Homework 3

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MA5233 Computational Mathematics

1 Problem

We have 2D time-dependent convection-diffusion equation:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla_x u = \frac{1}{10} \Delta u, \qquad \boldsymbol{x} = (x, y)^T \in \Omega, \quad t \in \mathbb{R}^+ \\
&u(\boldsymbol{x}, t) = 0, \qquad \boldsymbol{x} \in \partial\Omega, \quad t \in \mathbb{R}^+ \\
&u(\boldsymbol{x}, 0) = \sin(\pi(x^2 + y^2))[(x-1)^2 + y^2 - 9], \quad \boldsymbol{x} \in \Omega
\end{aligned}
$$

where $\boldsymbol{v} = (1, 2)^T$, the domain Ω is defined as following:

$$\Omega = \{(x, y) | x^2 + y^2 > 1 \text{ and } (x-1)^2 + y^2 < 9\}$$

And we will use the Finite Difference Method to solve the problem

2 Numerical Scheme

In this report we will use the center explicit difference method. For the centered explicit finite difference method we will use the a forward difference in time for the time derivative and a centered difference for the special derivative. We begin with our discretization of the independent variable x, y, t. Let

$$u_{m,n}^j = u(m\Delta x, n\Delta y, j\Delta t)$$

We expand the prime problem:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = \frac{1}{10}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Then:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{u_{m,n}^{j+1} - u_{m,n}^j}{\Delta t} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{u_{m+1,n}^j - 2u_{m,n}^j + u_{m-1,n}^j}{\Delta x^2} \\
\frac{\partial u}{\partial x} &= \frac{u_{m+1,n}^j - u_{m,n}^j}{\Delta x}
\end{aligned}
$$

Same for variable y. We subsititute the discretization into the prime problem:

$$
\begin{aligned}
\frac{u_{m,n}^{j+1} - u_{m,n}^j}{\Delta t} = \frac{1}{10}\left(\frac{u_{m+1,n}^j - 2u_{m,n}^j + u_{m-1,n}^j}{\Delta x^2} + \frac{u_{m,n+1}^j - 2u_{m,n}^j + u_{m,n-1}^j}{\Delta y^2}\right) \\
- \frac{u_{m+1,n}^j - u_{m-1,n}^j}{\Delta x} - 2\frac{u_{m,n+1}^j - u_{m,n-1}^j}{\Delta y}
\end{aligned}
$$

To simpilfy the notation, we let:

$$R_x = \frac{\Delta t}{10\Delta x^2}, R_y = \frac{\Delta t}{10\Delta y^2}, P_x = \frac{\Delta t}{\Delta x}, P_y = \frac{2\Delta t}{\Delta y}$$

Then the equation can be expressed as :

$$u_{m,n}^{j+1} = u_{m,n}^j + R_x(u_{m+1,n}^j - 2u_{m,n}^j + u_{m-1,n}^j) + R_y(u_{m,n+1}^j - 2u_{m,n}^j + u_{m,n-1}^j) - P_x(u_{m+1,n}^j - u_{m-1,n}^j) - P_y(u_{m,n+1}^j - u_{m,n-1}^j)$$

Grouping like terms we get:

$$\begin{aligned} u_{m,n}^{j+1} =& (1 - 2R_x - 2R_y)u_{m,n}^j + (R_x - P_x)u_{m+1,n}^j + (R_x + P_x)u_{m-1,n}^j \\ &+ (R_y - P_y)u_{m,n+1}^j + (R_y + P_y)u_{m,n-1}^j \\ =& \alpha u_{m,n}^j + \beta u_{m+1,n}^j + \lambda u_{m-1,n}^j + \gamma u_{m,n+1}^j + \omega u_{m,n-1}^j \end{aligned}$$

Where:

$$\begin{aligned} \alpha &= (1 - 2R_x - 2R_y) \\ \beta &= (R_x - P_x) \\ \lambda &= (R_x + P_x) \\ \gamma &= (R_y - P_y) \\ w &= (R_y + P_y) \end{aligned}$$

To gaurantee the method will converge, we should have following condition:

$$(1 - 2R_x - 2R_y) = 1 - 4R_x < 0 \Rightarrow R_x > \frac{1}{4}$$

3 Numerical Experiment

First we discrete the domain Ω:

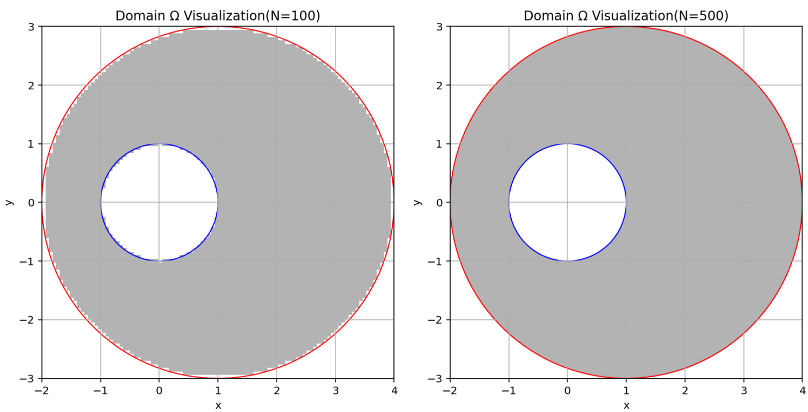

图 1. Discretization

In our experiments, we set $N=300$, and $dt=\frac{2}{(dx)^2}$, $dx=\frac{1}{300}$:

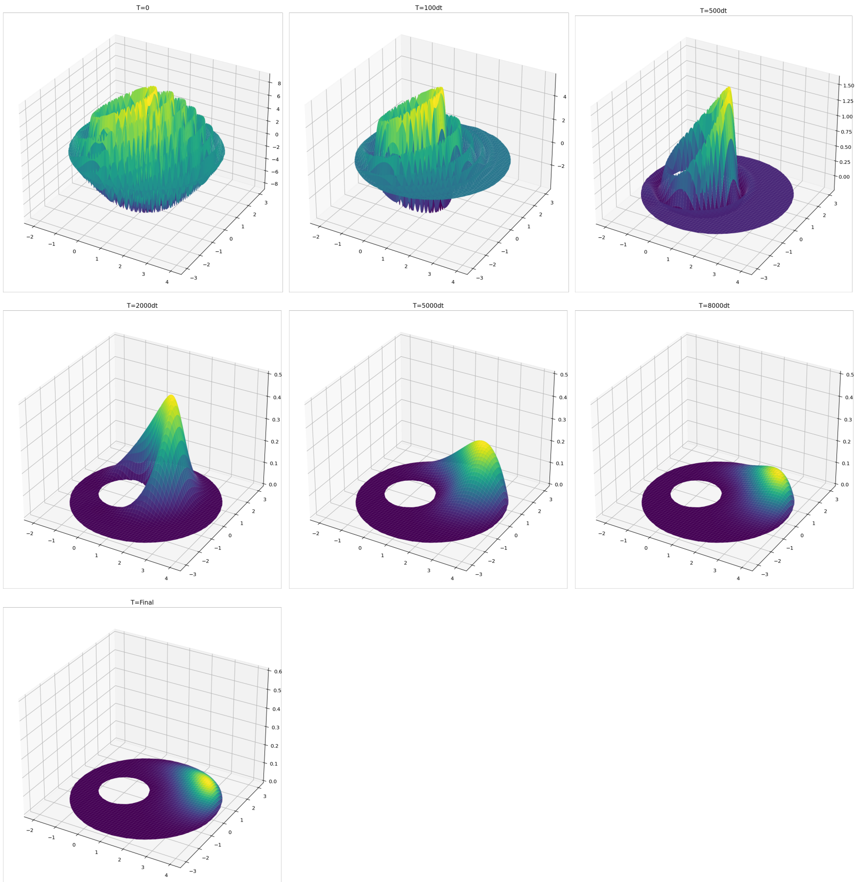

图 **2.**

As we can see, the solution tend to be stable(i.e converges to 0) as time goes.

Reference:

[1] Noam M. Buckman,The linear convection-diffusion equation in two dimensions. *Linear Partial Differential Equations,MIT*