Homework 1

BY ZHANG YUXIANG
A0246636X

1 Problem

We have the 2D Poisson Equation:

$$
\begin{aligned}
-\Delta u = 400(x^4 - y^4)s(20xy), \Omega = (0,1) \times (0,1)\\
u(x,0) = 0, x \in (0,1)\\
u(0,y) = 0, y \in (0,1)\\
u_y(x,1) = 20x(x^2-1)\cos(20x) - 2\sin(20x), x \in (0,1)\\
u_x(1,y) = 20y(1-y^2)\cos(20y) + 2\sin(20y), y \in (0,1)
\end{aligned}
$$

2 Finite Difference & Volume Method

2.1 Algorithm for FDM

The following process is almost the same between FDM and FVM. The only difference is the calculation method of source term. So we mainly focus on the FDM explaination

Let N denotes the number of nodes of a interval, $h = \frac{1}{N}$, we generate the node value as:

$$
\begin{aligned}
x = (h, 2h, \ldots, (N-1)h)\\
y = (h, 2h, \ldots, (N-1)h)
\end{aligned}
$$

Let V denote the mesh of x and y:

$$V = \{(x_i, y_j) \colon 1 \leqslant i, j \leqslant N\}$$

during the inverval $(h, 1-h)$. To solve the original problem we need to do discretization of primal problem, for $(x_i, y_j) \in V$, we have

$$
\begin{aligned}
\Delta u =& \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\\
\Delta u =& \frac{u(x_i + h, y_j) - 2u(x_i, y_j) + u(x_i - h, y_j)}{h^2} +\\
& \frac{u(x_i, y_j + h) - 2u(x_i, y_j) + u(x_i, y_j - h)}{h^2}\\
=& \frac{1}{h^2}(-4u(x_i, y_j) + u(x_i + h, y_j) + u(x_i - h, y_j) + u(x_i, y_j + h) + u(x_i, y_j - h))
\end{aligned}
$$

we have the following boundary condition:

1. $x_1 - h = 0$ or $y_1 - h = 0$, then $u(x_1 - h, y_j) = 0, u(x_i, y_1 - h) = 0$;
2. $x_N + h = 1$ or $y_N + h = 1$, we have:

$$
\begin{aligned}
u_x(1, y_j) = \lim_{h \to 0} \frac{u(1, y_j) - u(1-h, y_j)}{h}\\
\Rightarrow u(1, y_j) \approx u_x(1, y_j)h + u(x_N, y_j)
\end{aligned}
$$

Similarly for y we have:

$$u(x_i,1)\approx u_y(x_i,1)h+u(x_i,y_N)$$

for $x_1=h,\ y_1=h$,we have:

$$\Delta u(x_1,y_1)=\frac{1}{h^2}(-4u(x_1,y_j)+u(x_2,y_j)+u(x_1,y_j+h))$$

for $N>i>1$,$j=1$, we have:

$$\Delta u(x_i,y_1)=\frac{1}{h^2}(-4u(x_i,y_1)+u(x_i+h,y_1)+u(x_i-h,y_1)+u(x_i,y_1+h))$$

for $N-1>i,j>1$, we have:

$$\frac{-1}{h^2}(-4u(x_i,y_j)+u(x_i+h,y_j)+u(x_i-h,y_j)+u(x_i,y_j+h)+u(x_i,y_j-h))=f_{ij}$$

for $i=N,j<N$, we have:

$$\frac{-1}{h^2}(-3u(x_N,y_j)+u(x_N-h,y_j)+u(x_N,y_j+h)+u(x_N,y_j-h))=f(x_N,y_j)+\frac{u_x(x_N,y_j)}{h}$$

for $i=j=N$,we have:

$$\frac{-1}{h^2}(-2u(x_N,y_N)+u(x_N-h,y_N)+u(x_N,y_N-h))=f(x_N,y_N)+\frac{(u_x+u_y)(x_N,y_N)}{h}$$

for $i<N-1,j=N$, we have:

$$\frac{-1}{h^2}(-3u(x_i,y_N)+u(x_i-h,y_N)+u(x_i+h,y_N)+u(x_i,y_N-h))=f(x_N,y_j)+\frac{u_y(x_i,1)}{h}$$

Then we can solve the above linear equation system.The coefficient matrix for the system is:

$$\boldsymbol{Ax}=\boldsymbol{b}$$

We need define some base matrix:

$$B=\begin{pmatrix} -4 & 1 & 0 & \dots & 0\\ 1 & -4 & 1 & \dots & 0\\ 0 & 1 & -4 & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & 1\\ 0 & 0 & 0 & \dots & -3 \end{pmatrix}_{N\times N} \qquad Q=\begin{pmatrix} -3 & 1 & 0 & \dots & 0\\ 1 & -3 & 1 & \dots & 0\\ 0 & 1 & 3 & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & 1\\ 0 & 0 & 0 & \dots & -2 \end{pmatrix}_{N\times N}$$

So:

$$\boldsymbol{A}=\frac{-1}{h^2}\begin{pmatrix} B & I & & & \\ I & B & I & & \\ & I & \ddots & I & \\ & & I & B & I\\ & & & I & Q \end{pmatrix}_{N^2\times N^2}$$

And $\boldsymbol{b}$:

$$\lambda_j = \begin{pmatrix} f(x_1, y_j) \\ f(x_2, y_j) \\ \vdots \\ f(x_N, y_j) \end{pmatrix}_{N\times 1}, 1 \leqslant j \leqslant N$$

And

$$\kappa_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{u_x(x_N, y_j)}{h} \end{pmatrix}$$

for $j = N$, we have:

$$\kappa_N = \begin{pmatrix} \frac{u_y(x_1, 1)}{h} \\ \frac{u_y(x_2, 1)}{h} \\ \vdots \\ \frac{(u_x + u_y)(x_N, y_N)}{h} \end{pmatrix}$$

Therefore:

$$\boldsymbol{b} = \begin{pmatrix} \lambda_1 + \kappa_1 \\ \lambda_2 + \kappa_2 \\ \vdots \\ \lambda_N + \kappa_N \end{pmatrix}$$

By solveing the system, we get:

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

2.2 Algorithm for FVM

Since we use the same square mesh method in FVM as FDM. Meanwhile we notice that the $f(x, y)$ is continuous and integrable. So for each linear equation we calculate the volume as:

1.$i = 1$ and $j = 1$:

$$f(x_1, y_1) = \int_{V_c} f(x, y) dV = \frac{1}{\Delta x \Delta y} \int_0^{x_1} \int_0^{y_1} f(x, y) \mathrm{dxdy}$$

2.$i = 1$ and $N > j > 1$:

$$f(x_1, y_j) = \frac{1}{\Delta x \Delta y} \int_0^{x_1} \int_{y_{j-1}}^{y_j} f(x, y) \mathrm{dxdy}$$

3.$N > i > 1$ and $N > j > 1$:

$$f(x_i, y_j) = \frac{1}{\Delta x \Delta y} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x, y) \mathrm{dxdy}$$

4.$N > i > 1$ and $j = N$:

$$f(x_i, y_j) = \frac{1}{\Delta x \Delta y} \int_{x_{i-1}}^{x_i} \int_{y_N}^{1} f(x, y) \mathrm{dxdy}$$

5.$i = N$ and $j = N$:

$$f(x_i, y_j) = \frac{1}{\Delta x \Delta y} \int_{x_N}^{1} \int_{y_N}^{1} f(x, y) \mathrm{dxdy}$$

In our setting, $\frac{1}{\Delta x \Delta y} = \frac{1}{h^2}$

2.3 Numerical Result for FDM

The first part is the MSE Error

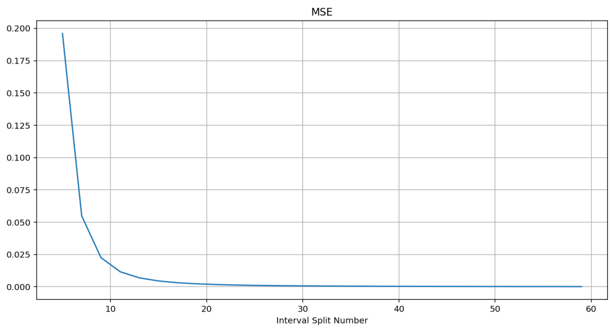

Figure 1. Error

Exact solution and Numerical Solution in 2D and 3D axis:

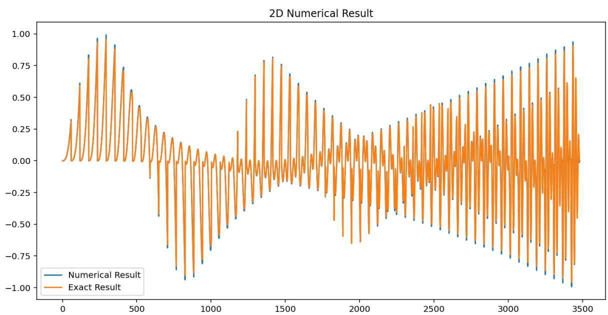

Figure 2. 2D Display

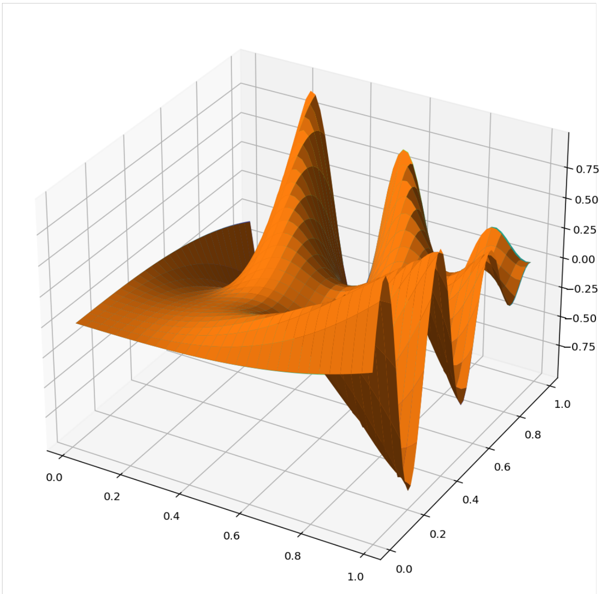

Figure 3. 3D

As we can see the convergence rate of FDM is fast and almost equal to the exact solution

2.4 Numerical Result for FVM

The first part is the MSE Error:

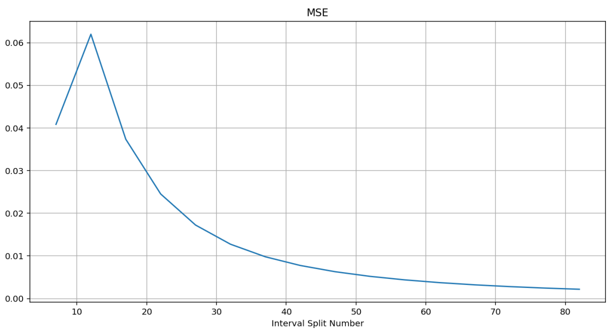

Figure 4. Convergency Rate

Exact solution and Numerical Solution in 2D and 3D axis:

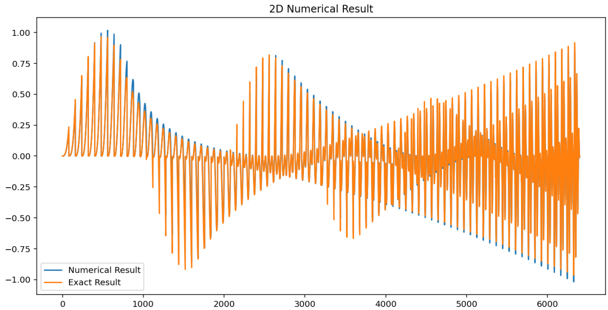

Figure 5. 2D

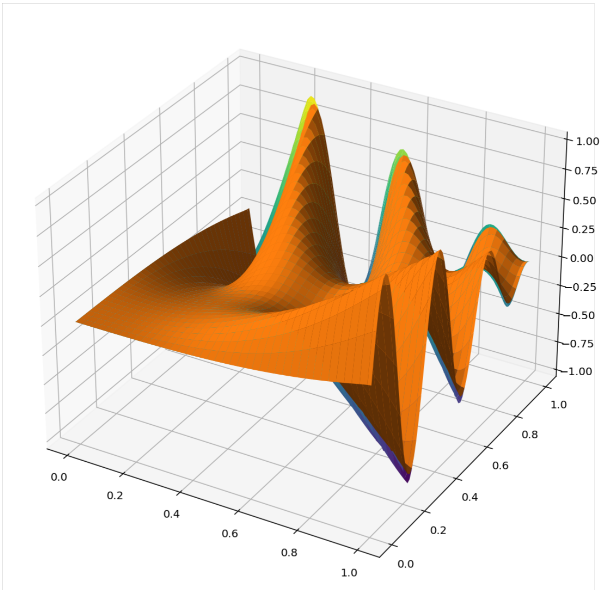

Figure 6. 3D

The figure shows rate the convergency rate of FVM is slower than FDM.

3 Finite Element Method

3.1 Algorithm

We cut the domain Ω into $M \times N$ pieces, in each small domain, we define

$$\varphi_{i,j} = \psi\left(\frac{x - x_i}{h_x}, \frac{y - y_j}{h_y}\right)$$

where

$$\psi(x,y)=\begin{cases}(1-x)(1-y), x\in[0,1], y\in[0,1]\\(1+x)(1-y), x\in[-1,0], y\in[0,1]\\(1+x)(1+y), x\in[-1,0], y\in[-1,0]\\(1-x)(1+y), x\in[0,1], y\in[-1,0]\end{cases}$$

Then we need to define $u(x_i,y_j)$(denote as $u_{i,j}$).

$$u_h=\sum_{i=0}^{M}\sum_{j=0}^{N}u_{i,j}\varphi_{i,j}$$

First we need to define the weak derivative, where $v\in V=\{v: v|_{\partial\Omega}=0\}$

$$\begin{cases}a(u,v)=\oint_\Omega\nabla u\cdot\nabla\text{vdxdy}\\f(v)=\oint_\Omega fv\,\text{dxdy}\end{cases}$$

We know the above equation to solve the number of $(M+1)\,(N+1)$ linear equation system. Meanwhile we know the value on the boundary. Therefore we need to calculate the value in the small domain.Let $v_{i,j}=\varphi_{i,j}$, where $1\leqslant i\leqslant M-1, 1\leqslant j\leqslant N-1$:

$$\begin{cases}a(u_h,\varphi_{i,j})=f(\varphi_{i,j}),1\leqslant i\leqslant M-1,1\leqslant j\leqslant N-1\\ \\u_h=u_0(x_i,y_j),(x_i,y_j)\in\partial\Omega\end{cases}$$

Since only few integrals are none zero on Ω,so:

$$\begin{cases}\int_\Omega(\sum_{j=0}^{N}u_{i,j}\nabla\varphi_{ij})\nabla\varphi_{\hat{i}\hat{j}}d\Omega=\int_\Omega f(\varphi_{\hat{i}\hat{j}})d\Omega\\ \\\sum_{m=0}^{\text{MN}-1}\int_{\Omega_m}(\sum_{j=0}^{N}u_{i,j}\nabla\varphi_{ij})\nabla\varphi_{\hat{i}\hat{j}}d\Omega=\sum_{m=0}^{\text{MN}-1}\int_\Omega f(\varphi_{\hat{i}\hat{j}})d\Omega\end{cases}$$

when $(x_i,y_j),(x_{\hat{i}},y_{\hat{j}})$ on the vertice Ω_m, we have $\int_{\Omega_m}\nabla\varphi_{ij}\nabla\varphi_{ij}d\Omega=0$. So:

$$\int_{\Omega_m}\left(\sum_{i=0}^{M}\sum_{j=0}^{N}u_{i,j}\nabla\varphi_{ij}\right)\nabla\varphi_{\hat{i}\hat{j}}d\Omega=\int_{\Omega_m}\sum_{\hat{i}=i-1}^{i+1}\sum_{\hat{j}=j-1}^{j+1}\nabla\varphi_{ij}\nabla\varphi_{\hat{i}\hat{j}}d\Omega$$

So the linear equation system coefficient satisfies:

$$a_{ij}=\sum_{i,j\in T_m}\oint_{T_m}\nabla\varphi_i\nabla\varphi_j\text{dxdy}+\sum_{i,j\in\partial\Omega_m}\oint_{\partial\Omega_m}\beta\varphi_i\varphi_jds$$

Where T_m is the m-th unit,i respresents the $i-$th node in the total MN nodes.Similarly, we can calculate the equation RHS:

$$b_i=\sum_{i\in T_m}\oint_{T_m}f\varphi_i\text{dxdy}+\sum_{i\in\partial\Omega_m}\oint_{\partial\Omega_m}\beta g\varphi_ids$$

I don't know how to use the terminal boundary condtion.

4 Reference

[1] F.Boyer. A Introduction to finite volume method for diffusion problem, 20.

www.math.univ-toulouse.fr/~fboyer/_media/exposes/mexique2013.pdf

[2] An introduction to the numerical solution of PDEs (finite element, finite difference methods).

https://blog.csdn.net/forrestguang/article/details/128067933?spm=1001.2014.3001.5502