Unit 11.4.

$u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$.

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$= i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

$$= i(u_2 v_3 - u_3 v_2) - j(u_1 v_3 - u_3 v_1) + k(u_1 v_2 - u_2 v_1)$$

<u>Example 1</u> let $u = \langle 1, 2, -2 \rangle$ and $v = \langle 3, 0, 1 \rangle$

Find (a) $u \times v$ (b) $v \times u$.

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

(b) $v \times u = -(u \times v)$

Example 2: show that $u \times u = 0$ for any vector u in 3-space.

$$u = u_1 i + u_2 j + u_3 k.$$

$$u \times u = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0$$

$$i \begin{vmatrix} u_2 & u_3 \\ u_2 & u_3 \end{vmatrix}$$

- $u \times v = -(v \times u)$.
- $u \times (v + w) = (u \times v) + (u \times w)$
- $(u+v) \times w = (u \times w) + (v \times w)$,
- $k(u \times v) = (ku) \times v$
  $= u \times (kv)$.

$i = 1, 0, 0$

$j = 0, 1, 0.$

$k = 0, 0, k$

$(a+b) \cdot c$

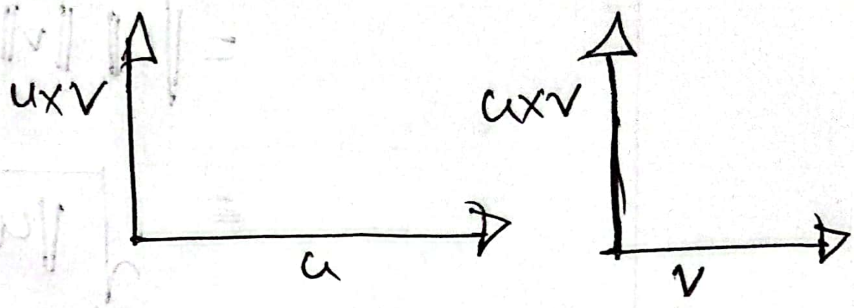
$ac + bc$

If u and v are vectors in 3-space, then :

(a) $u \cdot (u \times v) = 0$

(b) $v \cdot (u \times v) = 0$

$u \times v$ are



<u>Example 3</u> Find a vector that is orthogonal to both of the vectors $u = (2, -1, 3)$ and $v = (-7, 2, -1)$.

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix}$$

$$= i \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} - j \begin{vmatrix} 2 & 3 \\ -7 & -1 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ -7 & 2 \end{vmatrix}.$$

$$= -5i - 19j - 3k.$$

Proof

$$\|u\|\|v\|\sin\theta = \|u\|\|v\|\sqrt{1-\cos^2\theta}$$

$$= \|u\|\|v\|\sqrt{1-\frac{(u\cdot v)^2}{\|u\|^2\|v\|^2}}$$

$$= \sqrt{\|u\|^2\|v\|^2-(u\cdot v)^2}$$

$$= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2}$$

23 -32

$$= \sqrt{(u_2v_3-\cancel{v_3}\,u_3v_2)^2+(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2}$$

13 31

12 21

$$= \|u\times v\|$$

Example 4 Find the area of the triangle that is determined by the points $P_1(2,2,0)$, $P_2(-1,0,2)$ and $P_3(0,4,3)$.

$P_1P_2 = P_2\langle -1,0,2\rangle - P_1\langle 2,2,0\rangle$.

$P_1P_3 = P_3\langle 0,4,3\rangle - P_1\langle 2,2,0\rangle$

Area of parallelogram $= \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$

Area of triangle $= \frac{1}{2} \|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|$.

$$u\cdot(v\times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

(a) Let u, v and w be nonzero vectors in 3-space.

a) The volume V of the parallelepiped that has u, v and w as adjacent edges is

$$V = |u\cdot(v\times w)|$$

(b) $u\cdot(v\times w) = 0$ ~~if and~~ iff u, v and w lie in the same plane.

10. Find two unit vectors that are orthogonal to both.

$u = -7i + 3j + k \quad , \quad v = 2i + 4k.$

$$u \times v = \begin{vmatrix} i & j & k \\ -7 & 3 & 1 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= i \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} - j \begin{vmatrix} -7 & 1 \\ 2 & 4 \end{vmatrix} + k \begin{vmatrix} -7 & 3 \\ 2 & 0 \end{vmatrix}.$$

$u \times v = 12i + 30j - 6k.$

$$\|u \times v\| = \sqrt{(12)^2 + (30)^2 + (-6)^2}$$

$$= 6\sqrt{30}.$$

2 unit vectors are $= \pm\left(\frac{12}{6\sqrt{30}} i + \frac{30}{6\sqrt{30}} j - \frac{6}{6\sqrt{30}} k\right).$

$$= \pm\left(\frac{2}{\sqrt{30}} i + \frac{5}{\sqrt{30}} j - \frac{1}{\sqrt{30}} k\right).$$

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* 11. Find two unit vectors that are normal to the plane determined by the points $A(0,-2,1)$, $B(1,-1,-2)$ and $C(-1,1,0)$.

$$n = \vec{AB} \times \vec{AC}$$

$$= \langle 1-0,\ -1+2,\ -2-1 \rangle \times \langle -1, 3, -1 \rangle$$

$$= \langle 1, 1, -3 \rangle \times \langle -1, 3, -1 \rangle$$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = i \begin{vmatrix} 1 & -3 \\ 3 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= i(-1+9) - j(-1-3) + k(3+1)$$

$$= 8i + j + 4k$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{8^2 + {}^2 + 4^2} = 4\sqrt{6}$$

$$\pm \left(\frac{8}{4\sqrt{6}} i + \frac{4j}{4\sqrt{6}} + \frac{4}{4\sqrt{6}} k \right)$$

$$= \pm \left(\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} j + \frac{1}{\sqrt{6}} k \right)$$

$$= \pm \frac{1}{\sqrt{6}} \langle 2, 1, 1 \rangle$$

12. Find two unit vectors that are parallel to the yz-plane and are orthogonal to the vector $3i - j + 2k$.

parallel to the yz-plane, must be perpendicular to i

$$i \times (3i - j + 2k) = -2j - k.$$

$$= -(2j + k)$$

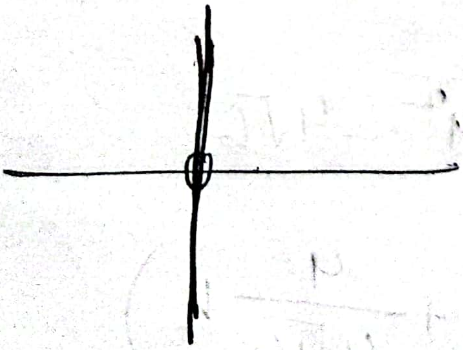
$$\|2j + k\| = \sqrt{2^2 + 1^2} = \sqrt{5}.$$

$$\pm (2j + k)\frac{1}{\sqrt{5}}.$$

17. $u \times v$ = area of parallelogram.

18. $u \times v$ = area of parallelogram.

19.



(29) ~~u =~~ Consider the parallelepiped with adjacent edges

$u = 3i + 2j + k.$

$v = i + j + 2k.$

$w = i + 3j + 3k.$

(a) Find the volume $u \cdot (v \times w) = \begin{vmatrix} \cancel{i} & \cancel{j} & \cancel{k} \\ 3 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{vmatrix}$

(b) Find the area of the face determined by u and w

$A = u \times w$

(c) Find the angle between u and the plane containing the face determined by v and w.

$u \cdot (v \times w) = \|u\| \|v \times w\| \cos\theta .$

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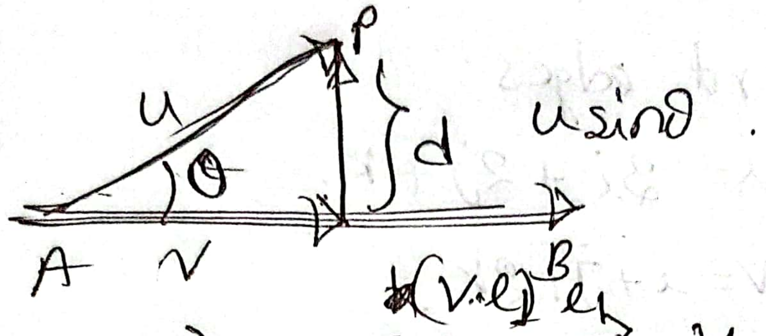


$$\operatorname{proj}_{\overrightarrow{AB}} \overrightarrow{AP} = \left(\overrightarrow{AP} \cdot \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}\right) \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}$$

$$= \left(\frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\|\overrightarrow{AB}\|^2}\right) \overrightarrow{AB}$$

$$d = v - \operatorname{proj}_{\overrightarrow{AB}} \overrightarrow{AP} = \overrightarrow{AP} - \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\|\overrightarrow{AB}\|^2} \cdot \overrightarrow{AB}.$$

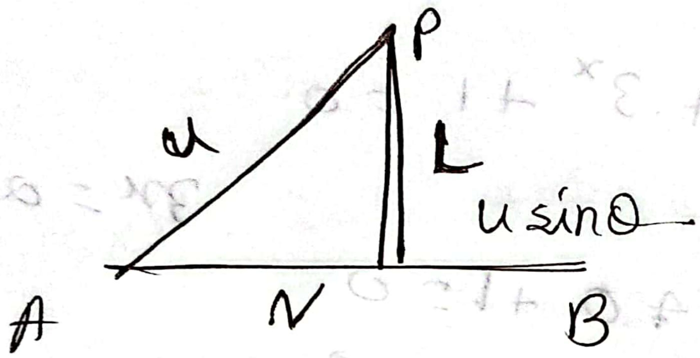
$\cancel{d}$

$$d = \|u\| \sin\theta = \frac{\|u\| \, \|v\| \sin\theta}{\|v\|}$$

$$= \frac{\|u \times v\|}{\|v\|}$$

31. Use the result in Exercise 30 to find the distance between the point P and the line through the points A and B.

a) $P(-3,1,2)$, $A(1,1,0)$, $B(-2,3,-4)$



$$d = u\sin\theta = \frac{\|u\|\sin\theta \cdot \|v\|}{\|v\|}$$

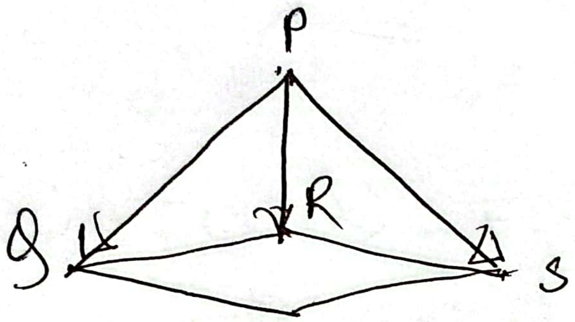
$$= \frac{\|u \times v\|}{\|v\|}$$

use that eqn.

33, Use the result of exercise 32 to find the volume of the tetrahedron with vertices.

like prism.

$P(-1,2,0)$, $Q(2,1,-3)$ $R(1,0,1)$, $S(3,-2,3)$.



volume of tetrahedron $= \frac{1}{6}\left(\vec{PQ} \cdot (\vec{PR} \times \vec{PS})\right)$