1.3 Dominating Sets

Definition 7. *Let $G = (V, E)$ be a graph. A set of vertices $D \subseteq V$ is called* ***dominating*** *with respect to G if every vertex in $V \setminus D$ is adjacent to a vertex in D.*

Theorem 8. *Suppose that $G = (V, E)$ is a graph with n vertices, and that $\delta(G) = \delta$, the minimum degree amongst G's vertices, is strictly positive. Then G contains a dominating set of size less than or equal to*

$$\frac{n \cdot (1 + \log(1 + \delta))}{1 + \delta}$$

Proof. Create a subset of G's vertices by choosing each $v \in V$ independently with probability p; call this subset X. Let Y be the collection of vertices in $V \setminus X$ without any neighbors in X; then, by definition, $X \cup Y$ is a dominating set for G.

What is the expected size of $|X \cup Y|$? Well; because they are disjoint subsets, we can calculate $|X \cup Y|$ by simply adding $|X|$ to $|Y|$:

$$\begin{aligned}
\mathbb{E}(|X|) &= \sum_{v \in V} \mathbb{E}\left(\mathbb{1}_{\{v \text{ is chosen}\}}\right) \\
&= p \cdot n, \text{ while} \\
\mathbb{E}(|Y|) &= \sum_{v \in V} \mathbb{E}\left(\mathbb{1}_{\{v \text{ is in } Y\}}\right) \\
&= \sum_{v \in V} \mathbb{E}\left(\mathbb{1}_{\{v \text{ isn't in } X, \text{ nor are any of its neighbors}\}}\right) \\
&= \sum_{v \in V} \mathbb{E}\left(1 - p\right)^{\deg(v)+1}, \text{(b/c we've made } \deg(v) + 1 \text{ choices independently)} \\
&\le \sum_{v \in V} (1 - p)^{\delta+1} \\
&= n(1 - p)^{\delta+1}.
\end{aligned}$$

This tells us that

$$\begin{aligned}
\mathbb{E}(|X \cup Y|) &\le np + n(1 - p)^{\delta+1} \\
&\le np + ne^{-p(\delta+1)},
\end{aligned}$$

which has a minimum at

$$p = \frac{\log(1+\delta)}{1+\delta}.$$

Thus, for such p, we can find a dominating set of size at most

$$\frac{n \cdot (1+\log(1+\delta))}{1+\delta},$$

as claimed. $\square$

Independent set

In a graph G, a subset of vertices $S \subset V$ is said to be an *independent set* if no two vertices $u, v \in S$ are adjacent in G. The problem of determining the size of the largest independent set in G is NP-hard. However, we can once again apply the probabilistic method to establish a good lower bound on the size of the maximum independent set for any graph.

Theorem 2.6. *Any graph $G = (V, E)$ contains an independent set $S \subset V$ such that $|S| \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$ where $\deg(v)$ is the number of vertices adjacent to v in G.*

Proof. Assign a random weight w_v to each vertex $v \in V$, choosing the weights independently and uniform from the interval $[0, 1]$. Call $v \in V$ a local minimum if $w_v < w_u$ for each vertex u adjacent to v. Since no two adjacent vertices can be local minima, the set of local minima form an independent set. Additionally, any vertex among v and its neighbors are equally likely to have minimum weight. Therefore, for each v, the probability that v is a local minimum is $\frac{1}{\deg(v)+1}$.

Now, we again use indicator variables for each vertex. Let X be the number of local minima, and X_v be an indicator random variable for the event that v is a local minimum. By linearity of expectation,

$$\mathbb{E}[X] = \sum_{v \in V} \mathbb{E}[X_v] = \sum_{v \in V} \frac{1}{\deg(v)+1}.$$

Therefore, there must be at least one independent set of that size. $\square$