Project 1

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1 Problem Statement

Consider following 3D equations defined on $\Omega = (-1,1) \times (-1,1) \times (-1,1)$:

$$\begin{aligned}
\frac{\partial F_1}{\partial x} &= \sigma\left(\frac{1}{6}\sum_{i=1}^{6} F_i - F_1\right)\\
-\frac{\partial F_2}{\partial x} &= \sigma\left(\frac{1}{6}\sum_{i=1}^{6} F_i - F_2\right)\\
\frac{\partial F_3}{\partial y} &= \sigma\left(\frac{1}{6}\sum_{i=1}^{6} F_i - F_3\right)\\
-\frac{\partial F_4}{\partial y} &= \sigma\left(\frac{1}{6}\sum_{i=1}^{6} F_i - F_4\right)\\
\frac{\partial F_5}{\partial z} &= \sigma\left(\frac{1}{6}\sum_{i=1}^{6} F_i - F_5\right)\\
-\frac{\partial F_6}{\partial z} &= \sigma\left(\frac{1}{6}\sum_{i=1}^{6} F_i - F_6\right)
\end{aligned}$$

Where $\sigma = 0.1, 1, 10, 100$ and the boundary conditions are:

$$\begin{aligned}
F_1(1,y,z) = F_b(y,z), F_3(x,-1,z) = F_b(x,z), F_5(x,y,-1) = F_b(x,y)\\
F_2(1,y,z) = F_4(x,1,z) = F_6(x,y,1) = 0
\end{aligned}$$

Where:

$$F_b(p,q) = \begin{cases} 1, & \text{if } |p| \leqslant 0.2 \text{ and } |q| \leqslant 0.2 \\ 0, & \text{otherwise} \end{cases}$$

2 Numerical Scheme

We introduce several different schemes to see the efficiency of the method.Let

$$F^{(h),i,j,k} = \frac{1}{|T^{i,j,k}|}\int_T F(x_i, y_j, z_k)\mathrm{dxdydz}$$

at iteration round h. Then we can apply the following scheme

2.1 Finite Difference Method with Relaxtion Fixed-Point Iteration Scheme

$$\varepsilon\big(F_1^{(h+1),i,j,k}-F_1^{(h),i,j,k}\big)+\frac{F_1^{(h+1),i,j,k}-F_1^{(h+1),i-1,j,k}}{\Delta x}=\sigma\Big(\frac{1}{6}(F_2+F_3+F_4+F_5+F_6)^{(h),i,j,k}-\frac{5}{6}F_1^{(h+1),i,j,k}\Big)$$

$$\varepsilon\big(F_2^{(h+1),i,j,k}-F_2^{(h),i,j,k}\big)-\frac{F_2^{(h+1),i+1,j,k}-F_2^{(h+1),i,j,k}}{\Delta x}=\sigma\Big(\frac{1}{6}(F_1+F_3+F_4+F_5+F_6)^{(h),i,j,k}-\frac{5}{6}F_2^{(h+1),i,j,k}\Big)$$

$$\varepsilon\big(F_3^{(h+1),i,j,k}-F_3^{(h),i,j,k}\big)+\frac{F_2^{(h+1),i,j,k}-F_2^{(h+1),i,j-1,k}}{\Delta y}=\sigma\Big(\frac{1}{6}(F_1+F_2+F_4+F_5+F_6)^{(h),i,j,k}-\frac{5}{6}F_3^{(h+1),i,j,k}\Big)$$

$$\varepsilon\big(F_4^{(h+1),i,j,k}-F_4^{(h),i,j,k}\big)-\frac{F_2^{(h+1),i,j+1,k}-F_2^{(h+1),i,j,k}}{\Delta y}=\sigma\Big(\frac{1}{6}(F_1+F_2+F_3+F_5+F_6)^{(h),i,j,k}-\frac{5}{6}F_4^{(h+1),i,j,k}\Big)$$

$$\varepsilon\big(F_3^{(h+1),i,j,k}-F_3^{(h),i,j,k}\big)+\frac{F_2^{(h+1),i,j,k}-F_2^{(h+1),i,j,k-1}}{\Delta z}=\sigma\Big(\frac{1}{6}(F_1+F_2+F_4+F_5+F_6)^{(h),i,j,k}-\frac{5}{6}F_5^{(h+1),i,j,k}\Big)$$

$$\varepsilon\big(F_4^{(h+1),i,j,k}-F_4^{(h),i,j,k}\big)-\frac{F_2^{(h+1),i,j,k+1}-F_2^{(h+1),i,j,k}}{\Delta y}=\sigma\Big(\frac{1}{6}(F_1+F_2+F_3+F_5+F_6)^{(h),i,j,k}-\frac{5}{6}F_4^{(h+1),i,j,k}\Big)$$

So we have the iteration rule as follows:

$$F_1^{(h+1),i,j,k}=\frac{\varepsilon F_1^{(h),i,j,k}+\frac{F_1^{(h+1),i-1,j,k}}{\Delta x}+\frac{\sigma}{6}(F_2+F_3+F_4+F_5+F_6)^{(h),i,j,k}}{\varepsilon+\frac{1}{\Delta x}+\frac{5}{6}\sigma}$$

$$F_2^{(h+1),i,j,k}=\frac{\varepsilon F_2^{(h),i,j,k}+\frac{F_2^{(h+1),i+1,j,k}}{\Delta x}+\frac{\sigma}{6}(F_1+F_3+F_4+F_5+F_6)^{(h),i,j,k}}{\varepsilon+\frac{1}{\Delta x}+\frac{5}{6}\sigma}$$

$$F_3^{(h+1),i,j,k}=\frac{\varepsilon F_3^{(h),i,j,k}+\frac{F_3^{(h+1),i,j-1,k}}{\Delta y}+\frac{\sigma}{6}(F_1+F_2+F_4+F_5+F_6)^{(h),i,j,k}}{\varepsilon+\frac{1}{\Delta y}+\frac{5}{6}\sigma}$$

$$F_4^{(h+1),i,j,k}=\frac{\varepsilon F_4^{(h),i,j,k}+\frac{F_3^{(h+1),i,j+1,k}}{\Delta y}+\frac{\sigma}{6}(F_1+F_2+F_3+F_5+F_6)^{(h),i,j,k}}{\varepsilon+\frac{1}{\Delta y}+\frac{5}{6}\sigma}$$

$$F_5^{(h+1),i,j,k} = \frac{\varepsilon F_5^{(h),i,j,k} + \frac{F_5^{(h+1),i,j,k-1}}{\Delta z} + \frac{\sigma}{6}(F_1 + F_2 + F_3 + F_4 + F_6)^{(h),i,j,k}}{\varepsilon + \frac{1}{\Delta z} + \frac{5}{6}\sigma}$$

$$F_6^{(h+1),i,j,k} = \frac{\varepsilon F_6^{(h),i,j,k} + \frac{F_5^{(h+1),i,j,k+1}}{\Delta z} + \frac{\sigma}{6}(F_1 + F_2 + F_3 + F_4 + F_5)^{(h),i,j,k}}{\varepsilon + \frac{1}{\Delta z} + \frac{5}{6}\sigma}$$

We can set:$F_m^{(0),i,j,k} = 0, m = 1, \ldots, 6$. And the residual R can be defined as:

$$\begin{aligned} R = &\sum_{i,j,k} \left| \frac{F_1^{i+1,j,k} - F_1^{i,j,k}}{\Delta x} - Q_1 \right| + \sum_{i,j,k} \left| -\frac{F_2^{i,j+1,k} - F_2^{i,j,k}}{\Delta x} - Q_2 \right| + \\ &\sum_{i,j,k} \left| \frac{F_3^{i,j+1,k} - F_3^{i,j,k}}{\Delta y} - Q_3 \right| + \sum_{i,j,k} \left| -\frac{F_4^{i,j+1,k} - F_4^{i,j,k}}{\Delta y} - Q_4 \right| + \\ &\sum_{i,j,k} \left| \frac{F_5^{i,j,k+1} - F_5^{i,j,k}}{\Delta z} - Q_5 \right| + \sum_{i,j,k} \left| -\frac{F_6^{i,j,k+1} - F_6^{i,j,k}}{\Delta z} - Q_6 \right| \end{aligned}$$

Where:

$$Q_m = \sigma \left(\frac{1}{6} \sum_{i=1}^{6} F_i - F_m \right)$$

Meanwhile, to show the method will converge to a stable solution, we calculate the residual between the 2 rounds:

$$\text{Dif} = \sum_{m=1}^{6} \sum_{i,j,k} \left| F_m^{(h+1)} - F_m^{(h)} \right|$$

2.2 Numerical Result

2.2.1 Residual

We set $\sigma = 0.1, 1, 10, 100, \varepsilon = 1.5\sigma$, and stop when residual converges to a value:

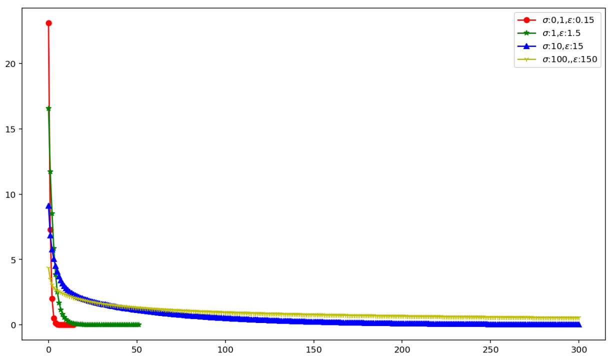

Figure 1. Difference Between Two Rounds

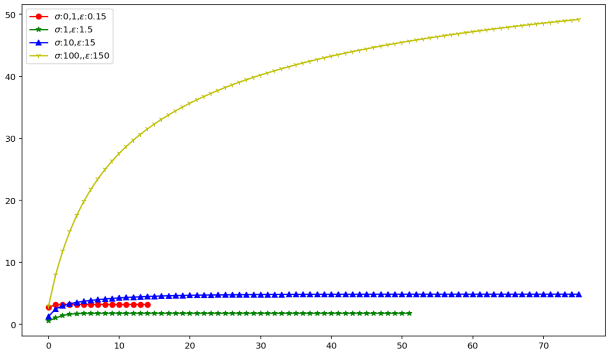

Figure 2. Residual

The 2 images shows the result tend to converge to a stable solution

2.2.2 Solution Contours

The following images display the solution contour of four different σ

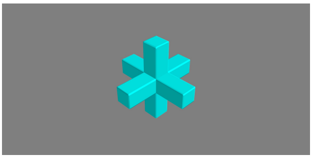

Figure 3. $\sigma = 0.1$

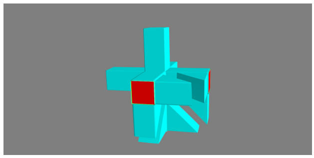

Figure 4. $\sigma = 1$

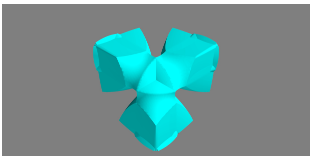

Figure 5. $\sigma = 10$

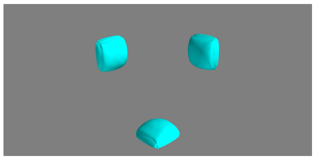

Figure 6. $\sigma = 100$