

NLA course project

”Risk analysis and high-dimensional integrals”

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1 Problem formulation

Problem statement is described in [2]. Let $c(s, t)$ denote the price at time t of a derivative of a stock (such as a European call option) when the price of the stock $S(t) = s$. Then c must satisfy the partial differential equation:

$$\begin{aligned} \frac{\partial c(s, t)}{\partial t} + rs \frac{\partial c(s, t)}{\partial s} + \frac{1}{2} \gamma^2 s^2 \frac{\partial^2 c(s, t)}{\partial s^2} &= rc(s, t), \quad s \in \mathbb{R}_+, \quad t \in [0; T'], \\ c(s, T') &= g(s), \quad s \in \mathbb{R}_+, \\ c(0, t) &= 0, \quad t \in [0; T']. \end{aligned} \tag{1}$$

Here r is the risk-free interest rate and γ is the volatility of the stock. This is called Black-Scholes partial differential equation. Different terminal conditions $g(s)$ corresponds to different types of derivatives (e.g. European call option).

Consider variable substitution of the following type: $x = \ln s$ and $w(x, t) = c(s, t)$. Then after sequence of elementary transformations one obtains:

$$\begin{aligned} \frac{\partial w(x, t)}{\partial t} + \left(r - \frac{\gamma^2}{2}\right) \frac{\partial w(x, t)}{\partial x} + \frac{1}{2} \gamma^2 \frac{\partial^2 w(x, t)}{\partial x^2} &= rw(x, t), \quad x \in \mathbb{R}, \quad t \in [0; T'], \\ w(x, T') &= g(e^x), \quad x \in \mathbb{R}, \\ w(-\infty, t) &= 0, \quad t \in [0; T']. \end{aligned} \tag{2}$$

To get rid of the drift term $\left(r - \frac{\gamma^2}{2}\right) \frac{\partial w(x, t)}{\partial x}$ we introduce

$$v(x, t) = e^{\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2})x} w(x, t), \tag{3}$$

and obtain

$$\begin{aligned} \frac{\partial v(x, t)}{\partial t} + \frac{1}{2} \gamma^2 \frac{\partial^2 v(x, t)}{\partial x^2} &= \left(r + \frac{1}{2\gamma^2} \left(r - \frac{\gamma^2}{2}\right)^2\right) v(x, t), \quad x \in \mathbb{R}, \quad t \in [0; T'], \\ v(x, T') &= e^{\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2})x} g(e^x), \quad x \in \mathbb{R}, \\ v(-\infty, t) &= 0, \quad t \in [0; T']. \end{aligned} \tag{4}$$

Introducing $\sigma = \frac{1}{2} \gamma^2$, $V(x, t) = V = r + \frac{1}{2\gamma^2} \left(r - \frac{\gamma^2}{2}\right)^2$ and $f(x) = e^{\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2})x} g(e^x)$ we have

$$\begin{aligned} -\frac{\partial v(x, t)}{\partial t} &= \sigma \frac{\partial^2 v(x, t)}{\partial x^2} - V(x, t) v(x, t), \quad x \in \mathbb{R}, \quad t \in [0; T'], \\ v(x, T') &= f(x), \quad x \in \mathbb{R}, \\ v(-\infty, t) &= 0, \quad t \in [0; T']. \end{aligned} \tag{5}$$

Finally, we set $u(x, t) = v(x, T' - t)$ and obtain

$$\begin{aligned}\frac{\partial u(x, t)}{\partial t} &= \sigma \frac{\partial^2 u(x, t)}{\partial x^2} - V(x, t)u(x, t), \quad x \in \mathbb{R}, \quad t \in [0; T'], \\ v(x, 0) &= f(x), \quad x \in \mathbb{R}, \\ v(-\infty, t) &= 0, \quad t \in [0; T'].\end{aligned}\tag{6}$$

The condition $v(-\infty, t) = 0$ will be satisfied automatically (prove!), so our partial differential equation is

$$\begin{aligned}\frac{\partial u(x, t)}{\partial t} &= \sigma \frac{\partial^2 u(x, t)}{\partial x^2} - V(x, t)u(x, t), \quad x \in \mathbb{R}, \quad t \in [0; T'], \\ v(x, 0) &= f(x), \quad x \in \mathbb{R}.\end{aligned}\tag{7}$$

This form is exactly one that is considered in Oseledets's paper.

So, when we have $g(s)$, γ , r , T' from our financial model, we just set

$$f(x) = e^{\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2})x} g(e^x), \quad \sigma = \frac{1}{2}\gamma^2, \quad V(x, t) = V = r + \frac{1}{2\gamma^2} \left(r - \frac{\gamma^2}{2} \right)^2,\tag{8}$$

solve (7) with this parameters, and reconstruct the desired function $c(s, t)$ as

$$c(s, t) = w(e^x, t) = e^{-\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2}) \ln s} v(\ln s, t) = e^{-\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2}) \ln s} u(\ln s, T' - t).\tag{9}$$