Project 1

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1 Problem Statement

$(x,y)\in\Omega=(0,1)\times(0,1)$, $\gamma=\frac{7}{5}$:

$$\begin{aligned}
&\frac{\partial\tilde{\rho}}{\partial t}+\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}=0\\
&\frac{\partial\tilde{u}}{\partial t}+\frac{\partial\tilde{p}}{\partial x}=0\\
&\frac{\partial\tilde{v}}{\partial t}+\frac{\partial\tilde{p}}{\partial y}=0\\
&\frac{\partial\tilde{p}}{\partial t}+\gamma\frac{\partial\tilde{u}}{\partial x}+\gamma\frac{\partial\tilde{v}}{\partial y}=0
\end{aligned}$$

The initial condition is:

$$\begin{aligned}
\tilde{u}(x,y,0)=\tilde{v}(x,y,0)=&0,\forall(x,y)\in\Omega\\
\tilde{\rho}(x,y,0)=\tilde{p}(x,y,0)=&\begin{cases}1, & \text{if }(x,y)\in B\\ 0, & \text{if }(x,y)\in\Omega\backslash B\end{cases}
\end{aligned}$$

Where $B=\{(x,y)|(x-x_0)^2+(y-y_0)^2\leqslant r^2\}$ and $\rho_0>0$.The boundary conditions are:

$$\tilde{u}(0,y,t)=\tilde{u}(1,y,t)=\tilde{v}(x,0,t)=\tilde{v}(x,1,t)=0$$

2 Numerical Scheme

In this part, we will use two different schemes to test the efficiency. We divide the computational domain Ω into a number of grid cells:

$$\Omega=\bigcup_{i=1}^{N}\Omega_i\,,\mathrm{int}(\Omega_i)\cap\mathrm{int}(\Omega_j)=\varnothing\quad\text{if } i\neq j$$

Then let:

$$\tilde{\rho}_{i,j}^t=\frac{1}{|\Delta x_i\Delta y_j|}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{i-1/2}}^{y_{i+1/2}}\tilde{\rho}(x,y,t)dydx$$

Same for $\tilde{u}_{i,j}^t,\tilde{v}_{i,j}^t,\tilde{p}_{i,j}^t$

2.1 Oridinary Finite Volume Method

$$\begin{aligned}
\frac{\tilde{\rho}_{i,j}^{t+1}-\tilde{\rho}_{i,j}^t}{\Delta t}+\frac{\tilde{u}_{i+1,j}^t-\tilde{u}_{i,j}^t}{\Delta x}+\frac{\tilde{v}_{i,j+1}^t-\tilde{v}_{i,j}^t}{\Delta y}=&0\\
\frac{\tilde{u}_{i,j}^{t+1}-\tilde{u}_{i,j}^t}{\Delta t}+\frac{\tilde{p}_{i+1,j}^t-\tilde{p}_{i,j}^t}{\Delta x}=&0\\
\frac{\tilde{v}_{i,j}^{t+1}-\tilde{v}_{i,j}^t}{\Delta t}+\frac{\tilde{p}_{i,j+1}^t-\tilde{p}_{i,j}^t}{\Delta y}=&0\\
\frac{\tilde{p}_{i,j}^{t+1}-\tilde{p}_{i,j}^t}{\Delta t}+\gamma\frac{\tilde{u}_{i+1,j}^t-\tilde{u}_{i,j}^t}{\Delta x}+\gamma\frac{\tilde{v}_{i,j+1}^t-\tilde{v}_{i,j}^t}{\Delta y}=&0
\end{aligned}$$

So we have iteration method:

$$
\begin{aligned}
\tilde{\rho}_{i,j}^{t+1} &= \tilde{\rho}_{i,j}^{t} - \left(\frac{\tilde{u}_{i+1,j}^{t} - \tilde{u}_{i,j}^{t}}{\Delta x} + \frac{\tilde{v}_{i,j+1}^{t} - \tilde{v}_{i,j}^{t}}{\Delta y}\right)\Delta t \\
\tilde{u}_{i,j}^{t+1} &= \tilde{u}_{i,j}^{t} - \left(\frac{\tilde{p}_{i+1,j}^{t} - \tilde{p}_{i,j}^{t}}{\Delta x}\right)\Delta t \\
\tilde{v}_{i,j}^{t+1} &= \tilde{v}_{i,j}^{t} - \left(\frac{\tilde{\rho}_{i,j+1}^{t} - \tilde{\rho}_{i,j}^{t}}{\Delta y}\right)\Delta t \\
\tilde{p}_{i,j}^{t+1} &= \tilde{p}_{i,j}^{t} - \gamma\Delta t\left(\frac{\tilde{u}_{i+1,j}^{t} - \tilde{u}_{i,j}^{t}}{\Delta x} + \frac{\tilde{v}_{i,j+1}^{t} - \tilde{v}_{i,j}^{t}}{\Delta y}\right)
\end{aligned}
$$

But the Oridinary Finite Volume Method has a very obvious disadvantage:The solution doesn't converge:it will become larger and larger as time evolves(numerical explosion,**CFL** Condition satisfies),just like this:

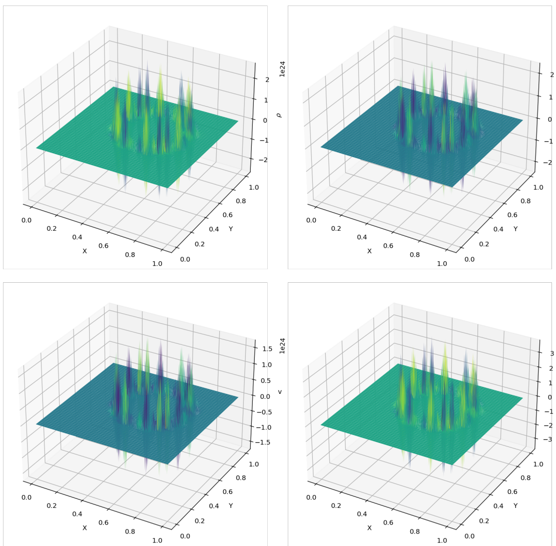

Figure 1. Numerical Explosion($t = 0.1$)

2.2 Upwind Scheme

Express the equation in matrix form:

$$
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \gamma & 0 \end{pmatrix}
$$

Then the problem can be expressed as:

$$\frac{\partial \mathbf{q}}{\partial t} + A\frac{\partial \mathbf{q}}{\partial x} + B\frac{\partial \mathbf{q}}{\partial y} = 0\,, \mathbf{q} = (\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p})$$

Define:

$$\mathbf{Q}^n_{i,j} = \frac{1}{|\Delta x_i \Delta y_j|}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{i-1/2}}^{y_{i+1/2}} \mathbf{q}(x, y, t_n)\,dy\,dx$$

First we diagnolized matrix:

$$A = U\Lambda U^{-1}, B = S\Sigma S^{-1}$$

Then:$A_+ = U\Lambda_+ U^{-1}$, $A_- = U\Lambda_- U^{-1}$, $B_+ = S\Sigma_+ S^{-1}$, $B_- = S\Sigma_- S^{-1}$. Then the scheme is expressed as follows:

$$\mathbf{Q}^{n+1}_{i,j} = \mathbf{Q}^n_{i,j} - \frac{\Delta t}{\Delta x}\Big(F^n_{i+\frac{1}{2},j} - F^n_{i-\frac{1}{2},j}\Big) - \frac{\Delta t}{\Delta y}\Big(G^n_{i,j+\frac{1}{2}} - G^n_{i,j-\frac{1}{2}}\Big)$$

Where:

$$\begin{aligned} F^n_{i+\frac{1}{2},j} &= A_+\mathbf{Q}^n_{i,j} + A_-\mathbf{Q}^n_{i+1,j};\; F^n_{i-\frac{1}{2},j} = A_+\mathbf{Q}^n_{i-1,j} + A_-\mathbf{Q}^n_{i,j} \\ G^n_{i,j+\frac{1}{2}} &= B_+\mathbf{Q}^n_{i,j} + B_-\mathbf{Q}^n_{i,j+1};\; G^n_{i,j-\frac{1}{2}} = B_+\mathbf{Q}^n_{i,j-1} + B_-\mathbf{Q}^n_{i,j} \end{aligned}$$

3 Numerical Solution of Upwind Scheme

We present the solution in the form of Image and 3D Plot at $t = 0.1, 0.2, 0.3, 0.4$.And we set number of discretiation as $Nx = Ny = 200$, dt $= 0.0005$

3.1 $r = \frac{1}{4}, (x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$

3.1.1 Image Show

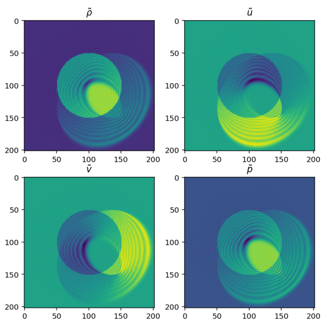

Figure 2. $t = 0.1$

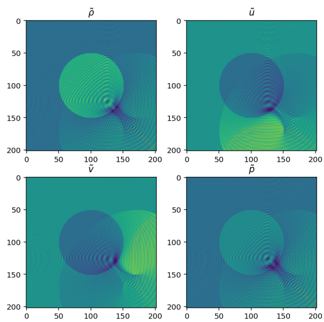

Figure 3. $t = 0.2$

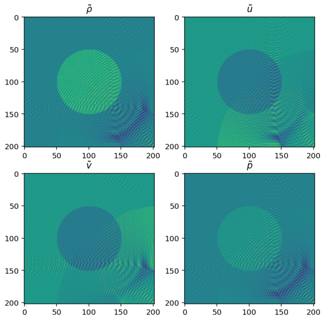

Figure 4. $t = 0.3$

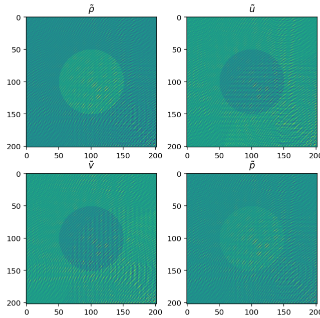

Figure 5. $t = 0.4$

3.1.2 3D Plot

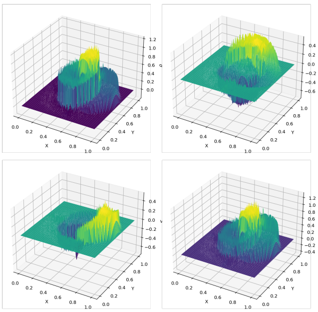

Figure 6. $t = 0.1$

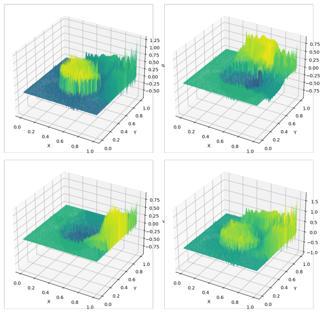

Figure 7. $t = 0.2$

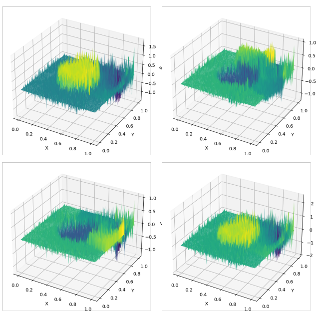

Figure 8. $t = 0.3$

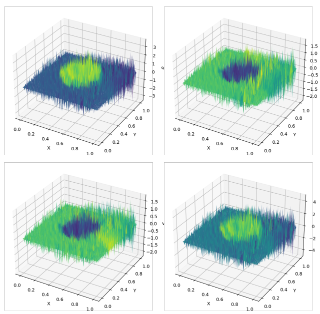

Figure 9. $t = 0.4$

3.2 $r = \frac{1}{5}, (x_0, y_0) = \left(\frac{1}{3}, \frac{1}{3}\right)$

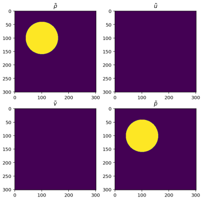

Figure 10. $t = 0$

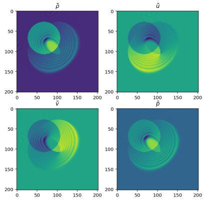

Figure 11. $t = 0.1$

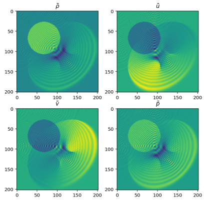

Figure 12. $t = 0.2$

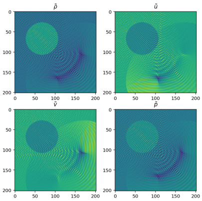

Figure 13. $t = 0.3$

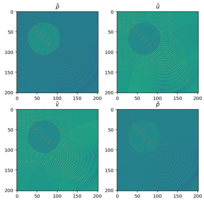

Figure 14. $t = 0.4$

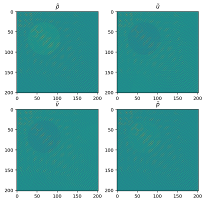

Figure 15. $t = 0.5$

We can conclude from the images:as time evolves, the four solution will be evenly distributed within the domain Ω. And at $t = 0.3$, the four waves all hit the four boundaries.