0113

Sri Lanka Institute of Information Technology

B.Sc. Special Honours Degree

in

Information Technology
(Computer Systems and Networking)

Final Examination

Year 1, Semester 1 (2010)

Advanced Topics in Mathematics (120)

Duration: 3 Hours

Friday, 30th April 2010

(Time: 9.00 a.m. – 12.00 noon.)

Instruction to Candidates:

- This paper has 6 Questions. Answer ALL Questions.
- Calculators are allowed but steps must be shown.
- Total Marks 100.
- Percentage towards final total: 70%
- This paper contains 3 pages excluding the Cover Page.

Question 1 **(15 marks)**

a) Prove the following using the <u>definition of the hyperbolic functions:</u> (6 marks)

i) $1-\tanh^2 x = \operatorname{sech}^2 x$

ii) $\coth(-x) = -\coth x$

iii) $\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}$

iv) $\frac{dy}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x.\coth x$

b) Evaluate:

$$\int_{ln 2}^{ln 3} \sinh^3 x \, dx$$ (5 marks)

c) If $\sinh^{-1} x$ is defined only for its positive values, prove that: (4 marks)

$$\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$$

Question 2 **(25 marks)**

a) Given that: $L[f(t)] = F(s)$, $L[g(t)] = G(s)$ and α, β are constants, <u>use the definition of the Laplace Transform</u> to show that: (5 marks)

(i) $L[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s)$

(ii) $L[f(t).e^{\alpha t}] = F(s-\alpha)$

What are these properties called respectively?

b) Given that $L[f(t)] = F(s)$, $f'(t)$ is the first derivative of the function $f(t)$ and $f(0)$ is the value of the function at $t = 0$, show that: (4 marks)

$$L[f'(t)] = sF(s) - f(0)$$

c) Find the Laplace Transforms of the following: (10 marks)

(i) $t^3 e^{-2t}$

(ii) $e^{-t}\cos^2 t$

(iii) $\cos t \cos 2t$

(iv) $\sin t \cos t$

(v) $\sin(\alpha t + \beta)$ where α and β are constants.

d) Find the Inverse Laplace Transform of the following: (6 marks)

(i) $\frac{s+4}{s(s-1)(s^2+4)}$

(ii) $\frac{s-1}{s^2-6s+25}$

Question 3 (14 Marks)

For an electric circuit with a constant voltage source E, the first order Ordinary Differential Equation in terms of the voltage across the capacitor $V_c(t)$ is given by:

$$R_1 C \frac{dV_c(t)}{dt} + \left(1 + \frac{R_1}{R_2}\right) V_c(t) = E$$

(Assume that the switch is closed at $t = 0$ and hence the voltage across the capacitor $V_c(t) = 0$ at time $t = 0$.)

(a) Solve the above equation by using Laplace Transforms **or** separation of variables method. (6 marks)

(b) If $R_1 = 2\Omega$, $R_2 = 6\Omega$, $C = 10$ F and $E = 12$V, find the value of $V_c(t)$ at $t = 3.5$ seconds and as $t \rightarrow \infty$. (3 marks)

(c) Find the general solution of the following ordinary differential equation (ODE) using the separation of variables: (5 marks)

$$\operatorname{sech} x \frac{dy}{dx} = e^{x-y}$$

Question 4 (17 Marks)

The voltages V_1, V_2 and V_3 resulting from an analysis of an electronic circuit are given by the following set of linear equations:

$$V_1 + 2V_2 - 3V_3 = 2$$
$$4V_1 + V_2 - V_3 = -3$$
$$2V_1 - 6V_2 + V_3 = 1$$

(a) Write the above set of equations in the form $A\underline{x} = \underline{b}$ where A is the matrix of coefficients. Hence show that the solutions can be obtained by: (3 marks)

$$\underline{x} = A^{-1}\underline{b}$$

(b) Find $adj(A)$, $det(A)$ and A^{-1}. Hence find the values of the voltages V_1, V_2 and V_3. (10 marks)

c) If A= $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, find A^2, A^3 and A^4. Hence show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

(4 marks)

Question 5 (13 Marks)

a) Write the Euler's Formula in relation to Complex numbers. Hence prove the following: (5 marks)

(i) $\cosh(i\,\theta) = \cos(\theta)$

(ii) $\sinh(i\theta) = i\sin(\theta)$

(iii) $\cos(i\,\theta) = \cosh(\theta)$

(iv) $\sin(i\,\theta) = i\sinh(\theta)$

b) Separate the following in to Real and Imaginary parts: (8 marks)

i. $\ln\,(x + iy)$
ii. $\sin\,(\alpha + i\frac{\pi}{4})$

Question 6 (16 Marks)

a) Find the partial derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ of the following functions: (6 marks)

i. $f(x, y) = x^2\cos(2xy)$
ii. $f(x, y) = (x - 3)^2 + (y + 2)^2 + 4$
iii. $f(x, y) = e^{-\lambda x}\sin(\lambda y) + e^{-\lambda y}\cos(\lambda y)$ where λ is a constant.

b) If $f(x, y, z) = 2\sin(x)\cos(y)\tan(z)$, find the mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. (4 marks)

c) If $f(x, y, z) = xyz + x + y + z + 1$, where $x(s,t) = st^2 - t$, $y(s,t) = t^2 + s - 1$ and $z(s,t) = e^{s+t}$, find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ using the Chain Rule. (6 marks)

LAPLACE TRANSFORM TABLE

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\} \quad = \int_0^\infty e^{-st} f(t)dt$	
*	α	$\frac{\alpha}{s}$,	$s > 0$
*	t^n, $\quad n$ an integer	$\frac{n!}{s^{n+1}}$,	$s > 0$
*	e^{at}	$\frac{1}{s-a}$,	$s > a$
*	$\sin bt$	$\frac{b}{s^2+b^2}$,	$s > 0$
*	$\cos bt$	$\frac{s}{s^2+b^2}$,	$s > 0$
*	$e^{at}f(t)$	$F(s-a)$	
	$e^{at}t^n \quad n$ an integer	$\frac{n!}{(s-a)^{n+1}}$,	$s > a$
	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$,	$s > a$
	$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$,	$s > a$
	$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$	$s > 0$
	$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$,	$s > 0$
	$u_c(t)f(t),\ c \geq 0$	$e^{-cs}\mathcal{L}\{f(t+c)\}(s)$	
	$u_c(t)f(t-c),\ c \geq 0^{**}$	$e^{-cs}\mathcal{L}\{f(t)\}(s)$	
	$y' = \dot{y} = \frac{dy}{dt}$	$sY(s) - y(0)$	
	$y'' = \ddot{y} = \frac{d^2y}{dt^2}$	$s^2Y(s) - sy(0) - \dot{y}(0)$	
*	$\sinh(\alpha t)$	$\frac{\alpha}{s^2-\alpha^2}$	
*	$\cosh(\alpha t)$	$\frac{s}{s^2-\alpha^2}$	

Table of Derivatives

Throughout this table, a and b are constants, independent of x.

$F(x)$	$F'(x) = \frac{dF}{dx}$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$
$f(x) - g(x)$	$f'(x) - g'(x)$
$af(x)$	$af'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(x)g(x)h(x)$	$f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
$\frac{1}{g(x)}$	$-\frac{g'(x)}{g(x)^2}$
$f\big(g(x)\big)$	$f'\big(g(x)\big)g'(x)$
1	0
a	0
x^a	ax^{a-1}
$g(x)^a$	$ag(x)^{a-1}g'(x)$
$\sin x$	$\cos x$
$\sin g(x)$	$g'(x)\cos g(x)$
$\cos x$	$-\sin x$
$\cos g(x)$	$-g'(x)\sin g(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
e^x	e^x
$e^{g(x)}$	$g'(x)e^{g(x)}$
a^x	$(\ln a)\, a^x$
$\ln x$	$\frac{1}{x}$
$\ln g(x)$	$\frac{g'(x)}{g(x)}$
$\log_a x$	$\frac{1}{x \ln a}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arcsin g(x)$	$\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\arctan g(x)$	$\frac{g'(x)}{1+g(x)^2}$
$\operatorname{arccsc} x$	$-\frac{1}{x\sqrt{1-x^2}}$
$\operatorname{arcsec} x$	$\frac{1}{x\sqrt{1-x^2}}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$

Table of Indefinite Integrals

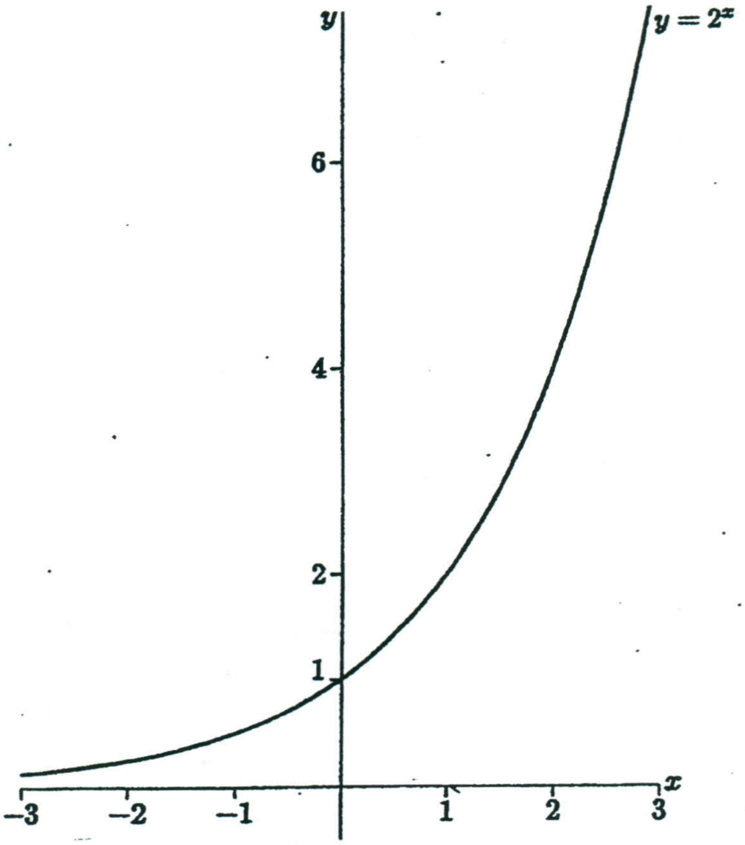
Throughout this table, a and b are given constants, independent of x and C is an arbitrary constant.

$f(x)$	$F(x) = \int f(x)\,dx$
$af(x) + bg(x)$	$a\int f(x)\,dx + b\int g(x)\,dx + C$
$f(x) + g(x)$	$\int f(x)\,dx + \int g(x)\,dx + C$
$f(x) - g(x)$	$\int f(x)\,dx - \int g(x)\,dx + C$
$af(x)$	$a\int f(x)\,dx + C$
$u(x)v'(x)$	$u(x)v(x) - \int u'(x)v(x)\,dx + C$
$f(y(x))y'(x)$	$F(y(x))$ where $F(y) = \int f(y)\,dy$
1	$x + C$
a	$ax + C$
x^a	$\frac{x^{a+1}}{a+1} + C$ if $a \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$g(x)^a g'(x)$	$\frac{g(x)^{a+1}}{a+1} + C$ if $a \neq -1$
$\sin x$	$-\cos x + C$
$g'(x)\sin g(x)$	$-\cos g(x) + C$
$\cos x$	$\sin x + C$
$\tan x$	$\ln\lvert\sec x\rvert + C$
$\csc x$	$\ln\lvert\csc x - \cot x\rvert + C$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$
$\cot x$	$\ln\lvert\sin x\rvert + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
e^x	$e^x + C$
$e^{g(x)}g'(x)$	$e^{g(x)} + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
a^x	$\frac{1}{\ln a}a^x + C$
$\ln x$	$x\ln x - x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{g'(x)}{\sqrt{1-g(x)^2}}$	$\arcsin g(x) + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a} + C$
$\frac{1}{1+x^2}$	$\arctan x + C$
$\frac{g'(x)}{1+g(x)^2}$	$\arctan g(x) + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\frac{x}{a} + C$
$\frac{1}{x\sqrt{1-x^2}}$	$\operatorname{arcsec} x + C$

Properties of Exponentials

In the following, x and y are arbitrary real numbers, a and b are arbitrary constants that are strictly bigger than zero and e is 2.7182818284, to ten decimal places.

1) $e^0 = 1, \; a^0 = 1$
2) $e^{x+y} = e^x e^y, \; a^{x+y} = a^x a^y$
3) $e^{-x} = \frac{1}{e^x}, \; a^{-x} = \frac{1}{a^x}$
4) $(e^x)^y = e^{xy}, \; (a^x)^y = a^{xy}$
5) $\frac{d}{dx}e^x = e^x, \; \frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}, \; \frac{d}{dx}a^x = (\ln a)\, a^x$
6) $\int e^x \, dx = e^x + C, \; \int e^{ax} \, dx = \frac{1}{a}e^{ax} + C$ if $a \neq 0$
7) $\lim_{x\to\infty} e^x = \infty, \; \lim_{x\to-\infty} e^x = 0$
   $\lim_{x\to\infty} a^x = \infty, \; \lim_{x\to-\infty} a^x = 0$ if $a > 1$
   $\lim_{x\to\infty} a^x = 0, \; \lim_{x\to-\infty} a^x = \infty$ if $0 < a < 1$
8) The graph of 2^x is given below. The graph of a^x, for any $a > 1$, is similar.



Properties of Logarithms

In the following, x and y are arbitrary real numbers that are strictly bigger than 0, a is an arbitrary constant that is strictly bigger than one and e is 2.7182818284, to ten decimal places.

1) $e^{\ln x} = x, \quad a^{\log_a x} = x, \quad \log_e x = \ln x, \quad \log_a x = \frac{\ln x}{\ln a}$
2) $\log_a\left(a^x\right) = x$, $\ln\left(e^x\right) = x$

   $\ln 1 = 0, \quad \log_a 1 = 0$

   $\ln e = 1, \quad \log_a a = 1$
3) $\ln(xy) = \ln x + \ln y, \quad \log_a(xy) = \log_a x + \log_a y$
4) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y, \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

   $\ln\left(\frac{1}{y}\right) = -\ln y, \quad \log_a\left(\frac{1}{y}\right) = -\log_a y,$
5) $\ln(x^y) = y \ln x, \quad \log_a(x^y) = y \log_a x$
6) $\frac{d}{dx} \ln x = \frac{1}{x}$, $\frac{d}{dx} \ln(g(x)) = \frac{g'(x)}{g(x)}$, $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$
7) $\int \frac{1}{x} dx = \ln|x| + C$, $\int \ln x \, dx = x \ln x - x + C$
8) $\lim\limits_{x\to\infty} \ln x = \infty$, $\lim\limits_{x\to 0} \ln x = -\infty$

   $\lim\limits_{x\to\infty} \log_a x = \infty$, $\lim\limits_{x\to 0} \log_a x = -\infty$
9) The graph of $\ln x$ is given below. The graph of $\log_a x$, for any $a > 1$, is similar.