Unit 12.2

$$\lim_{t \to a} r(t) = L \quad \text{--- (1)}$$

what it means for a vector-valued function $r(t)$ in 2-space or 3-space to approach a limiting vector L

$$\lim_{t \to a} r(t) = L$$

definition: is to position $r(t)$ and L with their initial points at the origin and interpret this limit to mean that the terminal point of $r(t)$ approaches the terminal point of L as t approaches a. ~~or~~

OR
The vector $r(t)$ approaches the vector L in both length and direction at t approaches a.

$$\lim_{t \to a} \| r(t) - L \| = 0$$

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$\boxed{\text{+}}$ $r(t) = \langle x(t), y(t) \rangle$

$r(t) = x(t)i + y(t)j$

$$\lim_{t\to a} r(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t) \rangle$$

$$= \lim_{t\to a} x(t)\, i + \lim_{t\to a} y(t)\, j.$$

$\boxed{\text{+}}$ $r(t) = \langle x(t), y(t), z(t) \rangle$

$= x(t)i + y(t)j + z(t)k.$ then

$$\lim_{t\to a} r(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle$$

$$\lim_{t\to a} r(t) = \lim_{t\to a} x(t)\, i + \lim_{t\to a} y(t)\, j + \lim_{t\to a} z(t)\, k$$

Example 1

Let $r(t) = t^2 i + e^t j - (2\cos \pi t)k$.

$$\lim_{t\to a} r(t) = \left(\lim_{t\to a} t^2\right)i + \left(\lim_{t\to a} e^t\right)j - \left(\lim_{t\to a} 2\cos\pi t\right)k$$

$$\lim_{t\to a} r(t) = 0^2 \cdot i + e^0 j - 2k.$$

$$= j - 2k.$$

Notation for vectors.

$$\lim_{t\to 0} r(t) = \lim_{t\to 0} \langle t^2, e^t, -2\cos\pi t \rangle$$

$$= \langle \lim_{t\to 0} t^2, \lim_{t\to 0} e^t, \lim_{t\to 0} (-2\cos\pi t) \rangle$$

$$= \langle 0, 1, -2 \rangle.$$

$r(t)$ to be continious at $t = a$ if

$$\lim_{t\to a} r(t) = r(a).$$

we can say $r(t)$ is continious on an interval I if it is continious at each point of I.

If $r(t)$ is a vector-valued function we define the <u>derivative</u> of r with respect to t.

$$r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}.$$

$r(t) = x(t)\, i + y(t)\, j.$

$$r'(t) = \frac{r(t+h) - r(t)}{h}$$

$$= \lim_{h \to 0} \frac{[x(t+h)\, i + y(t+h)\, j] - [x(t)\, i + y(t)\, j]}{h}$$

$$= \left(\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \right) i + \left(\lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \right) j$$

$$r'(t) = x'(t)\, i + y'(t)\, j$$

$$\boxed{\begin{aligned} \text{so, } \; r(t) &= x(t)\, i + y(t)\, j \\ r'(t) &= x'(t)\, i + y'(t)\, j. \end{aligned}}$$

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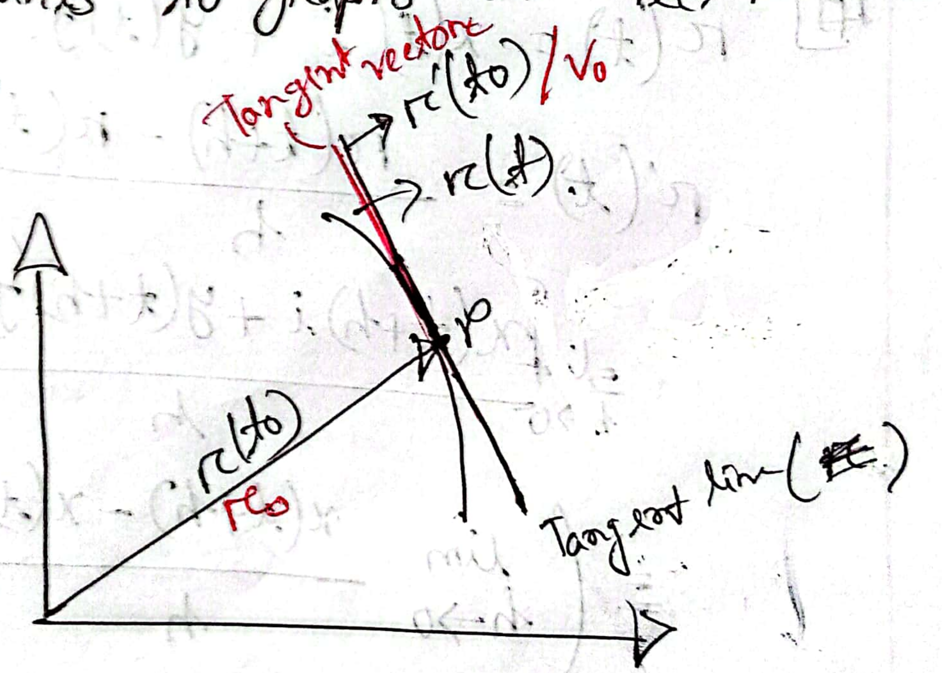
$$\frac{d}{dt}\cos t = -\sin t$$

Example Let $r(t) = t^2 i + e^t j - (2\cos\pi t)k$.

$$r'(t) = \frac{d}{dt}(t^2)i + \frac{d}{dt}(e^t)j - \frac{d}{dt}(2\cos\pi t)k$$

$$= 2t i + e^t j + (2\pi\cos\pi t)k.$$

Tangent lines to graphs of vector-valued functions



The tangent line to the graph of $r(t)$ at r_0 is given by the vector equation of the tangent line

$$\boxed{r = r_0 + t v_0}$$

Example 3

Find parametric eqns of the tangent line to the circular helix

$x = \cos t,\ y = \sin t,\ z = t$

where $t = t_0$, and use that result to find parametric eqns for the tangent line at the point where $t = \pi$.

The vector eqn of the helix is

$$r(t) = \cos t\, i + \sin t\, j + t k.$$

so we have

~~$r_{t_0} = r_0(t) = \cos$~~

$$r_0 = r(t_0) = \cos t_0\, i + \sin t_0\, j + t_0 k.$$

$$v_0 = r'(t_0) = (-\sin t_0)\, i + (\cos t_0)\, j + k.$$

vector eqn of the tangent line is

$$r = r_0 + t v_0$$

$$= \cos t_0\, i + \sin t_0\, j + t_0 k + t[(-\sin t_0) i + (\cos t_0) j + k]$$

$$= (\cos t_0 - t\sin t_0) i + (\sin t_0 + t\cos t_0) j + (t_0 + t) k.$$

Thus the parametric eqns of the tangent line at $t = t_0$ are

$x = \cos t_0 - t\sin t_0$, $y = \sin t_0 + t\cos t_0$ $z = t_0 + t$

In particular at $t = \pi$

$x = -1$, $y = -t$, $z = \pi + t$.

<u>Example 4.</u> Let $r_1(t) = (\tan^{-1} t)i + (\sin t)j + t^2 k$,

and $r_2(t) = (t^2 - t)i + (2t - 2)j + (\ln t)k$.

The graph of $r_1(t)$ and $r_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graph of $r_1(t)$ and $r_2(t)$ at the origin.

$\cancel{r(t_0) = r_0}$

$$r_1'(t) = \left(\frac{1}{1+t^2}\right)i + (\cos t)\,j + 2tk.$$

$$r_1'(0) = \left(\frac{1}{1}\right)i + (\cos 0^\circ)j + 2.0k.$$

$$= 1i + 1j + 0k = i + j.$$

$$r_2'(t) = (2t-1)\,i + 2j + \frac{1}{t}k.$$

$$r_2'(1) = (+1)\,i + 2j + k$$

$$= i + 2j + k.$$

$$\cos\theta = \frac{1+2+0}{\sqrt{1^2+1^2}\cdot\sqrt{1^2+2^2+1^2}}$$

$$= \frac{3}{\sqrt{2}\cdot\sqrt{6}}$$

$$\cos\theta = \frac{3}{\sqrt{12}}$$

$$\cancel{\cos}\,\theta = 30^\circ.$$

Derivatives of dot and cross products

$\frac{d}{dt}[r_1(t) \cdot r_2(t)]$

$= r_1(t) \frac{d}{dt}(r_2(t)) + r_2(t) \frac{d}{dt}[r_1(t)]$

$\frac{d}{dt}[r_1(t) \times r_2(t)]$.

$= r_1(t) \times \frac{d}{dt}[r_2(t)] + r_2(t) \times \frac{d}{dt}[r_1(t)]$

Definite integral of vector valued functions.

$$\int_a^b r(t)\,dt = \lim_{\Delta t_k \to 0} \sum_{k=1}^{n} r(t_k^*)\,\Delta t_k.$$

$r(t) = x(t)i + y(t)j.$

$$\boxed{\int_a^b r(t)\,dt = \cancel{\lim}\left(\int_a^b x(t)\,dt\right)i + \left(\int_a^b y(t)\,dt\right)j.}$$

$$\int_a^b r(t)\,dt = \left(\int_a^b x(t)\,dt\right)i + \left(\int_a^b y(t)\,dt\right)j + \left(\int_a^b z(t)\,dt\right)k.$$

Example 6 : Let $r(t) = t^2 i + e^t j - (2\cos\pi t)k$

$$\int_0^1 r(t)\,dt = \left(\int_0^1 t^2\,dt\right)i + \left(\int_0^1 e^t\,dt\right)j - \left(\int_0^1 2\cos\pi t\,dt\right)k.$$

$$= \left.\frac{t^3}{3}\right]_0^1 + \left.e^t\right]_0^1 - \left.\frac{2}{\pi}\sin\pi t\right]_0^1$$

$$= \frac{1}{3}i + (e-1)j.$$

$$\boxed{\square}\ \int_a^b [r_1(t) + r_2(t)]\,dt = \int_a^b r_1(t)\,dt + \int_a^b r_2(t)\,dt.$$

$\boxed{+}$ $\int_a^b [r_1(t) - r_2(t)]\, dt$

$= \int_a^b r_1(t)\, dt - \int_a^b r_2(t)\, dt$.

~~Antiderivative~~ for a

Example 7.

$\int (2t\,i + 3t^2 j)\, dt$

$= \left(\int 2t\, dt\right) i + \left(\int 3t^2\, dt\right) j$.

$= \left(\frac{2t^2}{2} + c_1\right) i + \left(\frac{3t^3}{3} + c_2\right) j$.

$= (t^2 + c_1)\, i + (t^3 + c_2) j$

$= \cancel{t^2 i + c_2 j + \frac{t}{3}}\ c_1 i + c_2 j$

$= t^2 i + t^3 j + \underbrace{c_1 i + c_2 j}_{c}$

$= t^2 i + t^3 j + c$.

Example 8.

Evaluate the definite integral.

$$\int_0^2 (2t\,i + 3t^2 j)\,dt$$

$$= \left(\int_0^2 2t\,dt\right) i + \left(\int_0^2 3t^2\,dt\right) j.$$

$$= \left(\frac{2t^2}{2}\Big]_0^2\right) i + \left(\frac{3t^3}{3}\Big]_0^2\right) j$$

$$= t^2\Big]_0^2 i + t^3\Big]_0^2$$

$$= 4i + 8j.$$

Example 9: Find $r(t)$ given that

$$r'(t) = \langle 3, 2t \rangle \text{ and } r(1) = \langle 2, 5 \rangle$$

$$r(t) = \int r'(t)\,dt = \int \langle 3, 2t \rangle dt.$$

$$= \langle 3t, t^2 \rangle + C.$$

$$r(1) = \langle 3, 1 \rangle + C = \langle 2, 5 \rangle$$

$$C = \langle 2-3, 5-1 \rangle = \langle -1, 4 \rangle$$

$$r(t) = \langle 3t, t^2 \rangle + \langle -1, 4 \rangle = \langle 3t-1, t^2+4 \rangle$$

(1-4) Find the limit.

1. $\lim_{t\to\infty} \left\langle \frac{t^2+1}{3t^2+2}, \frac{1}{t} \right\rangle$.

$= \left\langle \lim_{t\to\infty} \frac{t^2+1}{3t^2+2}, \lim_{t\to\infty} \frac{1}{t} \right\rangle$.

$= \left\langle \lim_{t\to\infty} \frac{\cancel{t^2}\left(1+\frac{1}{t^2}\right)}{\cancel{t^2}\left(3+\frac{2}{t^2}\right)}, 0 \right\rangle$.

$= \left\langle \frac{1}{3}, 0 \right\rangle$.

2. $\lim_{t\to 0^+} \left(\langle \sqrt{t}\, i + \frac{\sin t}{t} j \right)$

$= \left\langle \lim_{t\to 0^+} \sqrt{t}, \lim_{t\to 0} \frac{\sin t}{t} \right\rangle$.

$= \langle 0, 1 \rangle$ $\qquad \frac{\cos t}{1}$

3. $\lim_{t\to 2} \langle ti - 3j + t^2k \rangle$

$\langle \lim_{t\to 2} ti - \lim_{t\to 2} 3j + \lim_{t\to 2} t^2k \rangle$.

$= 2i - 3j + 4k$.

4. $\lim_{t\to 1} \langle \frac{3}{t^2}, \frac{\ln t}{t^2-1}, \sin 2t \rangle$.

$\langle \lim_{t\to 1} \frac{3}{t^2}, \lim_{t\to 1} \frac{\ln t}{t^2-1}, \lim_{t\to 1} \sin 2t \rangle$

$\frac{3}{1^2}, \frac{1}{t(2t)}, \sin 2$.

$= \langle 3, \frac{1}{2}, \sin 2 \rangle$.

5-6: Determine whether $r(t)$ is continous at $t=0$. Explain your reasoning.

5. a $r(t) = 3\sin t\, i - 2t j$.

$$\boxed{\lim_{t\to 0} r(t) = r(0)}$$

$\hookrightarrow$ continious.

9-10: Find $r'(t)$

$$r(t) = 4i - \cos t\, j.$$

$$r'(t) = \left(\frac{d}{dt}(4)\right) i - \left(\frac{d}{dt}\cos t\right) j.$$

$$= 0.i + \sin t\, j.$$

$$\boxed{r'(t) = 0.i + \sin t\, j.}$$

$$\frac{d}{dt}\left(\tan^{-1}t\right) = \frac{1}{1+t^2}$$

10. $r(t) = (\tan^{-1}t)i + t\cos t\, j - \sqrt{t}\, k.$

$$r'(t) = \frac{1}{1+t^2} + (\cos t + \sin t)j - \frac{1}{2\sqrt{t}}k.$$

11 - 14. Find the vector $r'(t_0)$; then sketch the graph of $r(t)$ in 2-space and draw the tangent vector $r'(t_0)$.

$r(t) = \langle t, t^2 \rangle$; $t_0 = 2$.

$r(t) = ti + t^2 j$

$r'(t) = i + 2tj.$

$\cancel{t_0 = 2}$

$r'(2) = \langle 1, 2t \rangle$

$= \langle 1, 4 \rangle$

$r(2) = \langle 2, 4 \rangle.$

12. $r(t) = t^3 i + t^2 j \;;\; t_0 = 1.$

$r'(t) = 3t^2 i + 2t j.$

$r'(1) = 3i + 2j.$

$r(1) = i + j$

13. $r(t) = \cancel{\sin}\ \sec t . i + \tan t\, j$

$r'(t) = \cancel{\tan^2}\ \sec t \tan t\ i + \sec^2 t\, j.$

$r'\left(\frac{\pi}{6}\right) = \sec\frac{\pi}{6} . \tan\frac{\pi}{6}\, i + \left(\sec\frac{\pi}{6}\right)^2 j.$

15-16 : Find the vector $r'(t_0)$; then sketch the graph of $r(t)$ in 3-space and draw the tangent vector $r'(t_0)$.

15. $r(t) = 2\sin t\, i + j + 2\cos t\, k$.

$v = \boxed{r'(t) = 2\cos t\, i \cancel{+ j} - 2\sin t\, k}$.

17-18: Use a graphing utility to generate the graph of $r(t)$ and the graph of the tangent line at t_0 on the same screen.

17. $r(t) = \sin \pi t\, i + t^2 j$; $t_0 = \frac{1}{2}$

$r_0 = r(t_0) = r(\frac{1}{2}) = \sin\frac{\pi}{2} i + (\frac{1}{2})^2 j$.

$= i + \frac{1}{4} j$.

$v_0 = r'(t_0) = \pi\cos\pi t\, i + 2t j$.

$= r(\frac{1}{2}) = \cancel{t}\, \pi\cos\frac{\pi}{2} i + 2 \cdot \frac{1}{2} j$

$= j$

$r = r_0 + t v_0$

$= i + \frac{1}{4} j + t j$

19-22: Find parametric eqns of the line tangent to the graph ~~of~~ of $r(t)$ at the point where $t = t_0$.

$$r(t) = t^2 i + (2 - \ln t) j; \; t_0 = 1.$$

$$r_0 = r(t_0) = t_0^2 i + (2 - \ln t_0) j$$

$$= 1^2 i + (2 - 0) j$$

$$= i + 2j.$$

$$v = r'(t) = 2t i - \frac{1}{t} j.$$

$$r'(1) = 2i - j.$$

$$r = r_0 + vt.$$

$$= i + 2j + (2i - j) t.$$

$$= i + 2it + 2j - jt.$$

$$= \cancel{i}(1 + 2t)i + \cancel{j}(2 - t) j$$

$$\left.\begin{aligned} x &= 1 + 2t \\ y &= 2 - t \end{aligned}\right\} \text{ parametric eqn.}$$

$2t \;\; x = 2t - 1.$

$-1 = 2t = 0.$

23-26 : Find a vector eqn of the line tangent to the graph of $r(t)$ at the point P_0 on the curve.

23. $r(t) = (2t - 1)\, i + \sqrt{3t + 4}\, j$; $P_0\,(-1, 2)$

$$r'(t) = 2i + \frac{3}{2\sqrt{3t+4}} \cdot j.$$

$x = 2t - 1.$

$-1 = 2t - 1.$

$2t = 0$

$\boxed{t = 0}$

$r_0 = r(0) = (-1)\, i + \sqrt{4}\; j \quad (-1)\, i + \sqrt{4}\, j$

$= -i + 2j. \qquad = -i + 2j$

$v = r'(t) = r'(0).$

$r = r_0 + vt.$

29-30. Calculate

$$\frac{d}{dt}[r_1(t)\cdot r_2(t)] \text{ and } \frac{d}{dt}[r_1(t)\times r_2(t)]$$

first by differentiating the product directly and then applying formulas

(6) and 7.

29. $r_1(t) = 2ti + 3t^2j + t^3k.$

$r_2(t) = t^4k.$

$$\frac{d}{dt}[r_1 t \cdot r_2(t)].$$

$$= r_1(t)\cdot\frac{d}{dt}(r_2(t)) + r_2(t)\cdot\frac{d}{dt}(r_1(t)).$$

$=$

31-34: Evaluate the infinite integral.

31. $\int (3i + 4tj)\, dt$.

$= \int (3\,dt)\, i + \int (4t\,dt)\, j$,

$= (3t)\, i + \left(\frac{4t^2}{2}\right) j$.

$= 3t\, i + 2t^2 j + C$

37. $\int_0^2 \overset{\text{magnitude}}{\|t\, i + t^2 j\|}\, dt$.

$r(t) = t\, i + t^2 j$

$r(t) = \langle t, t^2 \rangle$

The magnitude of the vector

$= \sqrt{t^2 + t^4}$

45. Solve the vector initial value problem for $y(t)$ by integrating and using the initial conditions to find the constants of integration.

45. $y'(t) = 2ti + 3t^2j,\ y(0) = i - j.$

$$y(t) = \int y'(t)\,dt$$

$$= \int (2ti + 3t^2j)$$

$$= \frac{2t^2}{2}i + \frac{3t^3}{3}j$$

$$y(t) = t^2 i + t^3 j + C$$

$$y(0) = 0$$

$$i - j = C$$

$$y(t) = (t^2+1)i + (t^3-1)j.$$

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46. $y'(t) = \cos t\, i + \sin t\, j$; $y(0) = i - j$.

$\int y'(t) = \int (\cos t) i + \int (\sin t) j$

$y(t) = \sin t\, i - \cos t\, j + c$

$y(0) = c$.

$i - j = c$.

$y(t) = \sin t\, i - \cos t\, j + i - j$.

$= (\sin t + 1)\, i - (\cos t + 1)\, j$,

47. $y''(t) = i + e^t j$, $y(0) = 2i$, $y'(0) = j$.

$y'(t) = t i + e^t j + c_1$.

$\cancel{y(t) = t^2}$

$y'(0) = j + c_1$.

$c_1 = 0$.

$y'(t) = t i + e^t j$.

$\int y'(t) = \frac{t^2}{2} i + e^t j + c_2$

$y(t) = \frac{t^2}{2} i + e^t j + c_2$.

$y(0) = j + c_2$

$2i = j + c_2$

$c_2 = 2i - j$

$y(t) = \left(\frac{1}{2}t^2 + 2\right)i + (e^t - 1)j$

(49) a) Find the points where the curve

$r = ti + t^2 j - 3tk$

intersects the plane $2x - y + z = -2$.

$2t - t^2 - 3t = -2$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

$t = -2, 1$

r এ বসায় পাই।

50. Find where the tangent line to the curve

$$r = e^{-2t} i + \cos t j + 3 \sin t k$$

at the point $(1,1,0)$ intersects the yz plane

→ tangent line এর x parametric eqn $= 0$ হলে t বের হবে।

$r' = -2e^{-2t} i - \sin t j + 3\cos t k$

$e^{-2t} = 1$, $\cos t = 1$, $3\sin t = 0$

$t = 0$, $t = 0$, $t = 0$

now solve.

51-52: Show that the graphs of $r_1(t)$ and $r_2(t)$ intersect at the point P. Find the nearest degree, the acute angle between the tangent lines to the graphs of $r_1(t)$ and $r_2(t)$ at the point P.

$r_1(t) = t^2 i + t j + 3t^3 k$.

$r_2(t) = (t-1) i + \frac{1}{4} t^2 j + (5-t) k$; $P(1, 1, 3)$

$t^2 = 1$, $t = 1$, $3t^3 = 3$.

$t = 1$.

$r_1(t) = r_1(1)$.

for $r_2(t)$

$t - 1 = 1$. $\frac{1}{4} t^2 = 1$. | $5 - t = 3$.

$t = 2$ $t^2 = 4$ | $t = 2$.

$r_2(2)$. $t = 2$

$\Rightarrow$

Ex calculate v_1 and v_2.

$$v_1 \cdot v_2 = \|v_1\| \cdot \|v_2\| \cdot \cos\theta$$

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \cdot \|v_2\|}$$