Proof.

We have, for the (original) Bergman kernel

$$k(x,y) = \frac{1}{\pi}\frac{1}{(1-x\overline{y})^2}$$

using $\frac{1}{(1-u)^2} = \sum\limits_{k\geqslant 0}(k+1)u^k$ for all $u \in T(r,R)$, we have

$$\sum_{k\geqslant 0}\frac{k+1}{\pi}x^k\overline{y}^k = \sum_{k\geqslant 0}\lambda_n^{r,R}\phi_n^{r,R}(x)\overline{\phi_n^{r,R}(y)}$$

taking $\phi_n^{r,R}$ and $\lambda_n^{r,R}$ as in the theorem statement. We have

$$\langle \phi_n^{r,R}, \phi_m^{r,R}\rangle_{L^2(T(r,R))} = \frac{1}{\pi}\sqrt{\frac{(n+1)(m+1)}{\lambda_n^{r,R}\lambda_m^{r,R}}}\int_{T(r,R)} z^n\overline{z}^m \mathrm{d}z$$

However,

$$\int_{T(r,R)} z^n\overline{z}^m\mathrm{d}z = \int_r^R\int_0^{2\pi} r^{n+m}e^{i\theta(n-m)}\mathrm{d}\theta r\mathrm{d}r = 2\pi\delta_{n,m}\int_r^R r^{n+m+1}\mathrm{d}r = 2\pi\delta_{n,m}\left[\frac{R^{2n+2}}{2n+2} - \frac{r^{2n+2}}{2n+2}\right]$$

Consequently,

$$\langle \phi_n^{r,R}, \phi_m^{r,R}\rangle_{L^2(T(r,R))} = \delta_{n,m}$$

The proof is thus complete. □

Remark 31. The previous proof actually shows that it is also feasible to compute the restriction to a domain of the form $\{z \in \mathbf{C}, |z| \in A\}$ where A is a Borel subset of $[0,1]$, yielding eigenvalues

$$\lambda_n^A = \int_A r^{2n+1}\mathrm{d}r,$$

with the same corresponding eigenfunctions up to a normalization factor.

V General results

To finish with, we have proven results for the Bergman DPP, the proofs of which used methods that turned out to be very general and apply to general DPPs. We list these results below and group them in this section.

As stated in Comment 16, our approach indicates that truncating restricted DPPs to a number of points equal to the expectation of their cardinality yields strong results such as Theorem 13 and Proposition 15. We here further study the deviation of this cardinality through the two following results.

Theorem 32. Let $\mathfrak{S}$ be a determinantal point process. Assume that its associated integral operator is trace-class.

Denote m the average number of points of $\mathfrak{S}$. m is finite, and we have, for $c \in]0,1[$