Course project «Optimization approaches to community detection»

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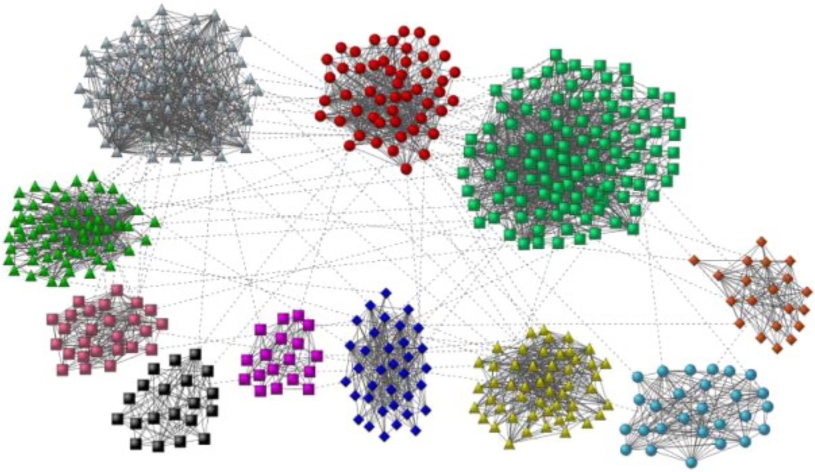
Plan

1. Introduction to community detection
2. Algorithms
    - Spectral method
    - Modularity-based method
    - Natural conjugate gradients method
    - Semidefinite relaxations
3. Experimental results

Example



Notations

Assumption

We consider **undirected unweighted** graphs **without loops** with n nodes. The nodes are enumerated as $\{1, \ldots, n\}$.
Graph is given by its $n \times n$ adjacency matrix A.

Goal of community detection

Find **partition** of nodes into **non-overlapping** clusters.
The number of clusters is k.
The clusters are denoted as $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$.

Subsets and Cuts

Measuring sizes of subsets

Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ be subsets of vertices. Then:

- $|\mathcal{C}_i| = \{\text{number of vertices in } \mathcal{C}_i\}$
- $vol(\mathcal{C}_i) = \sum\limits_{i \in \mathcal{C}_i} d_i$

MinCut

Define $W(\mathcal{C}_p, \mathcal{C}_q) := \sum\limits_{i \in \mathcal{C}_p, j \in C_q} a_{ij}$. Then MinCut problem is:

$$cut(\mathcal{C}_1, \dots, \mathcal{C}_k) = \frac{1}{2} \sum_{i=1}^{k} W(\mathcal{C}_i, \overline{\mathcal{C}_i}) \to \min_{\mathcal{C}_1, \dots, \mathcal{C}_k}$$

Balancing Cuts

RatioCut and Normalized Cut

The MinCut solution separates one individual vertex from the rest. The following objectives are considered:

$$RatioCut(\mathcal{C}_1, \dots, \mathcal{C}_k) = \sum_{i=1}^{k} \frac{cut(\mathcal{C}_i, \overline{\mathcal{C}_i})}{|\mathcal{C}_i|} \to \min_{\mathcal{C}_1, \dots, \mathcal{C}_k}$$

$$Ncut(\mathcal{C}_1, \dots, \mathcal{C}_k) = \sum_{i=1}^{k} \frac{cut(\mathcal{C}_i, \overline{\mathcal{C}_i})}{vol(\mathcal{C}_i)} \to \min_{\mathcal{C}_1, \dots, \mathcal{C}_k}$$

Balancing conditions lead to NP-hard problem. Spectral clustering is a way to solve relaxed versions of those problems

Relaxed problem

Types of Laplacians

- Unnormilized Laplacian: $L = D - A$
- Symmetric Laplacian: $L_{sym} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$
- Random walk Laplacian: $L_{rw} = D^{-1} L = I - D^{-1} A$

Idea

- Solving relaxed problem is equivalent to considering eigenvectors corresponding to k smallest eigenvalues of Laplacian that describe cluster properties of given graph

Related paper

Ulrike von Luxburg «A Tutorial on Spectral Clustering», 2007

Modularity-based method

Formulating an optimization problem

Modularity-based method

Natural conjugate gradients method

- Model parametrized by

  $z(i) \sim \text{Poly}(\pi), \quad i = \overline{1, n}$

  $P = \|p_{ij}\|_{i,j=\overline{1,k}}$ - probabilities of inter-cluster edges occurence

- Bayesian approach:

$$\pi \sim \text{Dirichlet}(\alpha)$$
$$p_{ii} \sim \text{Beta}(\beta), \quad i = \overline{1, k}$$
$$p_{ij} \ll 1, \quad \forall i \neq j$$

- $p(z, \pi, P|A)$ - true posterior with observed adjacency matrix A
- $\mathcal{Q}$ - family of feasible distributions

Natural conjugate gradients method

Formulating an optimization problem

$$\mathcal{L}(q) \equiv -\mathrm{KL}\left(q \| p(Z, \pi, P | A)\right) \longrightarrow \max_{q \in \mathcal{Q}}$$

- The problem can be reduced to unconstrained optimization:

$$\mathcal{L} = \mathcal{L}(\theta) \longrightarrow \max, \quad \theta \text{ - } n \times (k-1) \text{ matrix}$$

- Use conjugate gradients method in a statistical manifold
- Metrics is defined by a matrix

$$\mathcal{I}(\theta) - \text{Fischer information}$$

Semidefinite relaxations

Formulating an optimization problem

Semidefinite relaxations