NLA course project
"Risk analysis and high-dimensional integrals"

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Let $c(s,t)$ be the price of the option at time t when the market price is s. Then $c(s,t)$ satisfies the Black-Scholes PDE:

$$\begin{aligned}
&\frac{\partial c(s,t)}{\partial t} + rs\frac{\partial c(s,t)}{\partial s} + \frac{1}{2}\gamma^2 s^2 \frac{\partial^2 c(s,t)}{\partial s^2} = rc(s,t), \quad s \in \mathbb{R}_+, \;\; t \in [0;\, T'], \\
&c(s,T') = g(s), \quad s \in \mathbb{R}_+, \\
&c(0,t) = 0, \quad t \in [0;\, T'].
\end{aligned} \tag{1}$$

Here r is the risk-free interest rate and γ is the volatility of the stock.

This is called Black-Scholes partial differential equation. Different terminal conditions $g(s)$ correspond to different types of derivatives (e.g. European call option). The following $g(s)$ are considered (we consider only those cases when analytical solution can be derived).

1 European call option (vanilla call option)

For that case the terminal condition is the following:

$$g(s) = \max\{s - E, 0\} \tag{2}$$

where E is the exercise proce of the option. The corresponding analytical solution is (Black-Scholes Formula):

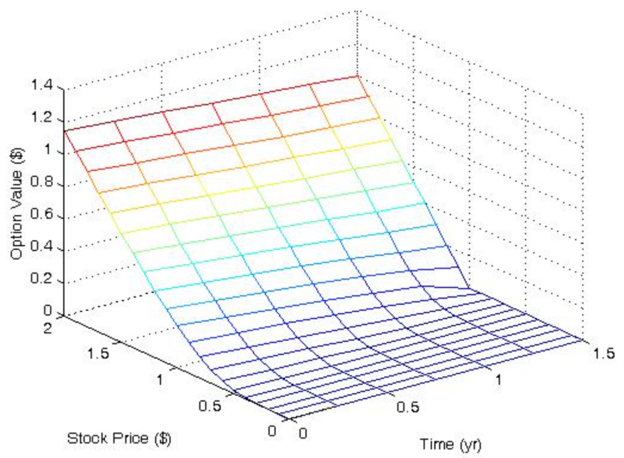
$$c(s,t) = s\Phi\left(d_+^{T'}(t)\right) - Ee^{-r(T'-t)}\Phi\left(d_-^{T'}(t)\right) \tag{3}$$

where:

$$d_+^{T'}(t) = \frac{\log(\frac{s}{E}) + (r + \frac{\gamma^2}{2})(T'-t)}{\gamma\sqrt{T'-t}} \tag{4}$$

$$d_+^{T'}(t) = \frac{\log(\frac{s}{E}) + (r - \frac{\gamma^2}{2})(T'-t)}{\gamma\sqrt{T'-t}} \tag{5}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy \tag{6}$$



2 European put option (vanilla put option)

For that case the terminal condition is the following:

$$g(s) = \max\{E - s, 0\} \tag{7}$$

For European put option the corresponding analytical solution is:

$$c(s,t) = Ee^{-r(T'-t)}\Phi(-d_-^{T'}) - s\Phi(-d_+^{T'}) \tag{8}$$

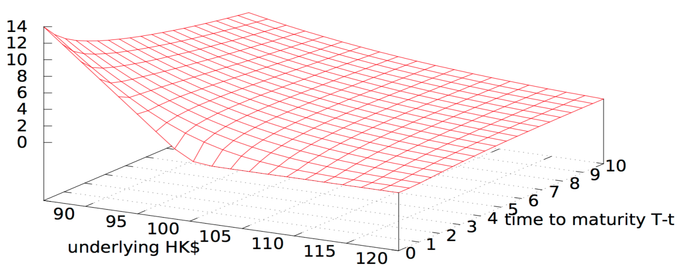


Fig. 5.3: Graph of the Black-Scholes put price function with strike price $K = 100$.

3 Cash-or-nothing call option

For that case the terminal condition is the following:

$$g(s) = \begin{cases} B & \text{if } s \geq E \\ 0 & \text{if } s < E \end{cases} \tag{9}$$

Analytical solution is given by:

$$c(s,t) = Be^{-r(T'-t)}\Phi(d_+^{T'}) \tag{10}$$

4 Cash-or-nothing put option

For that case the terminal condition is the following:

$$g(s) = \begin{cases} 0 & \text{if } s \geq E \\ B & \text{if } s < E \end{cases} \tag{11}$$

Analytical solution is given by:

$$c(s,t) = Be^{-r(T'-t)}\Phi(-d_+^{T'}) \tag{12}$$