

## Valuation of American Options

Among the seminal contributions to the mathematics of finance is the paper

F. Black and M. Scholes, “The pricing of options and corporate liabilities,” *Journal of Political Economy* 81 (1973), 637–659.

The Black-Scholes paper influenced a seminal paper

M.J. Brennan and E.S. Schwartz, “The valuation of American put options,” *Journal of Finance* 32 (1977), 449–462.

The latter does not explicitly identify the complementarity problem or the variational inequality problem *by name*, but the model does appear in the paper.

This presentation is based largely on the following very lucid book:

P. Wilmott, S. Howison, and J. Dewynne, *The Mathematics of Financial Derivatives*, Cambridge: Cambridge University Press, 1995.

Other books of interest on this topic are:

K.L. Chung and F. AitSahlia, *Elementary Probability Theory*, New York: Springer-Verlag, 2003.

J. Crank, *Free and Moving Boundary Problems*, Oxford: Oxford University Press, 1984.

F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, New York: Springer-Verlag 2003.

D.G. Luenberger, *Investment Science*, New York: Oxford University Press, 1998.

We shall begin with a series of definitions and then move on to the formulation of the problem of interest.

## A few definitions

We are concerned here with certain types of contracts called *options*. Such contracts entitle the holder to *buy* or to *sell* an asset (such as a share of stock) under specified conditions at a future time.

An option to buy is known as a *call option* whereas an option to sell is known as a *put option*.

The asset in question is called *the underlying asset* or simply *the underlying*.

The future time at which the buying or selling takes place is called the *expiration date* or *expiry*.

The option entitles the holder to buy or sell the underlying for an amount called the *exercise* (or *strike*) *price*.

## More definitions

The *holder* of a call (put) option has the *right* (but not the obligation) to buy (sell) the underlying at the specified strike price.

The *writer* has the obligation to sell (buy) the underlying at the strike price.

What has been described so far pertains to what is called a *European (call or put) option*.

An *American option* is one in which the holder has the right to exercise the option *on or before* the expiration date.

## Important questions:

1. What is the value of the option? That is, how much should one pay for the right it confers?
2. How can the writer of the option minimize the risk involved in the associated obligation?

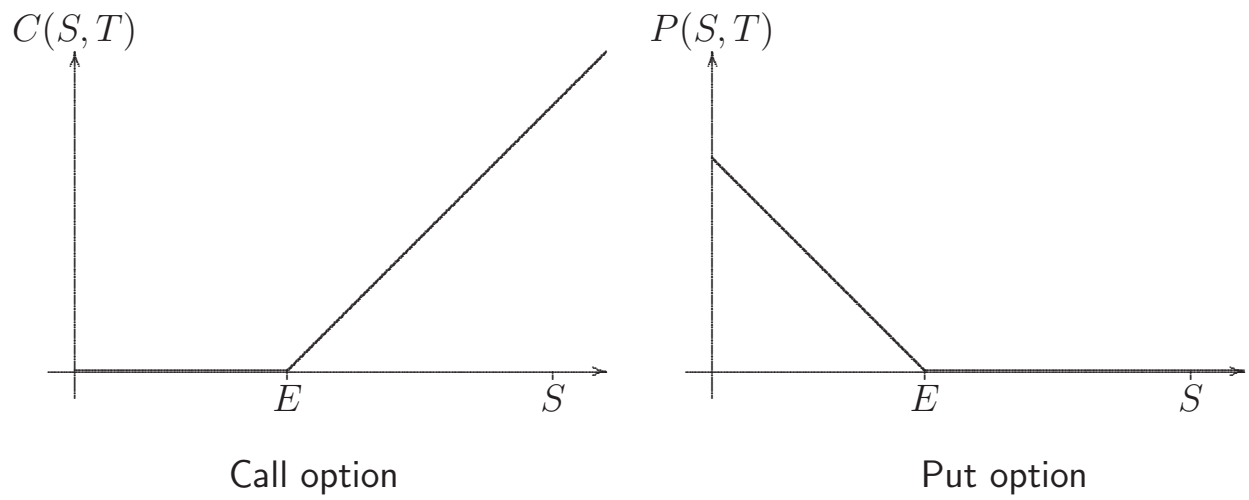
The right conferred by an option has to be paid for *in advance*, i.e. “up front.” The amount to be paid for the option is called the *premium*. *This should not be confused with either the current market price or the strike price of the underlying asset.*

Once this premium is paid, it induces an opportunity cost related to the (continuously compounded) risk-free interest rate. Hence the interest rate needs to be taken into account in determining the *value* of the option.

Being *derivative securities*, options are bought and sold. This is a good reason for wanting to determine their value.

## Some notation

$t$	time
$T$	the expiry (expiration date)
$E$	the exercise price of the option
$S$	market price of the underlying asset; $S = S(t)$
$V$	the value of the option; $V = V(S, t)$
$C$	the value of a call option; $C = C(S, t)$
$P$	the value of a put option; $P = P(S, t)$
$r$	the interest rate
$\sigma$	the <i>volatility</i> of the underlying asset — measure of the standard deviation of the returns of the asset
$\mu$	the <i>drift</i> of the underlying asset — measure of average rate of growth of the asset
$D(S, t)$	the <i>dividend rate</i> on the underlying asset — this is included in the Facchinei-Pang formulation



Values of options at expiry.

In  
 At  
 Out of

} the money

## The Black-Scholes option pricing equation

We consider first the simplest case: European options with

- no arbitrage
- no dividends
- no transaction costs

The price  $S$  of the underlying asset is assumed to follow a standard Brownian motion process (or Wiener process):

$$dS = \mu S dt + \sigma S dz$$

over the time interval  $[0, T]$ . In this equation,  $\mu > 0$  and  $\sigma > 0$  are the drift and volatility, respectively.



Let  $V(S, t)$  be the price of the derived security (i.e., option) at time  $t$  when the market price is  $S$ .

Then  $V(S, t)$  satisfies the *Black-Scholes equation*

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

This differential equation has infinitely many solutions, so we need to impose *boundary conditions*.

For European options:

The call price  $C(S, t)$  is required to satisfy

$$C(0, t) = 0 \text{ for all } t \in [0, T) \quad \text{and} \quad C(S, T) = \max\{S - E, 0\}.$$

The put price  $P(S, t)$  is required to satisfy

$$P(\infty, t) = 0 \quad \text{and} \quad P(S, T) = \max\{E - S, 0\}.$$

For American options:

The call price  $C(S, t)$  is required to satisfy

$$C(0, t) = 0 \text{ for all } t \in [0, T) \quad \text{and} \quad C(S, T) = \max\{S - E, 0\}.$$

The put price  $P(S, t)$  is required to satisfy

$$P(\infty, t) = 0 \quad \text{and} \quad P(S, T) = \max\{E - S, 0\}.$$

In addition to the preceding boundary conditions

$$C(S, t) \geq \max\{S - E, 0\},$$

and

$$P(S, t) \geq \max\{E - S, 0\}.$$

The condition for American call options is not needed in the present case since a call option on a stock that pays no dividend is never exercised early.

## American put options as free boundary problems

The problem at hand takes place on a cylinder:  $[0, \infty) \times [0, T]$ .

For each  $t \in [0, T]$  we want to split the  $S$  axis into two subintervals.

Doing so will divide the cylinder into two subregions.

The boundary between the regions will be given by a function  $S_f(t)$ .

Appropriate boundary conditions will hold on each of the subregions and on the boundary between them.

Since the location of the boundary between the two subregions is not known in advance, we have what is called a *free boundary problem*.

As it happens, this free boundary problem has an interpretation as a *differential inequality problem*.

Under the assumptions made above and after suitable discretization, this free boundary problem becomes an LCP.

First subregion:  $0 \leq S < S_f(t)$ .

For these values of  $S$ , *early exercise is optimal* and

$$P = E - S, \quad \frac{\partial P}{\partial t} + rS \frac{\partial P}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP < 0.$$

Second subregion:  $S_f(t) < S < \infty$ .

For these values of  $S$ , *early exercise is not optimal* and

$$P > E - S, \quad \frac{\partial P}{\partial t} + rS \frac{\partial P}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP = 0.$$

On the free boundary:  $S = S_f(t)$ .

The boundary conditions on  $S = S_f(t)$  are that  $P$  and its slope (which, in the jargon of the field, is called *delta*) are continuous and

$$P(S_f(t), t) = \max\{E - S_f(t), 0\} \quad \text{and} \quad \frac{\partial P}{\partial S}(S_f(t), t) = -1.$$

## A change of variables

In order to simplify the numerical solution of the free boundary problem posed above for the valuation of an American put option, one can carry out a complicated change of variables.

The variables  $S$  and  $t$  becomes  $x$  and  $\tau$ , respectively. In particular,

$$S = Ee^x \quad \text{and} \quad t = T - \tau/\frac{1}{2}\sigma^2.$$

Furthermore

$$k := r/\frac{1}{2}\sigma^2$$

and the function  $S = S_f(t)$  becomes  $x = x_f(t)$ .

The “payoff function”  $\max\{E - S, 0\}$  becomes

$$g(x, \tau) := e^{\frac{1}{2}(k+1)^2\tau} \max\{e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\}.$$

## The modified problem

The problem is now to find a solution of

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2} && \text{for } x > x_f(\tau) \\ u(x, \tau) &= g(x, \tau) && \text{for } x \leq x_f(\tau).\end{aligned}$$

The initial condition

$$u(x, 0) = g(x, 0) = \max\{e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\}$$

is imposed as is the behavioral property  $\lim_{x \rightarrow \infty} u(x, \tau) = 0$ .

For  $x \ll 0$ , early exercise is optimal, so  $u = g$ .

In addition, we have the constraint

$$u(x, \tau) \geq \max\{e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\}$$

and the requirement that  $u$  and  $\partial u / \partial x$  be continuous at  $x = x_f(t)$ .

For numerical reasons, we replace  $+\infty$  and  $-\infty$  by finite values  $x^+$  and  $-x^-$  where  $x^+ \gg 0$  and  $x^- \gg 0$ .

With this notation, we rephrase the boundary conditions as

$$u(x^+, \tau) = 0, \quad u(-x^-, \tau) = g(-x^-, \tau) \quad \text{for all } \tau.$$

This translates into the assumption that for large values of  $S$  we have  $P = 0$ , and for small values of  $S$ , we have  $P = S - E$ .

**Remark.** This formulation of the valuation problem for an American put option resembles that of an *obstacle problem* but with a time-dependent obstacle.

## Complementarity problem for valuation of American put options

$$\left( \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \right) \cdot (u(x, \tau) - g(x, \tau)) = 0$$

$$\left( \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \right) \geq 0, \quad (u(x, \tau) - g(x, \tau)) \geq 0$$

with boundary conditions

$$u(x, 0) = g(x, 0), \quad u(-x^-, \tau) = g(-x^-, \tau), \quad \text{and} \quad u(x^+, \tau) = 0$$

and the continuity of  $u$  and  $\frac{\partial u}{\partial x}$ .

When  $u = g$  it is optimal to exercise the option; when  $u > g$ , it is not.



## Numerical solution of the problem

The problem can be solved numerically by setting up a grid and using finite-difference approximations to the partial derivatives.

The free boundary  $x_f(\tau)$  can be discovered *after* the numerical solution is obtained.

For the put

$$u(x_f(\tau), \tau) = g(x_f(\tau), \tau), \text{ but } u(x_f(\tau), \tau) > g(x_f(\tau), \tau), x > x_f(\tau).$$

For the call

$$u(x_f(\tau), \tau) = g(x_f(\tau), \tau), \text{ but } u(x_f(\tau), \tau) > g(x_f(\tau), \tau), x < x_f(\tau).$$

The free boundary is defined by the points where  $u(x, \tau)$  first meets  $g(x, \tau)$ .

## Finite difference formulation

Using a regular finite mesh, truncate  $x$  so that

$$N^- \delta x \leq x = n \delta x \leq N^+ \delta x$$

where  $-N^- \gg 0$  and  $N^+ \gg 0$  are integers.

The discretization follows the Crank-Nicolson method. Let  $u_n^m := u(n\delta x, m\delta\tau)$ . Write

$$\frac{\partial u}{\partial \tau}(x, \tau + \delta\tau/2) = \frac{u_n^{m+1} - u_n^m}{\delta\tau} + O((\delta\tau)^2)$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x, \tau + \delta\tau/2) = & \frac{1}{2} \left( \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2} \right) \\ & + \frac{1}{2} \left( \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} \right) + O((\delta x)^2) \end{aligned}$$

Drop the terms  $O((\delta\tau)^2)$  and  $O((\delta x)^2)$  above.

The inequality  $\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$  is approximated by

$$\begin{aligned} u_n^{m+1} - \frac{1}{2}\alpha (u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) \\ \geq u_n^m + \frac{1}{2}\alpha (u_{n+1}^m - 2u_n^m + u_{n-1}^m) \end{aligned}$$

where

$$\alpha = \frac{\delta\tau}{(\delta x)^2}.$$

Now write  $g_n^m = g(n\delta x, m\delta\tau)$  for the discretized payoff function.

Then

$$u(x, \tau) \geq g(x, \tau) \text{ becomes } u_n^m \geq g_n^m \text{ for } m \geq 1.$$

The boundary conditions and initial conditions imply

$$u_{N-}^m = g_{N-}^m, \quad u_{N+}^m = g_{N+}^m, \quad u_n^0 = g_n^0.$$

Now let

$$Z_n^m := (1 - \alpha)u_n^m + \frac{1}{2}\alpha(u_{n+1}^m - u_{n-1}^m).$$

The approximation above becomes

$$(1 + \alpha)u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} - u_{n-1}^{m+1}) \geq Z_n^m.$$

Note that at time step  $(m+1)\delta\tau$  we can calculate  $Z_n^m$  explicitly since we already know the values of the  $u_n^m$ .

The complementarity condition above is approximated by

$$\left((1 + \alpha)u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} - u_{n-1}^{m+1}) - Z_n^m\right) \cdot (u_n^{m+1} - g_n^{m+1}) = 0.$$

## Toward the actual LCP

Write the vector of approximate values at time step  $m\delta\tau$  as

$$u^m = (u_{N^-+1}^m, \dots, u_{N^+-1}^m)$$

and the vector

$$g^m = (g_{N^-+1}^m, \dots, g_{N^+-1}^m).$$

Note that the omitted components  $u_{N^-}^m$  and  $u_{N^+}^m$  are determined by the boundary conditions.

Now define

$$\begin{aligned} b^m &= (b_{N^-+1}^m, \dots, b_0^m, \dots, b_{N^+-1}^m) \\ &= (Z_{N^-+1}^m, \dots, Z_0^m, \dots, Z_{N^+-1}^m) + \frac{1}{2}\alpha(g_{N^-}^{m+1}, 0, \dots, 0, g_{N^+}^{m+1}). \end{aligned}$$

Note that the last vector in the definition of  $b^m$  includes the boundary effects at  $n = N^-$  and  $n = N^+$ .

Let  $C = [c_{ij}]$  denote the matrix whose entries satisfy

$$c_{ij} = \begin{cases} 1 + \alpha & \text{if } i = j \\ -\frac{1}{2}\alpha & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that  $C$  is a symmetric, tridiagonal, diagonally dominant,  $\mathbf{Z}$ -matrix. Hence it is a Stieltjes matrix.

Our LCP is

$$\begin{aligned} Cu^{m+1} &\geq b^m, & u^{m+1} &\geq g^{m+1} \\ (u^{m+1} - g^{m+1}) \cdot (Cu^{m+1} - b^m) &= 0 \end{aligned}$$

We convert this to the traditional LCP form  $(q^m, M)$  by putting

$$M = C, \quad q^m = -b^m, \quad \text{and} \quad z^{m+1} = u^{m+1} - g^{m+1}.$$

Problems of this sort can be solved by a variety of methods, both direct and indirect. The latter are recommended for very large-scale problems in this class. One such method is called *projected SOR*.