Limits and Continuity

Text book: Calculus: Early Transcendental; Anton, Bivens and Davis, 10th Edition

Note: This lecture is designed based on Section 13.2 from the reference book

13.2 LIMITS AND CONTINUITY

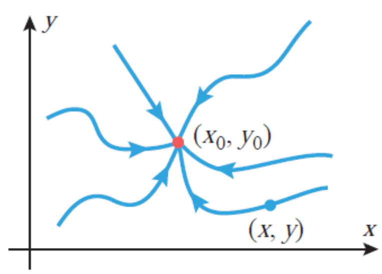
LIMITS ALONG CURVES

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left. For functions of two or three variables the situation is more complicated

because there are infinitely many different curves along which one point can approach another (Figure 13.2.1). Our first objective in this section is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).



▶ **Figure 13.2.1**

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,\, y) \to (x_0,\, y_0) \\ (\text{along } C)}} f(x, y) \quad \text{and} \quad \lim_{\substack{(x,\, y,\, z) \to (x_0,\, y_0,\, z_0) \\ (\text{along } C)}} f(x, y, z)$$

are defined by

$$\lim_{\substack{(x,\, y) \to (x_0,\, y_0) \\ (\text{along } C)}} f(x, y) = \lim_{t \to t_0} f(x(t), y(t)) \tag{1}$$

$$\lim_{\substack{(x,\, y,\, z) \to (x_0,\, y_0,\, z_0) \\ (\text{along } C)}} f(x, y, z) = \lim_{t \to t_0} f(x(t), y(t), z(t)) \tag{2}$$

To show a limit does not exist, it is still enough to find two paths along which the limits are not equal. In view of the number of possible paths, it is not always easy to know which paths to try. We give some suggestions here. You can try the following paths:

1. Horizontal line through (a, b), the equation of such a path is $y = b$.

2. Vertical line through (a, b), the equation of such a path is $x = a$.

3. Any straight line through (a, b) ,the equation of the line with slope m through (a, b) is $y = mx + b - am$.

4. Quadratic paths. For example, a typical quadratic path through $(0, 0)$ is $y = x^2$.

Remark 3.2.3 *There are several notation for this limit. They all represent the same thing, we list them below.*

1. $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$

2. $\displaystyle\lim_{\substack{x\to a \\ y\to b}} f(x, y) = L$

3. $f(x, y)$ *approaches* L *as* (x, y) *approaches* (a, b).

Example 3.2.4 *Consider the function* $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$. *Use a table of values to "guess"* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

We begin by making a table of values of $f(x,y)$ *for* (x,y) *close to* $(0,0)$.

$x \diagdown y$	**-1.0**	**-0.5**	**-0.2**	**0**	**0.2**	**0.5**	**1.0**
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1	0.999	0.986	0.829
0	0.841	0.990	1		1	0.990	0.841
0.2	0.829	0.986	0.999	1	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Looking at the table, we can estimate the limit along certain paths. For example, each column of the table gives the function values for a fixed y *value. In the column corresponding to* $y = 0$, *we have the values of* $f(x,0)$ *for values of* x *close to* 0, *from either direction. So we can estimate the limit along the path* $y = 0$. *In fact, the column corresponding to* $y = b$ *can be used to estimate the limit along the path* $y = b$. *Similarly, the row* $x = a$ *can be used to estimate the limit along the path* $x = a$. *The diagonal of the table from the top left to the bottom right correspond to values* $x = y$. *It can be used to estimate the limit along the path* $y = x$. *The other diagonal, from top right to bottom left corresponds to* $y = -x$. *So, it can be used to estimate the limit along the path* $y = -x$. *Looking at the table, it seems that the limit along any of the paths discussed appears to be* 1. *While this does not prove it for sure, as there are many more paths to consider, this gives us an indication that it might be. We can then try to use other methods we will discuss in the next sections to try to show the limit is indeed* 1. *It turns out this limit is indeed* 1.

Example 3.2.5 *Consider the function* $g(x,y)=\frac{x^2-y^2}{x^2+y^2}$. *Use a table of values to "guess"* $\lim\limits_{(x,y)\to(0,0)} g(x,y)$.

We begin by making a table of values of $g(x,y)$ *for* (x,y) *close to* $(0,0)$.

$x \diagdown y$	**−1.0**	**−0.5**	**−0.2**	**0**	**0.2**	**0.5**	**1.0**
−1.0	0	0.6	0.923	1	0.923	0.6	0
−0.5	−0.6	0	0.724	1	0.724	0	−0.6
−0.2	−0.923	−0.724	0	1	0	−0.724	−0.923
0	−1	−1	−1		−1	−1	−1
0.2	−0.923	−0.724	0	1	0	−0.724	−0.923
0.5	−0.6	0	0.724	1	0.724	0	−0.6
1	0	0.6	0.923	1	0.923	0.6	0

Looking at the table as indicated in the previous example, we see that the limit along the path $y=0$ *appears to be* 1 *while the limit along the path* $x=0$ *appears to be* -1. *This proves* $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ *does not exist.*

Theorem 3.2.7 (Properties of Limits of Functions of Several Variables) *We list these properties for functions of two variables. Similar properties hold for functions of more variables. Let us assume that* L, M, *and* k *are real numbers and that* $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$ *and* $\lim\limits_{(x,y)\to(a,b)} g(x,y)=M$, *then the following hold:*

1. *First, we have the obvious limits*

$$\begin{aligned}\lim_{(x,y)\to(a,b)} x &= a\\ \lim_{(x,y)\to(a,b)} y &= b\end{aligned}$$

   If c *is any constant,*

$$\lim_{(x,y)\to(a,b)} c=c$$

2. *Sum and difference rules:*

$$\lim_{(x,y)\to(a,b)} [f(x,y)\pm g(x,y)]=L\pm M$$

3. *Constant multiple rule:*

$$\lim_{(x,y)\to(a,b)} [kf(x,y)] = kL$$

4. *Product rule:*

$$\lim_{(x,y)\to(a,b)} [f(x,y)\, g(x,y)] = LM$$

5. *Quotient rule:*

$$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L}{M}$$

   provided $M \neq 0$.

6. *Power rule: If* r *and* s *are integers with no common factors, and* $s \neq 0$ *then*

$$\lim_{(x,y)\to(a,b)} [f(x,y)]^{\frac{r}{s}} = L^{\frac{r}{s}}$$

   provided $L^{\frac{r}{s}}$ *is a real number. If* s *is even, we assume* $L > 0$.

Theorem 3.2.8 *The above theorem applied to polynomials and rational functions implies the following:*

1. *To find the limit of a polynomial, we simply plug in the point.*

2. *To find the limit of a rational function, we plug in the point as long as the denominator is not* 0.

Example 3.2.10 *Find* $\lim_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2}$

Combining the rules mentioned above allows us to do the following

$$\begin{aligned} \lim_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2} &= \frac{1^2 1}{1^4+1^2} \\ &= \frac{1}{2} \end{aligned}$$

Example 3.2.12 *Find* $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y}$

We cannot plug in the point as we get 0 *in the denominator. We try to rewrite the fraction to see if we can simplify it.*

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y} &= \lim_{(x,y)\to(0,0)} \frac{(x-y)\left(x^2+xy+y^2\right)}{x-y} \\ &= \lim_{(x,y)\to(0,0)} \left(x^2+xy+y^2\right) \\ &= 0\end{aligned}$$

Example 3.2.13 *Find* $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Here, we cannot plug in the point because we get $\dfrac{0}{0}$*, an indeterminate form. Since this is a fraction which involves a radical, we multiply by the conjugate. We get:*

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{\left(x^2-xy\right)\left(\sqrt{x}+\sqrt{y}\right)}{\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)} \\ &= \lim_{(x,y)\to(0,0)} \frac{x(x-y)\left(\sqrt{x}+\sqrt{y}\right)}{x-y} \\ &= \lim_{(x,y)\to(0,0)} \frac{x\left(\sqrt{x}+\sqrt{y}\right)}{1} \\ &= 0\end{aligned}$$

Example 3.2.14 *Consider the function* $f(x,y) = \frac{y}{x+y-1}$*. The goal is to try to find* $\lim\limits_{(x,y)\to(1,0)} \frac{y}{x+y-1}$.

You may remember from Calculus I that in many cases, to compute a limit we simply plugged-in the point. If you try to do this here, you obtain $\frac{0}{0}$ *which is an indeterminate form. It does not mean the limit does not exist. It means that you need to study it further. We will do this by looking at the limit along various paths. As mentioned in the introduction, some obvious paths we might try are the path* $x = 1$ *and* $y = 0$.

1. *Limit along the path* $y=0$. *First, we find what the function becomes along this path. We will use the notation* $\left.\frac{y}{x+y-1}\right|_{y=0}$ *to mean* $\frac{y}{x+y-1}$ *along the path* $y=0$ *and* $\lim\limits_{\substack{(x,y)\to(1,0)\\ along\ y=0}} \frac{y}{x+y-1}$ *to mean* $\lim\limits_{(x,y)\to(1,0)} \frac{y}{x+y-1}$ *along the path* $y=0$. *We have:*

$$\begin{aligned}\left.\frac{y}{x+y-1}\right|_{y=0} &= \frac{0}{x-1} \\ &= 0\end{aligned}$$

Also, note that along the path $y=0$, y *is constant hence* $(x,y)\to(1,0)$ *can be replaced by* $x\to 1$. *Therefore*

$$\begin{aligned}\lim_{\substack{(x,y)\to(1,0)\\ along\ y=0}} \frac{y}{x+y-1} &= \lim_{x\to 1} 0 \\ &= 0\end{aligned}$$

2. *Limit along the path* $x=1$. *We have:*

$$\begin{aligned}\left.\frac{y}{x+y-1}\right|_{x=1} &= \frac{y}{1+y-1} \\ &= \frac{y}{y} \\ &= 1\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{\substack{(x,y)\to(1,0)\\ along\ x=1}} \frac{y}{x+y-1} &= \lim_{y\to 0} 1 \\ &= 1\end{aligned}$$

3. *Conclusion: The limits are different, therefore* $\lim\limits_{(x,y)\to(1,0)} \frac{y}{x+y-1}$ *does not exist.*

Example 3.2.15 *Consider the function* $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$. *The goal is to try to find* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

As mentioned in the introduction, some obvious paths we might try are the path $x=0$ *and* $y=0$. *Note that we can also combine both computations (finding what the function is along the path and finding the limit).*

1. Limit along the path $x = 0$. Along this path, we have

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ x=0}} \frac{x^2-y^2}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \left.\frac{x^2-y^2}{x^2+y^2}\right|_{x=0} \\
&= \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} \\
&= \lim_{(x,y)\to(0,0)} -1 \\
&= -1
\end{aligned}$$

2. Limit along the path $y = 0$. Along this path, we have

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ y=0}} y &= 0\frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \left.\frac{x^2-y^2}{x^2+y^2}\right|_{y=0} \\
&= \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} \\
&= \lim_{(x,y)\to(0,0)} 1 \\
&= 1
\end{aligned}$$

3. Conclusion: The limits are different, therefore $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Example 3.2.16 *Prove that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.*

First, we try the limit along the paths $x = 0$ and $y = 0$. The user will check that both limits are 0. Next, we try along the path $y = x$. We get

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ along\ y=x}} \frac{xy}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \left.\frac{xy}{x^2+y^2}\right|_{y=x} \\
&= \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} \\
&= \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} \\
&= \lim_{(x,x)\to(0,0)} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

We obtained a different limit. So, $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

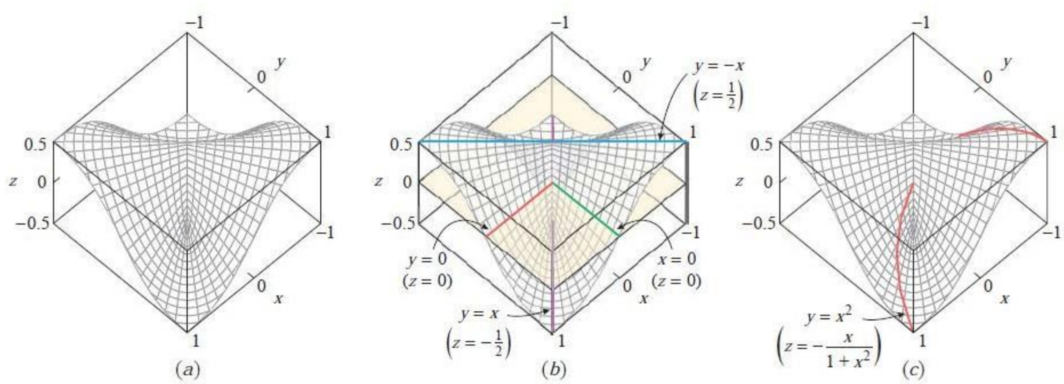
▸ Example 1 Figure 13.2.3*a* shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$

(d) the line $y = -x$ (e) the parabola $y = x^2$



▸ **Figure 13.2.3**

Solution (a). The x-axis has parametric equations $x = t$, $y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } y = 0)}} f(x, y) = \lim_{t \to 0} f(t, 0) = \lim_{t \to 0} \left(-\frac{0}{t^2}\right) = \lim_{t \to 0} 0 = 0$$

which is consistent with Figure 13.2.3*b*.

Solution (b). The y-axis has parametric equations $x = 0$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } x = 0)}} f(x, y) = \lim_{t \to 0} f(0, t) = \lim_{t \to 0} \left(-\frac{0}{t^2}\right) = \lim_{t \to 0} 0 = 0$$

which is consistent with Figure 13.2.3*b*.

Solution (c). The line $y = x$ has parametric equations $x = t$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } y = x)}} f(x, y) = \lim_{t \to 0} f(t, t) = \lim_{t \to 0} \left(-\frac{t^2}{2t^2}\right) = \lim_{t \to 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

Solution (d). The line $y = -x$ has parametric equations $x = t$, $y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } y = -x)}} f(x, y) = \lim_{t \to 0} f(t, -t) = \lim_{t \to 0} \frac{t^2}{2t^2} = \lim_{t \to 0} \frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

Solution (e). The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x, y) \to (0, 0) \\ (\text{along } y = x^2)}} f(x, y) = \lim_{t \to 0} f(t, t^2) = \lim_{t \to 0} \left(-\frac{t^3}{t^2 + t^4}\right) = \lim_{t \to 0} \left(-\frac{t}{1 + t^2}\right) = 0$$

This is consistent with Figure 13.2.3*c*, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1 + t^2}$$

superimposed on the surface. ◀

OPEN AND CLOSED SETS

Let C be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ. The set of points that are enclosed by the circle, but do not lie on the circle, is called the ***open disk*** of radius δ centered at (x_0, y_0), and the set of points that lie on the circle together with those enclosed by the circle is called the ***closed disk*** of radius δ centered at (x_0, y_0)

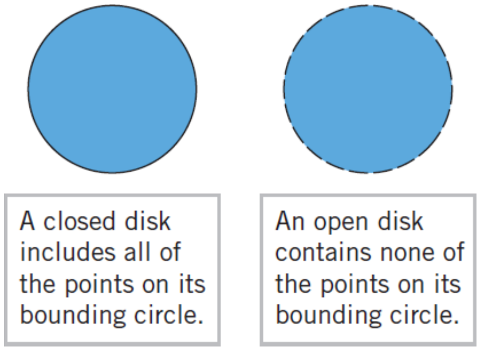


Figure 13.2.4

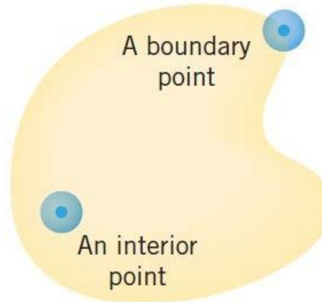


Figure 13.2.5

GENERAL LIMITS OF FUNCTIONS OF TWO VARIABLES

The statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

is intended to convey the idea that the value of $f(x, y)$ can be made as close as we like to the number L by restricting the point (x, y) to be sufficiently close to (but different from) the point (x_0, y_0). This idea has a formal expression in the following definition and is illustrated in Figure 13.2.6.

13.2.1 DEFINITION Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \tag{3}$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

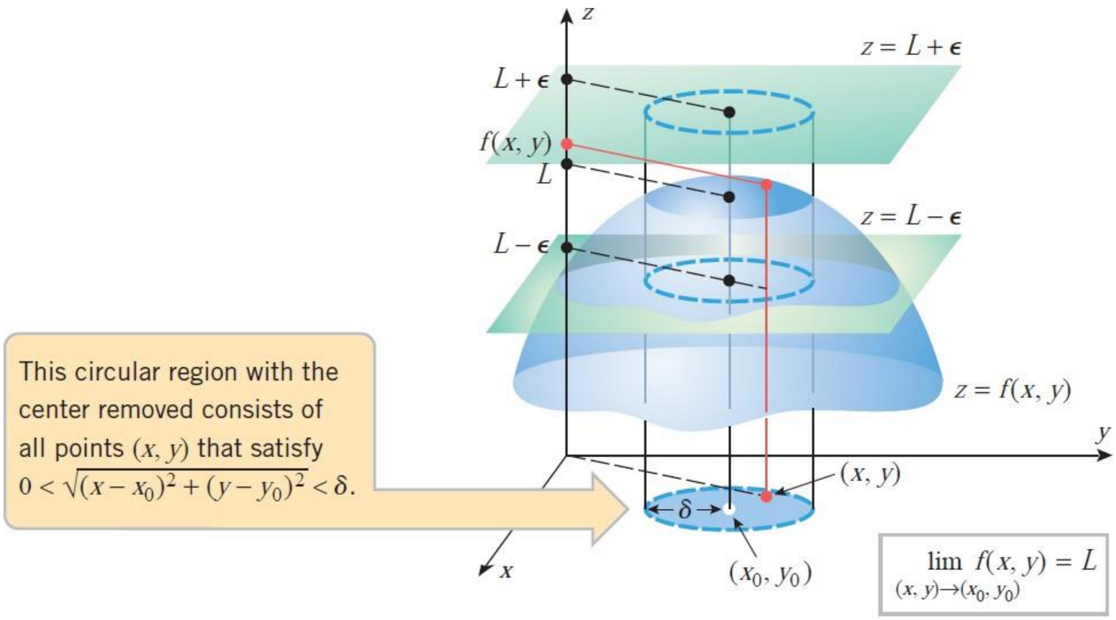


Figure 13.2.6

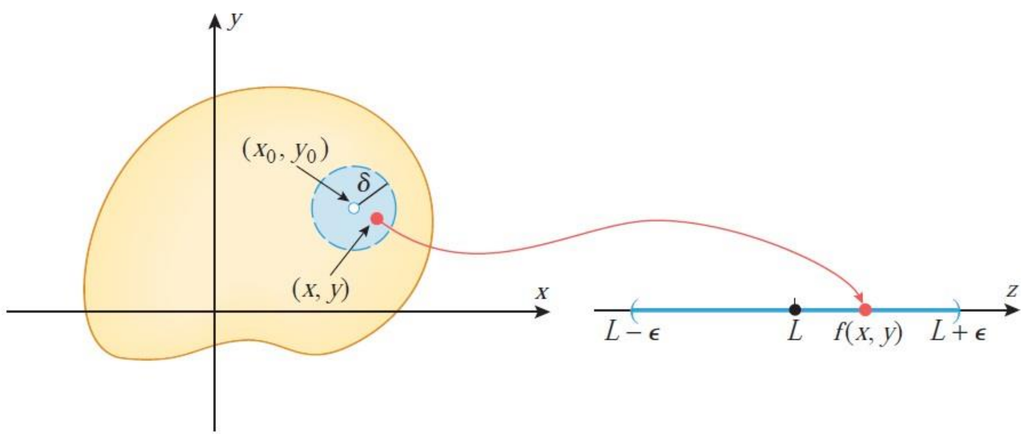


Figure 13.2.7

Example 2

$$
\begin{aligned}
\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] &= \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9 \\
&= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3 \left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9 \\
&= 5(1)^3(4)^2 - 9 = 71
\end{aligned}
$$

13.2.2 THEOREM

(*a*) *If $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$, then $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ along any smooth curve.*

(*b*) *If the limit of $f(x, y)$ fails to exist as $(x, y) \to (x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y) \to (x_0, y_0)$.*

▶ **Example 3** The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist because in Example 1 we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} -\frac{xy}{x^2+y^2} = -\frac{1}{2} \quad ◀$$

CONTINUITY

13.2.3 DEFINITION A function $f(x, y)$ is said to be ***continuous at*** $(\boldsymbol{x_0}, \boldsymbol{y_0})$ if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on D***, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

Definition 3.2.24 *A function $f(x, y)$ is said to be* ***continuous*** *at a point (a, b) if the following is true:*

1. *(a, b) is in the domain of f.*
2. *$\lim_{(x,y)\to(a,b)} f(x, y)$ exists.*
3. *$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$*

Theorem 3.2.27 *The following results are true for multivariable functions:*

1. *The sum, difference and product of continuous functions is a continuous function.*
2. *The quotient of two continuous functions is continuous as long as the denominator is not* 0.
3. *Polynomial functions are continuous.*
4. *Rational functions are continuous in their domain.*
5. *If* $f(x, y)$ *is continuous and* $g(x)$ *is defined and continuous on the range of* f, *then* $g(f(x, y))$ *is also continuous.*

13.2.4 THEOREM

(*a*) *If* $g(x)$ *is continuous at* x_0 *and* $h(y)$ *is continuous at* y_0, *then* $f(x, y) = g(x)h(y)$ *is continuous at* (x_0, y_0).

(*b*) *If* $h(x, y)$ *is continuous at* (x_0, y_0) *and* $g(u)$ *is continuous at* $u = h(x_0, y_0)$, *then the composition* $f(x, y) = g(h(x, y))$ *is continuous at* (x_0, y_0).

(*c*) *If* $f(x, y)$ *is continuous at* (x_0, y_0), *and if* $x(t)$ *and* $y(t)$ *are continuous at* t_0 *with* $x(t_0) = x_0$ *and* $y(t_0) = y_0$, *then the composition* $f(x(t), y(t))$ *is continuous at* t_0.

Example 3.2.28 *Is* $f(x, y) = x^2y + 3x^3y^4 - x + 2y$ *continuous at* $(0, 0)$? *Where is it continuous?*
$f(x, y)$ *is a polynomial function, therefore it is continuous on* $\mathbb{R}^2$. *In particular, it is continuous at* $(0, 0)$.

Example 3.2.29 *Where is* $f(x, y) = \frac{2x-y}{x^2+y^2}$ *continuous?*
f *is the quotient of two continuous functions, therefore it is continuous as long as its denominator is not* 0 *that is on* $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Example 3.2.30 *Where is* $f(x, y) = \frac{1}{x^2-y}$ *continuous?*
As above, f *is the quotient of two continuous functions. Therefore, it is continuous as long as its denominator is not* 0. *The denominator is* 0 *along the parabola* $y = x^2$. *Therefore,* f *is continuous on* $\left\{(x, y) \in \mathbb{R}^2 \mid y \neq x^2\right\}$.

Example 3.2.31 *Find where* $\tan^{-1}\left(\frac{xy^2}{x+y}\right)$ *is continuous.*
Here, we have the composition of two functions. We know that $\tan^{-1}$ *is continuous on its domain, that is on* $\mathbb{R}$*. Therefore,* $\tan^{-1}\left(\frac{xy^2}{x+y}\right)$ *will be continuous where* $\frac{xy^2}{x+y}$ *is continuous. Since* $\frac{xy^2}{x+y}$ *is the quotient of two polynomial functions, therefore it will be continuous as long as its denominator is not* 0*, that is as long as* $y \neq -x$*. It follows that* $\tan^{-1}\left(\frac{xy^2}{x+y}\right)$ *is continuous on* $\left\{(x,y) \in \mathbb{R}^2 : y \neq -x\right\}$.

Example 3.2.32 *Find where* $\ln\left(x^2+y^2-1\right)$ *is continuous.*
Again, we have the composition of two functions. $\ln$ *is continuous where it is defined, that is on* $\{x \in \mathbb{R} : x > 0\}$*. So,* $\ln\left(x^2+y^2-1\right)$ *will be continuous as long as* x^2+y^2-1 *is continuous and positive.* x^2+y^2-1 *is continuous on* $\mathbb{R}^2$*, but* $x^2+y^2-1>0$ *if and only if* $x^2+y^2>1$*, that is outside the circle of radius* 1*, centered at the origin. It follows that* $\ln\left(x^2+y^2-1\right)$ *is continuous of the portion of* $\mathbb{R}^2$ *outside the circle of radius* 1*, centered at the origin.*

Example 3.2.33 *Where is* $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \textit{if} \quad (x,y) \neq (0,0) \\ 0 & \textit{at} \quad (0,0) \end{cases}$ *continuous?*
Away from $(0,0)$*,* f *is a rational function always defined. So, it is continuous. We still need to investigate continuity at* $(0,0)$*. In an earlier example, we found that* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ *did not exist. Therefore,* f *is continuous everywhere except at* $(0,0)$.

Example 3.2.34 *Where is* $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \textit{if} \quad (x,y) \neq (0,0) \\ 0 & \textit{at} \quad (0,0) \end{cases}$ *continuous?*
Away from $(0,0)$*,* f *is a rational function always defined. So, it is continuous. We still need to investigate continuity at* $(0,0)$*. In an earlier example, we found that* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$*. Therefore,* f *is also continuous at* $(0,0)$*. It follows that* f *is continuous everywhere.*

▶ **Example 4** Use Theorem 13.2.4 to show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution. The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by part (a) of Theorem 13.2.4, the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane. Since $3x^2y^5$ is continuous at every point in the xy-plane and $\sin u$ is continuous at every real number u, it follows from part (b) of Theorem 13.2.4 that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous everywhere. ◀

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

▶ **Example 5** Evaluate $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2}$.

Solution. Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5} \quad ◀$$

▶ **Example 6** Since the function

$$f(x, y) = \frac{x^3y^2}{1-xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$. ◀

LIMITS AT DISCONTINUITIES

Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y)\to(0, 0)$ along any smooth curve (Figure 13.2.9). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0 \cdot \infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.

► **Example 7** Find $\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$.

Solution. Let (r, θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2 + y^2}$, so that $r \to 0^+$ if and only if $(x, y) \to (0, 0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2) &= \lim_{r\to 0^+} r^2 \ln r^2 \\
&= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \text{This converts the limit to an indeterminate form of type } \infty/\infty. \\
&= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \text{L'Hôpital's rule} \\
&= \lim_{r\to 0^+} (-r^2) = 0
\end{aligned}$$ ◄

Practice Problem:

Chapter 13.2

1,3,5,7,9-12,13,15,23-26