MATH 303 SPRING 2025 — HOMEWORK 2

Due on Saturday, March 29, 2025, 20:00

Upload your solutions as a single PDF file to Gradescope before the deadline and assign the correct PDF page to each question on Gradescope. The uploaded solutions must also contain the questions, not only your answers.

Homework assignments will be graded based on:

- Correctness, completeness, and quality of the mathematical arguments and computations **(90%)**.
- Quality of the presentation **(5%)**. Using LaTeX is optimal. Equally acceptable are very clearly handwritten solutions on a tablet (using an app such as GoodNotes or Notability), or very clearly handwritten solutions in a neatly scanned PDF. Handwritten solutions with smudges or nonlinear presentation or improperly scanned (e.g., photos directly converted to PDF) are not acceptable.
- In all cases, PDF page numbers must be assigned to each answer on Gradescope **(5%)**.

Question 1 (25 points)

Consider the differential equation

$$x' = \cos x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{1.1}$$

(a) Show that Eq. (1.1) has a unique solution for any initial condition $x(0) = x_0$.

(b) Define $y = \sin x$ and obtain a differential equation for y in the form $y' = g(y)$. What is the range of possible values for y?

(c) Solve the initial value problem $y' = g(y)$, $y(0) = y_0$ for y_0 in the range of possible values for y.

(d) Determine the flow $\phi(t, x_0)$ for the system in Eq. (1.1).

(e) Confirm that the flow determined in the previous subquestion generates Eq. (1.1), that is, verify that $\partial\phi/\partial t\ (0, x_0) = \cos x_0$.

Solution

(a) The conditions of the uniqueness and existence theorem are satisfied since $f(x) = \cos x$ and $\partial f/\partial x = -\sin x$ are continuous for all $x \in \mathbf{R}$ and, of course, $x_0 \in \mathbf{R}$.

(b) We compute

$$y' = \cos x\, x' = (\cos x)^2 = 1 - (\sin x)^2 = 1 - y^2.$$

Since $y = \sin x$ and $x \in [-\pi/2, \pi/2]$ we have that $y \in [-1, 1]$.

(c) We solve the equation $y' = 1 - y^2$. We have

$$\frac{\mathrm{d}y}{1 - y^2} = \mathrm{d}t.$$

Then

$$\int \frac{\mathrm{d}y}{1 - y^2} = \operatorname{arctanh} y = t + c,$$

that is, $y = \tanh(t + c)$. The initial condition $y(0) = y_0$ gives $c = \operatorname{arctanh} y_0$ if $y_0 \in (-1, 1)$. Therefore, the solution is

$$y(t) = \tanh(t + \operatorname{arctanh} y_0),\ \ y_0 \in (-1, 1).$$

If $y_0 = \pm 1$, then the corresponding solution is $y(t) = \pm 1$.

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(d) The flow $\phi(t, x_0)$ is the solution to the initial value problem $x' = \cos x$, $x(0) = x_0$. We have $y_0 = \sin x_0$ and $x(t) = \arcsin y(t)$. Therefore,

$$\phi(t, x_0) = \arcsin(\tanh(t + \operatorname{arctanh}(\sin x_0))), \ x_0 \in (-\pi/2, \pi/2),$$

and $\phi(t, \pm\pi/2) = \pm\pi/2$.

(e) For $x_0 \in (-\pi/2, \pi/2)$ we write $b = \operatorname{arctanh}(\sin x_0)$ and we compute

$$\frac{\partial \phi}{\partial t}(t, x_0) = \frac{1}{(1 - \tanh^2(t + b))^{1/2}} \frac{1}{\cosh^2(t + b)}.$$

We have

$$\cosh(b) = \cosh(\operatorname{arctanh}(\sin x_0)) = \frac{1}{\sqrt{1 - \sin^2 x_0}} = \frac{1}{\cos x_0},$$

and

$$\tanh(b) = \tanh(\operatorname{arctanh}(\sin x_0)) = \sin x_0.$$

Therefore,

$$\frac{\partial \phi}{\partial t}(0, x_0) = \frac{1}{(1 - \sin^2 x_0)^{1/2}} \cos^2 x_0 = \cos x_0.$$

For $x_0 = \pm\pi/2$ we compute $\partial\phi/\partial t \ (0, \pm\pi/2) = 0 = \cos(\pm\pi/2)$. Therefore, in all cases we find $\partial\phi/\partial t \ (0, x_0) = \cos x_0$.

Question 2 (15 points)

Let $\phi(t, x_0)$ denote the flow of the autonomous equation $x' = f(x)$. We assume that f is complete and that f, $\partial f/\partial x$ are continuous for all $x \in \mathbf{R}$. Show that the flow of the equation $x' = -f(x)$ is $\psi(t, x_0) = \phi(-t, x_0)$.

Solution

We have that $\psi(0, x_0) = \phi(0, x_0) = x_0$. Moreover,

$$\frac{\partial \psi}{\partial t}(t, x_0) = \frac{\partial}{\partial t}[\phi(-t, x_0)] = -\frac{\partial \phi}{\partial t}(-t, x_0) = -f(\phi(-t, x_0)) = -f(\psi(t, x_0)).$$

This shows that $\psi(t, x_0)$ is the unique solution to the initial value problem $x' = -f(x)$, $x(0) = x_0$ and thus it is the flow for the equation $x' = -f(x)$.

More detailed derivation of $\partial\psi/\partial t \ (t, x_0) = - \ \partial\phi/\partial t \ (-t, x_0)$. We have

$$\begin{aligned}
\frac{\partial \psi}{\partial t}(t, x_0) &\overset{\text{def}}{=} \lim_{\varepsilon \to 0} \frac{\psi(t + \varepsilon, x_0) - \psi(t, x_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\phi(-t - \varepsilon, x_0) - \phi(-t, x_0)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\phi(-t + \varepsilon, x_0) - \phi(-t, x_0)}{-\varepsilon} = - \lim_{\varepsilon \to 0} \frac{\phi(-t + \varepsilon, x_0) - \phi(-t, x_0)}{\varepsilon} \\
&\overset{\text{def}}{=} -\frac{\partial \phi}{\partial t}(-t, x_0),
\end{aligned}$$

where to pass from the last limit on the first line to the first limit on the second line we used that, for any expression $g(\varepsilon)$ for which $\lim_{\varepsilon \to 0} g(\varepsilon)$ exists, we have $\lim_{\varepsilon \to 0} g(-\varepsilon) = \lim_{\varepsilon \to 0} g(\varepsilon)$.

One more remark: the expression $\partial/\partial t \ [\phi(-t, x_0)]$ appearing earlier is, by definition, equal to

$$\frac{\partial}{\partial t}[\phi(-t, x_0)] = \lim_{\varepsilon \to 0} \frac{\phi(-(t + \varepsilon), x_0) - \phi(-t, x_0)}{\varepsilon}.$$

Chain rule. To apply the chain rule without any shortcuts, define the function $F : \mathbf{R}^2 \to \mathbf{R}^2$ given by $F(t, x) = (-t, x)$. Then, $\psi = \phi \circ F$ and we have

$$\left[\frac{\partial \psi}{\partial t}(t, x_0), \frac{\partial \psi}{\partial x}(t, x_0)\right] = \left[\frac{\partial \phi}{\partial t}(F(t, x_0)), \frac{\partial \phi}{\partial x}(F(t, x_0))\right] \begin{bmatrix} \partial F_1/\partial t \ (t, x_0) & \partial F_1/\partial x \ (t, x_0) \\ \partial F_2/\partial t \ (t, x_0) & \partial F_2/\partial x \ (t, x_0) \end{bmatrix},$$

where $F_1(t,x) = -t$, $F_2(t,x) = x$, and thus

$$\left[\frac{\partial\psi}{\partial t}(t,x_0), \frac{\partial\psi}{\partial x}(t,x_0)\right] = \left[\frac{\partial\phi}{\partial t}(-t,x_0), \frac{\partial\phi}{\partial x}(-t,x_0)\right] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, once more

$$\frac{\partial\psi}{\partial t}(t,x_0) = -\frac{\partial\phi}{\partial t}(-t,x_0).$$

Question 3 (25 points)

Consider the differential equation

$$x' = ax - x^3, \quad x \in \mathbf{R}. \tag{3.1}$$

(a) Find the equilibria and draw the phase line of the system in Eq. (3.1) for $a < 0$, $a = 0$, $a > 0$. Determine the stability of the equilibria directly from the phase line.

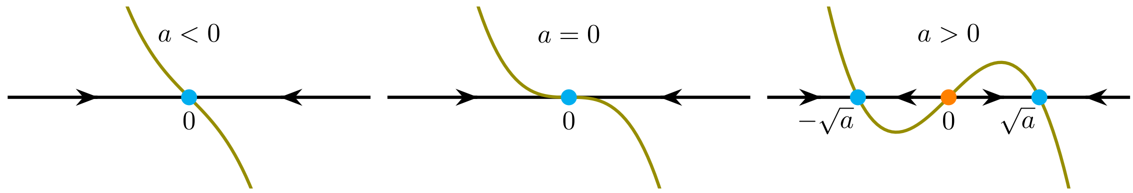
(b) Use linear stability analysis to determine the stability of the equilibria for $a \neq 0$. Why you cannot use linear stability analysis when $a = 0$?

(c) Draw the bifurcation diagram of Eq. (3.1), that is, draw a plot on the (a,x) plane showing the positions of the equilibria as a function of a. Use different colors (e.g., blue and red) or different types of lines (e.g., solid and dashed) to distinguish stable from unstable equilibria.

(d) Show that the conditions of fold bifurcation theorem are not satisfied at $a = 0$.

Solution

(a) Write $f(a,x) = ax - x^3$. The equilibria are solutions of the equation $f(a,x) = x(a - x^2) = 0$. Therefore, $x = 0$ is an equilibrium for all a. Moreover, for $a > 0$ there are two additional equilibria $x = \pm\sqrt{a}$, while for $a \leq 0$ there are no additional equilibria. The phase lines are shown below for different values of a.



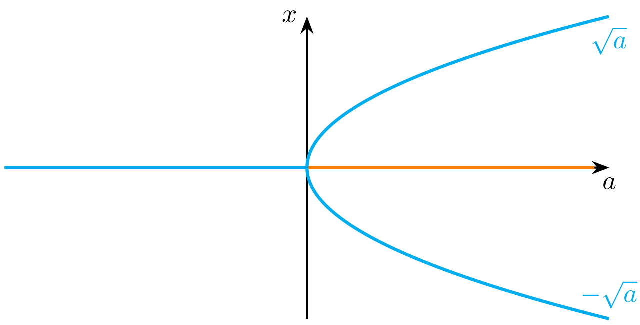
For $a \leq 0$ we have $f(a,x) > 0$ for $x < 0$, $f(a,0) = 0$, and $f(a,x) < 0$ for $x > 0$. This implies that $x = 0$ corresponds to a stable equilibrium when $a < 0$.

For $a > 0$ we have $f(a,x) = x(x - \sqrt{a})(x + \sqrt{a})$. Therefore, $f(a,x) > 0$ for $x < -\sqrt{a}$ or $0 < x < \sqrt{a}$, while $f(a,x) < 0$ for $-\sqrt{a} < x < 0$ or $x > \sqrt{a}$. This implies that the equilibria $\pm\sqrt{a}$ are stable, while the equilibrium 0 is unstable.

(b) For the linear stability analysis we compute $f_x(a,x) = a - 3x^2$. That is, $f_x(a,0) = a$, implying that for $a > 0$, the equilibrium $x = 0$ is unstable, while for $a < 0$ it is stable. For the additional equilibria $\pm\sqrt{a}$ that exist for $a > 0$ we have $f_x(a, \pm\sqrt{a}) = -2a < 0$. This means that the equilibria $\pm\sqrt{a}$ are stable.

For $a = 0$, there is only one equilibrium at $x = 0$ and we have $f_x(0,0) = 0$. Therefore, in this case the linearized equation cannot be used to determine the stability.

(c) We draw on the (a,x) plane the positions of the equilibria. We mark the asymptotically stable equilibria with the cyan color and the unstable equilibria with orange.



(d) We check that the first two conditions are satisfied: $f(0,0) = f_x(0,0) = 0$. However, $f_a(0,0) = 0$ and $f_{xx} = 0$, showing that the last two conditions of the theorem are not satisfied.

Question 4 (25 points)

Consider the differential equation

$$\theta' = a - \sin\theta, \quad \theta \in S^1, \tag{4.1}$$

where θ represents the angle on the circle $S^1 = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$.

(a) Find the equilibria and draw the phase space of the system in Eq. (4.1) for $0 \le a < 1$, $a = 1$, $a > 1$. Determine the stability of the equilibria directly from the phase space.

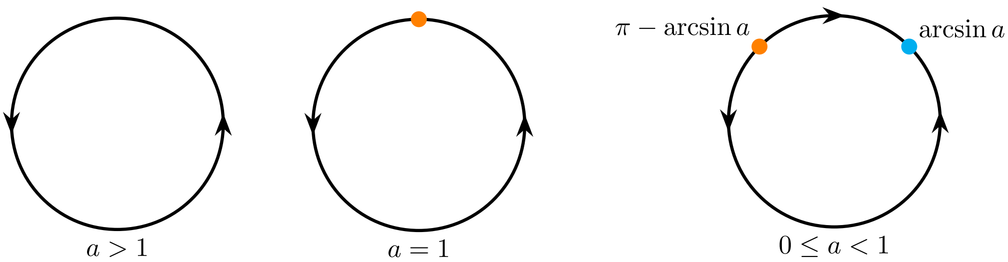
(b) Use linear stability analysis to determine the stability of the equilibria for $0 \le a < 1$ or $a > 1$.

(c) Draw the bifurcation diagram of Eq. (4.1), that is, draw a plot on the (a, θ) plane showing the positions of the equilibria as a function of a — to represent S^1 use the interval $[0, 2\pi)$. Use different colors (e.g., blue and red) or different types of lines (e.g., solid and dashed) to distinguish stable from unstable equilibria.

(d) Show that the conditions of the fold bifurcation theorem are satisfied at $a = 1$.

Solution

(a) Write $f(a,\theta) = a - \sin\theta$. The equation $\sin\theta = a$ has no solutions when $a > 1$, it has a single solution $\theta_0 = \pi/2$ when $a = a_0 = 1$, and it has two solutions $\arcsin(a)$ and $\pi - \arcsin(a)$ when $0 \le a < 1$. We draw the phase space for these cases in the following pictures.



For $a > 1$ we have $f(a,\theta) > 0$ for all $\theta \in S^1$. Therefore, the solution curve moves counterclockwise as shown in the picture above. For $a = 1$ we have $f(1,\theta) > 0$ for all $\theta \ne \pi/2$ and $f(1, \pi/2) = 0$. From the phase space we see that the equilibrium $\theta_0 = \pi/2$ is unstable. Finally, for $a > 1$ we observe that $\arcsin a$ is stable, while $\pi - \arcsin a$ is unstable.

(b) We compute $f_\theta(a,\theta) = -\cos\theta$. For $a > 1$ there are no equilibria. For $0 \le a < 1$ we find

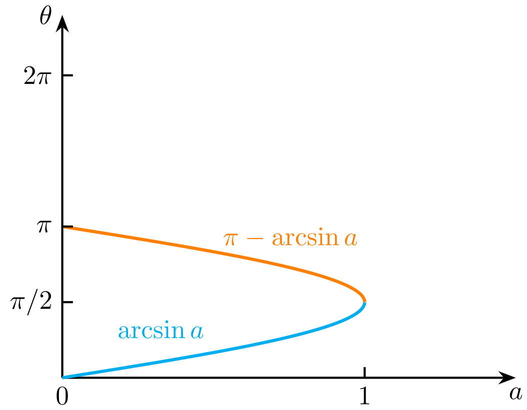
$$f_\theta(a, \arcsin a) = -\cos(\arcsin a) = -\sqrt{1 - a^2} < 0,$$

thus the equilibrium $\arcsin a$ is stable. Moreover, we find

$$f_\theta(a, \pi - \arcsin a) = -\cos(\pi - \arcsin a) = \cos(\arcsin a) = \sqrt{1 - a^2} > 0,$$

thus the equilibrium $-\arcsin a$ is unstable.

(c) We draw on the (a, θ) plane the positions of the equilibria. We mark the stable equilibria with the cyan color and the unstable equilibria with orange.



(d) We check that $f(1, \pi/2) = 0$, $f_\theta(1, \pi/2) = -\cos(\pi/2) = 0$, $f_a(1, \pi/2) = 1 \neq 0$, and $f_{\theta\theta}(1, \pi/2) = \sin(\pi/2) = 1 \neq 0$.

Total points: 90