According to observations (see Figure 1), we may conjecture that, conditionnally on the set $I = \{n \geqslant 0, B_n = 1\}$ of active Bernoulli random variables, the law of the set of moduli $\{|X_k|, k \in I\}$ of the points for the *restricted* Bergman determinantal point process is exactly the law of the set

$$\{U_k^{1/(2k)}, k \in I\}$$

where $(U_k)_{k\geqslant 1}$ is a sequence of independent, uniform in $[0, R]$ random variables.

We then have the following.

Proposition 28. Let $m := \min_{1\leqslant k\leqslant n}(|X_k|)$ denote the smallest radius among the n first points of the Bergan point process (as always, "first" in the sense of Mercer's decomposition).

We have, for all $x \in [0, 1]$

$$\mathbf{P}(m \leqslant x) = 1 - \prod_{k=1}^{n}(1 - x^{2k})$$

Proof.

We have

$$\mathbf{P}(U_k^{1/(2k)} \geqslant x) = \mathbf{P}(U_k \geqslant x^{2k}) = 1 - x^{2k}$$

According to Theorem 26, we have

$$\mathbf{P}(m \geqslant x) = \prod_{k=1}^{n} \mathbf{P}(U_k^{1/(2k)} \geqslant x) = \prod_{k=1}^{n}(1 - x^{2k})$$

The proof is complete. □

Remark 29. As a corollary, we have

$$\mathbf{P}(m \leqslant x) \underset{x\to 0}{\sim} x^2$$

because the polynomial involved is even and vanishes at zero. This suggests that restricting to a annulus instead of a ball could be enough. Though it is to be mentionned that this has not yet been implemented in [15], we construct this restriction.

Theorem 30. Denote $T(r, R)$ the compact annulus centered at 0 with inner radius r and outer radius R. Mercer's decomposition of the kernel $k_{r,R}(x, y)$ of the Bergman determinantal point process restricted to $T(r, R)$ is

$$k_{r,R}(x, y) = \sum_{n\geqslant 0} \lambda_n^{r,R} \phi_n^{r,R}(x)\overline{\phi_n^{r,R}(y)}$$

where the eigenvalues are

$$\lambda_k^R = R^{2k+2} - r^{2k+2}$$

and the eigenfunctions

$$\phi_k^R : x \mapsto \sqrt{\frac{k+1}{\pi(R^{2k+2} - r^{2k+1})}}x^k$$