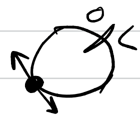
Quantization of the momentum of a particle in a box with periodic boundary conditions

($\hbar=1$)

$$-\frac{1}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$



Assume $V(x)=0$ (free particle) and

$$\left.\begin{aligned}\psi(0) &= \psi(L)\\ \psi'(0) &= \psi'(L)\end{aligned}\right\} \text{ in a 1-D ring}$$

$$\Downarrow$$

$$\psi(x+L) = \psi(x)$$

The solution to the previous equation in

$$\psi(x) = A e^{i\sqrt{2mE}\,x} + B e^{-i\sqrt{2mE}\,x}, \quad E = \frac{k^2}{2m}$$

$$= A e^{ikx} + B e^{-ikx}$$

$$\left.\begin{aligned}\psi(0)=\psi(L) &\Rightarrow A+B = A e^{ikL} + B e^{-ikL}\\ \psi'(0)=\psi'(L) &\Rightarrow A-B = A e^{ikL} - B e^{-ikL}\end{aligned}\right\}$$

$$\cancel{2A} = \cancel{2A}\, e^{ikL} \rightarrow e^{ikL} = 1 \Rightarrow \boxed{k = \frac{2\pi}{L} n}, \quad n \in \mathbb{Z}$$

This can be generalized to 3D, and we have $\vec{k} = \frac{2\pi}{L}\vec{n}$, $\vec{n}=(n_x,n_y,n_z)$

Then, the spectrum of 2 particles will be,

$$E_n = \sqrt{m_1^2+k_1^2} + \sqrt{m_2^2+k_2^2} \underset{m_1=m_2}{=} 2\sqrt{m^2+k^2} = 2\sqrt{m^2 + \left(\frac{2\pi}{L}\right)^2|\vec{n}|^2}$$

$|\vec{n}|^2 = 0,1,2,3,4,5,6,8\ldots$