"In the name of Almighty Allah (Lord), Most Gracious, Most Merciful"

Partial Derivatives & Applications

[redacted]

Course Title: Multivariable Calculus (III)

Text book: Calculus: Early Transcendental; Anton, Bivens and Davis, 10th Edition

Note: This lecture is designed based on Section 13.3 from the reference book

13.3 PARTIAL DERIVATIVES

13.3.1 DEFINITION If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f, then the ***partial derivative of f with respect to x*** at (x_0, y_0) [also called the ***partial derivative of z with respect to x*** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \tag{1}$$

Similarly, the ***partial derivative of f with respect to y*** at (x_0, y_0) [also called the ***partial derivative of z with respect to y*** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \tag{2}$$

Now let's take a quick look at some of the possible alternate notations for partial derivatives. Given the function $z = f(x, y)$ the following are all equivalent notations,

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(f(x,y)\right) = z_x = \frac{\partial z}{\partial x} = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(f(x,y)\right) = z_y = \frac{\partial z}{\partial y} = D_y f$$

For the fractional notation for the partial derivative notice the difference between the partial derivative and the ordinary derivative from single variable calculus.

$$f(x) \qquad \Rightarrow \qquad f'(x) = \frac{df}{dx}$$

$$f(x,y) \qquad \Rightarrow \qquad f_x(x,y) = \frac{\partial f}{\partial x} \quad \& \quad f_y(x,y) = \frac{\partial f}{\partial y}$$

Example: Find the first order partial derivatives of the following functions

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

(b) $w = x^2 y - 10y^2 z^3 + 43x - 7\tan(4y)$

(c) $h(s,t) = t^7 \ln(s^2) + \dfrac{9}{t^3} - \sqrt[7]{s^4}$

(d) $f(x,y) = \cos\left(\dfrac{4}{x}\right)\mathbf{e}^{x^2 y - 5y^3}$

Solution

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

Let's first take the derivative with respect to *x* and remember that as we do so all the *y*'s will be treated as constants. The partial derivative with respect to *x* is,

$$f_x(x,y) = 4x^3$$

Notice that the second and the third term differentiate to zero in this case. It should be clear why the third term differentiated to zero. It's a constant and we know that constants always differentiate to zero. This is also the reason that the second term differentiated to zero. Remember that since we are differentiating with respect to *x* here we are going to treat all *y*'s as constants. That means that terms that only involve *y*'s will be treated as constants and hence will differentiate to zero.

Now, let's take the derivative with respect to *y*. In this case we treat all *x*'s as constants and so the first term involves only *x*'s and so will differentiate to zero, just as the third term will. Here is the partial derivative with respect to *y*.

$$f_y(x,y) = \frac{3}{\sqrt{y}}$$

(b) $w = x^2 y - 10y^2 z^3 + 43x - 7\tan(4y)$

With this function we've got three first order derivatives to compute. Let's do the partial derivative with respect to *x* first. Since we are differentiating with respect to *x* we will treat all *y*'s and all *z*'s as constants. This means that the second and fourth terms will differentiate to zero since they only involve *y*'s and *z*'s.

This first term contains both *x*'s and *y*'s and so when we differentiate with respect to *x* the *y* will be thought of as a multiplicative constant and so the first term will be differentiated just as the third term will be differentiated.

Here is the partial derivative with respect to *x*.

$$\frac{\partial w}{\partial x} = 2xy + 43$$

Let's now differentiate with respect to *y*. In this case all *x*'s and *z*'s will be treated as constants. This means the third term will differentiate to zero since it contains only *x*'s while the *x*'s in the first term and the *z*'s in the second term will be treated as multiplicative constants. Here is the derivative with respect to *y*.

$$\frac{\partial w}{\partial y} = x^2 - 20yz^3 - 28\sec^2(4y)$$

Finally, let's get the derivative with respect to *z*. Since only one of the terms involve *z*'s this will be the only non-zero term in the derivative. Also, the *y*'s in that term will be treated as multiplicative constants. Here is the derivative with respect to *z*.

$$\frac{\partial w}{\partial z} = -30y^2z^2$$

(c) $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$

With this one we'll not put in the detail of the first two. Before taking the derivative let's rewrite the function a little to help us with the differentiation process.

$$h(s,t) = t^7 \ln(s^2) + 9t^{-3} - s^{\frac{4}{7}}$$

Now, the fact that we're using *s* and *t* here instead of the "standard" *x* and *y* shouldn't be a problem. It will work the same way. Here are the two derivatives for this function.

$$h_s(s,t) = \frac{\partial h}{\partial s} = t^7\left(\frac{2s}{s^2}\right) - \frac{4}{7}s^{-\frac{3}{7}} = \frac{2t^7}{s} - \frac{4}{7}s^{-\frac{3}{7}}$$

$$h_t(s,t) = \frac{\partial h}{\partial t} = 7t^6 \ln(s^2) - 27t^{-4}$$

Remember how to differentiate natural logarithms.

$$\frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)}$$

(d) $f(x,y) = \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y - 5y^3}$

Now, we can't forget the product rule with derivatives. The product rule will work the same way here as it does with functions of one variable. We will just need to be careful to remember which variable we are differentiating with respect to.

Let's start out by differentiating with respect to *x*. In this case both the cosine and the exponential contain *x*'s and so we've really got a product of two functions involving *x*'s and so we'll need to product rule this up. Here is the derivative with respect to *x*.

$$f_x(x,y) = -\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)\mathbf{e}^{x^2y-5y^3} + \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}(2xy)$$
$$= \frac{4}{x^2}\sin\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3} + 2xy\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$$

Also, don't forget how to differentiate exponential functions,

$$\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$$

$$f_y(x,y) = \left(x^2 - 15y^2\right)\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$$

Example: Find the 1st order partial derivatives

(a) $z = \dfrac{9u}{u^2+5v}$

(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

Solution

(a) $z = \dfrac{9u}{u^2+5v}$

We also can't forget about the quotient rule. Since there isn't too much to this one, we will simply give the derivatives.

$$z_u = \frac{9\left(u^2+5v\right)-9u(2u)}{\left(u^2+5v\right)^2} = \frac{-9u^2+45v}{\left(u^2+5v\right)^2}$$
$$z_v = \frac{(0)\left(u^2+5v\right)-9u(5)}{\left(u^2+5v\right)^2} = \frac{-45u}{\left(u^2+5v\right)^2}$$

In the case of the derivative with respect to *v* recall that *u*'s are constant and so when we differentiate the numerator we will get zero!

(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

Now, we do need to be careful however to not use the quotient rule when it doesn't need to be used. In this case we do have a quotient, however, since the *x*'s and *y*'s only appear in the numerator and the *z*'s only appear in the denominator this really isn't a quotient rule problem.

$$g_x(x,y,z) = \frac{\sin(y)}{z^2} \qquad\qquad g_y(x,y,z) = \frac{x\cos(y)}{z^2}$$

Now, in the case of differentiation with respect to *z* we can avoid the quotient rule with a quick rewrite of the function. Here is the rewrite as well as the derivative with respect to *z*.

$$g(x,y,z) = x\sin(y)z^{-2}$$

$$g_z(x,y,z) = -2x\sin(y)z^{-3} = -\frac{2x\sin(y)}{z^3}$$

Interpretations of Partial Derivatives

The first interpretation we've already seen and is the more important of the two. As with functions of single variables partial derivatives represent the rates of change of the functions as the variables change. As we saw in the previous section, $f_x(x,y)$ represents the rate of change of the function $f(x,y)$ as we change *x* and hold *y* fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change *y* and hold *x* fixed.

Example 1 Determine if $f(x,y) = \dfrac{x^2}{y^3}$ is increasing or decreasing at $(2,5)$,

(a) if we allow *x* to vary and hold *y* fixed.
(b) if we allow *y* to vary and hold *x* fixed.

Solution
(a) If we allow *x* to vary and hold *y* fixed.

In this case we will first need $f_x(x,y)$ and its value at the point.

$$f_x(x,y) = \frac{2x}{y^3} \qquad \Rightarrow \qquad f_x(2,5) = \frac{4}{125} > 0$$

So, the partial derivative with respect to *x* is positive and so if we hold *y* fixed the function is increasing at $(2,5)$ as we vary *x*.

(b) If we allow *y* to vary and hold *x* fixed.

For this part we will need $f_y(x,y)$ and its value at the point.

$$f_y(x,y) = -\frac{3x^2}{y^4} \qquad \Rightarrow \qquad f_y(2,5) = -\frac{12}{625} < 0$$

Here the partial derivative with respect to *y* is negative and so the function is decreasing at $(2,5)$ as we vary *y* and hold *x* fixed.

Higher Order Partial Derivatives

Example 1 Find all the second order derivatives for $f(x,y) = \cos(2x) - x^2\mathbf{e}^{5y} + 3y^2$.

Solution

We'll first need the first order derivatives so here they are.

$$f_x(x,y) = -2\sin(2x) - 2x\mathbf{e}^{5y}$$
$$f_y(x,y) = -5x^2\mathbf{e}^{5y} + 6y$$

Now, let's get the second order derivatives.

$$f_{xx} = -4\cos(2x) - 2\mathbf{e}^{5y}$$
$$f_{xy} = -10x\mathbf{e}^{5y}$$
$$f_{yx} = -10x\mathbf{e}^{5y}$$
$$f_{yy} = -25x^2\mathbf{e}^{5y} + 6$$

Clairaut's Theorem

Suppose that *f* is defined on a disk *D* that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on this disk then,

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example 2 Verify Clairaut's Theorem for $f(x,y) = x\mathbf{e}^{-x^2y^2}$.

Solution

We'll first need the two first order derivatives.

$$f_x(x,y) = \mathbf{e}^{-x^2y^2} - 2x^2y^2\mathbf{e}^{-x^2y^2}$$
$$f_y(x,y) = -2yx^3\mathbf{e}^{-x^2y^2}$$

Now, compute the two mixed second order partial derivatives.

$$f_{xy}(x,y) = -2yx^2\mathbf{e}^{-x^2y^2} - 4x^2y\mathbf{e}^{-x^2y^2} + 4x^4y^3\mathbf{e}^{-x^2y^2} = -6x^2y\mathbf{e}^{-x^2y^2} + 4x^4y^3\mathbf{e}^{-x^2y^2}$$
$$f_{yx}(x,y) = -6yx^2\mathbf{e}^{-x^2y^2} + 4y^3x^4\mathbf{e}^{-x^2y^2}$$

Sure enough they are the same.

Example 3 Find the indicated derivative for each of the following functions.

(a) Find f_{xxyzz} for $f(x,y,z) = z^3y^2\ln(x)$

(b) Find $\dfrac{\partial^3 f}{\partial y\partial x^2}$ for $f(x,y) = \mathbf{e}^{xy}$

Solution

(a) Find f_{xxyzz} **for** $f(x,y,z) = z^3 y^2 \ln(x)$

In this case remember that we differentiate from left to right. Here are the derivatives for this part.

$$f_x = \frac{z^3 y^2}{x}$$

$$f_{xx} = -\frac{z^3 y^2}{x^2}$$

$$f_{xxy} = -\frac{2z^3 y}{x^2}$$

$$f_{xxyz} = -\frac{6z^2 y}{x^2}$$

$$f_{xxyzz} = -\frac{12zy}{x^2}$$

(b) Find $\dfrac{\partial^3 f}{\partial y \partial x^2}$ **for** $f(x,y) = \mathbf{e}^{xy}$

Here we differentiate from right to left. Here are the derivatives for this function.

$$\frac{\partial f}{\partial x} = y\mathbf{e}^{xy}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2\mathbf{e}^{xy}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = 2y\mathbf{e}^{xy} + xy^2\mathbf{e}^{xy}$$

Practice Problem:

Chapter-13.3:

Problems: 58, 61, 67, 69,71, 74, 76,86, 89, 90.

Thank you for your attention!