# Energy method and Lyapunov's stability theorem

## MATH 303 ODE and Dynamical Systems

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# Energy method

# Mechanical systems

We consider planar systems of the form

$$\begin{aligned} x' &= y \\ y' &= f(x) \end{aligned}$$

**Remark.** Such systems appear commonly in physics (classical mechanics) and they are often called **mechanical systems**. However, even if we get a system of this type that does not arise from a physics problem we can still use the same techniques to analyze its phase portrait.

# Potential and energy conservation

Given the function $f(x)$ we define the **potential** $U(x)$ as an antiderivative of $-f(x)$, that is, $U'(x) = -f(x)$. Then the **energy** function

$$E(x, y) = \frac{1}{2}y^2 + U(x)$$

is a conserved quantity.

We can check this directly as follows:

$$\begin{aligned}\frac{d}{dt}[E(x(t), y(t))] &= y(t)\, y'(t) + \frac{dU}{dx}(x(t))\, x'(t) \\ &= y(t)\, f(x(t)) + (-f(x(t)))y(t) = 0.\end{aligned}$$

Alternatively, we can find the form of $E(x, y)$ by reducing the system to one dimension. We find the separable reduced equation

$$\frac{dy}{dx} = \frac{f(x)}{y},$$

which can be integrated to give

$$\frac{1}{2}y^2 = c + \int f(x)dx.$$

Solving for $c$ we find

$$c = E(x, y) := \frac{1}{2}y^2 + \int (-f(x))\, dx = \frac{1}{2}y^2 + U(x),$$

where $U(x)$ is any anti-derivative of $-f(x)$, that is, $U'(x) = -f(x)$.

# Level sets

The fact that $E(x, y)$ is a conserved quantity allows us to easily derive the phase portrait of the original system.

Suppose that we fix the value of $E(x, y)$ to a value $h$ and we want to draw the corresponding level set $E(x, y) = h$.

Solving the equation $h = \frac{1}{2}y^2 + U(x)$ for $y$ we get

$$y = \pm\sqrt{2(h - U(x))}.$$

# Remarks

1. The expression for $y$ makes sense only when $h - U(x) \geq 0$. Therefore, the graph of $y$ is defined only over the subsets of the $x$-axis where $U(x) \leq h$.

2. Since we have $y = \pm\sqrt{2(h - U(x))}$ the level sets consists of two parts, one above the $x$-axis ($y \geq 0$) and one below the $x$-axis ($y \leq 0$), and each one of these parts is the reflection of the other.

3. The intersections of the level set $E(x, y) = h$ with the $x$-axis are the points $(x,0)$ where $E(x,0) = U(x) = h$.

4. The absolute value $\left|y\right|$ on a given level set is larger for larger values of $h - U(x)$.

# Example

Consider the system

$$x' = y,$$
$$y' = -a^2x, \quad a > 0.$$

In this case, $f(x) = -a^2x$ and therefore we can take $U(x) = \dfrac{1}{2}a^2x^2$. The energy is

$$E(x, y) = \frac{1}{2}y^2 + \frac{1}{2}a^2x^2.$$

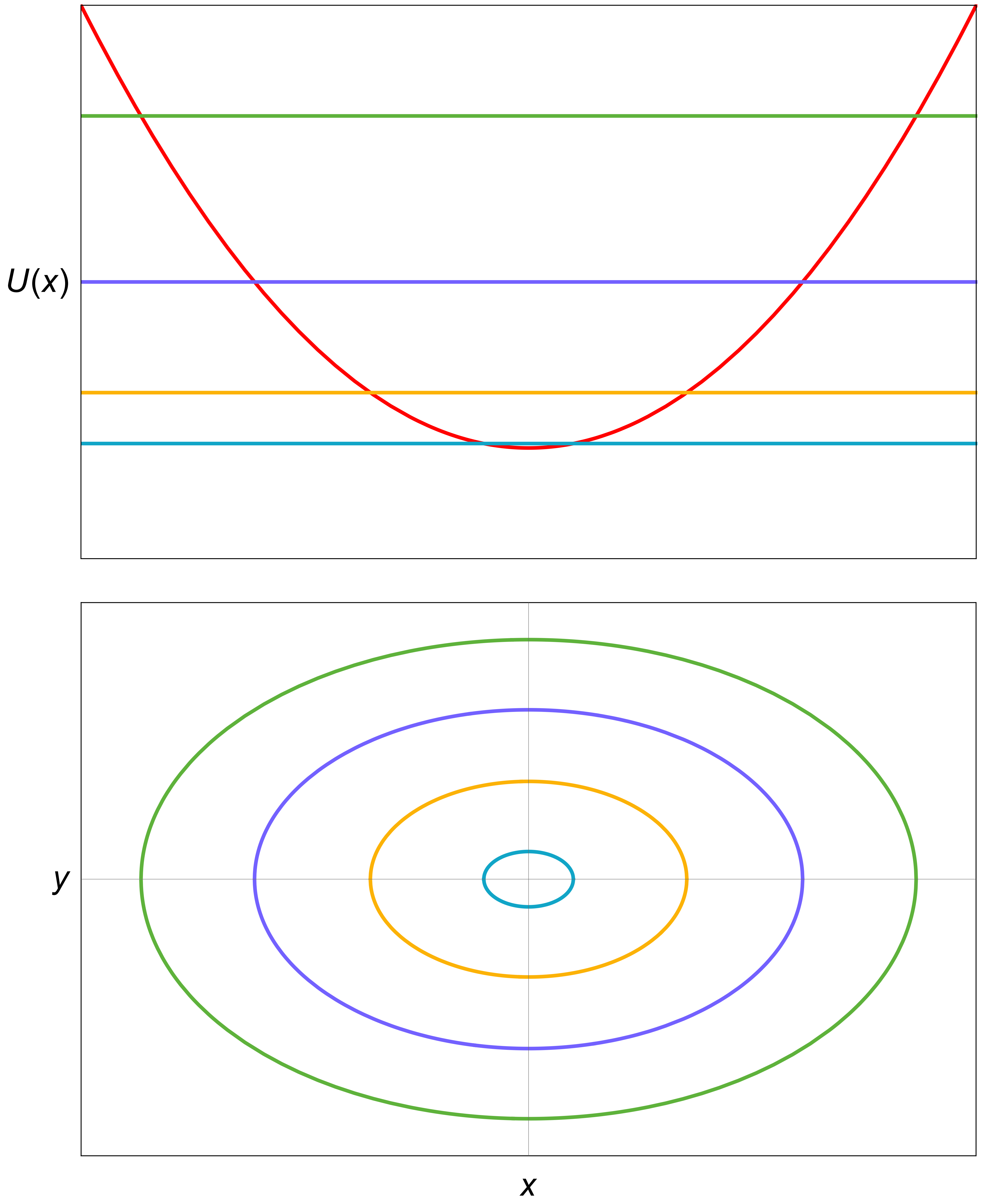
Clearly, the level sets are ellipses for $h > 0$, a single point for $h = 0$, and are not defined for $h < 0$. We will temporarily ignore this fact and try to use the remarks from the previous slide to draw the phase portrait.

At the right we have drawn the graph of the potential $U(x)$ together with different values of $h$ (top) and the corresponding level curves of $E(x, y)$ (bottom).

Note that for each value of $h$ the corresponding level curve is defined only in the region where $U(x)$ is below $h$.

All level curves are symmetric under reflections through the $x$-axis (changing $y$ to $-y$).

Also, the level curves are furthest away from the $x$-axis at $x = 0$ since at that point the distance between $U(x)$ and $h$ is maximal.



# One more remark

Recall that the level set $E(x, y) = h$ intersects the $x$-axis at points $x_i$ with $U(x_i) = h$. We will show that if $U'(x_i) \neq 0$ then the corresponding level curve (if defined on one side of $x_i$) has "infinite slope", that is, it is vertical there.

Consider the upper side of the level curve so that $y = \sqrt{2(h - U(x))}$. We find

$$\frac{dy}{dx} = -\frac{U'(x)}{\sqrt{2(h - U(x))}}.$$

Suppose that for $x \leq x_i$ we have $U(x) \leq U(x_i) = h$ and $U'(x_i) > 0$. Then

$$\lim_{x \to x_i^-} \frac{dy}{dx} = -\lim_{x \to x_i^-} \frac{U'(x)}{\sqrt{2(h - U(x))}} = -\infty.$$

# Example

Consider the system

$$
\begin{aligned}
x' &= y, \\
y' &= b^2 x, \quad b > 0.
\end{aligned}
$$

In this case, $f(x) = b^2 x$ and therefore we can take $U(x) = -\dfrac{1}{2}b^2x^2$. The energy is

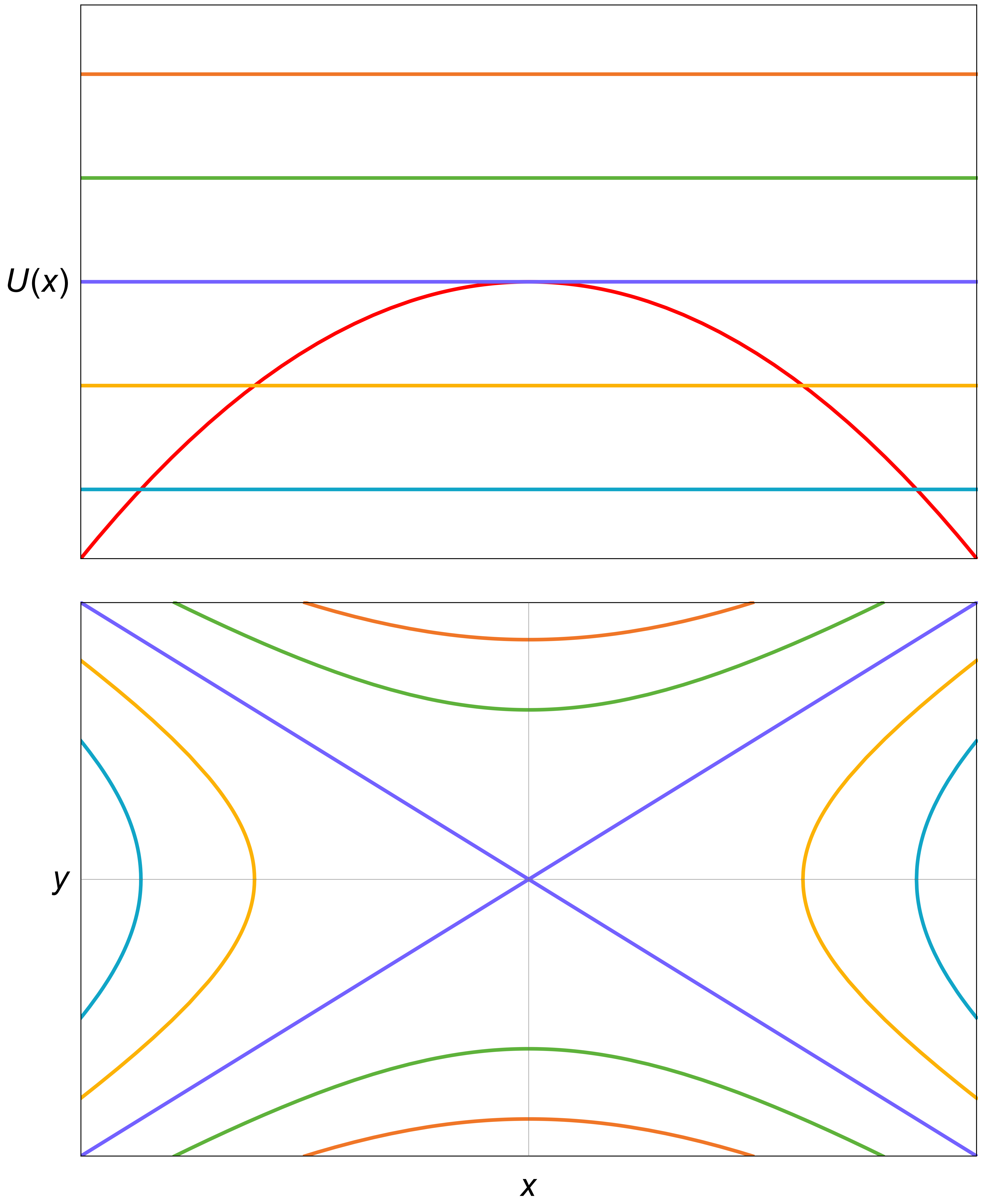
$$E(x, y) = \frac{1}{2}y^2 - \frac{1}{2}b^2x^2.$$

The level sets are hyperbolas when $h \neq 0$. For $h = 0$ we have the straight lines $y = \pm bx$.

At the right we have drawn the graph of the potential $U(x)$ together with different values of $h$ (top) and the corresponding level curves of $E(x, y)$ (bottom).

Note again that for each value of $h$ the corresponding level curve is defined only in the region where $U(x)$ is below $h$.

The level curve for $h = 0$ (blue) does not become vertical at the origin. This happens because $U'(0) = 0$ and therefore the previous argument does not apply.



# Equilibria

The equilibria of the system

$$x' = y$$
$$y' = f(x)$$

are the points $(x_e, 0)$ for which $f(x_e) = 0$. Since $U'(x) = -f(x)$ we conclude that at an equilibrium we have $U'(x_e) = 0$.

This means that critical points of the potential $U(x)$ correspond to equilibria of the system. In particular, maxima and minima of $U(x)$ correspond to equilibria.

# Linearization

The Jacobian matrix corresponding to $\boldsymbol{F} = \langle y, f(x) \rangle$ is given by

$$D\boldsymbol{F}(x, y) = \begin{bmatrix} 0 & 1 \\ f'(x) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -U''(x) & 0 \end{bmatrix}.$$

We assume that $f(x)$ and $f'(x)$ are continuous functions in a neighborhood of a point $x_e$ with $f(x_e) = 0$, and that $U''(x_e) \neq 0$.

We distinguish two cases. First, if $U''(x_e) > 0$ then the potential has a minimum at $x_e$.

The eigenvalues of the corresponding linear system are $\pm i\sqrt{U''(x_e)}$ and therefore the equilibrium for the linear system is a center. The Hartman-Grobman theorem cannot be used in this case.

The Taylor series of the potential $U(x)$ up to quadratic terms is

$$U(x) \approxeq U(x_e) + \frac{1}{2}U''(x_e)(x - x_e)^2.$$

This quadratic expression is essentially the expression $U(x) = \dfrac{1}{2}a^2x^2$ that we met in an earlier example.

Even though the linearization theorem does not allow us to determine the stability of the equilibrium, because $E(x, y)$ is a conserved quantity and near the equilibrium it is approximately

$$E(x, y) \approxeq U(x_e) + \frac{1}{2}y^2 + \frac{1}{2}U''(x_e)(x - x_e)^2$$

we conclude that the level curves near $(x_e, 0)$ are approximate ellipses.

Second, if $U''(x_e) < 0$ then the potential has a maximum at $x_e$.

The eigenvalues of the corresponding linear system are $\pm\sqrt{-U''(x_e)}$ and therefore the equilibrium for the linear system is a saddle. The linearization theorem then ensures that the equilibrium $(x_e, 0)$ is also a saddle for the full system.

We can work similarly as for the case $U''(x_e) > 0$ to write

$$U(x) \approxeq U(x_e) + \frac{1}{2}U''(x_e)(x - x_e)^2.$$

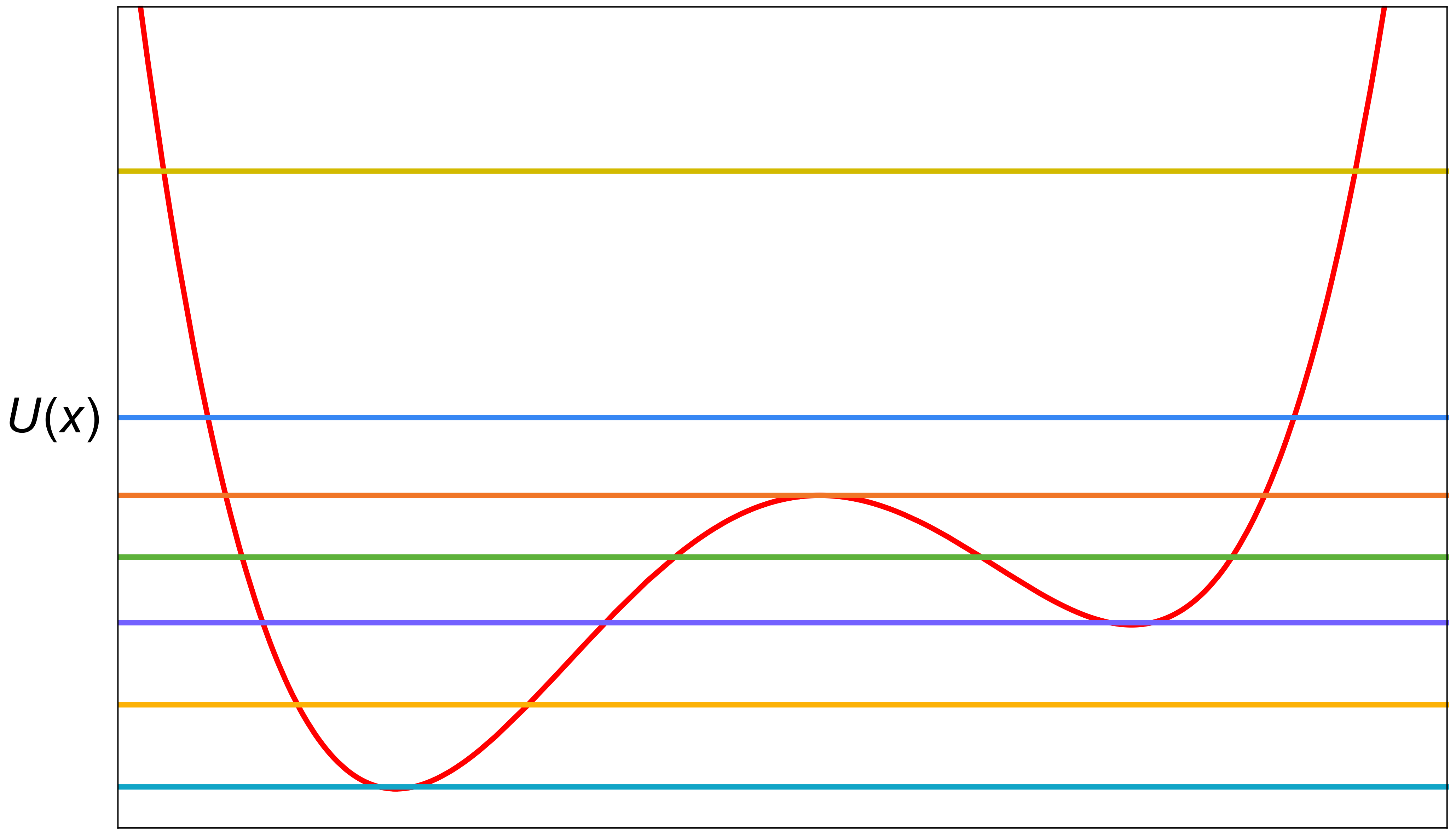
Since $U''(x_e) < 0$ this quadratic expression is essentially the same as the expression $U(x) = -\frac{1}{2}b^2x^2$ that we also met in an earlier example.

# Practice problem

For the potential function

$$U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{10}x$$

draw the corresponding phase portrait.

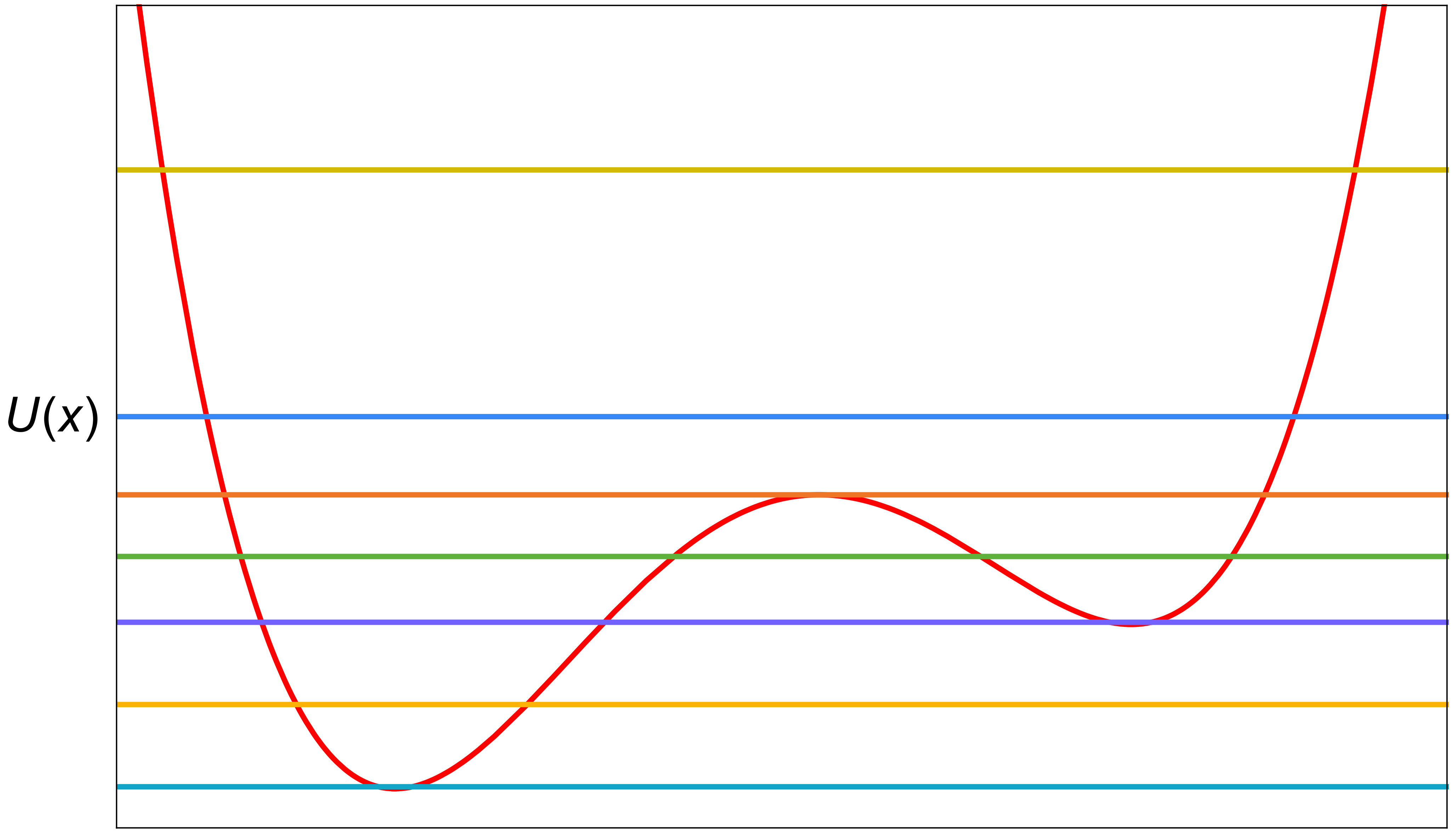


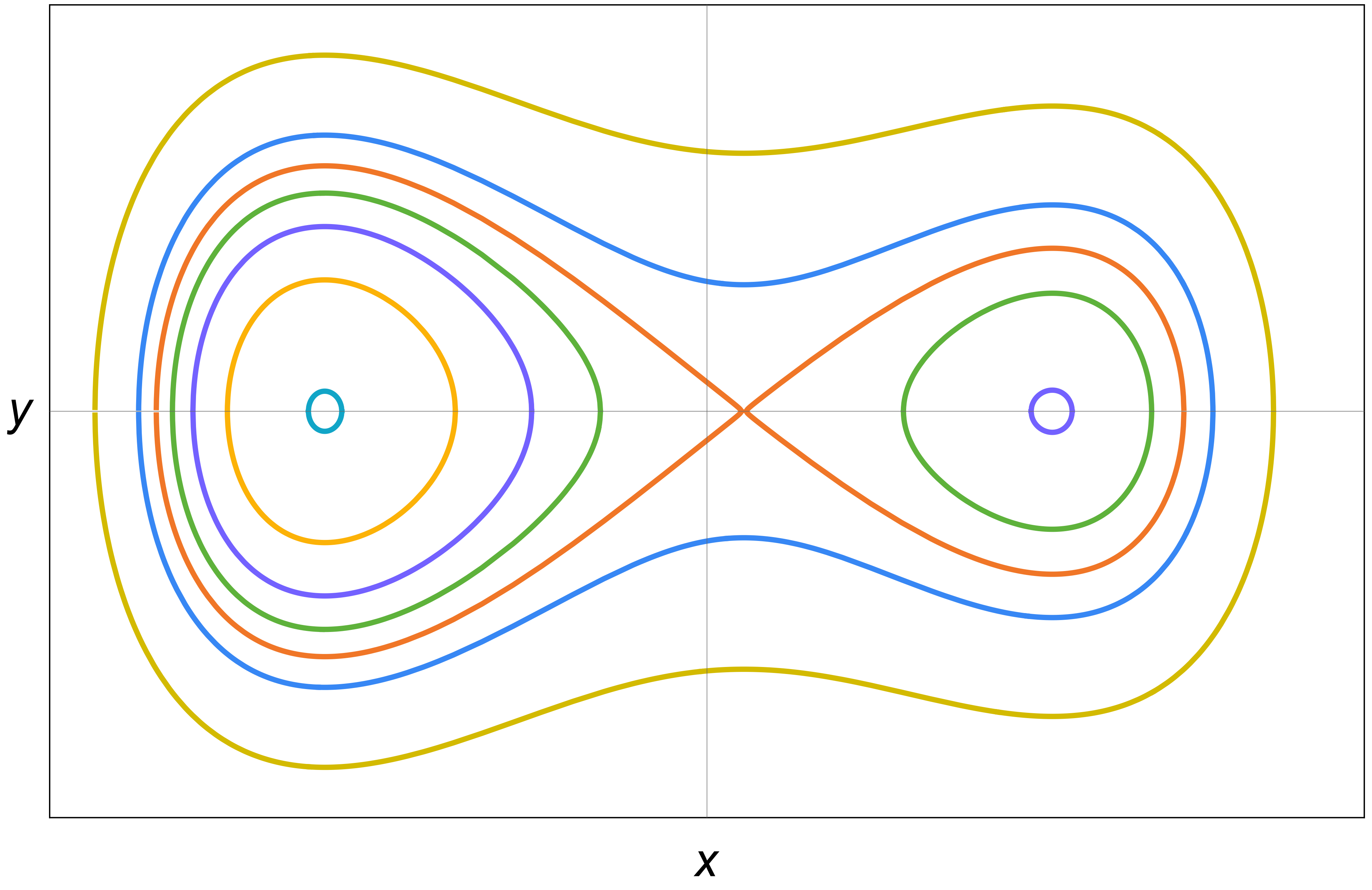
# Answer

The graph of

$$U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{10}x$$

and the corresponding phase portrait are shown at the right.



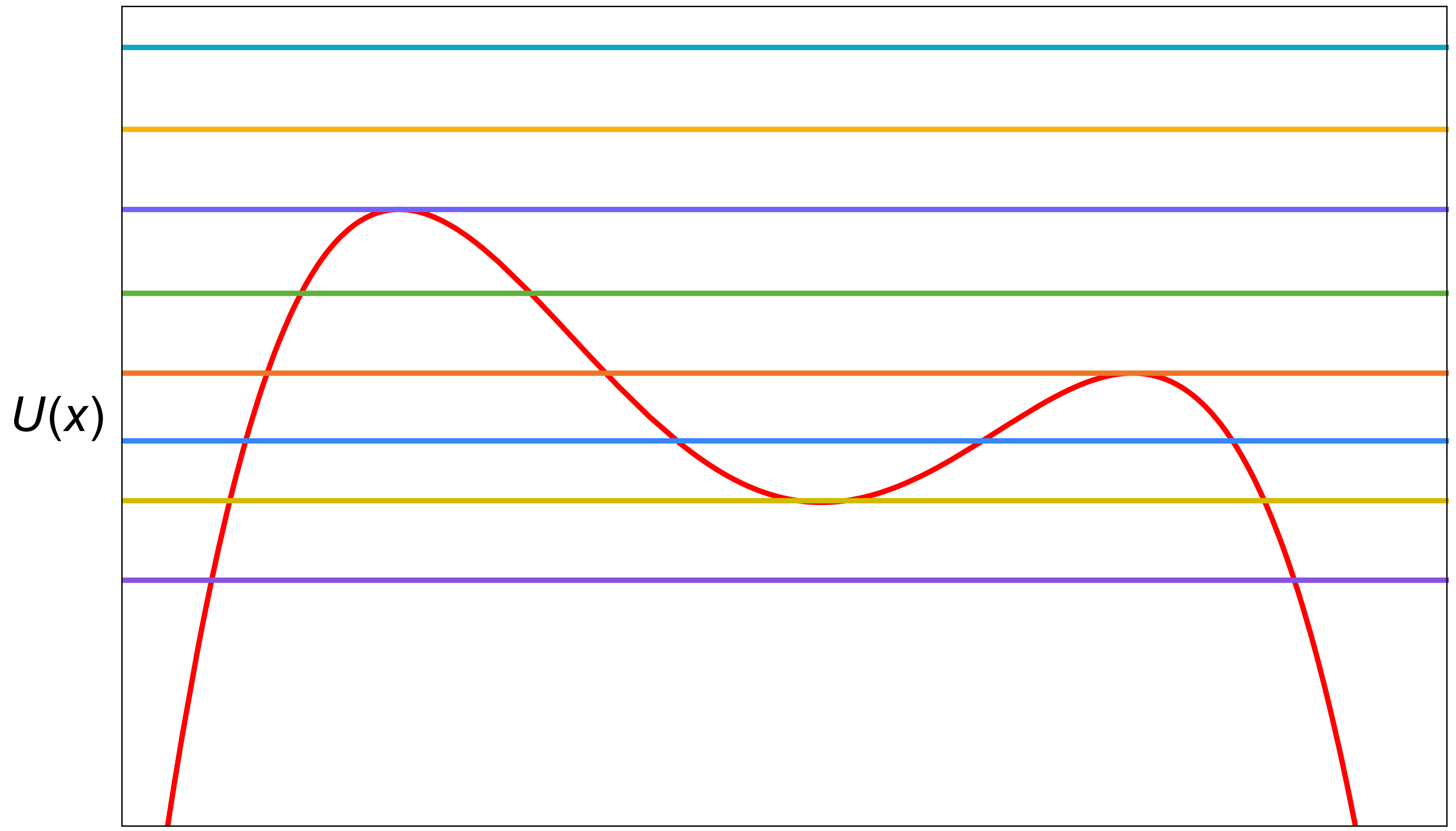



# Practice problem

For the potential function

$$U(x) = -\frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{10}x,$$

draw the corresponding phase portrait.



# Answer

The graph of

$$U(x) = -\frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{10}x$$

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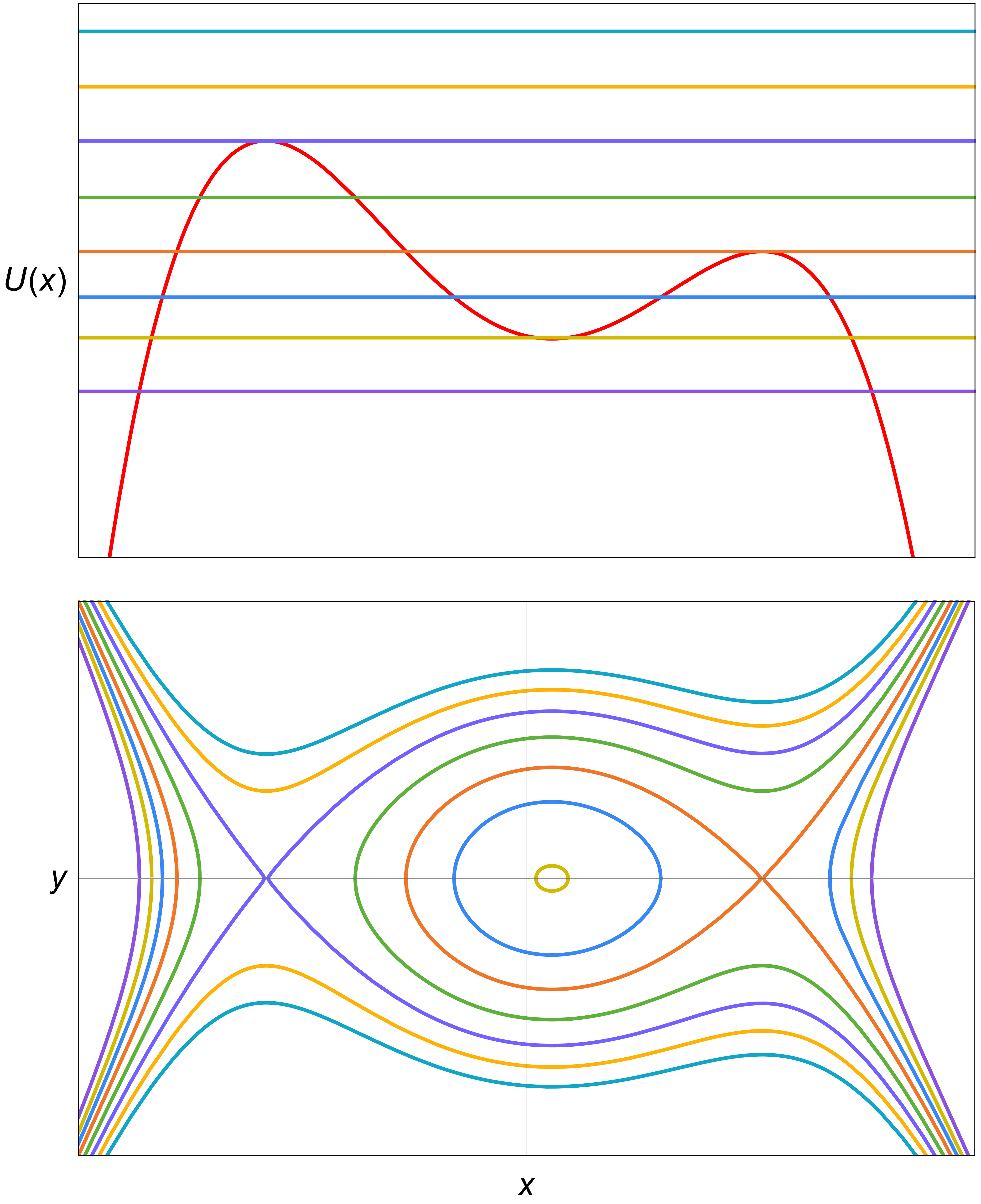


# Practice problem

Consider the equation

$$x'' = -\sin x,$$

describing the motion of a pendulum.

Determine the potential and draw the phase portrait for $x \in [-\pi, \pi]$.

# Answer

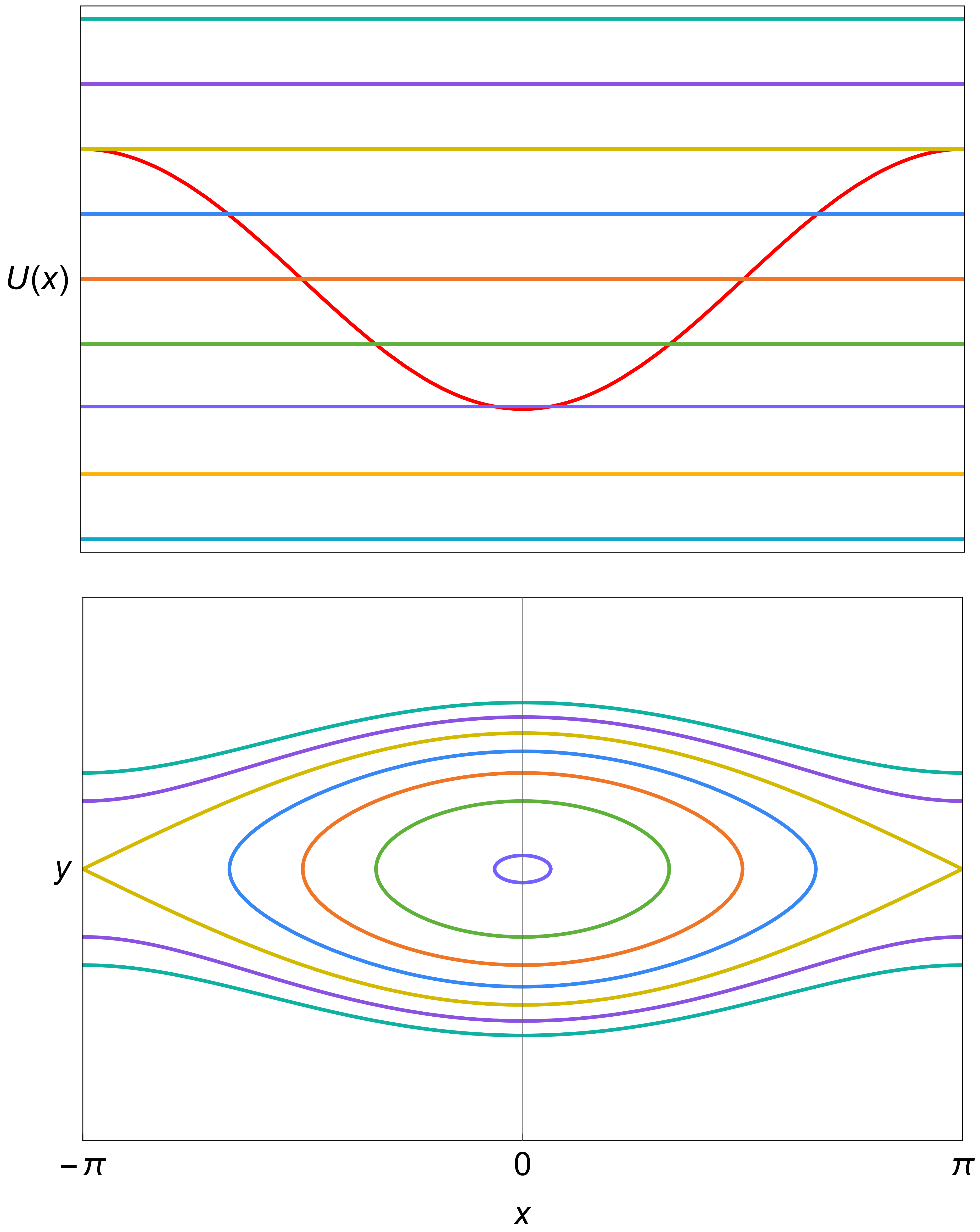
The equations describing the motion of a pendulum are

$$x' = y,$$
$$y' = -\sin x\,.$$

In this case, $f(x) = -\sin x$ and therefore we can take $U(x) = -\cos x$. The energy is

$$E(x, y) = \frac{1}{2}y^2 - \cos x.$$

The level curves $E(x, y) = h$ for different values of $h$ are shown at the right.



# Lyapunov's Method

# Isolated equilibria

**Definition.** An equilibrium $\boldsymbol{x}_e$ of a planar system is an **isolated equilibrium** if there is an open disk $D = B_\delta(\boldsymbol{x}_e)$ which does not contain any other equilibria.

# Positive / Negative (semi-)definite

**Definition.** Let $D$ be an open disk centered at $\mathbf{0}$ and consider a function $W(\boldsymbol{x})$ which is continuous in $D$ and satisfies $W(\mathbf{0}) = 0$.

- If $W(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in D_* = D \backslash \{\mathbf{0}\}$ then $W$ is **positive definite** in $D$.
- If $W(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in D$ then $W$ is **positive semidefinite** in $D$.
- If $W(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \in D_* = D \backslash \{\mathbf{0}\}$ then $W$ is **negative definite** in $D$.
- If $W(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in D$ then $W$ is **negative semidefinite** in $D$.

**Remark.** Clearly $W$ is positive (semi-)definite in $D$ if and only if its opposite, $-W$, is negative (semi-)definite in $D$.

# Examples

The function $W(x, y) = x^2 + y^2$ is positive definite on $\mathbb{R}^2$.

The function $W(x, y) = x^2y^2$ is positive semi-definite on $\mathbb{R}^2$.

# Derivative along a solution curve

Consider a planar system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ and a real-valued function $V(\boldsymbol{x})$. Let $\boldsymbol{x}(t) = (x_1(t), x_2(t))$ be a solution curve of the given planar system. Then

$$\frac{d}{dt}[V(\boldsymbol{x}(t))] = \frac{d}{dt}[V(x_1(t), x_2(t))] = \frac{\partial V}{\partial x_1}(\boldsymbol{x}(t))\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}(\boldsymbol{x}(t))\frac{dx_2}{dt}$$

$$= \frac{\partial V}{\partial x_1}(\boldsymbol{x}(t))f_1(\boldsymbol{x}(t)) + \frac{\partial V}{\partial x_2}(\boldsymbol{x}(t))f_2(\boldsymbol{x}(t))\,.$$

Given a function $V(\mathbf{x})$ and a planar system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ define a new function $\dot{V}(\boldsymbol{x})$ by

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial x_1}(\boldsymbol{x})f_1(\boldsymbol{x}) + \frac{\partial V}{\partial x_2}(\boldsymbol{x})f_2(\boldsymbol{x}).$$

With this notation we have

$$\frac{d}{dt}[V(\boldsymbol{x}(t))] = \dot{V}(\boldsymbol{x}(t)).$$

We call $\dot{V}$ the **derivative of $V$ along the solution curves of $\boldsymbol{f}$**. Moreover, notice that

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial x_1}(\boldsymbol{x}) f_1(\boldsymbol{x}) + \frac{\partial V}{\partial x_2}(\boldsymbol{x}) f_2(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \cdot \nabla V(\boldsymbol{x}),$$

or $\dot{V} = \boldsymbol{f} \cdot \nabla V$.

The last relation shows that $\dot{V}$ is the directional derivative of $V$ along the vector field $\boldsymbol{f}$.

# Lyapunov's Stability Theorem

**Theorem.** Consider a planar system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ for which $\mathbf{0}$ is an isolated equilibrium.

(a) If there is a function $V(\boldsymbol{x})$ which is positive definite in an open disk $D$ centered at $\mathbf{0}$ while $\dot{V}(\boldsymbol{x})$ is negative definite in $D$ then $\mathbf{0}$ is **asymptotically stable**.

(b) If there is a function $V(\boldsymbol{x})$ which is positive definite in an open disk $D$ centered at $\mathbf{0}$ while $\dot{V}(\boldsymbol{x})$ is negative semidefinite in $D$ then $\mathbf{0}$ is **stable**.

**Remark.** A function $V$ satisfying the conditions of (either part of) this theorem is called a **Lyapunov function**.

# Example

Consider the system

$$x' = y - xy^2 - x^3,\ y' = -x - x^2y - y^3,$$

for which $(0,0)$ is an isolated equilibrium.

Let $V(x, y) = x^2 + y^2$. Then $V$ is positive definite on $\mathbb{R}^2$ since $V$ is continuous on $\mathbb{R}^2$, $V(0,0) = 0$, and $V(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2 \backslash (0,0)$. Moreover,

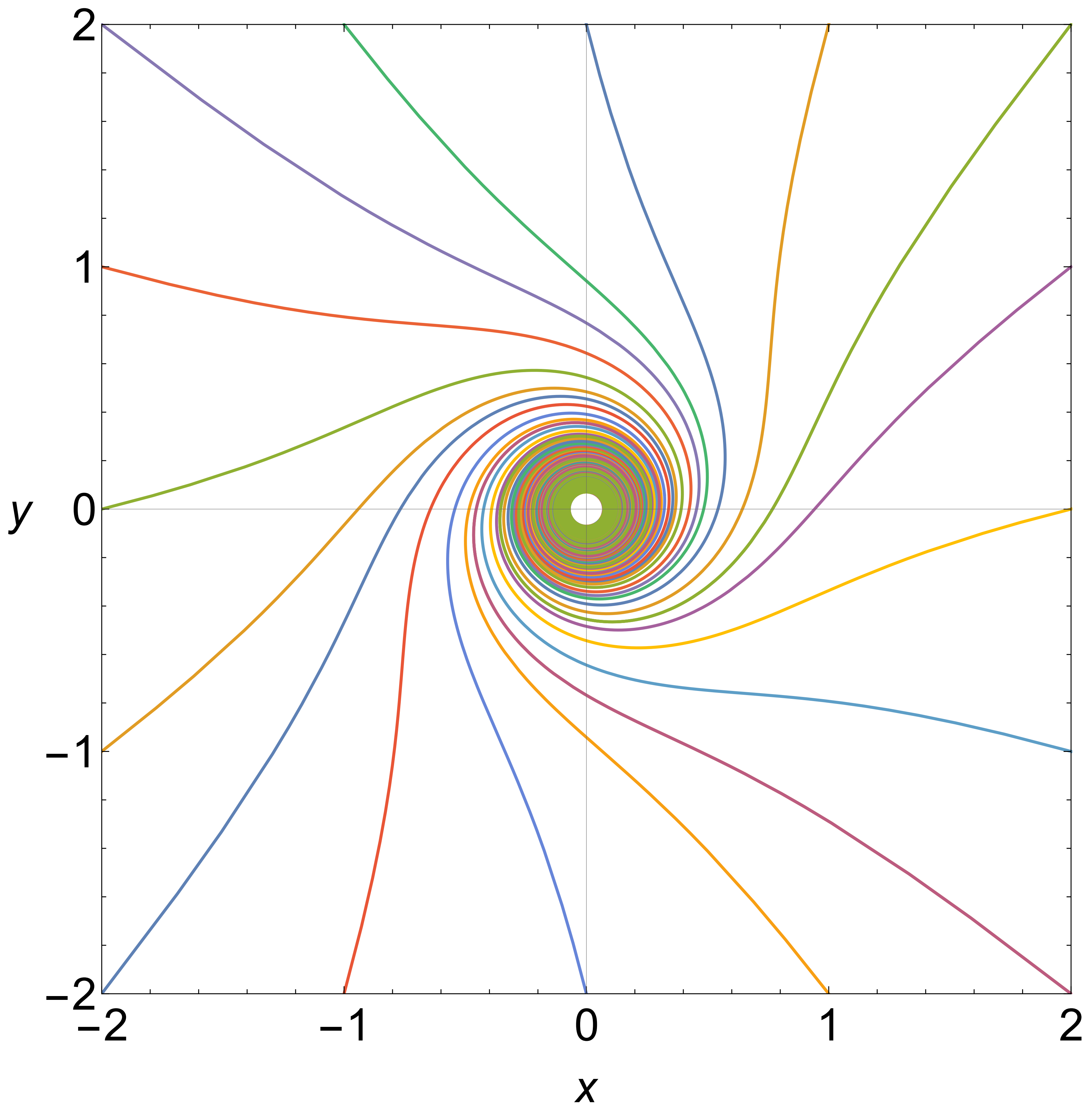
$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} \cdot (y - xy^2 - x^3) + \frac{\partial V}{\partial y} \cdot (-x - x^2y - y^3) \\ &= 2x(y - xy^2 - x^3) + 2y(-x - x^2y - y^3) = -2(x^4 + y^4)\,. \end{aligned}$$

Therefore, $\dot{V}$ is negative definite on $\mathbb{R}^2$ since $\dot{V}$ is continuous on $\mathbb{R}^2$, $\dot{V}(0,0) = 0$, and $\dot{V}(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \backslash (0,0)$.

Applying Lyapunov's Stability Theorem we conclude that $(0,0)$ is asymptotically stable.

**Remark.** The phase portrait of the system is shown at the right.

**Remark.** The linearization of this system is $x' = y,\ y' = -x$ which corresponds to a center. Therefore, the Hartman-Grobman theorem cannot be used in this case to determine stability.



# Question

Consider the system $x' = -2y^3$, $y' = x - 3y^3$. Apply Lyapunov's stability theorem with $V(x, y) = x^2 + y^4$ to determine the stability of the origin.

# Solution

First, $(0,0)$ is an isolated equilibrium.

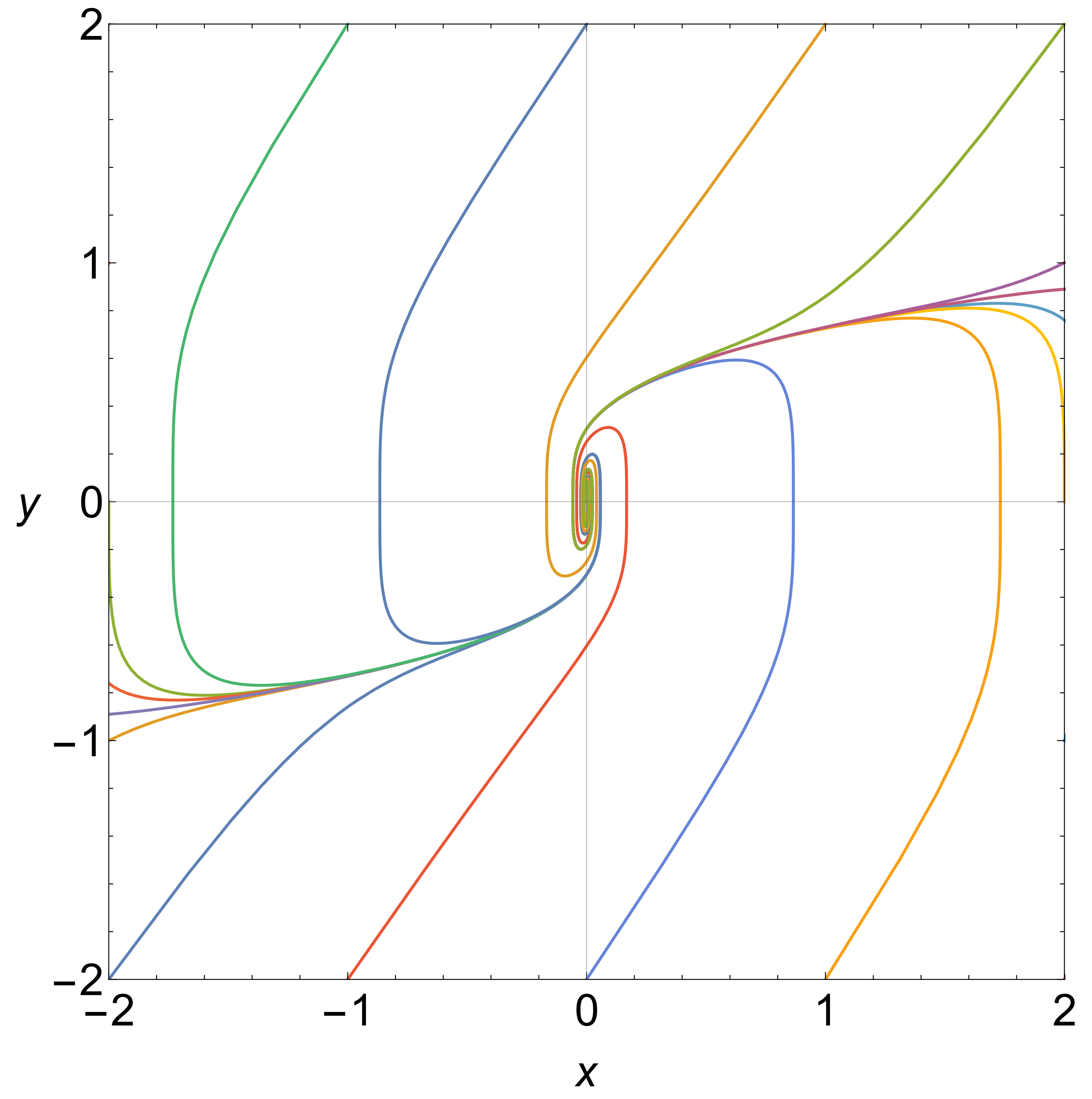
The function $V(x, y) = x^2 + y^4$ is positive definite and continuous on $\mathbb{R}^2$ with $V(0,0) = 0$. Then, we have

$$\dot{V} = \frac{\partial V}{\partial x} \cdot (-2y^3) + \frac{\partial V}{\partial y} \cdot (x - 3y^3) = -4xy^3 + 4y^3(x - 3y^3) = -12y^6 \leq 0.$$

We also have $\dot{V}(0,0) = 0$ and $\dot{V}$ is continuous on $\mathbb{R}^2$, therefore $\dot{V}$ is negative semidefinite on $\mathbb{R}^2$. Therefore, according to the theorem, the origin is stable.

Actually, the origin is asymptotically stable but the theorem cannot ensure this.

The phase portrait of the system is shown at the right.



# Example

We consider the pendulum with damping:

$$x' = y, \quad y' = -by - \sin x,$$

where $b > 0$. Recall that when $b = 0$ the energy

$$E(x, y) = \frac{1}{2}y^2 - \cos x$$

is a conserved quantity. In our case with $b > 0$ we have

$$\dot{E} = y\frac{dy}{dt} + \sin x\frac{dx}{dt} = -by^2 - y\sin x + y\sin x = -by^2 \leq 0.$$

We want to use $E(x, y)$ to show that the isolated equilibrium $(0,0)$ is stable.

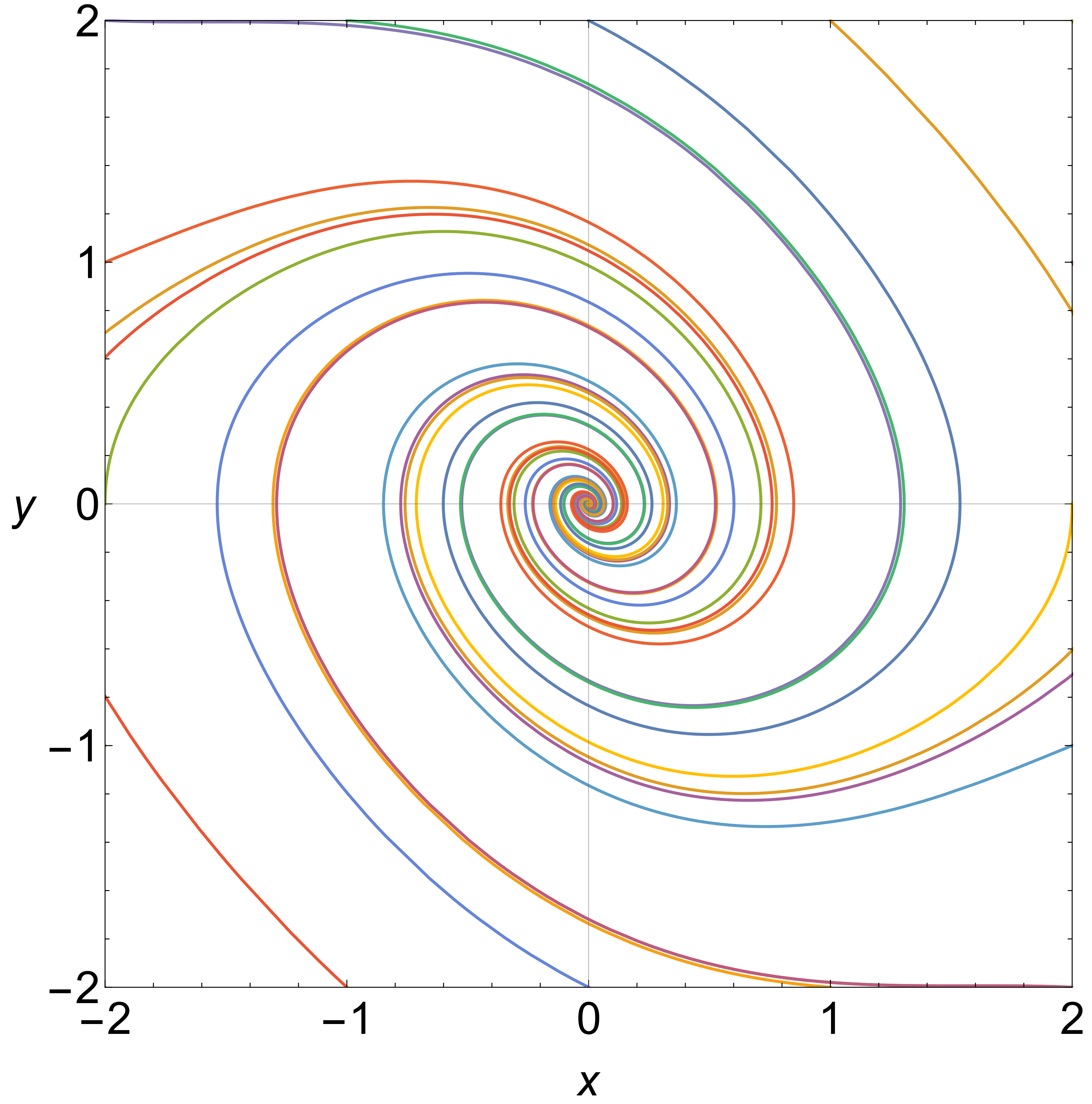
Since we have $E(0,0) = -1$ we choose

$$V(x, y) = E(x, y) + 1 = \frac{1}{2}y^2 + (1 - \cos x).$$

Then $V(0,0) = 0$ and $V(x, y)$ is continuous on $\mathbb{R}^2$. Moreover, $V(x, y)$ is positive definite on $(-\pi, \pi) \times \mathbb{R}$ and certainly there is a disk $D$ centered at $(0,0)$ and contained in this set, for example, the open disk with radius $\pi$. Therefore, $V(x, y)$ is positive definite in $D$.

Moreover, $\dot{V}(x, y) = -by^2$ is continuous on $\mathbb{R}^2$, has $\dot{V}(0,0) = 0$, and $\dot{V}(x, y) = -by^2 \leq 0$ for all $(x, y) \in D$. Therefore, $\dot{V}$ is negative semidefinite on $D$. Then Lyapunov's Stability Theorem allows us to conclude that $(0,0)$ is (at least) stable.

Actually, again the equilibrium is not just stable but asymptotically stable. The phase portrait of the system near the origin is shown at the right.



# Lyapunov's Stability Theorem

**Theorem.** Consider a planar system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ for which $\mathbf{0}$ is an isolated equilibrium.

(a) If there is a function $V(\boldsymbol{x})$ which is positive definite in an open disk $D$ centered at $\mathbf{0}$ while $\dot{V}(\boldsymbol{x})$ is negative definite in $D$ then $\mathbf{0}$ is **asymptotically stable**.

(b) If there is a function $V(\boldsymbol{x})$ which is positive definite in an open disk $D$ centered at $\mathbf{0}$ while $\dot{V}(\boldsymbol{x})$ is negative semidefinite in $D$ then $\mathbf{0}$ is **stable**.

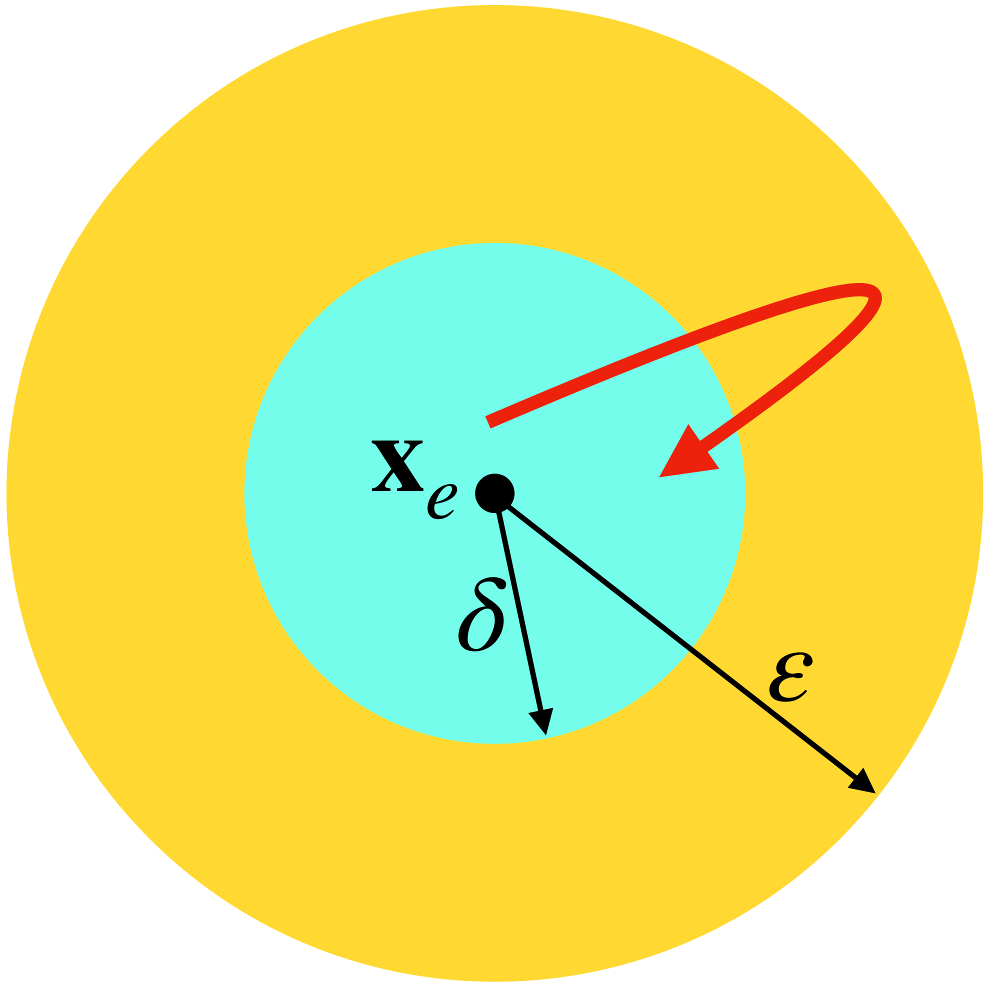
**Remark.** A function $V$ satisfying the conditions of (either part of) this theorem is called a **Lyapunov function**.

# Proof of Lyapunov's Stability Theorem

**(b)** If there is a function $V(\boldsymbol{x})$ which is positive definite in an open disk $D$ centered at $\mathbf{0}$ while $\dot{V}(\boldsymbol{x})$ is negative semidefinite in $D$ then $\mathbf{0}$ is stable.

First, recall the definition of a stable equilibrium.

**Definition.** An equilibrium $\boldsymbol{x}_e$ of $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is **stable** if for every $\varepsilon > 0$ there is $\delta > 0$ such that any solution $\boldsymbol{x}(t)$ with initial condition $\boldsymbol{x}(0) \in B_\delta(\boldsymbol{x}_e)$ satisfies $\boldsymbol{x}(t) \in B_\varepsilon(\boldsymbol{x}_e)$ for all $t \geq 0$.



# Remark

In the proof we use several times the following result.

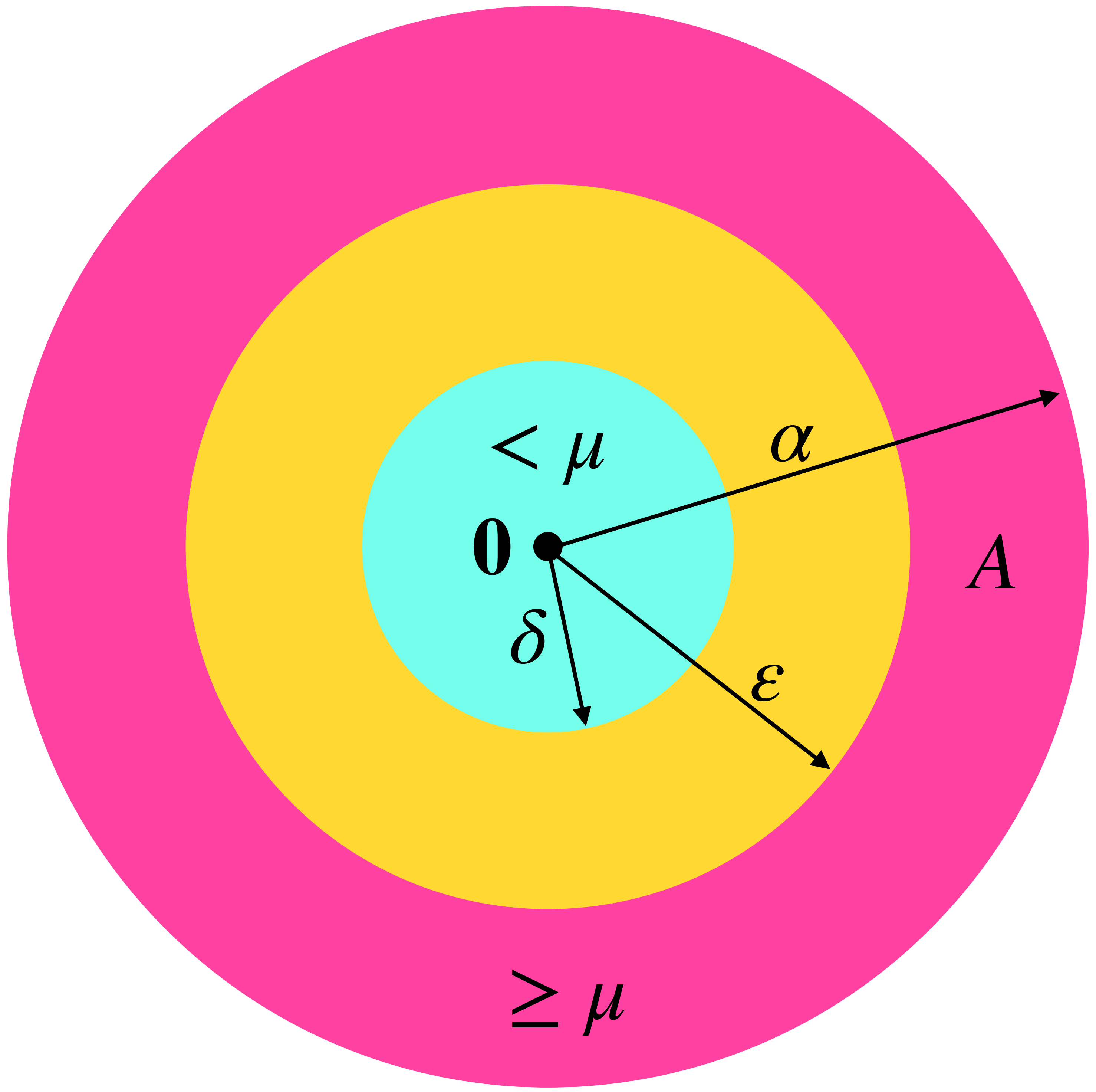
**Theorem.** If a function $g : B \to \mathbb{R}$ is continuous on the closed and bounded set $B$, then $g$ **attains its maximum and minimum values** on $B$, that is, there are $\boldsymbol{y} \in B$ and $\boldsymbol{z} \in B$ such that

$$\min_{\boldsymbol{x} \in B} g(\boldsymbol{x}) = g(\boldsymbol{y}) \leq g(\boldsymbol{x}) \leq g(\boldsymbol{z}) = \max_{\boldsymbol{x} \in B} g(\boldsymbol{x}).$$

To show that the origin is stable we need for any given $B_\varepsilon(\mathbf{0})$ to find a neighborhood $B_\delta(\mathbf{0})$ such that $\boldsymbol{x}(0) \in B_\delta(\mathbf{0})$ implies $\boldsymbol{x}(t) \in B_\varepsilon(\mathbf{0})$ for all $t \geq 0$.

Consider the closed disk of radius $\alpha > 0$ and centered at $\mathbf{0}$, $\bar{B}_\alpha(\mathbf{0}) \subseteq D$. Then, for any $\varepsilon < \alpha$ consider the annulus

$$A = \{(x, y) : \varepsilon^2 \leq x^2 + y^2 \leq \alpha^2\} = \bar{B}_\alpha(\mathbf{0}) \backslash B_\varepsilon(\mathbf{0}).$$




Let $\mu = \min_{\boldsymbol{x} \in A} V(\boldsymbol{x}) > 0$.

Select $\delta$ such that $0 < \delta < \varepsilon$ and $V(\boldsymbol{x}) < \mu$ for all $\boldsymbol{x} \in B_\delta(\mathbf{0})$. Such $\delta$ exists since $V(\mathbf{0}) = 0$ and $V$ is continuous.

Suppose that $\boldsymbol{x}(t)$ is a solution with $\boldsymbol{x}(0) \in B_\delta(\mathbf{0})$ which does not stay in $B_\varepsilon(\mathbf{0})$ for all $t \geq 0$. That is, there is $t_1 > 0$ such that $\|\boldsymbol{x}(t_1)\| = \varepsilon$, which also implies, $\boldsymbol{x}(t_1) \in A$.

Since $\dot{V}$ is negative semidefinite, we have

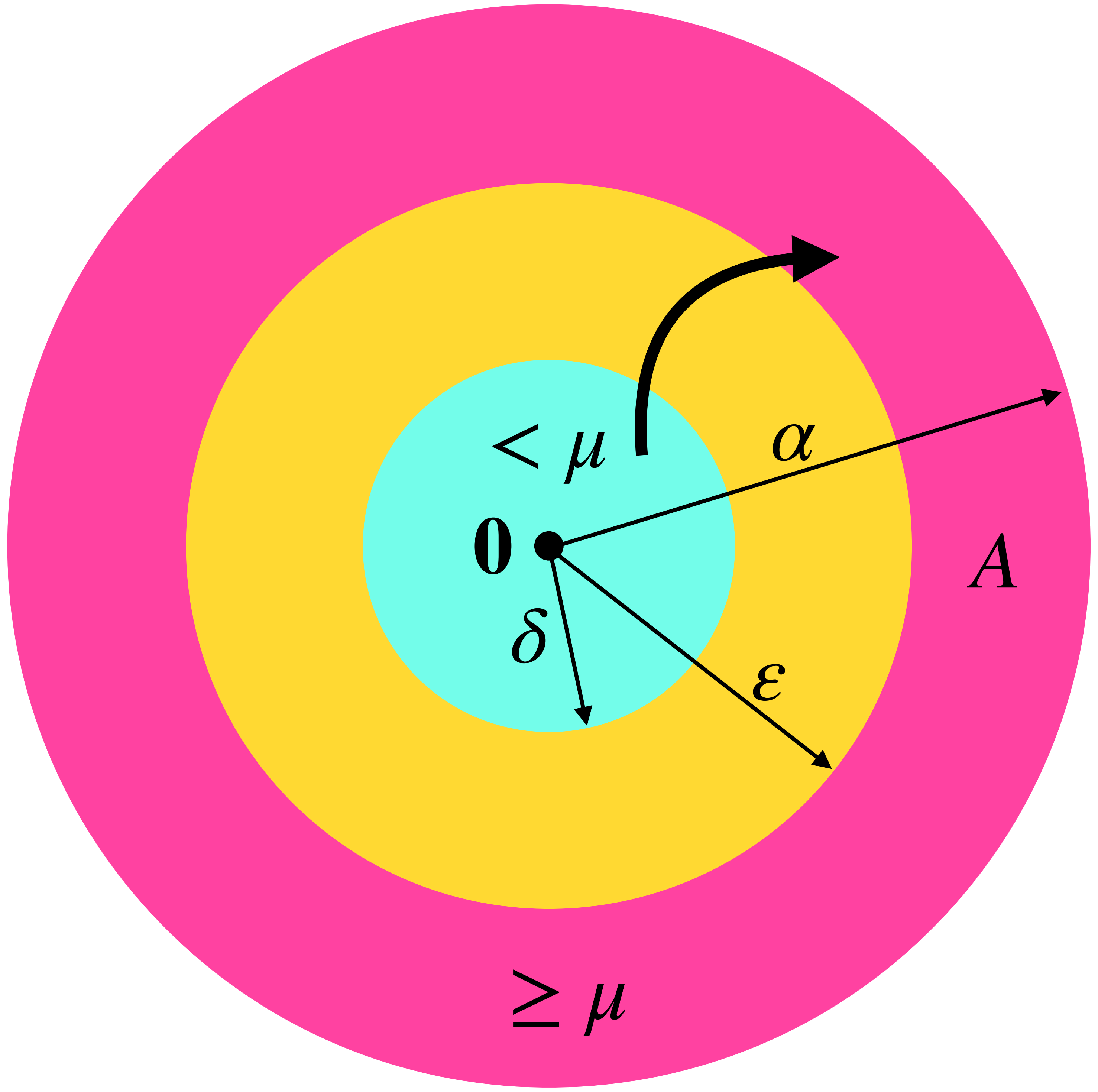
$$V(\boldsymbol{x}(t)) - V(\boldsymbol{x}(0)) = \int_0^t \frac{d}{ds}[V(\boldsymbol{x}(s))]ds$$
$$= \int_0^t \dot{V}(\boldsymbol{x}(s))ds \leq 0,$$

for all $t \in [0,t_1]$.

Therefore, $V(\boldsymbol{x}(t_1)) \leq V(\boldsymbol{x}(0))$.

However, we have $V(\boldsymbol{x}(0)) < \mu$ since $\boldsymbol{x}(0) \in B_\delta(\mathbf{0})$, and we have $V(\boldsymbol{x}(t_1)) \geq \mu$ since $\boldsymbol{x}(t_1) \in A$.

That is, $V(\boldsymbol{x}(t_1)) \geq \mu > V(\boldsymbol{x}(0))$, which is a contradiction.



# Questions

How would the proof fail if $V$ is positive semidefinite instead of positive definite?

How is the fact that $V$ is positive definite being used in the proof?

How is the fact that $\dot{V}$ is negative semidefinite being used in the proof?

Is the fact that $V$ is continuous important?

Is the fact that $V(\mathbf{0}) = 0$ important?

Is the fact that $\dot{V}$ is continuous important?

**(a)** If there is a function $V(\boldsymbol{x})$ which is positive definite in an open disk $D$ centered at $\mathbf{0}$ while $\dot{V}(\boldsymbol{x})$ is negative definite in $D$ then $\mathbf{0}$ is asymptotically stable.

Recall the definition of asymptotically stable equilibrium.

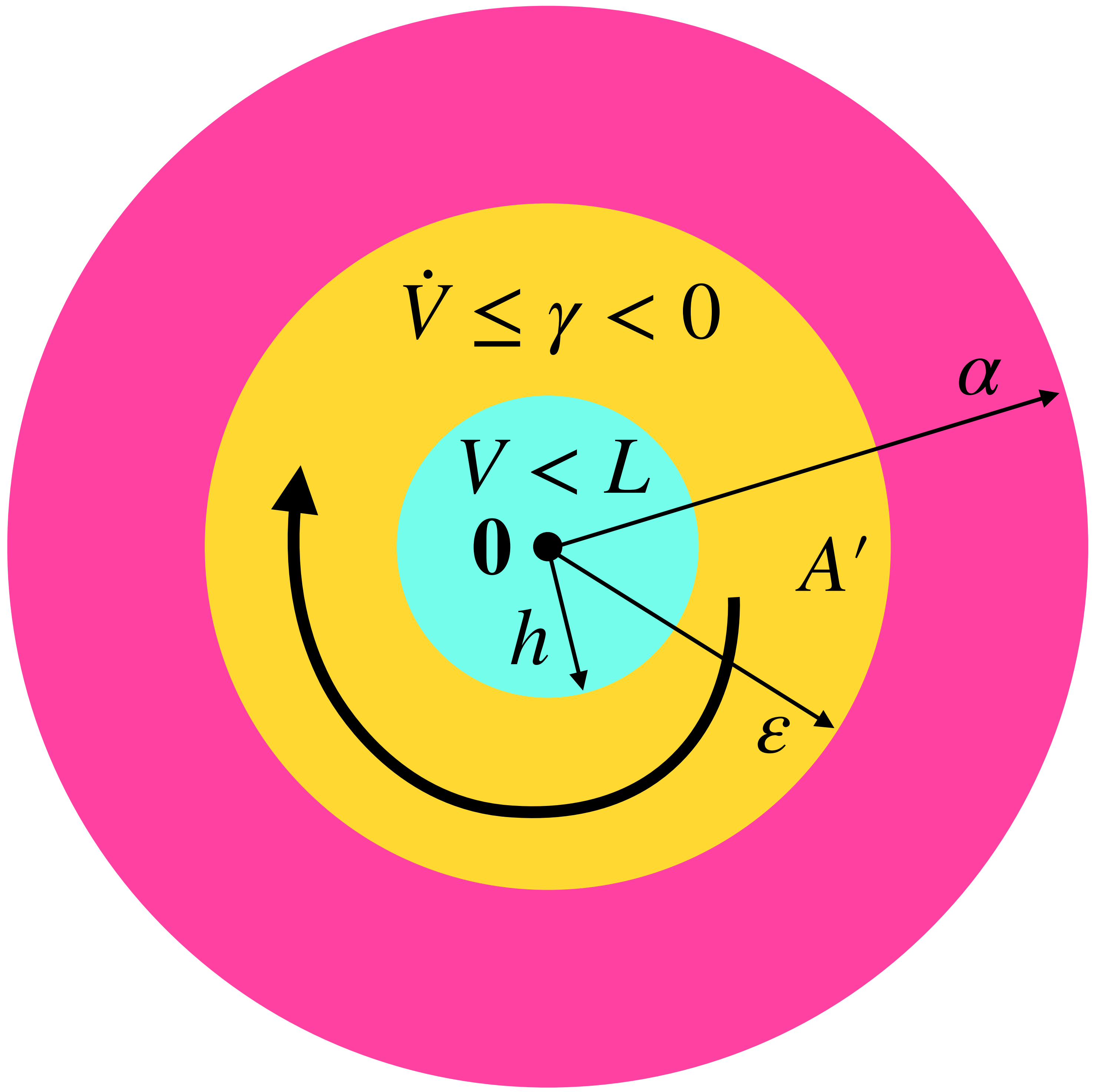
**Definition.** An equilibrium $\boldsymbol{x}_e$ of $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is **asymptotically stable** if it is stable and if there is $\eta > 0$ such that for all solutions $\boldsymbol{x}(t)$ with initial condition $\boldsymbol{x}(0) \in B_\eta(\boldsymbol{x}_e)$ we have $||\boldsymbol{x}(t) - \boldsymbol{x}_e|| \to 0$ as $t \to \infty$.

Since $\dot{V}$ is negative definite, it is also negative semidefinite, so we can conclude from (a) that the equilibrium is stable. Therefore, we only need to find $B_\eta(\mathbf{0})$ such that if $\boldsymbol{x}(0) \in B_\eta(\mathbf{0})$ then $||\boldsymbol{x}(t)|| \to 0$ as $t \to \infty$.

Since $V$ is continuous and positive definite we have that $\|\boldsymbol{x}(t)\| \to 0$ as $t \to \infty$ if and only if $V(\boldsymbol{x}(t)) \to 0$ as $t \to \infty$. Therefore, we only need to find $B_\eta(\mathbf{0})$ such that if $\boldsymbol{x}(0) \in B_\eta(\mathbf{0})$ then $V(\boldsymbol{x}(t)) \to 0$ as $t \to \infty$.

Let $\varepsilon = \alpha/2$, choose $0 < \delta < \varepsilon$ as in the proof of (a), and choose $\eta = \delta$. Consider a solution $\boldsymbol{x}(t)$ with $\boldsymbol{x}(0) \in B_\delta(\mathbf{0})$. Because of stability, and the choice we made for $\delta$, we have that $\boldsymbol{x}(t) \in B_\varepsilon(\mathbf{0})$ for all $t \geq 0$.

Therefore, the value of $V(\boldsymbol{x}(t))$ is strictly decreasing. If we assume that it does not approach $0$ as $t \to \infty$, then there must be $L > 0$ such that $V(\boldsymbol{x}(t)) \geq L$ for all $t \geq 0$.



Since $V$ is continuous and $V(\mathbf{0}) = 0$, there is $h > 0$ such that $V(\boldsymbol{x}) < L$ for all $\boldsymbol{x} \in B_h(\mathbf{0})$. Therefore, for all $t \geq 0$ we have

$$\boldsymbol{x}(t) \in A' = \{\boldsymbol{x} : h \leq \|\boldsymbol{x}\| \leq \varepsilon\}.$$

Let $\gamma$ be the maximum value of $\dot{V}$ on $A'$. Since $\dot{V}$ is negative definite in $D \supseteq A'$ and $\mathbf{0} \notin A'$ we have $\gamma < 0$. Then

$$V(\boldsymbol{x}(t)) - V(\boldsymbol{x}(0)) = \int_0^t \frac{d}{ds}[V(\boldsymbol{x}(s))]ds = \int_0^t \dot{V}(\boldsymbol{x}(s))ds \leq \gamma t,$$

which gives $V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x}(0)) + \gamma t$.

Therefore, for large enough $t$ we get $V(\boldsymbol{x}(t)) < 0$ which is a contradiction since $V$ is positive definite.