NLA course project
"Risk analysis and high-dimensional integrals"

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December 10, 2016

1 Problem formulation

Problem statement is described in [2]. Let $c(s,t)$ denote the price at time t of a derivative of a stock (such as a European call option) when the price of the stock $S(t) = s$. Then c must satisfy the partial differential equation:

$$\begin{aligned}
&\frac{\partial c(s,t)}{\partial t} + rs\frac{\partial c(s,t)}{\partial s} + \frac{1}{2}\gamma^2 s^2 \frac{\partial^2 c(s,t)}{\partial s^2} = rc(s,t), \quad s \in \mathbb{R}_+, \ \ t \in [0;\ T'],\\
&c(s,T') = g(s), \quad s \in \mathbb{R}_+,\\
&c(0,t) = 0, \quad t \in [0;\ T'].
\end{aligned} \tag{1}$$

Here r is the risk-free interest rate and γ is the volatility of the stock. This is called Black-Scholes partial differential equation. Different terminal conditions $g(s)$ corresponds to different types of derivatives (e.g. European call option).

Consider variable substitution of the following type: $x = \ln s$ and $w(x,t) = c(s,t)$. Then after sequence of elementary transformations one obtains:

$$\begin{aligned}
&\frac{\partial w(x,t)}{\partial t} + \left(r - \frac{\gamma^2}{2}\right)\frac{\partial w(x,t)}{\partial x} + \frac{1}{2}\gamma^2 \frac{\partial^2 w(x,t)}{\partial x^2} = rw(x,t), \quad x \in \mathbb{R}, \ \ t \in [0;\ T'],\\
&w(x,T) = g(e^x), \quad x \in \mathbb{R},\\
&w(-\infty,t) = 0, \quad t \in [0;\ T'].
\end{aligned} \tag{2}$$

To get rid of the drift term $\left(r - \frac{\gamma^2}{2}\right)\frac{\partial w(x,t)}{\partial x}$ we introduce

$$v(x,t) = e^{-\frac{1}{\gamma^2}(r-\frac{\gamma^2}{2})} w(x,t), \tag{3}$$

and obtain

$$\begin{aligned}
&\frac{\partial v(x,t)}{\partial t} + \frac{1}{2}\gamma^2 \frac{\partial^2 v(x,t)}{\partial x^2} = \left(r + \frac{1}{2\gamma^2}\left(r - \frac{\gamma^2}{2}\right)^2\right) v(x,t), \quad x \in \mathbb{R}, \ \ t \in [0;\ T'],\\
&v(x,T') = e^{-\frac{1}{\gamma^2}(r-\frac{\gamma^2}{2})} g(e^x), \quad x \in \mathbb{R},\\
&v(-\infty,t) = 0, \quad t \in [0;\ T'].
\end{aligned} \tag{4}$$

Introducing $\sigma = \frac{1}{2}\gamma^2$, $V(x,t) = V = r + \frac{1}{2\gamma^2}\left(r - \frac{\gamma^2}{2}\right)^2$ and $f(x) = e^{-\frac{1}{\gamma^2}(r-\frac{\gamma^2}{2})} g(e^x)$ we have

$$\begin{aligned}
&-\frac{\partial v(x,t)}{\partial t} = \sigma \frac{\partial^2 v(x,t)}{\partial x^2} - V(x,t)v(x,t), \quad x \in \mathbb{R}, \ \ t \in [0;\ T'],\\
&v(x,T') = f(x), \quad x \in \mathbb{R},\\
&v(-\infty,t) = 0, \quad t \in [0;\ T'].
\end{aligned} \tag{5}$$

Finally, we set $u(x,t) = v(x, T - t)$ and obtain

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = \sigma \frac{\partial^2 u(x,t)}{\partial x^2} - V(x,t)u(x,t), \quad x \in \mathbb{R}, \ \ t \in [0;\ T'], \\
&v(x,0) = f(x), \quad x \in \mathbb{R}, \\
&v(-\infty, t) = 0, \quad t \in [0;\ T'].
\end{aligned} \tag{6}
$$

The condition $v(-\infty, t) = 0$ will be satisfied automatically (prove!), so our partial differential equation is

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = \sigma \frac{\partial^2 u(x,t)}{\partial x^2} - V(x,t)u(x,t), \quad x \in \mathbb{R}, \ \ t \in [0;\ T'], \\
&v(x,0) = f(x), \quad x \in \mathbb{R}.
\end{aligned} \tag{7}
$$

This form is exactly one that is considered in Oseledets's paper.

So, when we have $g(s)$, γ, r from our financial model, we just set

$$
f(x) = e^{-\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2})} g(e^x), \ \ \sigma = \frac{1}{2}\gamma^2, \quad V(x,t) = V = r + \frac{1}{2\gamma^2}\left(r - \frac{\gamma^2}{2}\right)^2, \tag{8}
$$

solve (7) with this parameters, and reconstruct the desired function $c(s,t)$ as

$$
c(s,t) = w(e^x, t) = e^{\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2})} v(e^x, t) = e^{\frac{1}{\gamma^2}(r - \frac{\gamma^2}{2})} u(e^x, T' - t). \tag{9}
$$