SELF-DRIVE VERSUS EXTERNAL FORCING

Modelization and simulation of biosystems

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Exercise 3.1

Choose one of the systems below and study its synchronization properties. In all cases, think of natural situations that might be represented by those systems. If you find examples that are close but do not fit, discuss why.

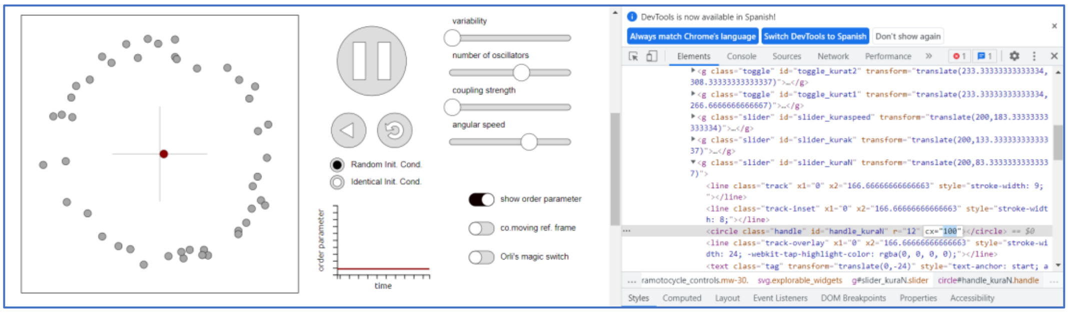
Self-drive versus external forcing. Implement two ensembles of 100 Kuramoto oscillators, one coupled through their common average phase Y and another through an external driving phase Wt. Choose between a Gaussian distribution g(w) of natural frequencies centered at p, a uniform distribution between 0 and 2p or a bimodal distribution with peaks at p/2 and 3p/2 and (equal) dispersions of your choice.

0. **First approach**

   First step I have taken is getting familiar with Kuramoto model. For this task, I have played with [Kuramoto simulator](), then I set asked parameter using web tools shown below.

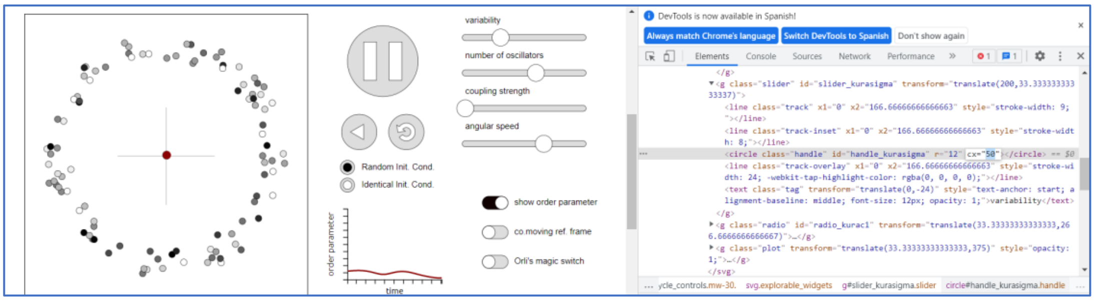
Ensemble 1:

- Varying = 0
- Nodes number = 100
- Coupling strength = 0
- Angular speed = 111



Ensemble 2:

- Varying = 50
- Nodes number = 100
- Coupling strength = 0
- Angular speed = 111



1. **Study the synchronization transition through the order parameter r as K varies in the two ensembles. Use W=p as external forcing. Are there quantitative differences between the two transitions? Why? How may the transition change if a different g(w) is used?**

Order parameter describes the strength of synchrony of the oscillators in the system. This value is always between 0 and 1. Consequently, if r = 0, oscillators are scattered uniformly around the circle. For r = 1, every oscillator will be on phase, and therefore, moving in synchrony.

Order parameter or phase-coherence formula:

$$r = \frac{1}{N}\sqrt{\left(\sum_j \sin\theta_j\right)^2 + \left(\sum_j \cos\theta_j\right)^2}$$

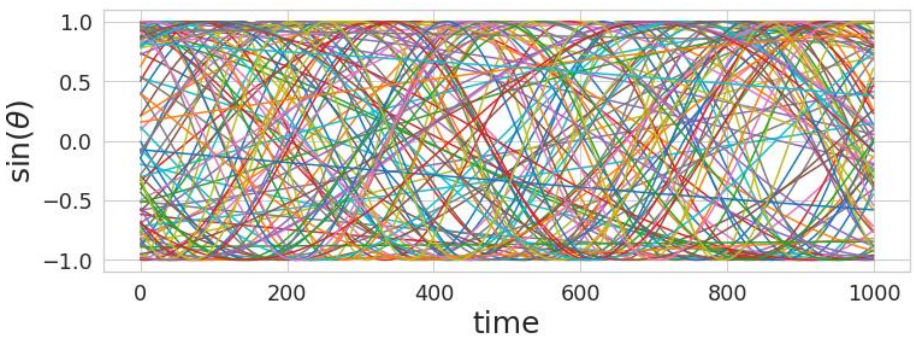
External forcing formula:

$$\dot{\theta}_i = \omega_i + K\sin(\Omega t - \theta_i)$$

Where K is the coupling strength and Ω is the external average phase

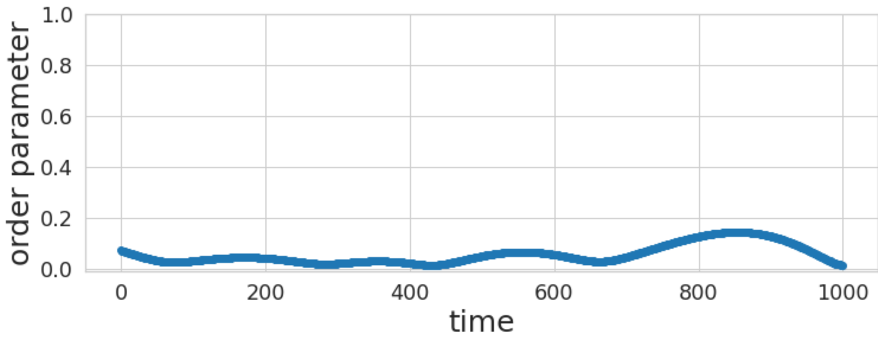
How does synchronization transition change as K varies?

a. **Ensembles 1:**
As we can remember *Ensemble 1* had 100 nodes in their common phase average (natural phase) and there wasn't any external forcing which tries to synchronize them, they don't have the same angular frequency.



As there are not external forcing and needer external driving, phases oscillate, without following any pattern.

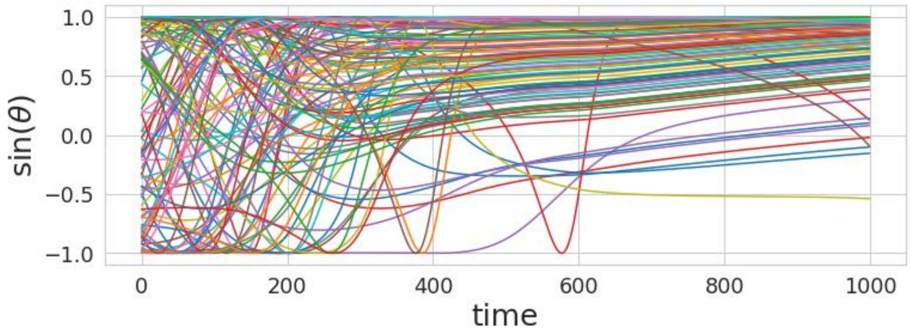
Another aspect we need to take into account is how order parameter change through time. Here we can see its evolution:



As order parameter is the mean of every node phase plotted before, and as we know, there isn't any external forcing to synchronize oscillators, it maintains kind of stable near 0 (oscillators are not in phase).

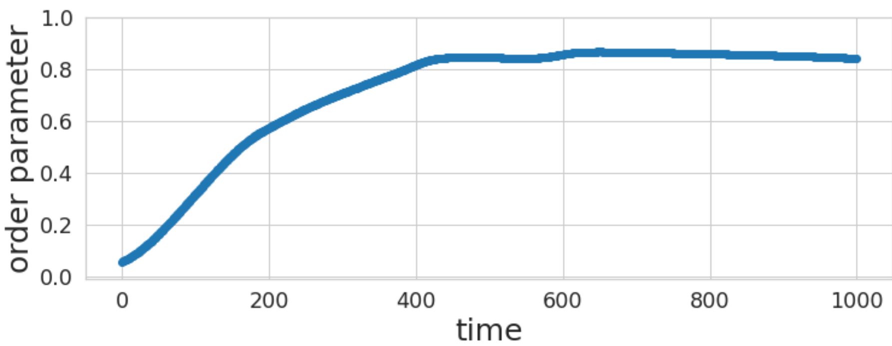
But what happens if an external force is introduce in the model?

I have taken just for instance an external forcing with value 2.5. Let's see how model develops.



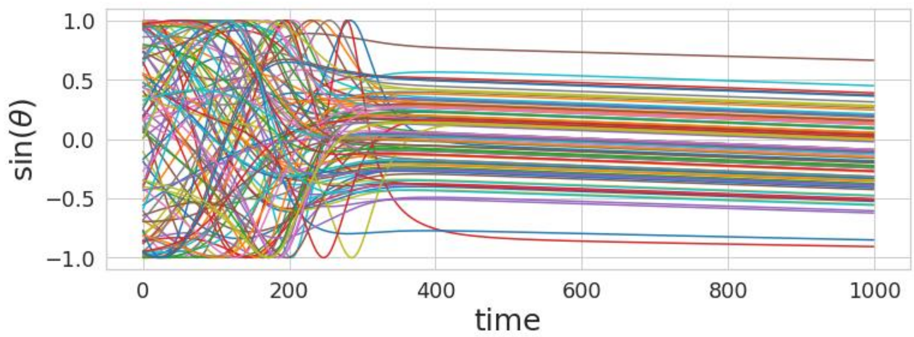
By applying external forces, we synchronize oscillators. That means oscillators turns to have the same angular frequency. Focusing on the graph, we can appreciate a development on the parallel position of the phase lines of each oscillator.

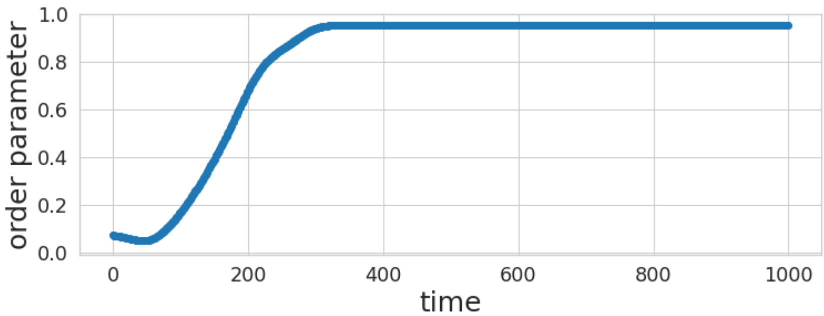
Let's see now how order parameter has developed with these changes:



Therefore, order parameter has also change. At last model (with no external forcing), order parameter keeps kind of stable. Now, it is greatly appreciable that order parameter increases as far as it almost turns to be one. In other words, every node is turning to be synchronize.

If K is increased just a bit, it is finally observable that order parameter value is one and every oscillator phase is parallel to each other (same angular frequency):

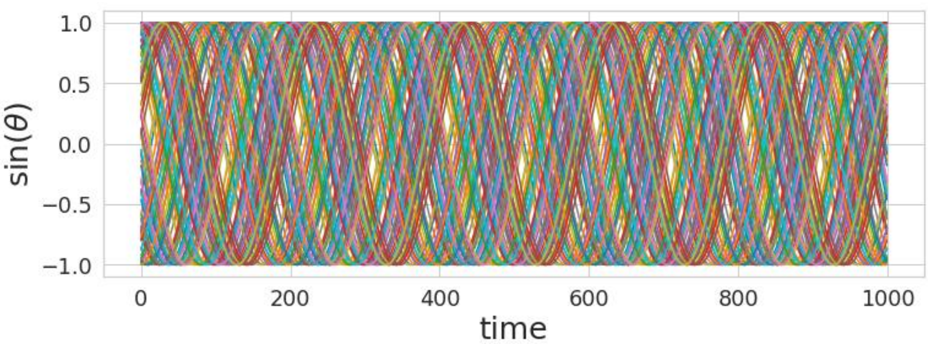




b. Ensembles 2:

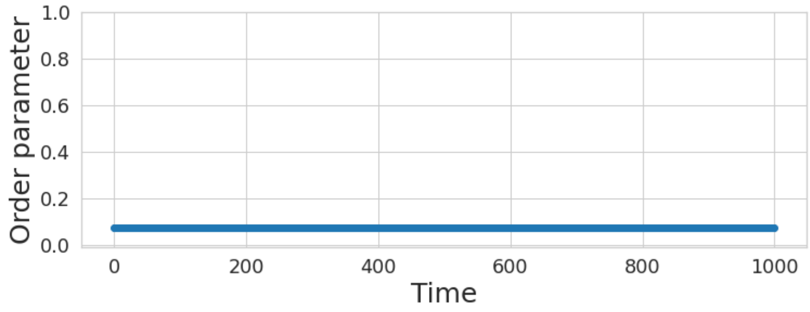
As we can remember *Ensemble 2* had 100 nodes and their phase average has been changed by an external driving phase (π). As last ensemble, there wasn't any external forcing which tries to synchronize them.

Here you can observe how different node's phase change through time:



In contrast with first ensemble, in this sample we can observe different oscillations in phases. The main reason for these changes is the external driving phase which I have set to π. It is also important to highlight that although all of them have the same angular frequency (lines at graph are parallel), they are not in phase.

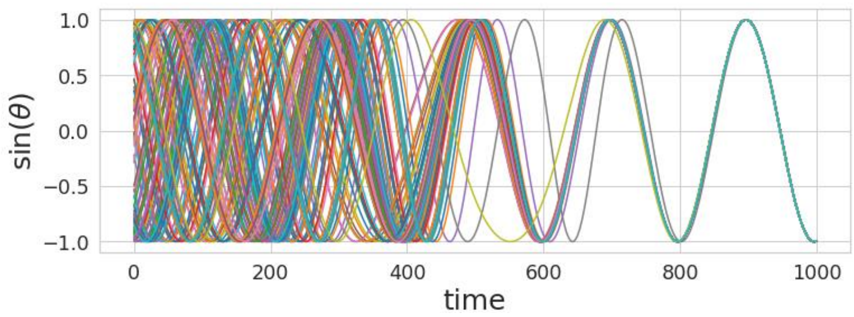
Let's see now how order parameter has developed:



Although we have introduced different phases in the model, order parameter keeps completely stable (phases are proportional to each other so mean doesn't change). In addition, there is not any external forcing which tries to synchronize the system, and therefore, it doesn't increase.

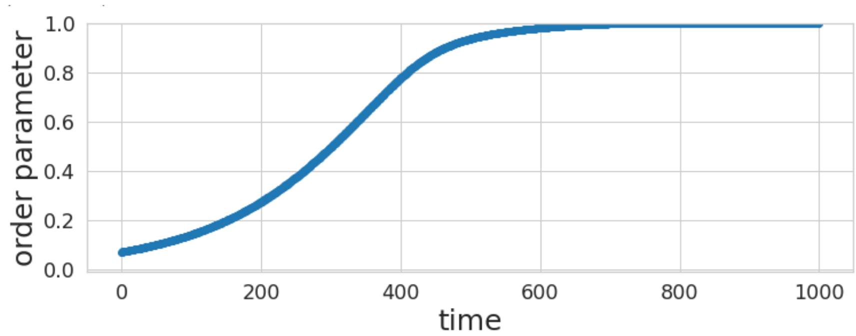
But what happens if an external force is introduce in the model?

I have taken just for instance an external forcing with value 2.5. Let's see how model develops.



At the initial model, nodes shared angular frequency but they were not in phase. By applying an external force, phases turn to be in synchrony. That's why at the end of the model every phase is overlap to each other. Finally, every oscillator shares angular frequency and phase with each other.

As predicable, order parameter will also change to turn from 0 to 1 (nodes in phase)



Conclusions: Are there quantitative differences between the two transitions?

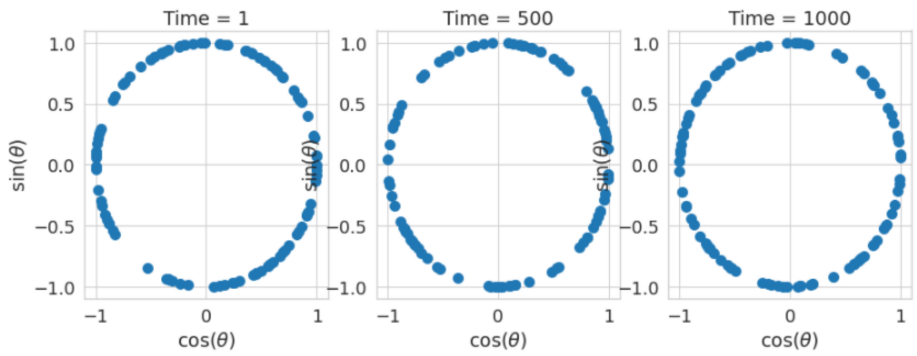
If we compare both ensembles, we can appreciate that both of them turn to be kind of synchronize. However, as oscillators of the first model did not have a proportional phase, they only synchronize its angular frequency. On the other hand, second assembler, as at the beginning they were out of phase but on a proportional way, applying an external forcing they turn to be synchronize talking about phase and angular frequency.

2. **Plot the distribution of phase differences between all pairs of oscillators for different values of K, before and after the transition. Discuss their shape**

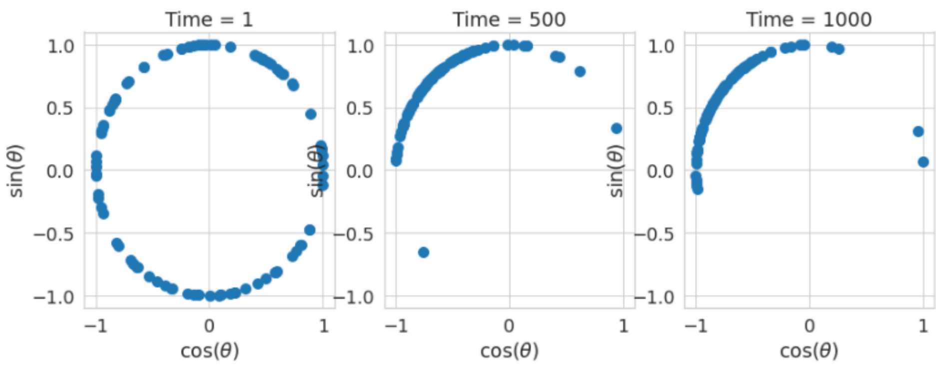
   As I have already shown how each node phase develops in each scenario, now let's see how nodes seems to be in the circle:

   a. **Ensemble 1:**

      The first set of images is for a non-external forcing (k) scenario. As I have explained before, there is no synchronization and therefore, oscillators don't overlap.

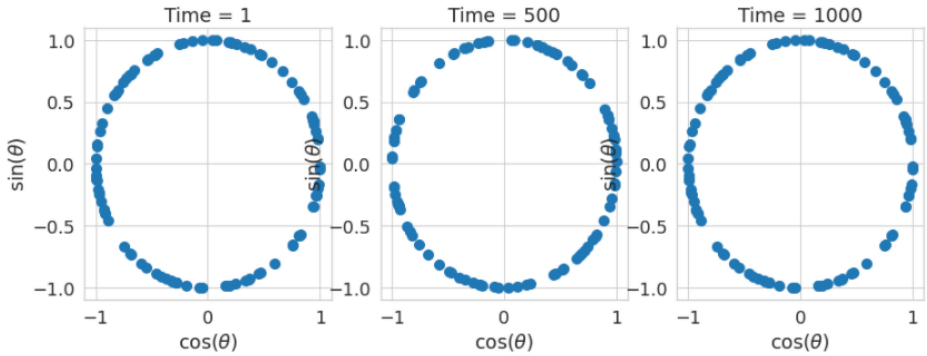


These three images, represent the model with an external force applied. As it is observable, oscillators turn to be overlap and therefore almost synchronize at time 1000.

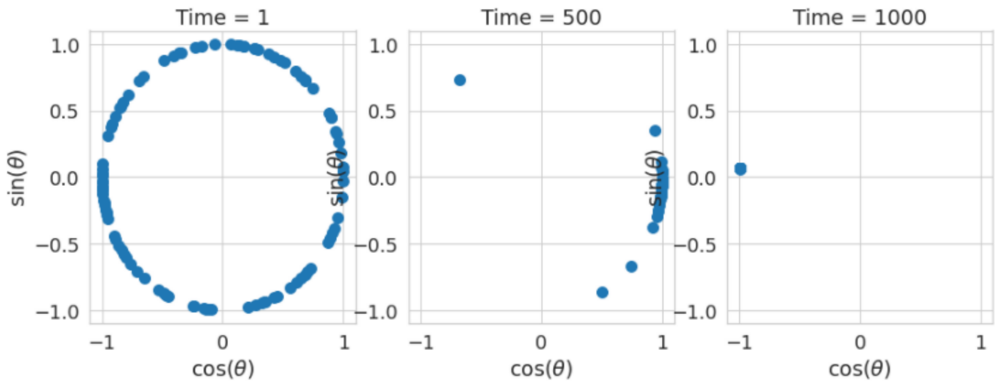


b. Ensemble 2:

The first set of images is for a non-external forcing (k) scenario. As I have explained before, there is no synchronization and therefore, oscillators don't overlap.



These three images, represent the model with an external force applied. As it is observable, starts to synchronize each other until they overlap at time 1000.



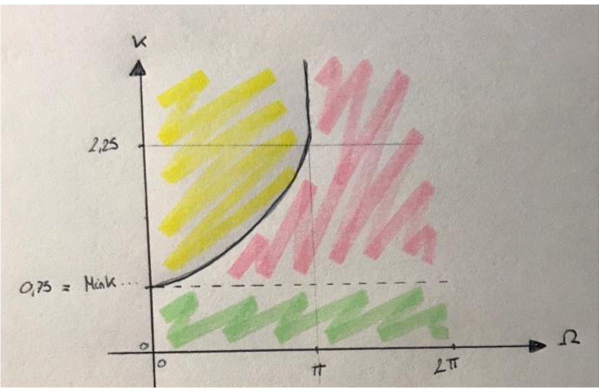
3. Vary Ω and K to estimate the regions where the frequencies of all oscillators are synchronized (Arnold tongues).

In mathematics, particularly in dynamical systems, Arnold tongues are a pictorial phenomenon that occur when visualizing how the rotation number of a dynamical system, or other related invariant property thereof, changes according to two or more of its parameters.

By experimenting with Kuramoto model, I have arrived to the conclusion that high K and low Ω turn the system to a synchronized stage. On the other hand, high values of Ω and low values of K maintains the system unsynchronized. For other cases, the behavior of the system is not sure, therefore it can be considered chaotic.

I have obtained a vertical asymptote at $\Omega = \pi$ and the lowest value for k to turn the system synchrony is K=0.75.

The result graph of these experiments can be seen below:



Yellow = synchronized
Green = unsynchronized
Pink = chaotic