① Nalezněte partikulární řešení diferenciální rovnice

$$y' = axy + x^3 a\cos\left(\frac{a\pi}{2}\right) - y^3 x a \sin\left(\frac{a\pi}{2}\right) \qquad y(0) = a$$

$a = 1 + 3 = 4$

$y' = 4xy + 4x^3$

$y' - 4xy = 4x^3$

$y' - 4xy = 0$

$\frac{dy}{dx} - 4xy = 0 \quad \Big| \cdot \frac{dx}{y}$

$\frac{1}{y} dy = 4x\, dx$

$\ln|y| = 2x^2 + c$

$y = e^{2x^2 + c}$

$y = e^{2x^2} \cdot e^c \quad | e^c = C$

$y = Ce^{2x^2}$

$y = C(x) \cdot e^{2x^2}$

$(C(x) \cdot e^{2x^2})' - 4x\,C(x)e^{2x^2} = 4x^3$

$C'(x)e^{2x^2} \cancel{+ C(x)4xe^{2x^2}} \cancel{- 4xC(x)e^{2x^2}} = 4x^3$

$C'(x) = \frac{4x^3}{e^{2x^2}}$

$$C(x) = \int C'(x)\,dx = \int 4x^3 \cdot e^{-2x^2}\,dx = \left| \begin{array}{l} x^2 = t \\ 2x\,dx = dt \\ dx = \frac{dt}{2x} \end{array} \right| = \int 2t e^{-2t}\,dt = \left| \begin{array}{ll} u = 2t & u' = 2 \\ v' = e^{-2t} & v = -\frac{1}{2}e^{-2t} \end{array} \right| =$$

$$= -\frac{2te^{-2t}}{2} - \int -e^{-2t}\,dt = -\frac{t}{e^{2t}} - \frac{1}{2e^{2t}} + c = -\frac{x^2}{e^{2x^2}} - \frac{1}{2e^{2x^2}} + c = -\frac{2x^2+1}{2e^{2x^2}} + c$$

$$y(x) = \left(-\frac{2x^2+1}{2e^{2x^2}} + c\right) \cdot e^{2x^2} = -x^2 - \frac{1}{2} + ce^{2x^2}$$

$y(0) = 4$

$-0^2 - \frac{1}{2} + c \cdot e^{2 \cdot 0^2} = 4$

$c = 4{,}5$

$$\underline{\underline{y(x) = -x^2 - \frac{1}{2} + 4{,}5e^{2x^2}}}$$

② Napište obecné řešení lineární diferenciální rovnice

$$y'' - 2ay' + \left(a^2 + 4\sin\frac{a\pi}{2}\right)y = 4e^{(a+2)x}$$

$a = 6$

$y'' - 12y' + 36y = 4e^{8x}$

$\lambda^2 - 12\lambda + 36 = 0$

$(\lambda - 6)(\lambda - 6) = 0$

$\lambda_{1,2} = 6 \;\rightarrow\; y_1 = e^{6x} \;;\; y_2 = xe^{6x}$

$y_H = C_1 e^{6x} + C_2 x e^{6x}$

$4e^{8x}$

$\alpha = 8 \;,\; k = 0$

$Y = e^{8x} \cdot x^0 \cdot A = Ae^{8x}$

$Y' = Ae^{8x} \cdot 8$

$Y'' = 64Ae^{8x}$

$Y'' - 12Y' + 36Y = 4e^{8x}$

$64Ae^{8x} - 96Ae^{8x} + 36Ae^{8x} = 4e^{8x}$

$64A - 96A + 36A = 4$

$4A = 4$

$A = 1 \;\rightarrow\; Y = e^{8x}$

$y = y_H + Y$

$\underline{\underline{y = C_1 e^{6x} + C_2 x e^{6x} + e^{8x}}}$

③ Určete parametr p tak, aby funkce $u(x,y)$ byla reálnou částí vhodné holomorfní funkce $f(z)$. Dále určete derivaci $f'(z)$ a možné funkce $f(z)$.

$$u(x,y) = \sin(ax) \cdot \left(e^{py} + \cos\left(\frac{a\pi}{2}\right)e^{-py}\right)$$

$a = 7$

$$u(x,y) = \sin(7x) \cdot e^{py}$$

$u'_x = 7e^{py}\cos(7x)$

$u''_{xx} = -49e^{py}\sin(7x)$

$u'_y = pe^{py}\sin(7x)$

$u''_{yy} = p^2e^{py}\sin(7x)$

$u''_{xx} + u''_{yy} = 0$

$-49e^{py}\sin(7x) + p^2e^{py}\sin(7x) = 0$

$p^2 - 49 = 0$

$p = \sqrt{49}$

$p = \pm 7$

Pro $p = 7$:

$u'_x = v'_y = 7e^{7y}\cos(7x)$

$u'_y = -v'_x = -7e^{7y}\sin(7x)$

$v = \int v'_x\,dx = \int -7e^{7y}\sin(7x) = -7e^{7y}\int \sin 7x\,dx =$

$= \left| \begin{matrix} 7x = t \\ dx = \frac{dt}{7} \end{matrix} \right| = -e^{7y}\int \sin t\,dt = e^{7y}\cos(7x) + h(y)$

$v'_y = v'$

$\cancel{7e^{7y}\cos(7x)} = \cancel{7e^{7y}\cos(7x)} + h'_y(y)$

$h'_y(y) = 0$

$h(y) = \int 0\,dy = C$

$v(x,y) = e^{7y}\cos(7x) + C$

$f(x+jy) = u(x,y) + jv(x,y) = e^{7y}\sin(7x) + j(e^{7y}\cos(7x) + C)$

$x = z;\ y = 0$

$\underline{f(z) = \sin(7z) + j(\cos(7z) + C)}$

$\underline{f'(z) = 7\cos(7z) - 7j\sin(7z)}$

$$u'_x = v'_y = 7e^{-7y}\cos(7x)$$

$$u'_y = -v'_x = -7e^{-7y}\sin(7x)$$

$$v = \int v'_x\,dx = \int 7e^{-7y}\sin(7x)\,dx = -7e^{-7y}\int \sin(7x)\,dx = \left| \begin{array}{l} 7x = t \\ dx = \frac{dt}{7} \end{array} \right| = e^{-7y}\int \sin(t)\,dt =$$

$$= -e^{-7y}\cos(7x) + h(y)$$

$$v'_y = v'_?$$

$$\cancel{7e^{-7y}\cos(7x)} = \cancel{7e^{-7y}\cos(7x)} + h'_y(y)$$

$$h'_y(y) = 0 \qquad v(x,y) = -e^{7y}\cos(7x) + C$$

$$h'_y(y) = \int 0\,dy = C$$

$$f(x+jy) = u(x,y) + jv(x,y) = e^{-7y}\sin(7x) + j(-\cos(7x)e^{-7y} + C)$$

$$x = z;\ y = 0$$

$$\underline{\underline{f(z) = \sin(7z) + j(C - \cos(7z))}}$$

$$\underline{\underline{f'_{(z)} = 7\cos(7z) + 7j\sin(7z)}}$$

④ Vypočtěte integrál $\int_{\Gamma} f(z)\,dz$

$$f(z) = z\sin z \qquad \Gamma(t) = t\pi e^{j\pi t} \qquad t \in \langle 0; a \rangle$$

$$a = 12 \rightarrow t \in \langle 0; 12 \rangle$$

$$f(x+jy) = \underbrace{(x+jy)(\sin(x+jy)}_{j v(x,y)}$$

$$v'_x = \sin(x+jy) + (x+jy)\cos(x+jy) = -u'_y$$

$$v'_y = j\sin(x+jy) + (x+jy)\cos(x+jy)j = u'_x$$

$$u''_{yy} = -\cos(x+jy)j - j\cos(x+jy) + (x+jy)\sin(x+jy)j$$

$$u''_{xx} = j\cos(x+jy) + \cos(x+jy)j - (x+jy)\sin(x+jy)j$$

$$u''_{xx} + u''_{yy} = 0$$

$$\cancel{-\cos(x+jy)j} - \cancel{j\cos(x+jy)} + \cancel{(x+jy)\sin(x+jy)j} + \cancel{j\cos(x+jy)} + \cancel{\cos(x+jy)j} - \cancel{(x+jy)\sin(x+jy)j} = 0$$

$0 = 0 \rightarrow$ funkce je holomorfní

$$f(z) = z\sin z$$

$$F(z) = \int z\sin z\,dz = \begin{vmatrix} u = z & u' = 1 \\ v' = \sin z & v = -\cos z \end{vmatrix} = -z\cos z - \int -\cos z\,dz = -z\cos z + \sin z$$

$$t \in \langle 0; 12 \rangle$$

$$z_1 = 0$$

$$z_2 = 12\pi e^{12j\pi}$$

$$\int_{\Gamma} z\sin z\,dz = \Big[-z\cos z + \sin z\Big]_0^{12\pi e^{12j\pi}} = \left(-12\pi e^{12j\pi}\cdot\cos(12\pi e^{12j\pi}) + \sin(12\pi e^{12j\pi})\right) - \sin(0) =$$

$$= \sin(12\pi e^{12j\pi}) - 12\pi e^{12j\pi}\cos(12\pi e^{12j\pi}) =$$

$$e^{12j\pi} = (-1)^{12} = 1$$

$$= \sin(12\pi) - 12\pi\cos(12\pi) = 0 - 12\pi\cdot 1 = \underline{\underline{-12\pi}}$$

⑤ Vypočtěte $\oint_{\Pi} f(z)\,dz$, přes kladně orientovanou křivku Π.

$$f(z) = \frac{1}{z^{1-(-1)^a} \cdot (z^2 - ja^2)(z^2 + (-1)^a a^2)} \qquad \Pi: |z-a| = a\cdot\sqrt{3}$$

$$a = 10 \qquad f(z) = \frac{1}{(z^2 - 100j)(z^2+100)} \qquad \Pi: |z-10| = 10\sqrt{3}$$

Singulární body: $z^2 + 100 = 0$ $\quad z_{1,2} = \pm 10j$

$z^2 - 100j = 0$ $\quad z = 10\sqrt{j}$ $\quad \sqrt{j} = \pm\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)$

$z_{1,2} = \pm(5\sqrt{2} + 5j\sqrt{2})$

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$$f(z) = \frac{1}{(z-5\sqrt{2}-5j\sqrt{2})(z+5\sqrt{2}+5j\sqrt{2})(z-10j)(z+10j)}$$

$$\operatorname*{res}_{z=10j} f(z) = \lim_{z\to 10j} \cancel{(z-10j)} \cdot \frac{1}{(z-5\sqrt{2}-5j\sqrt{2})(z+5\sqrt{2}+5j\sqrt{2})\cancel{(z-10j)}(z+10j)} =$$

$$= \frac{1}{(10j-5\sqrt{2}-5j\sqrt{2})(10j+5\sqrt{2}+5j\sqrt{2})(20j)} = \frac{1}{4000} + \frac{1}{4000}j$$

$$\operatorname*{res}_{z=-10j} f(z) = \frac{1}{(-10j-5\sqrt{2}-5j\sqrt{2})(-10j+5\sqrt{2}+5j\sqrt{2})(-20j)} = -\frac{1}{4000} - \frac{1}{4000}j$$

$$\operatorname*{res}_{z=5\sqrt{2}+5j\sqrt{2}} f(z) = \frac{1}{(5\sqrt{2}+5j\sqrt{2}+5\sqrt{2}+5j\sqrt{2})(5\sqrt{2}+5j\sqrt{2}-10j)(5\sqrt{2}+5j\sqrt{2}+10j)} = \frac{1}{(10\sqrt{2}+10j\sqrt{2})(100+100j)} =$$

$$= \frac{1}{1000\sqrt{2}+1000j\sqrt{2}+1000j\sqrt{2}-1000\sqrt{2}} = \frac{1}{2000j\sqrt{2}}$$

$$\operatorname*{res}_{z=-5\sqrt{2}-5j\sqrt{2}} f(z) = \frac{1}{(-5\sqrt{2}-5j\sqrt{2}-5\sqrt{2}-5j\sqrt{2})(-5\sqrt{2}-5j\sqrt{2}+10j)(-5\sqrt{2}-5j\sqrt{2}-10j)} = \frac{1}{(-10\sqrt{2}-10j\sqrt{2})(100+100j)} =$$

$$= \frac{1}{-1000\sqrt{2}-1000j\sqrt{2}-1000j\sqrt{2}+1000\sqrt{2}} = -\frac{1}{2000j\sqrt{2}}$$

$\Gamma: \; |z-10| = 10\sqrt{3}$

$z_1 = -10j$

$|-10j-10| = 10\sqrt{2} < 10\sqrt{3} \rightarrow$ leží uvnitř křivky

$z_2 = 10j$

$|10j-10| = 10\sqrt{2} < 10\sqrt{3} \rightarrow$ leží uvnitř křivky

$z_3 = 5\sqrt{2} + 5j\sqrt{2}$

$|5\sqrt{2} + 5j\sqrt{2} - 10| \doteq 7{,}65 < 10\sqrt{3} \rightarrow$ leží uvnitř křivky

$z_4 = -5\sqrt{2} - 5j\sqrt{2}$

$|-5\sqrt{2} - 5j\sqrt{2} - 10| \doteq 18{,}5 > 10\sqrt{3} \rightarrow$ leží vně křivky

$$\int_\Gamma f(z)\,dz = 2\pi j\left(\operatorname*{res}_{z=-10j} f(z) + \operatorname*{res}_{z=10j} f(z) + \operatorname*{res}_{z=5\sqrt{2}+5j\sqrt{2}} f(z)\right) = 2\pi j\left(\cancel{-\frac{1}{4000}} \cancel{-\frac{1}{4000j}} \cancel{+\frac{1}{4000}} \cancel{+\frac{1}{4000}j} + \frac{1}{2000j\sqrt{2}}\right)$$

$$= \frac{2\pi j}{2000j\sqrt{2}} = \frac{\pi}{1000\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \underline{\underline{\frac{\pi\sqrt{2}}{2000}}}$$

(6) Pomocí Laplaceovy transformace vyřešte:

$$y'(t) + 2(a-1)y(t) + \left((a-1)^2 + \sin\left(\frac{(a-1)\pi}{2}\right)\right)\int_0^t y(s)\,ds = 2e^{-at} \qquad y(0) = -a\sin\left(\frac{(a-1)\pi}{2}\right)$$

$a = 5$

$$y'(t) + 8y(t) + 16\int_0^t y(s)\,ds = 2e^{-5t} \qquad y(0) = 0$$

$$\mathcal{L}\{y'(t)\} = pF(p)$$

$$\mathcal{L}\{8y(t)\} = 8F(p)$$

$$\mathcal{L}\left\{16\int_0^t y(s)\,ds\right\} = 16\,\frac{F(p)}{p}$$

$$\mathcal{L}\{2e^{-5t}\} = \frac{2}{p+5}$$

$$pF(p) + 8F(p) + 16\,\frac{F(p)}{p} = \frac{2}{p+5}$$

$$F(p)\cdot\left(p + 8 + \frac{16}{p}\right) = \frac{2}{p+5}$$

$$F(p) = \frac{2}{p+5}\cdot\frac{1}{\underbrace{p+\frac{16}{p}+8}} = \frac{2p}{(p+5)(p+4)(p+4)}$$

$$\frac{p^2}{p} + \frac{16}{p} + \frac{8p}{p} = \frac{1}{p}\cdot(p+4)^2$$

$$\frac{2p}{(p+5)(p+4)(p+4)} = \frac{A}{p+5} + \frac{B}{p+4} + \frac{C}{(p+4)^2}$$

$$2p = A(p^2+8p+16) + B(p^2+9p+20) + C(p+5)$$

$$2p = Ap^2 + A8p + A16 + Bp^2 + B9p + B20 + Cp + C5$$

$A + B = 0 \rightarrow A = -B$

$8A + 9B + C = 2 \qquad A = -10$

$16A + 20B + 5C = 0$

$B + C = 2 \rightarrow B = 2 - C$

$B = 10$

$4B + 5C = 0$

$8 - 4C + 5C = 0$

$C = -8$

$$F(p) = -\frac{10}{p+5} + \frac{10}{p+4} - \frac{8}{(p+4)^2} = -10\,\frac{1}{p+5} + 10\,\frac{1}{p+4} - 8\,\frac{1}{(p+4)^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{-10\,\frac{1}{p+5}\right\} + \mathcal{L}^{-1}\left\{10\,\frac{1}{p+4}\right\} + \mathcal{L}^{-1}\left\{-8\,\frac{1}{(p+4)^2}\right\}$$

$$\underline{\underline{y(t) = -10e^{-5t} + 10e^{-4t} - 8te^{-4t}}}$$