NLA course project
"Risk analysis and high-dimensional integrals"

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Let $V(S,t)$ be the price of the option at time t when the market price is S. Then $V(S,t)$ satisfies the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

Here the following notations will be used:

$$t - \text{time}$$

$$T - \text{the expiry (expiration date)}$$

$$E - \text{the exercise price of the option}$$

$$S - \text{market price of the underlying asset; } S = S(t)$$

$$V - \text{the value of the option; } V = V(S,t)$$

$$C - \text{the value of a call option; } C = C(S,t)$$

$$P - \text{the value of a put option; } P = P(S,t)$$

$$r - \text{the interest rate}$$

$$\sigma - \text{the volatility of the underlying asset}$$

This PDE has infinitely many solutions, so one needs boundary conditions. Let's consider some of possible types.

1 European options

The call price $C(S,t)$ is required to satisfy

$$C(0,t) = 0 \ \forall t \in [0,T]$$

$$C(S,T) = \max\{S - E, 0\}$$

The put price $P(S,t)$ is required to satisfy:

$$P(\infty,t) = 0$$

$$P(S,T) = \max\{E - S, 0\}$$

Pay-off call option of type $(S-E)_+$ is also called vanilla call option ($(E-S)_+$ is known as vanilla put option). These kinds of options can be computed analytically.

For European call option one obtains (Black-Scholes formula):

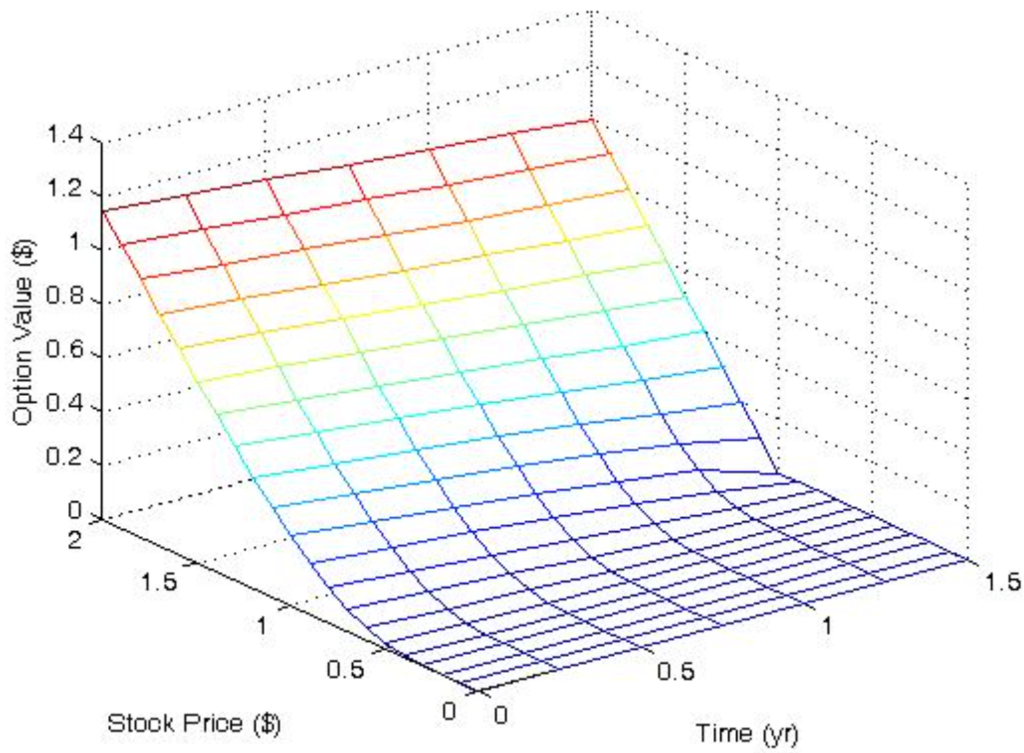
$$V_C(S,t) = S\Phi\left(d_+^T(t)\right) - Ee^{-r(T-t)}\Phi\left(d_-^T(t)\right)$$

where:

$$d_+^T(t) = \frac{\log(\frac{S}{E}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_+^T(t) = \frac{\log(\frac{S}{E}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$$



For European put option one obtains:

$$V_P(S,t) = Ee^{-r(T-t)}\Phi(-d_-^T) - S\Phi(-d_+^T)$$

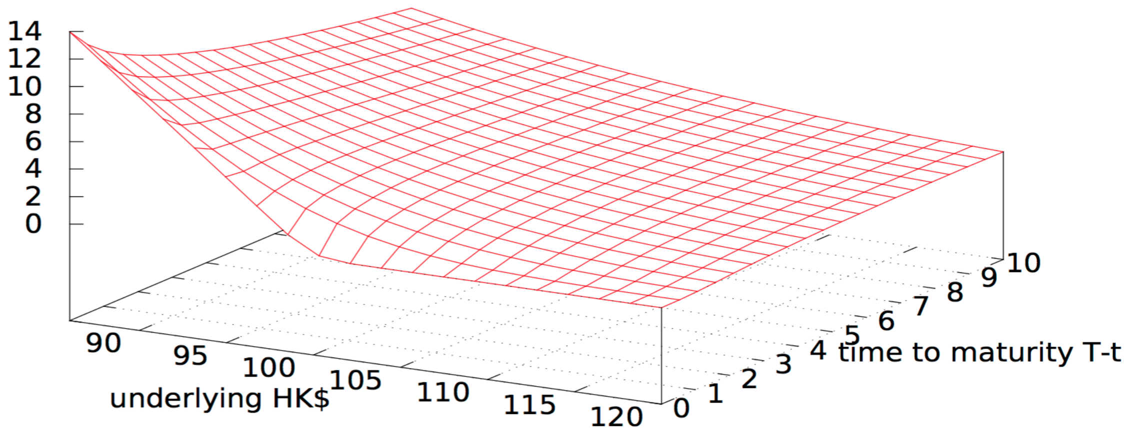


Fig. 5.3: Graph of the Black-Scholes put price function with strike price $K = 100$.

2 American options

In addition to those conditions presented for European options, one has to add additional constraints:

$$C(S,t) \geq \max\{S - E, 0\}$$
$$P(S,t) \geq \max\{E - S, 0\}$$

These kinds of options can not be computed analytically.

3 Cash-or-nothing option

The corresponding pay-off function is:

$$V_{con}(S,T) = \begin{cases} B & \text{if } S > E \\ 0 & \text{if } S \leq E \end{cases}$$

Analytical solution is given by:

$$V_{con}(S,t) = Be^{-r(T-t)}\Phi(d_+^T)$$