Proof. According to the fundamental theorem of DPPs (Theorem 7 in this paper), this expectation is $\sum_{n\geqslant 0} \lambda_n^R$ where λ_n^R denotes the n-th eigenvalue of the restricted Ginibre integral operator.

We have (see [8]) $\lambda_n^R = \frac{\gamma(n+1,R^2)}{n!}$ where γ stands for the incomplete gamma function $\gamma(n,x) = \int_0^x t^{x-1}e^{-t}\mathrm{d}t$.

We have

$$\sum_{n=0}^{\infty} \lambda_n^R = \sum_{n=0}^{\infty} \frac{1}{n!}\gamma(n+1,R^2) = \sum_{n=0}^{\infty} \frac{1}{n!}\int_0^{R^2} t^{n+1-1}e^{-t}\mathrm{d}t = \int_0^{R^2} \sum_{n=0}^{\infty} \frac{1}{n!}t^n e^{-t}\mathrm{d}t = \int_0^{R^2} \mathrm{d}t = R^2$$

as wanted. □

Remark 21. The behavior of the rational function $\frac{R^2}{1+R}$ inside the exponential function is to be nicely bounded in the neighborhood of $R \to 1$. In other words, the bound (the deviation) is indeed exponential (exponentially small) in β. For instance, for $R \in [0.9, 1]$, we have $g(R) = \frac{R^2}{1+R} \geq 0.42$, so we get $\leq \dots \exp(-0.98\beta)$. For $R \in [0.99, 1]$, $g(R) \geq 0.493$, so $\leq \dots \exp(-0.986\beta)$, and so on. (Since $g(R)$ is continuous and vanishes at 0, it cannot be bounded from below on $[0, 1]$ by a constant $K_0 > 0$. This is why here have to consider subintervals that do not touch 0 and see the bounds on those).

Remark 22. We have $N_R \xrightarrow[R\to 1^-]{} +\infty$. This comforts in thinking that the number of points is well-chosen, since as $R \to 1^-$, we seem to find a restriction that comes closer and closer to the original Bergman point process.

The following results further confirm this idea.

Proposition 23. Denoting $\mathfrak{G}^R$ the law of the restricted Bergman to a compact ball of radius R centered at the origin and $\mathfrak{G}_\alpha^R$ the law of its truncation to α points We have

$$\mathfrak{G}_N^R \xrightarrow[N\to\infty]{} \mathfrak{G}^R$$

in distribution.

Proof. The goal is to reduce this result to Theorem 25. Denote $||\cdot||_\infty$ the norm of uniform convergence. Let $\Lambda \subset \mathcal{B}(0, R)$ be compact. Denote k_N^R the truncated kernel to N points and k^R the asymptotic one. We have

$$\left\|k_N^R - k^R\right\|_{\infty,\Lambda} = \left\|\frac{1}{\pi}\sum_{k=N}^{\infty}(k+1)x^k\overline{y}^k\right\|_{\infty,\Lambda}$$

Observe that $\Lambda \subset \mathcal{B}(0, R)$ implies that

$$\begin{aligned}\|k_{N,R} - k_R\|_{\infty,\Lambda} &\leqslant \frac{1}{\pi}\sum_{k=N}^{\infty}(k+1)\left\|x^k\overline{y}^k\right\|_{\infty,\Lambda} \\ &\leqslant \frac{1}{\pi}\sum_{k=N}^{\infty}(k+1)R^{2k} \xrightarrow[N\to\infty]{} 0\end{aligned}$$

The proof is complete □.