Course project «High-dimensional integrals for option pricing»

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December 16, 2016

Plan

Introduction to option pricing

Options

- Contract which gives the owner of the option the right, but not the obligation, to buy or sell an underlying asset at a specified price on a specified date, depending on the form of the option.
- Two basic types of options: the call and the put.
- Call - gives the holder the right to buy an asset
- Put - gives the holder the right to sell an asset

Notations

- s - stock price
- $c(s, t)$ - general option value

Black-Scholes equation

Black-Scholes equation

$$\frac{\partial c(s,t)}{\partial t} + rs\frac{\partial c(s,t)}{\partial s} + \frac{1}{2}\gamma^2 s^2 \frac{\partial^2 c(s,t)}{\partial s^2} = rc(s,t),$$
$$c(s,T') = g(s), \qquad s \in \mathbb{R}_+,$$
$$c(0,t) = 0, \qquad t \in [0;\ T'].$$

Substitution

- New variable: $x = \ln s$
- New initial condition: $f(x) = e^{\frac{1}{\gamma^2}(r-\frac{\gamma^2}{2})x} g(e^x)$,
- New coefficients: $\sigma = \frac{1}{2}\gamma^2$, $V(x,t) = V = r + \frac{1}{2\gamma^2}\left(r - \frac{\gamma^2}{2}\right)^2$,
- New solution: $u(x,t) = e^{\frac{1}{\gamma^2}(r-\frac{\gamma^2}{2})x} c(e^x, T'-t)$

Diffusion equation

One-dimensional reaction-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \sigma \frac{\partial^2 u(x,t)}{\partial x^2} - V(x,t)u(x,t), \quad t \in [0;\ T'],$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}.$$

Fast method for solving this equation was proposed in the paper:
"A low-rank approach to the computation of path integrals",
M. Litsarev, I. Oseledets, 2015.

Idea of the method

- The analytical solution is given by the Feynman-Kac formula

$$u(x, T) = \int\limits_{C\{x,0;T\}} f(\xi(T)) e^{-\int\limits_0^T V(\xi(\tau), T-\tau) d\tau} \mathcal{D}_\xi$$

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- Time discretization reduces this formula to n-dimensional integral ($n \gg 1$)
- But it can be computed as n one-dimensional convolutions efficiently

Definitions

- E - exercise price of the option.
- B - fixed value
- $d_{+}^{T'} = \frac{\log(\frac{s}{E})+(r+\frac{\gamma^2}{2})(T'-t)}{\gamma\sqrt{T'-t}}$
- $d_{-}^{T'} = \frac{\log(\frac{s}{E})+(r-\frac{\gamma^2}{2})(T'-t)}{\gamma\sqrt{T'-t}}$
- $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$

Experiments: european put option

Terminal conditions:

$$g(s) = \max\{E - s, 0\}$$

Analytical solution

$$c(s,t) = Ee^{-r(T'-t)}\Phi(-d_-^{T'}) - s\Phi(-d_+^{T'})$$

Experiments: european put option

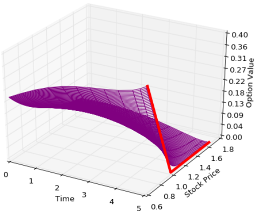

Figure: Numerical solution

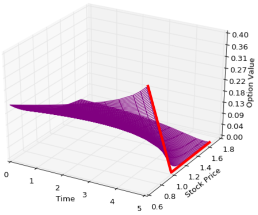

Figure: Analytical solution

Experiments: Cash-or-nothing call option

Terminal conditions:

$$g(s) = \begin{cases} B, & s \geq E \\ 0, & s < E \end{cases}$$

Analytical solution

$$c(s, t) = Be^{-r(T'-t)}\Phi(d_-^{T'})$$

Experiments: Cash-or-nothing call option

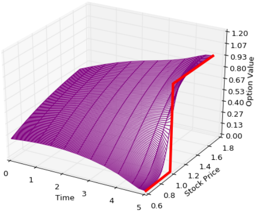

Figure: Numerical solution

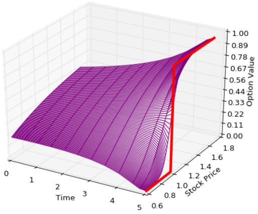

Figure: Analytical solution

Discussion

Unbounded terminal condition

- If n is big, numerical solution $\to +\infty$.
  Explanation:
  substitution $x = \ln s$ and one speciality of the algorithm.
- If n is small, numerical solution is not accurate and cross approximation is not useful.

Bounded terminal condition

- No such failure with big n as in the previous case.
- The bigger n, the more accurate solution.
- Cross approximation is useful: allows to reduce complexity.

More detailed analysis on that in our report.

Thanks for your attention!