STA 2105 : ESTIMATION THEORY

Course content.

i) Point estimation - method of moments
- Maximum likelihood Estimation

ii) Properties of estimators
- unbiasedness
- consistency
- Efficiency.
- mean square error.
- Sufficiency
- Completeness
- Cramer-Rao inequality (single parameter)

iii) Joint sufficiency, minimal sufficiency. Minimum variance unbiased estimators (MVUE)

iv) Asymptotic properties of maximum likelihood Estimators, bootstrap method, confidence interval, for unknown parameters, CI for mean and variance of normal distribution binomial and poisson distributions. CI for 2 sample situations involving normal distribution binomial distribution and poisson distribution
CI for difference in mean and variance for paired data.

Expectation
At the end of the course you should
i) Describe the method of moments for constructing estimators of population parameters and apply it.
ii) Describe the MLE method for constructing estimators of population and apply
iii) Define efficiency, consistency, mean square error and sufficiency of estimators

iv) Define the property of unbiased estimator and apply.
v) Define the mean square error of an estimator and apply. to compare estimators
vi) Describe the asymptotic behaviour of MLE(s) and apply
vii) Use the bootstrap method to estimate parameter characteristics.
viii) Define the confidence interval for non parameter using a sampling distribution,
ix) Calculate the CI for 2 sample situations involving the normal binormial and poisson distribution.
x) Calculate the CI for difference in means involving paired data.
xi) Calculate the CI of mean and variance of a normal distibution

Pre-requisite.
sta 130S : Probability and statistics 1

Referrence
Theory of point of estimation by lehman
class notes

Point Estimation

Statistical estimation refers to the process of deriving estimators for population parameters with the use of sample data. (observation)

It is divided into 2

i) Point estimation

ii) Interval estimation.

Interval estimation

It is the evaluation of population parameters eg mean by the use of interval or a range of data (values)

Point estimation.

Refers to an estimation of population parameters given as a single number. ie values taken by a point estimator which are referred to as a point estimate.

tions a) Population - This refers to the entire group of people or objects that a researcher wants to measure. ie the group under study.

b) Sample - It is a representative subset of a population

c) Population Parameter. Refers to a descriptive measurement of a population under study eg the population size mean body weight, mean body temp

- Population parameters describes or summarises a specific aspect of population under study.

d) Estimator This refers to a function for calculating a population parameter estimate by the use of sample data.

e) Estimate - values taken by the estimator

Example

Sample mean can be taken as a point estimator of a population mean.

An estimator of a population parameter θ is denoted by $\hat{\theta}$

Therefore $\bar{X}$ is an estimator of λ in a poisson distribution $\Rightarrow \bar{X} = \hat{\lambda}$

Method Of Moments

Step 1 Let X have a pdf $f(x;\underline{\theta})$ where $\underline{\theta} = (\theta_1, \theta_2 \ldots \theta_n)$ are unknown parameters

We take a random sample $X_1, X_2, \ldots X_n$ and calculate the true moments about the origin ie about 0 (zero)

$$u_1' = E(x) = E(x-0)$$
$$u_2' = E(x^2) = E(x-0)^2$$
$$u_3' = E(x^3) = E(x-0)^3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$u_k' = E(x^k) = E(x-0)^k$$

Recall $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$= \int_{-\infty}^{\infty} x \cdot f(x;\underline{\theta})\,dx$$

$$= \int_{-\infty}^{\infty} x^k \cdot f(x;\underline{\theta})\,dx$$

$$= \underbrace{\int_{-\infty}^{\infty} x^k \cdot f(x, \theta_1, \theta_2, \theta_3 \ldots \theta_n)\,dx}_{\text{A function of } \underline{\theta}}$$

$$= \Phi(\theta_1, \theta_2 \ldots \theta_n)$$

We get the true moments depending on the number of parameters in the distribution . ie if a distribution has 1 parameter we ~~can~~ calculate ~~create~~ the first moment and if the distribution have 2 parameters we calculate the first 2 moments

Step 2 Calculate the sample moments about the origin that is about zero (0)

$$M_1' = \sum_{i=1}^{n} \frac{X_i}{n} = \bar{X}$$

$$M_2' = \sum_{i=1}^{n} \frac{X_i^2}{n}$$

$$M_3' = \sum_{i=1}^{n} \frac{X_i^3}{n}$$

$$\vdots$$

$$M_k' = \sum_{i=1}^{n} \frac{X_i^k}{n}$$

This results to k simultaneous equations in $\theta_1, \theta_2, \ldots \theta_k$

Step 3 Equate the true sample moment to the sample moments

ie $u_1' = m_1'$

$u_2' = m_2'$

$\vdots$

$u_k' = m_k'$ - Solve the simultaneous equation

Let the solutions of the simultaneous equations be $\hat{\theta}_1, \hat{\theta}_2 \ldots \hat{\theta}_k$
Then these are the estimate of $\theta_1, \theta_2, \ldots \theta_k$ by the method of moments.

Example 1

let ~~f(x)~~ $f(x;\theta) = \frac{2}{\theta^2}(\theta - x)$ for $-\infty < \theta < \infty$

$0 < x < \theta$

If x_1, x_2 upto x_n is a random sample from ~~x~~
obtain the estimates of θ ~~for~~ by the method of moments

Step 1

i) $u_1' = E(x) = E(x-0)$

Recall

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x;\theta)\,dx$$

$$= \int_0^\theta x \cdot \frac{2}{\theta^2}(\theta - x)\,dx$$

$$= \frac{2}{\theta^2}\int_0^\theta x \cdot (\theta - x)\,dx$$

$$= \frac{2}{\theta^2}\int_0^\theta (x\theta - x^2)\,dx$$

Recall

$$\int x^k\,dx = \frac{x^{k+1}}{k+1}$$

$$= \frac{2}{\theta^2}\left[\frac{\theta x^2}{2} - \frac{x^3}{3}\right]_0^\theta$$

$$\frac{2}{\theta^2}\left[\left[\frac{\theta(\theta)^2}{2} - \frac{\theta^3}{3}\right] - \left[\frac{\theta(0)^2}{2} - \frac{0^3}{3}\right]\right]$$

$$\frac{2}{\theta^2}\left[\left[\frac{\theta^3}{2} - \frac{\theta^3}{3}\right] - 0\right]$$

$$\frac{3\theta^3 - 2\theta^3}{6} = \frac{\theta^3}{6}$$

$$\frac{\cancel{2}^{1}}{\cancel{\theta^2}_{1}}\left[\frac{\cancel{\theta^3}^{\theta}}{\cancel{6}_{3}}\right] = \cancel{\frac{\theta}{3}} = \frac{\theta}{3} = \mu_1'$$

Step 2
Get the sample moment about 0

$$m_1' = \sum_{i=1}^{n} \frac{x_i}{n} = \bar{x}$$

Step 3
Equate : true moment to sample

$$\frac{\theta}{3} = \bar{x}$$

step 4

$$\cancel{3} \times \frac{\theta}{\cancel{3}} = \bar{x} \times 3$$

$$\theta = 3\bar{x}$$

By the method of moments obtain the estimator of parameter θ in an exponential population defined by

$$f(x;\theta)=\{ 1/\theta \; e^{-x/\theta}$$

$$f(x;\theta)= \{ 1/\theta \quad e^{-x/\theta}$$

Step 1 Get the first true moment about 0

$$u_1' = E(x-0) = E(x) = \int x \cdot f(x) dx$$

$$\int_0^\infty x \cdot 1/\theta \; e^{-x/\theta} dx$$

$$1/\theta \left[\int_0^\infty x \cdot e^{-x/\theta} dx \right.$$

By Partial Integration we use formula $\int u dv = uv - \int v du$

let $u = x$ $\qquad dv = e^{-x/\theta} dx$

$\frac{du}{dx} = 1 \qquad du = dx \qquad v = e^{-x/\theta} \times \frac{\theta}{1} = -\theta e^{-x/\theta}$

$$\frac{1}{\theta}\left[x \cdot -\theta e^{-x/\theta} - \int -\theta e^{-x/\theta} dx \right]$$

$$\frac{1}{\theta}\left[-x\theta e^{-x/\theta} + \int \theta e^{-x/\theta} dx \right] = \frac{1}{\theta}\left[-x\theta e^{-x/\theta} + \theta \int e^{-x/\theta} dx \right]$$

$$\frac{1}{\theta}\left[-x\theta e^{-x/\theta} + \theta \left(e^{-x/\theta} \div \frac{\theta}{1} \right) \right. = \frac{1}{\theta}\left[-x\theta e^{-x/\theta} + \theta^2 \left(e^{-x/\theta} \right) \right]_0^\infty$$

$$1/\theta \left[0 - - \theta^2 \right] = 1/\theta \left[\theta^2 \right]$$

$$u_1' = \frac{\theta^2}{\theta} = \theta$$

Step 2.

Get the 1^{st} sample moment about 0

$$M_1' = \sum_{i=1}^{n} \frac{X_i}{n} = \bar{X}$$

Step 3

Equate first true moment about 0 to first sample moment about 0

$$\theta = \bar{X}$$

Step 4

Estimator $(\theta) = \hat{\theta} = \bar{X}$

By the method of moments obtain the estimator of lambda in a poisson population

$$f(X : \lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!} \quad \text{for } X = 0, 1, 2, \ldots \infty$$

Step 1

$$u_1' = E(x) = E(x-0)$$

$$E(x) = \sum_{i=0}^{\infty} x \cdot f(x; \theta$$

$$= \sum_{i=0}^{\infty} x \cdot e^{-\lambda} \frac{\lambda^{x}}{x!}$$

$$e^{-\lambda} \sum_{i=0}^{\infty} x \cdot \frac{\lambda^{x}}{x!}$$

$$e^{-\lambda} \sum_{i=0}^{\infty} \frac{\cancel{x} \lambda^{x}}{\cancel{x}(x-1)!}$$

$$e^{\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{x}}{(x-1)!}$$

$$e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{x}}{(x-1)!}$$

$$e^{-\lambda} \left[\frac{\lambda}{1} + \frac{\lambda^{2}}{1!} + \frac{\lambda^{3}}{2!} + \cdots \right]$$

$$\lambda e^{-\lambda} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^{2}}{2!} + \cdots \right]$$

Recall the exponential series

$$e^{x} = \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$$

$$\lambda e^{-x}\left[e^{\lambda}\right]$$

$$\lambda e^{-\lambda+\lambda} = \lambda e^{0} = \lambda$$

Step 2 first sample moment

$$m_1' = \sum_{i=1}^{\infty} \frac{x_i}{n} = \bar{x}$$

Step 3 $\hat{\lambda} = \bar{x}$

Exercise

* Let X be normally distributed with mean u and variance δ^2 ie $X \sim N(u, \delta^2)$ and $x_1, x_2 \ldots x_n$ be a random sample of X. Obtain the estimation of u and δ^2 by the method of moments

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u

Soln

Step 1

Get the true moment about 0

$$u_1 = E(x-0) = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$u_1' = \int_{-\infty}^{\infty} x \frac{1}{\delta\sqrt{2\pi}} e^{-1/2\left(\frac{x-u}{\delta}\right)^2} dx$$

$$u_1' = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{1}{2}\left(\frac{x-u}{\delta}\right)^2} dx$$

use substitution method

let $Y = \frac{x-u}{\delta}$

Make x the subject of the formular

$$Y\delta = \frac{\cancel{x-u}}{\cancel{\delta}} \quad x - u \Rightarrow$$

$$x = u + Y\delta$$

Changing limits

$$-\infty < x < \infty \quad -\infty\, Y < \infty$$

$$\infty < u < \infty$$

$$Y = \frac{x-u}{\delta} \quad \frac{dy}{dx} = \frac{1}{\delta}$$

$$\frac{dy}{dx} = \frac{1}{\delta} = \delta dy = dx$$

$$u_1' = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} u + \delta y\, e^{-\frac{1}{2}y^2}\, \delta dy$$

$$u_1' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u + \delta y e^{-\frac{1}{2}y^2} dy$$

$$u_1' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-1/2 Y^2} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta Y e^{-1/2 Y^2} dy$$

$$= \frac{u}{2\pi} \underbrace{\int_{-\infty}^{\infty} e^{-1/2 Y^2} dy}_{\sqrt{2\pi}} + \frac{\delta}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} Y e^{-1/2 Y^2} du}_{\text{Odd function}}$$

& $\frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y e^{-1/2 Y^2} dy$ - Odd function

An odd function a, a function $f(-x) = -f(x)$

An even function is a function in which
$f(x) = f(-x)$

An odd function multiplied by an even function becomes an odd function

$e^{-1/2 Y^2}$ is an even function

let $Y = 2$

$e^{-1/2(-2^2)} = e^{-1/2(4)} = e^{-2}$

An integral of an odd function under the limits $-\infty$ to ∞ is 0

$$u_1' = \frac{u}{\sqrt{2\pi}} \times \sqrt{2\pi} \times \left(\frac{\delta}{2\pi} \times 0 \right)$$

$$u_1' = u + 0 = u$$

Step 2

Get the sample moment about 0

$$M_1' = \sum_{i=1}^{n} \frac{X_i}{n} = \bar{X}$$

Step 3

Equat true moment to sample moment

$$u = \bar{X}$$

Step 4

$$\hat{u} = \bar{X}$$

Estimator for δ^2

We get the 2^{nd} true moment about 0

$$u_2' = E(X-0)^2 = E(X^2)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 fx$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\delta\sqrt{2\pi}} e^{-1/2\left(\frac{x-u}{\delta}\right)^2} dx$$

$$\frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-1/2\left(\frac{x-u}{\delta}\right)^2} dx$$

Using substitution Method.

$$Y = \frac{x-u}{\delta}$$

$$Y\delta = x - u$$

$$x = Y\delta + u$$

but x is squared

$$x^2 = (u + \delta y)^2$$
$$= u^2 + u\delta y + u\delta y + \delta^2 y^2$$
$$= u^2 + 2u\delta y + \delta^2 y^2$$

The new integral

$$\frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} (u^2 + 2u\delta y + \delta^2 y^2) e^{-\frac{1}{2} y^2} \, dx$$

$$y = \frac{x+u}{\delta} \qquad \frac{dy}{dx} = \frac{1}{\delta} \Rightarrow \underline{dx = \delta dy}$$

$$\therefore \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 + 2u\delta y + \delta^2 y^2 \cdot e^{-\frac{1}{2} y^2} \cdot \delta dy$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}y^2} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2u\delta y e^{-\frac{1}{2}y^2} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta^2 y^2 e^{-\frac{1}{2}y^2} dy$$

$$\frac{u^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy}_{\sqrt{2\pi}} + \frac{2u\delta}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} y e^{-\frac{1}{2}y^2} dy}_{0} + \frac{\delta^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} y^2 e^{-\frac{1}{2}y^2} dy}_{\sqrt{2\pi}}$$

$$= \frac{u^2}{\sqrt{2\pi}} \times \sqrt{2\pi} + \cancel{\frac{2u\delta}{\sqrt{2\pi}} \times 0} + \frac{\delta^2}{\sqrt{2\pi}} \times \sqrt{2\pi}$$

$$u_2' = u^2 + \delta^2$$

Step 2
Get the second sample moment about 0

$$M_2' = \sum_{i=1}^{n} \frac{X_i^2}{n}$$

Step 3
Equate the second true moment about 0 to the second sample moment

$$u_2' = M_2'$$

$$u^2 + \delta^2 = \sum_{i=1}^{n} \frac{X_i^2}{n}$$

$$\cancel{\delta}\ \delta^2 = \sum_{i=1}^{n} X_i^2 - u^2$$

but $u = \bar{X}$

$$\delta^2 = \sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

$$= 1/n \sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

$$\delta^2 = 1/n \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\delta^2 = 1/n \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Exercise.

Estimate the parameter λ in the exponential function

$$f(x:\lambda) = \lambda e^{-\lambda x}$$

Exercise.

a) X is a binomial random variable with parameter n known and parameter p is unknown. Given a random sample of n observation of X, compute by the method of moment the estimator of p

b) What would be the estimator for n and p when both are unknown.

Step 1

$$\mu_1' = E(X-0) = E(X) = \sum_{i=1}^{n} x \cdot f(x)$$

$$f(x, n, p) = {}^nC_x \, p^x q^{n-x} \quad \text{for } x = 0, 1, 2 \ldots n$$

$$\mu_1' = \sum_{x=0} x \cdot {}^nC_x \, p^x q^{n-x}$$

$${}^nC_x = \frac{n!}{(n-x)!\, x!}$$

$$= \sum_{x=0}^{n} x \cdot p^x q^{n-x} \frac{n!}{(n-x)!\, x!}$$

$$= \sum_{x=0}^{n} \cancel{x} \cdot p^x q^{n-x} \frac{n!}{(n-x)!\, \cancel{x}(x-1)!}$$

$$= \sum_{x=1}^{n} p^x q^{n-x} \frac{n(n-1)!}{(n-x)!\,(x-1)!}$$

$$= \sum_{x=1}^{n} p \cdot p^{x-1} q^{n-x} \frac{n(n-1)!}{(n-x)!\,(x-1)!}$$

$$= np \sum_{x=1}^{n} p^{x-1} q^{n-x} \frac{(n-1)!}{(n-x)!\,(x-1)!}$$

$$np \underbrace{\left[p^0 q^{n-1} \frac{(n-1)!}{(n-1)!\,(0)!} + p q^{n-2} \frac{(n-1)!}{(n-2)!\,(1)!} + p^2 q^{n-3} \frac{(n-1)!}{(n-3)!\,2!} + \ldots p^{n-1} q^{n-n} \frac{(n-1)!}{(n-n)!\,(n-1)!} \right.}_{(p+q)^{n-1}}$$

$np(p+q)^{n-1}$ $\quad p+q=1$

$np(1)^{n-1}$

$= np$

Step 2

Get the sample moment about 0

$$m_1' = \sum_{i=1}^{n} \frac{x_i}{n} = \bar{x}$$

Step 3

Equate first true moment about 0 to first moment about 0

$$np = \bar{x}$$

$$p = \frac{\bar{x}}{n}$$

Step 4

Get the estimator of p where p = ~~prop~~ probability of success

$$\hat{p} = \frac{\bar{x}}{n}$$

Parameter Estimation

i) method of moments - Normal, exponential $\cdot$ Id 2 $\cdot$ Poisson & binomial

ii) The maximum likelihood method (MLE)

We estimate n when both n and

step 1

Get the true population moment about 0.

N get the second true population moment ie u_2'

$$u_2' = E(X-0)^2 = E(X^2)$$

$$= \int_{-\infty}^{\infty} x^2 f(x) \text{ for discrete case}$$

$$\sum_{\text{all } x} x^2 \cdot f(x)$$

$$= \sum_{x=0}^{n} x^2 \cdot {}^nC_x \cdot p^x q^{n-x}$$

Note $x^2 \to x(x+1) - x \,/\, x^2 = x(x-1) + x$

$$= \sum_{x=0}^{n} X(X-1) + X \; {}^nC_x \; p^x q^{n-x}$$

$$\sum_{x=0}^{n} X(X-1) + X \cdot \frac{n!}{(n-x)x!} p^x q^{n-x}$$

$$\sum_{x=0}^{n} x(x-1)\frac{n!}{(n-x)(x!)} p^x q^{n-x} + \sum_{x=0}^{n} x \frac{n!}{(n-x)(x!)} \cdot p^x q^{n-x}$$

$$\sum_{x=0}^{n} \frac{\cancel{x(x-1)}\, n(n-1)(n-2)!}{(n-x)\, x(x-1)(x-2)!} p^x q^{n-x} + \sum_{x=0}^{n} x \frac{n!}{(n-x)(x!)} p^x q^{n-x}$$

$$\sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(n-x)(x-2!} p^x q^{n-x} + np$$

$$n(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!\,(n-2)!} p^x q^{n-x} + np$$

$$n(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)(x-2)!} p^2 p^{x-2} q^{n-x} + np$$

$$n(n-1) p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)(x-2)!} p^{x-2} q^{n-x} + np$$

$$n(n-1) p^2 \left[\frac{(n-2)!}{(n-2)(2-2)!} p^{2-2} q^{n-2} + \frac{(n-2)!}{(n-3)(3-2)!} p^{3-2} q^{n-3} \right.$$

$$+ \frac{(n-2)!}{(n-4)(4-2)!} p^{4-2} q^{n-4} + \dots \frac{(n-2)!}{(n-n)(n-2)!} p^{(n-2)} q^{n-n} + np$$

$$(p+q)^{n-2} = 1^{n-2} = 1$$

$$n(n-1)p^2 (1) + np$$

Step 2 ; Get the second sample moment about 0

$$m_2' = \sum_{i=1}^{n} \frac{x_i^2}{n}$$

Recall

$S^2 = E(x^2) - (E(x))^2$

Make $E(x^2)$ the subject.

$E(x^2) = S^2 + (E(x))^2$

$E(x^2) = n(n-1)p^2 + np = S^2 + (E(x))^2$

Recall $E(x) = np$

$= n(n-1)p^2 + np = S^2 + (np)^2$

$n(n-1)p^2 + np = S^2 + n^2p^2$

$(n^2 - n)p^2 + np = S^2 + n^2p^2$

$n^2p^2 - np^2 + np = S^2 + n^2p^2$

$S^2 = np - np^2$

$S^2 = np(1-p)$

Recall $np = E(x) = \bar{x}$ $\qquad np = \bar{x}$: make p the subje

$p = \frac{\bar{x}}{n}$

$n\left(\frac{\bar{x}}{n}\right)\left(1 - \frac{\bar{x}}{n}\right)$

$\bar{x}\left(1 - \frac{\bar{x}}{n}\right) \quad = \bar{x} - \frac{\bar{x}^2}{n} = S^2$

$-\frac{\bar{x}^2}{n} = S^2 - \bar{x} \qquad \frac{1}{\bar{x}^2} \times \frac{\bar{x}^2}{n} = \bar{x} - S^2 \times \frac{1}{\bar{x}^2}$

$\frac{1}{n} = \frac{\bar{x} - S^2}{\bar{x}^2} \qquad n = \frac{\bar{x}^2}{\bar{x} - S^2}$:.

$$\therefore \hat{n} = \frac{\bar{x}^2}{\bar{x} - s^2}$$

A gamma distribution is denoted by $\Gamma \alpha B$ for parameter α & B

$$\Gamma \alpha B = B^{\alpha} x^{\alpha-1} e^{-B/x} \quad \text{for } 0 < x \leq \infty$$

Gamma properties

i) Recursive forms

ii) Factorial form

i) Recursive Properties

a) $\Gamma \alpha = \int_{\delta}^{\delta} x^{\alpha-1} e^{-x} \, dx$

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} \, dx$$

$x e^{-x}$

ii) $\Gamma n = n-1 \, \Gamma n-1$

iii) $\Gamma n+1 = n \Gamma n$ very useful

iv) $\Gamma n+2 = n+1 \, \Gamma n+1$

v) $\Gamma n-2 = n-3 \, \Gamma n-3$

vi) $\Gamma 1 = 1$

vii) $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

Example 1

By recursive properties of gamma, calculate

a) $\Gamma 4$ b) $\Gamma 7$

a) $\Gamma 4 = n-1\,\Gamma n-1$

$\Gamma 4 = 3\,\Gamma 3$

$= 3 \times 2\,\Gamma 2$

$= 3 \times 2 \times 1\,\Gamma 1$

$= 3 \times 2 \times 1 \times 1 = 6$

b) $\Gamma 7 = n-1\,\Gamma n-1$

$\Gamma 7 = 6\,\Gamma 6$

$= 6 \times 5\,\Gamma 5$

$= 6 \times 5 \times 4\,\Gamma 4$

$= 6 \times 5 \times 4 \times 3\,\Gamma 3$

$= 6 \times 5 \times 4 \times 3 \times 2\,\Gamma 2$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1\,\Gamma 1$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$

ii) Factorial Properties

The gamma function is a generalisation of the factorial function

If n is a positive natural number ie $(1, 2, 3, \ldots)$

$\Gamma n = (n-1)!$

Example

By factorial property of gamma function find

a) $\Gamma 4$

b) $\Gamma 7$

Solution

$\Gamma 4 = (4-1)! = (3)! = 3 \times 2 \times 1 = \underline{6}$

$\Gamma 7 = (7-1)! = (6)! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

This factorial form is very powerful. One can estimate the gamma of any positive natural number by a second.

The gamma properties can also be used to get the gamma of fraction and negative numbers.

For example.

Solution

$\Gamma n = n-1 \, \Gamma n-1$

$\Gamma 3/2 = 3/2 - 1 \, \Gamma 3/2 - 1$

$= 1/2 \, \Gamma 1/2$

but $\Gamma 1/2 = \sqrt{\pi}$

$= 1/2 \sqrt{\pi} = \dfrac{\sqrt{\pi}}{2}$

Example 2.

$\Gamma -1/2$

Recall the rule $\Gamma n+1 = n \Gamma n$.

make Γn the subject

$\dfrac{\Gamma n+1}{n} = \Gamma n$

let $n = -1/2$

$= \dfrac{\Gamma -1/2 + 1}{-1/2} = \Gamma 1/2$

$= \dfrac{\Gamma 1/2}{-1/2} = -2 \, \Gamma 1/2 = -2\sqrt{\pi}$

Maximum likelihood Estimation Method (MLE)

Let $f(x)$ be the pdf of a population with parameter θ where θ is unknown

Let X_1, X_2 upto X_n be a random sample from the population

The likelihood function denoted as $L(X_1, X_2, X_3 \ldots X_n, \theta) = f(X_1, \theta) \cdot f(X_2, \theta) \ldots f(X_n, \theta) = \prod_{i=1}^{n} f(X, \theta)$

To get the maximum likelihood estimator of parameter θ we choose a value of θ for which $L(X_1, X_2, X_3 \ldots X_n, \theta)$ is maximum

This estimate is a solution to the following

i) $\frac{d}{d\theta} L(X_1, X_2 \ldots X_n, \theta) = 0$

ii) $\frac{d^2}{d\theta^2} L(X_1, X_2 \ldots X_n, \theta) < 0$

Alternatively the estimate is a solution to

~~the~~ i) $\frac{d}{d\theta} \log L(X_1, X_2, X_3 \ldots X_n, \theta) = 0$

ii) $\frac{d^2}{d\theta^2} \log L(X_1, X_2, X_3 \ldots X_n, \theta) < 0$

This is because the likelihood and log likelihood function increase and decrease in the same manner.

If a population has k parameters ie $\theta_1, \theta_2, \theta_3 \ldots \theta_k$ the maximum likelihood estimators of $\theta_1, \theta_2, \theta_3 \ldots \theta_k$ ~~is a~~ can be found by solving the following equations.

$\frac{d}{d\theta_i} \log L = 0$ for $i = 1, 2 \ldots k$

ii) $\frac{d^2}{d\theta^2} \log L < 0$ for $i = 1, 2, \ldots k$

Example.

Find the MLE of P in a point binomial distribution

A point binomial distribution is a binomial distribution with $n = 1$

$$f(x, P) = P^x (1-P)^{1-x}$$

for $x = 0, 1$

Let $X_1\ X_2 \ldots X_n$ be a random sample distribution

$$L(X_1, X_2 \ldots X_n, P) = f(x_1 P) \times f(X_2, P)$$
$$\ldots f(X_n, P)$$

$$L(X_1, X_2, X_n, P) = P^{x_1} (1-P)^{1-x_1} \times P^{x_2} (1-P)^{1-x_2} \ldots \times P^{x_n} (1-P)^{1-x_n}$$

$$\prod_{i=1}^{n} f(x_i, \theta)$$

$$L(X_1\ X_2 - X_n, P) = P^{\Sigma x_i} (1-P)^{n - \Sigma x_i}$$

Likelihood function

$$L(X_1\ X_2 \cdot X_n, P) = P^{\Sigma x_i} (1-P)^{n - \Sigma x_i}$$

Recall $n \times \bar{X} = \Sigma \frac{X_i}{n} \times n$

$$\Sigma X_i = n\bar{X}$$

$$L(X_1\ X_2, X_n, P) = P^{n\bar{x}} (1-P)^{n - n\bar{x}}$$

Log likelihood function

$$\log L = \log P^{n\bar{x}} (1-P)^{n - n\bar{x}}$$

$$\log L = n\bar{x} \log P - n - n\bar{x} \log(1-P)$$

i) $\frac{d}{dp} \log L = \frac{n\bar{x}}{p} + \frac{(-1)n - n\bar{x}}{1-p}$

$= \frac{n\bar{x}}{p} + \left(\frac{n\bar{x} - n}{1-p}\right) = 0$

$\frac{n\bar{x}}{p} - \frac{n - n\bar{x}}{1-p}$

$\frac{(1-p)n\bar{x} - p(n - n\bar{x})}{p(1-p)} = 0$

$(1-p)n\bar{x} - p(n - n\bar{x}) = 0$

$n\bar{x} - n\bar{x}p - np + n\bar{x}p = 0$

$n\bar{x} - np = 0$

$\frac{n\bar{x}}{n} = \frac{np}{n} \qquad p = \frac{n\bar{x}}{n}$

$\hat{p} = \bar{x}$

verify that it is maximum.

$\frac{d}{dp} = \frac{n\bar{x}}{p} - \frac{n - n\bar{x}}{1-p}$

$\frac{d}{dp} \cdot \frac{n\bar{x}}{p} - \frac{n - n\bar{x}}{1-p}$

Recall the power rule

$\frac{d}{dx} x^n = nx^{n-1}$

$\frac{d}{dp} n\bar{x}p^{-1} - n - n\bar{x}(1-p)^{-1}$

$-1n\bar{x}p^{-2} - (-1)(-1)\, n - n\bar{x}(1-p)^{-2}$

$-n\bar{x}p^{-2} - n - n\bar{x}(1-p)^{-2}$

$$= \frac{-n\bar{x}}{p^2} - \frac{n-n\bar{x}}{(1-p)^2}$$

X is a poisson random variable with parameter lamb
find the MLE of lambda. λ