Chapter 5
The Black-Scholes PDE

In this chapter we review the notions of assets, self-financing portfolios, risk-neutral measures, and arbitrage in continuous time. We also derive the Black-Scholes PDE for self-financing portfolios, and we solve this equation using the heat kernel method.

5.1 Continuous-Time Market Model

Let $(A_t)_{t\in\mathbb{R}_+}$ be the riskless asset given by

$$\frac{dA_t}{A_t} = r dt, \quad t \in \mathbb{R}_+, \quad i.e. \quad A_t = A_0 \, \mathrm{e}^{rt}, \quad t \in \mathbb{R}_+, \tag{5.1}$$

where $r > 0$ is the risk-free interest rate.

For $t > 0$, let the risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ be defined as

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+. \tag{5.2}$$

By Proposition 4.8 we have

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \qquad t \in \mathbb{R}_+.$$

5.2 Self-Financing Portfolio Strategies

Let ξ_t and η_t denote the (possibly fractional) quantities invested at time t, respectively in the assets S_t and A_t, and let

$$\bar{\xi}_t = (\eta_t, \xi_t), \qquad \bar{S}_t = (A_t, S_t), \qquad t \in \mathbb{R}_+,$$

denote the associated portfolio and asset price processes. The value of the portfolio V_t at time t is given by

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+. \tag{5.3}$$

Our description of portfolio strategies proceeds in four steps which correspond to different interpretations of the self-financing condition.

Portfolio update

The portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from (η_t, ξ_t) to $(\eta_{t+dt}, \xi_{t+dt})$, *i.e.*

$$\bar{\xi}_t \cdot \bar{S}_{t+dt} = A_{t+dt}\eta_t + S_{t+dt}\xi_t = A_{t+dt}\eta_{t+dt} + S_{t+dt}\xi_{t+dt} = \bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}, \tag{5.4}$$

which is the continuous-time equivalent of the self-financing condition already encountered in the discrete setting of Chapter 2, see Definition 2.1. A major difference with the discrete-time case of Definition 2.1, however, is that the continuous-time differentials dS_t and $d\xi_t$ do not make pathwise sense as the stochastic integral is defined by an L^2 limit, cf. Proposition 4.6, or by convergence in probability.

Portfolio value	$\bar{\xi}_t \cdot \bar{S}_t$ $\longrightarrow$	$\bar{\xi}_t \cdot \bar{S}_{t+dt}$ =	$\bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}$ $\longrightarrow$	$\bar{\xi}_{t+dt} \cdot \bar{S}_{t+2dt}$
Asset value	S_t	S_{t+dt}	S_{t+dt}	S_{t+2dt}
Time scale	t	$t+dt$	$t+dt$	$t+2dt$
Portfolio allocation	ξ_t	ξ_t	ξ_{t+dt}	ξ_{t+dt}

Fig. 5.1: Illustration of the self-financing condition (5.4).

Portfolio re-allocation

Equivalently, Condition (5.4) can be rewritten as

$$A_{t+dt}d\eta_t + S_{t+dt}d\xi_t = 0, \tag{5.5}$$

or

$$S_{t+dt}(\xi_{t+dt} - \xi_t) = A_{t+dt}(\eta_t - \eta_{t+dt}), \tag{5.6}$$

i.e. when one sells a (possibly fractional) quantity $\eta_t - \eta_{t+dt} > 0$ of the riskless asset A_{t+dt} between the time periods $[t, t+dt]$ and $[t+dt, t+2dt]$ for a total amount $A_{t+dt}(\eta_t - \eta_{t+dt})$, one should entirely spend this income to buy a quantity $\xi_{t+dt} - \xi_t > 0$ of the risky asset for an amount $S_{t+dt}(\xi_{t+dt} - \xi_t) > 0$.

Similarly, if one sells a quantity $-d\xi_t > 0$ of the risky asset S_{t+dt} between the time periods $[t, t+dt]$ and $[t+dt, t+2dt]$ for a total amount $-S_{t+dt}d\xi_t$, one should entirely use this income to buy a quantity $d\eta_t > 0$ of the riskless asset for an amount $A_{t+dt}d\eta_t > 0$, *i.e.*

$$A_{t+dt}d\eta_t = -S_{t+dt}d\xi_t.$$

Itô calculus version

Condition (5.6) can be rewritten as

$$S_t(\xi_{t+dt} - \xi_t) + A_{t+dt}(\eta_{t+dt} - \eta_t) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) = 0,$$

i.e.

$$S_t d\xi_t + A_t d\eta_t + dA_t \cdot d\eta_t + dS_t \cdot d\xi_t = 0. \tag{5.7}$$

Using the relation

$$(A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = dA_t \cdot d\eta_t = rA_t(dt \cdot d\eta_t) \simeq 0$$

in the sense of the Itô calculus by the Itô Table 4.1, Relation (5.7) rewrites as

$$A_t d\eta_t + S_t d\xi_t + dS_t \cdot d\xi_t = 0 \tag{5.8}$$

in differential notation.

Portfolio differential

In practice we will use the following definition for the self-financing portfolio property.

Proposition 5.1. *A portfolio allocation $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ with price*

$$V_t = \xi_t S_t + \eta_t A_t, \qquad t \in \mathbb{R}_+,$$

is self-financing if and only if the relation

$$dV_t = \eta_t dA_t + \xi_t dS_t \tag{5.9}$$

holds.

Proof. We check that by Itô's calculus we have

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t + A_t d\eta_t + S_t d\xi_t + d\eta_t \cdot dA_t + d\xi_t \cdot dS_t \\ &= \eta_t dA_t + \xi_t dS_t + A_t d\eta_t + S_t d\xi_t + d\xi_t \cdot dS_t, \end{aligned}$$

since $d\eta_t \cdot dA_t = rA_t dt \cdot d\eta_t = 0$, hence Condition (5.8) rewrites as (5.9), which is equivalent to (5.4) and (5.5). □

Let

$$\widetilde{V}_t = e^{-rt} V_t \qquad \text{and} \qquad X_t = e^{-rt} S_t$$

respectively denote the discounted portfolio value and discounted risky asset prices at time $t \geq 0$. We have

$$\begin{aligned} dX_t &= d(e^{-rt} S_t) \\ &= -r e^{-rt} S_t dt + e^{-rt} dS_t \\ &= -r e^{-rt} S_t dt + \mu e^{-rt} S_t dt + \sigma e^{-rt} S_t dB_t \\ &= X_t((\mu - r)dt + \sigma dB_t). \end{aligned}$$

In the next lemma we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum over time of discounted profits and losses (number of risky assets ξ_t times discounted price variation dX_t).

The following lemma is the continuous-time analog of Lemma 3.1.

Lemma 5.2. *Let $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ be a portfolio strategy with value*

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+.$$

The following statements are equivalent:

i) the portfolio strategy $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ is self-financing,

ii) we have

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u dX_u, \qquad t \in \mathbb{R}_+. \tag{5.10}$$

Proof. Assuming that (i) holds, the self-financing condition and (5.1)-(5.2) show that

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t \qquad t \in \mathbb{R}_+, \end{aligned}$$

hence

$$e^{-rt} dV_t = r e^{-rt} V_t dt + (\mu - r) e^{-rt} \xi_t S_t dt + \sigma e^{-rt} \xi_t S_t dB_t, \quad t \in \mathbb{R}_+,$$

and

$$\begin{aligned} d\widetilde{V}_t &= d\left(e^{-rt} V_t\right) \\ &= -r e^{-rt} V_t dt + e^{-rt} dV_t \\ &= (\mu - r)\xi_t e^{-rt} S_t dt + \sigma \xi_t e^{-rt} S_t dB_t \\ &= (\mu - r)\xi_t X_t dt + \sigma \xi_t X_t dB_t \end{aligned}$$

$$= \xi_t dX_t, \qquad t \in \mathbb{R}_+,$$

i.e. (5.10) holds by integrating on both sides as

$$\widetilde{V}_t - \widetilde{V}_0 = \int_0^t d\widetilde{V}_u = \int_0^t \xi_u dX_u, \qquad t \in \mathbb{R}_+.$$

(ii) Conversely, if (5.10) is satisfied we have

$$\begin{aligned}
dV_t &= d(\mathrm{e}^{rt}\widetilde{V}_t) \\
&= r\,\mathrm{e}^{rt}\widetilde{V}_t dt + \mathrm{e}^{rt} d\widetilde{V}_t \\
&= r\,\mathrm{e}^{rt}\widetilde{V}_t dt + \mathrm{e}^{rt}\xi_t dX_t \\
&= rV_t dt + \mathrm{e}^{rt}\xi_t dX_t \\
&= rV_t dt + \mathrm{e}^{rt}\xi_t X_t((\mu - r)dt + \sigma dB_t) \\
&= rV_t dt + \xi_t S_t((\mu - r)dt + \sigma dB_t) \\
&= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\
&= \eta_t dA_t + \xi_t dS_t,
\end{aligned}$$

hence the portfolio is self-financing according to Definition 5.1. □

As a consequence of (5.10), the hedging problem of a claim C with maturity T is reduced to that of finding the representation of the discounted claim $\tilde{C} = \mathrm{e}^{-rT}C$ as a stochastic integral:

$$\tilde{C} = \widetilde{V}_0 + \int_0^T \xi_u dX_u.$$

Note also that (5.10) shows that the value of a self-financing portfolio can be written as

$$V_t = \mathrm{e}^{rt}V_0 + (\mu - r)\int_0^t \mathrm{e}^{r(t-u)}\xi_u S_u du + \sigma\int_0^t \mathrm{e}^{r(t-u)}\xi_u S_u dB_u, \quad t \in \mathbb{R}_+. \tag{5.11}$$

5.3 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the discrete and two-step models. In the sequel we will only consider *admissible* portfolio strategies whose total value V_t remains nonnegative for all times $t \in [0, T]$.

Definition 5.3. *A portfolio strategy* $(\xi_t, \eta_t)_{t\in[0,T]}$ *with price* $V_t = \xi_t S_t + \eta_t A_t$, $t \in \mathbb{R}_+$, *constitutes an arbitrage opportunity if all* three *following conditions are satisfied:*

i) $V_0 \leq 0$,
ii) $V_T \geq 0$,

iii) $\mathbb{P}(V_T > 0) > 0$.

Roughly speaking, (ii) means that the investor wants no loss, (iii) means that he wishes to sometimes make a strictly positive gain, and (i) means that he starts with zero capital or even with a debt.

Next we turn to the definition of risk-neutral measures in continuous time. Recall that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, *i.e.*

$$\mathcal{F}_t = \sigma(B_u \ : \ 0 \leq u \leq t), \qquad t \in \mathbb{R}_+.$$

Definition 5.4. *A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if it satisfies*

$$\mathbb{E}^*[S_t|\mathcal{F}_u] = \mathrm{e}^{r(t-u)}S_u, \qquad 0 \leq u \leq t, \tag{5.12}$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

From the relation

$$A_t = \mathrm{e}^{r(t-u)}A_u, \qquad 0 \leq u \leq t,$$

we interpret (5.12) by saying that the expected return of the risky asset S_t under $\mathbb{P}^*$ equals the return of the riskless asset A_t. The discounted price X_t of the risky asset is defined by

$$X_t = \mathrm{e}^{-rt}S_t = \frac{S_t}{A_t/A_0}, \qquad t \in \mathbb{R}_+,$$

i.e. A_t/A_0 plays the role of a *numéraire* in the sense of Chapter 12.

Definition 5.5. *A continuous time process $(Z_t)_{t\in\mathbb{R}_+}$ of integrable random variables is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if*

$$\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s, \qquad 0 \leq s \leq t.$$

Note that when $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale, Z_t is in particular $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$.

As in the discrete case, the notion of martingale can be used to characterize risk-neutral measures.

Proposition 5.6. *The measure $\mathbb{P}^*$ is risk-neutral if and only if the discounted price process $(X_t)_{t\in\mathbb{R}_+}$ is a martingale* under $\mathbb{P}^*$.

Proof. If $\mathbb{P}^*$ is a risk-neutral measure we have

$$\begin{aligned}
\mathbb{E}^*[X_t|\mathcal{F}_u] &= \mathbb{E}^*[\mathrm{e}^{-rt}S_t|\mathcal{F}_u] \\
&= \mathrm{e}^{-rt}\,\mathbb{E}^*[S_t|\mathcal{F}_u] \\
&= \mathrm{e}^{-rt}\,\mathrm{e}^{r(t-u)}S_u
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-ru} S_u \\
&= X_u, \qquad 0 \le u \le t,
\end{aligned}
$$

hence $(X_t)_{t\in\mathbb{R}_+}$ is a martingale. Conversely, if $(X_t)_{t\in\mathbb{R}_+}$ is a martingale then

$$
\begin{aligned}
\mathbb{E}^*[S_t|\mathcal{F}_u] &= \mathrm{e}^{rt}\,\mathbb{E}^*[X_t|\mathcal{F}_u] \\
&= \mathrm{e}^{rt} X_u \\
&= \mathrm{e}^{r(t-u)} S_u, \qquad 0 \le u \le t,
\end{aligned}
$$

hence the measure $\mathbb{P}^*$ is risk-neutral according to Definition 5.4. □

As in the discrete time case, $\mathbb{P}^*$ would be called a risk-premium measure if it satisfied

$$
\mathbb{E}^*[S_t|\mathcal{F}_u] > \mathrm{e}^{r(t-u)} S_u, \qquad 0 \le u \le t,
$$

meaning that by taking risks in buying S_t, one could make an expected return higher than that of

$$
A_t = \mathrm{e}^{r(t-u)} A_u, \qquad 0 \le u \le t.
$$

Next we note that the first fundamental theorem of asset pricing also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

Theorem 5.7. *A market is* without *arbitrage opportunity if and only if it admits at least one (equivalent) risk-neutral measure* $\mathbb{P}^*$.

Proof. cf. [51] and Chapter VII-4a of [104]. □

5.4 Market Completeness

Definition 5.8. *A contingent claim with payoff* C *is said to be attainable if there exists a (self-financing) portfolio strategy* $(\eta_t, \xi_t)_{t\in[0,T]}$ *such that*

$$
C = V_T.
$$

In this case the price of the claim at time t will be equal to the value V_t of any self-financing portfolio hedging C.

Definition 5.9. *A market model is said to be* complete *if every contingent claim* C *is attainable.*

The next result is a continuous-time restatement of the second fundamental theorem of asset pricing.

Theorem 5.10. *A market model without arbitrage is complete if and only if it admits only one (equivalent) risk-neutral measure* $\mathbb{P}^*$.

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Proof. cf. [51] and Chapter VII-4a of [104]. □

In the Black-Scholes model one can show the existence of a unique risk-neutral measure, hence the model is without arbitrage and complete.

5.5 The Black-Scholes PDE

We start by deriving the Black-Scholes Partial Differential Equation (PDE) for the price of a self-financing portfolio. Note that the drift parameter μ in (5.2) is absent of the PDE (5.13) and is not involved in the Black-Scholes formula (5.18).

Proposition 5.11. *Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy such that*

(i) $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the value $V_t := \eta_t A_t + \xi_t S_t$, $t \in \mathbb{R}_+$, takes the form

$$V_t = g(t, S_t), \qquad t \in \mathbb{R}_+,$$

for some $g \in \mathcal{C}^{1,2}((0,\infty)\times(0,\infty))$.

Then the function $g(t,x)$ satisfies the Black-Scholes PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t,x), \quad x > 0, \quad t \in [0,T], \tag{5.13}$$

and ξ_t is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t), \qquad t \in \mathbb{R}_+. \tag{5.14}$$

Proof. First, note that the self-financing condition (5.9) implies

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \end{aligned} \tag{5.15}$$

$t \in \mathbb{R}_+$. We now rewrite (4.25) under the form of an Itô process

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

as in (4.19), by taking

$$u_t = \sigma S_t, \quad \text{and} \quad v_t = \mu S_t, \qquad t \in \mathbb{R}_+.$$

The application of Itô's formula Theorem 4.7 to $V_t = g(t, S_t)$ leads to

$$\begin{aligned} dg(t, S_t) &= v_t \frac{\partial g}{\partial x}(t, S_t)dt + u_t \frac{\partial g}{\partial x}(t, S_t)dB_t \\ &\quad + \frac{\partial g}{\partial t}(t, S_t)dt + \frac{1}{2}|u_t|^2 \frac{\partial^2 g}{\partial x^2}(t, S_t)dt \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + \mu S_t \frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t)dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t)dB_t. \end{aligned} \tag{5.16}$$

By respective identification of the terms in dB_t and dt in (5.15) and (5.16) we get

$$\begin{cases} r\eta_t A_t dt + \mu \xi_t S_t dt = \dfrac{\partial g}{\partial t}(t, S_t)dt + \mu S_t \dfrac{\partial g}{\partial x}(t, S_t)dt + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t, S_t)dt, \\ \xi_t S_t \sigma dB_t = S_t \sigma \dfrac{\partial g}{\partial x}(t, S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rV_t - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t, S_t) + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t, S_t), \\ \xi_t = \dfrac{\partial g}{\partial x}(t, S_t), \end{cases}$$

i.e.

$$\begin{cases} rg(t, S_t) = \dfrac{\partial g}{\partial t}(t, S_t) + rS_t \dfrac{\partial g}{\partial x}(t, S_t) + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t, S_t), \\ \xi_t = \dfrac{\partial g}{\partial x}(t, S_t). \end{cases} \tag{5.17}$$

□

The derivative giving ξ_t in (5.14) is called the Delta of the option price, cf. Proposition 5.13 below.

The amount invested on the riskless asset is

$$\eta_t A_t = V_t - \xi_t S_t = g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t),$$

and η_t is given by

$$\begin{aligned} \eta_t &= \frac{V_t - \xi_t S_t}{A_t} \\ &= \frac{g(t, S_t) - S_t \dfrac{\partial g}{\partial x}(t, S_t)}{A_t} \end{aligned}$$

$$= \frac{g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t)}{A_0 \,\mathrm{e}^{rt}}.$$

In the next proposition we add a terminal condition $g(T, x) = f(x)$ to the Black-Scholes PDE in order to hedge claim C of the form $C = f(S_T)$.

Proposition 5.12. *The price of any self-financing portfolio of the form $V_t = g(t, S_t)$ hedging an option with payoff $C = f(S_T)$ satisfies the Black-Scholes PDE*

$$\boxed{\begin{cases} rg(t, x) = \dfrac{\partial g}{\partial t}(t, x) + rx\dfrac{\partial g}{\partial x}(t, x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t, x), \\ \\ g(T, x) = f(x). \end{cases}}$$

Forward contracts

When $C = S_T - K$ is the (linear) payoff of a forward contract, *i.e.* $f(x) = x - K$, the Black-Scholes PDE admits the easy solution

$$g(t, x) = x - K\,\mathrm{e}^{-r(T-t)}, \qquad x > 0, \quad t \in [0, T],$$

and the Delta of the option price is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1, \qquad t \in [0, T],$$

cf. Exercise 5.4. The forward contract can be realized by the option issuer as follows:

a) At time t, receive the option premium $S_t - \mathrm{e}^{-r(T-t)}K$ from the option buyer.
b) Borrow $\mathrm{e}^{-r(T-t)}K$ from the bank, to be refunded at maturity.
c) Buy the risky asset using the amount $S_t - \mathrm{e}^{-r(T-t)}K + \mathrm{e}^{-r(T-t)}K = S_t$.
d) Hold the risky asset until maturity (do nothing, constant portfolio strategy).
e) At maturity T, hand in the asset to the option holder, who gives the price K in exchange.
f) Use the amount $K = \mathrm{e}^{r(T-t)}\,\mathrm{e}^{-r(T-t)}K$ to refund the bank of the sum $\mathrm{e}^{-r(T-t)}K$ borrowed at time t.

For a future contract expiring at time T we take $K = S_0\,\mathrm{e}^{rT}$ and the contract is usually quoted at time t using the forward price $\mathrm{e}^{r(T-t)}(S_t - K\,\mathrm{e}^{-r(T-t)}) = \mathrm{e}^{r(T-t)}S_t - K = \mathrm{e}^{r(T-t)}S_t - S_0\,\mathrm{e}^{rT}$, or simply using $\mathrm{e}^{r(T-t)}S_t$. Future contracts are “marked to market” at each time step via a cash flow between both parties, ensuring that the absolute difference $|\mathrm{e}^{r(T-t)}S_t - K|$ has been credited to the buyer's account if it is positive, or to the seller's account if it is negative.

Black-Scholes formula for European call options

Recall that in the case of a European call option with strike price K the payoff function is given by $f(x) = (x-K)^+$ and the Black-Scholes PDE reads

$$\begin{cases} rg_c(t,x) = \dfrac{\partial g_c}{\partial t}(t,x) + rx\dfrac{\partial g_c}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g_c}{\partial x^2}(t,x) \\ g_c(T,x) = (x-K)^+. \end{cases}$$

In Sections 5.6 and 5.7 we will prove that the solution of this PDE is given by the *Black-Scholes* formula

$$g_c(t,x) = \mathrm{Bl}(K,x,\sigma,r,T-t) = x\Phi\big(d_+^T(t)\big) - K\,\mathrm{e}^{-r(T-t)}\Phi\big(d_-^T(t)\big), \qquad (5.18)$$

with

$$d_+^T(t) = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_-^T(t) = \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad (5.19)$$

cf. Proposition 5.16 below, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

denotes the standard Gaussian distribution function, with

$$d_+^T(t) = d_-^T(t) + \sigma\sqrt{T-t}.$$

The following script is an implementation of the Black-Scholes formula for European call options in R.

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
BSCall}
```

Table 5.1: The Black-Scholes call function in R.

One can easily check that

$$\lim_{t\nearrow T} d_+^T(t) = \lim_{t\nearrow T} d_-^T(t) = \begin{cases} +\infty, & x > K, \\ -\infty, & x < K, \end{cases}$$

which allows us to recover the boundary condition

$$g_c(T,x) = \left\{ \begin{array}{ll} x\Phi(+\infty) - K\Phi(+\infty) = x - K, & x > K \\ x\Phi(-\infty) - K\Phi(-\infty) = 0, & x < K \end{array} \right\} = (x-K)^+$$

at $t = T$. Similarly we can check that

$$\lim_{T\to\infty} d_-^T(t) = \begin{cases} +\infty, & r > \sigma^2/2, \\ -\infty, & r < \sigma^2/2, \end{cases}$$

and $\lim_{T\to\infty} d_+^T(t) = +\infty$, hence

$$\lim_{T\to\infty} \mathrm{Bl}(K, S_t, \sigma, r, T-t) = S_t, \qquad t \in \mathbb{R}_+.$$

Figure 5.2 presents an interactive graph of the Black call price function, *i.e.* the solution

$$(t,x) \longmapsto g_c(t,x) = x\Phi\big(d_+^T(t)\big) - K\,\mathrm{e}^{-r(T-t)}\Phi\big(d_-^T(t)\big)$$

of the Black-Scholes PDE for a call option.

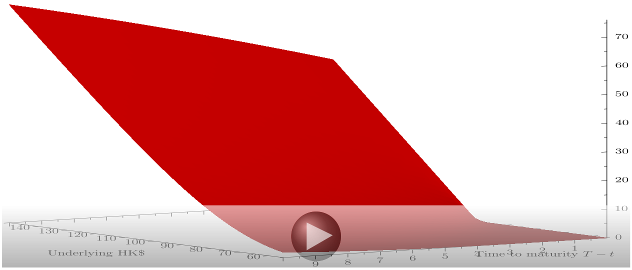

Fig. 5.2: Graph of the Black-Scholes call price function with strike $K = 100$.*

* Right click on the figure for interaction and "Full Screen Multimedia" view (works in Acrobat reader on the entire pdf file).

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The next proposition is proved by direct differentiation of the Black-Scholes function, and will be recovered later using a probabilistic argument in Proposition 6.10 below.

Proposition 5.13. *The Black-Scholes Delta of a European call option is given by*

$$\xi_t = \Phi\big(d_+^T(t)\big) \in [0,1], \tag{5.20}$$

where $d_+^T(t)$ is given by (5.19).

Proof. By (5.18) we have

$$\begin{aligned}
\frac{\partial g_c}{\partial x}(t,x) &= \frac{\partial}{\partial x}\left(x\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&\quad -K\,\mathrm{e}^{-r(T-t)}\frac{\partial}{\partial x}\left(\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= \Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad +x\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K\,\mathrm{e}^{-r(T-t)}\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad +\frac{1}{\sqrt{2\pi}\sigma\sqrt{T-t}}\exp\left(-\frac{1}{2}\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)^2\right) \\
&\quad -\frac{K\,\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}\sigma x\sqrt{T-t}}\exp\left(-\frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)^2\right) \\
&= \Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned} \tag{5.21}$$

□

Black-Scholes formula for European put options

Similarly, in the case of a European put option with strike price K the payoff function is given by $f(x)=(K-x)^+$ and the Black-Scholes PDE reads

$$\begin{cases}
rg_p(t,x) = \dfrac{\partial g_p}{\partial t}(t,x) + rx\dfrac{\partial g_p}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2x^2\dfrac{\partial^2 g_p}{\partial x^2}(t,x), \\[2ex]
g_p(T,x) = (K-x)^+,
\end{cases}$$

with explicit solution

$$g_p(t,x) = K\,\mathrm{e}^{-r(T-t)}\Phi\big(-d_-^T(t)\big) - x\Phi\big(-d_+^T(t)\big),$$

as illustrated in Figure 5.3.

The following script is an implementation of the Black-Scholes formula for European put options in R.

```
BSPut <- function(S, K, r, T, sigma)
{d1 = (log(S/K)+(r-sigma^2/2)*T)/(sigma*sqrt(T))
d2 = d1 - sigma * sqrt(T)
BSPut = K*exp(-r*T) * pnorm(-d2) - S*pnorm(-d1)
BSPut}
```

Table 5.2: The Black-Scholes put function in R.

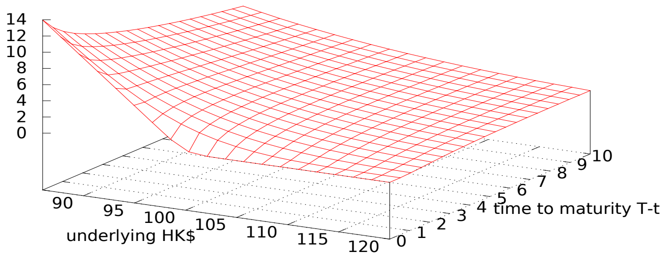

Fig. 5.3: Graph of the Black-Scholes put price function with strike price $K = 100$.

Note that the call-put parity relation

$$g(t,S_t) = x - K\,\mathrm{e}^{-r(T-t)} = g_c(t,S_t) - g_p(t,S_t), \qquad 0 \le t \le T, \qquad (5.22)$$

is satisfied here.

Numerical examples

In Figure 5.4 we consider the stock price of HSBC Holdings (0005.HK) over one year:
Consider a call option issued by Societe Generale on 31 December 2008 with strike price K=$63.704, maturity T = October 05, 2009, and an entitlement ratio of 100, meaning that one option contract is divided into 100 *warrants*, cf. page 6. The next graph gives the time evolution of the Black-Scholes portfolio price

$$t \longmapsto g_c(t,S_t)$$

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Fig. 5.4: Graph of the stock price of HSBC Holdings.

driven by the market price $t \longmapsto S_t$ of the underlying risky asset as given in Figure 5.4, in which the number of days is counted from the origin and not from maturity.

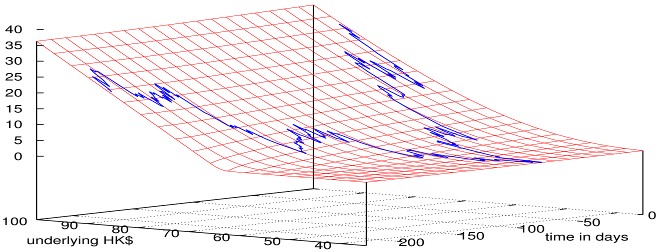

Fig. 5.5: Path of the Black-Scholes price for a call option on HSBC.

As a consequence of Proposition 5.13, in the Black-Scholes model the amount invested in the risky asset is

$$S_t \xi_t = S_t \Phi\big(d_+^T(t)\big) = S_t \Phi\left(\frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geq 0,$$

which is always positive, *i.e.* there is no short selling, and the amount invested on the riskless asset is

$$\eta_t A_t = -K A_0 \, \mathrm{e}^{-r(T-t)} \Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \leq 0,$$

which is always negative, *i.e.* we are constantly borrowing money, as noted in Figure 5.6.

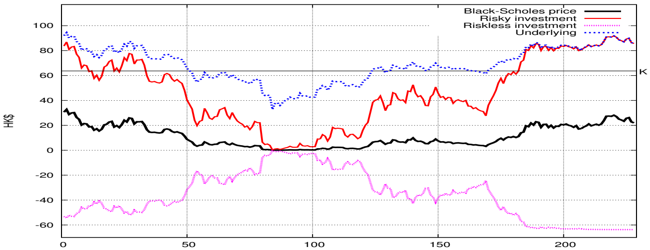

Fig. 5.6: Time evolution of a hedging portfolio for a call option on HSBC.

For one more example, we consider a put option issued by BNP Paribas on 04 November 2008 with strike price K=$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593, cf. page 6. In the next Figure 5.7 the number of days is counted from the origin and not from maturity.

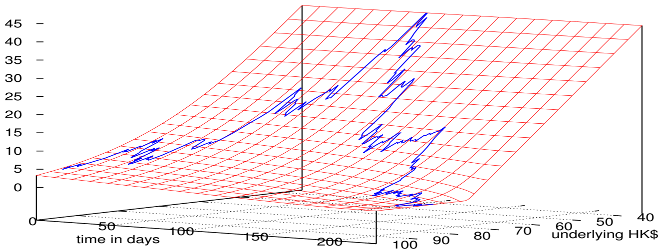

Fig. 5.7: Path of the Black-Scholes price for a put option on HSBC.

In the case of a Black-Scholes put option the Delta is given by

$$\xi_t = -\Phi\big(-d_+^T(t)\big) \in [-1,0],$$

and the amount invested on the risky asset is

$$-S_t\Phi\big(d_+^T(t)\big) = -S_t\Phi\left(-\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \leq 0,$$

i.e. there is always short selling, and the amount invested on the riskless asset is

$$K\,\mathrm{e}^{-r(T-t)}\Phi\left(-\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geq 0,$$

which is always positive, *i.e.* we are constantly investing on the riskless asset.

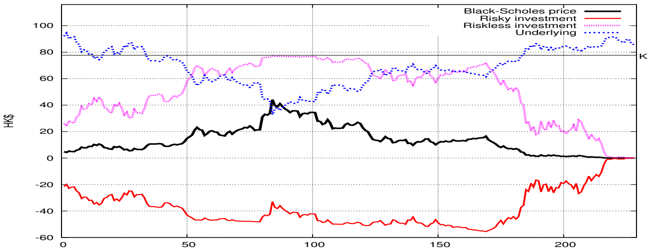

Fig. 5.8: Time evolution of the hedging portfolio for a put option on HSBC.

5.6 The Heat Equation

In this section we study the *heat equation*

$$\frac{\partial g}{\partial t}(t, y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t, y)$$

which is used to model the diffusion of heat over time through solids. Here, the data of $g(x, t)$ represents the temperature measured at time t and point x. We refer the reader to [109] for a complete treatment of this topic.

Fig. 5.9: Time-dependent solution of the heat equation.*

In Section 5.7 this equation will be shown to be equivalent to the Black-Scholes PDE after a change of variables. In particular this will lead to the explicit solution of the Black-Scholes PDE.

Proposition 5.14. *The heat equation*

* The animation works in Acrobat reader on the entire pdf file.

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ \\ g(0,y) = \psi(y) \end{cases} \tag{5.23}$$

with initial condition

$$g(0,y) = \psi(y)$$

has the solution

$$g(t,y) = \int_{-\infty}^{\infty} \psi(z)\,\mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}}, \qquad t > 0. \tag{5.24}$$

Proof. We have

$$\begin{aligned}
\frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi(z)\,\mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \int_{-\infty}^{\infty} \psi(z) \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-(y-z)^2/(2t)}}{\sqrt{2\pi t}}\right) dz \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z)\left(\frac{(y-z)^2}{t^2} - \frac{1}{t}\right) \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z)\frac{\partial^2}{\partial z^2}\mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z)\frac{\partial^2}{\partial y^2}\mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} \psi(z)\,\mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).
\end{aligned}$$

On the other hand it can be checked that at time $t = 0$,

$$\lim_{t\to 0}\int_{-\infty}^{\infty} \psi(z)\,\mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} = \lim_{t\to 0}\int_{-\infty}^{\infty} \psi(y+z)\,\mathrm{e}^{-z^2/(2t)} \frac{dz}{\sqrt{2\pi t}} = \psi(y),$$

$y \in \mathbb{R}$. □

Let us provide a second proof of Proposition 5.14 using stochastic calculus and Brownian motion. Note that under the change of variable $x = z - y$ we have

$$\begin{aligned}
g(t,y) &= \int_{-\infty}^{\infty} \psi(z)\,\mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \int_{-\infty}^{\infty} \psi(y+x)\,\mathrm{e}^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}} \\
&= \mathbb{E}[\psi(y+B_t)],
\end{aligned}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Applying Itô's formula we have

$$\begin{aligned}\mathbb{E}[\psi(y+B_t)] &= \psi(y) + \mathbb{E}\left[\int_0^t \psi'(y+B_s)dB_s\right] + \frac{1}{2}\,\mathbb{E}\left[\int_0^t \psi''(y+B_s)ds\right] \\ &= \psi(y) + \frac{1}{2}\int_0^t \mathbb{E}\left[\psi''(y+B_s)\right] ds \\ &= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\,\mathbb{E}\left[\psi(y+B_s)\right] ds,\end{aligned}$$

since the expectation of the stochastic integral is zero. Hence

$$\begin{aligned}\frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\,\mathbb{E}[\psi(y+B_t)] \\ &= \frac{1}{2}\frac{\partial^2}{\partial y^2}\,\mathbb{E}\left[\psi(y+B_t)\right] \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).\end{aligned}$$

Concerning the initial condition we check that

$$g(0,y) = \mathbb{E}[\psi(y+B_0)] = \mathbb{E}[\psi(y)] = \psi(y).$$

The expression $g(t,y) = \mathbb{E}[\psi(y+B_t)]$ provides a probabilistic interpretation of the heat diffusion phenomenon based on Brownian motion.

5.7 Solution of the Black-Scholes PDE

In this section we will solve the Black-Scholes PDE by the kernel method of Section 5.6 and a change of variables. This solution method uses a transformation of variables (5.26) which involves the time inversion $t \longmapsto T-t$ on the interval $[0,T]$, so that the terminal condition at time T in the Black-Scholes equation (5.25) becomes an initial condition at time $t=0$ in the heat equation (5.28).

Proposition 5.15. *Assume that $f(t,x)$ solves the Black-Scholes PDE*

$$\begin{cases} rf(t,x) = \dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 f}{\partial x^2}(t,x), \\ \\ f(T,x) = (x-K)^+, \end{cases} \tag{5.25}$$

with terminal condition $h(x) = (x-K)^+$. Then the function $g(t,y)$ defined by

$$g(t,y) = \mathrm{e}^{rt} f\left(T-t,\, \mathrm{e}^{\sigma y + (\sigma^2/2 - r)t}\right) \tag{5.26}$$

solves the heat equation (5.23) *with initial condition*

$$g(0,y) = h(\mathrm{e}^{\sigma y}), \qquad y \in \mathbb{R}, \tag{5.27}$$

i.e.

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ \\ g(0,y) = h(\mathrm{e}^{\sigma y}). \end{cases} \tag{5.28}$$

Proof. Letting $s = T - t$ and $x = \mathrm{e}^{\sigma y + (\sigma^2/2 - r)t}$ we have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= r\,\mathrm{e}^{rt} f(T-t,\, \mathrm{e}^{\sigma y + (\sigma^2/2-r)t}) - \mathrm{e}^{rt}\frac{\partial f}{\partial s}\big(T-t,\, \mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\big) \\ &\quad + \left(\frac{\sigma^2}{2} - r\right)\mathrm{e}^{rt}\,\mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\frac{\partial f}{\partial x}\big(T-t,\, \mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\big) \\ &= r\,\mathrm{e}^{rt} f(T-t,x) - \mathrm{e}^{rt}\frac{\partial f}{\partial s}(T-t,x) + \left(\frac{\sigma^2}{2} - r\right)\mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T-t,x) \\ &= \frac{1}{2}\mathrm{e}^{rt} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(T-t,x) + \frac{\sigma^2}{2}\mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T-t,x), \end{aligned} \tag{5.29}$$

where on the last step we used the Black-Scholes PDE. On the other hand we have

$$\frac{\partial g}{\partial y}(t,y) = \sigma\,\mathrm{e}^{rt}\,\mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\frac{\partial f}{\partial x}\big(T-t,\, \mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\big)$$

and

$$\begin{aligned} \frac{1}{2}\frac{\partial g^2}{\partial y^2}(t,y) &= \frac{\sigma^2}{2}\mathrm{e}^{rt}\,\mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\frac{\partial f}{\partial x}\big(T-t,\, \mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\big) \\ &\quad + \frac{\sigma^2}{2}\mathrm{e}^{rt}\,\mathrm{e}^{2\sigma y + 2(\sigma^2/2-r)t}\frac{\partial^2 f}{\partial x^2}\big(T-t,\, \mathrm{e}^{\sigma y + (\sigma^2/2-r)t}\big) \\ &= \frac{\sigma^2}{2}\mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T-t,x) + \frac{\sigma^2}{2}\mathrm{e}^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T-t,x). \end{aligned} \tag{5.30}$$

We conclude by comparing (5.29) with (5.30), which shows that $g(t,x)$ satisfies the heat equation (5.23) with initial condition

$$g(0,y) = f(T,\, \mathrm{e}^{\sigma y}) = h(\mathrm{e}^{\sigma y}).$$

$\square$

In the next proposition we recover the Black-Scholes formula (5.18) by solving the PDE (5.25). The Black-Scholes will also be recovered by probabilistic arguments and the computation of an expectation in Proposition 6.5.

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Proposition 5.16. *When $h(x) = (x-K)^+$, the solution of the Black-Scholes PDE* (5.25) *is given by*

$$f(t,x) = x\Phi\big(d_+^T(t)\big) - K\,\mathrm{e}^{-r(T-t)}\Phi\big(d_-^T(t)\big),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-y^2/2}dy, \qquad x\in\mathbb{R},$$

and

$$d_+^T(t) = \frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_-^T(t) = \frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Proof. By inversion of (5.26) with $s = T-t$ and $x = \mathrm{e}^{\sigma y+(\sigma^2/2-r)t}$ we get

$$f(s,x) = \mathrm{e}^{-r(T-s)}g\left(T-s, \frac{-(\sigma^2/2-r)(T-s)+\log x}{\sigma}\right).$$

Hence using the solution (5.24) and Relation (5.27) we get

$$\begin{aligned}
f(t,x) &= \mathrm{e}^{-r(T-t)}g\left(T-t, \frac{-(\sigma^2/2-r)(T-t)+\log x}{\sigma}\right)\\
&= \mathrm{e}^{-r(T-t)}\int_{-\infty}^{\infty}\psi\left(\frac{-(\sigma^2/2-r)(T-t)+\log x}{\sigma}+z\right)\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&= \mathrm{e}^{-r(T-t)}\int_{-\infty}^{\infty}h\left(x\,\mathrm{e}^{\sigma z-(\sigma^2/2-r)(T-t)}\right)\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&= \mathrm{e}^{-r(T-t)}\int_{-\infty}^{\infty}\left(x\,\mathrm{e}^{\sigma z-(\sigma^2/2-r)(T-t)}-K\right)^+\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&= \mathrm{e}^{-r(T-t)}\int_{\frac{(-r+\sigma^2/2)(T-t)+\log(K/x)}{\sigma}}^{\infty}\left(x\,\mathrm{e}^{\sigma z-(\sigma^2/2-r)(T-t)}-K\right)\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&= x\,\mathrm{e}^{-r(T-t)}\int_{-d_-^T(t)\sqrt{T-t}}^{\infty}\mathrm{e}^{\sigma z-(\sigma^2/2-r)(T-t)}\,\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&\quad -K\,\mathrm{e}^{-r(T-t)}\int_{-d_-^T(t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&= x\int_{-d_-^T(t)\sqrt{T-t}}^{\infty}\mathrm{e}^{\sigma z-\sigma^2(T-t)/2-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&\quad -K\,\mathrm{e}^{-r(T-t)}\int_{-d_-^T(t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&= x\int_{-d_-^T(t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-(z-\sigma(T-t))^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}\\
&\quad -K\,\mathrm{e}^{-r(T-t)}\int_{-d_-^T(t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2\pi(T-t)}}
\end{aligned}$$

$$= x \int_{-d_-^T(t)\sqrt{T-t}-\sigma(T-t)}^{\infty} \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2\pi(T-t)}}$$
$$-K\,\mathrm{e}^{-r(T-t)} \int_{-d_-^T(t)\sqrt{T-t}}^{\infty} \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2\pi(T-t)}}$$
$$= x \int_{-d_-^T(t)-\sigma\sqrt{T-t}}^{\infty} \mathrm{e}^{-z^2/2} \frac{dz}{\sqrt{2\pi}} - K\,\mathrm{e}^{-r(T-t)} \int_{d_-^T(t)}^{\infty} \mathrm{e}^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$$
$$= x\left(1 - \Phi\big(-d_+^T(t)\big)\right) - K\,\mathrm{e}^{-r(T-t)}\left(1 - \Phi\big(-d_-^T(t)\big)\right)$$
$$= x\Phi\big(d_+^T(t)\big) - K\,\mathrm{e}^{-r(T-t)}\Phi\big(d_-^T(t)\big),$$

where we used the relation

$$1 - \Phi(a) = \Phi(-a), \qquad a \in \mathbb{R}.$$

□

Exercises

Exercise 5.1

a) Solve the Black-Scholes PDE

$$rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t) \tag{5.31}$$

with terminal condition $g(x,T) = x^2$.

Hint: Try a solution of the form $g(x,t) = x^2 f(t)$, and find $f(t)$.

b) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = \mathrm{e}^{rt}$ in the portfolio with value

$$V_t = g(S_t,t) = \xi_t S_t + \eta_t A_t$$

hedging the contract with payoff S_T^2 at maturity.

Exercise 5.2

a) Solve the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t \tag{5.32}$$

in terms of $\alpha, \sigma > 0$, and the initial condition S_0.

b) Write down the Black-Scholes PDE satisfied by the function $C(t,x)$, where $C(t,S_t)$ is the price at time $t \in [0,T]$ of the contingent claim with payoff $\phi(S_T) = \exp(S_T)$.

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c) Solve the Black-Scholes PDE of Question (b).

Hint: Search for a solution of the form

$$C(t,x) = \exp\left(-r(T-t) + xh(t) + \frac{\sigma^2}{4r}(h^2(t)-1)\right), \qquad (5.33)$$

where $h(t)$ is a function to be determined, with $h(T) = 1$.

d) Compute the strategy $(\zeta_t, \eta_t)_{t\in[0,T]}$ that hedges the contingent claim with payoff $\exp(S_T)$.

Exercise 5.3 On December 18, 2007, a call warrant has been issued by Fortis Bank on the stock price S of the MTR Corporation with maturity $T = 23/12/2008$, strike price $K =$ HK\$ 36.08 and entitlement ratio=10. Recall that in the Black-Scholes model, the price at time t of a European claim on the underlying asset S_t, with strike price K, maturity T, interest rate r and volatility σ is given by the Black-Scholes formula as

$$f(t, S_t) = S_t \Phi\big(d_+^T(t)\big) - K\,\mathrm{e}^{-r(T-t)}\Phi\big(d_-^T(t)\big),$$

where

$$d_-^T(t) = \frac{(r-\sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_+^T(t) = d_-^T(t) + \sigma\sqrt{T-t}.$$

Recall that

$$\frac{\partial f}{\partial x}(t, S_t) = \Phi\big(d_+^T(t)\big),$$

cf. Proposition 5.13.

a) Using the values of the Gaussian cumulative distribution function, compute the Black-Scholes price of the corresponding call option at time $t =$November 07, 2008 with $S_t =$ HK\$ 17.200, assuming a volatility $\sigma = 90\% = 0.90$ and an *annual* risk-free interest rate $r = 4.377\% = 0.04377$,

b) Still using the values of the Gaussian cumulative distribution function, compute the quantity of the risky asset required in your portfolio at time $t =$November 07, 2008 in order to hedge one such option at maturity $T = 23/12/2008$.

c) Figure 1 represents the Black-Scholes price of the call option as a function of $\sigma \in [0.5, 1.5] = [50\%, 150\%]$.

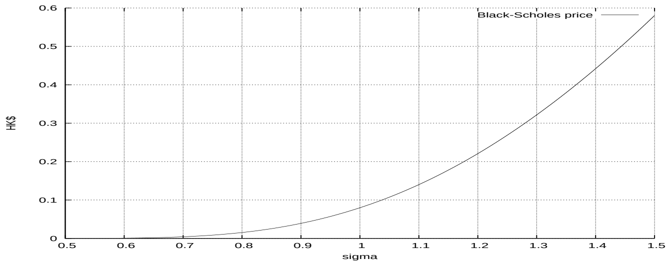

Fig. 5.10: Option price as a function of the volatility σ.

Knowing that the closing price of the warrant on November 07, 2008 was HK\$ 0.023, which value can you infer for the implied volatility σ at this date ?

Exercise 5.4 Forward contracts. Recall that the price $\pi_t(C)$ of a claim $C = h(S_T)$ of maturity T can be written as $\pi_t(C) = g(t, S_t)$, where the function $g(t, x)$ satisfies the *Black-Scholes PDE*

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(T,x) = h(x), \end{cases} \tag{1}$$

with terminal condition $g(T, x) = h(x)$.

a) Assume that C is a forward contract with payoff

$$C = S_T - K,$$

at time T. Find the function $h(x)$ in (1).

b) Find the solution $g(t, x)$ of the above PDE and compute the price $\pi_t(C)$ at time $t \in [0, T]$.
Hint: search for a solution of the form $g(t, x) = x - \alpha(t)$ where $\alpha(t)$ is a function of t to be determined.

c) Compute the quantity

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t)$$

of risky assets in a self-financing portfolio hedging C.

Exercise 5.5

a) Solve the Black-Scholes PDE

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$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(t,x) \tag{5.34}$$

with terminal condition $g(T,x) = 1$.

Hint: Try a solution of the form $g(t,x) = f(t)$ and find $f(t)$.

b) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = \mathrm{e}^{rt}$ in the portfolio with value

$$V_t = g(t,S_t) = \xi_t S_t + \eta_t A_t$$

hedging the contract with payoff \$1 at maturity.

Similar exercise: Repeat the above questions with the terminal condition $g(T,x) = x$.

Exercise 5.6 Forward contracts revisited. Consider a risky asset whose price S_t is given by $S_t = S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Consider a forward contract with maturity T and payoff $S_T - \kappa$.

a) Compute the price C_t of this claim at any time $t \in [0,T]$.

b) Compute a hedging strategy for the option with payoff $S_T - \kappa$.

Exercise 5.7 Computation of Greeks. Consider an underlying asset whose price $(S_t)_{t\in\mathbb{R}_+}$ is given by a stochastic differential equation of the form

$$dS_t = rS_t dt + \sigma(S_t)dW_t,$$

where $\sigma(x)$ is a Lipschitz coefficient, and an option with payoff function ϕ and price

$$C(x,T) = \mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\Big|S_0 = x\right],$$

where $\phi(x)$ is a twice continuously differentiable ($\mathcal{C}^2$) function, with $S_0 = x$. Using the Itô formula, show that the sensitivity

$$\mathrm{Theta}_T = \frac{\partial}{\partial T}\left(\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\Big|S_0 = x\right]\right)$$

of the option price with respect to maturity T can be expressed as

$$\begin{aligned}\mathrm{Theta}_T = &-r\,\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\Big|S_0 = x\right] + r\,\mathrm{e}^{-rT}\,\mathbb{E}\left[S_t\phi'(S_T)\Big|S_0 = x\right]\\ &+\frac{1}{2}\,\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi''(S_T)\sigma^2(S_T)\Big|S_0 = x\right].\end{aligned}$$

Exercise 5.8 Black-Scholes PDE with dividends. Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled as

$$dS_t = (\mu - D)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $D > 0$ is a continuous-time dividend rate. By absence of arbitrage, the payment of a dividend entails a drop in the stock price by the same amount.

a) Write down the corresponding Black-Scholes PDE for the price $g(t, S_t)$ of a European call option.
b) Compute the price at time $t \in [0, T]$ of the European call option in a market with dividend rate D.

Exercise 5.9 Show that the Black-Scholes PDE of Proposition 5.11 can be recovered from the induction relation (3.18) when the number N of time steps tends to infinity, using the renormalizations $r_N := rT/N$ and

$$a_N := (1 + r_N)\,\mathrm{e}^{-\sigma\sqrt{T/N}} - 1, \quad b_N := (1 + r_N)\,\mathrm{e}^{\sigma\sqrt{T/N}} - 1, \quad N \geq 1,$$

of Section 3.6.

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