int_{x_{i-3/2}}^{x_{i-1/2}} q(x,t_n)\mathrm{dx}+\int_{x_{i+1/2}}^{x_{i+3/2}} q(x,t_n)\mathrm{dx}-2\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_{n+1})\mathrm{dx}\right)\\
&-\frac{a}{2\Delta x^2}\left(\int_{x_{i+1/2}}^{x_{i+3/2}} q(x,t_n)\mathrm{dx}-\int_{x_{i-3/2}}^{x_{i-1/2}} q(x,t_n)\mathrm{dx}\right)\\
=&\frac{1}{2\Delta x\Delta t}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} q(x-\Delta x,t_n)\mathrm{dx}+\int_{x_{i-1/2}}^{x_{i+1/2}} q(x+\Delta x,t_n)\mathrm{dx}-2\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_{n+1})\mathrm{dx}\right)\\
&-\frac{a}{2\Delta x^2}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} q(x+\Delta x,t_n)\mathrm{dx}-\int_{x_{i-1/2}}^{x_{i+1/2}} q(x-\Delta x,t_n)\mathrm{dx}\right)
\end{aligned}
$$

By Taylor's expansion:

$$
\begin{aligned}
q(x-\Delta x,t_n)=&q(x,t_n)-\Delta x q_x(x,t_n)+\frac{1}{2}(\Delta x)^2 q_{xx}(\xi_1(x),x)\\
q(x+\Delta x,t_n)=&q(x,t_n)+\Delta x q_x(x,t_n)+\frac{1}{2}(\Delta x)^2 q_{xx}(\xi_2(x),x)\\
q(x,t_{n+1})=&q(x,t_n)+\Delta t q_t(x,t_n)+\frac{1}{2}(\Delta t)^2 q_{tt}(x,\tau(x))
\end{aligned}
$$

Substitute into formula, we get:

$$
\begin{aligned}
&\frac{1}{2\Delta x\Delta t}\int_{x_{i-1/2}}^{x_{i+1/2}}\left[\frac{1}{2}(\Delta x)^2(q_{xx}(\xi_1(x),x)+q_{xx}(\xi_2(x),x))-2\Delta t q_t(x,t_n)-(\Delta t)^2 q_{tt}(x,\tau(x)))\right]\mathrm{dx}\\
=&\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\Delta x}{4\Delta t}(q_{xx}(\xi_1(x),x)+q_{xx}(\xi_2(x),x))-\frac{1}{\Delta x}q_t(x,t_n)-\frac{\Delta t}{2\Delta x}q_{tt}(x,\tau(x))\right)\mathrm{dx}\\
&\frac{a}{2\Delta x^2}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} q(x+\Delta x,t_n)\mathrm{dx}-\int_{x_{i-1/2}}^{x_{i+1/2}} q(x-\Delta x,t_n)\mathrm{dx}\right)\\
=&\frac{a}{2\Delta x^2}\int_{x_{i-1/2}}^{x_{i+1/2}} 2\Delta x q_x(x,t_n)+\frac{1}{2}(\Delta x)^2(q_{xx}(\xi_2(x),x)-q_{xx}(\xi_1(x),x))\mathrm{dx}\\
=&\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{a}{\Delta x}q_x(x,t_n)+\frac{a}{4}\left(q_{xx}(\xi_2(x),x)-q_{xx}(\xi_1(x),x)\right)\right)\mathrm{dx}
\end{aligned}
$$

Sum two parts, and apply $q_{tt}=a^2 q_{xx}$, $q_t+aq_x=0$, we get:

$$
\begin{aligned}
&\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\Delta x}{4\Delta t}(q_{xx}(\xi_1(x),x)+q_{xx}(\xi_2(x),x))-a^2\frac{\Delta t}{2\Delta x}q_{xx}(x,\tau(x))-\frac{a}{4}\left(q_{xx}(\xi_2(x),x)-q_{xx}(\xi_1(x),x)\right)\right)\mathrm{dx}\\
=&\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\Delta x}{4\Delta t}+\frac{a}{4}\right)q_{xx}(\xi_1(x),x)+\left(\frac{\Delta x}{4\Delta t}-\frac{a}{4}\right)q_{xx}(\xi_2(x),x)-\frac{a\nu}{2}q_{xx}(x,\tau(x))dx\\
\leqslant&\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{\Delta x}{2\Delta t}-\frac{a\nu}{2}\right)\max_{x,t}\{q_{xx}(x,t)\}dx\\
=&\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\frac{a}{2\nu}-\frac{a\nu}{2}\right)dx\cdot\max_{x,t}\{q_{xx}(x,t)\}\\
=&\left(\frac{a}{2\nu}-\frac{a\nu}{2}\right)\Delta x\max_{x,t}\{q_{xx}(x,t)\}\leqslant Ca\Delta x\max_{x,t}\{q_{xx}(x,t)\}
\end{aligned}
$$

Then:

$$
\left|\frac{\mathcal{N}(Q_i^n)-q(x,t_{n+1})}{\Delta t}\right|\leqslant Ca\Delta x\max_{x,t}\{q_{xx}(x,t)\}
$$

Let

$$
\tau_\infty^n=\max_{x,t}\left|\frac{\mathcal{N}(Q_i^n)-q(x,t_{n+1})}{\Delta t}\right|\leqslant Ca\Delta x\max_{x,t}\{q_{xx}(x,t)\}
$$

Let $E^n=Q^n-q^n$, we have:

$$
\begin{aligned}
E(Q^{n+1})=\|Q^{n+1}-q^{n+1}\|=&\|\mathcal{N}(Q^n)-q^{n+1}\|\\
\leqslant&\|\mathcal{N}(Q^n)-\mathcal{N}(q^n)\|+\|\mathcal{N}(q^n)-q^{n+1}\|\\
\leqslant&\|\mathcal{N}(q^n+E^n)-\mathcal{N}(q^n)\|+\Delta t\tau^n=\|\mathcal{N}(E^n)\|+\Delta t\tau^n\\
\|\mathcal{N}(E^n)\|=&\left\|\frac{1}{2}\left(1-\frac{a\Delta t}{\Delta x}\right)E^n+\frac{1}{2}\left(1+\frac{a\Delta t}{\Delta x}\right)E^n\right\|\\
\leqslant&\frac{1}{2}\left(1-\frac{a\Delta t}{\Delta x}\right)\|E^n\|+\frac{1}{2}\left(1+\frac{a\Delta t}{\Delta x}\right)\|E^n\|=\|E^n\|
\end{aligned}
$$

Thus

$$\begin{aligned}\|E^{n+1}\| \leqslant & \|E^n\| + \Delta t \tau^n \\ \leqslant & \|E^0\| + n \Delta t C a \Delta x \max_{x,t} \{q_{xx}(x,t)\} \\ = & \|E^0\| + a C T \Delta x \max_{x,t} \{q_{xx}(x,t)\}\end{aligned}$$

So, the scheme is first order in Δx

(c)

$$\begin{aligned}&\mathcal{L}(Q_{i-1}^n, Q_{i+1}^n) = \frac{1}{2}\left(1 - \frac{a\Delta t}{\Delta x}\right) Q_{i+1}^n + \frac{1}{2}\left(1 + \frac{a\Delta t}{\Delta x}\right) Q_{i-1}^n \\ &\mathcal{A} = \frac{1}{\Delta t}\log\left(\frac{1}{2}\left(1 + \frac{a\Delta t}{\Delta x}\right)\exp(\Delta x \partial_x) + \frac{1}{2}\left(1 - \frac{a\Delta t}{\Delta x}\right)\exp(-\Delta x \partial_x)\right) \\ &\exp(\pm \Delta x \partial_x) = 1 \pm \Delta x \partial_x + \frac{1}{2}\Delta x^2 \partial_x^2 \pm \frac{1}{6}\Delta x^3 \partial_x^3\end{aligned}$$

By taylor expansion to $O(\Delta x^3)$, *we have:*

$$\begin{aligned}\mathcal{A} \approx & \frac{1}{\Delta t}\log\left(\left(\frac{1}{2} - \frac{a\Delta t}{2\Delta x}\right)\left(1 + \Delta x \partial_x + \frac{1}{2}\Delta x^2 \partial_x^2 + \frac{1}{6}\Delta x^3 \partial_x^3\right) + \left(\frac{1}{2} + \frac{a\Delta t}{2\Delta x}\right)\left(1 - \Delta x \partial_x + \frac{1}{2}\Delta x^2 \partial_x^2 - \frac{1}{6}\Delta x^3 \partial_x^3\right)\right) \\ = & \frac{1}{\Delta t}\log\left(1 - \frac{a\Delta t}{\Delta x}\Delta x \partial_x + \frac{1}{2}\Delta x^2 \partial_x^2 - \frac{a\Delta t}{6}\Delta x^2 \partial_x^3\right) \\ \approx & \frac{1}{\Delta t}\left(-\frac{a\Delta t}{\Delta x}\Delta x \partial_x + \frac{1}{2}\Delta x^2 \partial_x^2 - \frac{a\Delta t}{6}\Delta x^2 \partial_x^3\right)\end{aligned}$$

So the modified equation is:

$$\begin{aligned}\partial_t g = & -a\partial_x g + \frac{1}{2\Delta t}\Delta x^2 g_{xx} - \frac{a}{6}\Delta x^2 g_{xxx} \\ \Rightarrow \partial_t g + a\partial_x g = & \frac{\Delta x^2}{2\Delta t} g_{xx} - \frac{a\Delta x^2}{6} g_{xxx}\end{aligned}$$

Q3

$$\begin{aligned}\frac{\partial q}{\partial t} + \frac{\partial(q^3 - q)}{\partial x} = & \frac{\partial q}{\partial t} + (3q^2 - 1)\frac{\partial q}{\partial x} \\ q(x,0) = & 2H(x) - 1\end{aligned}$$

The characteritic line satisfies:

$$\begin{aligned}\frac{x - x_0(x,t)}{t - 0} = \frac{\partial x}{\partial t} = & 3(q(x,t))^2 - 1 = 3(q(x,0))^2 - 1 = 2 \\ x_0(x,t) = & x - 2t\end{aligned}$$

By Rankine-Hugoniot Condition:

$$s = \frac{f'(q_-) - f'(q_+)}{q_- - q_+} = 0$$

So one solution is:

$$q(x,t) = 2H(x - 2t) - 1 = \left\{\begin{array}{ll} -1 & x < 2t \\ 1 & x > 2t \end{array}\right.$$

The check the entropy condition, $f(q) = q^3 - q$, $f'(q) = 3q^2 - 1$. *For* $q_- < q_+$, *for any* $q \in (q_-, q_+)$,*we have:*

$$\begin{aligned}\frac{f(q) - f(q_-)}{q - q_-} = & \frac{q^3 - q}{q + 1} = q(q - 1) \\ \frac{f(q) - f(q_+)}{q - q_+} = & \frac{q^3 - q}{q - 1} = q(q + 1)\end{aligned}$$

For $q \in (-1, 0)$, *we have* $\frac{f(q) - f(q_-)}{q - q_-} \geqslant s \geqslant \frac{f(q) - f(q_+)}{q - q_+}$, *but for* $q \in (0, 1)$, *the entropy condition doesn't holds*

Since $q_- < q_+$, by (Evans,Chap.3 Theorem), we have another solution:

$$q(x,t)=\begin{cases} -1 & x<-t \\ \operatorname{sgn}(x)\cdot\sqrt{\frac{x/t+1}{3}} & -t<x<2t \\ 1 & x>2t \end{cases}$$

And by theorem, the solution satisfies the entropy condition.

Q4

(a)

Through entropy inequality, we have:

$$\frac{\partial}{\partial t}\eta(q)+\frac{\partial}{\partial x}\psi(q)\leqslant 0 \tag{1}$$

Since $\tilde{Q}_{i+1/4}^{n+1}=Q_i^n-\frac{2\Delta t}{\Delta x}[\tilde{F}_{i+1/2}^n-f(Q_i^n)]$, $\tilde{Q}_{i-1/4}^{n+1}=Q_i^n-\frac{2\Delta t}{\Delta x}[f(Q_i^n)-\tilde{F}_{i-1/2}^n]$, we have:

$$\begin{aligned}\tilde{Q}_{i-1/4}^{n+1}=&\frac{1}{\Delta x/2}\int_{x_i-1/2}^{x_i}q(x,t_{n+1})\mathrm{dx}\\ \tilde{Q}_{i+1/4}^{n+1}=&\frac{1}{\Delta x/2}\int_{x_i}^{x_{i+1/2}}q(x,t_{n+1})\mathrm{dx}\end{aligned}$$

Where $\varphi(q)=\eta'(q)f'(q)$. By Jensen's inequality

$$\eta(\tilde{Q}_{i+1/4}^{n+1})\leqslant\frac{1}{\Delta x/2}\int_{x_i}^{x_{i+1/2}}\eta(Q_i^n)\mathrm{dx}$$

By integral form of (1), we get:

$$\begin{aligned}\eta(\tilde{Q}_{i+1/4}^{n+1})\leqslant&\eta(Q_i^n)-2\frac{\Delta t}{\Delta x}[\tilde{\Psi}_{i+1/2}^n-\psi(Q_i^n)]\\ \eta(\tilde{Q}_{i-1/4}^{n+1})\leqslant&\eta(Q_i^n)-2\frac{\Delta t}{\Delta x}[\psi(Q_i^n)-\tilde{\Psi}_{i+1/2}^n]\end{aligned}$$

Where $\tilde{\Psi}_{i+1/2}^n$ is Godunov's numerical flux

(b)

We can write $\hat{Q}_{i+1/4}^{n+1},\hat{Q}_{i-1/4}^{n+1}$ as:

$$\begin{aligned}\hat{Q}_{i+1/4}^{n+1}=&\frac{1}{2}(Q_i^n+Q_{i+1}^n)-\frac{\Delta t}{\Delta x}(f(Q_{i+1}^n)-f(Q_i^n))\\ \hat{Q}_{i-1/4}^{n+1}=&\frac{1}{2}(Q_i^n+Q_{i-1}^n)-\frac{\Delta t}{\Delta x}(f(Q_i^n)-f(Q_{i-1}^n))\end{aligned}$$

Then the Lax-Friedrich Scheme can be viewed as the two-cell averaging of two noninteracting Riemann problems:

$$Q_i^{n+1}=\frac{1}{2\Delta x}\left(\int_{x_{i-1/2}}^{x_{i+1/2}}q(x,t_{n+1})\mathrm{dx}+\int_{x_{i+1/2}}^{x_{i+3/2}}q(x,t_{n+1})\mathrm{dx}\right)$$

So we integrating the inequality of (1):

$$\begin{aligned}\eta(Q_i^{n+1})\leqslant&\eta(Q_i^n)-\frac{\Delta t}{\Delta x}(\Psi(Q_{i+1/2}^n)-\Psi(Q_{i-1/2}^n))\\ \Psi(Q_{i+1/2}^n)=&\frac{\psi(Q_{i+1}^n)+\psi(Q_{i-1}^n)}{2}-\frac{\Delta x}{\Delta t}\frac{\eta(Q_{i+1}^n)-\eta(Q_i^n)}{2}\end{aligned}$$

So

$$\begin{aligned}\eta(\hat{Q}_{i-1/4}^{n+1})\leqslant&\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}}\eta(q(x,t_{n+1}))\mathrm{dx}\\ \leqslant&\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}}\eta(q(x,t_n))\mathrm{dx}-\frac{\Delta t}{\Delta x}\cdot\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(\psi(q(x_{i+1/2},t))-\psi(q(x_{i-1/2},t)))\mathrm{dt}\\ \leqslant&\eta(Q_i^n)-\frac{2\Delta t}{\Delta x}(\psi(Q_i^n)-\tilde{\Psi}_{i-1/2}^n)\end{aligned}$$

(c)

Let $\gamma = \frac{\Delta t}{\Delta x}$*, then we can rewrtie the E-Scheme as:*

$$\begin{aligned}
Q_i^{n+1} =& \frac{1}{2}((1-\theta_{i-1/2}^{n+1})\tilde{Q}_{i-1/4}^{n+1} + \theta_{i-1/2}^{n+1}\hat{Q}_{i-1/4}^{n+1} + (1-\theta_{i+1/2}^{n+1})\tilde{Q}_{i+1/4}^{n+1} + \theta_{i+1/2}^{n+1}\hat{Q}_{i+1/4}^{n+1}) \\
=& \frac{(1-\theta_{i-1/2}^{n+1})}{2}\tilde{Q}_{i-1/4}^{n+1} + \frac{\theta_{i-1/2}^{n+1}}{2}\hat{Q}_{i-1/4}^{n+1} + \frac{(1-\theta_{i+1/2}^{n+1})}{2}\tilde{Q}_{i+1/4}^{n+1} + \frac{\theta_{i+1/2}^{n+1}}{2}\hat{Q}_{i+1/4}^{n+1} \\
=& \frac{(1-\theta_{i-1/2}^{n+1})}{2}(Q_i^n - 2\gamma[f(Q_i^n) - \tilde{F}_{i-1/2}^n]) + \frac{(1-\theta_{i+1/2}^{n+1})}{2}(Q_i^n - 2\gamma[\tilde{F}_{i+1/2}^n - f(Q_i^n)]) \\
&+ \frac{\theta_{i+1/2}^{n+1}}{2}\left(\frac{1}{2}(Q_i^n + Q_{i+1}^n) - \gamma(f(Q_{i+1}^n) - f(Q_i^n))\right) + \frac{\theta_{i-1/2}^{n+1}}{2}\left(\frac{1}{2}(Q_i^n + Q_{i-1}^n) - \gamma(f(Q_i^n) - f(Q_{i-1}^n))\right) \\
=& Q_i^n + \frac{\theta_{i+1/2}^{n+1}}{4}(Q_{i+1}^n - Q_i^n) - \frac{\theta_{i-1/2}^{n+1}}{4}(Q_i^n - Q_{i-1}^n) - \frac{\gamma}{2}\theta_{i+1/2}^{n+1}(f(Q_{i+1}^n) + f(Q_i^n) - 2\tilde{F}_{i+1/2}^n) + \\
& \frac{\gamma}{2}\theta_{i-1/2}^{n+1}(f(Q_{i-1}^n) + f(Q_i^n) - 2\tilde{F}_{i-1/2}^n) - \gamma(\tilde{F}_{i+1/2}^n - \tilde{F}_{i-1/2}^n)
\end{aligned}$$

Simplifying the scheme, we get:

$$\begin{aligned}
Q_i^{n+1} - Q_i^n =& \frac{\theta_{i+1/2}^{n+1}}{2}\left[\frac{1}{2}(Q_{i+1}^n - Q_i^n) - \gamma(f(Q_{i+1}^n) + f(Q_i^n) - 2\tilde{F}_{i+1/2}^n)\right] \\
& - \frac{\theta_{i-1/2}^{n+1}}{2}\left[\frac{1}{2}(Q_i^n - Q_{i-1}^n) - \gamma(f(Q_{i-1}^n) + f(Q_i^n) - 2\tilde{F}_{i+1/2}^n)\right] \\
& - \gamma(\tilde{F}_{i+1/2}^n - \tilde{F}_{i-1/2}^n)
\end{aligned}$$

Since $\mathrm{TV}(Q_i^{n+1}) < \mathrm{TV}(Q_i^n)$*, we get:*

$$\begin{aligned}
\left|\frac{1}{2}(Q_{i+1}^n - Q_i^n) - \gamma(f(Q_{i+1}^n) + f(Q_i^n) - 2\tilde{F}_{i+1/2}^n)\right| \geqslant& 0 \\
\left|\frac{1}{2}(Q_i^n - Q_{i-1}^n) - \gamma(f(Q_{i-1}^n) + f(Q_i^n) - 2\tilde{F}_{i+1/2}^n)\right| \geqslant& 0
\end{aligned}$$

Which equavilent to:

$$\frac{\Delta t}{\Delta x}(f(Q_{i+1}) + f(Q_i) - 2F_{i+1/2}) \leqslant \frac{1}{2}|(Q_{i+1} - Q_i)|$$

(d)

From previous, we show:

$$Q_i^{n+1} == \frac{(1-\theta_{i-1/2}^{n+1})}{2}\tilde{Q}_{i-1/4}^{n+1} + \frac{\theta_{i-1/2}^{n+1}}{2}\hat{Q}_{i-1/4}^{n+1} + \frac{(1-\theta_{i+1/2}^{n+1})}{2}\tilde{Q}_{i+1/4}^{n+1} + \frac{\theta_{i+1/2}^{n+1}}{2}\hat{Q}_{i+1/4}^{n+1}$$

By convexity of $\eta(\cdot)$*:*

$$\begin{aligned}
\eta(Q_i^{n+1}) \leqslant& \frac{(1-\theta_{i-1/2}^{n+1})}{2}\eta(\tilde{Q}_{i-1/4}^{n+1}) + \frac{(1-\theta_{i+1/2}^{n+1})}{2}\eta(\tilde{Q}_{i+1/4}^{n+1}) + \frac{\theta_{i-1/2}^{n+1}}{2}\eta(\hat{Q}_{i-1/4}^{n+1}) + \frac{\theta_{i+1/2}^{n+1}}{2}\eta(\hat{Q}_{i+1/4}^{n+1}) \\
\leqslant& \frac{(1-\theta_{i-1/2}^{n+1})}{2}\left(\eta(Q_i^n) - 2\frac{\Delta t}{\Delta x}[\psi(Q_i^n) - \tilde{\Psi}_{i-1/2}^n]\right) + \frac{(1-\theta_{i+1/2}^{n+1})}{2}\left(\eta(Q_i^n) - 2\frac{\Delta t}{\Delta x}[\tilde{\Psi}_{i+1/2}^n - \psi(Q_i^n)]\right) \\
&+ \frac{\theta_{i-1/2}^{n+1}}{2}\left(\eta(Q_i^n) - 2\frac{\Delta t}{\Delta x}[\psi(Q_i^n) - \hat{\Psi}_{i-1/2}^n]\right) + \frac{\theta_{i+1/2}^{n+1}}{2}\left(\eta(Q_i^n) - 2\frac{\Delta t}{\Delta x}[\hat{\Psi}_{i+1/2}^n - \psi(Q_i^n)]\right) \\
=& \eta(Q_i^n) - \frac{\Delta t}{\Delta x}(\bar{\Psi}_{i+1/2}^n - \bar{\Psi}_{i-1/2}^n)
\end{aligned}$$

Where:$\bar{\Psi}_{i\pm1/2}^n = (1-\theta_{i\pm1/2}^{n+1})\tilde{\Psi}_{i\pm1/2}^n + \theta_{i\pm1/2}^{n+1}\hat{\Psi}_{i\pm1/2}^n$. *Therefore, E-Schemes satisfies the entropy condition.*