Frequency Domain Processing

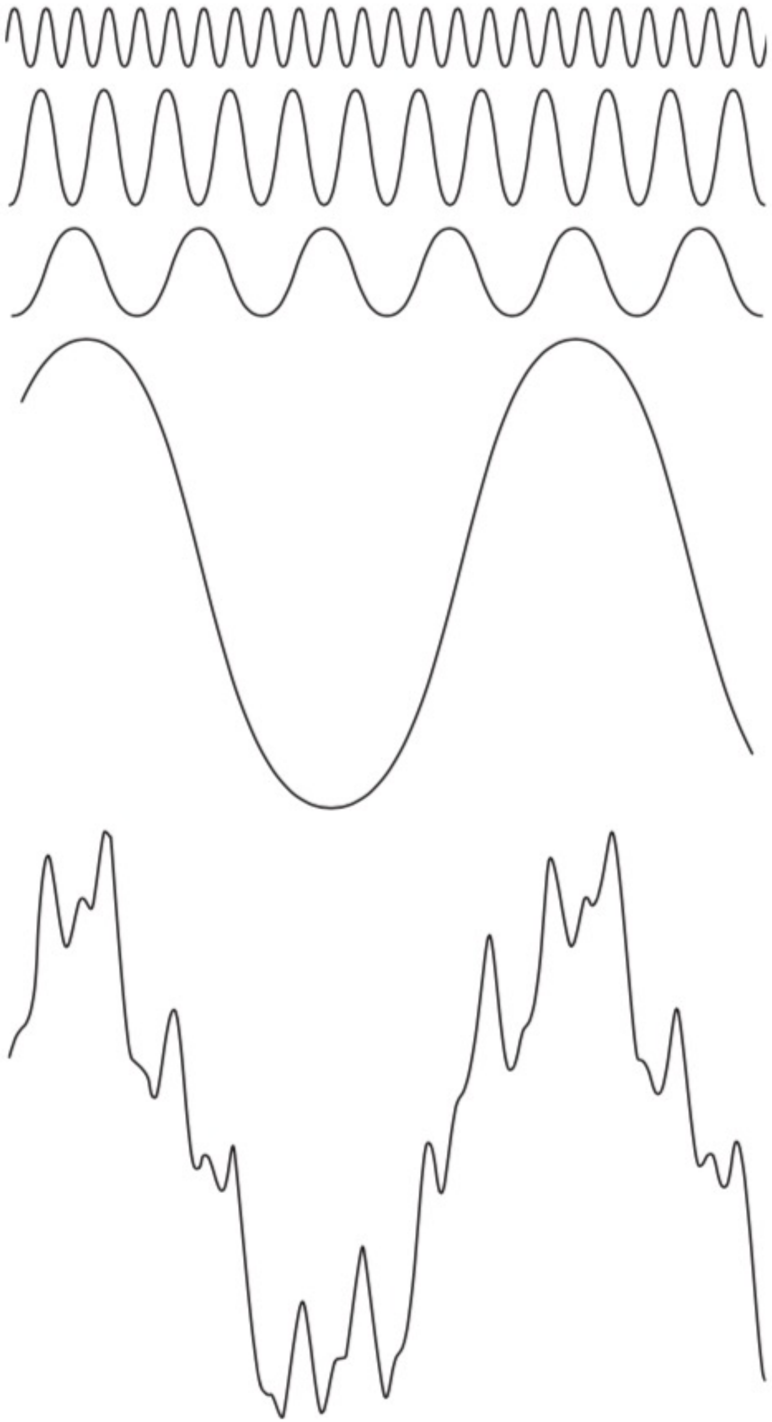
Digital Image Processing

Contents

Basics of Fourier Series & Transform

History

- French mathematician, Jean Baptiste Joseph Fourier
  - La Theorie Analitique de la Chaleur, 1822
  - Any periodic function can be expressed as sum of sines and/or cosines off different frequencies, each multiplied by a different coefficient
  - Fourier series
- Non-periodic function (finite)
  - Fourier transform



Preliminary Concepts

- Complex numbers
  - $C = R + jI$
  - Where $j = \sqrt{-1}$
  - R: real part
  - I: imaginary part
- Conjugate
  - $C^* = R - jI$

Preliminary Concepts

- In Polar coordinate
  - $C = |C|(\cos\theta + j\sin\theta)$
  - Where $|C| = \sqrt{R^2 + I^2}$, (length, distance from origin)
  - $\tan\theta = \left(\frac{I}{R}\right), or\ \theta = \arctan\frac{I}{R}$
  - $\theta \in [-\pi, \pi]$ (in C, atan2 function)

Preliminary Concepts

- Euler formula
  - $C = |C|(\cos\theta + j\sin\theta) = |C|e^{j\theta}$
  - Where $e^{j\theta} = \cos\theta + j\sin\theta$
- Complex function
  - $F(u) = R(u) + jI(u)$
  - $|F(u)| = \sqrt{R(u)^2 + I(u)^2}$

Impulses and Their Sifting Property

- Unit impulse
  - $\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$
  - $\int_{-\infty}^{\infty} \delta(t) dt = 1$

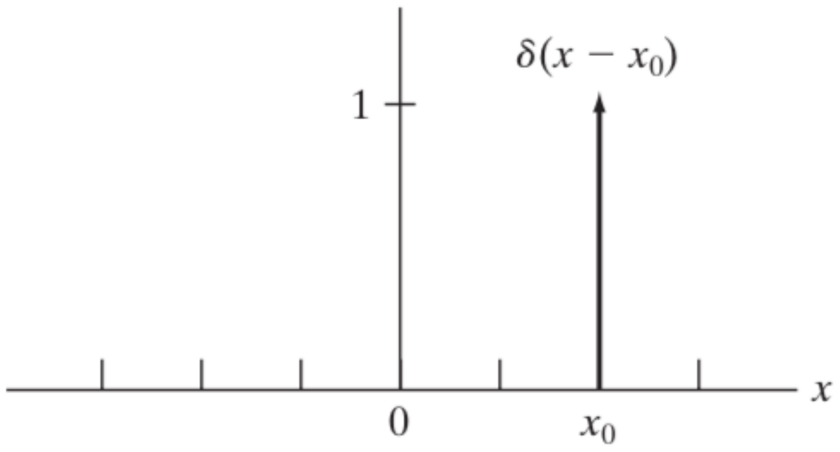
- Sifting property
  - $\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0)$
  - Yielding a $f(t)$ at the location of the impulse
  - $\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$

Impulses and Their Sifting Property

- For discrete variable x
  - $\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$
  - $\sum_{x=-\infty}^{\infty} \delta(x) = 1$

- Sifting property
  - $\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$

  - Yielding a $f(x)$ at the location of the impulse
  - $\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$



Impulse Train

- Impulse Train
  - Sum of many periodic impulses (ΔT units apart)
  - $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$

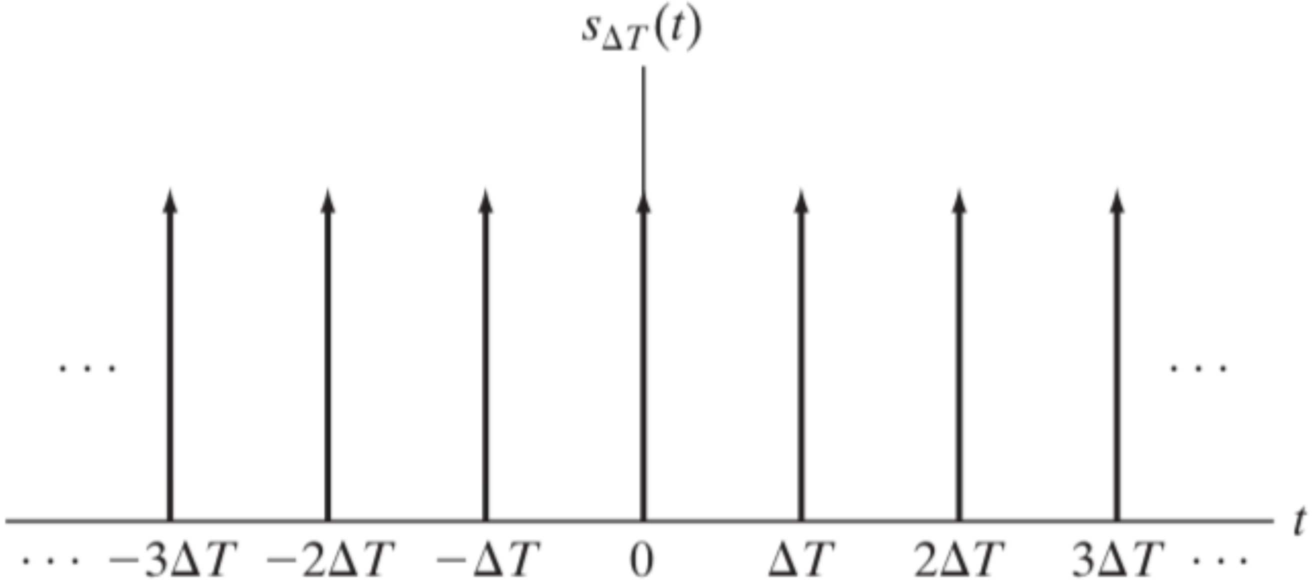
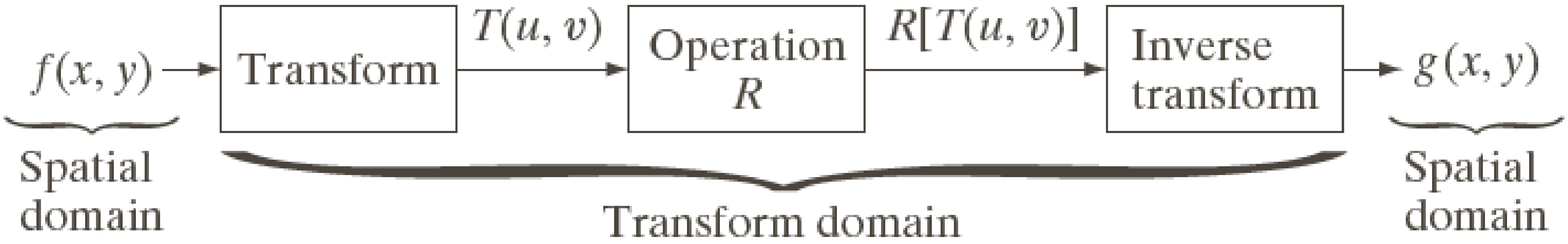


Image Transforms

- Image transform approach
  - Transform the image (in spatial domain) into another domain
  - Perform some operation
  - Apply inverse transform back to spatial domain



Fourier Series

- Basic concept
  - A periodic function $f(t)$ can be expressed with sum of cosines and sines
    - T is period (We are only interested in $\left[-\frac{T}{2}, \frac{T}{2}\right]$)
    - A sine function with the largest period : $\sin\left(2\pi\frac{t}{T}\right)$
  - $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n\frac{t}{T}} = \sum_{n=-\infty}^{\infty} c_n \left(\cos 2\pi n\frac{t}{T} + j\sin 2\pi n\frac{t}{T}\right)$
  - The coefficients for each sine/cosine function would be
  - $c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi n\frac{t}{T}}dt$ for $n = 0, \pm 1, \pm 2, \cdots$

Fourier Transform

- Fourier transform
  - $\Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)]dt$
  - $\Im$ is a function of μ, which is also a continuous variable
  - $F(\mu) = \Im\{f(t)\}$
- Inverse transform
  - $f(t) = \Im^{-1}\{F(\mu)\}$
  - $f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$
    - Multiplying with coefficients and adding all cosine/sine functions

In Human Language

- Inverse transform
  - $f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$

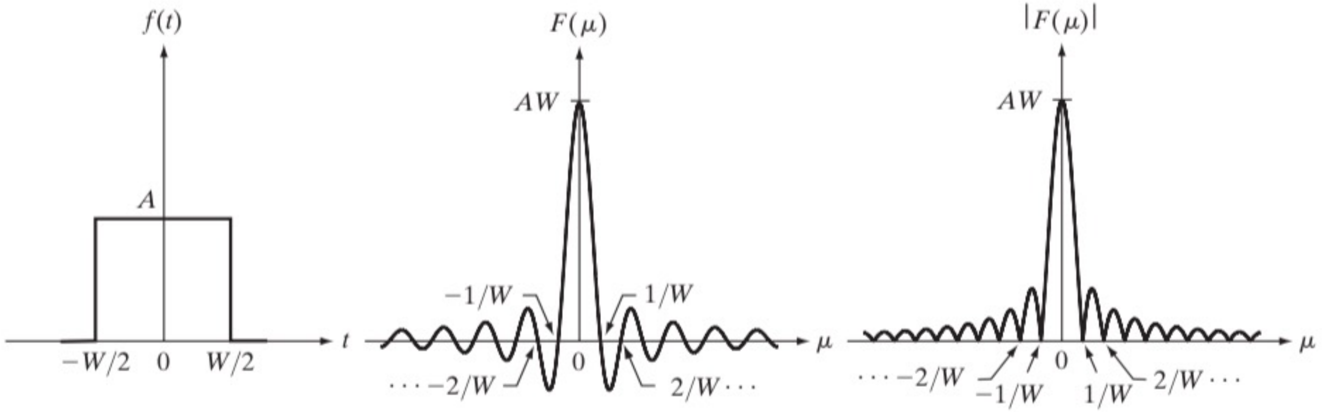
  - $e^{j2\pi\mu t} = \cos 2\pi\mu t + j \sin 2\pi\mu t$
  - $\cos 2\pi\mu t$ is a cosine function of t
  - One cycle is $[0, 1/\mu]$
  - If μ is large, the function is cycling faster.
  - Therefore, μ is frequency of $\cos 2\pi\mu t$
  - => $\cos 2\pi\mu t$ is the cosine function with frequency, μ
  - $f(t)$ is mixture of cosine function with various frequencies
  - $F(\mu)$ means how much a cosine function of frequency, μ ($\cos 2\pi\mu t$) contribute to $f(t)$

In Human Language

- Forward transformation
  - $F(\mu) = \Im\{f(t)\}$=$\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)]dt$
  - How much a cosine/sine function of frequency, μ is contained in the function $f(t)$

Fourier Transform: Example

- A simple function: $f(t) = \begin{cases} A & \text{if } -\frac{W}{2} < x < \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$
  - $F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-\frac{W}{2}}^{\frac{w}{2}} A e^{-j2\pi\mu t} dt = -\frac{A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-\frac{W}{2}}^{\frac{W}{2}}$
  - $= \frac{A}{j2\pi\mu}\left[e^{j2\pi\mu W} - e^{-j2\pi\mu W}\right] = AW\frac{\sin(\pi\mu W)}{\pi\mu W}$
    - $(\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j})$
  - Usually $F(\mu)$ is complex, thus display $|F(\mu)|$



Fourier Transform of a Unit Impulse

- Fourier transform of a unit impulse
    - $F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu 0} = e^0 = 1$

- Impulse located at $t = t_0$
    - $F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu t_0} = \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)$

Fourier Transform of a Unit Impulse

- Impulse train ($s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$)
  - $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$
  - Where $c_n = \frac{1}{\Delta T}\int_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} S_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T} e^0 = \frac{1}{\Delta T}$
  - Therefore, $s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$
- Transform
  - $\Im\left(e^{j\frac{2\pi n}{\Delta T}t}\right) = \delta\left(\mu - \frac{n}{\Delta T}\right)$
  - So, $S(\mu) = \Im\big(s_{\Delta T}(t)\big) = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$
  - $= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \Im\left(e^{j\frac{2\pi n}{\Delta T}t}\right) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$

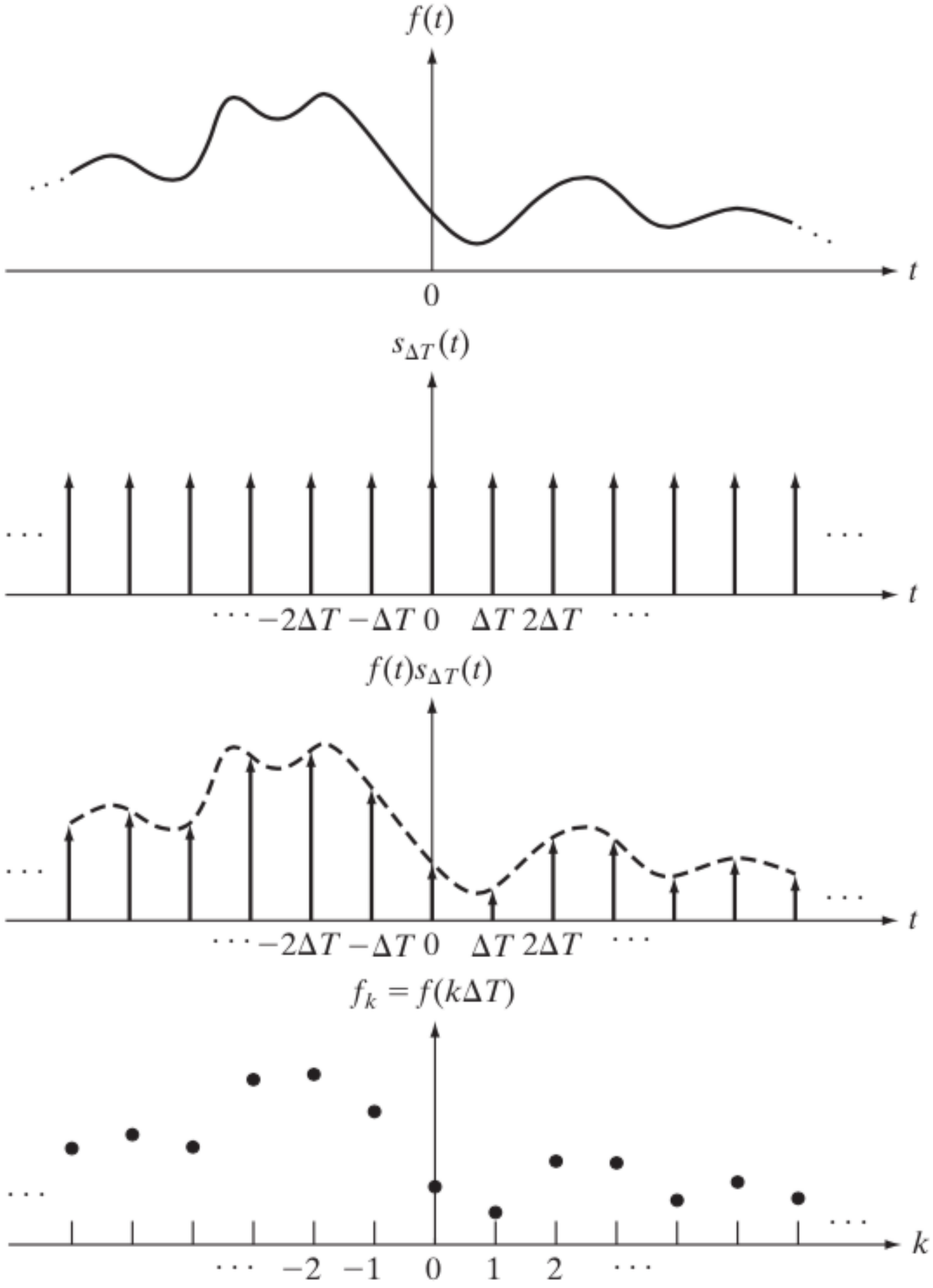
Convolution

- Definition
  - $f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$
  - Very similar to spatial filtering
- Fourier transform of convolution
  - $\Im\{f(t) * h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right]e^{-j2\pi\mu t}dt$
  - $= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\, e^{-j2\pi\mu t}dt\right]d\tau = \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi\mu t}dt\right]d\tau$
    - $H(\mu) = \Im\{h(t-\tau)\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi\mu t}e^{-j2\pi\mu\tau}dt = H(\mu)e^{-j2\pi\mu\tau}$
  - $= \int_{-\infty}^{\infty} f(\tau)H(\mu)e^{-j2\pi\mu\tau}d\tau = H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}d\tau = H(\mu)F(\mu)$
- Therefore
  - $f(t) * h(t) \Leftrightarrow H(\mu)F(\mu)$

Sampling & the Fourier Transform of Sampled Functions

Sampling

- Sampling function
  - $\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum f(t)\delta(t - n\Delta T)$
  - $f_k = f(k\Delta T)$

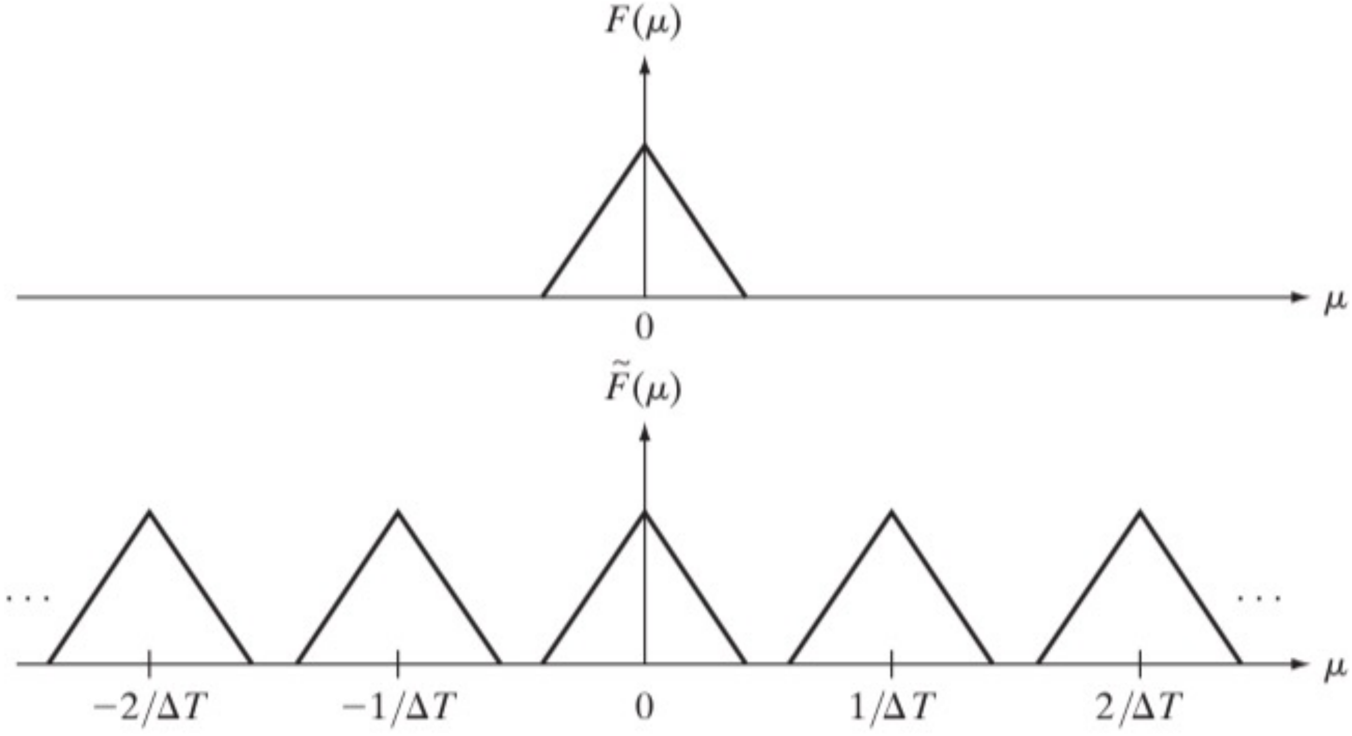


Fourier Transform of Sampling

- Fourier transform of sampling
  - $\tilde{F}(\mu) = \Im\{\tilde{f}(t)\} = \Im\{f(t)s_{\Delta T}(t)\} = F(\mu) * S(\mu)$
  - $S(\mu) = \frac{1}{\Delta T}\sum \delta\left(\mu - \frac{n}{\Delta T}\right)$

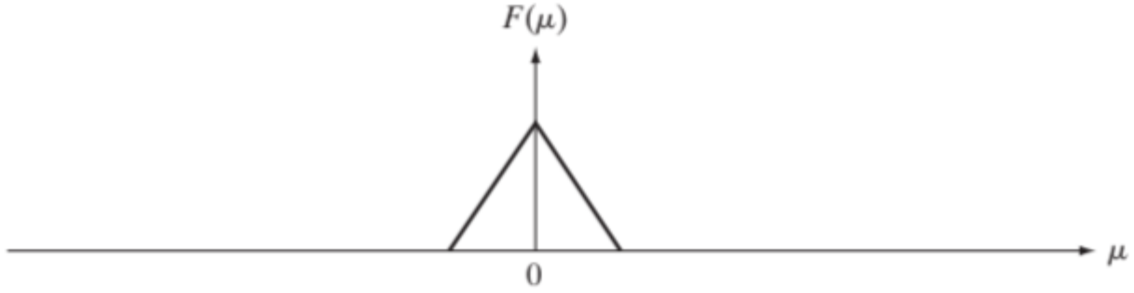
Fourier Transform of Sampled Functions

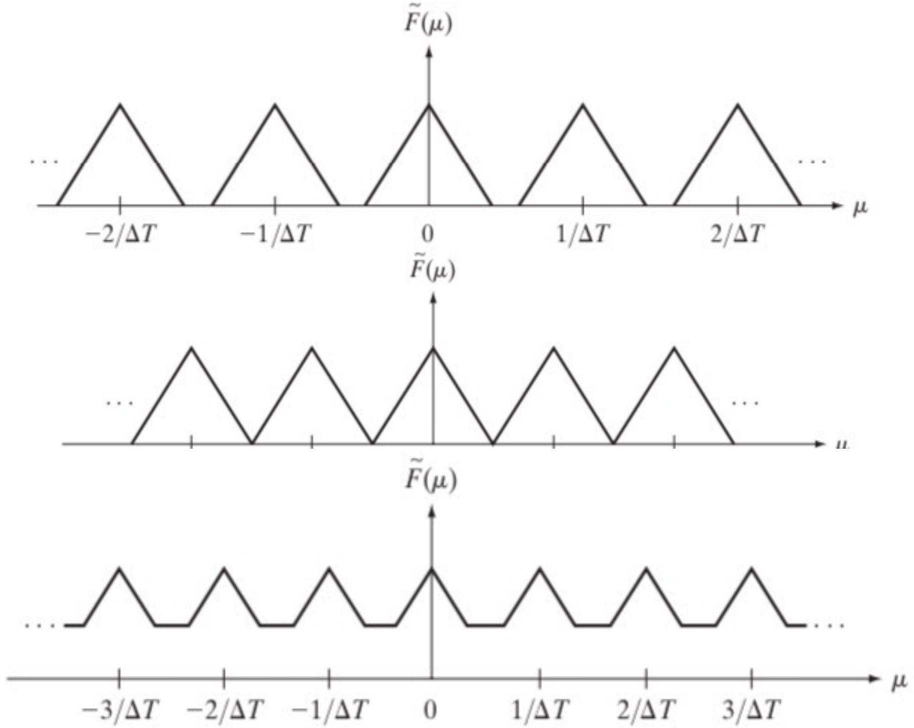
- Fourier transform of a sampled function
  - $\tilde{F}(\mu) = F(\mu) * S(\mu) = \cdots = \frac{1}{\Delta T} \sum F\left(\mu - \frac{n}{\Delta T}\right)$
  - $\tilde{F}(\mu)$ is an infinite, periodic sequence of copies of $F(\mu)$



Sampling Frequency

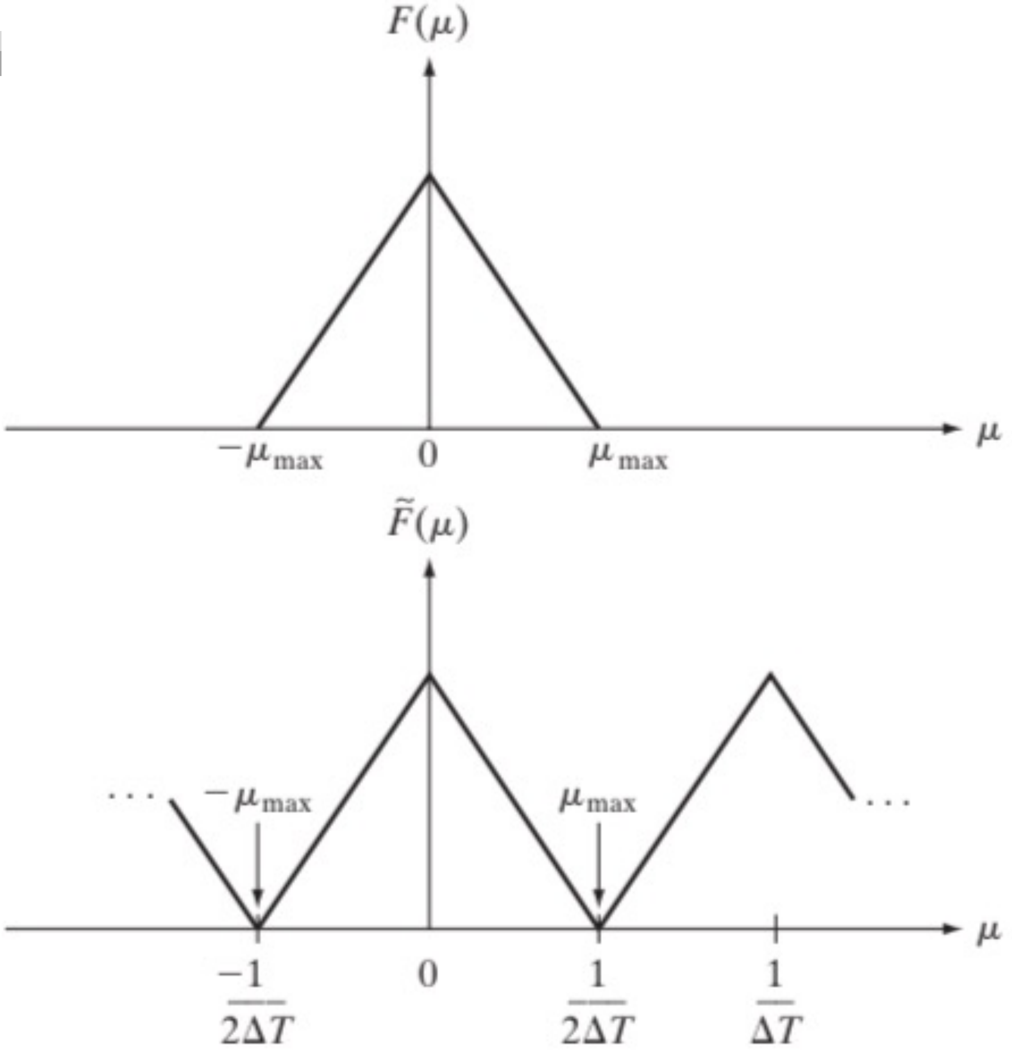
- Sampling frequency
  - $\tilde{F}(\mu) = F(\mu) * S(\mu) = \cdots = \frac{1}{\Delta T} \sum F\left(\mu - \frac{n}{\Delta T}\right)$
  - Fourier transform of $f(t)$ sampled with sample rate $\frac{1}{\Delta T}$





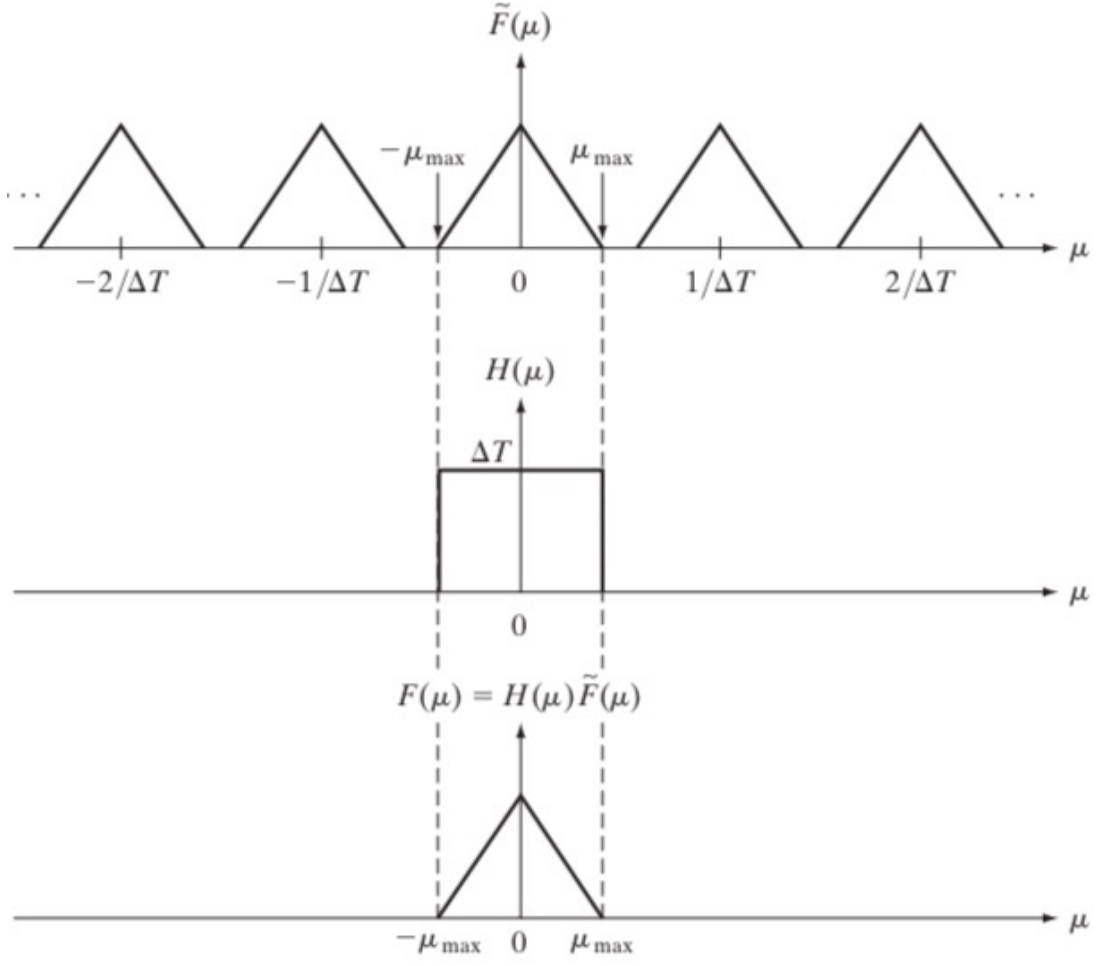
Sampling Theorem

- Band limited function $F(\mu)$
  - Maximum frequency: μ_{max}
- After sampling
  - $\tilde{F}(t)$ becomes periodic: $\frac{1}{\Delta T}$
- To keep the original shape of F,
  - $\frac{1}{\Delta T} > 2\mu_{max}$
  - Called Nyquist rate



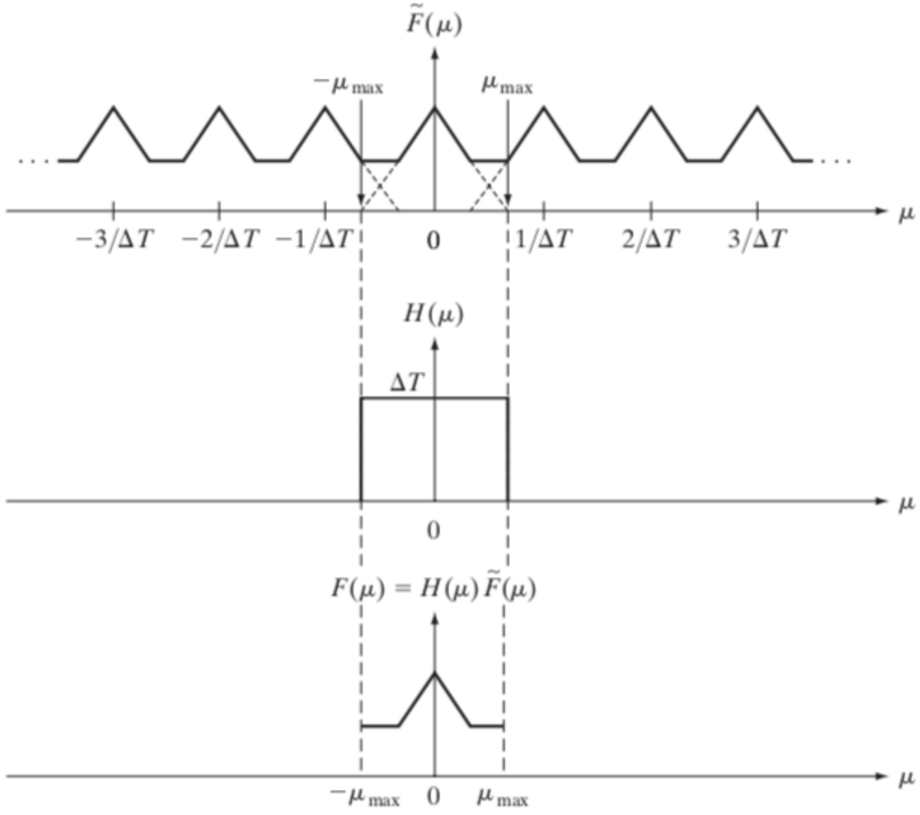
Recovering Function from Samples

- Low-pass Filter
  - $H(\mu) = \begin{cases} \Delta T & -\mu_{max} \leq \mu \leq \mu_{max} \\ 0 & \text{otherwise} \end{cases}$
  - $F(\mu) = H(\mu)\tilde{F}(\mu)$
  - $f(t) = \mathfrak{I}^{-1}\{F(\mu)\} = \int F(\mu) e^{j2\pi\mu t} d\mu$

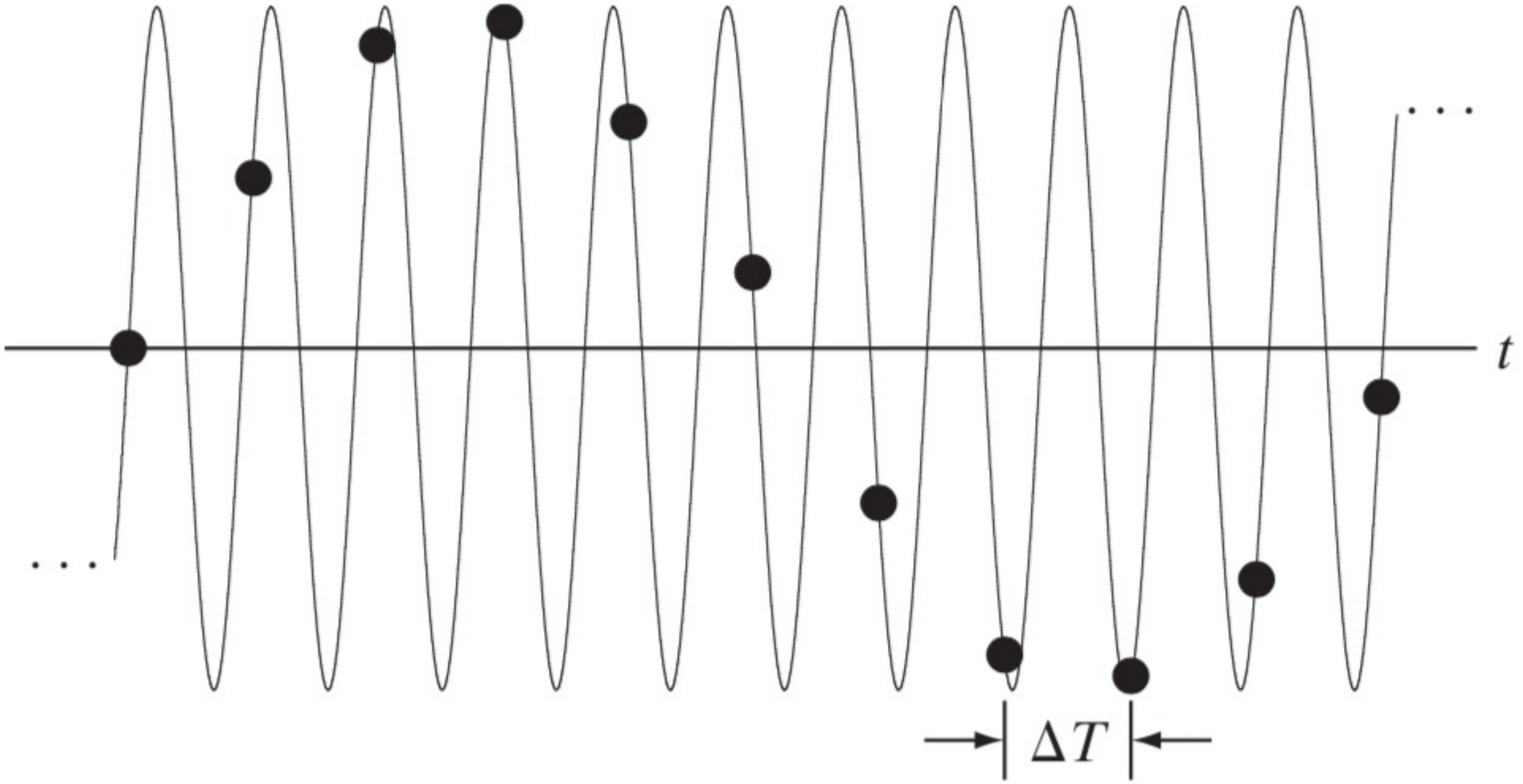


Aliasing

- What if
  - a band-limited function is sample at a rate less than twice its highest frequency?
  - => aliasing (a false identity)



Aliasing



Moiré Effect

Discrete Fourier Transform

Derivation

- Original Fourier transform
  - $\tilde{F}(\mu) = \int \tilde{f}(t)e^{-j2\pi\mu t}dt = \int \sum f(t)\delta(t - n\Delta T)e^{-j2\pi\mu t}\,dt$
  - $= \sum \int f(t)\delta(t - n\Delta T)e^{-j2\pi\mu t}dt = \sum f_n e^{-j2\pi\mu n\Delta T}$
- FFT of Discrete function
  - Sum of basis (cosine/sine) function multiplied by the sampled function

Discrete Fourier Transform (DFT)

- Discrete cosine transform
  - $\tilde{F}(\mu) = \sum f_n e^{-j2\pi\mu n\Delta T}$
  - Continuous function
  - Infinite range
- Given M samples of f
  - only M samples of $\tilde{F}(\mu)$ over 0 to $\frac{1}{\Delta T}$ is required
  - $\mu = \frac{m}{M\Delta T}, m = 0,1,2, \ldots M - 1$
  - Therefore, $F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi\frac{mn}{M}}$

Inverse Discrete Fourier Transform

- Forward DFT (as previous slide)
  - $F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi\frac{mn}{M}}$
- Inverse of DFT
  - $f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi\frac{mn}{M}}$
- Periodicity
  - $F(u) = F(u + kM)$
  - $f(x) = f(x + kM)$

Extension to Two Variables

2-D Impulses and Sifting Property

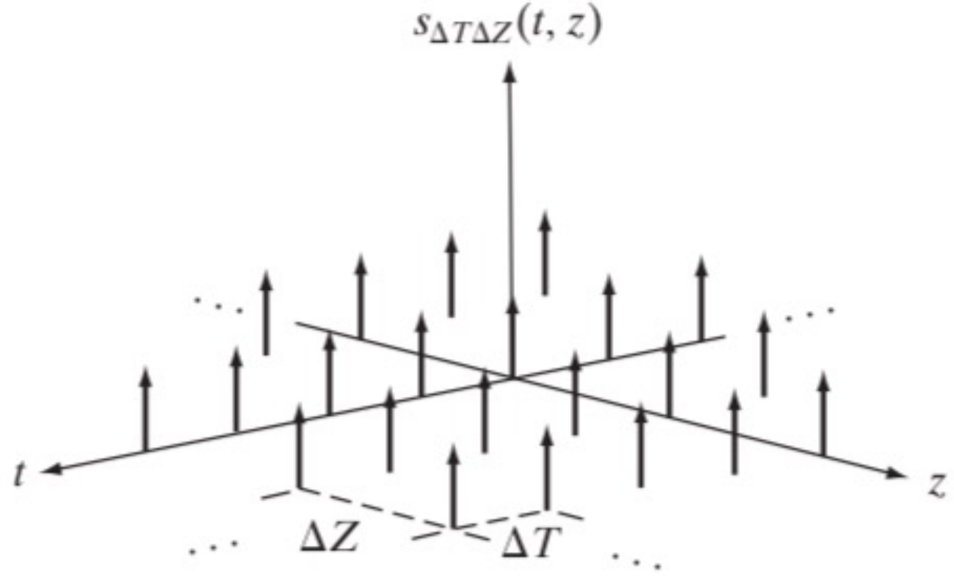
- Impulses
  - $\delta(t,z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$
  - $\int\int \delta(t,z)dt\,dz = 1$
- Sifting property
  - $\int\int f(t,z)\delta(t,z)dt\,dz = f(0,0)$
  - $\int\int f(t,z)\delta(t-t_0,z-z_0)dt\,dz = f(t_0,z_0)$
- Discrete case
  - $\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$
  - $\sum\sum f(x,y)\delta(x-x_0,y-y_0) = f(x_0,y_0)$

2D Continuous Fourier Transform Pair

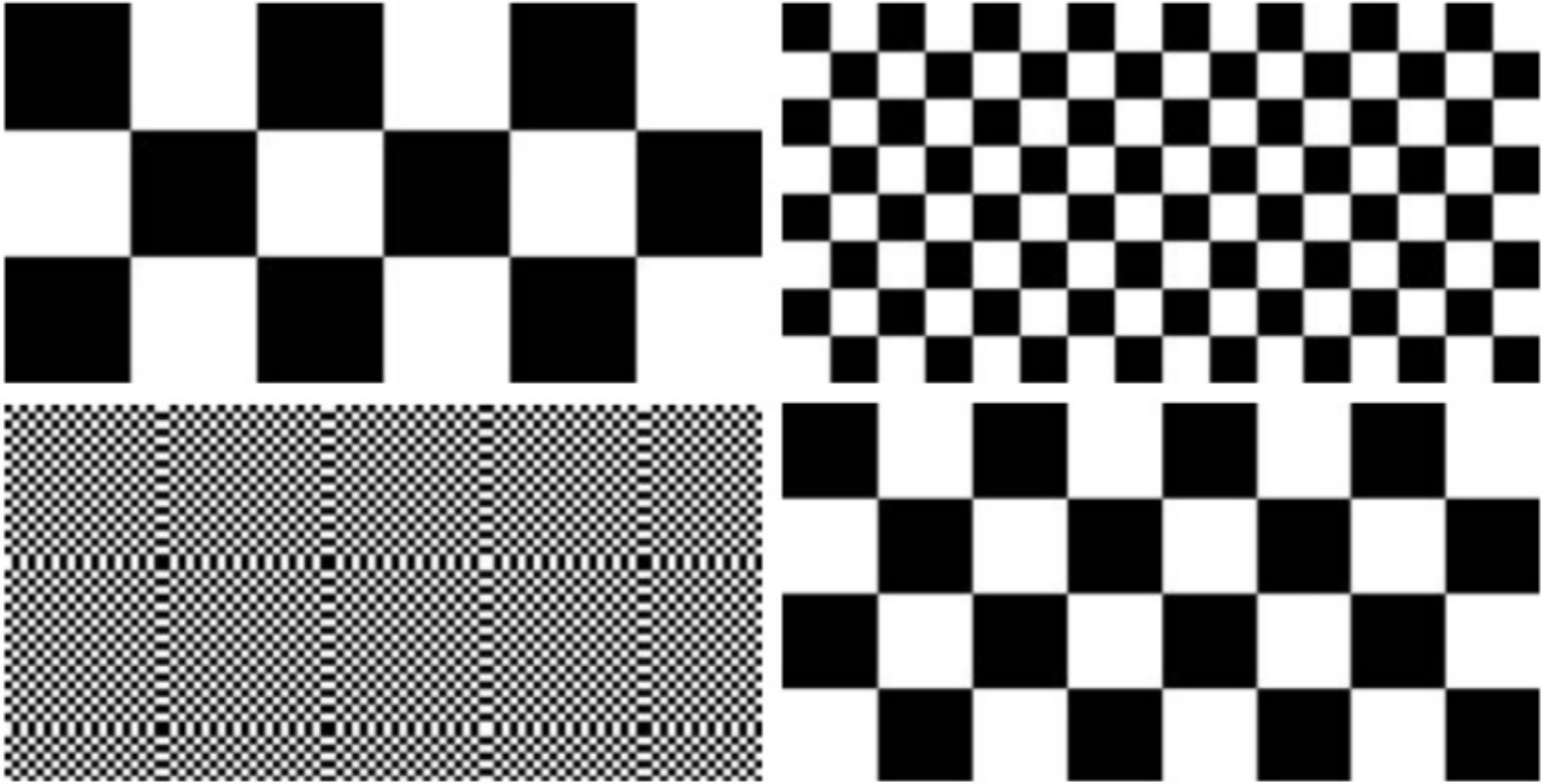
- 2D Fourier transform
  - $F(\mu, \nu) = \int \int f(t, z) e^{-j2\pi(\mu t + \nu z)} dt\, dz$
  - $f(t, z) = \int \int F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu\, d\nu$

Two-Dimensional Sampling

- Impulse train
  - $s_{\Delta T \Delta Z}(t, z) = \sum \sum \delta(t - m\Delta T, z - n\Delta Z)$
- Band-limited function
  - $F(\mu, \nu) = 0$ for $|\mu| \geq \mu_{max}$ and $|\nu| \geq \nu_{max}$
- Sampling theorem
  - $\Delta T < \frac{1}{2\mu_{max}}, \Delta Z < \frac{1}{2\nu_{max}}$
  - $\frac{1}{\Delta T} > 2\mu_{max}, \frac{1}{\Delta Z} > 2\nu_{max}$



Aliasing in Images



Aliasing in Image

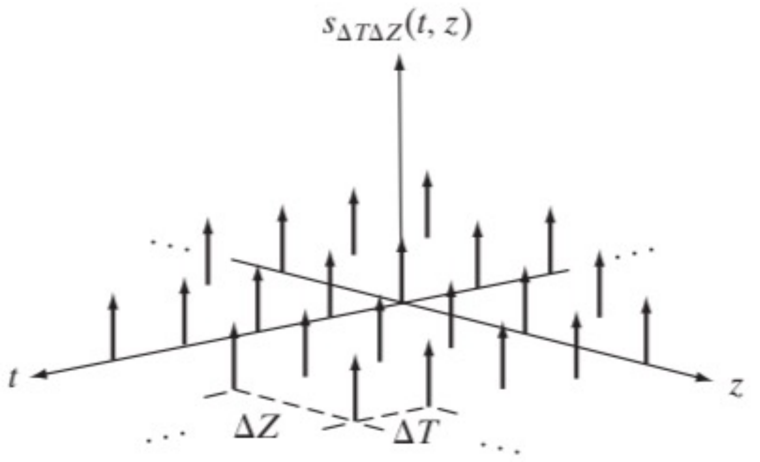
2D Discrete Fourier Transform

- 2D DFT
  - $F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$
- 2D IDFT
  - $f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$

Properties of 2D DFT

Spatial & Frequency Intervals

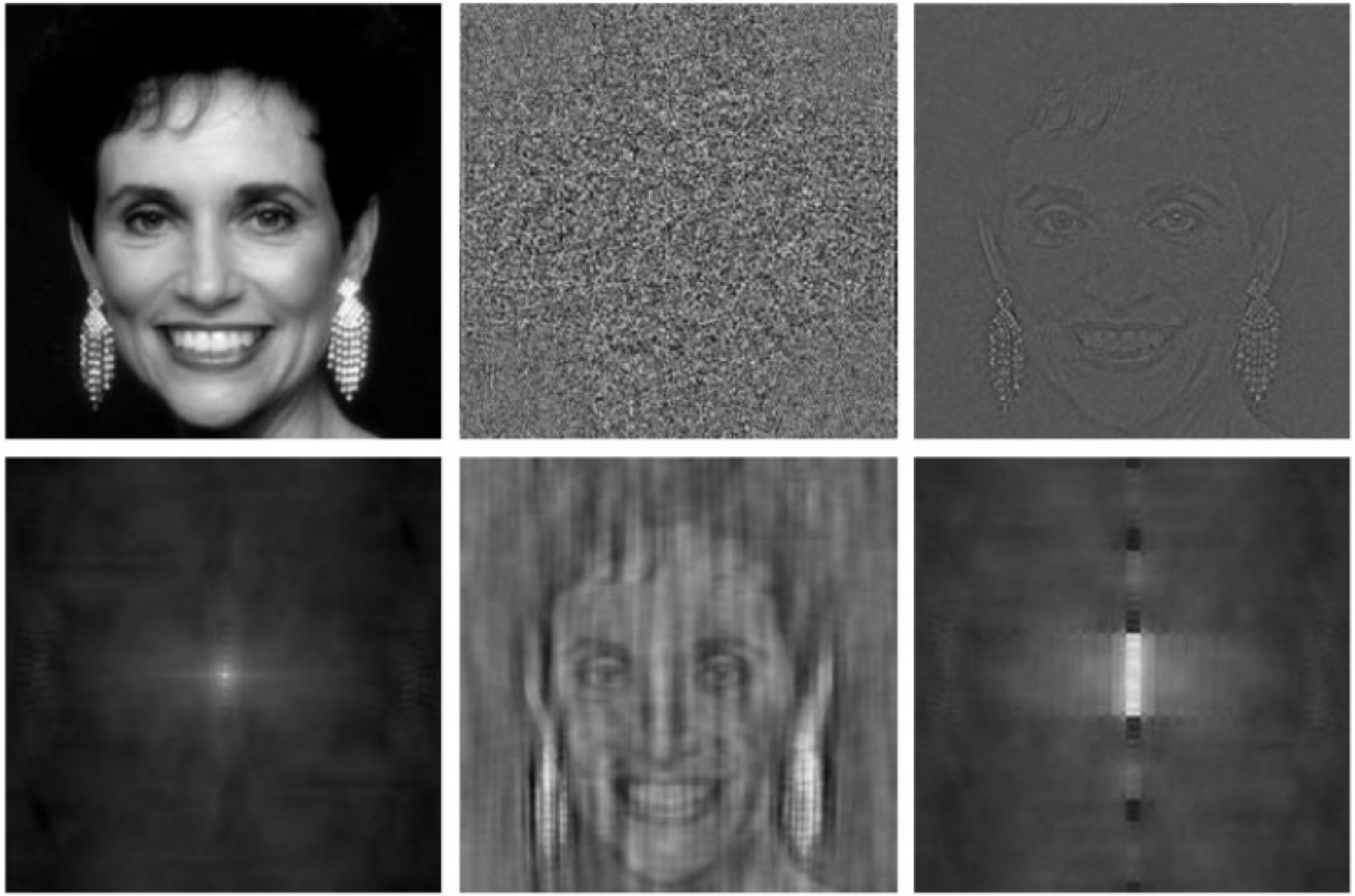
- Sampling
  - $f(x, y)$ is sampled $M \times N$ at $\Delta x, \Delta y$
  - The frequency density: $\Delta \mu = \frac{1}{M \Delta \mathrm{x}}, \Delta \nu = \frac{1}{N \Delta y}$



Power Spectrum & Phase Angle

- Power spectrum
  - Squared magnitude (Power) of each spectrum
  - $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$
- Phase angle
  - $F(u,v) = |F(u,v)|e^{j\phi(u,v)}$
  - $\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$

Power Spectrum & Phase Angle



Translation and Rotation

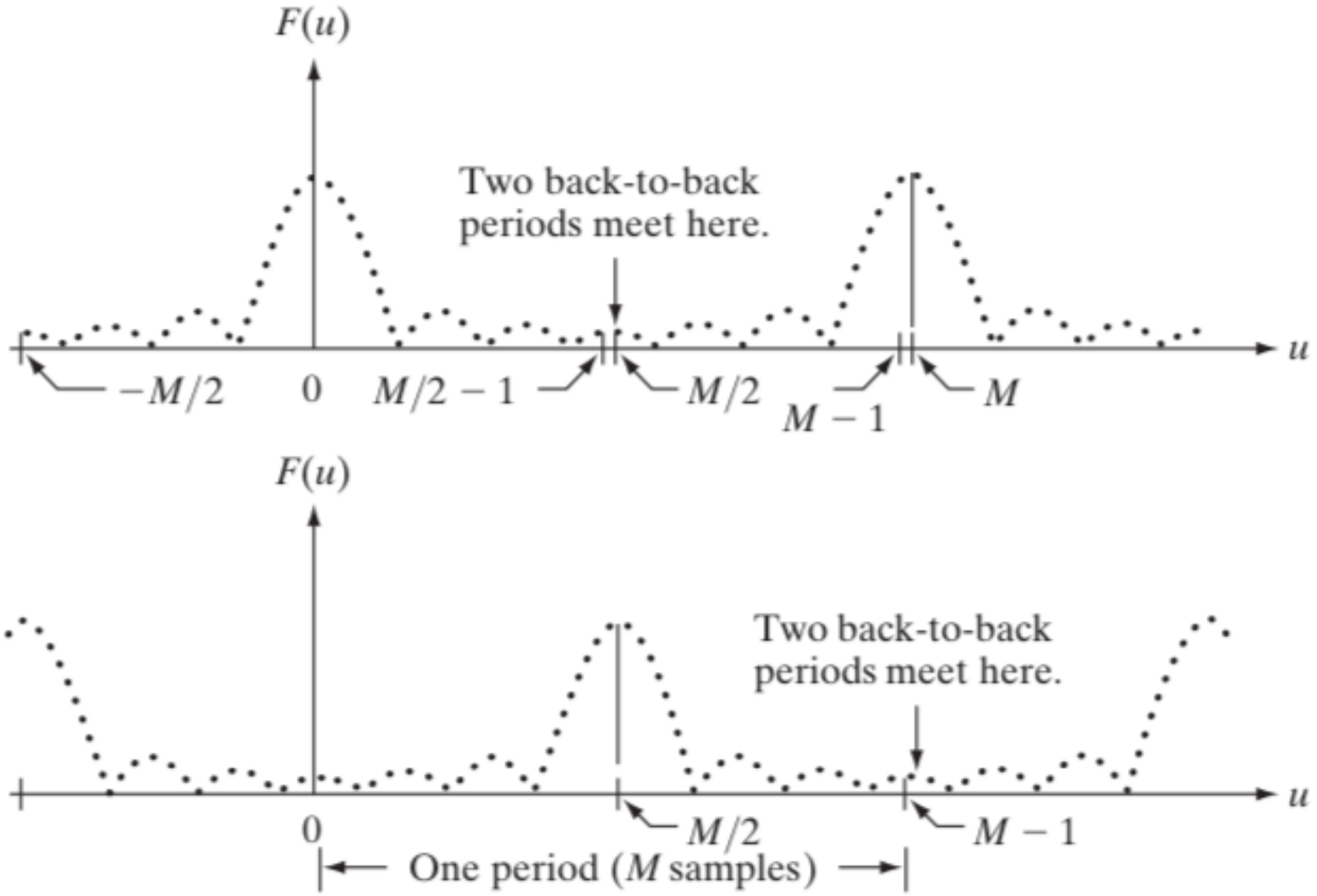
- Translation
  - The original equations
  - $f(x, y) \Leftrightarrow F(\mu, \nu)$
  - $f(x - x_0, y - y_0) \Leftrightarrow F(\mu, \nu)e^{-j2\pi\left(\frac{\mu x_0}{M}+\frac{\nu y_0}{N}\right)}$
  - f$(x, y)e^{j2\pi\left(\frac{\mu_0 x}{M}+\frac{\nu_0 y}{N}\right)} = F(\mu - \mu_0, \nu - \nu_0)$
- Rotation
  - $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \phi + \theta_0)$, $(x = r\cos\theta, y = r\sin\theta,\ \mu = \omega\cos\phi, \nu = \omega\sin\phi)$
  - Therefore, if the f is rotated, F is also rotated by the same angle

Periodicity

- DFT is periodic in $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ direction infinitely
  - $F(\mu,\nu) = F(\mu + k_1M, \nu) = F(\mu, \nu + k_2N) = F(u + k_1M, \nu + k_2N)$
- So IDFT is
  - $f(x,y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1, y + k_2N)$
- Shifting F by $M/2$ (1D example)
  - Translation: $f(x)e^{j2\pi\left(\frac{\nu_0 x}{M}\right)} \Leftrightarrow F(u - u_0)$
  - Shifting by $M/2$: $f(x)(-1)^x \Leftrightarrow F\left(u - \frac{M}{2}\right)$

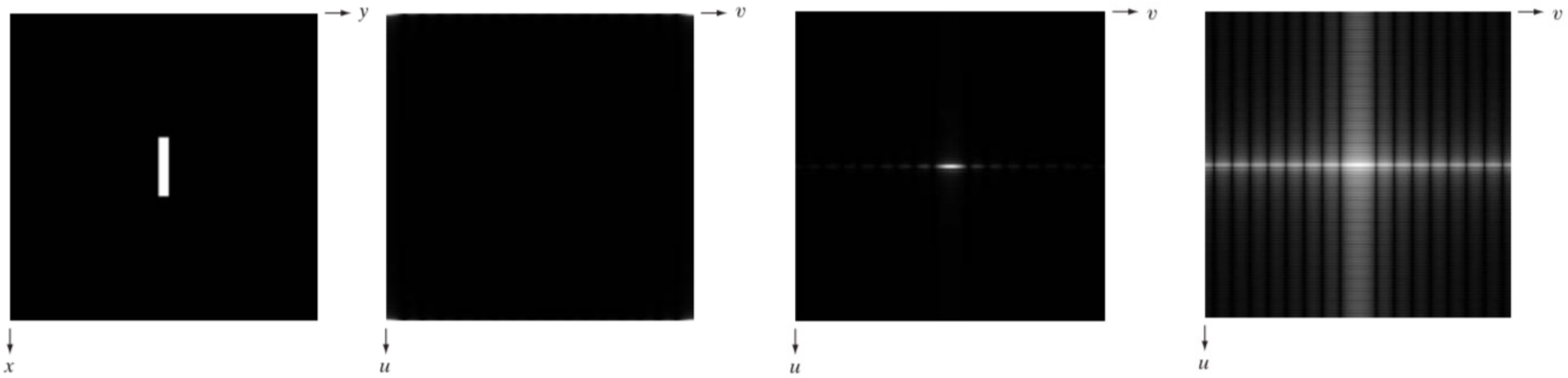
Periodicity

- Centering

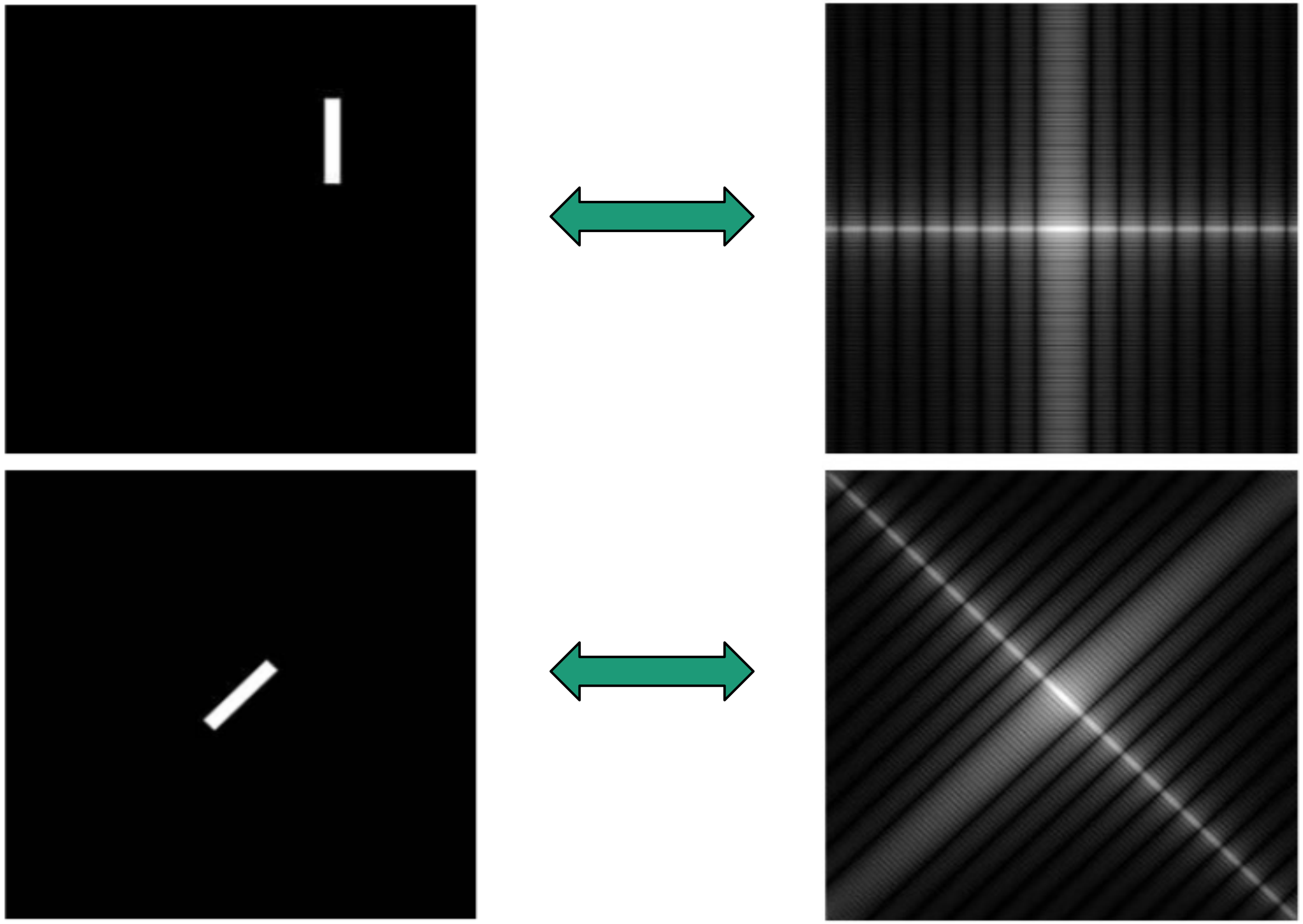


Centering

- Translate the input image by (M/2, N/2)
  - $f(x, y)(-1)^{x+y} \Leftrightarrow F\left(u - \frac{M}{2}, v - \frac{N}{2}\right)$



Translation & Rotation



Convolution Theorem

- 2D circular convolution
  - $f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$
- And
  - $f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$

Summary

- DFT $F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$
- IDFT $f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$
- Polar representation $F(u,v) = |F(u,v)|e^{j\phi(u,v)}$
- Spectrum $|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{\frac{1}{2}}$
- Phase angle $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
- Power spectrum $P(u,v) = |F(u,v)|^2$
- Average value $\bar{f}(x,y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = \frac{1}{MN}F(0,0)$
- Convolution $f(x,y) * h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$

  $\Leftrightarrow F(u,v)H(u,v)$
- Translation $f(x,y)e^{j2\pi\left(\frac{u_0 x}{M}+\frac{v_0 y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)$

  $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}$
- Rotation $f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \phi+\theta_0)$

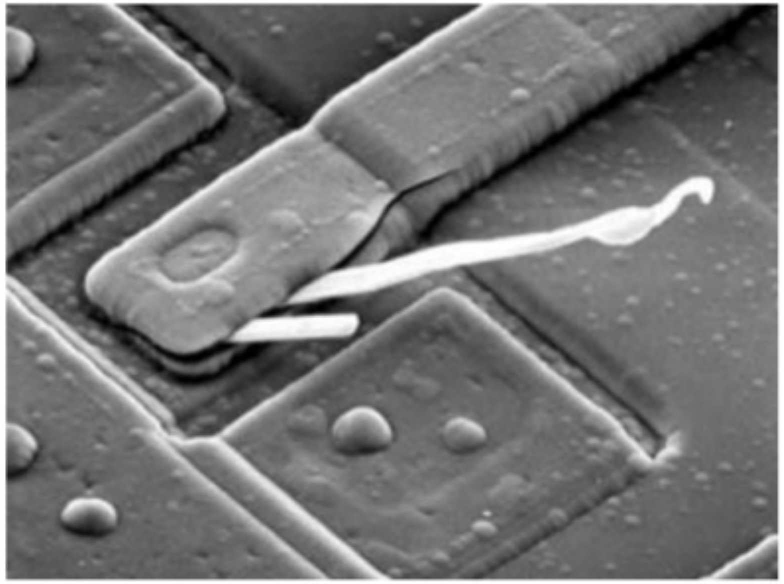
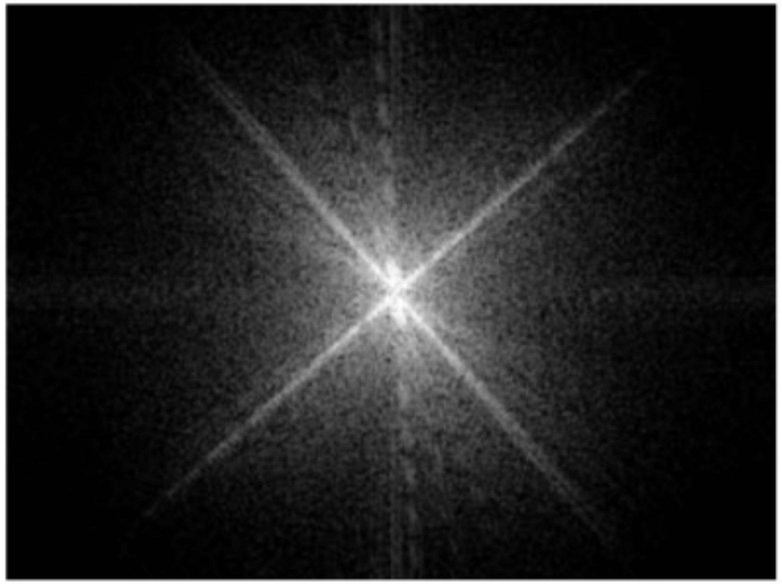
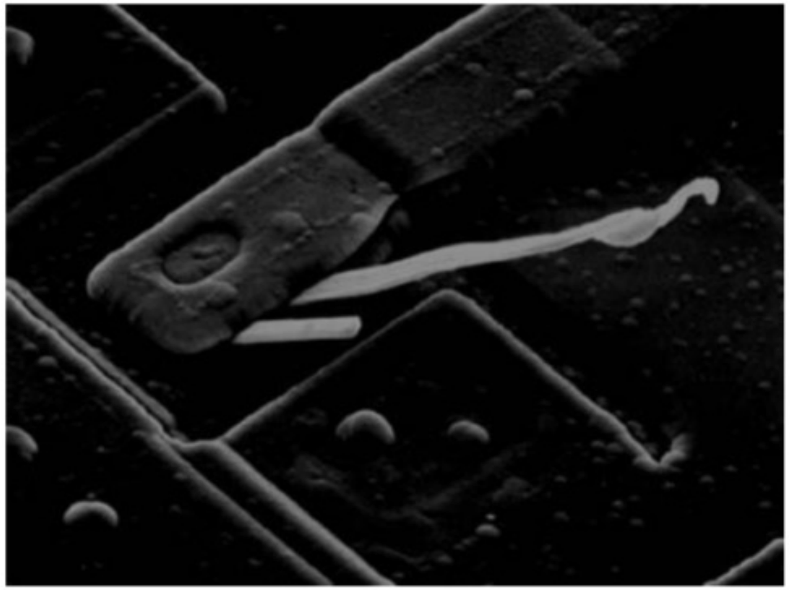
Basics of Filtering in the Frequency Domain

Fundamentals

- Filtering equation
  - $g(x,y) = \Im^{-1}[H(u,v)F(u,v)]$
  - $H(u,v)$ is filter function
  - Which is corresponding to convolution
- Meaning
  - $H(u,v)$ can either enhance the power of some frequency components or
  - reduce the power of some frequency components

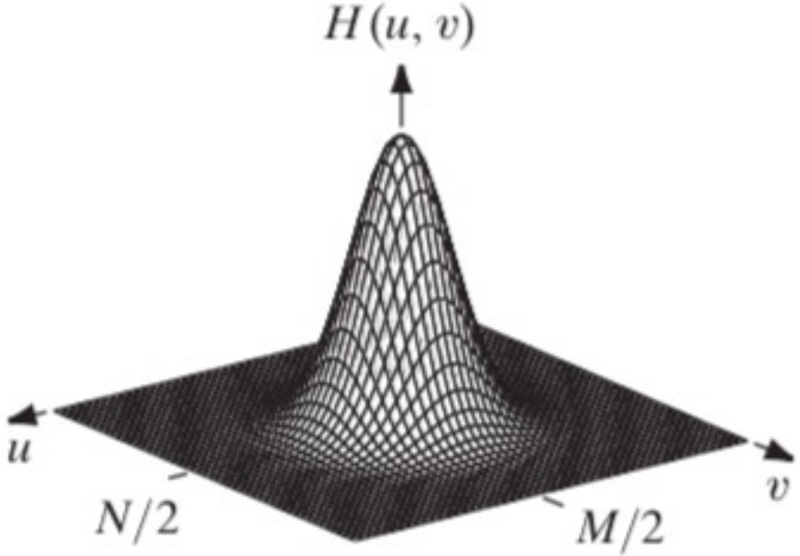
Eliminating DC Component

- Direct current component
  - Average of all pixels
  - The lowest frequency term ($u = v = 0$)

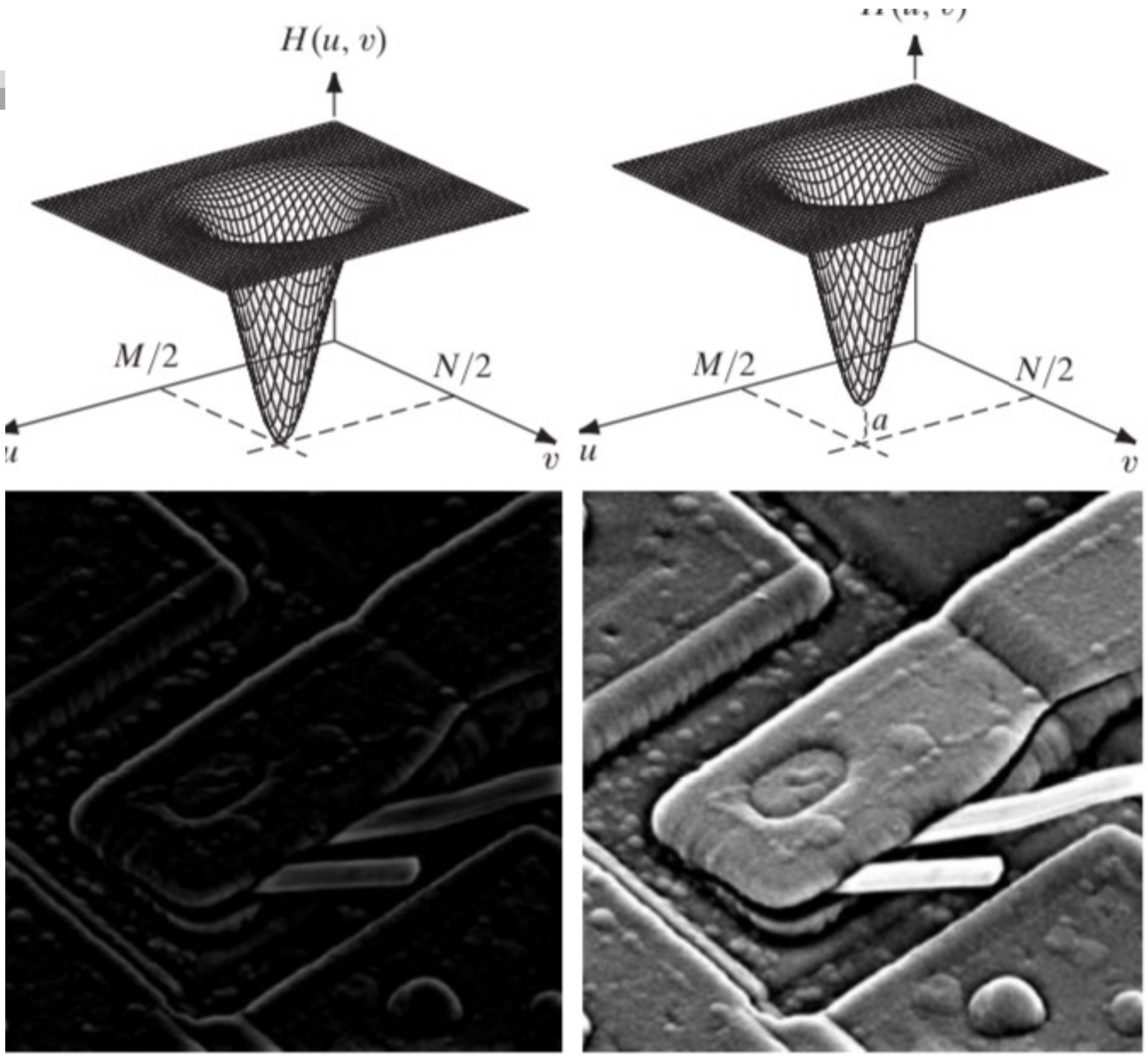
Lowpass Filter

- Attenuation of high frequency
  - While passing low frequency component
  - Providing blurry result



Highpass Filter

- Leaving only high frequency
- Offsetting

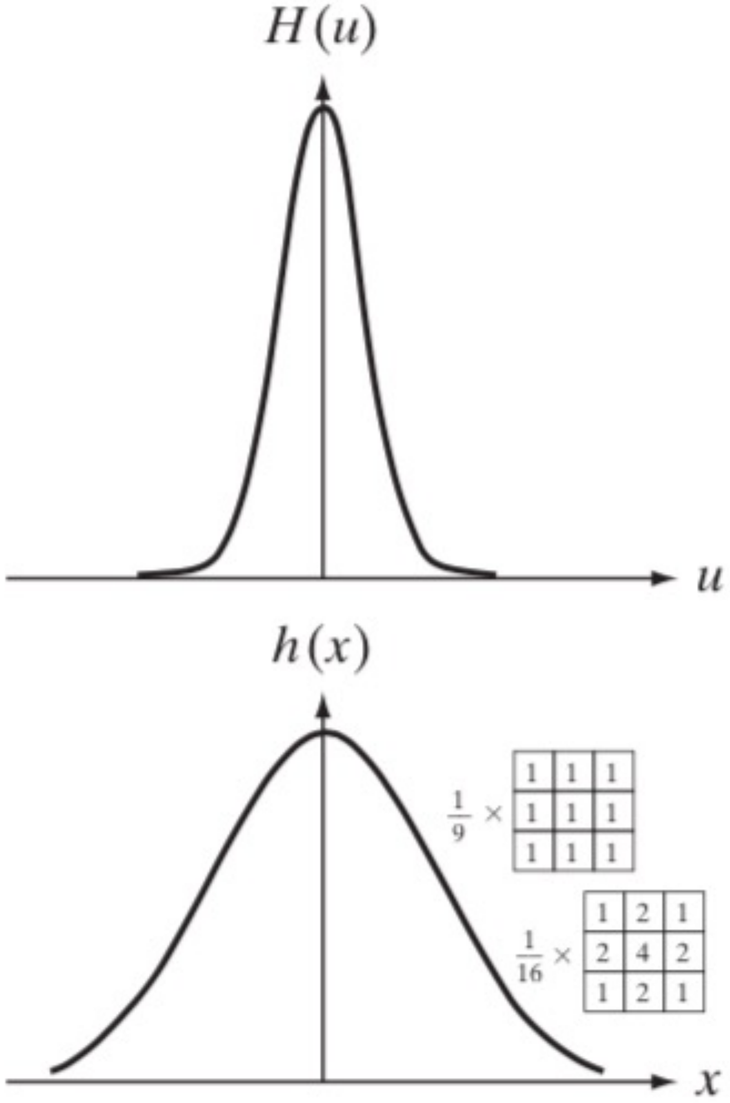


Relation btw Spatial & Frequency Domain

- Filter
  - $H(u, v) \Leftrightarrow h(x, y)$
  - Because this filter obtained from the response of frequency domain filter to an Impulse $h(x, y)$: impulse response filter
  - Because $H(u, v)$ is discrete and finite: finite impulse response (FIR) filter

FIR Filter

- 1D Gaussian Example
  - $H(u) = Ae^{-u^2/2\sigma^2}$
  - $h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2x^2}$
  - Both are real and Gaussian



FIR Filter

- Complex filter
  - Difference of Gaussian
  - $H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$
  - Where $A \geq B$ and $\sigma_1 > \sigma_2$
  - $h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2 x^2}$

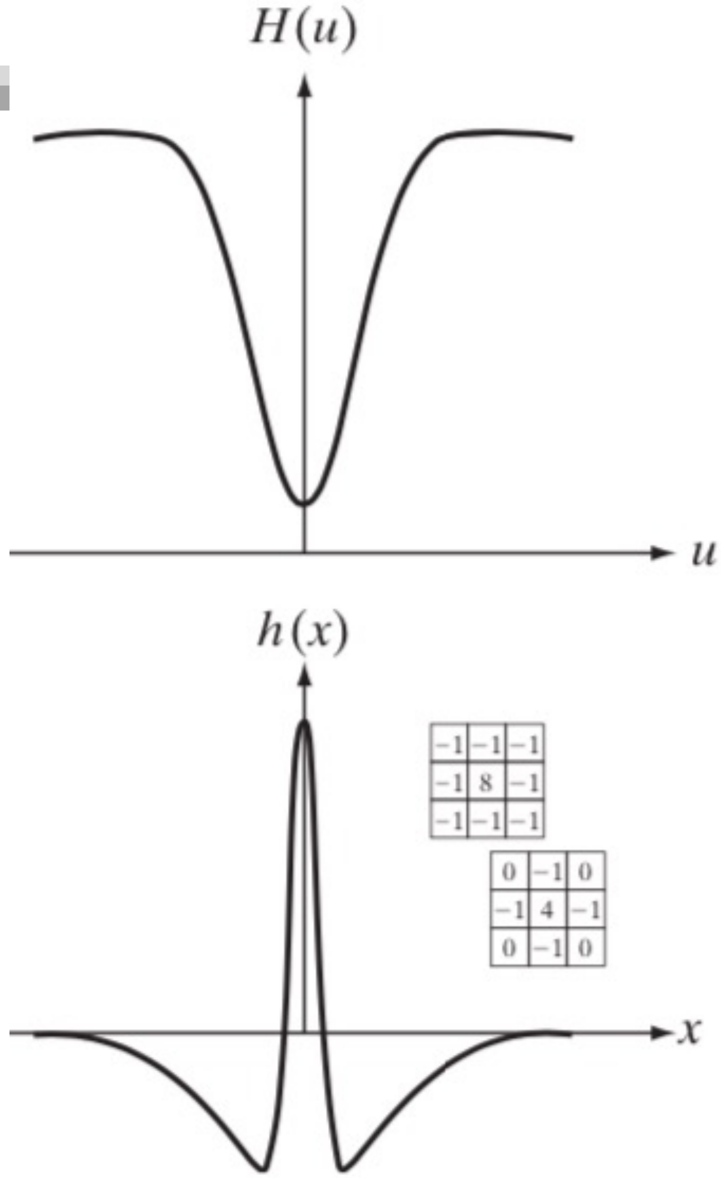
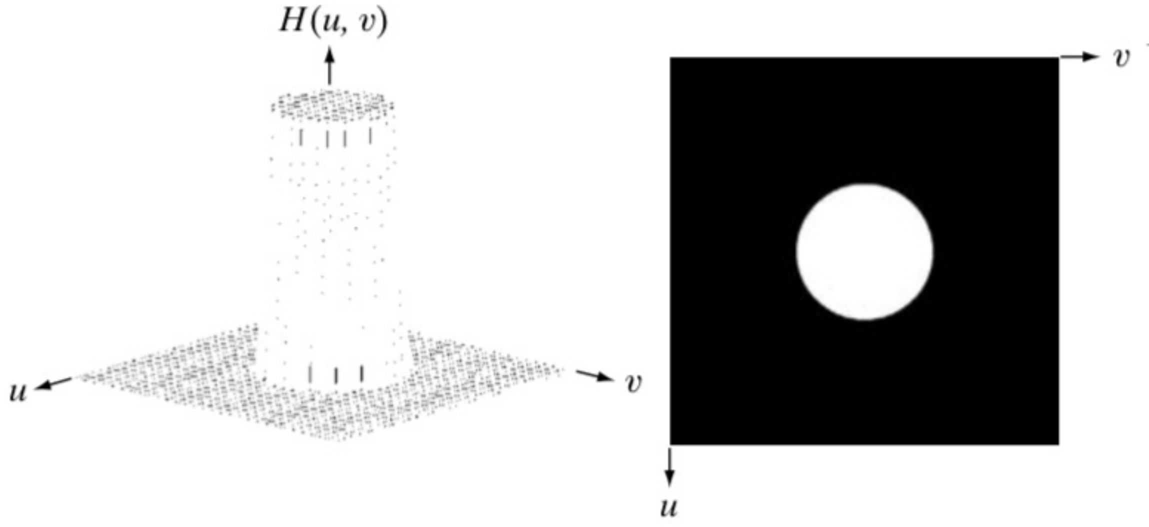
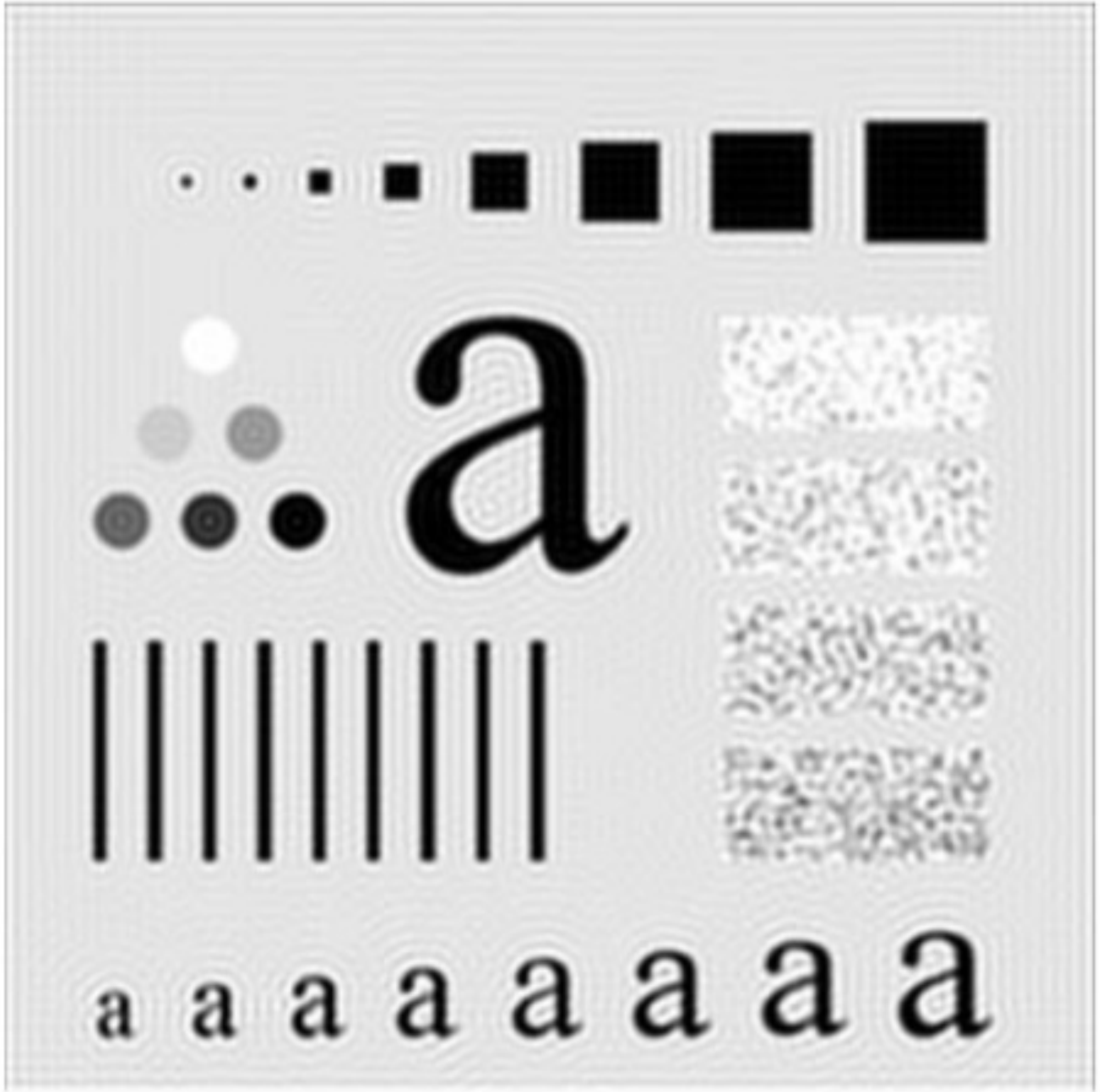


Image Smoothing in Frequency Domain

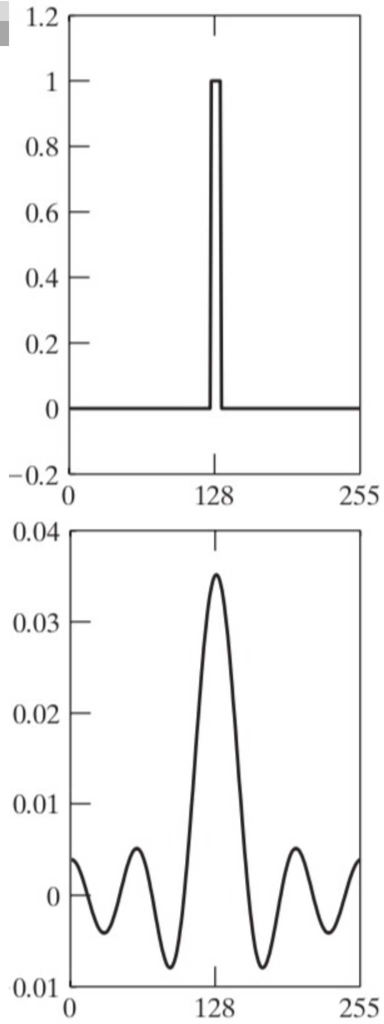
- Ideal lowpass filters
  - Leaving only some (low) frequency
  - Zero-outing higher frequency
  - $H(u,v) = \begin{cases} 1 & if\ D(u,v) \leq D_0 \\ 0 & if\ D(u,v) > D_0 \end{cases}$
  - $D(u,v) = \left[\left(u - \frac{M}{2}\right)^2 + \left(v - \frac{N}{2}\right)^2\right]^{1/2}$
  - D_0 is cutoff frequency





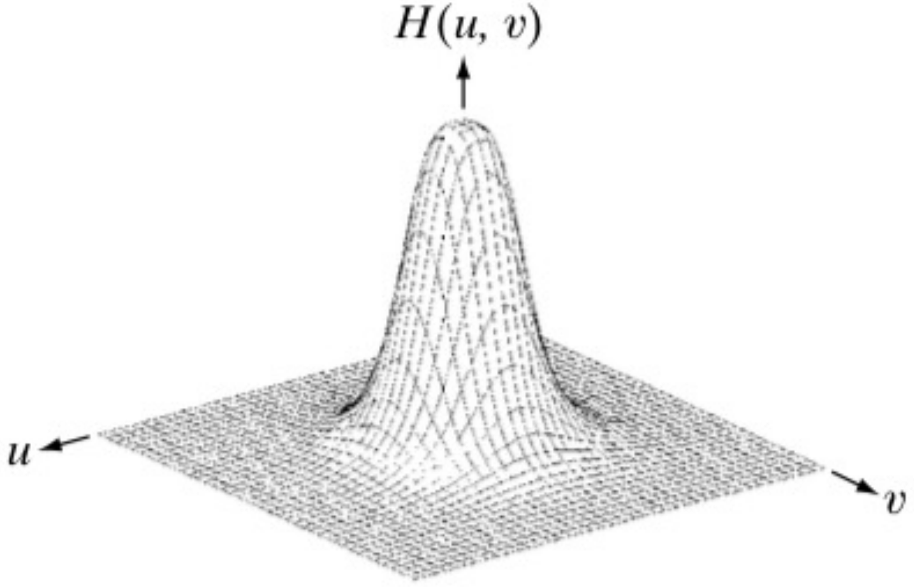
Ideal Lowpass Filtering

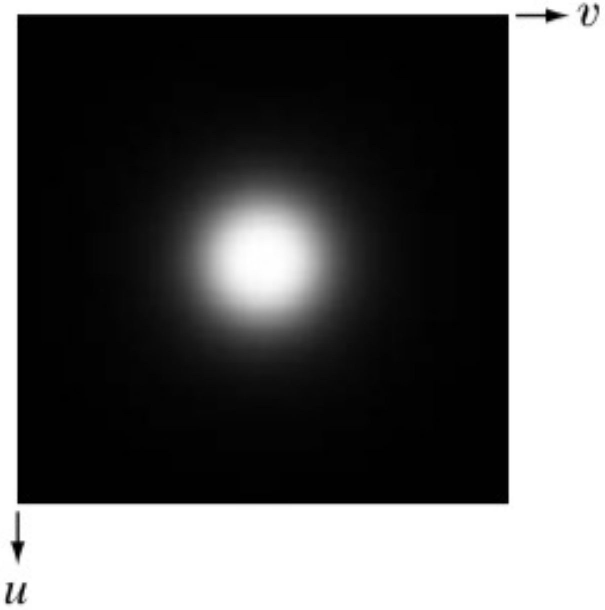
- Ringing properties

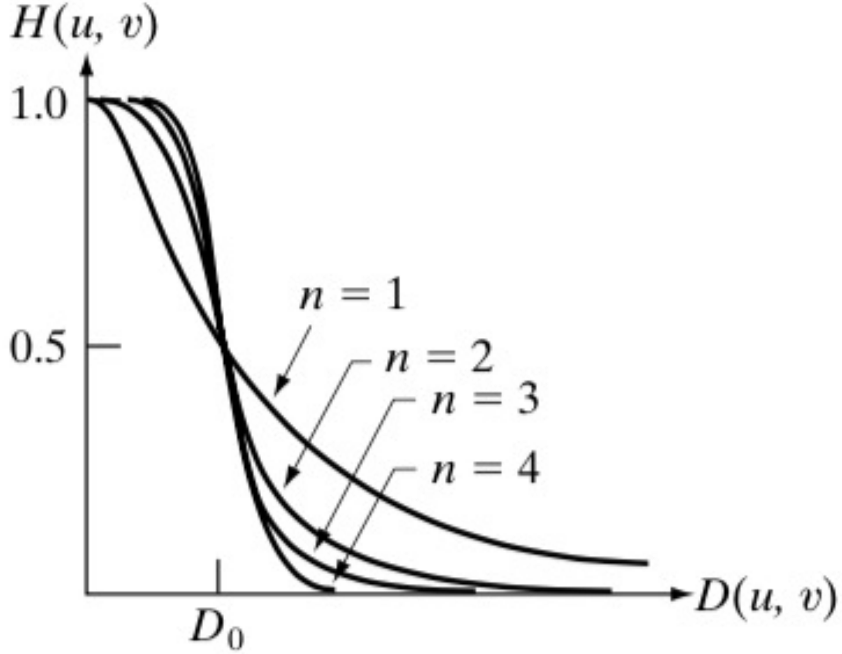


Butterworth Lowpass Filters

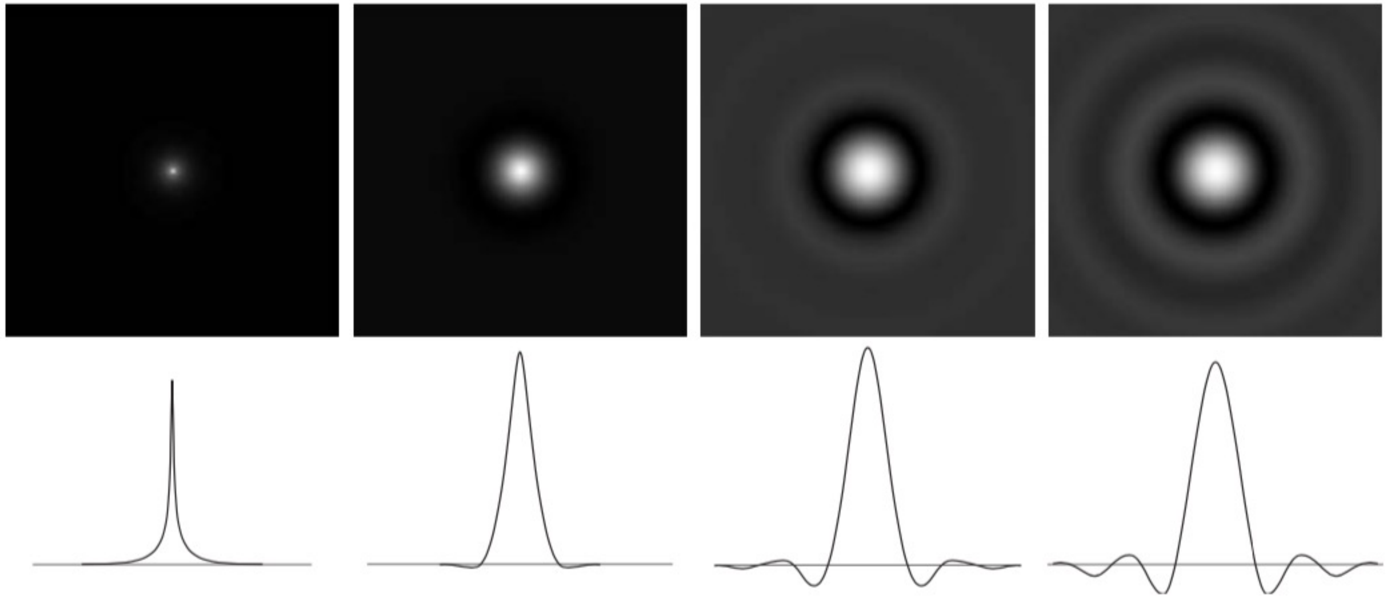
- Definition
  - $H(u,v) = \frac{1}{1+\left[\frac{D(u,v)}{D_0}\right]^{2n}}$
  - Smoother boundary





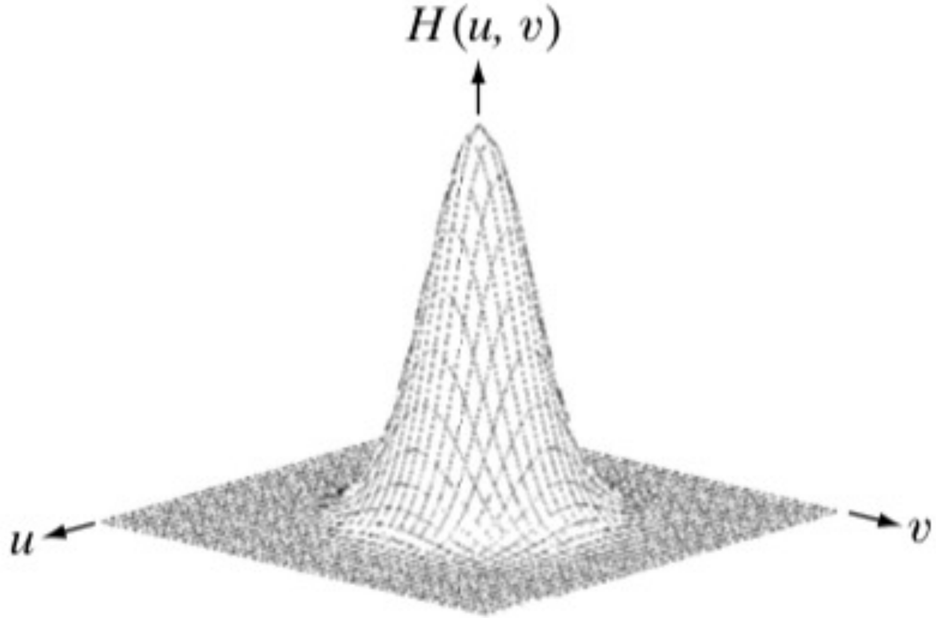


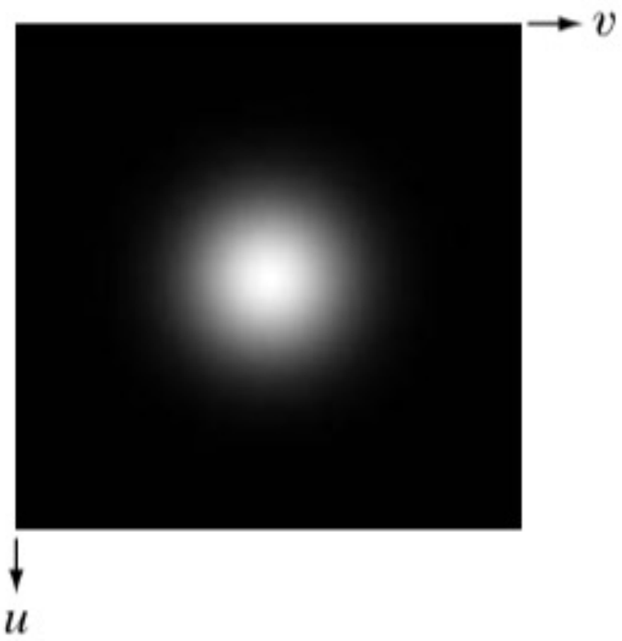

Spatial Representation of BLPF

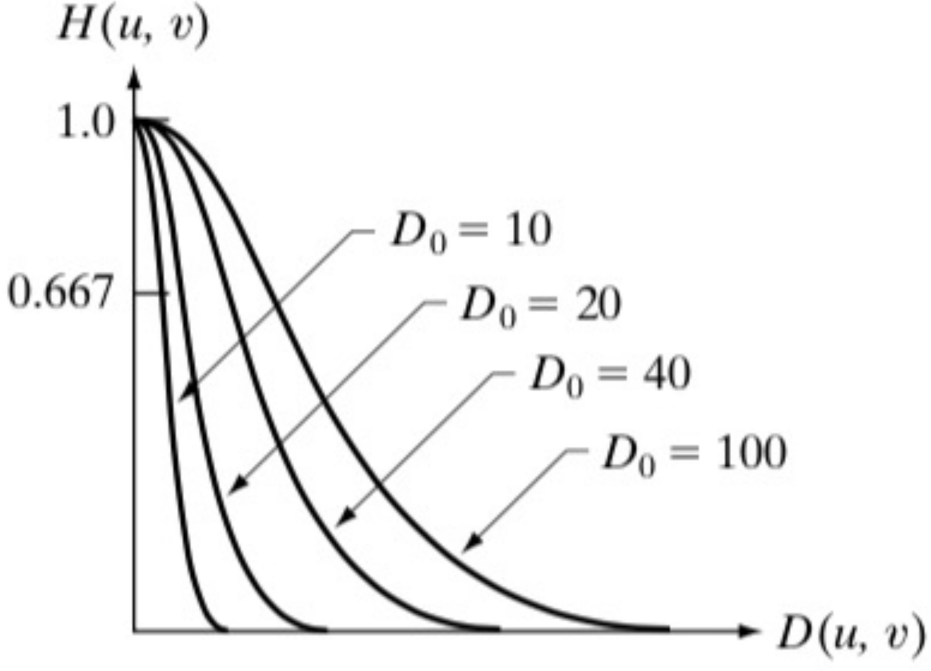


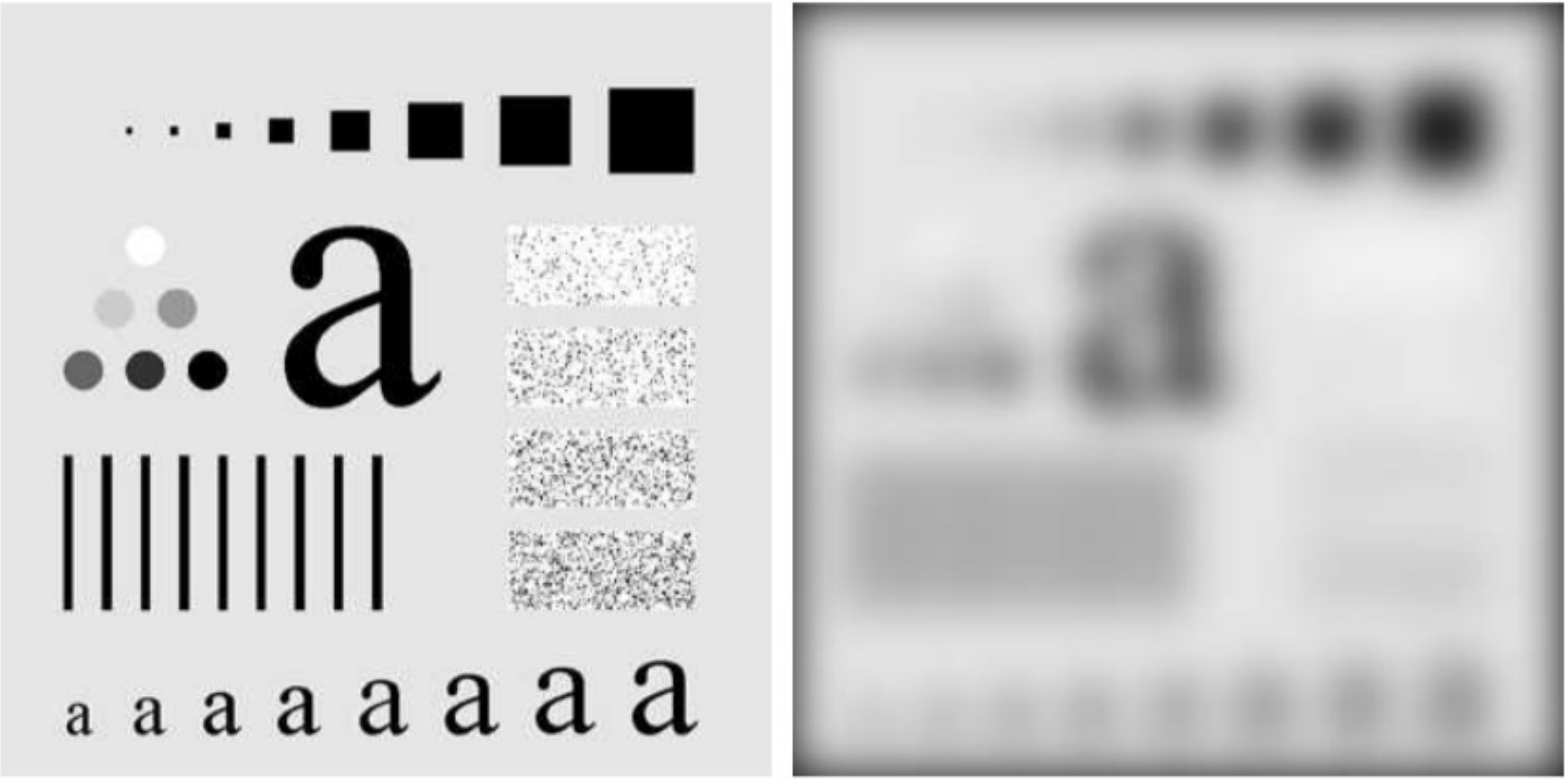
Gaussian Lowpass Filters

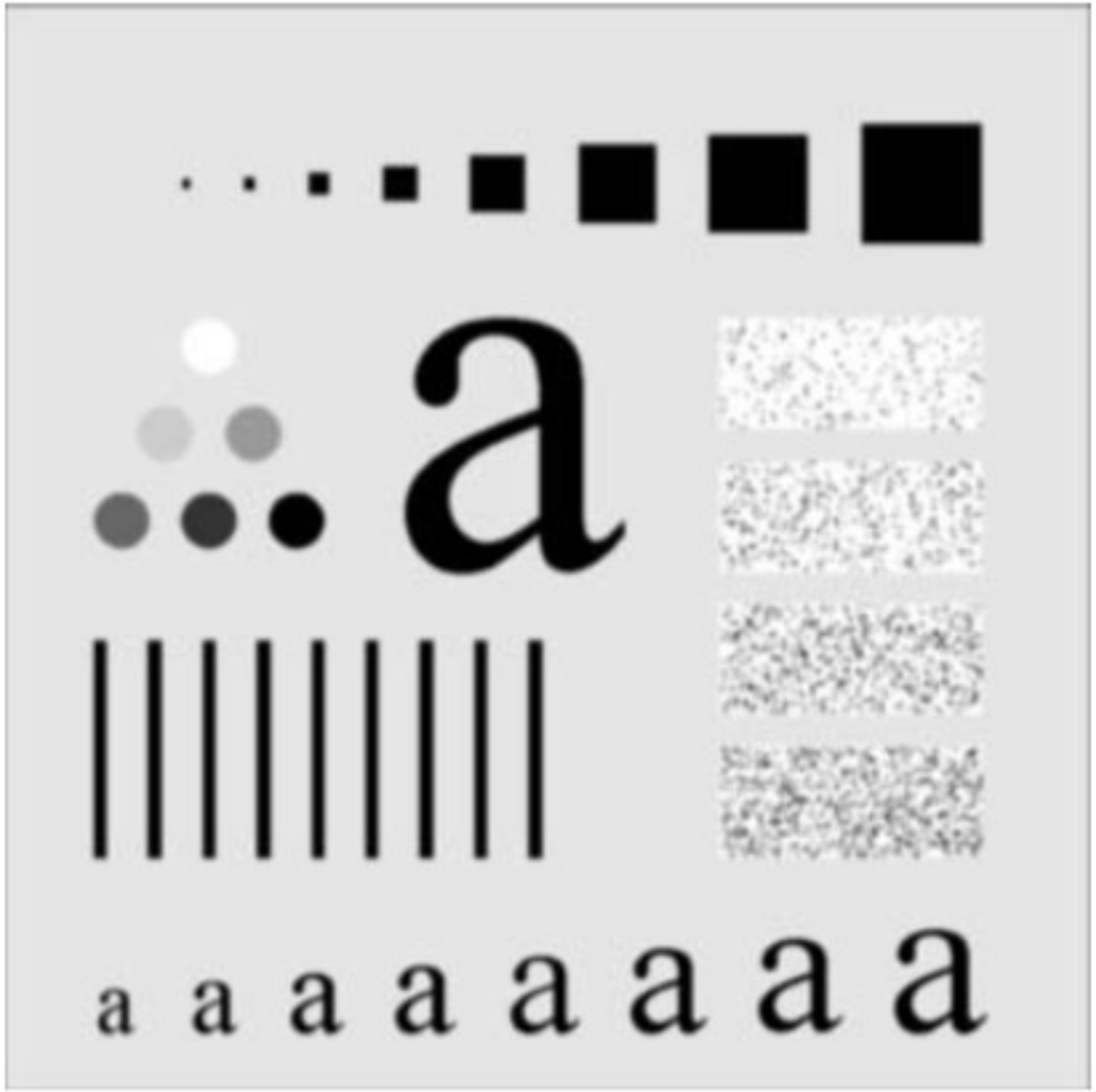
- Gaussian filter
  - $H(u,v) = e^{-D^2(u,v)/2\sigma^2}$
  - Usually $\sigma = D_0$

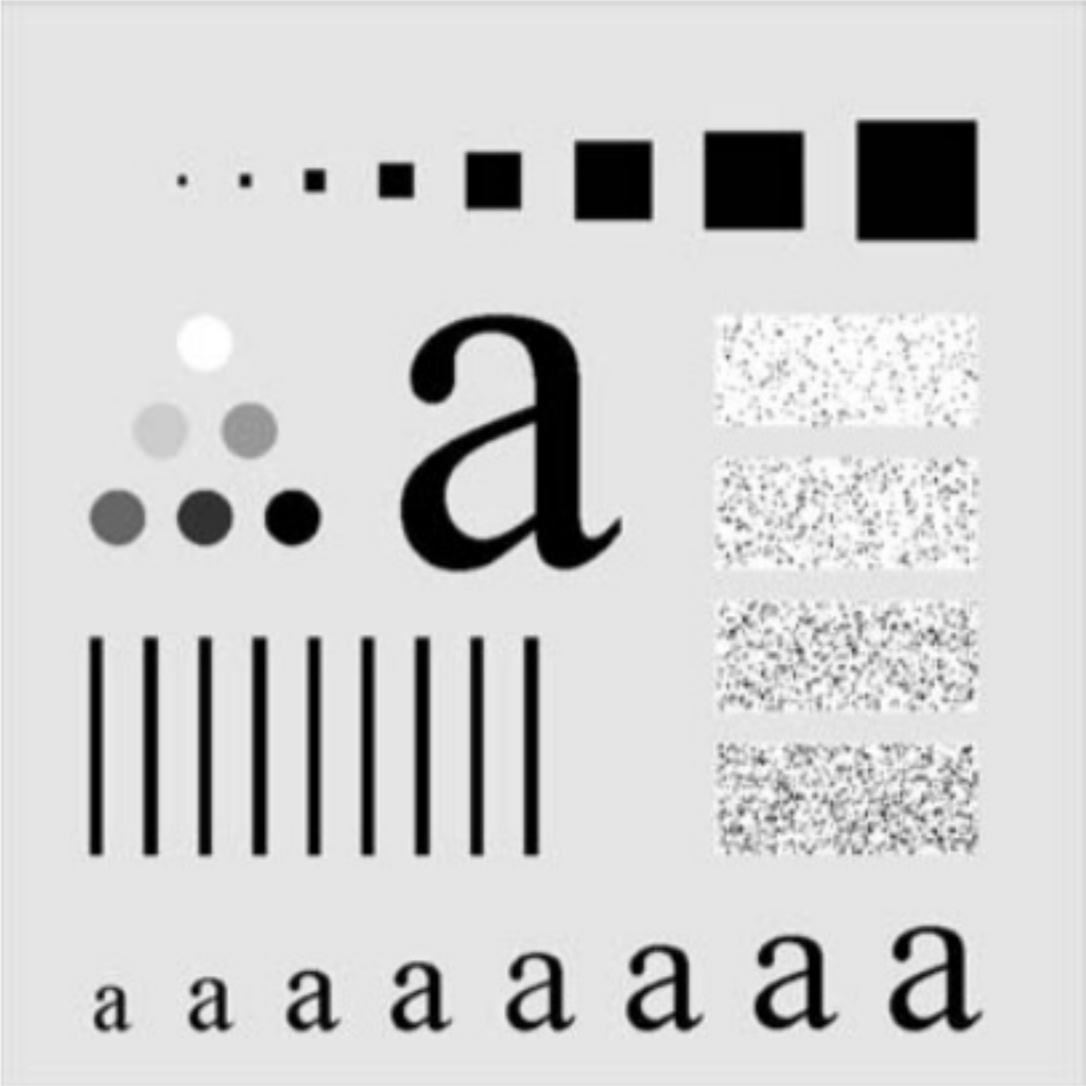










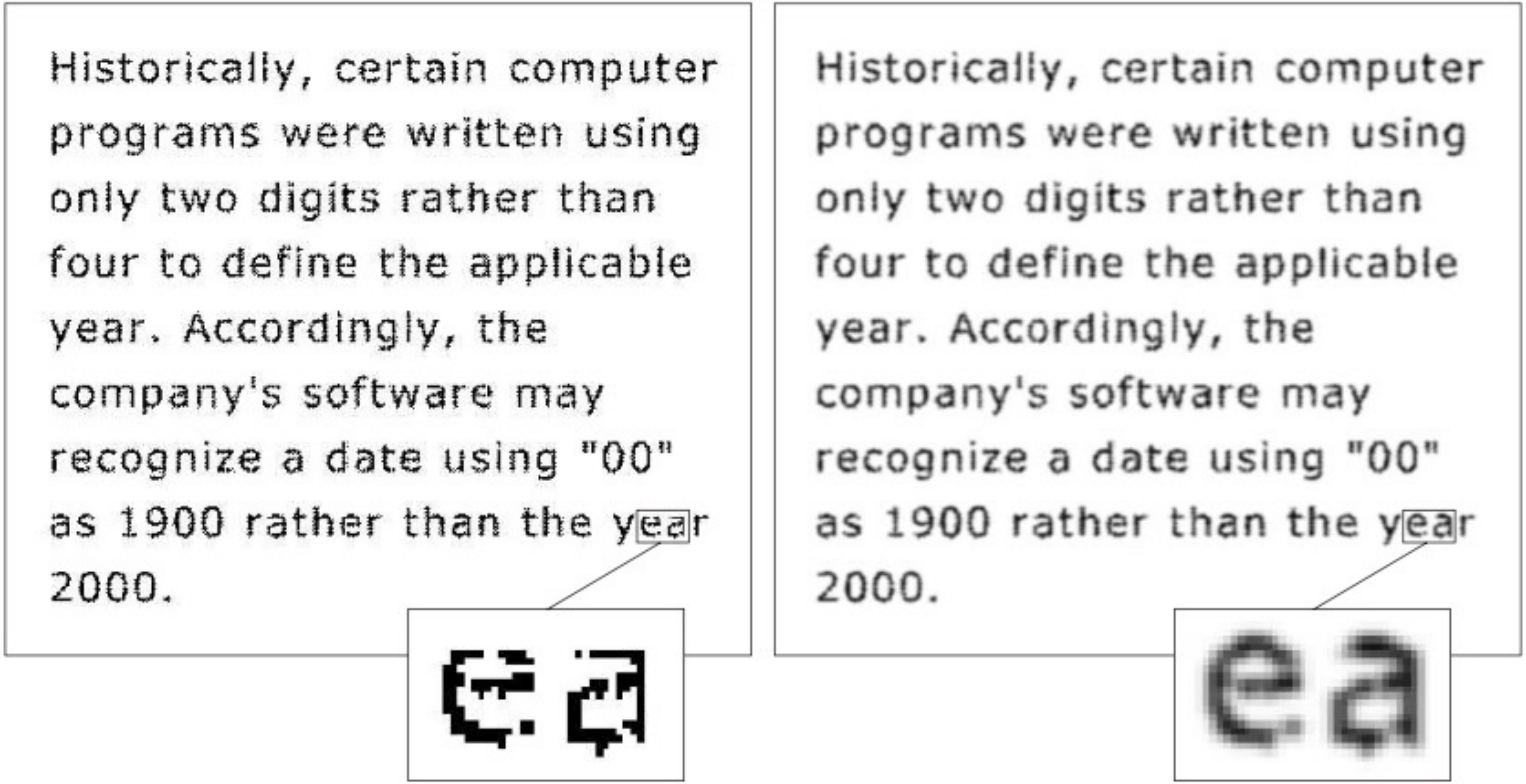
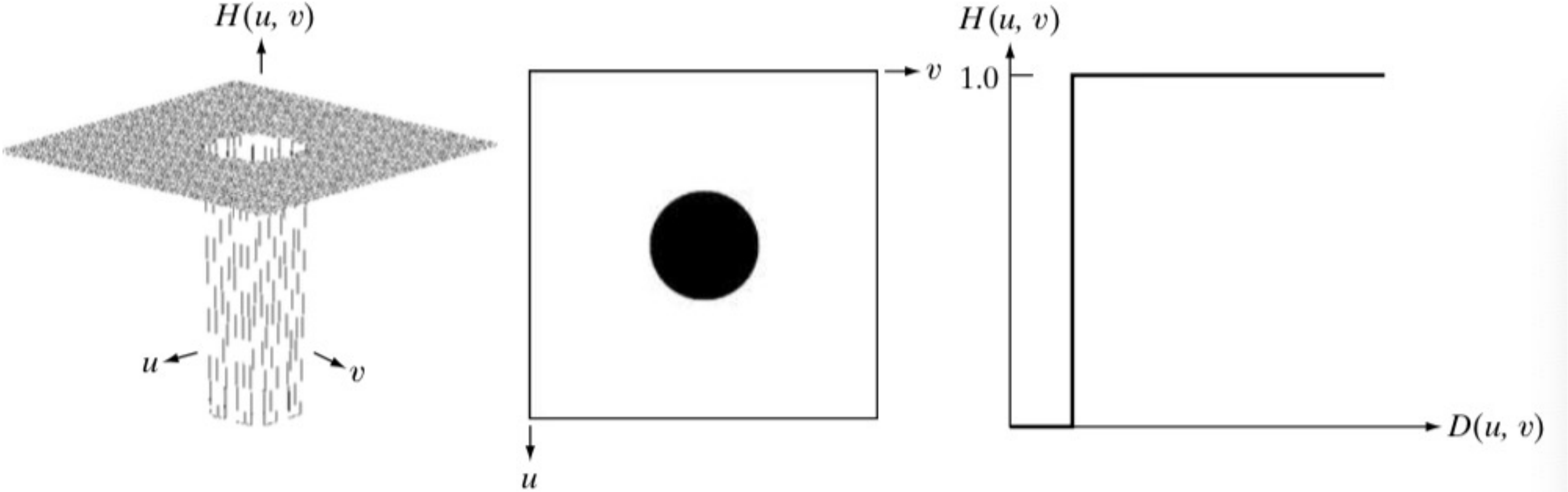

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.
ea
Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.
ea

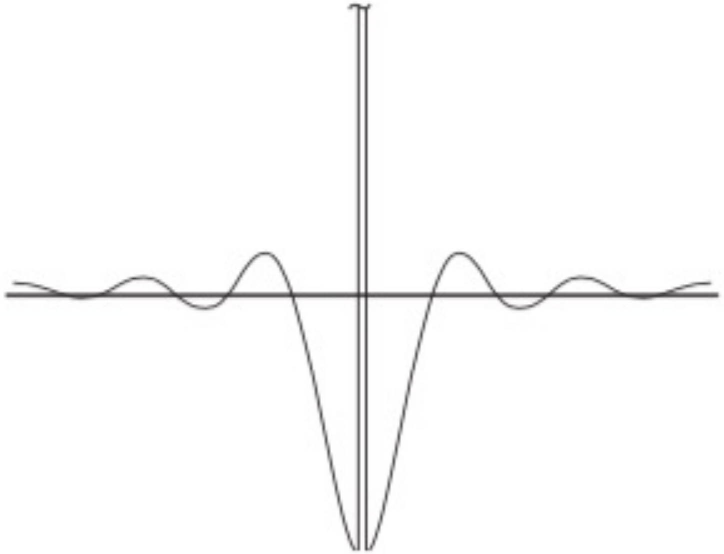
Image Sharpening in Frequency Domain

- Cutting off low frequency
  - $H_{HP}(u,v) = 1 - H_{LP}(u,v)$
- Ideal highpass filters
  - $H(u,v) = \begin{cases} 0 & if\ D(u,v) \leq D_0 \\ 1 & if\ D(u,v) > D_0 \end{cases}$



Ideal Highpass Filters

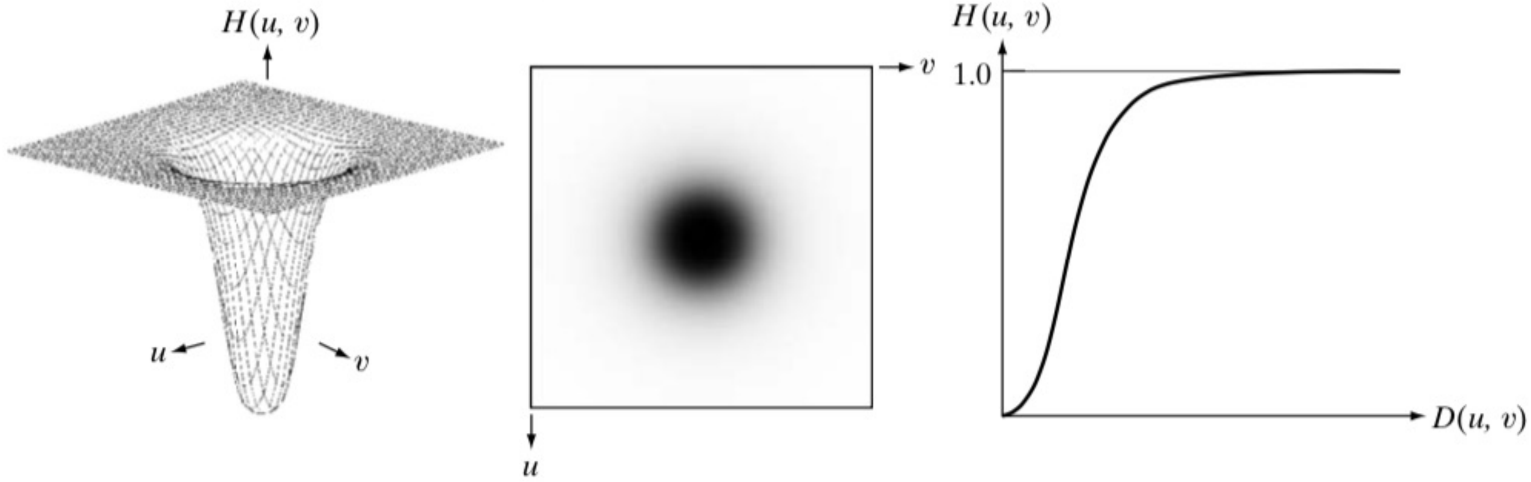
- Spatial representation

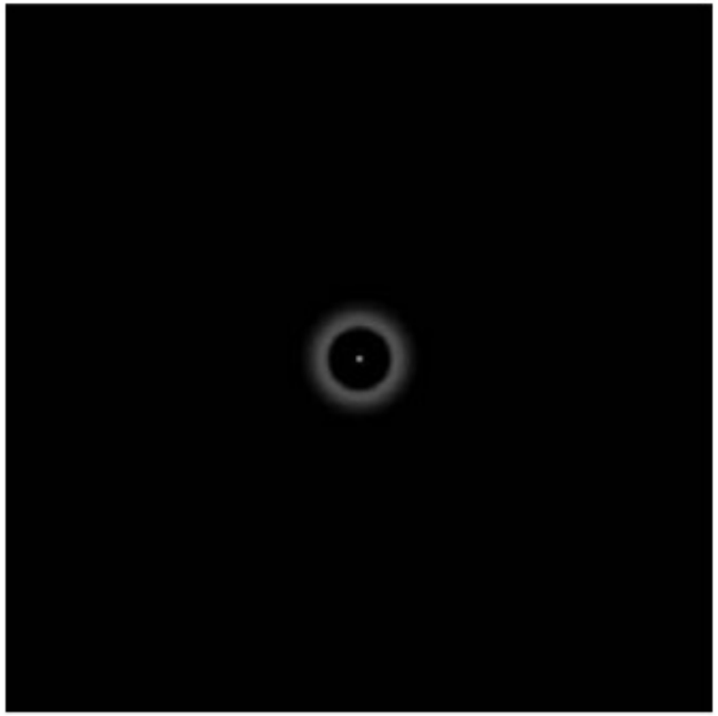


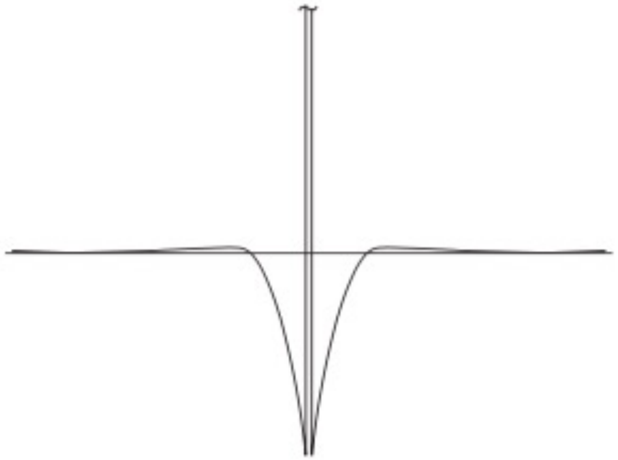
Ideal Highpass Filters

Butterworth Highpass Filters

- Definition
  - $H(u,v) = \frac{1}{1+[D_0/D(u,v)]^{2n}}$

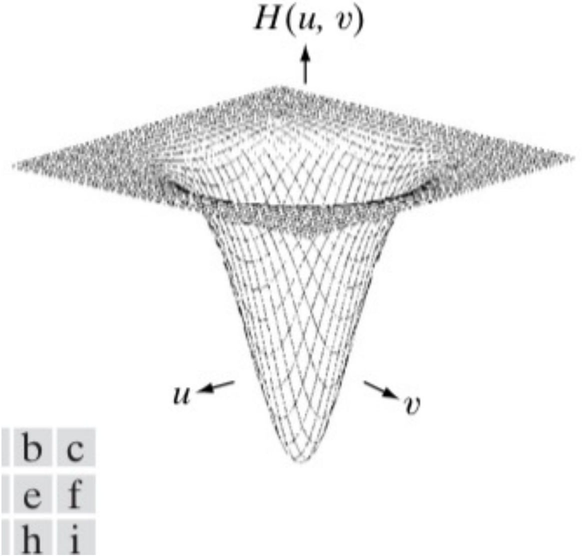


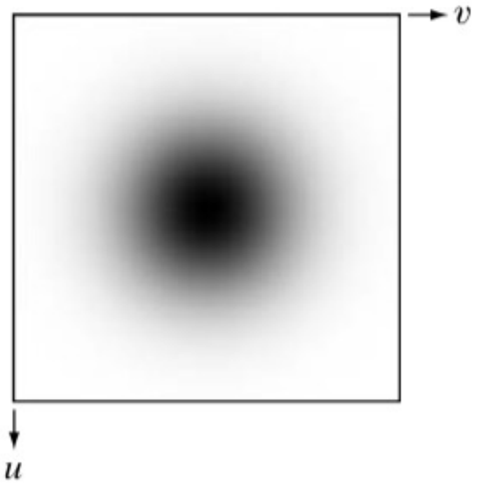


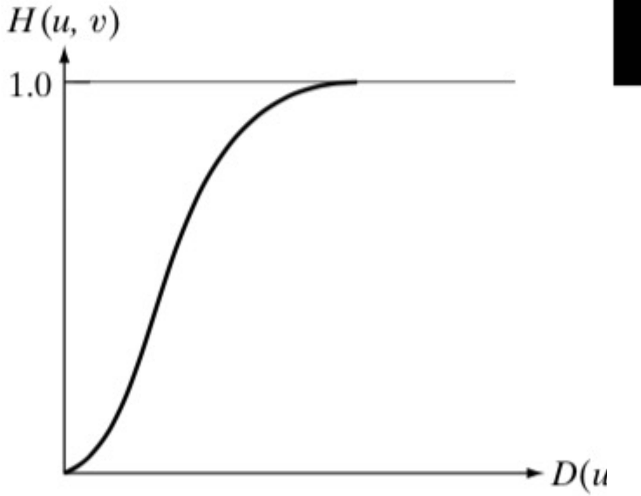


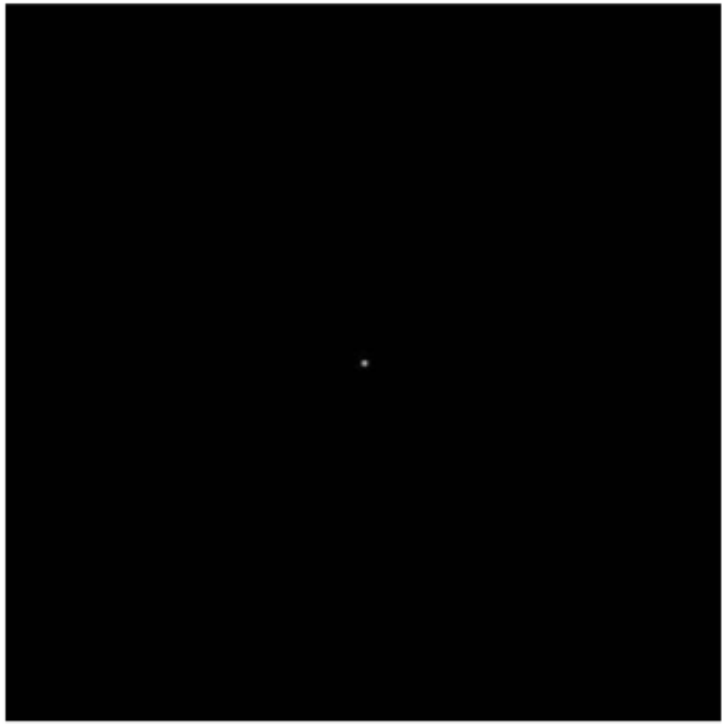
Gaussian Highpass Filters

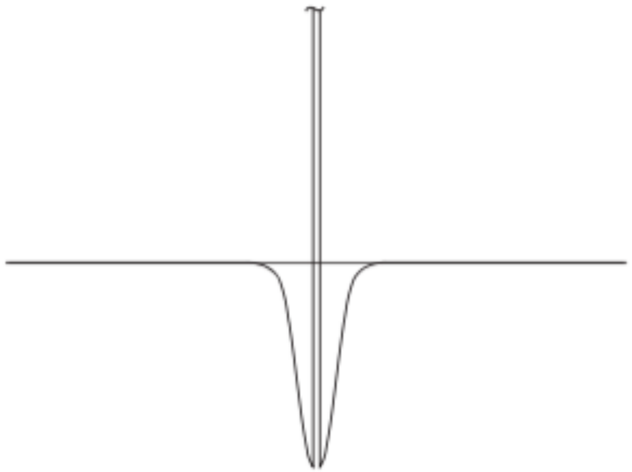
- Definition
    - $H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$











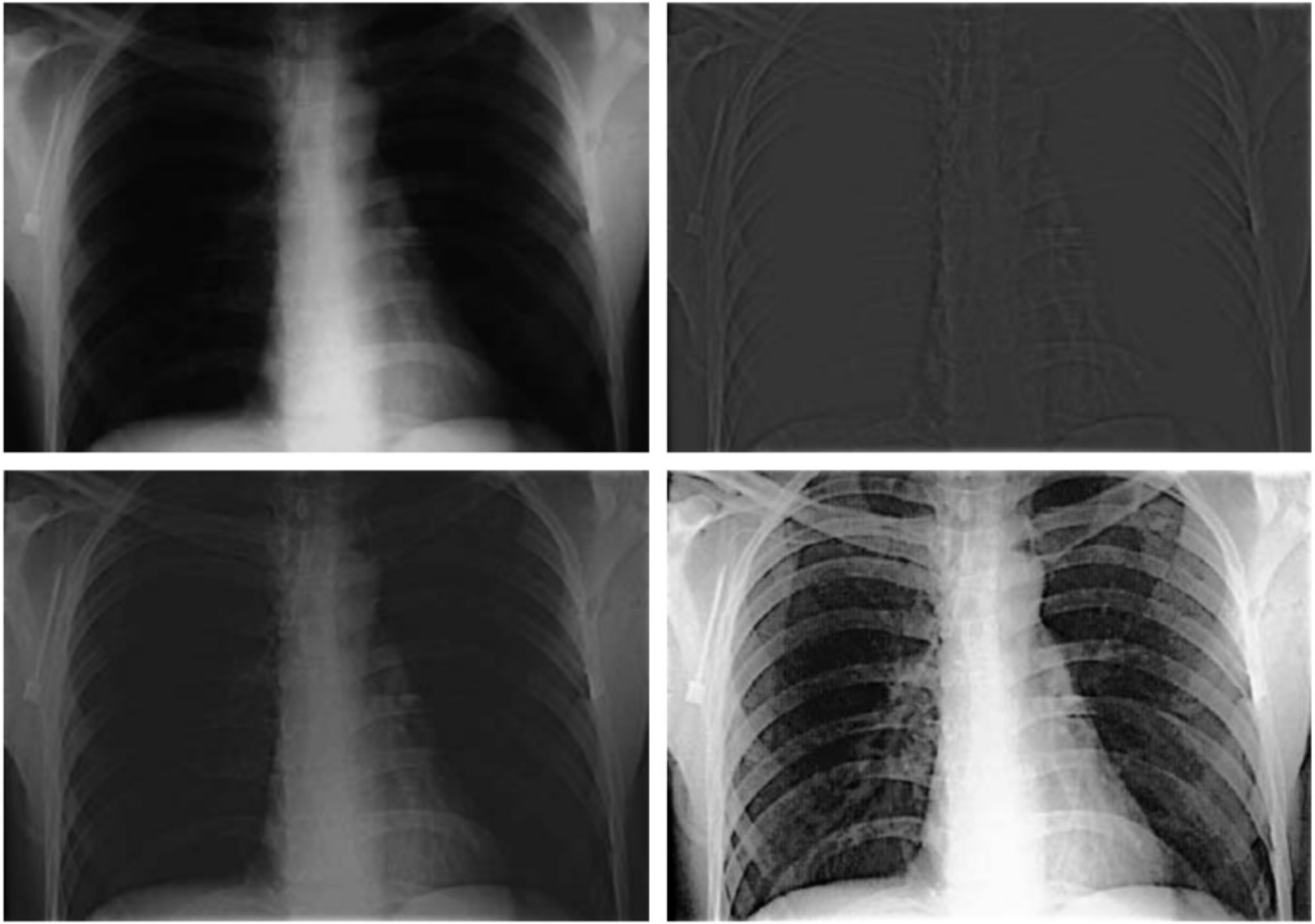
Laplacian in the Frequency Domain

- Laplacian filter
  - $H(u, v) = -4\pi^2(u^2 + v^2)$
  - $\nabla^2 f(x, y) = \mathfrak{I}^{-1}\{H(u, v)F(u, v)\}$
- Sharpening with Laplacian
  - $g(x, y) = f(x, y) + c\nabla^2 f(x, y)$

$$\begin{aligned} &= \mathfrak{I}^{-1}\{F(u, v) - H(u, v)F(u, v)\} \\ &= \mathfrak{I}^{-1}\{[1 - H(u, v)]F(u, v)\} \\ &= \mathfrak{I}^{-1}\{[1 + 4\pi^2 D^2(u, v)]F(u, v)\} \end{aligned}$$


Unsharp Masking

- Unsharp mask
  - $g_{mask}(x,y) = f(x,y) - f_{LP}(x,y)$
  - Where $f_{LP}(x,y) = \mathfrak{I}^{-1}[H_{LP}(u,v)F(u,v)]$
- Unsharp masking
  - $g(x,y) = f(x,y) + k \cdot g_{mask}(x,y)$
    $= \mathfrak{I}^{-1}\{\left[1 - k \cdot \left(1 - H_{LP}(u,v)\right)\right]F(u,v)\}$
    $= \mathfrak{I}^{-1}\{[1 + k \cdot H_{HP}(u,v)]\, F(u,v)\}$
- Frequency emphasizing filter
  - $g(x,y) = \mathfrak{I}^{-1}\{[k_1 + k_2 \cdot H_{HP}(u,v)]\, F(u,v)\}$

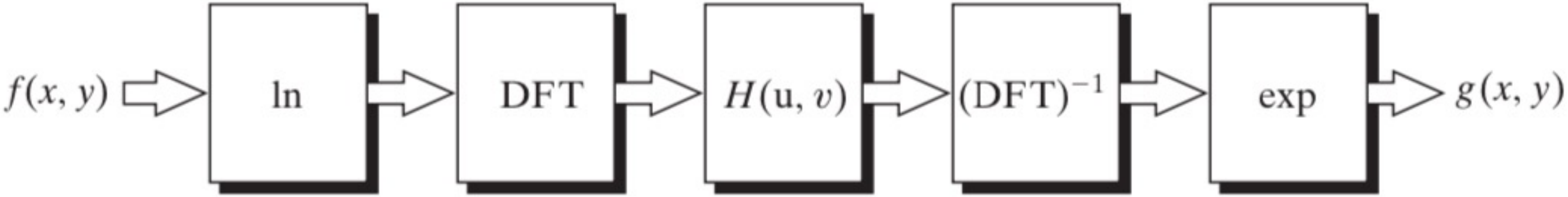


Homomorphic Filtering

- Illumination-reflectance model
  - Image multiplication
  - $f(x, y) = i(x, y)r(x, y)$
- However,
  - $\Im[f(x, y)] \neq \Im[i(x, y)]\Im[r(x, y)]$
- Logarithm
  - $\ln f(x, y) = \ln i(x, y) + \ln r(x, y)$
- Then
  - $\Im\{\ln f(x, y)\} = \Im\{\ln i(x, y)\} + \Im\{\ln r(x, y)\}$
  - $Z(u, v) = F_i(u, v) + F_r(u, v)$

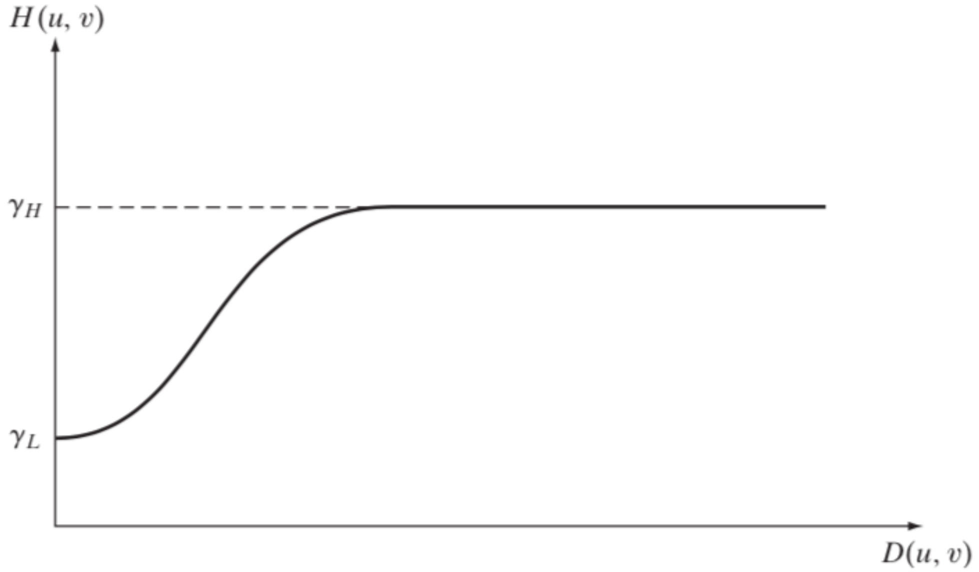
Homomorphic Filtering

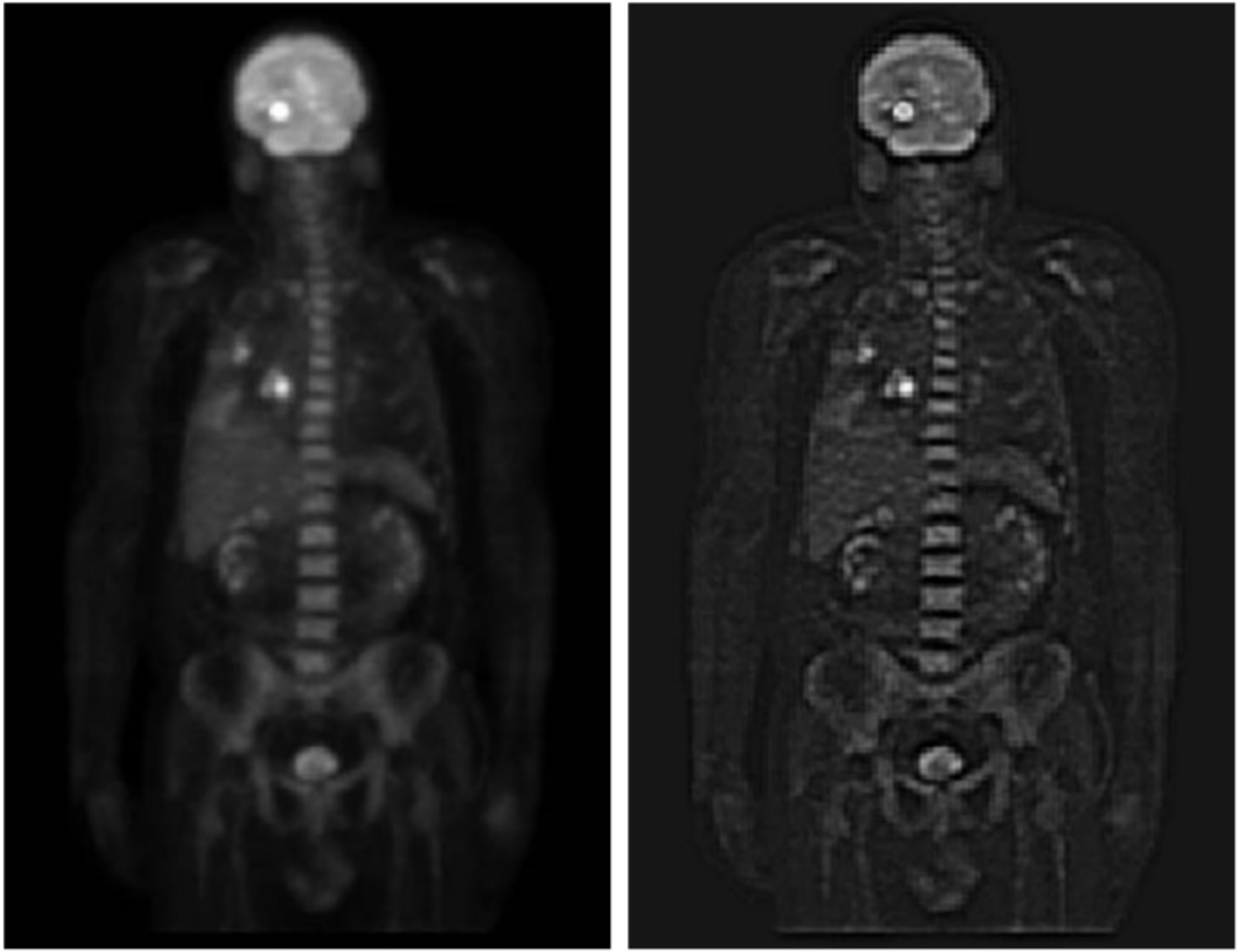
- Filtering Z using H
  - $S(u, v) = H(u, v)Z(u, v) = H(u, v)F_i(u, v) + H(u, v)F_r(u, v)$
- Back to spatial domain
  - $s(x, y) = \Im^{-1}\{S(u, v)\} = \Im^{-1}\{H(u, v)F_i(u, v)\} + \Im^{-1}\{H(u, v)F_r(u, v)\}$
- Therefore
  - $g(x, y) = e^{s(x,y)}$



Illumination-Reflectance Model

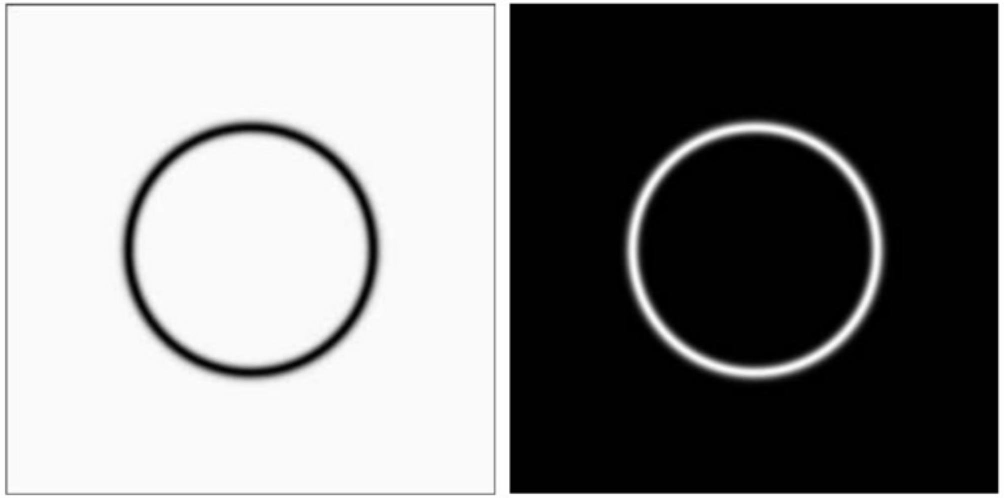
- Characteristics
  - Illumination: slow spatial variation
  - Reflectance: vary abruptly
- Controllable illumination-reflectance model
  - $H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\left[\frac{D^2(u,v)}{D_0^2}\right]}\right] + \gamma_L$



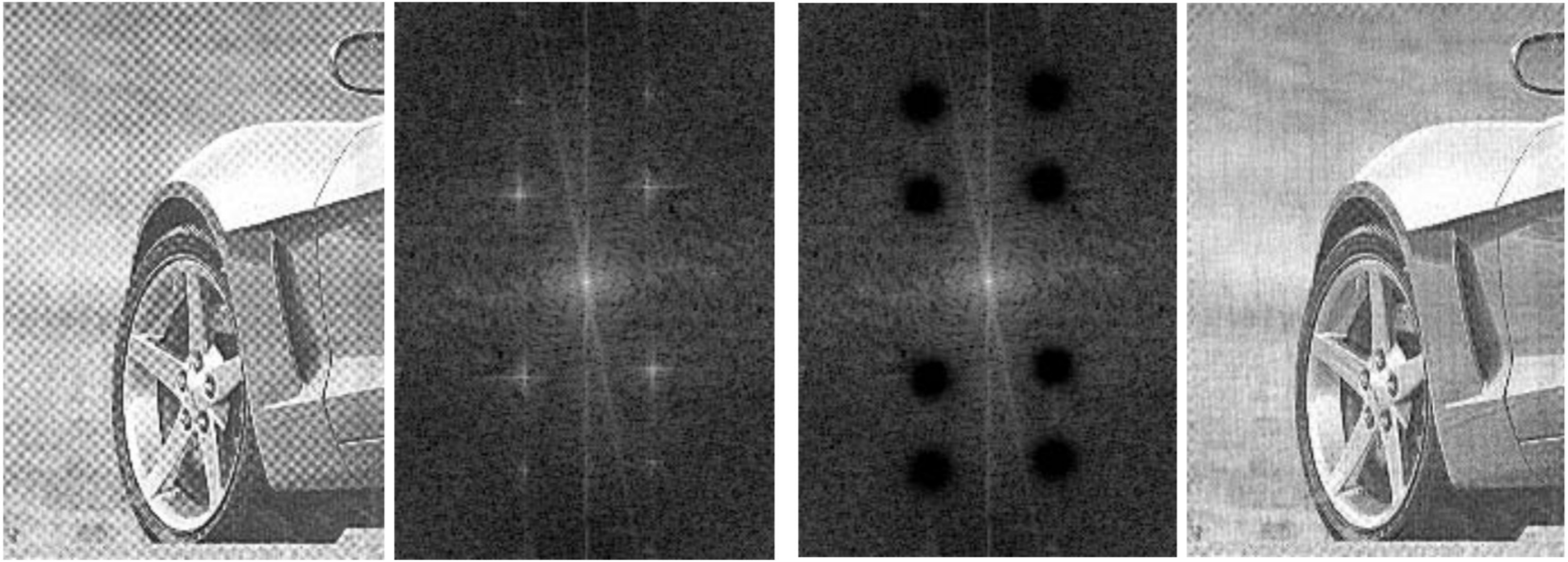


Selective Filtering

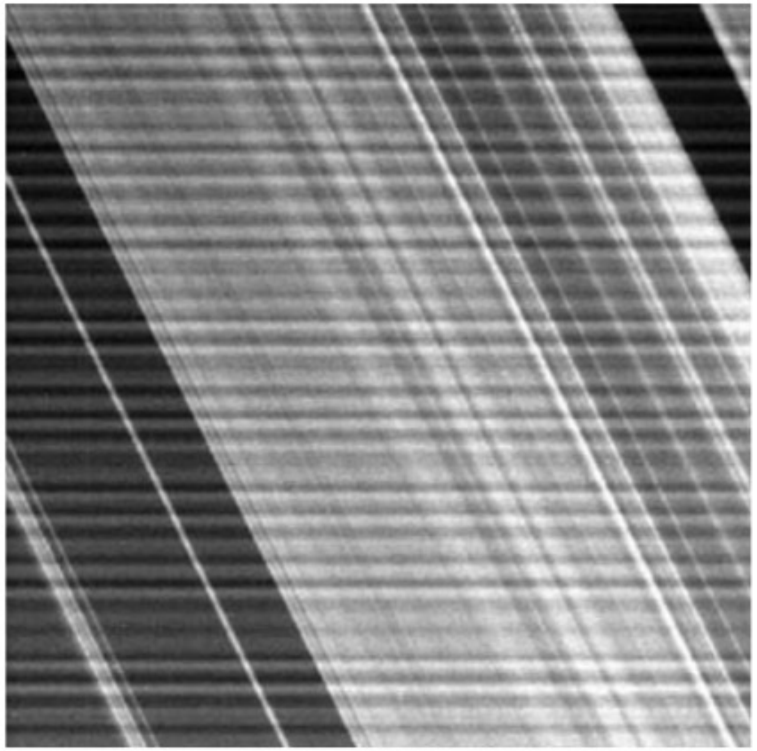
- Bandreject and Bandpass filters
  - Filtering out only some specific frequency
  - Or leave only some specific frequency
- Notch filters
  - $H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v)H_{-k}(u,v)$

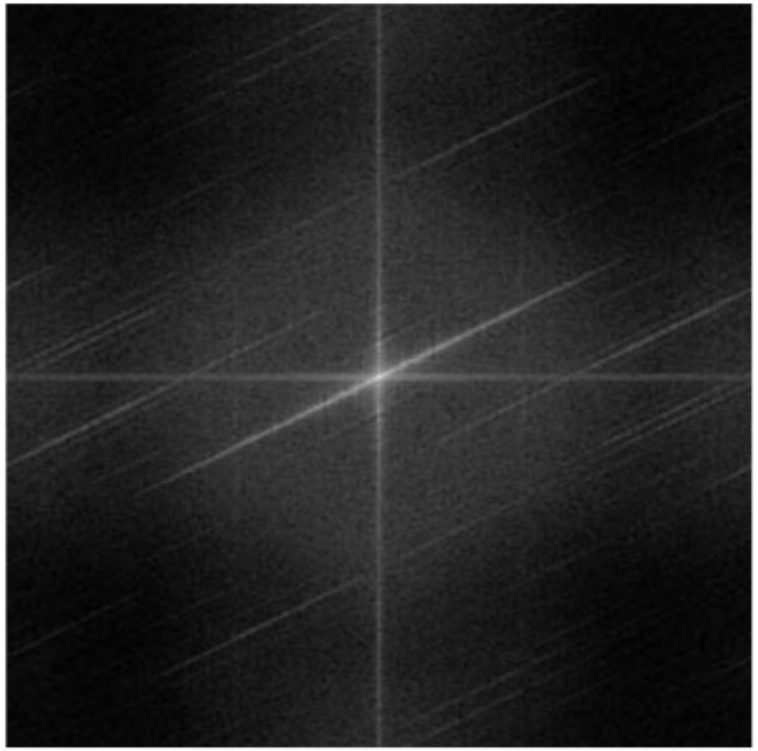


Notch Filter

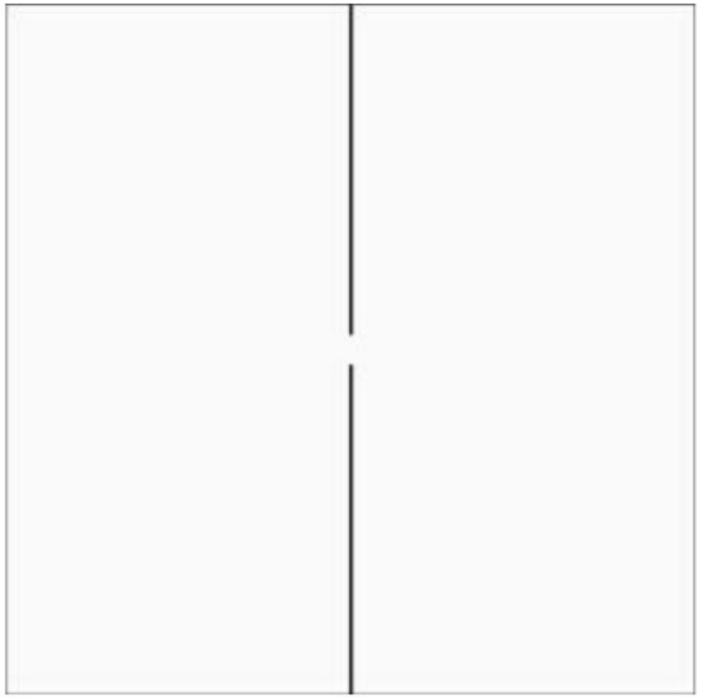


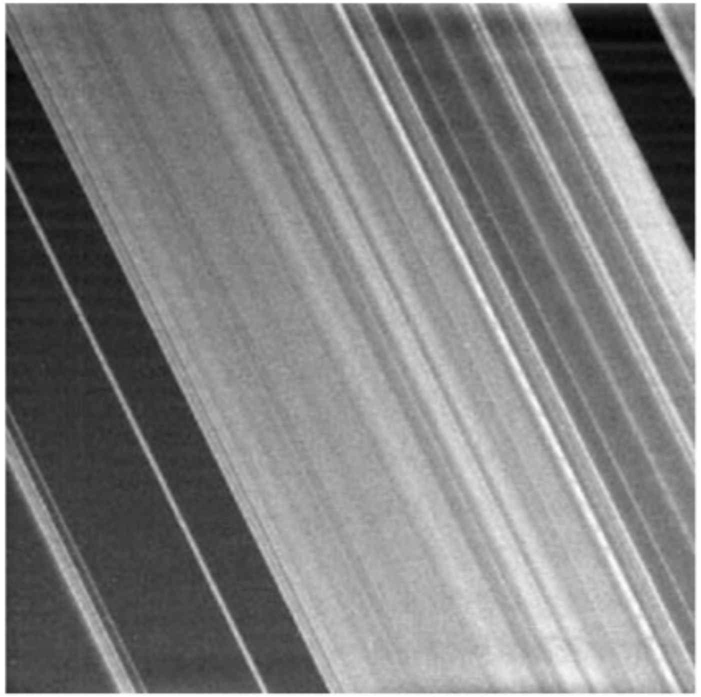
Notch Filter



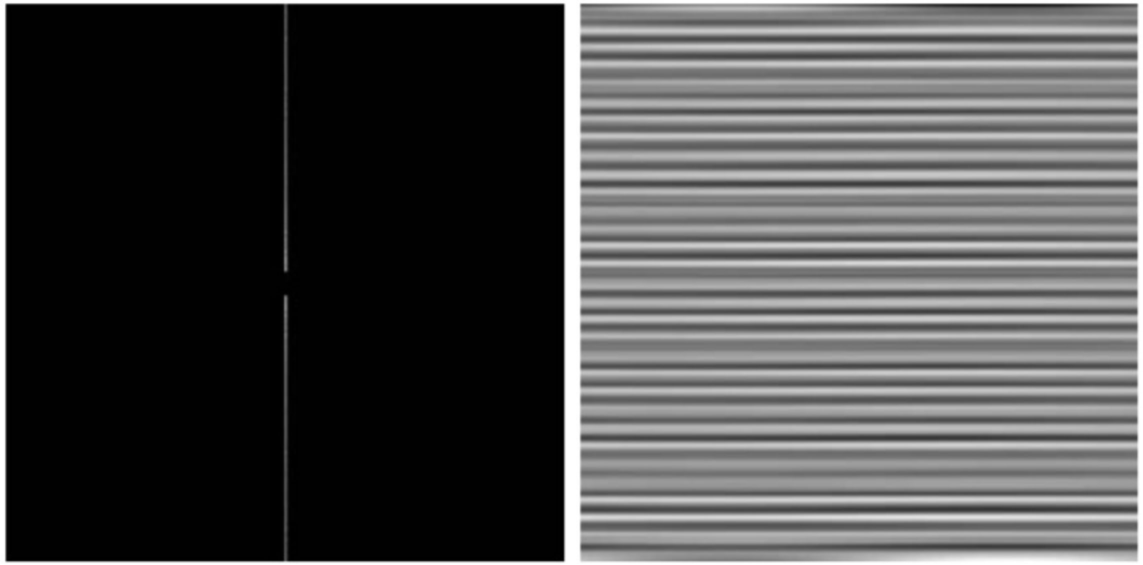


Notch Filter





Notch Filter



Summary

- Fourier Transform
  - Spatial domain -> frequency domain
  - Each pixel in $F(u, v)$ represents the frequency response of the input image
- Convolution
  - A spatial filter can be transformed to the frequency domain for faster convolution
- Frequency domain filters
  - Smoothing: filtering out high frequency components
  - Sharpening: enhancing high frequency components
  - Other filters: homomorphic filter, notch filter, …