CSL 101 - Discrete Mathematics
Indian Institute of Technology Bhilai
Tutorial Sheet 6

1. For any undirected graph, show that the number of odd degree vertices must be even.

2. Prove that For any tree, the number of leaf nodes is at least 2.

3. Show that in any graph, there must be at least two vertices with the same degrees.

4. Let G be a graph where every vertex has degree at least d. Show that G contains a path of length d.

5. For any undirected weighted graph $G = (V, E, W)$, where $W : E \rightarrow R^+$, a shortest path between any two vertices u and v is defined as the path from u to v for which the sum of the weights of the edges on the path is minimum. Suppose that $P(u, v)$ be the shortest path from u to v and x and y are two vertices on P. Then show that the subpath of P that connects x and y is also a shortest path from x to y.

6. How many spanning trees are there in an n-wheel (W_n: a graph with n "outer" vertices in a cycle, each connected to an $(n + 1)^{\text{st}}$ "hub" vertex), when $n \geq 3$?

7. Prove that in every tree, any two paths with maximum length have a node in common.

8. Prove that either a graph G or its complement $\overline{G}$ (includes only those edges which are not in G) is connected.

   Concept:*Degree sequence of a graph is the list of degree of all the vertices of the graph. Usually we list the degrees in nonincreasing order, that is from largest degree to smallest degree.*

9. A sequence $d_1, d_2, \ldots, d_n$ is called *graphic* if it is the degree sequence of a simple graph.
   (a) Show that there is a simple graph with vertices $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$.
   (b) Show that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.

   Degree Sequence

10. Fleury's algorithm constructs Euler circuits by first choosing an arbitrary vertex of a connected multigraph, and then forming a circuit by choosing edges successively. Once an edge is chosen, it is removed. Edges are chosen successively so that each edge begins where the last edge ends, and so that this edge is not a cut edge unless there is no alternative. Prove that Fleury's algorithm always produces an Euler circuit.

11. The distance between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The *radius* of a graph is the minimum over all vertices v of the maximum distance from v to another vertex. The *diameter* of a graph is the maximum distance between two distinct vertices.

(a) Show that if the diameter of the simple graph G is at least four, then the diameter of its complement $\overline{G}$ is no more than two.

(b) If the diameter of the simple graph G is at least three, then the diameter of its complement G is no more than three.