This is all very nice and convenient, but only applies to one DPP law, the Ginibre DPP. In this paper, we extend this work, by constructing restricted (onto balls of radius R) and restricted-trucated versions of the Bergman DPP, which will henceforth be at the very core of our study. We find the compromise as to the right number of points depending on the chosen radius R to truncate our restricted DPP and provide the theoretical results that ensure that the use of the simulation algorithm presented in [8] produces a result that is not blatantly far from the (theoretical and non-feasible) simulation of the restricted Bergman DPP. In other words, the truncation error is small when choosing the right number of points, hence providing an answer to the fifth open question asked in [6] for this DPP. Some of our results are far from specific to the Bergman DPP and apply in the general case.

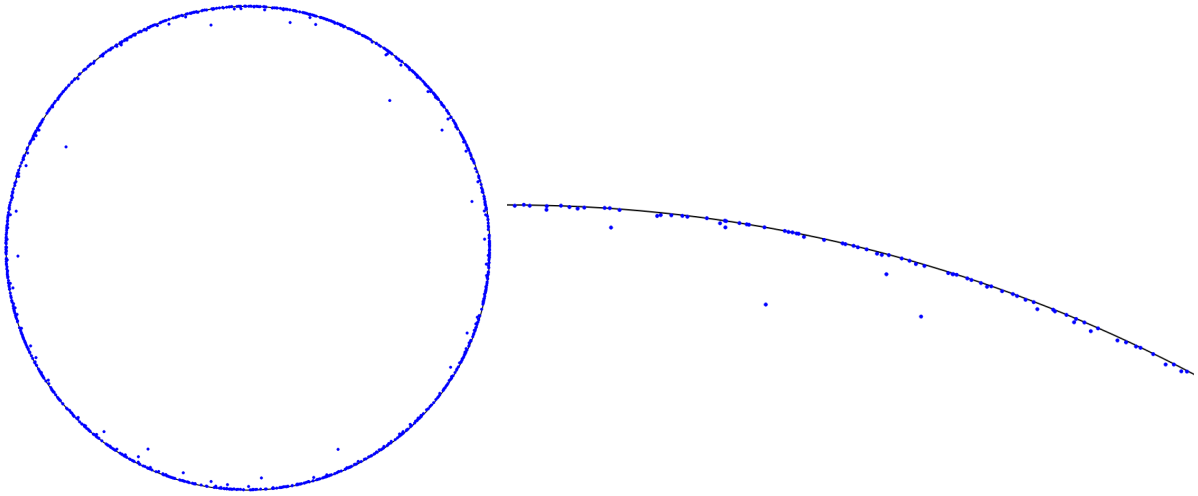

Figure 1: Left : Simulation of the Bergman DPP restricted to a radius of 0.9995, using [15]. Number of points : 985. Right : A zoom on the boundary

II Determinantal point processes

Let E be a Polish space, equipped with the Borel σ-algebra generated by the open subsets of E. In the sequel, λ is a reference Radon measure on this measurable space. Denote $\mathfrak{N}$ the space of locally finite subsets in E, also called the configuration space :

$$\mathfrak{N} = \{\xi \subset E, \forall \Lambda \subset E \text{ compact}, |\Lambda \cap \xi| < \infty\}.$$

It is equipped with the topology of vague convergence, also called weak topology, which is the weakest topology such that for all continuous and compactly supported functions f on E, the mapping $\xi \mapsto \langle f, \xi \rangle := \sum_{x \in \xi} f(x)$ is continuous. Once topologized, it naturally comes with a σ-algebra. Elements of $\mathfrak{N}$, i.e., locally finite configurations on E, are identified with atomic Radon measures on E.

Next, considering a compact subset $\Lambda \subset E$, we may consider the set $\mathfrak{N}_\Lambda = \{\xi \subset \Lambda, |\xi| < \infty\}$ of finite configurations on Λ, equipped with the trace σ-algebra.

A point process is then any random variable valued in $\mathfrak{N}$.

Definition 2. Let η be a point process on E. For $n \geqslant 1$, the n-th factorial moment of η is the point process defined on E^n as the set of n-tuples of points of η :

$$M_n^\eta = \{(x_1, ..., x_n) \mid x_1, ..., x_n \in \eta\}.$$

Definition 3. The n-th factorial moment measure of η is the mesure defined by

$$\mu_n^\eta(B_1 \times ... \times B_n) = \mathbb{E}(M_n^\eta(B_1 \times ... \times B_n)),$$

where $B_1, ..., B_n$ are measurable subsets of E. This relation defines a unique measure on E^n equipped with the product σ-algebra.

Definition 4. If μ_n^η admits a Radon-Nikodym derivative ρ_n with respect to $\lambda^{\otimes n}$, the former is called n-th correlation function of μ. In other words, we have, assuming their existence

$$\rho_n(x_1, ..., x_n) = \frac{\mathrm{d}(\mu_n^\eta)}{\mathrm{d}(\lambda^{\otimes n})}(x_1, ..., x_n)$$

Equivalently, a point process η is said to have correlation functions $(\rho_n)_{n\geqslant 1}$ if for all disjoint bounded Borel subsets $A_1, ..., A_n$ of E, we have

$$\mathbb{E}\left[\prod_{k=1}^{n} \xi(A_k)\right] = \int_{A_1 \times ... \times A_n} \rho_n(x_1, ..., x_n) \mathrm{d}\lambda^{\otimes n}.$$

Correlation functions are symmetric and characterized the correlation functions. ρ_1 is interpreted as the average density of particles with respect to λ. More generally,

$$\rho(x_1, ..., x_n)\mathrm{d}\lambda(x_1)...\mathrm{d}\lambda(x_n),$$

represents the probability of the event « For each $k \in [\![1, n]\!]$, there is a (at least one) point at the vicitiny $\mathrm{d}x_k$ of x_k ». See [4] for an introduction to this topic.

For any compact set $\Lambda \subset E$, we denote by $L^2(\Lambda, \lambda)$ the Hilbert space of complex-valued square integrable functions with respect to the restriction of the Radon measure λ on Λ, equipped with the usual inner-product. Recall that an integral operator $K : L^2(E, \lambda) \to L^2(E, \lambda)$ is said to have kernel $k : E^2 \to \mathbb{C}$ if K is written

$$K : f \mapsto \int k(x, y) f(y) \mathrm{d}\lambda(y).$$

If k is in $L^2(E^2, \lambda^{\otimes 2})$, the operator K is a linear bounded operator on $L^2(E, \lambda)$.

Definition 5. Let now be a compact subset $\Lambda \subset E$, the *restriction* of the integral operator K to Λ is the integral operator

$$K^\Lambda : f \in L^2(\Lambda, \lambda) \to \int_\Lambda k(x, y) f(y) \mathrm{d}\lambda(y) \in L^2(\Lambda, \lambda),$$

and K^Λ is then a compact operator.

Definition 6. The operator K is said to be Hermitian or self-adjoint if its kernel k verifies $k(x, y) = \overline{k(y, x)}$ for $\lambda^{\otimes 2}$-almost every $(x, y) \in E^2$.

Equivalently, this means that the integral operators K^Λ are self-adjoint for any compact set $\Lambda \subset E$. If K^Λ is self-adjoint, by the spectral theorem for self-adjoint and compact operators we have that $L^2(\Lambda, \lambda)$ has an orthonormal basis $(\phi_j^\Lambda)_{j\geqslant 0}$ of eigenfunctions of K^Λ. The corresponding eigenvalues $(\lambda_j^\Lambda)_{j\geqslant 0}$ have finite multiplicity (except possibly the zero eigenvalue) and the only possible accumulation point of the eigenvalues is zero. In that case, Mercer's theorem indicates that the kernel k^Λ of K^Λ can be written

$$k^\Lambda(x,y) = \sum_{n\geqslant 0} \lambda_n^\Lambda \phi_n^\Lambda(x)\overline{\phi_n^\Lambda(y)},$$

where the $(\phi_k^\Lambda)_{k\geqslant 0}$ form a Hilbert basis of $L^2(E,\lambda)$ composed of eigenfunctions of K^Λ.

Recall that K is positive if its spectrum is included in $\mathbf{R}^+$, and is of trace-class if

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty,$$

its trace is then $\mathrm{Tr}(K) := \sum\limits_{n=1}^{\infty} \lambda_n$.

Definition 7. If K^Λ is of trace-class for all compacts Λ, K is said to be locally trace-class.

Hypothesis 8. Throughout this paper, our kernels will be self-adjoint, locally trace class, with spectrum contained in $[0,1]$.

We refer to [2] and [3] for further developments on these notions.

Definition 9. A locally finite and simple point process on E is a determinantal point process if its correlation functions with respect to the reference Radon measure λ on E exist and are of the form

$$\rho_n(x_1,...,x_n) = \det(k(x_i,x_j))_{1\leqslant i,j\leqslant n},$$

where k satisfies Hypothesis 8.

The dynamics of DPPs are described by the following fundamental theorem.

Theorem 10. Under the aforementionned assumptions, consider Mercer's decomposition of the kernel k of the determinantal point process η:

$$k(x,y) = \sum_{k=1}^{n} \lambda_k \phi_k(x)\overline{\phi_k(y)}.$$

Here, the eigenvalues are all in $[0,1]$ and can all be chosen in $(0,1]$; n is equal to the rank of K, which can be either finite or infinite. The (ϕ_n) form a Hilbert basis is the space $L^2(E,\lambda)$. Consider then a sequence of independent Bernoulli random variables $(B_k)_{1\leqslant k\leqslant n}$, and consider the random kernel

$$k_B(x,y) = \sum_{k=1}^{n} B_k \phi_k(x)\overline{\phi_k(y)},$$

Then the point process η_B with (random) kernel k_B has the same law as that of k :

$$k \overset{\text{Law}}{=} k_B.$$

Corollary 11. If the n above is finite, then the point process has a finite number of points almost surely. By independence, the Borel-Cantelli lemmas imply that the number of points is almost surely finite if and only if K is of trace class. Since the integral operators that we will consider will all be locally of trace-class, the restrictions of our DPPs to compact subsets will all have a finite number of points almost surely.

For more details, we refer to [10] and [18].

Definition 12. The Bergman determinantal point process is the determinantal point process on the open unit disc centered at the origin of the complex plane with kernel

$$k(x,y) = \frac{1}{\pi}\frac{1}{(1-x\overline{y})^2}.$$

See [17] and [11] for a thorough study.

The key motivation of this article is the algorithm that was introduced and studied in [7], [8]. It is recalled down below. Its goal is to simulate a DPP restricted to a ball centered at the origin and of radius R. It assumes that the set

$$I = \{n \geqslant 0, B_n = 1\},$$

of "active" Bernoulli random variables has been computed, and computes the positions of the points $(X_k)_{k\in I}$. Recall that this makes sense because restricting a DPP to a compact reduces the number of points to almost surely finite.

Since simulating a countable infinity of Bernoulli random variables is unfeasible, we introduce the truncation to N points of the restricted DPP, which is defined by its kernel

$$k_N^\Lambda = \sum_{k=0}^{N-1} \lambda_n^\Lambda \phi_n^\Lambda(x)\overline{\phi_n^\Lambda(y)},$$

which is the only one that's numerically feasible.

III The Bergman DPP on a disc

The big question is now to know to what extent this truncation greatly affects the simulation of the DPP law (or if the error is only a minor issue). We provide the theoretical results that answer the last question in the negative for the Bergman DPP (and partially for general DPPs). This is the question that was asked in [6].

Notation. Throughout this paper and unless expressly stated otherwise, R and r denote real numbers in the interval $(0,1)$ such that $r < R$.

Let us first contruct the restriction of the Bergman DPP to a compact ball centered at the origin with radius R.

Proposition 13. Denote $\mathcal{B}(0,R)$ the compact ball centered at 0 with radius R. Mercer's decomposition of the kernel $k^R(x,y)$ of the Bergman determinantal point process restricted to $\mathcal{B}(0,R)$ is

$$k^R(x,y) = \sum_{n\geqslant 0} \lambda_n^R \phi_n^R(x)\overline{\phi_n^R(y)},$$