CSL253 - Theory of Computation

Tutorial 7

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Question 3

Let B be the set of all infinite sequences over $\{0, 1\}$. Show that B is uncountable, using a proof by diagonalization.

Solution

In order to show that B, the set of all infinite sequences over $\{0, 1\}$ is uncountable, we show that no correspondence exists between $\mathbb{N}$ and B. The proof is by contradiction. Suppose that a correspondence f existed between $\mathbb{N}$ and B. We need to show that f fails to work as it is supposed to work. For it to be a correspondence, f must pair all the members of $\mathbb{N}$ with all the members of B uniquely. But we will find an x in B that is not paired with anything in $\mathbb{N}$, which will be our contradiction.

The way to find this x is to actually construct it. We choose each digit of x to make x different from one of the elements in B that is paired with an element in $\mathbb{N}$. In the end we are sure that x is different from any element in B that is paired.

We can illustrate this idea by giving an example. Suppose the correspondence f exists. Let $f(1) = 0000...$, $f(2) = 1000...$, $f(3) = 0100...$, and so on, just to make up some values for f. Then f pairs the number 1 with 0000..., the number 2 with 1000..., and so on. The following table shows a few values of a hypothetical correspondence f between $\mathbb{N}$ and B.

n	$f(n)$
1	0000...
2	1000...
3	0100...
4	1101...
$\vdots$	$\vdots$

We construct the desired x by giving its binary representation. Our objective is to ensure that $x \neq f(n)$ for any n. To ensure that $x \neq f(1)$, we let the first digit of x be anything different from the first binary bit of $f(1) = 0000....$ Hence we let it be 1 as this is the only option. To ensure that $x \neq f(2)$, we let the second binary bit of x be anything different from the second binary bit of $f(2) = 1000....$ Hence we let the second binary bit of x be 1. The third binary bit of $f(3) = 0100...$ is 0. Hence the third binary bit of x will be 1. The fourth binary bit of $f(4) = 1101...$ is 1. Hence the fourth binary bit of x will be 0. Continuing in this way down the diagonal of the table for f, we obtain all the digits of x, as shown in the following table. We know that x is not $f(n)$ for any n because it differs from $f(n)$ in the nth binary bit.

n	$f(n)$
1	$\underline{0}000...$
2	$1\underline{0}00...$
3	$01\underline{0}0...$
4	$110\underline{1}...$
$\vdots$	$\vdots$

$$x = 1110...$$

As we can see with $x = 1110...$, $x \neq f(n)$ for every value of $n \in \mathbb{N}$. But as x is a member of B, there should be a pairing between x and some value in $\mathbb{N}$ for the correspondence f to exist. Hence by proof of contradiction no such correspondence between $\mathbb{N}$ and B exists and hence B is uncountable.