The Probabilistic Method

The probabilistic method is a remarkable technique for proving the existence of combinatorial objects with specified properties. It is based on probability theory but, surprisingly, it can be used for proving theorems that have nothing to do with probability. The usual approach can be described as follows.

We would like to prove the existence of a combinatorial object with specified properties. Unfortunately, an explicit construction of such a "good" object does not seem feasible, and maybe we do not even need a specific example; we just want to prove that something "good" exists. Then we can consider a random object from a suitable probability space and calculate the probability that it satisfies our conditions. If we prove that this probability is strictly positive, then we conclude that a "good" object must exist; if all objects were "bad", the probability would be zero.

Let us start with an example illustrating how the probabilistic method works in its basic form.

We close here with one last example of the probabilistic method:

Theorem 2 *If G is a graph, then G contains a bipartite subgraph with at least $E/2$ edges.*

Proof. Pick a subset of G's vertices, T, uniformly at random (i.e. select T by flipping a coin for each of G's vertices, and placing vertices in T iff our coin comes up heads.) Let $B = V(G) \setminus T$.

Call an edge $\{x, y\}$ of $E(G)$ **crossing** iff exactly one of x, y lie in T, and let X be the random variable defined by

$$X(T) = \text{number of crossing edges for } T.$$

Then, we have that

$$X(T) = \sum X_{x,y}(T),$$

where $X_{x,y}(T)$ is the 0-1 random variable defined by $X_{x,y}(T) = 1$ if $\{x, y\}$ is an edge of G that's crossing, and 0 otherwise.

The expectation $\mathbb{E}(X_{x,y})$ is clearly $1/2$, because we chose x and y to be in T at random. Thus, by the linearity of expectation, we have that

$$\mathbb{E}(X) = \sum \mathbb{E}(X_{x,y}) = E/2.$$

so the expected number of crossing edges for a random subset of G is $E/2$. Thus, there must be some $T \subset V(G)$ such that $X(T) \geq E/2$; taking the collection of crossing edges this set creates then gives us a bipartite graph (B, T) with $\geq E/2$ edges in it.