Algorithm 1 Sampling of the locations of the points given the set I of active Bernoulli random variables

Input: R, I
Output: $(X_i)_{i\in I}$
Let $\varphi_I^R(x) = (\varphi_i^R(x), i \in I)$
Draw X_1 from the distribution with density $\|\varphi_I(x)\|^2_{\mathbf{C}^{|I|}}/|I|$
$e_1 \leftarrow \frac{\varphi_I^R(X_1)}{\|\varphi_I^R(X_1)\|_{\mathbf{C}^{|I|}}}$
for $i \leftarrow 2$ to $|I|$ **do**
  Draw X_i from the distribution with density

$$p_i(x) = \frac{1}{|I| - i + 1}\left(\|\varphi_I^R(x)\|^2_{C^{|I|}} - \sum_{k=1}^{i-1} |\langle e_k, \varphi_I^R(x)\rangle|^2\right)$$

  $u_i \leftarrow \varphi_I^R(X_i) - \sum_{k=1}^{i-1} \langle e_k, \varphi_I^R(X_i)\rangle e_k$
  $e_i \leftarrow \frac{u_i}{\|u_i\|_{\mathbf{C}^{|I|}}}$
end for

where the eigenvalues are

$$\lambda_k^R = R^{2k+2},$$

and the eigenfunctions

$$\phi_k^R : x \mapsto \sqrt{\frac{k+1}{\pi}}\frac{1}{R^{k+1}}x^k.$$

Proof.

We have, for the (original) Bergman kernel

$$k(x,y) = \frac{1}{\pi}\frac{1}{(1-x\overline{y})^2},$$

using $\frac{1}{(1-u)^2} = \sum_{k\geqslant 0}(k+1)u^k$ for all $u \in \mathcal{B}(0,R)$, we have

$$\begin{aligned}\sum_{k\geqslant 0}\frac{k+1}{\pi}x^k\overline{y}^k &= \sum_{k\geqslant 0} R^{2k+2}\frac{k+1}{\pi}\frac{x^k}{R^{k+1}}\frac{\overline{y}^k}{R^{k+1}} \\ &= \sum_{k\geqslant 0}\lambda_n^R\phi_n^R(x)\overline{\phi_n^R(y)},\end{aligned}$$

taking ϕ_n^R and λ_n^R as in the theorem statement. Because we are restricting onto $L^2(\mathcal{B}(0,R),\lambda)$, we ought to consider the inner product on this space and not $L^2(\mathcal{B}(0,1),\lambda)$. We have

$$\langle\phi_n^R, \phi_m^R\rangle_{L^2(\mathcal{B}(0,R))} = \frac{\sqrt{(n+1)(m+1)}}{\pi}\frac{1}{R^{n+m+2}}\int_{\mathcal{B}(0,R)} z^n\overline{z}^m \mathrm{d}z.$$

However,

$$\int_{\mathcal{B}(0,R)} z^n\overline{z}^m\mathrm{d}z = \int_0^R\int_0^{2\pi} r^{n+m}e^{i\theta(n-m)}\mathrm{d}\theta r\mathrm{d}r = 2\pi\delta_{n,m}\int_0^R r^{n+m+1}\mathrm{d}r = 2\pi\delta_{n,m}\frac{R^{2n+2}}{2n+2}.$$