A Markovian Bus Model:

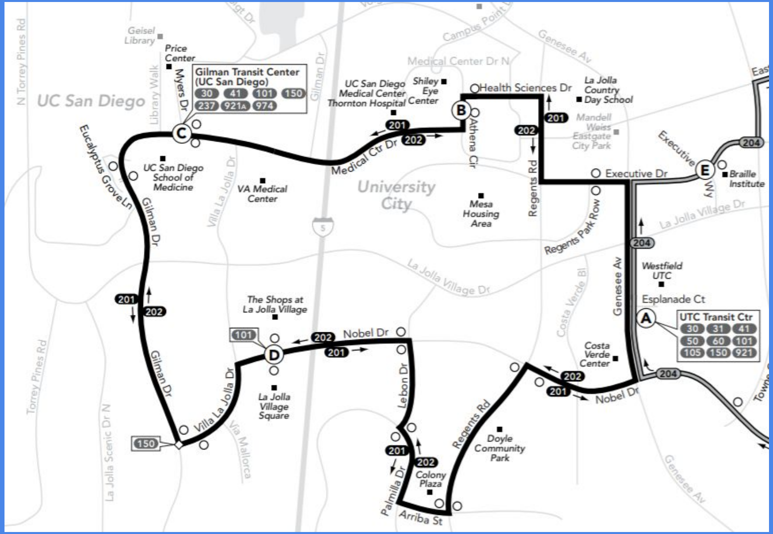
How Timely Does The 202 Need To Be?

Introduction

- Arriving to school on time is stressful when the bus is consistently late and overcrowded with students

- The 202 bus route is one of the primary routes that students rely on

- We will be using a Markov chain to model the timeliness of the 202 bus



Questions

What does the timeliness of the bus's full route and individual stops look like, when we alter the probabilities in the Markov transition matrix?

When emphasizing the delay in the stops where most students board, how late can the bus be, so that it still arrives mostly on time at school?

Background

Relevant Research

- “Model and Algorithm for Solving School Bus Problem” by Taehyeong Kim and Bum-jin Park
- “Optimization Model for Multi-Vehicle-Type Structure in Urban Bus Systems” by Shuzhi Zhao, et al.
- “Optimal Time Point Configuration of a Bus Route - A Markovian Approach” by W. Klumpenhouwer and S.C. Wirasinghe

Klumpenhouwer and Wirasinghe

- Constructed a transition matrix representing the probability of the bus arriving on time at a stop, given the timeliness of the bus at the previous stop

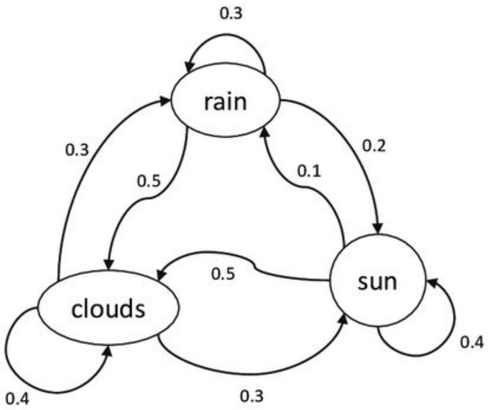
- Timeliness measured in minutes

$$\mathscr{P}_{i,i+1} = \begin{array}{c} \delta(i+1) \\ \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 1/2 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ 2 \\ 1 \\ 0 \;\; \delta(i) \\ -1 \\ -2 \end{array}$$

Model Description

What is a Markov Chain?

- A **Markov Chain** is a stochastic model describing a sequence of events in which the probability of each event is dependent only on the state attained in the previous event
- **Transition matrices** are matrices containing information on the probability of transitioning from one step to the next



$$
\begin{array}{r c}
 & \begin{array}{ccc} \text{clouds} & \text{rain} & \text{sun} \end{array} \\
\begin{array}{r} \text{clouds} \\ \text{rain} \\ \text{sun} \end{array} & \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.5 & 0.3 & 0.2 \\ 0.5 & 0.1 & 0.4 \end{pmatrix}
\end{array}
$$

Computing the End State

- Given a starting state vector, we can use the transition matrix ($A_{i,i+1}$) to obtain the ending state vector after n steps in the chain
- **Ending state vector (p_n)** = a vector that sums to 1, representing the probabilities of being in each state after a certain amount of steps

$$p_n = p_{n-1} * A_{n-1,n}$$

$$p_n = p_{n-2} * A_{n-2,n-1} * A_{n-1,n}$$

…

$$p_n = p_0 * A_{0,1} * A_{1,2} * \dots * A_{n-1,n}$$

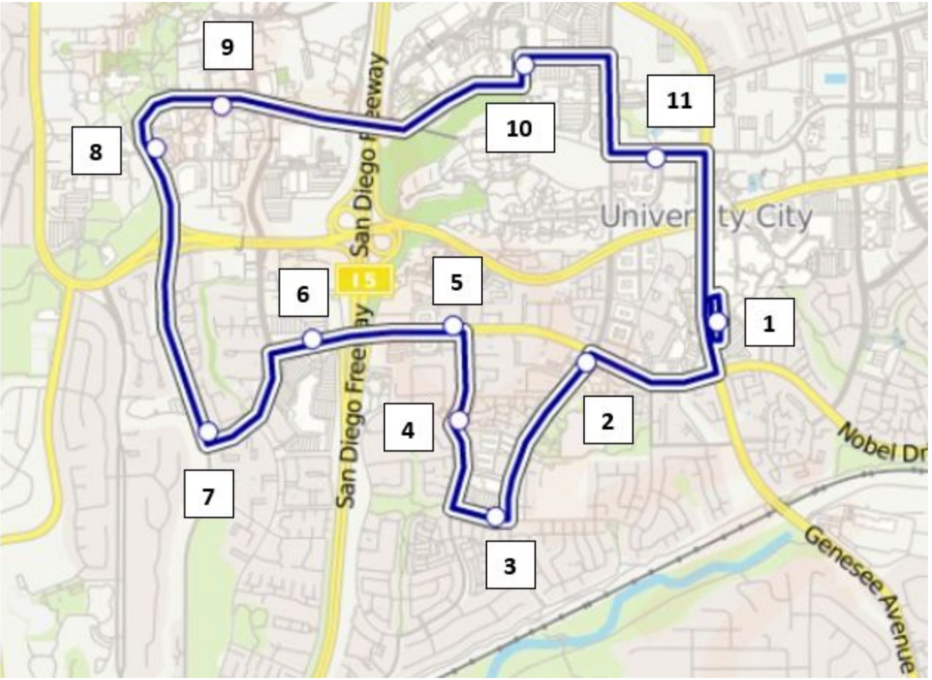
If the transition matrices are all equal, then:

$$p_n = p_0 * A^n$$

The 202 Model

$$
A_{i,i+1} = \begin{array}{c} \delta(i+1) \\ \begin{bmatrix} p_{early,early} & p_{early,on\ time} & p_{early,late} \\ p_{on\ time,early} & p_{on\ time,on\ time} & p_{on\ time,late} \\ p_{late,early} & p_{late,on\ time} & p_{late,late} \end{bmatrix} \end{array} \delta(i)
$$

$$
A = \begin{bmatrix} 0 & A_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & A_{2,3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{11,1} & 0 & 0 & \cdots & 0 \end{bmatrix}
$$



The 202 Model

Assumptions:

- Markov Property
- Time-Homogeneous Property
- The bus is always on time at the start of the route

How we will use it:

- Create various hypothetical scenarios
- Apply functions to the transition matrices that allow for better insight into the behavior of the bus
    - Computing end state after a full route
    - Computing end state after each stop
    - Stationary Distribution

Stationary Distribution

- The **stationary distribution** of a Markov chain is a probability distribution that remains unchanged in the Markov chain as the sequence of events progresses

- In other words, given a transition matrix A and row vector π:

$$\pi = \pi * A \quad \leftrightarrow \quad \pi^T = A^T * \pi^T$$

- Equivalent to A^T having an eigenvector π^T with eigenvalue 1

- **Solve for eigenvectors of A^T** → Obtain stationary distribution of A

Model Analyses

Question 1

What does the timeliness of the bus's full route and individual stops look like, when we alter the probabilities in the Markov transition matrix?

$$Scenario\ 1 : A_{i,i+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Scenario\ 2 : A_{i,i+1} = \begin{bmatrix} .9 & .05 & .05 \\ .05 & .9 & .05 \\ .05 & .05 & .9 \end{bmatrix}$$

$$Scenario\ 3 : A_{i,i+1} = \begin{bmatrix} .8 & .1 & .1 \\ .1 & .8 & .1 \\ .1 & .1 & .8 \end{bmatrix} \quad Scenario\ 4 : A_{i,i+1} = \begin{bmatrix} .7 & .15 & .15 \\ .15 & .7 & .15 \\ .15 & .15 & .7 \end{bmatrix}$$

$$Scenario\ 5 : A_{i,i+1} = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix} \quad Scenario\ 6 : A_{i,i+1} = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$$

$$Scenario\ 7 : A_{i,i+1} = \begin{bmatrix} .4 & .3 & .3 \\ .3 & .4 & .3 \\ .3 & .3 & .4 \end{bmatrix}$$

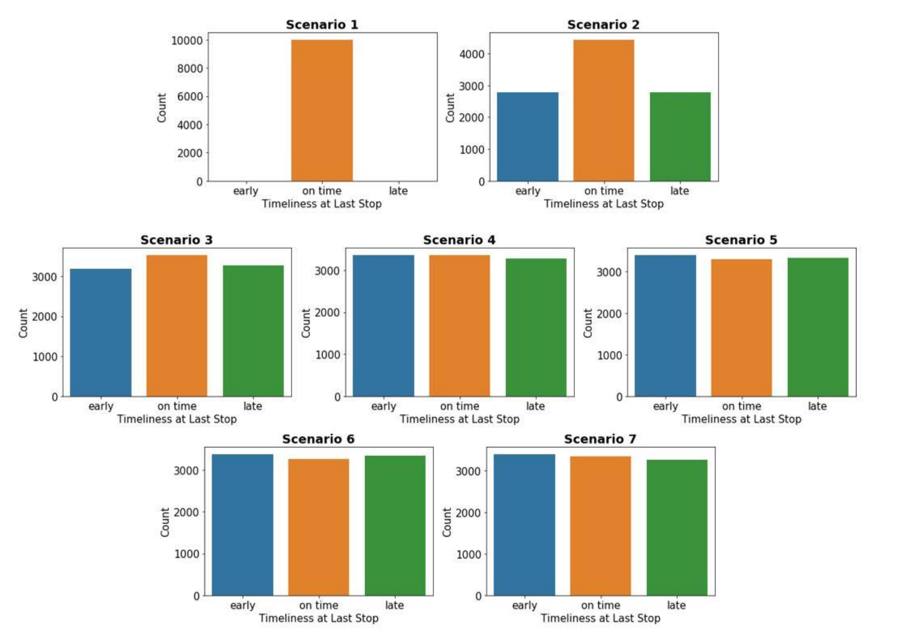
Function 1

Procedure:

- Simulate 10,000 full bus routes
- Track end state

Results:

- By scenario 3, the probability of the bus returning to stop 1 is equally likely to be on time, early, or late
- Initial timeliness is not carried over



Function 2

Procedure:

- Find stationary distribution of A_{11} where

$$A_{11} = A_{1,2} * A_{2,3} * \dots * A_{10,11} * A_{11,1}$$

Results:

- Scenario 1 is the identity matrix, so has numerous stationary distributions
- Scenario 2-7: $\pi = [⅓\ ⅓\ ⅓]$

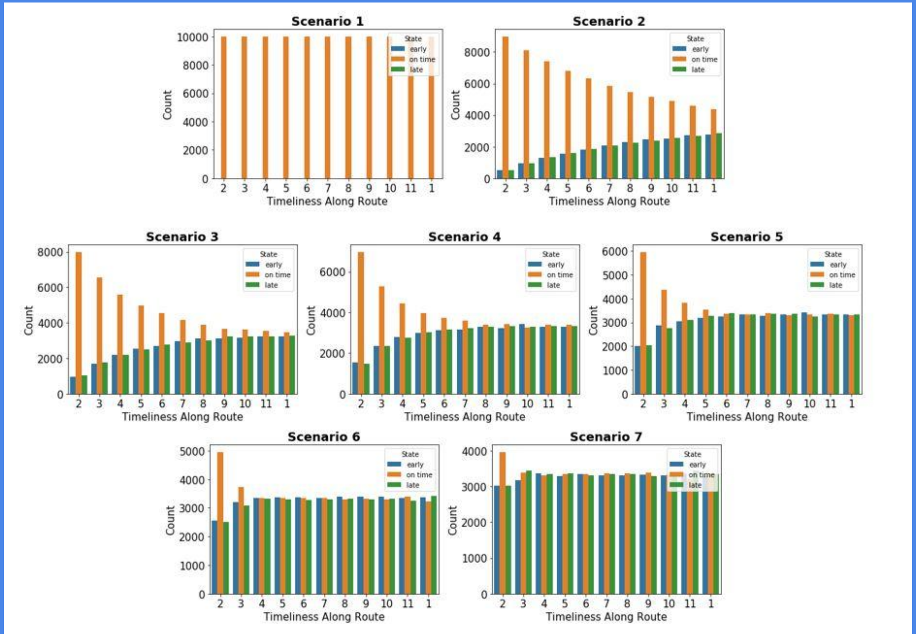
Function 3

Procedure:

- Simulate 10,000 full bus routes
- Track state after each stop

Results:

- We get a much clearer idea of exactly how much the timeliness degrades at each stop, depending on the transition matrix used



Question 2

When emphasizing the delay in the stops where most students board, how late can the bus be, so that it still arrives mostly on time at school?

$$A_{i,i+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & .9 & .1 \\ 0 & .2 & .8 \end{bmatrix} \quad where\ i \in \{1, 2, 5, 6, 7\}$$

$$Scenario\ 1 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .1 & .9 \\ 0 & .1 & .9 \\ 0 & .1 & .9 \end{bmatrix} \qquad Scenario\ 2 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .2 & .8 \\ 0 & .2 & .8 \\ 0 & .2 & .8 \end{bmatrix}$$

$$Scenario\ 3 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .3 & .7 \\ 0 & .3 & .7 \\ 0 & .3 & .7 \end{bmatrix} \qquad Scenario\ 4 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .4 & .6 \\ 0 & .4 & .6 \\ 0 & .4 & .6 \end{bmatrix}$$

$$Scenario\ 5 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .5 & .5 \\ 0 & .5 & .5 \\ 0 & .5 & .5 \end{bmatrix}$$

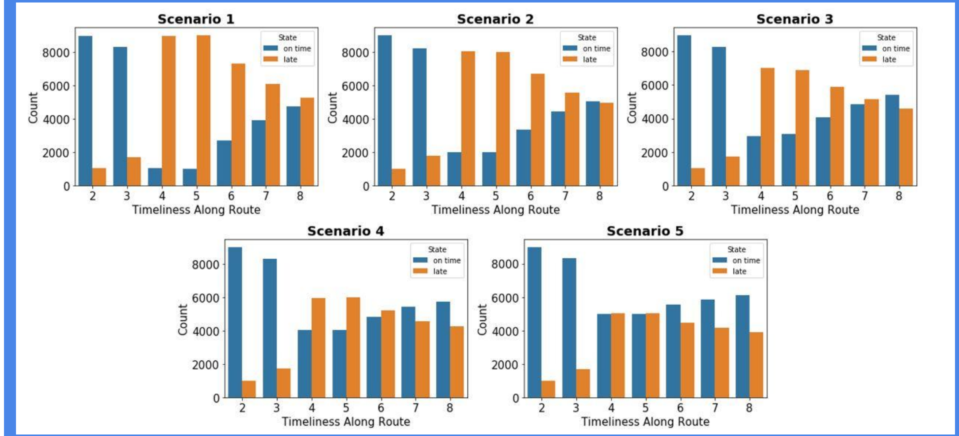
Function

Procedure:

- Look at subset - stops 2 to 8
- Simulate 10,000 full bus routes
- Track state after each stop

Results:

- Scenario 3, 4, and 5 are able to counteract the lateness from stops 3 and 4, with the timeliness from the stops after



Conclusion

Question 1

Function 1 & 2:

- In the long term, the bus will always drift towards having equal probability of being early, on time, or late

Function 3:

- We can predict that a bus having at least a **70% chance of remaining in the same state** and at worst a **30% chance of switching states**, will have a significantly **higher** chance of being **on time** for the majority of the route

Question 2

- For the bus to have a higher chance of being on time rather than late to school, the bus must have at least a **30% chance of arriving on time** to the next stop and at most a **70% chance of arriving late** to the next stop, after stop 3 and 4.

Possible Extensions

- Serve as a template for other bus routes
- Utilize data to create actual transition matrices
- Adding more scenarios
  - Different types of delays
    - Weather
    - Road conditions
    - Disabled persons boarding
  - Different times of the day
    - The 202 is more heavily used in the morning and early afternoon

Concluding Remark

Although this model is rather simplistic, it nevertheless provides valuable insight into how much of a delay the bus can afford, while still achieving its main purpose in transporting students to school.

The journey towards creating a reliable and stress-free bus system is more than possible—it's necessary.

References

Kim, Taehyeong, and Bum-jin Park. “Model and Algorithm for Solving School Bus Problem.” Journal of Emerging Trends in Computing and Information Sciences, CIS Journal, Aug. 2013, www.semanticscholar.org/paper/Model-and-Algorithm-for-Solving-School-Bus-Problem-Kim-Park/12dbc7defec4db5cdf9f0e3dd233833fcacd61f9.

Klumpenhouwer, W., and S.C. Wirasinghe. “Optimal Time Point Configuration of a Bus Route - A Markovian Approach.” Transportation Research Part B: Methodological, Elsevier, 13 Sept. 2018, www.sciencedirect.com/science/article/pii/S019126151730989X.

Zhao, Shuzhi, et al. “Optimization Model for Multi-Vehicle-Type Structure in Urban Bus Systems.” Mathematical Problems in Engineering, Hindawi, 9 Apr. 2017, www.hindawi.com/journals/mpe/2017/7914318/.