$$g(R) = \frac{R^2}{1+R}$$

Proof.

Consider the coupling of $(\xi^R_{\beta N_R}, \xi^R)$ such that the first random variable is a subset of the second one consisting in the point whose indexes are $n \leqslant \beta N_R$ in Mercer's decomposition. We then have

$$\mathcal{W}_{KR}(\mathfrak{S}^R, \mathfrak{S}^R_{\beta N_R}) \leqslant \sum_{k=\beta N_R+1}^{\infty} R^{2k+2}.$$

Introducing $\varepsilon = 1 - R$,

$$\begin{aligned}
\mathcal{W}_{KR}(\mathfrak{S}^R, \mathfrak{S}^R_{\beta N_R}) &\leqslant \frac{(1-\varepsilon)^2}{\varepsilon(2-\varepsilon)}(1-\varepsilon)^{2\beta N_R} \\
&\leqslant \frac{(1-\varepsilon)^2}{\varepsilon(2-\varepsilon)} e^{2\beta N_R \log(1-\varepsilon)} \\
&\leqslant \frac{(1-\varepsilon)^2}{\varepsilon(2-\varepsilon)} \exp\left(2\beta \frac{(1-\varepsilon)^2}{\varepsilon(2-\varepsilon)} \log(1-\varepsilon)\right) \\
&\leqslant \frac{(1-\varepsilon)^2}{\varepsilon(2-\varepsilon)} \exp\left(-2\beta \frac{(1-\varepsilon)^2}{2-\varepsilon}\right).
\end{aligned}$$

The proof is complete. □

Remark 17. As a corollary of the previous proof, the two point processes coincide with high probability. The previous proof shows that :

Proposition 18. We have

$$\mathbf{P}(\mathfrak{S}^R \neq \mathfrak{S}^R_{\beta N_R}) \leqslant N_R e^{-2\beta \frac{R^2}{1+R}}.$$

Proof. Introducing the Bernoulli random variables from Theorem 7, we have

$$\mathbf{P}(\mathfrak{S}^R \neq \mathfrak{S}^R_{\beta N_R}) \leqslant \mathbf{P}(\exists k > \beta N_R, B_k = 1) \leqslant \sum_{k > \beta N_R} \mathbf{P}(B_k = 1) = \sum_{k > \beta N_R} \lambda_k^R \leqslant N_R e^{-2\beta \frac{R^2}{1+R}}$$

as wanted. □

Remark 19. In [8], the authors have chosen to truncate the restricted Ginibre DPP to $N_R = R^2$ points as seen in result (1) that shows that the truncation error is exponentially small in the deviation c from N_R. Though the authors of [8] have described this as a "well-known *observation*", it is interesting to observe that this R^2 is by no means random and can be theoretically forecasted. This is because turns out to correspond exactly to the expectation of the number of points of the Ginibre DPP. See for yourself : though very interesting to be pointed out, the following proposition is to be found nowhere in the litterature. It motivates to truncate to the expectation and to study the deviation that we have written in Theorem 13.

Proposition 20. The expected number of points $\mathbf{E}[|\mathfrak{S}^G_R|]$ that will come out from the restricted Ginibre DPP to $\mathcal{B}(0, R)$ is exactly R^2 (here, R can be any positive real number).