EXPONENTIAL MAP ANALYSIS

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1. **Identify the parameter regions that produce meaningful dynamics. For the logistic equation, we found that the parameter *r* must be between 0 and 4. This may be different for other maps.**

Exponential map: $x_{n+1} = x * e^{r(1-x_n)}$

If x_{n+1} becomes negative, the population Will become extinct. Therefore, we must look for solutions with $x_{n+1} > 0$

The only one scenario where we can find this situation is the next one:

1. $\boldsymbol{x_n > 0}$: if x_n is positive, x_{n+1} will be always positive.

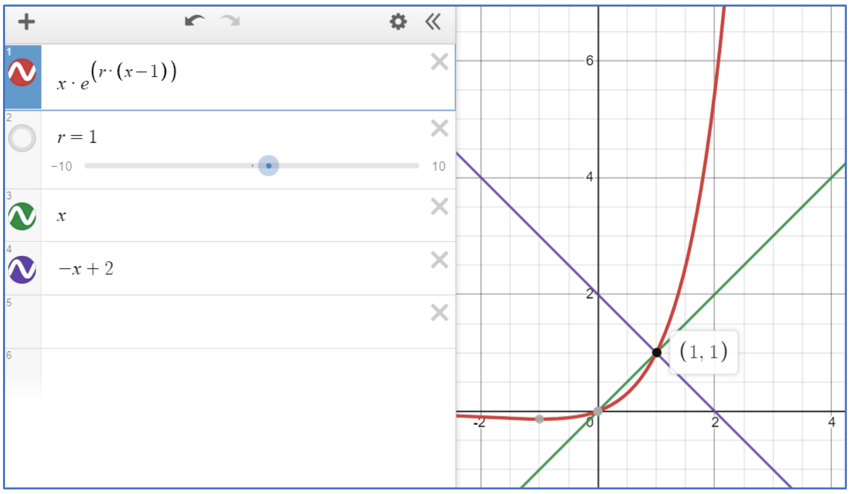
2. **Calculate fixed points. Estimate their stability. Determine the values of the parameter characterizing the different dynamics. If you find equations hard to solve, you can use numerical root-finding methods (e.g. Newton-Raphson, bisection, etc)**

1. Stationary state (x^*)

$$x_{n+1} = x_n$$
$$x^* = x^* * e^{r(1-x^*)}$$
$$\ln(1) = r * (1 - x^*)$$

$$0 = r(1 - x^*)$$
$$x^* = 1$$

*We only obtain 1 fixed point due to de linear grade of the equation



Here we can observe the fixed point (1,1) where 45º line (green line) cross the graphic. In addition, the tangent (purple line) is 45º (lower or equal to 45º implies a fixed point)

2. Estimating stability:

As we know, the stability condition is $|f(x^*)'| < 1$

Therefore:

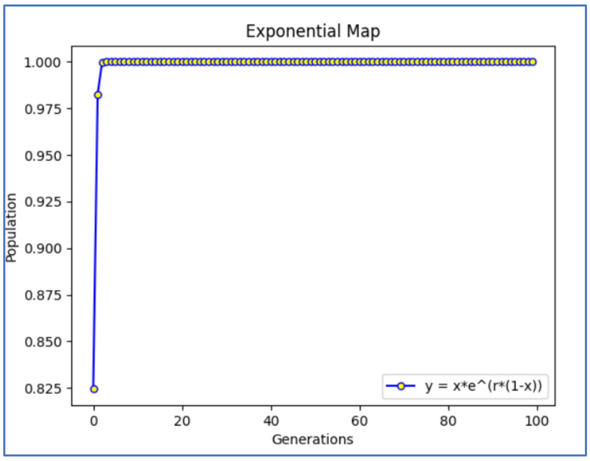
$$f(x) = x * e^{r(1-x_n)}$$
$$f'(x) = 1 * e^{r(1-x_n)} + \text{x} * (-r * e^{r(1-x_n)})$$

$f'(x) = e^{r(1-x_n)} - rx * e^{r(1-x_n)}$
$f'(x) = e^{r(1-x_n)} * (1 - rx)$

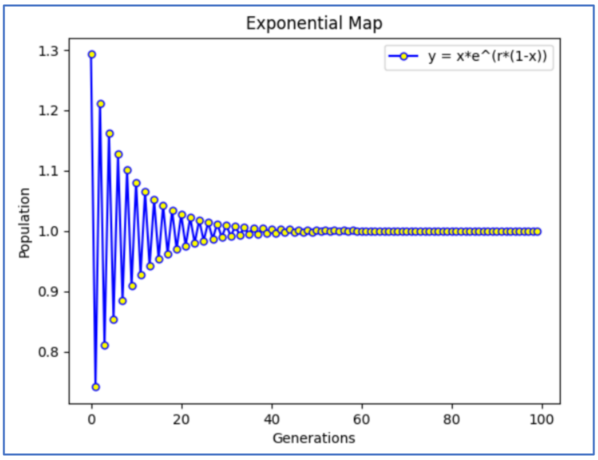
$f'(x) = |e^{r(1-x_n)} * (1 - rx)| < 1$

For x^*:
$f'(x^*) = |e^{r(1-1)} * (1 - r * 1)| < 1$
$f'(x^*) = |1 * (1 - r)| < 1$
$f'(x^*) = |1 - r| < 1$
$f'(x^*) = |r| < 2$
So: 0<=r<2

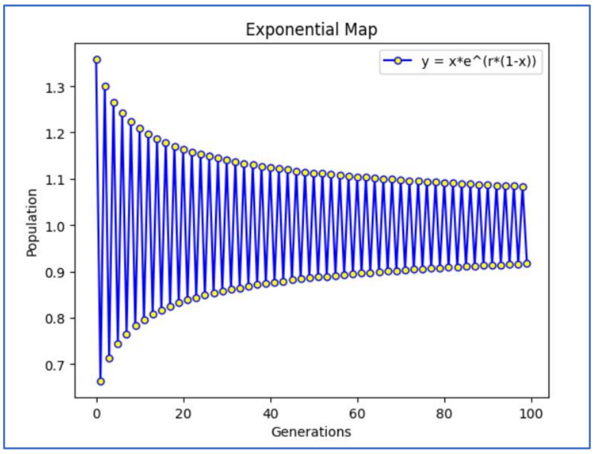
3. **Plot the dynamics of the system for different values of the relevant parameter. Try to identify regimes with a stable fixed point, period 2, 4 or other, and chaos.**
   By a python script y tried it for several values for r:
   X is always 0.5



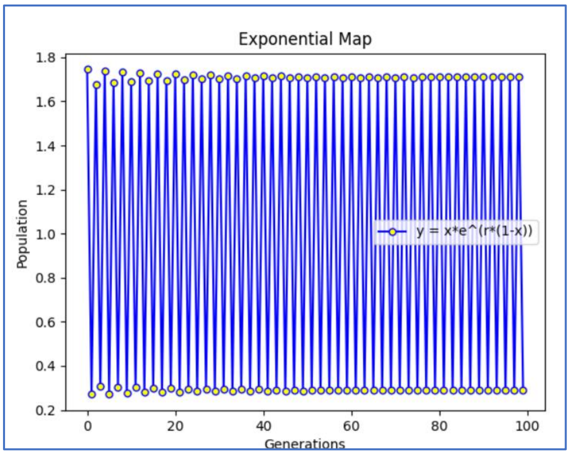
R=1



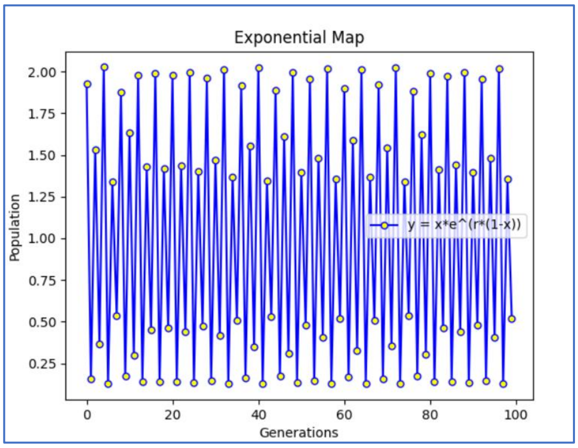
R= 1.9



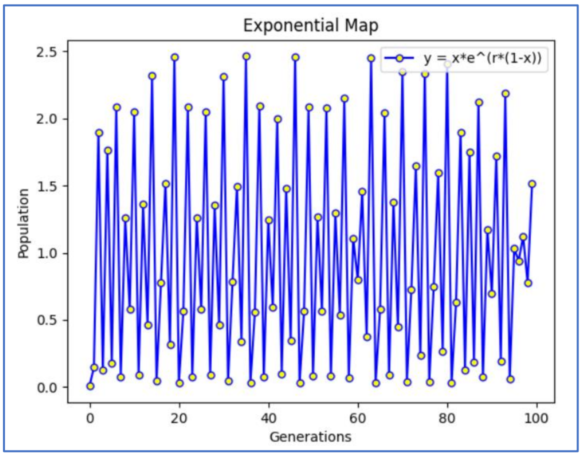
R=2



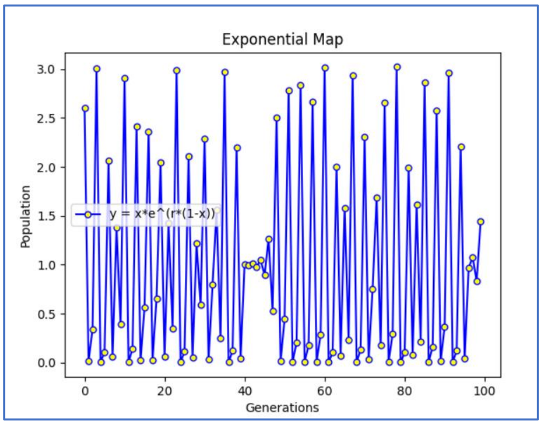
R=2.5



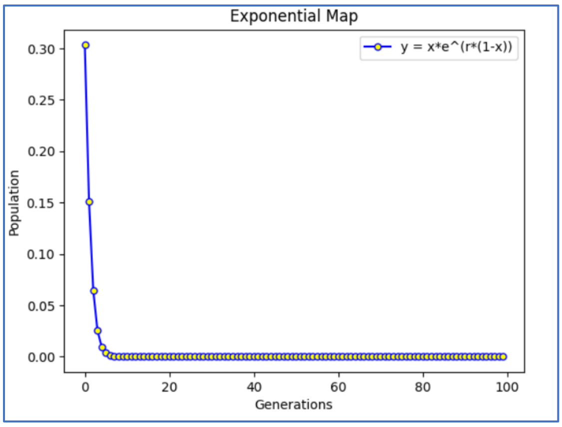
R=2.7



R=3



R=3.3

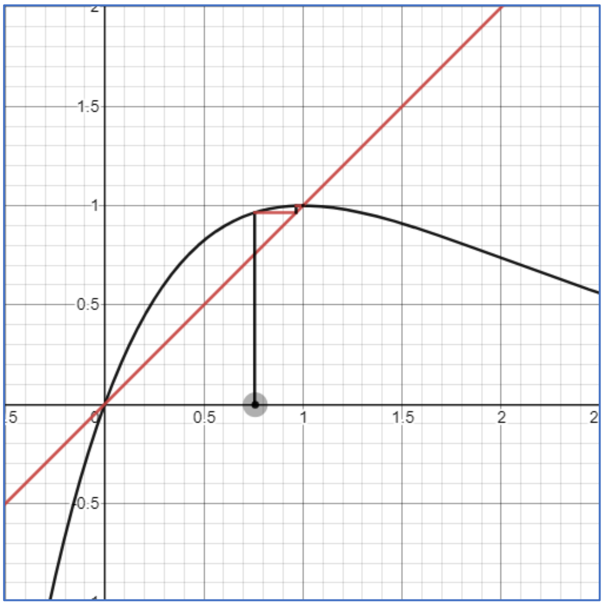


R=-1

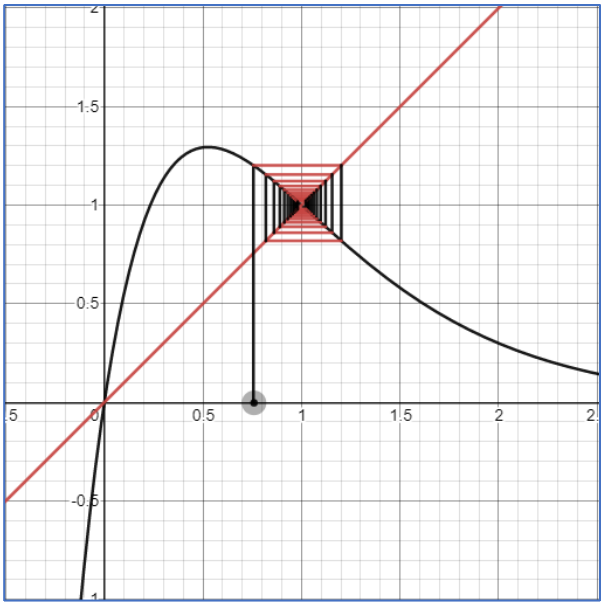
Conclusions:

- For r<0: population will become extinct
- For 0<r<2: Final value will be stable
- For 2<r<=2.5: population will oscillate between 2 values (period 2)
- For 2.6<r<=2.7: population will oscillate between 4 values (period 4)
- For 2.7<r<=3: chaotic behaviour
- For 3<r<=3.2: population will oscillate between 2 values (period 2)
- For r>=3.3: chaotic behaviour

4. Plot examples of the return map (also cobweb plot or graph for xn+1 versus xn) for different regimes.

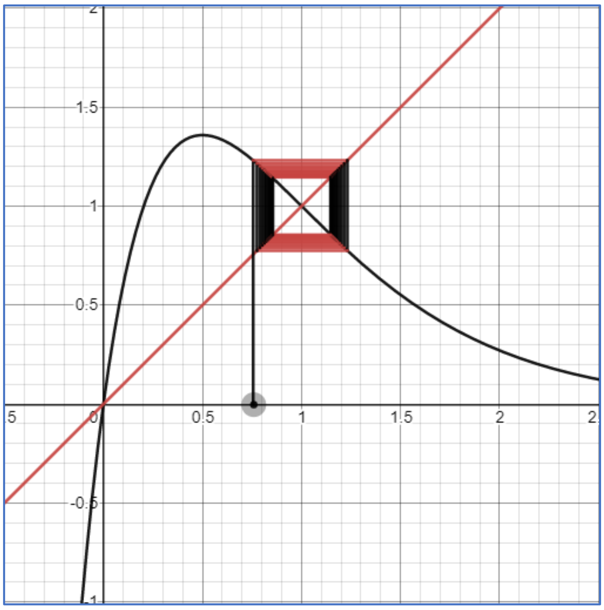


r=1

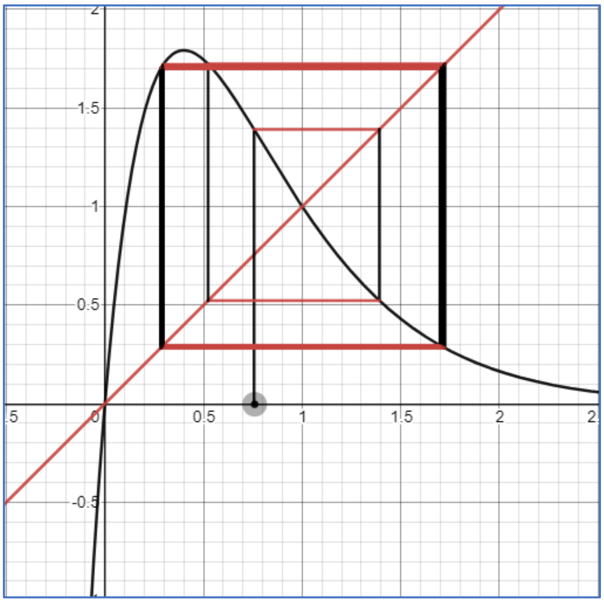


r=1.9

In both examples we can see that there is only one stable result. Concretely (1.0,1.0)



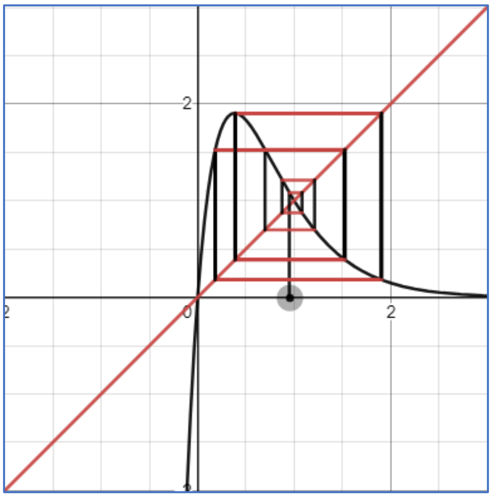
r=2



r=2.5

For this interval we can verify that there are 2 final results which are repeated over the time.

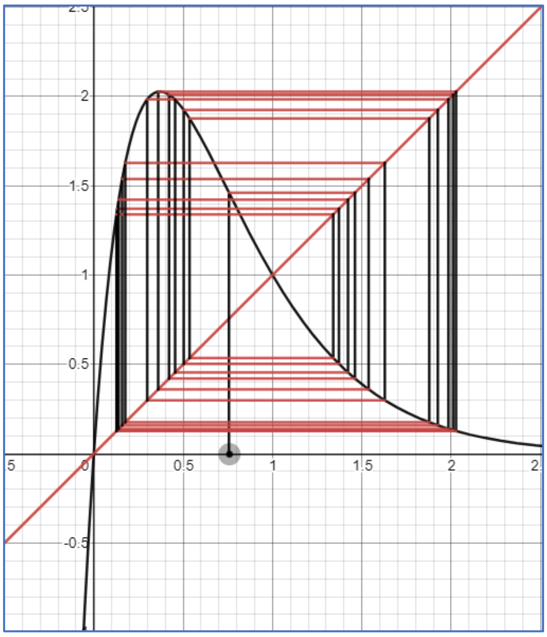
Therefore, for values between 2 and 2.5 there is a 2-period.



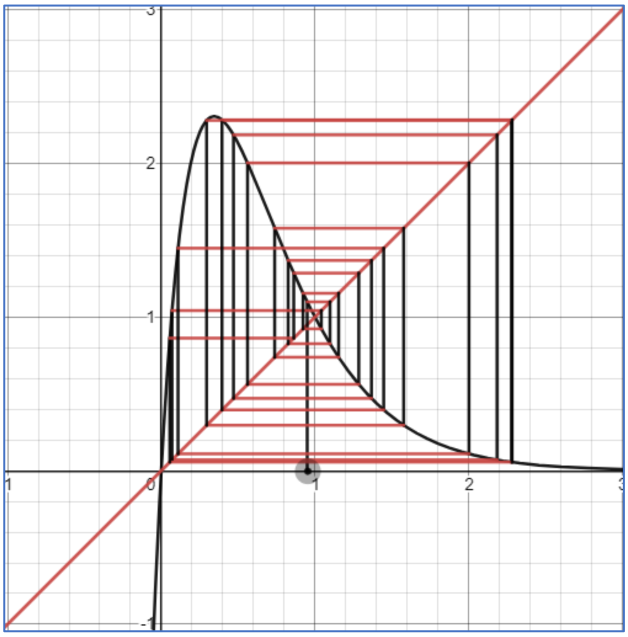
r=2.6

This 4-period interval is quite small but in this plot we can appreciate the 4 values

which result by using r=2.6

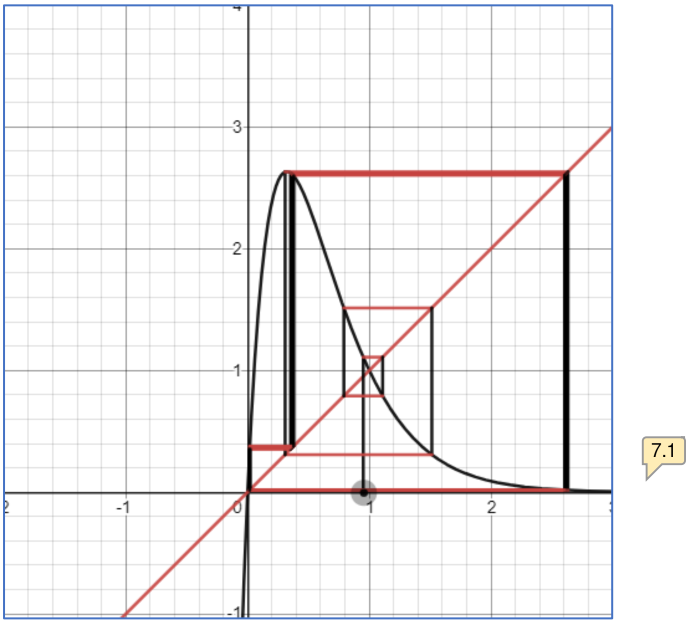


r=2.7



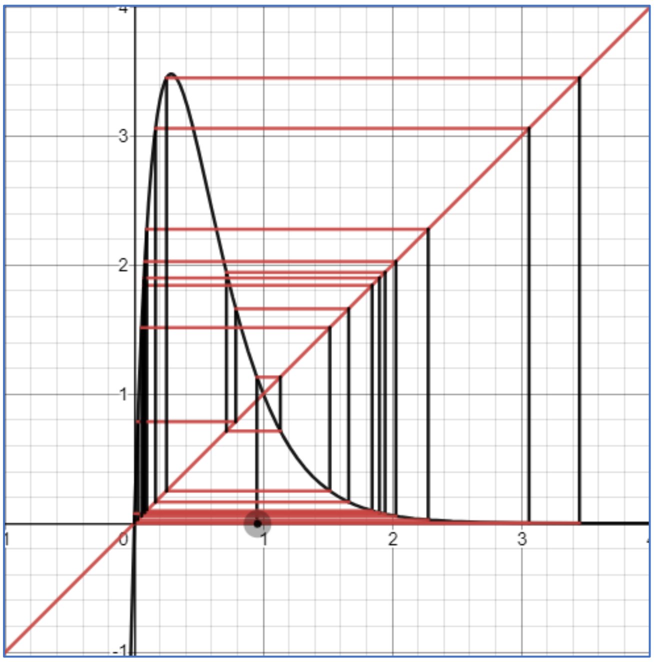
r=2.9

For r between 2.7 and 3 we obtain a chaotic behaviour. As we can see, there are several values (more than 4)



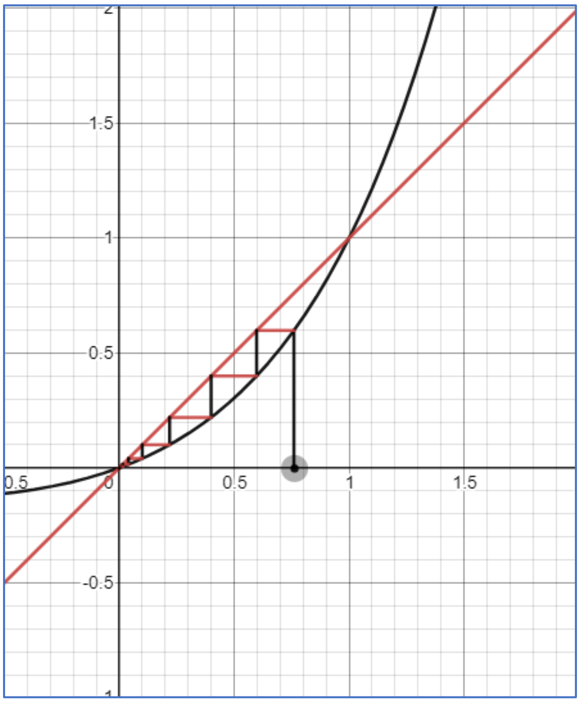
r= 3.1

This is a very particular case. Later, we will see at the bifurcation graph that there is a small interval which turn to be 2-period after being chaotic for the previous interval. Here, for r = 3.1 it is easy to see the 2 final results.



r=3.5

After the last explained 2-period interval, the graph becomes again chaotic. Here we can see its multiple results again.



r=-1

Although it is not much relevant, for r<=0 (as I explained in the first point) will end in 0, the hole population will become extinct

5. **Can you solve the equation for period-2 dynamics? Can you try to calculate its stability?**
   1. Period 2 equation

      Exponential map:

      $x * e^{(r*(1-x))}$

      Second period implies $f(f(x))$, therefore:

      $$(x * e^{(r*(1-x))}) * e^{\left(r*\left(1-(x*e^{(r*(1-x))})\right)\right)}$$
      $$x * e^{r-rx} * e^{r-rxe^{r-rx}}$$
      $$x * e^{r-rx} * e^{r-rx} * e^{e^{r-rx}}$$
      $$x * e^{(r-rx)^2} * e^{(e)^{r-rx}}$$
      $$x * e^{2r-2rx} * e^{er-erx}$$
      $$x * e^{r(2-2x)} * e^{r(e-ex)}$$
      $$x * e^{r^2} * e^{2-2x} * e^{e-ex}$$
      $$x * e^{2r} * e^{2-2x} * e^{e-ex}$$
      $$x = \frac{1}{e^{2r*(e^{2-2x})*e^{(e-ex)}}}$$
      $$x = \frac{1}{e^{(4r-4rx)*(e-ex)}}$$
      $$x = \frac{1}{e^{4erx^2-8erx+4er}}$$

$$x = \frac{1}{e^{4er*(x^2-x+1)}}$$
$$x = e^{-4er*(x^2-x+1)}$$

2. Estimating stability:

   As we know, the stability condition is $|f(x^*)'| < 1$
   Therefore:

   $$f(x) = e^{-4er*(x^2-x+1)}$$

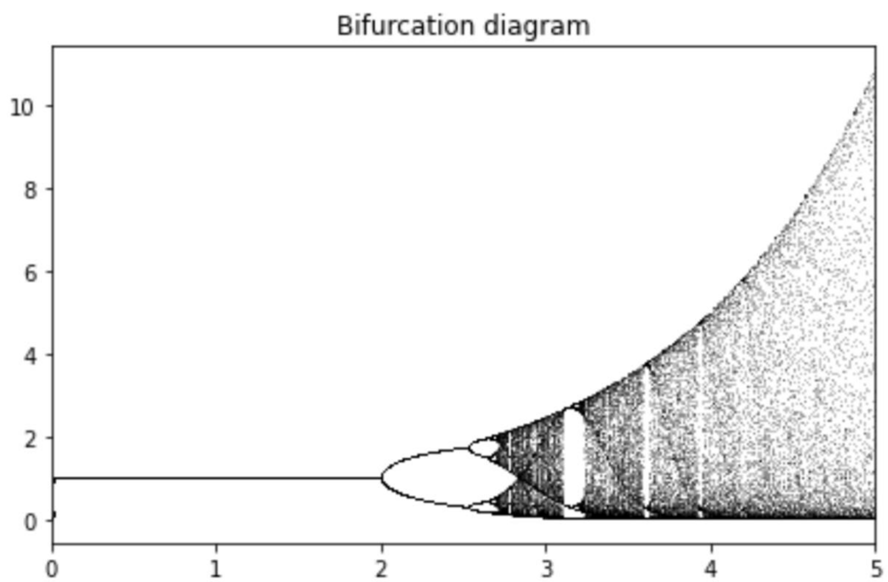
   $$f'(x) = (-8erx + 8er) * e^{-4er*(x^2-x+1)}$$

   $$f'(x) = |(-8erx + 8er) * e^{-4er*(x^2-x+1)}| < 1$$

   For x^*:
   $$f'(x^*) = |(-8erx^* + 8er) * e^{-4er*\left(x^{*2}-x^*+1\right)})| < 1$$
   $$f'(x^*) = |(-8er * (1-1) * e^{-4er*(1^2-1+1)}| < 1$$
   $$f'(x^*) = |0| < 1$$

6. Plot the bifurcation diagram (discard transients and represent values of xn for a number of time-steps at each value of r).

By a python script, I have made the bifurcation diagram which I shown below.



Here, we can appreciate which we have been discussing through this assignment. As we can see, the first bifurcation is around r = 2. At this point it starts a 2-period interval until r= 2.5.

For r = 2.5 until approx. 2.7 the graph turns to by 4-period. Then it turns to be chaotic as we cannot see how many results are there (at least more than 4)

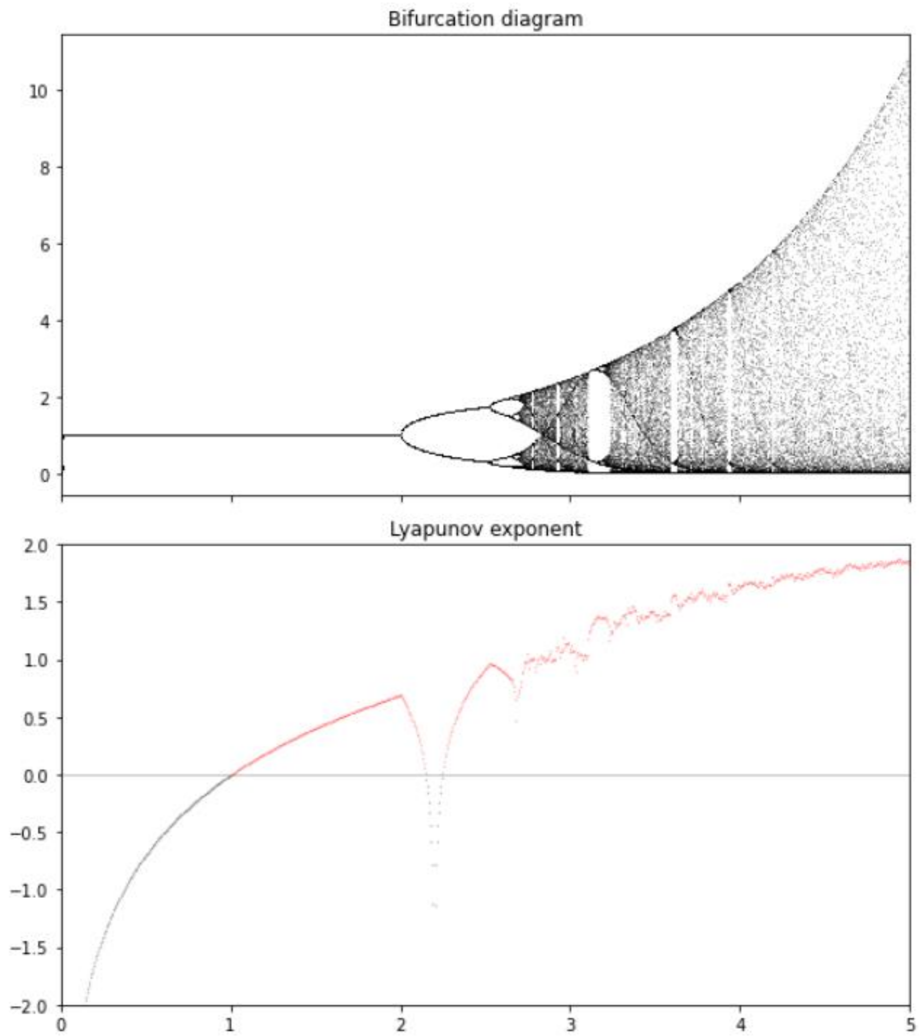
It is interesting to observe the graph value for r=3.1. If we look at the development of the graph, this point should preserve the chaotic state which occurs in previous intervals. However, we can see that at the [3.0-3.2] interval the graph turns to be 2-period again. After this interval, it returns to have a chaotic behaviour.

7. **Calculate the Lyapunov exponent for different values of r (advanced).**

The Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. It is usually use at the chaotic theory studies.

A we can see, the graph peaks matches with the r values where bifurcations occurs.

With a python script I have plotted the Lyapunov exponent for 0<r<5:



Índice de comentarios

2.1 This is correct. Additionally, you could have specificed that this holds for any r>0.

2.2 And also x*=0

2.3 The exponent has the wrong sign: you are actually plotting r = -1

4.1 Which plot shows that?

7.1 If you look carefully you will notice that the period is 3.

10.1 If you look closely you will see that there is period 3.

10.2 I think that the Lyapunov exponent is shifted 1 unit up: the chaotic region should coincide with values of the exponent >0, but here it seems it coincides with values >1.