EXPONENTIAL MAP ANALYSIS

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1. **Identify the parameter regions that produce meaningful dynamics. For the logistic equation, we found that the parameter *r* must be between 0 and 4. This may be different for other maps.**

Exponential map: $x_{n+1} = x * e^{r(1-x_n)}$

If x_{n+1} becomes negative, the population Will become extinct. Therefore, we must look for solutions with $x_{n+1} > 0$

The only one scenario where we can find this situation is the next one:

1. $\boldsymbol{x_n > 0}$: if x_n is positive, x_{n+1} will be always positive.

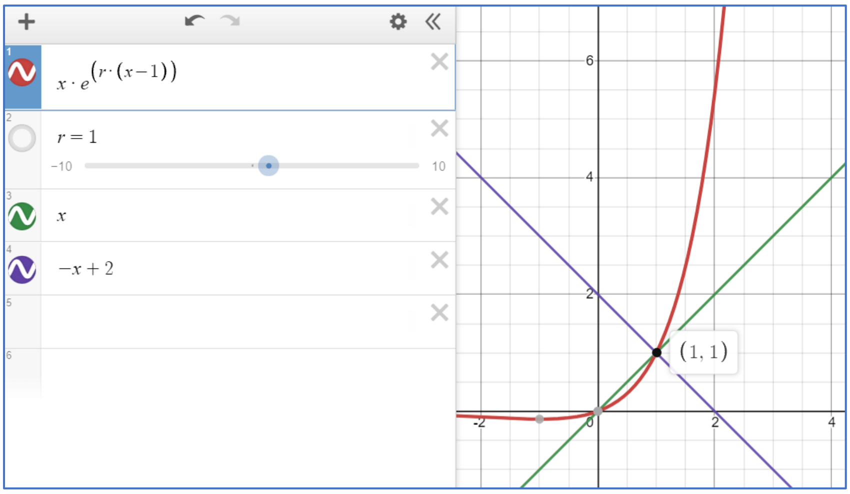
2. **Calculate fixed points. Estimate their stability. Determine the values of the parameter characterizing the different dynamics. If you find equations hard to solve, you can use numerical root-finding methods (e.g. Newton-Raphson, bisection, etc)**

1. Stationary state (x^*)

$$x_{n+1} = x_n$$
$$x^* = x^* * e^{r(1-x^*)}$$
$$\ln(1) = r * (1 - x^*)$$

*We only obtain 1 fixed point due to de linear grade of the equation

$$0 = r(1 - x^*)$$
$$x^* = 1$$



Here we can observe the fixed point (1,1) where 45º line (green line) cross the graphic. In addition, the tangent (purple line) is 45º (lower or equal to 45º implies a fixed point)

2. Estimating stability:

As we know, the stability condition is $|f(x^*)'| < 1$

Therefore:

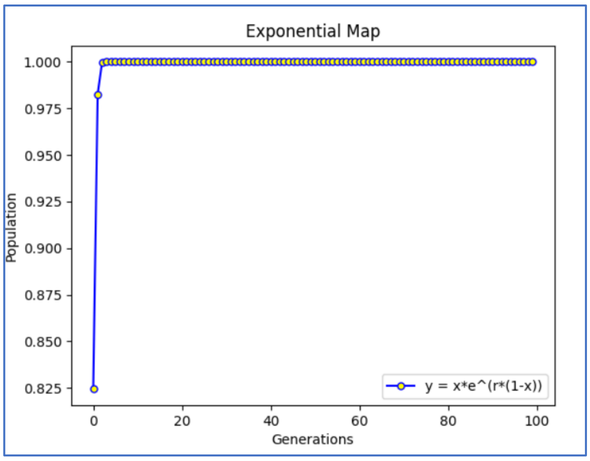
$$f(x) = x * e^{r(1-x_n)}$$
$$f'(x) = 1 * e^{r(1-x_n)} + \mathrm{x} * (-r * e^{r(1-x_n)})$$

$$f'(x) = e^{r(1-x_n)} - rx * e^{r(1-x_n)}$$
$$f'(x) = e^{r(1-x_n)} * (1 - rx)$$

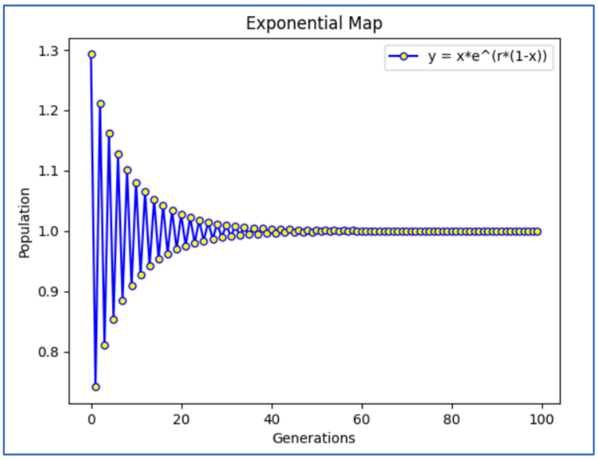
$$f'(x) = |e^{r(1-x_n)} * (1 - rx)| < 1$$

For x^*:
$$f'(x^*) = |e^{r(1-1)} * (1 - r * 1)| < 1$$
$$f'(x^*) = |1 * (1 - r)| < 1$$
$$f'(x^*) = |1 - r| < 1$$
$$f'(x^*) = |r| < 2$$
So: 0<=r<2

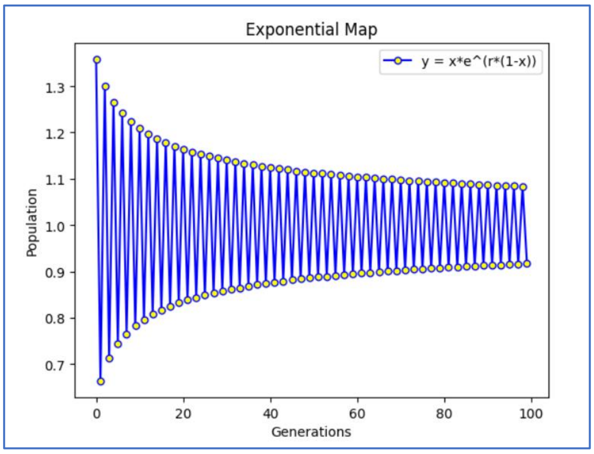
3. **Plot the dynamics of the system for different values of the relevant parameter. Try to identify regimes with a stable fixed point, period 2, 4 or other, and chaos.**
   By a python script y tried it for several values for r:
   X is always 0.5



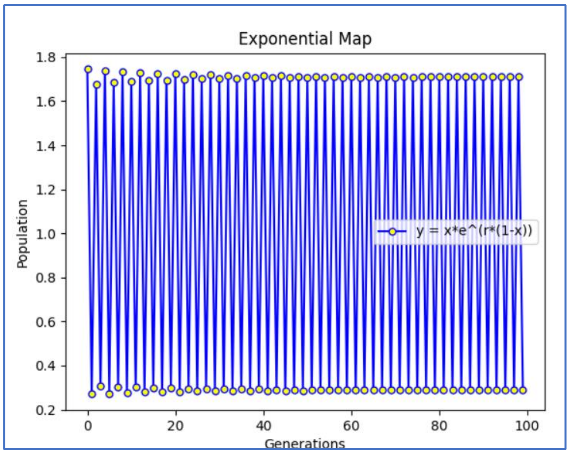
R=1



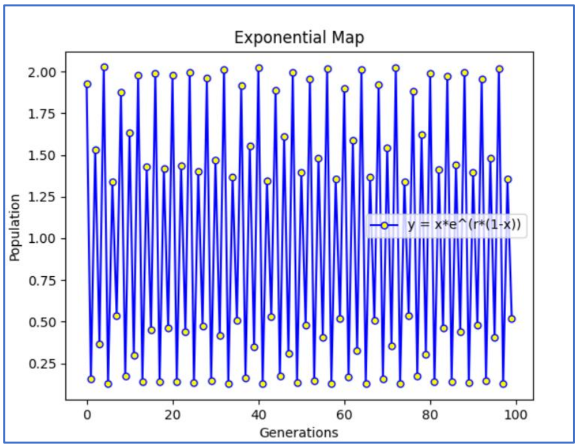
R= 1.9



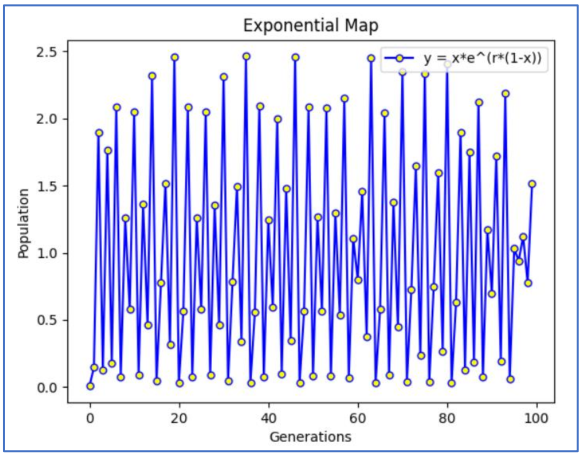
R=2



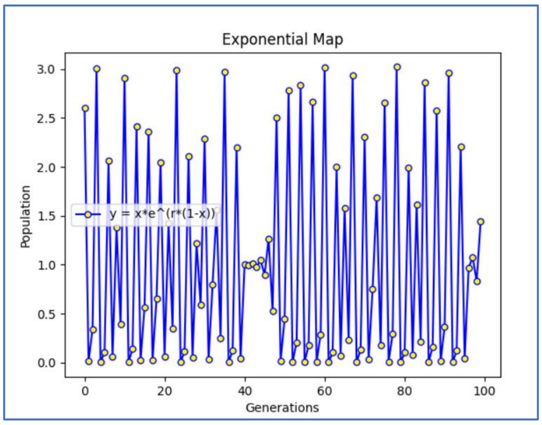
R=2.5



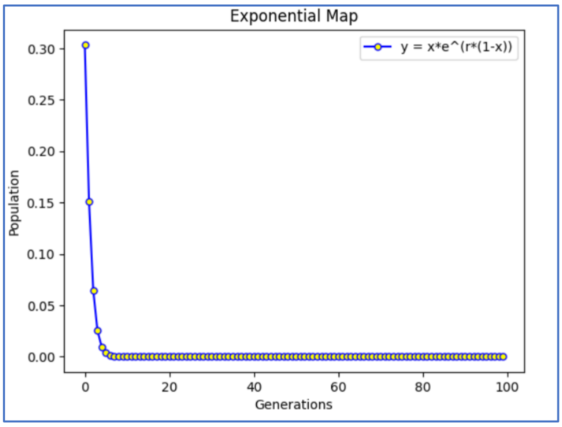
R=2.7



R=3



R=3.3

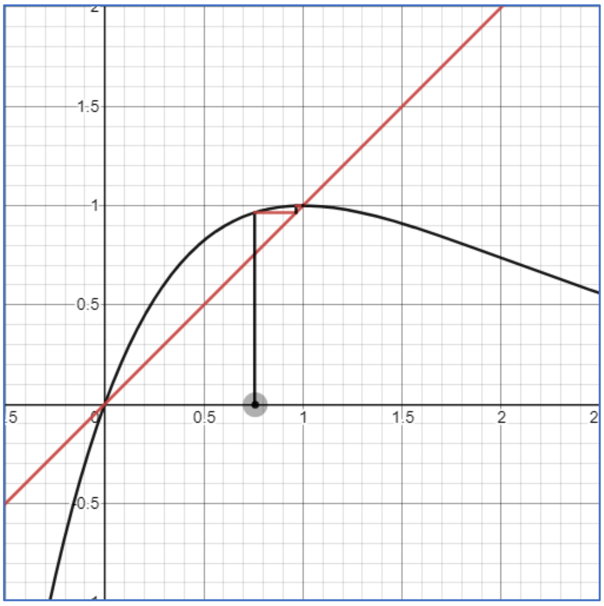


R=-1

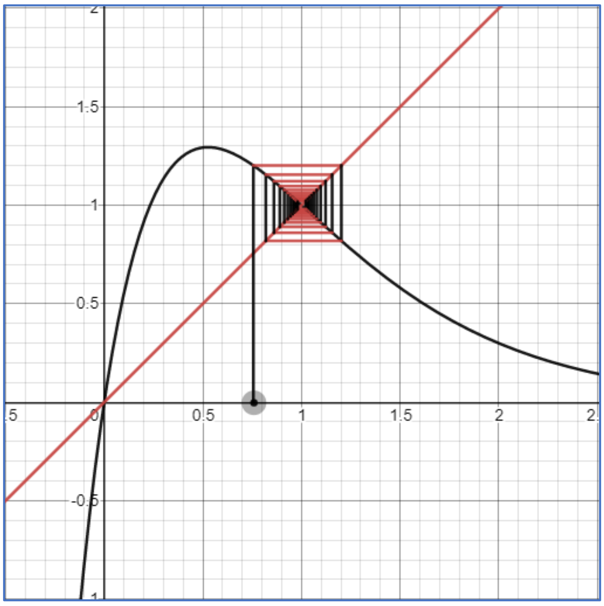
Conclusions:

- For r<0: population will become extinct
- For 0<r<2: Final value will be stable
- For 2<r<=2.5: population will oscillate between 2 values (period 2)
- For 2.6<r<=2.7: population will oscillate between 4 values (period 4)
- For 2.7<r<=3: chaotic behaviour
- For 3<r<=3.2: population will oscillate between 2 values (period 2)
- For r>=3.3: chaotic behaviour

4. Plot examples of the return map (also cobweb plot or graph for xn+1 versus xn) for different regimes.

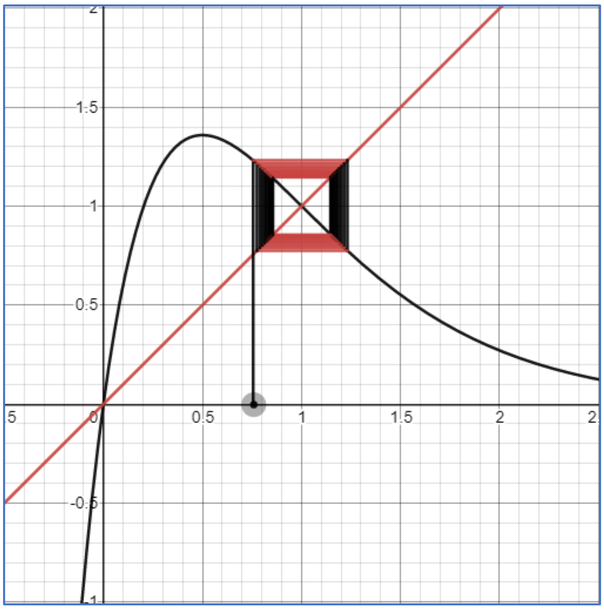


r=1

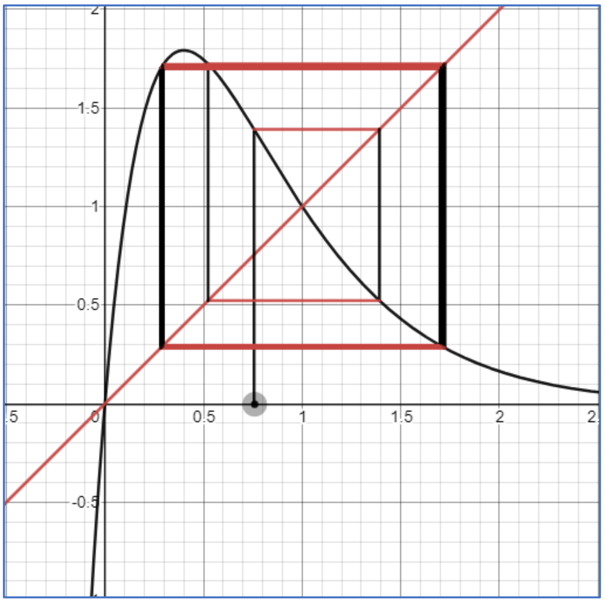


r=1.9

In both examples we can see that there is only one stable result. Concretely (1.0,1.0)



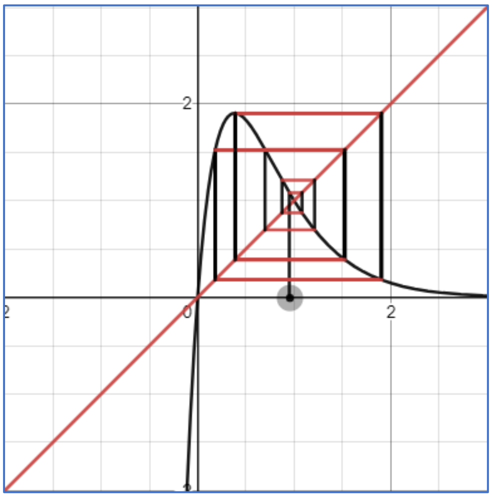
r=2



r=2.5

For this interval we can verify that there are 2 final results which are repeated over the time.

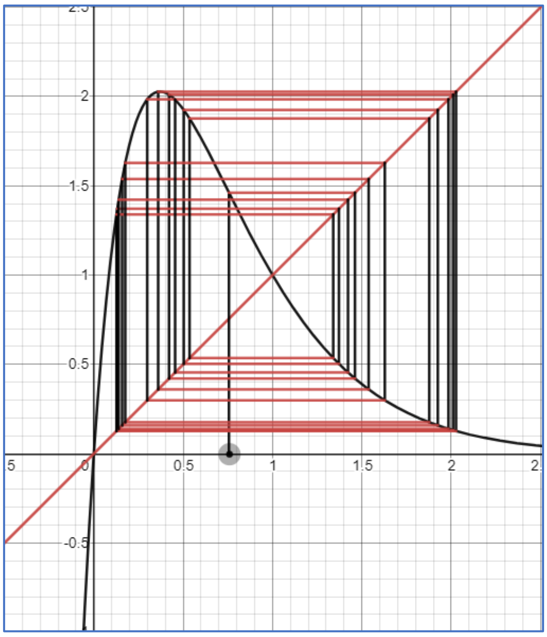
Therefore, for values between 2 and 2.5 there is a 2-period.



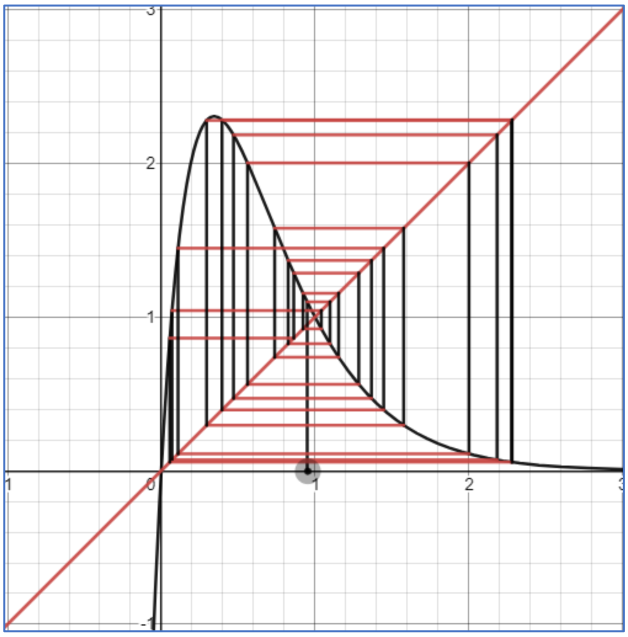
r=2.6

This 4-period interval is quite small but in this plot we can appreciate the 4 values

which result by using r=2.6

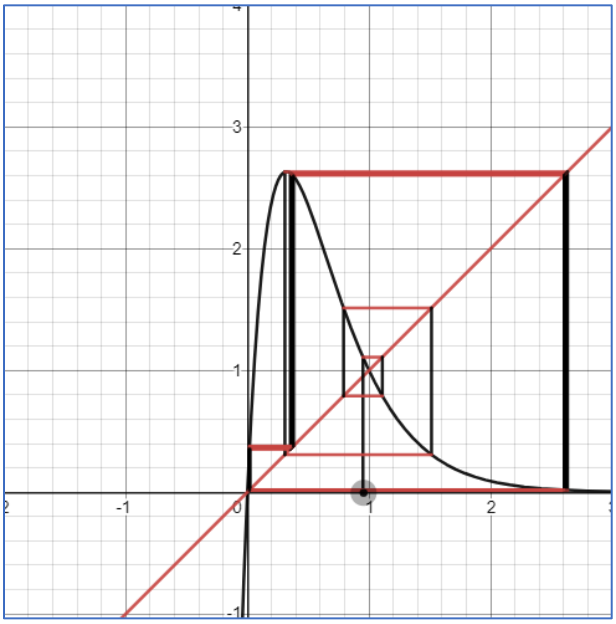


r=2.7



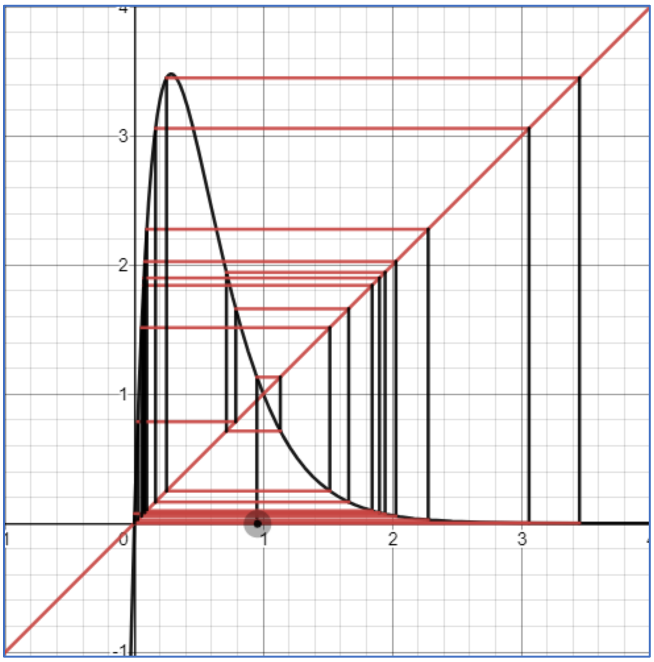
r=2.9

For r between 2.7 and 3 we obtain a chaotic behaviour. As we can see, there are several values (more than 4)



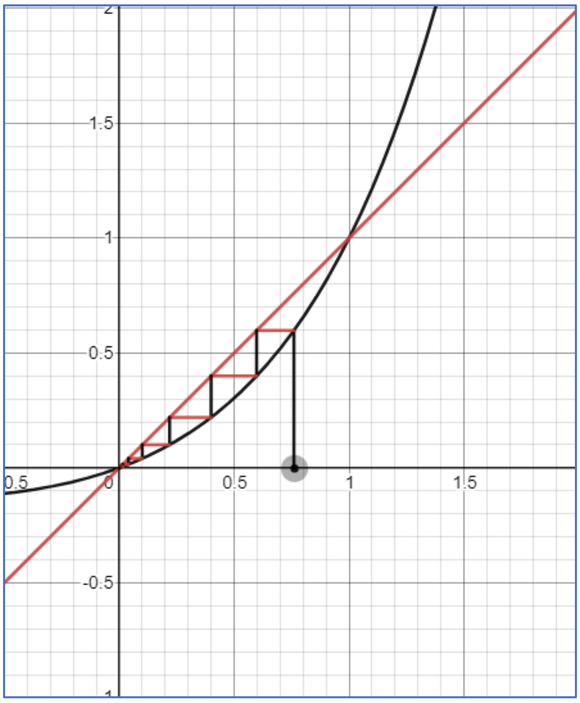
r= 3.1

This is a very particular case. Later, we will see at the bifurcation graph that there is a small interval which turn to be 2-period after being chaotic for the previous interval. Here, for r = 3.1 it is easy to see the 2 final results.



r=3.5

After the last explained 2-period interval, the graph becomes again chaotic. Here we can see its multiple results again.



r=-1

Although it is not much relevant, for r<=0 (as I explained in the first point) will end in 0, the hole population will become extinct

5. **Can you solve the equation for period-2 dynamics? Can you try to calculate its stability?**
   1. Period 2 equation

      Exponential map:

      $x * e^{(r*(1-x))}$

      Second period implies $f(f(x))$, therefore:

      $(x * e^{(r*(1-x))}) * e^{\left(r*\left(1-\left(x*e^{(r*(1-x))}\right)\right)\right)}$

      $x * e^{r-rx} * e^{r-rxe^{r-rx}}$

      $x * e^{r-rx} * e^{r-rx} * e^{e^{r-rx}}$

      $x * e^{(r-rx)^2} * e^{(e)^{r-rx}}$

      $x * e^{2r-2rx} * e^{er-erx}$

      $x * e^{r(2-2x)} * e^{r(e-ex)}$

      $x * e^{r^2} * e^{2-2x} * e^{e-ex}$

      $x * e^{2r} * e^{2-2x} * e^{e-ex}$

      $x = \dfrac{1}{e^{2r*(e^{2-2x})*e^{(e-ex)}}}$

      $x = \dfrac{1}{e^{(4r-4rx)*(e-ex)}}$

      $x = \dfrac{1}{e^{4erx^2-8erx+4er}}$

$$x = \frac{1}{e^{4er*(x^2-x+1)}}$$
$$x = e^{-4er*(x^2-x+1)}$$

2. Estimating stability:

As we know, the stability condition is $|f(x^*)'| < 1$
Therefore:

$$f(x) = e^{-4er*(x^2-x+1)}$$

$$f'(x) = (-8erx + 8er) * e^{-4er*(x^2-x+1)}$$

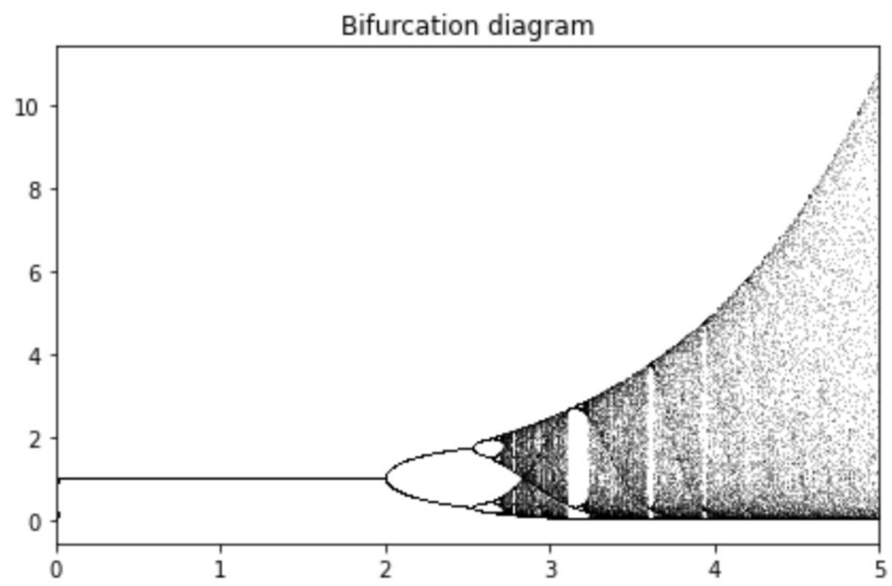
$$f'(x) = |(-8erx + 8er) * e^{-4er*(x^2-x+1)}| < 1$$

For x^*:

$$f'(x^*) = |(-8erx^* + 8er) * e^{-4er*\left(x^{*2}-x^*+1\right)})| < 1$$
$$f'(x^*) = |(-8er * (1-1) * e^{-4er*(1^2-1+1)}| < 1$$
$$f'(x^*) = |0| < 1$$

6. Plot the bifurcation diagram (discard transients and represent values of xn for a number of time-steps at each value of r).

By a python script, I have made the bifurcation diagram which I shown below.



Here, we can appreciate which we have been discussing through this assignment. As we can see, the first bifurcation is around r = 2. At this point it starts a 2-period interval until r= 2.5.

For r = 2.5 until approx. 2.7 the graph turns to by 4-period. Then it turns to be chaotic as we cannot see how many results are there (at least more than 4)

It is interesting to observe the graph value for r=3.1. If we look at the development of the graph, this point should preserve the chaotic state which occurs in previous intervals. However, we can see that at the [3.0-3.2] interval the graph turns to be 2-period again. After this interval, it returns to have a chaotic behaviour.

7. **Calculate the Lyapunov exponent for different values of r (advanced).**

The Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. It is usually use at the chaotic theory studies.

A we can see, the graph peaks matches with the r values where bifurcations occurs.

With a python script I have plotted the Lyapunov exponent for 0<r<5: