A Markovian Bus Model:
How Timely Does The 202 Need To Be?

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1 Introduction

The stress to get to school on time was one felt by thousands of students at the University of California, San Diego. In March 2020, that all changed when COVID-19 hit and forced almost every in-person meeting into a virtual setting. However, once the virus is under control and UCSD opens its campus again, most students will have to return to relying on the bus for their transportation needs. Having a historically bad reputation for being late, the bus system at UCSD needs to be studied and optimized now. This way, when campus operations return to full capacity, students will be able to rely on punctual transportation and mitigate yet another source of stress in their lives.

One of the primary bus routes that students use is the 202 route. This route is responsible for transporting passengers from surrounding neighborhoods and shopping centers to the UCSD campus. We will be using the Markov Chain to model the behavior of the bus traveling from one stop to the next.

The first question we consider is: What does the timeliness of the bus's full route and individual stops look like when we alter the probabilities in the Markov transition matrix? To answer this, we will study the behavior of the full bus route by looking at various hypothetical probability distributions and simulating thousands of routes. With this information, we will look at the state of timeliness (early, on time, or late) as the bus returns to the last stop, as well as the stationary distribution of this timeliness. Next, we will use the previous simulations to study the behavior of the individual bus stops by recording the state of timeliness at each stop and comparing the overall trend across multiple scenarios.

The second question we consider is: When emphasizing the delay in the stops where most students board, how late can the bus be, so that it still arrives mostly on time at school? To answer this, we will focus on a subset of the full route that includes everything from the start of the route until the first stop located at UCSD. We will keep a fixed transition matrix that favors being on time for most of the stops, and only vary the two matrices that correspond to the stops having notable delays. After running many simulations, we will be able to see the overall trend across different probability distributions and see which scenarios allow the bus to still be on time when arriving to school.

The paper will be structured as follows. In Section 2, previous relevant models and classwork will be presented and compared. In Section 3, the Markovian bus model will be developed, and the associated functions used to understand this model will be explained. The computations and analyses of the various functions will be discussed in Section 4. Lastly, in Section 5, conclusions and possible extensions will be offered.

2 Background

There has been abundant research on the topic of modeling and optimizing bus systems. Kim and Park (2013) tackled the school bus routing problem by proposing a model for the route in the form of a mixed-integer programming problem and solving it by using a heuristic algorithm based on harmony search [1]. Zhao et al. (2017) proposed an optimization model for multi-vehicle-type structure in urban bus systems, in order to reduce the cost and increase the quality [2]. Zhao et al. also utilized the Markov method to analyze and predict the development trend of vehicle structure.

The model presented in this paper is inspired by Klumpenhouwer and Wirasinghe's (2018) approach to using the Markov chain to model a bus's state of timeliness [3]. They constructed a transition matrix that represents the probability of the bus arriving on time at a stop, given the timeliness of the bus at the previous stop. The timeliness of the bus was measured in minutes, such that the resulting matrix was a 5 by 5 matrix with the values -2, -1, 0, 1, and 2, corresponding to the bus arriving at the stop two minutes early, one minute early, and so on, up to two minutes late. The authors researched further and also created an optimal time point configuration for a bus route, which would minimize cost to passengers, cost of an early bus, cost of a late bus, and operating costs.

The model developed in this paper for the 202 bus route differs from Klumpenhouwer and Wirasinghe's approach in that it only includes the states early, on time, and late in the transition matrix, as opposed to keeping track of the specific minutes. This is because the data for the 202 bus route is not easily accessible, and hence, it is more practical to construct a simpler version of the states of timeliness that could be extrapolated in the future. Minimizing cost will also not be a component of this model because the complete cost structure of the bus system in La Jolla is not publicly available. Instead, the model will focus on studying the timeliness and the various probability distributions that influence this.

3 Model Description

A Markov chain is a stochastic model describing a sequence of events in which the probability of each event is dependent only on the state attained in the previous event. We will specifically be using a time-homogeneous Markov chain, meaning that the probability of any state transition is independent of time.

State transitions in a Markov chain are represented by transition matrices. A transition matrix $A_{s,s+1}$ is a matrix containing information on the probability of transitioning from step s to step s+1. Hence, $A_{s,s+1}$ is given by:

$$(A_{s,s+1})_{i,j} = P(\ \delta(s+1) = j \mid \delta(s) = i\ )$$

where $\delta(s)$ represents the state attained at step s. Each row of the transition matrix is a probability vector and sums to one. Furthermore, the product of subsequent transition matrices represents the transition of different states along a sequence of events, thereby creating a chain. Given a row vector p_0 representing the starting state of the chain (and whose entries sum to one), we can calculate the probability of the ending state p_n after n steps. This can be seen in the following equations:

$$\begin{aligned}
&p_n = p_{n-1} * A_{n-1,n} \\
&p_n = p_{n-2} * A_{n-2,n-1} * A_{n-1,n} \\
&\cdots \\
&p_n = p_0 * A_{0,1} * A_{1,2} * \cdots * A_{n-1,n}
\end{aligned}$$

Also, if the steps of the sequence all have equal transition matrix A, then:

$$p_n = p_0 * A * A * \cdots * A = p_0 * A^n$$

Now that we have introduced the general setup of a Markov chain, we will shift to the setup of our particular Markov chain model of the 202 bus route. There is a total of eleven stops on this route. The transition matrix will include information regarding which stop the bus is at, and the state in which it arrived there: early, on time, or late. We are assuming that the probability of the bus arriving early, on time, or late at the next stop, is solely dependent on the timeliness of the bus at the previous stop. In other words, factors such as the driver, number of passengers, and traffic conditions will not be accounted for in this model. The transition matrix corresponding to the probability of the state of timeliness of the bus arriving at stop i+1, $\delta(i+1)$, given the state of timeliness at stop i, $\delta(i)$, is structured as follows:

$$\begin{array}{cc} & \delta(i+1) \\ A_{i,i+1} = & \begin{bmatrix} p_{early,early} & p_{early,on\ time} & p_{early,late} \\ p_{on\ time,early} & p_{on\ time,on\ time} & p_{on\ time,late} \\ p_{late,early} & p_{late,on\ time} & p_{late,late} \end{bmatrix} \delta(i) \end{array}$$

where each entry in the matrix is filled with the respective probability. In addition, we will use a block matrix A, comprised of the eleven matrices corresponding to the bus stops, to represent the bus's full route:

$$A = \begin{bmatrix} 0 & A_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & A_{2,3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{11,1} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We will be creating multiple versions of these transition matrices containing different probabilities, and then applying three analyses to each model: observing the state at the end of a full route for numerous simulations, observing the state at each individual bus stop for numerous simulations, and lastly, computing the stationary distribution of the last stop's timeliness. The latter of these analyses require us to introduce another concept.

The stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as the sequence of events progresses. In other words, given a transition matrix A and a row vector π, we have:

$$\pi = \pi * A$$

By transposing either side, this can be rewritten as:

$$\pi^T = A^T * \pi^T$$

which is equivalent to the transition matrix A^T having an eigenvector π^T with eigenvalue 1. Thus, by solving for the eigenvectors of A^T, we can compute the stationary distribution of A.

Lastly, we should take note that these transition matrices are hypothetical and not derived from actual data. Data exists for the 202 bus's real time of arrival on the app OneBusAway, but unfortunately this data is not accurate. Depending on the current state of the bus, the app will alter the past times of arrival that were originally recorded. If accurate data is available in the future, the probabilities in the transition matrices could be computed using the data.

Also, the transition matrices could record the minutes in the states of timeliness rather than just assigning early, on time, or late. Then, similar analyses to what we have already proposed could be done. Furthermore, data could be used to validate the model. For instance, one could compare the actual end states after many full routes with the Markov chain's predictions, and see if the results match up. If they do not, this could signify that there may be an external factor that is worth considering to add into the model. For our purposes, the structure of the Markov chain model will remain simplistic so that it can be used as a basis for future applications with real data.

4 Model Analyses

As mentioned at the start of this paper, we considered two particular questions to study the behavior of the bus. The first is: What does the timeliness of the bus's full route and individual stops look like, when we alter the probabilities in the Markov transition matrix? To answer this, we looked at seven hypothetical scenarios of transition matrices. We start with a 'perfect' scenario where the bus is always on time and gradually alter the matrices so that a more unstable arrival is represented. The scenarios are as follows:

$$\mathit{Scenario}\ 1 : A_{i,i+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathit{Scenario}\ 2 : A_{i,i+1} = \begin{bmatrix} .9 & .05 & .05 \\ .05 & .9 & .05 \\ .05 & .05 & .9 \end{bmatrix}$$

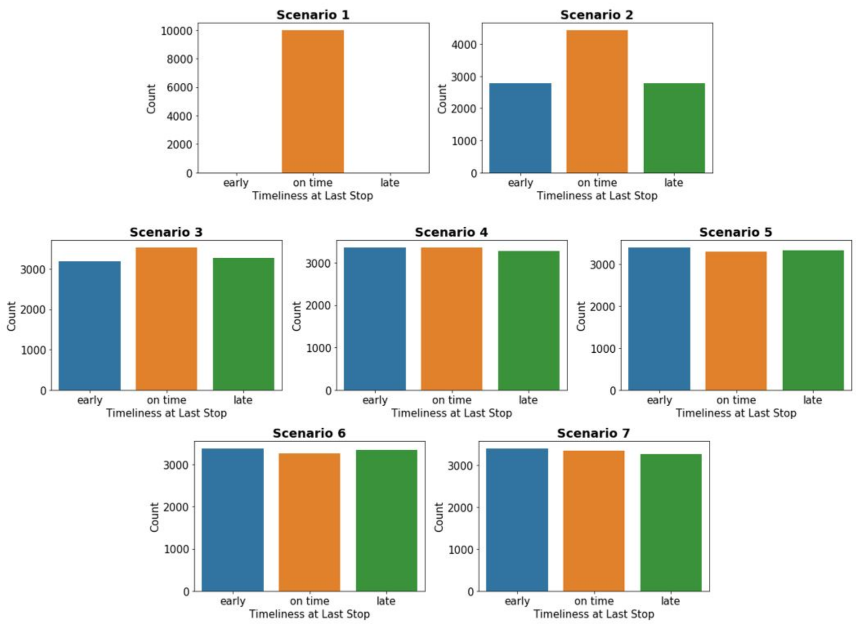
$$\mathit{Scenario}\ 3 : A_{i,i+1} = \begin{bmatrix} .8 & .1 & .1 \\ .1 & .8 & .1 \\ .1 & .1 & .8 \end{bmatrix} \quad \mathit{Scenario}\ 4 : A_{i,i+1} = \begin{bmatrix} .7 & .15 & .15 \\ .15 & .7 & .15 \\ .15 & .15 & .7 \end{bmatrix}$$

$$\mathit{Scenario}\ 5 : A_{i,i+1} = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix} \quad \mathit{Scenario}\ 6 : A_{i,i+1} = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$$

$$\mathit{Scenario}\ 7 : A_{i,i+1} = \begin{bmatrix} .4 & .3 & .3 \\ .3 & .4 & .3 \\ .3 & .3 & .4 \end{bmatrix}$$

Three functions are applied to each of these scenarios. In the first function, we simulate multiple full bus routes and track the probability of the end state. We assume that at stop 1, the bus arrived on time since that is the starting point of the route. Then, we use the block matrix A, as described in the previous section, and multiply our starting state vector by the matrix A eleven times (for the eleven submatrices of bus stops).

The resulting product equals a vector that represents the probability of the state at the end of the route, i.e. the probability that the bus returned to stop 1 early, on time, or late. Using these probabilities, we randomly choose and record a state for that route. After repeating this procedure 10,000 times, we create a countplot illustrating the number of times the bus returned to stop 1 early, on time, and late. The plots for each of the seven scenarios are shown below:



By scenario 3, we can already see that after one full route, the probability of the bus returning to its initial starting point is about equally likely to be early, on time, or late. This demonstrates that although the transition matrix may represent a bus that favors an on time arrival, much can change along a sequence of stops. Hence, the initial timeliness is not carried over to the end of the route.

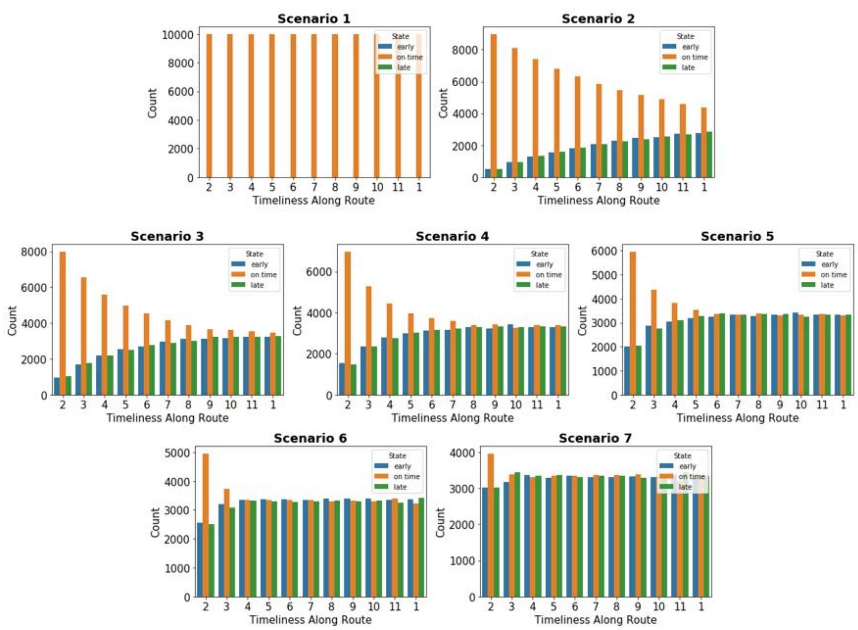
The second function we apply to the seven scenarios is finding the stationary distribution of the state of the bus after a full route. In order to do this, we create a new matrix A_{11} that will serve as our transition matrix for this purpose. A_{11} is defined as follows:

$$A_{11} = A_{1,2} * A_{2,3} * \cdots * A_{10,11} * A_{11,1}$$

After obtaining A_{11}, we find the eigenvector with associated eigenvalue 1, and normalize this vector to get the stationary distribution. Because the first of our seven scenarios is the identity matrix, there are numerous eigenvectors with eigenvalue 1, giving rise to numerous associated stationary distributions. Hence, it does not make sense to discuss the stationary distribution in this scenario. However, for the rest of the six scenarios we obtain a stationary distribution and interestingly enough, they are all equal to the vector: $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. This means that if the bus drove through the full route numerous times, the probability of the bus returning to the initial stop would be equally distributed across early, on time, and late, and would remain so for all subsequent routes. The stationary distribution could also be interpreted as the distribution in which the initial state no longer affects the states in the subsequent sequence of events.

This was also seen in the previous function where we saw that the end states after several simulations were about equally likely for almost all scenarios. These two functions together emphasize that not much can be said about the long-term behavior of the bus's timeliness. As the events in the Markov chain progress, it is difficult to obtain a precise forecast. Therefore, our third function will concentrate on tracking the behavior of the individual bus stops.

The procedure for the third function is very similar to the first function. The only difference is that instead of working with the block matrix A, and recording the state at the end of the full route, we will be using each individual transition matrix representing each bus stop, and recording the state after every bus stop. Again, we perform 10,000 simulations of a full route, and then create a countplot for each scenario as shown below:



By recording the state after each stop, we get a much clearer idea of exactly how much the timeliness degrades at each stop, depending on the constructed transition matrix that was used. Obviously, scenario 1 stays on time for the entire route because the transition matrix ensures that the bus cannot switch between states of timeliness. In scenario 2, the bus stays mostly on time throughout the entire route as well. In scenario 3, we start to see the probabilities of early, on time, and late become equally distributed in the later stops. However, from stop 2 to about stop 7, there exists a significant difference between the states in that the bus is more likely to be on time than not. Continuing on to scenario 4, this significant difference in probabilities can be seen until stop 6. In scenario 5, until stop 4; in scenario 6, until stop 3; and in scenario 7, only until stop 2.

With this information, we can decide how punctual the bus should be so that it arrives mostly on time at particular bus stops. For instance, if the first four stops on the route are the most impacted by the amount of passengers boarding, then the actual transition matrix of the bus would need to be at least as on time as scenario 5 is.

This leads us into our second question: When emphasizing the delay in the stops where most students board, how late can the bus be, so that it still arrives mostly on time at school? The 202 route starts at UTC which we have defined as stop 1. Since we assumed earlier that stop 1 will always be on time, the first stop affected by the transition matrix is stop 2. Historically, stop 3, Arriba St & Regents Rd, and stop 4, Lebon Dr & Palmilla Dr, are the most impacted stops because the majority of UCSD students who rely on the bus live in the neighborhoods surrounding these two locations. Stop 8, Gilman Dr & Eucalyptus Grove Ln, is the first stop located at the UCSD campus and also the final stop for a large portion of passengers who are on their way to school. With this background in mind, we will look at a subset of the full route, namely stop 2 to stop 8, and create new hypothetical scenarios that emphasize the delays at stops 3 and 4 while using a transition matrix for the other stops that favor the bus being on time.

For the stops not including the delay caused by stops 3 and 4, they will have a fixed transition matrix with an 80% chance of remaining in the same state and a 20% chance of switching to a nearby state as so:

$$A_{i,i+1} = \begin{bmatrix} .8 & .2 & 0 \\ .1 & .8 & .1 \\ 0 & .2 & .8 \end{bmatrix} \quad where\ i \in \{1, 2, 5, 6, 7\}$$

However, we primarily would like to focus on if the bus arrived on time or late. Because of this, we will assume that if the bus arrives to a stop early, then it will wait so that it leaves on time. Therefore, we combine the two categories, early and on time, and place the sum in the on time category as so:

$$A_{i,i+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & .9 & .1 \\ 0 & .2 & .8 \end{bmatrix} \quad where\ i \in \{1, 2, 5, 6, 7\}$$

Now that we have our fixed matrix, we define the transition matrix for stops

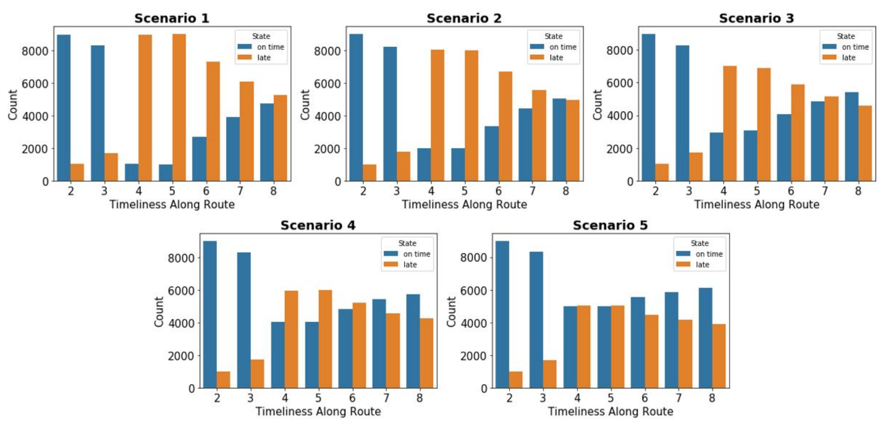
3 and 4. There will be a total of five scenarios as shown below:

$$Scenario\ 1 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .1 & .9 \\ 0 & .1 & .9 \\ 0 & .1 & .9 \end{bmatrix} \quad Scenario\ 2 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .2 & .8 \\ 0 & .2 & .8 \\ 0 & .2 & .8 \end{bmatrix}$$

$$Scenario\ 3 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .3 & .7 \\ 0 & .3 & .7 \\ 0 & .3 & .7 \end{bmatrix} \quad Scenario\ 4 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .4 & .6 \\ 0 & .4 & .6 \\ 0 & .4 & .6 \end{bmatrix}$$

$$Scenario\ 5 : A_{3,4} = A_{4,5} = \begin{bmatrix} 0 & .5 & .5 \\ 0 & .5 & .5 \\ 0 & .5 & .5 \end{bmatrix}$$

Just like the third function that was mentioned to answer the first question, we will apply this same function for this question, only with different scenarios and a subset of the full route. After recording each stop's timeliness for 10,000 simulations, we obtain the following plots:



For scenario 1, the delay caused by stops 3 and 4 is so drastic that the bus cannot get back on track by stop 8. As we can see, there is a higher chance that the bus will arrive to school late compared to on time. In scenario 2, the delay is still quite large, causing about an equal probability that the bus will be either on time or late when it reaches school. In scenario 3, although the transition matrix for stops 3 and 4 is still heavily weighted towards being late (70% versus 30% on time probability), we see that the bus is able to counteract that lateness with the timeliness of the stops in between. By stop 8, there is a higher chance that the bus will be on time than late. Lastly, for both scenario 4 and 5, there is a significantly higher chance that the bus will be on time rather than late by the time it arrives at school.

5 Conclusion

After applying three functions to our Markov chain model of the 202 bus, we were able to grasp a better understanding of the bus's overall timeliness. The first two functions discussed demonstrated that in the long term, the bus will always drift to having equal probability of being early, on time, or late. However, there are many other factors that the model did not account for, which would have illustrated a different long-term picture. For instance, after every full route, the bus driver normally takes a break or switches drivers. Hence, it only makes sense to look at one drive of the route at a time instead of analyzing long-term patterns such as the stationary distribution.

On the other hand, in the third function discussed we were able to see more specifically at what point along the route the bus's timeliness started to degrade. We saw that in scenario 1, 2, 3, and 4, the bus had a higher chance of remaining on time for the majority of the route. In scenario 5, 6, and 7, the bus's timeliness degraded before hitting the halfway point on the route. From our results, we can predict that a bus having at least a 70% chance of remaining in the same state and at worst a 30% chance of switching states, will have a significantly higher chance of being on time for the majority of the route.

After studying the general behavior of the 202 bus, we looked closer at the subset of stops, stops 2 to 8, in order to test how timely the bus needs to be so that it arrives to school on time. While using fixed transition matrices to represent most of the stops, we varied the matrices that emphasized the delays caused by students boarding at stops 3 and 4. We found that for the bus to have a higher chance of being on time rather than late to school, the bus must have at least a 30% chance of arriving on time to the next stop and at most a 70% chance of arriving late to the next stop, after stops 3 and 4.

This model could be extended in many ways. Because we looked at hypothetical scenarios, it could serve as a template for other bus routes besides just the 202 in La Jolla. If accurate data is available, that could be utilized to compute the actual transition matrices instead of relying on hypothetical ones. If data is not available, this model could still be extended by adding an abundant of scenarios and obtaining resulting statistics that better illustrate the behavior of the bus. These scenarios could focus on different types of delays such as weather, road conditions, or disabled persons boarding. The model could also include cases for different times of the day, as the 202 is more heavily used in the morning until early afternoon when students are on their way to school, as opposed to late at night when most students are already at school or have returned home.

Although our Markov chain model of the 202 bus route is rather simplistic, it nevertheless provides valuable insight into how much of a delay the bus can afford, while still achieving its main purpose in transporting students to school.

With these findings, the journey towards creating a reliable and stress-free bus system is more than possible—it's necessary.

References:

[1] Kim, Taehyeong, and Bum-jin Park. "Model and Algorithm for Solving School Bus Problem." Journal of Emerging Trends in Computing and Information Sciences, CIS Journal, Aug. 2013, www.semanticscholar.org/paper/Model-and-Algorithm-for-Solving-School-Bus-Problem-Kim-Park/12dbc7defec4db5cdf9f0e3dd233833fcacd61f9.

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