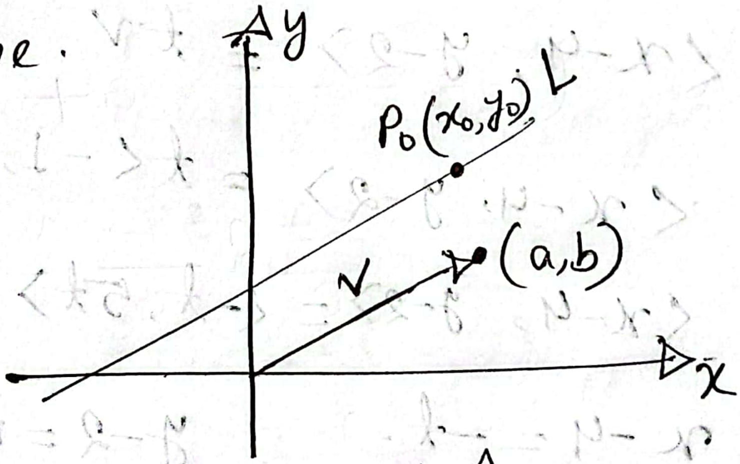
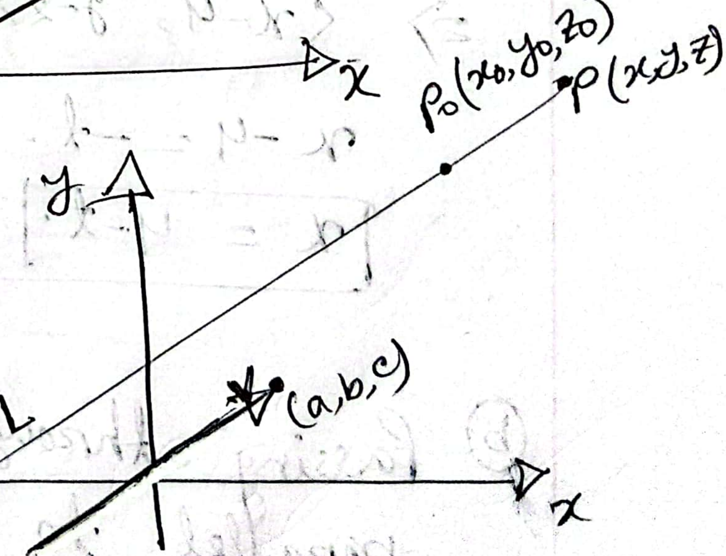
Chapter 11.5.

Lines determined by a point and a vector.

A Line in 2-space or 3-space can be determined uniquely by specifying a point on the line and a non-zero vector parallel to the line.





$\overrightarrow{P_0P} = tv.$

$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle$

$\Rightarrow \langle x - x_0, y - y_0, z - z_0 \rangle = \langle ta, tb, tc \rangle$

$x - x_0 = ta$	$y - y_0 = tb.$	$z - z_0 = tc$
$x = x_0 + at$	$y = y_0 + tb$	$z = z_0 + tc.$

Example 1. Find parametric equations of the line

(a) Passing through (4,2) and parallel to $v\langle -1,5\rangle$

$$\langle x-4, y-2\rangle = tv.$$

$$\Rightarrow \langle x-4, y-2\rangle = t\langle -1,5\rangle.$$

$$\Rightarrow \langle x-4, y-2\rangle = \langle -t, 5t\rangle$$

$x-4=-t.$ $\qquad$ $y-2=5t.$

$\boxed{x = 4-t.}$ $\qquad$ $\boxed{y = 5t+2}$

(b) Passing through (1, 2, -3) and parallel to $v = 4i+5j-7k.$

$\langle x-1, y-2, z+3\rangle = t\langle 4, 5, -7\rangle$.

$\Rightarrow \langle x-1, y-2, z+3\rangle = \langle 4t, 5t, -7t\rangle$

$x-1 = 4t$ | $y-2 = 5t$. | $z+3 = -7t$.
$\boxed{x = 1+4t.}$ | $\boxed{y = 2+5t}$ | $\boxed{z = -7t-3.}$

(c) Passing through the origin in 3-space and parallel to $v = \langle 1, 1, 1\rangle$.

$\langle x-0, y-0, z-0\rangle = \langle t, t, t\rangle$

$x-0 = t$. $y-0 = t$. $z-0 = t$.
$\boxed{x = t}$ $\boxed{y = t}$ $\boxed{z = t.}$

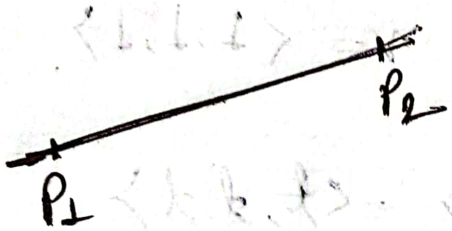
Example: 2

(a) Find parametric eqns of the line L passing through the points $P_1(2,4,-1)$ and $P_2(5,0,7)$.

$$\overrightarrow{P_1P_2} \equiv \langle 5-2,\ 0-4,\ 7+1 \rangle$$

$= \langle 3,-4,8 \rangle$ is parallel to L.

and the point $P_1(2,4,-1)$ lies on L.



$$\cancel{\langle x-3,\ y+4,\ z-8 \rangle = 2}$$

$$\langle x-2,\ y-4,\ z+1 \rangle = t\langle 3,-4,8 \rangle$$

$$\langle x-2,\ y-4,\ z+1 \rangle = \langle 3t,\ -4t,\ 8t \rangle$$

$x-2 = 3t$ $\quad$ $y-4 = -4t$ $\quad$ $z+1 = 8t$

$\boxed{x = 2+3t}$ $\quad$ $\boxed{y = 4-4t}$ $\quad$ $\boxed{z = 8t-1}$

(b) Where does the line intersect the xy-plane.?

intersects the ~~line~~ xy plane at the point $z=0$

$8t-1=0.$

$t=\frac{1}{8}.$

Substituting the value of t in the

$(x,y,z)=\left(\frac{19}{8}, \frac{7}{2}, 0\right).$

Example 3

let L_1 and L_2 be the lines

$L_1: x=1+4t, \quad y=5-4t \quad z=-1+5t.$

$L_2: x=2+8t, \quad y=4-3t, \quad z=5+t.$

a) Are the lines parallel?

For L_1 the parallel vector is $\langle 4, -4, 5\rangle$

For L_2 the parallel vector is $\langle 8, -3, 1\rangle$

This lines are not parallel since neither is a scalar multiple of the other.

(b) Do the lines intersects?

$x_0 = 1 + 4t_1$, $y_0 = 5 - 4t_1$, $z_0 = -1 + 5t_1$.

and

$x_0 = 2 + 8t_2$, $y_0 = 4 - 3t_2$, $z_0 = 5 + t_2$.

This leads to three conditions on t_1 and t_2.

$$1 + 4t_1 = 2 + 8t_2$$

$$5 - 4t_1 = 4 - 3t_2$$

$$-1 + 5t_1 = 5 + t_2$$

The lines intersect if there are values of t_1 and t_2 that satisfy all three equations, and the lines don't intersect if there are no such values.

$$t_2 = 0, \; t_1 = \frac{1}{4}$$

don't satisfy the 3rd eqn, so the lines don't intersect.

Example 4.

Find parametric equations describing the line segment joining the point $P_1(2,4,-1)$ and $P_2(5,0,7)$.

$\overrightarrow{P_1P_2} = \langle 5-2,\ 0-4,\ 7+1 \rangle = \langle 3,-4,8 \rangle.$

$\langle x-2,\ y-4,\ z+1 \rangle = t\langle 3,-4,8 \rangle.$

$\Rightarrow \langle x-2,\ y-4,\ z+1 \rangle = \langle 3t, -4t, 8t \rangle$

$x-2 = 3t.$ $\quad y-4 = -4t.$ $\quad z+1 = 8t.$

$\boxed{x = 2+3t}$ $\quad \boxed{y = 4-4t.}$ $\quad z = -1+8t$

Example 5.

The equation

$$\overset{x}{\langle x, y, z \rangle} = \overset{x_0}{\langle -1, 0, 2 \rangle} + \overset{vt.}{t\langle 1, 5, -4 \rangle}.$$

13. If L_1 and L_2 intersect at a point (x,y,z) - then there exists a single value of t such that,

$L_0: x = x_0 + a_0 t$, $y = y_0 + b_0 t$, $z = z_0 + c_0 t$.

$L_1: x = x_1 + a_1 t$, $y = y_1 + b_1 t$, $z = z_1 + c_1 t$.

are satisfied :

$\underbrace{(x_0, y_0, z_0)}_{}^{P_0}$ $\quad$ $\underbrace{(a_0, b_0, c_0)}_{}^{v}$.

15. The line through $(-5,2)$ that is parallel to $2i - 3j$.

$\cancel{(x+5)}$

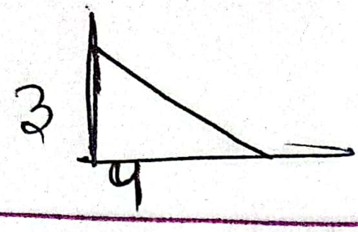
$\langle x+5, y-2 \rangle = t\langle 2, -3 \rangle$.

$\langle x+5, y-2 \rangle = \langle 2t, -3t \rangle$.

$\boxed{x+5 = 2t.}$ $\quad$ $y - 2 = -3t$.

$x = 2t - 5$. $\quad$ $\boxed{y = 2 - 3t.}$

CamScanner



17. The line that is tangent to the cicle $x^2+y^2=25$ at the point $(3,-4)$.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{-3}{-4} = \frac{3}{4}$$

$$v = 4i + 3j$$

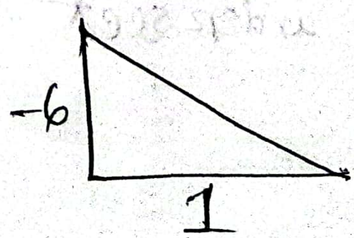
$$\boxed{x = 3 + 4t}$$

$$\boxed{y = -4 + 3t}$$

18. The line that is tangent to the parabola $y=x^2$ at the point $(-2,4)$.

$$\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{-6}{1}$$



$$\vec{v} = i - 6j$$

$$\boxed{x = -2 + t} \quad \boxed{y = 4 - 6t}$$

19. The line through $(-1,2,4)$ that is parallel to $2i-4j+k$.

$$\boxed{x=-1+2t.}$$

$$\boxed{y=2-4t.}$$

$$\boxed{z=4+t}$$

20. The line through $(2,-1,5)$ that is parallel to $\langle -1,2,7\rangle$.

$$\boxed{x=2-t.}$$

$$\boxed{y=-1+2t.}$$

$$\boxed{z=5+7t}.$$

23. Where does the line $x=1+3t,\ y=2-t$ intersect

23. Where does the line $x = 1+3t,\ y = 2-t$ intersect

ⓐ the x-axis

$y = 0$

$2 - t = 0$

$t = 2.$

$x = 1 + 3 \times 2 = 7.$

$y = 2 - 2 = 0.$ $(7, 0).$

ⓑ ~~1+3t~~ the y-axis

$1 + 3t = 0.$

$3t = -1.$

$t = -\frac{1}{3}.$

$x = \left(0, \frac{7}{3}\right)$

27. Where does the line $\langle x, y\rangle = \langle 4t, 3t\rangle$ intersect the circle $x^2+y^2=25$..

$$(4t)^2 + (3t)^2 = 25.$$

$$25t^2 = 25.$$

$$t^2 = 1.$$

$\boxed{t = \pm 1.}$ the line intersects the circle at $\pm(4,3)$.

29. Show that the lines L_1 and L_2 intersect, and find their point of intersect.

L_1: $x = 2+t$, $y = 2+3t$, $z = 3+t$.

L_2: $x = 2+t$, $y = 3+4t$, $z = 4+2t$.

$$2+3t = 3+4t.$$

$$3+t = 4+2t.$$

$t = -1$ satisfy both eqn. so the lines intersect at point $(x,y,z) = (2+t,\ 2+3t,\ 3+t)$
$= (1, -1, 2)$.

35-36: Determine whether the points P_1, P_2 and P_3 lie on the same line.

35. $P_1(6, 9, 7)$, $P_2(9, 2, 0)$, $P_3(0, -5, -3)$.

$$\overrightarrow{P_1P_2} = \langle 9-6,\ 2-9,\ 0-7 \rangle$$

$$= \langle 3, -7, -7 \rangle$$

$$\overrightarrow{P_2P_3} = \langle 0-9,\ -5-2,\ -3-0 \rangle$$

$$= \langle -9, -7, -3 \rangle$$

These vectors are not parallel so the points do not lie on the same line.

36. $\overrightarrow{P_1P_2} = \langle 2,-4,-4\rangle$, $\overrightarrow{P_2P_3} = \langle 1,-2,-2\rangle$

$\overrightarrow{P_1P_2} = 2\overrightarrow{P_2P_3}$

so the vectors are parallel and the points lie on the same line.

37. Show that the lines L_1 and L_2 are the same.

$L_1 : x = 3-t,\ y = 1+2t.$

$L_2 : x = -1+3t,\ y = 9-6t.$

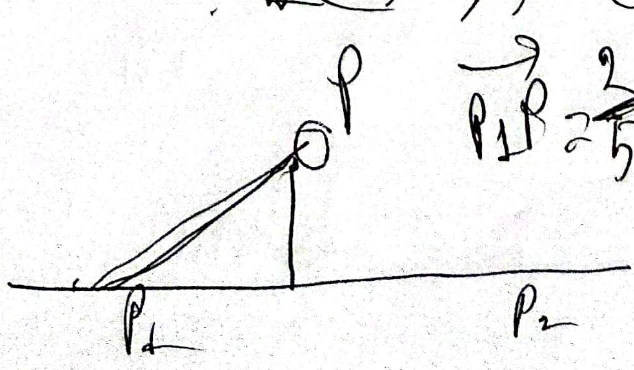
bad mein karungi.

45. $P_1(3,6),\ P_2(8,-4)$ $x - x_0 = \frac{2}{5}\cdot 5$

$x - x_0 = 2$

$\overrightarrow{P_1P} = \frac{2}{5}\overrightarrow{P_1P_2}\langle 8-3, -10\rangle$

$\langle 5,-10\rangle$.



46. Find the point on the line segment joining $P_1(1,4,-3)$ and $P_2(1,5,-1)$ that is $\frac{2}{3}$ of the way from P_1 to P_2.

$\overrightarrow{P_1P_2} = \langle 1-1, 5-4, -1+3 \rangle$

$= \langle 0, 1, 2 \rangle$

$\overrightarrow{P_1P} = \langle x-1, y-4, z+3 \rangle$

$\overrightarrow{P_1P} = \frac{2}{3} \overrightarrow{P_1P_2}$

$\langle x-1, y-4, z+3 \rangle = \frac{2}{3} \langle 0, 1, 2 \rangle$

$\Rightarrow \langle x-1, y-4, z+3 \rangle = \langle 0, \frac{2}{3}, \frac{4}{3} \rangle$

$x-1=0$, $x=1$ | $y-4=\frac{2}{3}$, $y=\frac{2}{3}+4=\frac{14}{3}$ | $z+3=\frac{4}{3}$, $z=\frac{4}{3}-3$, $z=\frac{4-9}{3}=-\frac{5}{3}$

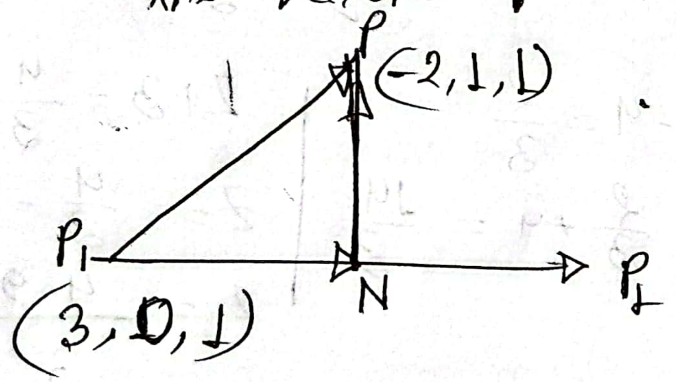
The point $(1, \frac{14}{3}, -\frac{5}{3})$.

47-48 Use the method in exercise 32 of section 11.3 to find the distance from (orthogonal) the point P to the line L, and then check your answer using the method in exercise 30 of section 11.4.

(47) $P(-2,1,1)$

$L: x=3-t,\ y=t,\ z=1+2t.$

the vector $v=\langle -1,1,2\rangle$



$\overrightarrow{P_1P} = \langle -2-3,1-0,1-1\rangle = \langle -5,1,0\rangle.$

(V.e) $\text{Proj}_{\overrightarrow{P_1P_2}}\overrightarrow{P_1P} = \langle -5,1,0\rangle \cdot \dfrac{\langle -1,1,2\rangle}{\left(\sqrt{1^2+1^2+2^2}\right)^2} \cdot \langle -1,1,2\rangle.$

$= \dfrac{5+1}{\sqrt{6}} \langle -1,1,2\rangle.$

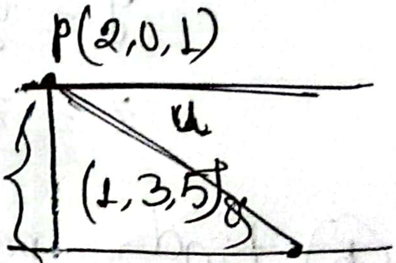
$\|\overrightarrow{PP_1} - \text{Proj}_{\overrightarrow{P_1P_2}}\overrightarrow{P_1P}\| = PN$

49 - 50. 49, Show that the lines L_1 and L_2 are parallel, and find the distance between them.

$L_1 : x = 2 - t,\ y = 2t,\ z = 1 + t.$ $P(2, 0, 1)$

$L_2 : x = 1 + 2t,\ y = 3 - 4t,\ z = 5 - 2t.$ $Q(1, 3, 5)$.

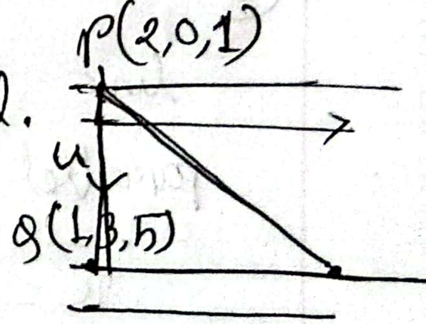
$V_1 = \langle -1, 2, 1 \rangle$.



$V_2 = \langle 2, -4, -2 \rangle$.

$V_2 = -2V_1$.

so V_1 and V_2 are parallel.

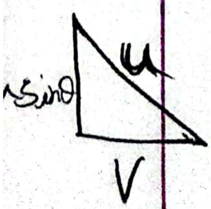


$\overrightarrow{PQ} = \langle 1 - 2,\ 3 - 0,\ 5 - 1 \rangle$.

$u = \langle -1, 3, 4 \rangle$.

$V = \frac{1}{2} \vec{V_2} = \frac{1}{2} \langle 2, -4, -2 \rangle$

$= \langle 1, -2, -1 \rangle$.



~~u×v~~ $d = u \sin\theta$

$= \frac{\|\vec{u}\| \sin\theta \|\vec{v}\|}{\|v\|} = \frac{\|u \times v\|}{\|v\|}$

51. a) Find parametric eqns for the line through the points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$.

vector $\vec{PQ} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$x = x_0 + t(x_1 - x_0)$

$y = y_0 + t(y_1 - y_0)$

$z = z_0 + t(z_1 - z_0)$.

(b) Find parametric equations for the line through the point (x_1, y_1, z_1) and parallel to the line.

$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$.

vector $\langle a, b, c \rangle$

$$\boxed{\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}} \longrightarrow \text{Parametric eqn.}$$

CS CamScanner

52. $t = t = t$.

53. a) Describe the line whose symmetric equations are

$$\frac{x-1}{2} = \frac{y+3}{4} = \frac{z-5}{1}.$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

vector $\langle a,b,c\rangle = \langle 2,4,1\rangle$.

point $(x_0,y_0,z_0) = (1,-3,-5)$.

55. Let L_1 and L_2 be the lines whose parametric eqns are

L_1: $x = 1+2t$, $y = 2-t$, $z = 4-2t$ $\Rightarrow \vec{u}$

L_2: $x = 9+t$, $y = 5+3t$, $z = -4-t$. $\Rightarrow \vec{v}$

(b) acute angle between L_1 and L_2

$\vec{u} = \langle 2,-1,-2\rangle$. $\quad \vec{u}\cdot\vec{v} = |u||v|\cos\theta$.

$\vec{v} = \langle 1,3,-1\rangle$.

⊙ Perpendicular with the two lines

$u \times v = 7i + 7k$ and hence so is $\boxed{i+k}$

passes through the point $(7,-1,-2)$.

$x = x_0 + at$.

$x = 7 + 7t$.

$y = -1$.

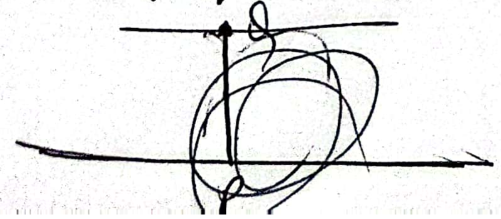
$z = -2 + 7t$.

57-58: Find parametric eqns of the line that contains the point P and intersects the line L at a right angle and find the distance between P and L.

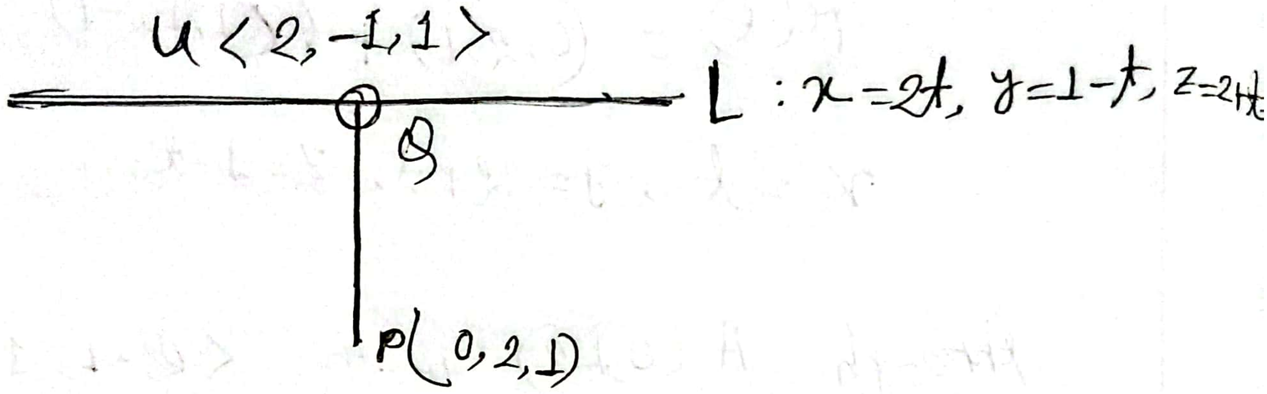
57. $P(0,1,2)$ $u = j - k$

$L: x = 2t,\ y = 1 - t,\ z = 2 + t$.

vector, $\vec{v} = \langle 2,-1,1 \rangle$ $(0,1,2)$ $Q(0,1,2)$.



57.



$L: x = 2t,\ y = 1 - t,\ z = 2 + t.$

$(0, 1, 2)$ vector $u(2, -1, 1)$

$Q \quad L: x = 0 + 2t,\ y = 1 - t,\ z = 2 + t.$

$Q(2t, 1-t, 2+t).$

$\vec{QP} = P - Q = (0 - 2t,\ 2 - 1 + t,\ 1 - 2 - t)$

$= (-2t, 1+t, -1-t).$

$\vec{QP} \cdot u = 0.$

$(-2t, 1+t, -1-t) \cdot (2, -1, 1) = 0.$

$t = -\frac{1}{3}$

$x = 2t = -\frac{2}{3}$.

$y = 1 - t = 1 - \left(-\frac{1}{3}\right) = \frac{4}{3}$

$z = 2 + t = 2 - \frac{1}{3} = \frac{5}{3}$.

$Q\left(-\frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$.

$P(0, 2, 1)$.

$\vec{PQ} = \langle -\frac{2}{3}, \frac{4}{3} - 2, \frac{5}{3} - 1 \rangle$

$\vec{PQ} = \langle -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$.

$= \langle -1, -1, 1 \rangle$

$\|\vec{PQ}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$

$= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}}$

$= \sqrt{\frac{12}{9}} = \frac{2\sqrt{3}}{3}$.