Introduction to Machine Learning
Homework 8: Convolutional Neural Networks

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Submit answers to only problems 1–3. But, make sure you know how to do all problems.

1. Let X and W be arrays,

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 0 \\ 0 & 3 & 3 & 3 & 0 \\ 0 & 3 & 2 & 3 & 0 \\ 0 & 3 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Let Z be the 2D convolution (without reversal):

$$Z[i,j] = \sum_{k_1,k_2} W[k_1,k_2]X[i+k_1, j+k_2]. \tag{1}$$

Assume that the arrays are indexed starting at $(0,0)$.

(a) What are the limits of the summations over k_1 and k_2 in (1)?

(b) What is the size of the output $Z[i,j]$ if the convolution is computed only on the *valid* pixels (i.e. the pixel locations (i,j) where the summation in (1) does not exceed the boundaries of W or X).

(c) What is the largest positive value of $Z[i,j]$ and state one pixel location (i,j) where that value occurs.

(d) What is the largest negative value of $Z[i,j]$ and state one pixel location (i,j) where that value occurs.

(e) Find one pixel location where $Z[i,j] = 0$.

2. Suppose that a convolutional layer of a neural network has an input tensor $X[i,j,k]$ and computes an output via a convolution and ReLU activation,

$$\begin{aligned} Z[i,j,m] &= \sum_{k_1}\sum_{k_2}\sum_{n} W[k_1,k_2,n,m]X[i+k_1,j+k_2,n] + b[m], \\ U[i,j,m] &= \max\{0, Z[i,j,m]\}. \end{aligned}$$

for some weight kernel $W[k_1,k_2,n,m]$ and bias $b[m]$. Suppose that X has shape (48,64,10) and W has shape (3,3,10,20). Assume the convolution is computed on the *valid* pixels.

(a) What are the shapes of Z and U?

(b) What are the number of input channels and output channels?

(c) How many multiplications must be performed to compute the convolution in that layer?

(d) If W and b are to be learned, what are the total number of trainable parameters in the layer?

3. Suppose that a convolutional layer in some neural network is described as a linear convolution followed by a sigmoid activation,

$$
\begin{aligned}
Z[i,j,m] &= \sum_{k_1}\sum_{k_2}\sum_{n} W[k_1,k_2,n,m]X[i+k_1,j+k_2,n] + b[m], \\
U[i,j,m] &= 1/(1+\exp(-Z[i,j,m])).
\end{aligned}
$$

where $X[i,j,n]$ is the input of the layer and $U[i,j,m]$ is the output. Suppose that during back-propagation, we have computed the gradient $\partial J/\partial U$ for some loss function J. That is, we have computed the components $\partial J/\partial U[i,j,m]$. Show how to compute the following:

(a) The gradient components $\partial J/\partial Z[i,j,m]$.

(b) The gradient components $\partial J/\partial W[k_1,k_2,n,m]$.

(c) The gradient components $\partial J/\partial X[i,j,n]$.

4. In the previous problem, we considered a single sample. Suppose there were a mini-batch of samples.

(a) How would you represent Z and U for the mini-batch case?

(b) Re-write the equations for Z and U.

(c) Re-compute the gradients.