Optimal transport and the Bergman determinantal point process

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Abstract

We study the Bergman determinantal point process in a theoretical point of view aiming at its simulation. We construct restricted and restricted-truncated variants of the Bergman kernel and show optimal transport inequalities involving the Kantorovitch-Rubinstein Wasserstein distance to show to what extent it is fair to truncate the restriction of this point process to a compact ball of radius $1-\varepsilon$. We also investigate the deviation of the number of points of the projected Bergman determinantal point process, indicate which number of points looks like an optimal choice, and provide upper bounds to its deviation, providing an answer to an open question asked in [2].

I Introduction

Introduced for the first time in [8], determinantal point processes (DPPs) are a particular class of point process, the correlations functions of which can be written as determinants. After having been thoroughly studied from a probabilistic point of view, (see [13] and [12] for instance, among many others), they have been used to model fermion particles [14]. They have suprisingly and remarkably appeared in the study of two very elegant mathematical problems : the first one is the point process defined as the set of eigenvalues of random $n \times n$ matrices with independent complex standard gaussian entries. It turns out that, as n goes to infinity, this point process converges in law to a point process that can be described [4] as a determinantal point process, the kernel of which was thereafter called the Ginibre kernel. The second one is the point process defined as the set of zeroes of Gaussian Analytical Functions (GAF), which are analytical functions (i.e., power series), the coefficients of which are all independent complex standard gaussian random variables : as it turns out this point process can also be described as a determinantal point process, the kernel of which is this time the Bergman kernel (see [6], [11]). DPPs also appear and are used in the study of a vide variety of different mathematical problems, such as self-avoiding walks, random spanning trees or random integer partitions, see [1], [10], [15]. More recently, some peers attempted to apply them to machine learning [16], and to model phenomena arising in the field of networking ([9], [17], [18]). The fact that these processes exhibit repulsion between particles is the core property that makes them very fitting for many applications.

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The two aforementionned kernels, respectively Bergman and Ginibre kernels, benefit from several interesting properties. While the former, taking its values in the unit disc of the complex plane, is invariant with respect to rotations, the latter, which is valued in the whole complex plane, is invariant with respect to rotations and translations. Whether it be for quantum mechanics, machine learning or networking modelling, for pratical uses it is necessary to benefit from a pratical simulation algorithm. This work has been initiated in [5] wherein the authors give a practical algorithm for the simulation of DPPs. The simulation procedure which is hinted in [4] was fully developed in [7].

The main problem of these algorithms is that, strictly speaking, they unfeasible due to the fact that the point processes involved exhibit an infinite number of points almost surely, and that, in the case of the Ginibre kernel, the points scatter throughout the whole plane. In [3], the authors construct projected and trucated variants of the Ginibre point process, that allow simulation to become closer to feasible. More precisely, the Ginibre point process is therein projected onto a ball of arbitrary radius and centered at the origin. The number of points then becomes almost surely finite. However, the stochastic dynamic of DPPs implies that computing the number of points that the point process requires to simulate a countable infinity of Bernoulli random variables, which is still unfeasible. The only option left is then to *truncate* the DPPs to a fixed (and finite) number of points, and brush off the remaining ones. But then, is it possible to quantify the approximation error that's inevitably produced with this method ? To what extent is this new probability measure close to the original one ? Theoretical results were shown in [3], providing answers to this question. For instance, the authors have shown that denoting $\mathfrak{S}_R$ and $\mathfrak{S}_R^N$ the Ginibre DPP projected to a ball of radius R centered at the origin of the complex plane and its truncation to N points respectively, then when choosing to truncate this DPP to $N_R = (R+c)^2$ points, we have for all $c > 0$

$$\mathcal{W}_{KR}(\mathfrak{S}_R, \mathfrak{S}_R^{N_R}) \leqslant \sqrt{\frac{2}{\pi}} R e^{-c^2}$$

whenever $R > c$, where $\mathcal{W}_{KR}$ denotes the Kantorovitch-Rubinstein (or Wassertein-1) distance on the set of probability measures on $\mathbb{C}$. This results shows that the (Wasserstein) distance between the two probability measures is exponentially small in c^2 as the number $N_R = (R+c)^2$ *deviates* from R^2, and implies that the two DPPs $\mathfrak{S}_R, \mathfrak{S}_R^{N_R}$ actually coïncide with high probability. As a result, truncating to R^2 points looks like a very good choice.

This is all very nice and convenient, but only applies to one DPP law, the Ginibre DPP. In this paper, we extend this work, by constructing projected (onto balls of radius R) and projected-trucated versions of the Bergman DPP, which will henceforth be at the very core of our study. We find the right number of points depending on the chosen radius R to truncate our projected DPP and provide the theoretical results that ensure that the use of the simulation algorithm presented in [3] is not blatantly different from the original - in other words that the truncation error is small when choosing the right number of points, hence providing an answer to the fifth open question asked in [2] for this DPP. Some of our results are far from specific to the Bergman DPP and apply in the general case.