Project 2

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1 Problem Statement

Our purpose is to solve 1-D Euler Equations System:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0\\
&\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p)}{\partial x}=0\\
&\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho u v)}{\partial x}=0\\
&\frac{\partial e}{\partial t}+\frac{\partial(u(e+p))}{\partial x}=0
\end{aligned}$$

1 Solve Riemann Problem with Initial Condition:

$$p\,(x,0)=1, u(x,0)=0, v(x,0)=\begin{cases}-1, & x>0\\ 1, & x<0\end{cases}, \rho(x,0)=\begin{cases}1 & x>0\\ 3 & x<0\end{cases}$$

2 Solve the Woodward-Collela blast wave Problem

$$\rho(x,0)=1, u(x,0)=0, , p(x,0)=\begin{cases}1000, & \text{if } x\in(0,0.1)\\ 0.01, & \text{if } x\in(0.1,0.9)\\ 100, & \text{if } x\in(0.9,1)\end{cases}, v(x,0)=\begin{cases}-10, & x\in(0,0.5)\\ 20, & x\in(0.5,1)\end{cases}$$

With reflective boundary conditions:

$$u(0,t)=u(1,t)=0$$

2 Numerical Scheme

We use the HLLC method to solve the Riemann Problem. Before our method, we first introduce the HLL Method. The Godunov Scheme can be expressed as:

$$\mathbf{Q}_i^{n+1}=\mathbf{Q}_i-\frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1/2}-\mathbf{F}_{i-1/2})$$

Where

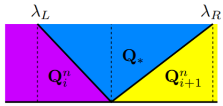
$$\mathbf{Q}_i^n=\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}}\mathbf{q}(x,t_n)\,\mathrm{dx}$$

2.1 HLL Flux

Consider a Riemann Problem:

$$\begin{aligned}
&\frac{\partial \mathbf{q}}{\partial t}+\frac{\partial f(\mathbf{q})}{\partial x}=0\\
&\mathbf{q}(x,0)=\begin{cases}\mathbf{Q}_i^n, & x<0\\ \mathbf{Q}_{i+1}^n, & x>0\end{cases}
\end{aligned}$$

We assume that the region in the center and both sides are connected by waves with velocities:λ_L and λ_R:



Where:

$$\lambda_L = \min\{\lambda(\mathbf{Q}_i^n), \lambda(\mathbf{Q}_{i+1}^n)\}, \lambda_R = \max\{\lambda(\mathbf{Q}_i^n), \lambda(\mathbf{Q}_{i+1}^n)\}$$

Obviously:$\lambda_L < 0, \lambda_R > 0$. So we have $\mathbf{F}_{i+1/2}^{HLL} = \mathbf{f}(\mathbf{Q}_{i+1}^n)$ and $\mathbf{F}_{i+1/2}^{HLL} = \mathbf{f}(\mathbf{Q}_i^n)$. So we only need to consider $\lambda_L < 0 < \lambda_R$. Consider the intergral form of the PDE:

$$\int_{x_L}^{x_R} \mathbf{q}(x,t)\,dx = \int_{x_L}^{x_R} \mathbf{q}(x,0)\,dx + \int_0^T \mathbf{F}(\mathbf{q}(x_L,t))\,dt - \int_0^T \mathbf{F}(\mathbf{q}(x_R,t))\,dt$$

By direct calculation:

$$\int_{x_L}^{x_R} \mathbf{q}(x,t)\,dx = x_R\mathbf{Q}_{i+1}^n - x_R\mathbf{Q}_i^n + T(\mathbf{f}(\mathbf{Q}_i^n) - \mathbf{f}(\mathbf{Q}_{i+1}^n))$$

On the other hand, we have:

$$\begin{aligned}\int_{x_L}^{x_R} \mathbf{q}(x,t)\,dx =& \int_{x_L}^{T\lambda_L} \mathbf{q}(x,t)\,dx + \int_{T\lambda_L}^{T\lambda_R} \mathbf{q}(x,t)\,dx + \int_{T\lambda_R}^{x_R} \mathbf{q}(x,t)\,dx \\ =& (T\lambda_L - x_L)\mathbf{Q}_i^n + \int_{T\lambda_L}^{T\lambda_R} \mathbf{q}(x,t)\,dx + (T\lambda_R - x_R)\mathbf{Q}_{i+1}^n\end{aligned}$$

Combining two equations, we get:

$$\int_{T\lambda_L}^{T\lambda_R} \mathbf{q}(x,t)\,dx = \frac{\lambda_R\mathbf{Q}_{i+1}^n - \lambda_L\mathbf{Q}_i^n + \mathbf{f}(\mathbf{Q}_i^n) - \mathbf{f}(\mathbf{Q}_{i+1}^n)}{\lambda_R - \lambda_L}$$

As $T \to 0^+$, we get the approximated estimator $\mathbf{Q}^*$:

$$\mathbf{Q}^* = \frac{\lambda_R\mathbf{Q}_{i+1}^n - \lambda_L\mathbf{Q}_i^n + \mathbf{f}(\mathbf{Q}_i^n) - \mathbf{f}(\mathbf{Q}_{i+1}^n)}{\lambda_R - \lambda_L}$$

Then:

$$\tilde{\mathbf{Q}}(x,t) = \begin{cases} \mathbf{Q}_i^n, & x < \lambda_L t \\ \mathbf{Q}^*, & \lambda_L t < x < \lambda_R t \\ \mathbf{Q}_{i+1}^n, & x > \lambda_R t \end{cases}$$

Let $\mathbf{F}^*$ denotes the flux in star region, by Rankine-Hugoniot condition:

$$\begin{aligned}\mathbf{F}^* =& \mathbf{f}(\mathbf{Q}_i^n) + \lambda_L(\mathbf{Q}^* - \mathbf{Q}_i^n) \\ \mathbf{F}^* =& \mathbf{f}(\mathbf{Q}_{i+1}^n) + \lambda_R(\mathbf{Q}^* - \mathbf{Q}_{i+1}^n)\end{aligned}$$

Then we solved:

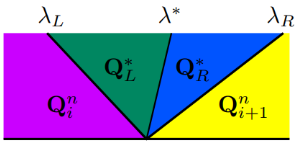
$$\mathbf{F}^* = \frac{\lambda_R\mathbf{f}(\mathbf{Q}_i^n) - \lambda_L\mathbf{f}(\mathbf{Q}_{i+1}^n) + \lambda_L\lambda_R(\mathbf{Q}_{i+1}^n - \mathbf{Q}_i^n)}{\lambda_R - \lambda_L}$$

Then the flux of HLL can be defined as:

$$\mathbf{F}_{i+1/2}=\begin{cases}\mathbf{f}(\mathbf{Q}_i^n), & \lambda_L \geqslant 0\\ \mathbf{F}^*, & \lambda_L<0<\lambda_R\\ \mathbf{f}(\mathbf{Q}_{i+1}^n), & \lambda_R \leqslant 0\end{cases}$$

2.2 HLLC Solver

Consider a three-wave approximate Riemann solution:



By the Rankine-Hugoniot condition:

$$\begin{aligned}\mathbf{F}_L^* - \mathbf{f}(\mathbf{Q}_i^n) &= \lambda_L(\mathbf{Q}_L^* - \mathbf{Q}_i^n)\\ \mathbf{F}_R^* - \mathbf{F}_L^* &= \lambda^*(\mathbf{Q}_R^* - \mathbf{Q}_L^*)\\ \mathbf{f}(\mathbf{Q}_{i+1}^n) - \mathbf{F}_R^* &= \lambda_R(\mathbf{Q}_{i+1}^n - \mathbf{Q}_R^*)\end{aligned}$$

For Euler's equation, by the property of the equation itself, the velocity and pressure are actually continuous:

$$u_L^* = u_R^* = u^*, p_L^* = p_R^* = p^*$$

Meanwhile, at the discontinuity:

$$\lambda^* = u^*$$

Cancelling $\mathbf{F}_L^*$ and $\mathbf{F}_R^*$, we get:

$$\begin{aligned}\mathbf{f}(\mathbf{Q}_{i+1}^n) - \mathbf{f}(\mathbf{Q}_i^n) - \lambda^*(\mathbf{Q}_R^* - \mathbf{Q}_L^*) &= \lambda_L(\mathbf{Q}_L^* - \mathbf{Q}_i^n) + \lambda_R(\mathbf{Q}_{i+1}^n - \mathbf{Q}_R^*)\\ (\lambda_R - \lambda^*)\mathbf{Q}_R^* + (\lambda^* - \lambda_L)\mathbf{Q}_L^* &= \lambda_R\mathbf{Q}_{i+1}^n - \lambda_L\mathbf{Q}_i^n - [\mathbf{f}(\mathbf{Q}_{i+1}^n) - \mathbf{f}(\mathbf{Q}_i^n)]\end{aligned}$$

The first two equations are:

$$\begin{aligned}(\lambda_R - \lambda^*)\rho_R^* + (\lambda^* - \lambda_L)\rho_L^* &= \lambda_R\rho_{i+1}^n - \lambda_L\rho_i^n - (\rho_{i+1}^n u_{i+1}^n - \rho_i^n u_i^n)\\ (\lambda_R - \lambda^*)\rho_R^*\lambda^* + (\lambda^* - \lambda_L)\rho_L^*\lambda^* &= \lambda_R\rho_{i+1}^n u_{i+1}^n - \lambda_L\rho_i^n u_i^n - \\ &\quad [\rho_{i+1}^n(u_{i+1}^n)^2 + p_{i+1}^n - \rho_i^n(u_i^n)^2 - p_i^n]\end{aligned}$$

We get:

$$\lambda^* = \frac{\lambda_R\rho_{i+1}^n u_{i+1}^n - \lambda_L\rho_i^n u_i^n - [\rho_{i+1}^n(u_{i+1}^n)^2 + p_{i+1}^n - \rho_i^n(u_i^n)^2 - p_i^n]}{\lambda_R\rho_{i+1}^n - \lambda_L\rho_i^n - (\rho_{i+1}^n u_{i+1}^n - \rho_i^n u_i^n)}$$

Then:

$$\begin{aligned}\mathbf{F}_L^* - \mathbf{f}(\mathbf{Q}_i^n) &= \lambda_L(\mathbf{Q}_L^* - \mathbf{Q}_i^n) \Rightarrow \rho_L^* = \frac{\lambda_L - u_i^n}{\lambda_L - \lambda^*}\rho_i^n\\ \mathbf{f}(\mathbf{Q}_{i+1}^n) - \mathbf{F}_R^* &= \lambda_R(\mathbf{Q}_{i+1}^n - \mathbf{Q}_R^*) \Rightarrow \rho_L^* = \frac{\lambda_R - u_{i+1}^n}{\lambda_R - \lambda^*}\rho_{i+1}^n\end{aligned}$$

We solve p_L^* and p_R^*:

$$[\rho_L^*(\lambda^*)^2+p_L^*]-[\rho_i^n(u_i^n)^2+p_i^n]=\lambda_L(\rho_L^*\lambda^*-\rho_i^n u_i^n)$$
$$\Downarrow$$
$$\begin{aligned}p_L^* &= \rho_i^n(u_i^n-\lambda^*)(u_i^n-\lambda_L)+p_i^n\\ p_R^* &= \rho_{i+1}^n(u_{i+1}^n-\lambda^*)(u_{i+1}^n-\lambda_L)+p_{i+1}^n\end{aligned}$$

Similar, we get also get e_L^* and e_R^*:

$$\begin{aligned}e_L^* &= \frac{\lambda_L-u_i^n}{\lambda_L-\lambda^*}e_i^n+\frac{\lambda^*-u_i^n}{\lambda_L-\lambda^*}p_i^n+\rho_i^n\frac{\lambda^*(\lambda^*-u_i^n)(\lambda_L-u_i^n)}{\lambda_L-\lambda^*}\\ e_R^* &= \frac{\lambda_R-u_{i+1}^n}{\lambda_R-\lambda^*}e_{i+1}^n+\frac{\lambda^*-u_{i+1}^n}{\lambda_R-\lambda^*}p_{i+1}^n+\rho_{i+1}^n\frac{\lambda^*(\lambda^*-u_{i+1}^n)(\lambda_R-u_{i+1}^n)}{\lambda_R-\lambda^*}\end{aligned}$$

Then the scheme can be expressed as follows:

$$\mathbf{Q}_i^{n+1}=\mathbf{Q}_i-\frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1/2}-\mathbf{F}_{i-1/2})$$

The numerical flux is:

$$\mathbf{F}_{i+1/2}=\begin{cases}\mathbf{f}(\mathbf{Q}_i^n) & \lambda_L\geqslant 0\\ \mathbf{f}(\mathbf{Q}_i^n)+\lambda_L(\mathbf{Q}_L^*-\mathbf{Q}_i^n) & \lambda_L<0<\lambda^*\\ \mathbf{f}(\mathbf{Q}_{i+1}^n)+\lambda_R(\mathbf{Q}_R^*-\mathbf{Q}_{i+1}^n) & \lambda^*\leqslant 0<\lambda_R\\ \mathbf{f}(\mathbf{Q}_{i+1}^n) & \lambda_R\leqslant 0\end{cases}$$

Where:

$$\begin{aligned}\mathbf{Q}_L^* &= (\ \rho_L^*\quad \rho_L^*\lambda^*\quad \rho_L^*v_i^n\quad e_L^*\ )^T\\ \mathbf{Q}_R^* &= (\ \rho_R^*\quad \rho_R^*\lambda^*\quad \rho_R^*v_{i+1}^n\quad e_R^*\ )^T\end{aligned}$$

2.3 Speed Estimate

In 2.2 we set:

$$\lambda_L=\min\{\lambda(\mathbf{Q}_i^n),\lambda(\mathbf{Q}_{i+1}^n)\},\lambda_R=\max\{\lambda(\mathbf{Q}_i^n),\lambda(\mathbf{Q}_{i+1}^n)\}$$

Which is also equal to:

$$\lambda_L=\min\{u_i^n-c_i^n,u_{i+1}^n-c_{i+1}^n\},\lambda_R=\max\{u_i^n+c_i^n,u_{i+1}^n+c_{i+1}^n\}$$

These estimates make use of data values only, are exceedingly simple but are not recommended for practical computations.There are some other ways to choose speed

2.3.1 Average Eigenvalues

$$\lambda_L=\tilde{u}-\tilde{a},\lambda_R=\tilde{u}+\tilde{a}$$

Where:

$$\tilde{u}=\frac{\sqrt{\rho_i^n}u_i^n+\sqrt{\rho_{i+1}^n}u_{i+1}^n}{\sqrt{\rho_i^n}+\sqrt{\rho_{i+1}^n}},\tilde{a}=\sqrt{(\gamma-1)\left(\tilde{H}-\frac{1}{2}\tilde{u}\right)},\tilde{H}=\frac{\sqrt{\rho_i^n}H_i^n+\sqrt{\rho_{i+1}^n}H_{i+1}^n}{\sqrt{\rho_i^n}+\sqrt{\rho_{i+1}^n}}$$

Here H is defined as:

$$H = \frac{E+p}{\rho}$$

Motivated by the Roe eigenvalues,Einfeldt proposed the estimates:

$$\lambda_L = \bar{u} - \bar{d}, \lambda_R = \bar{u} + \bar{d}$$

where:

$$\bar{d} = \frac{\sqrt{\rho_i^n} a_i^n + \sqrt{\rho_{i+1}^n} a_{i+1}^n}{\sqrt{\rho_i^n} + \sqrt{\rho_{i+1}^n}} + \eta_2 (u_{i+1}^n - u_i^n)^2$$
$$\eta_2 = \frac{1}{2} \frac{\sqrt{\rho_i^n}\sqrt{\rho_{i+1}^n}}{\left(\sqrt{\rho_i^n} + \sqrt{\rho_{i+1}^n}\right)^2}$$

2.3.2 Pressure–Based Wave Speed Estimates

$$\lambda_L = u_i^n - a_i^n q_i^n, \lambda_R = u_{i+1}^n + a_{i+1}^n q_{i+1}^n$$

Where:

$$q_K = \begin{cases} 1, & p^* \leqslant p_K \\ \sqrt{1 + \frac{\gamma+1}{2\gamma}\left(\frac{p^*}{p_K} - 1\right)}, & p^* > p_K \end{cases}$$

Where $K = L, p_K = p_i^n; K = R, p_K = p_{i+1}^n$

3 Numerical Result

3.1 For Riemann Problem

We present the result at $t = 1, 2, 3$:

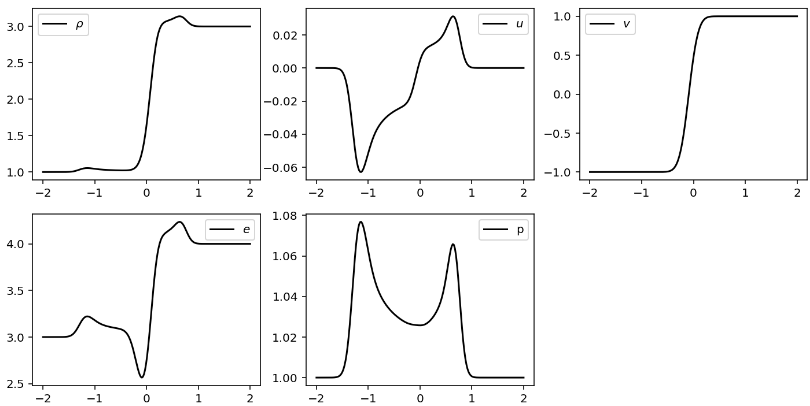


Figure 1. $t = 1$

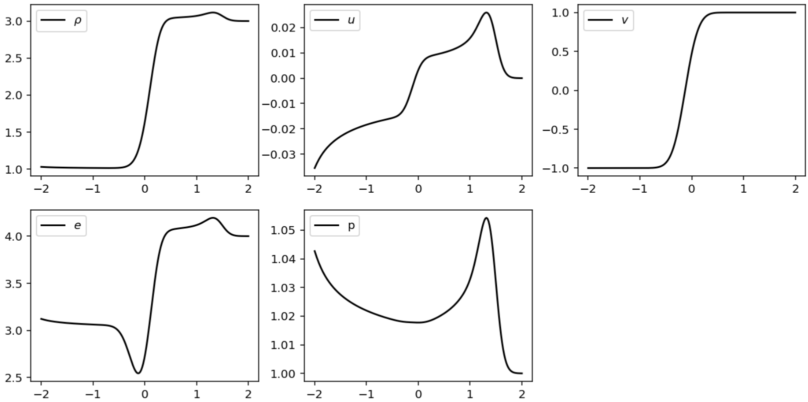


Figure 2. $t = 2$

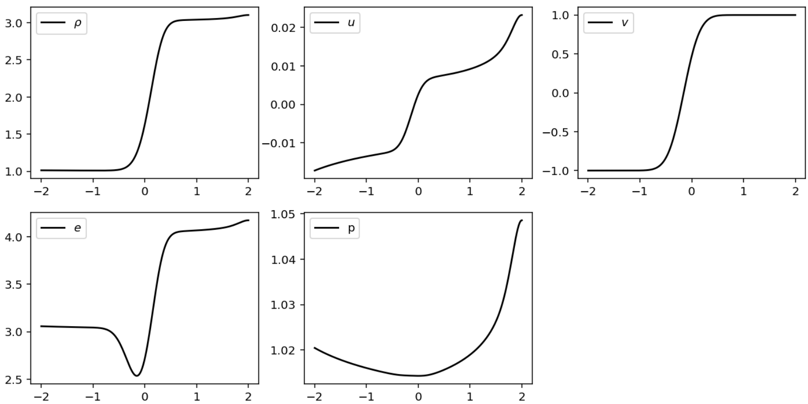


Figure 3. $t = 3$

We test the result under different speed estimate, and find the speed estimate has litte impact on final result(take $t = 1$ as example:

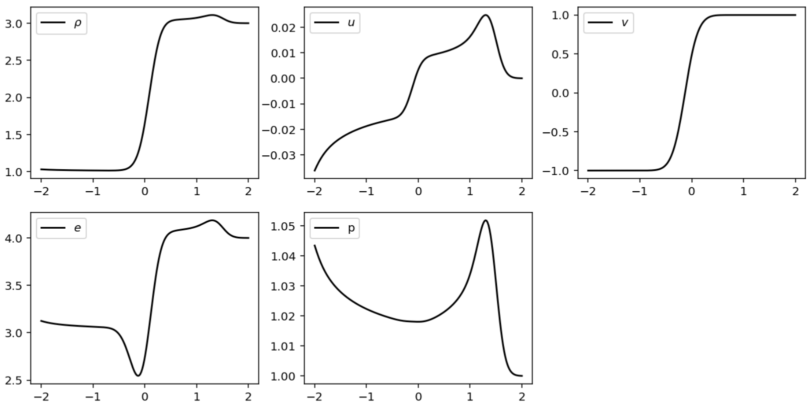


Figure 4. $t = 2$ Under Average Eigenvalues Speed

3.2 For Woodward-Collela blast wave Problem

We present the numerical result at $t = 0.01, 0.02, 0.03, 0.038$

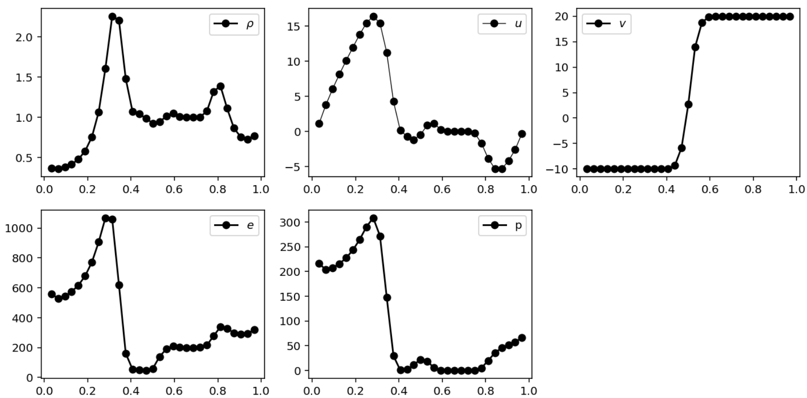


Figure 5. $t = 0.01$

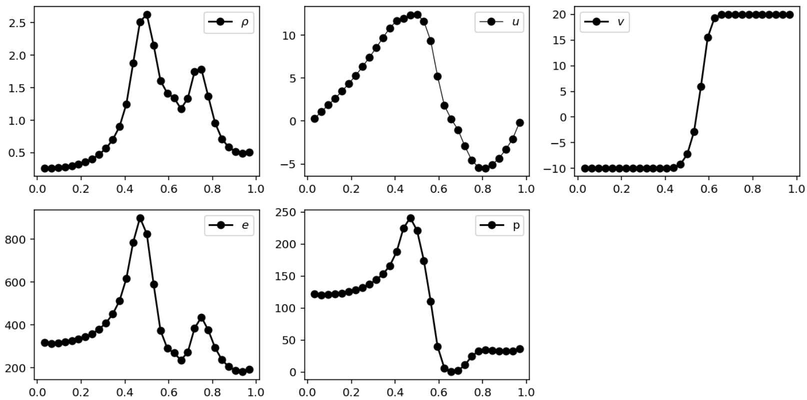


Figure 6. $t = 0.02$

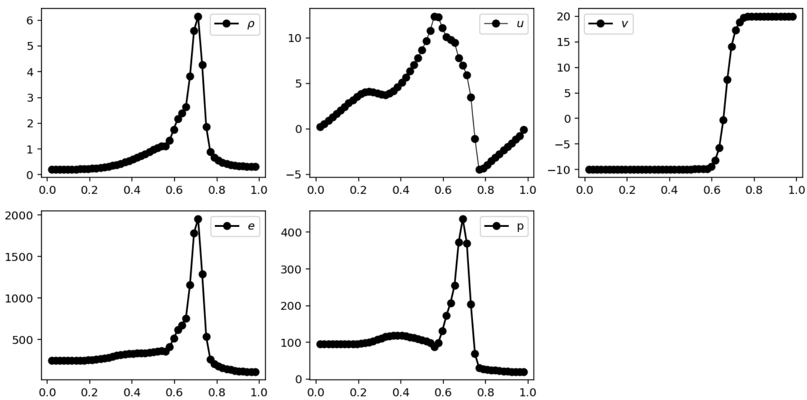


Figure 7. $t = 0.03$

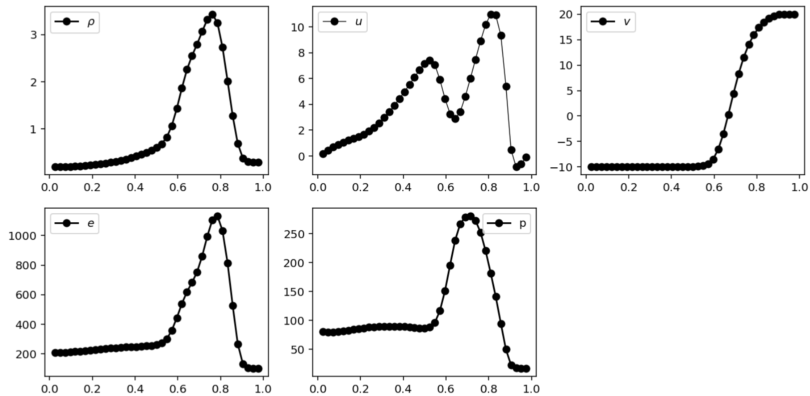


Figure 8. $t = 0.038$

Bibliography

[1] Toro(2009). Riemann Solvers and Numerical Methods for Fluid Dynamics,a Pratical Introduciton. Springer