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1 Introduction

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We want to solve the following equation for $F(t) = F_0 cos(\omega t)$:

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \tag{1}$$

Equation 1 is a second order linear differential equation. According to theorem 3.5.2 in Boyce [1]:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \tag{2}$$

With $Y(t)$, any solution to the nonhomogeneous differential equation and with $y_1(t)$ and $y_2(t)$ that form a fundamental set of solutions to the homogeneous differential equation.

$$\ddot{y}(t) + \frac{\omega}{Q}\dot{y}(t) + \omega^2 y(t) = 0 \tag{3}$$

We first solve $y(t)$ by seeing that, for constant values of ω,m and Q, a solution for equation 3 is $y(t) = e^{r\,t}$. Applying to equation 3 yields:

$$(r^2 + \frac{\omega}{Q}r + \omega^2)y(t) = 0 \quad \Rightarrow \quad r^2 + \frac{\omega}{Q}r + \omega^2 = 0 \tag{4}$$

We find two solutions for r:

$$r_1 = \frac{-1}{2}\left(\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right) \quad , \quad r_2 = \frac{1}{2}\left(-\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right)$$

We use this to define:

$$y_1(t) = e^{r_1 t} \quad , \quad y_2(t) = e^{r_2 t}$$

r_1 and r_2 will be complex numbers for $\mid Q \mid > \frac{1}{2}$ meaning that differential equation 1 will lead to a (damped) oscillator. Just as expected.

The method to find $Y(t)$ is the method of undetermined coefficients described in section 3.5 of Boyce [1].
We assume the $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

We find $a_1 = 0$ and $a_2 = \frac{F_0 Q}{\omega^2 m}$ (see appendix). Therefore we find:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

By imposing the initial conditions on the latter result (see appendix) we obtain:

$$c_1 = \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0(2\,r_2 - r_1)\right)$$

$$c_2 = \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2\right)$$

3 question 2

4 question 3

4.1 Analytical method

We want to solve equation 5 when a amplitude modulated external force is applied, this force follows from equation 6.

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \tag{5}$$

$$F(t) = F_0 t\frac{T-t}{T^2} = F_0\frac{t}{T} - F_0\frac{t^2}{T^2} \tag{6}$$

Since this equation 6 is relatively simple, it can be solved using the method of undetermined coefficients as outlined in paragraph 3.5 of Boyce [1].
The homogenous form has already been solved in the previous questions, the resulting roots are shown below in equation 7 and the solution in 8.

$$r_1 = -1/2\left(\frac{\omega}{Q} + \sqrt{\frac{\omega^2}{Q^2} - 4\omega^2}\right), r_2 = 1/2\left(-\frac{\omega}{Q} + \sqrt{\frac{\omega^2}{Q^2} - 4\omega^2}\right) \tag{7}$$

$$y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t} \tag{8}$$

Now for the particular solution $y_p(t)$ we use the aforementioned method, the derivation is show below. We start by assuming that $y_p(t)$ is of the shape:

$$y_p(t) = c_1 + c_2 \cdot t + c_3 \cdot t^2$$

If we then differentiate $y_p(t)$ two times and substitute the result into equation 5 we get the following:

$$y_p'(t) = c_2 + 2c_3 \cdot t, y_p"(t) = 2c_3$$

$$my"(t) + \frac{m\omega}{Q}y'(t) + m\omega^2 y(t) = F_0\frac{t}{T} - F_0\frac{t^2}{T^2}$$

$$m(2a_3) + \frac{m\omega}{Q}(a_2 + 2a_3 t) + m\omega^2(a_1 + a_2 t + a_3 t^2) = F_0\frac{t}{T} - F_0\frac{t^2}{T^2}$$

If we then equate the terms in front of the functions and it's derivatives we get the following:

$$m\omega^2 a_1 + \frac{m\omega a_2}{Q} + 2ma_3 = 0$$

$$m\omega^2 a_2 + \frac{2m\omega a_3}{Q} = \frac{F_0}{T}$$

$$m\omega^2 a_3 = -\frac{F_0}{T^2}$$

Solving the system of equations and substituting back into $y_p(t)$ yields:

$$a_1 = \frac{F_0(2Q^2 - QT\omega - 2)}{mQ^2T^2\omega^4}$$

$$a_2 = \frac{F_0(QT\omega + 2)}{mQT^2\omega^3}$$

$$a_3 = \frac{-F_0}{mT^2\omega^2}$$

$$y_p(t) = \frac{F_0 t}{mT\omega^2}(1 - t/T) + \frac{F_0}{mQT\omega^3}\left[\frac{2(t+Q)}{T} - \frac{2}{QT\omega} - 1\right] \tag{9}$$

4.2 Numerical method

To solve equation 5 numerically we first have to split the second order differential equations into a system of two first order equations. We do this by substituting two new time dependant functions for y, namely $u(t)$ and $v(t)$ equal to $y(t)$ and $y'(t)$ respectively. We can then derive the following system:

$$\begin{cases} \text{u(t) = y(t)} \\ \text{v(t) = y'(t)} \end{cases}$$

So that their derivatives become:

$$\begin{cases} \text{u'(t) = y'(t) = v(t)} \\ \text{v'(t) = y"(t)} \end{cases}$$

If we then substitute in these equations into the second order differential equation we get the following system:

$$\begin{aligned} u'(t) &= y'(t) = v(t) \\ v'(t) &= y''(t) \\ F(t) &= my''(t) + m\frac{\omega}{Q}y'(t) + m\omega^2 y(t) \end{aligned}$$

$$\begin{aligned} v'(t) &= y''(t) = 1/m \cdot F(t) - \omega/Q \cdot y'(t) - \omega^2 \cdot y(t) \\ v'(t) &= 1/m \cdot F(t) - \omega/Q \cdot v(t) - \omega^2 \cdot u(t) \end{aligned}$$

So that we now have the following system of of first-order linear differential equations:

$$v'(t) = 1/m \cdot F(t) - \omega/Q \cdot v(t) - \omega^2 \cdot u(t) \tag{10}$$

$$u'(t) = v(t) \tag{11}$$

This system can be easily solved numerically using the following python code:

```
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp as solve

"""
|                          code in between lines                         |
"""

#Defining constants
m = 1            #Mass in kilogramme
Q = 1           #Quality factor
F_naught = 1     #force in newton
freqT = 0.1      #wT dimensionless
tmax = 10        #Maximum time to elapse

#Variables
T = 1
```

```
freq = freqT/T

#Defining our timepoints
time = np.linspace(0,tmax,100)

def force(t):
    if t < T:
        return F_naught*t*(T-t)/(T**2)
    else:
        return 0


def system(t , func_array):
    u_prime = func_array[1]
    v_prime = 1/m * force(t) - freq/Q*func_array[1] -freq**2 *func_array[0]

    #function that takes in a vector [u, v]^T and returns [u' , v']^T

    return [u_prime, v_prime]

solution = solve(system, (0,tmax), [0,0], t_eval=time)
"""                    ^       ^       ^      \ Timepoints to evaluate at.
                       |       |        \ The initial values for u(t) and v(t)
                       |        \ The timespace in which system needs evaluating.
                        \ The system defining our system of first order ODE's
"""
plt.plot(solution.t,solution.y[0])
plt.show()
```

5 conclusion

References

[1] W. E. Boyce, R. C. Dirpima, and D. B. Meade, *Elementary Differential Equations and Boundary Value Problems*. John Wiley and Sons, Inc., 11 ed., 2017.

6 Appendix

6.1 Question 1

6.1.1 Solving $Y(t)$

In order to solve the particular solution to the differential equation we assume $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

Differentiating with respect to t once and twice yields:

$$\dot{Y}(t) = -a_1\omega\ sin(\omega t) + a_2\omega\ cos(\omega t)$$
$$\ddot{Y}(t) = -a_1\omega^2\ cos(\omega t) - a_2\omega^2\ sin(\omega t)$$

If we substitute this for $x(t)$ in equation 1 and with $F(t) = F_0\ cos(\omega t)$ we find:

$$F(t)m = \ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t)$$
$$F_0\ cos(\omega t) = m\left[-a_1\omega^2\ cos(\omega t) - a_2\omega^2\ sin(\omega t)\right] + m\frac{\omega}{Q}\left[-a_1\omega\ sin(\omega t) + a_2\omega\ cos(\omega t)\right] + m\omega^2\left[a_1\ cos(\omega t) + a_2\ sin(\omega t)\right]$$
$$F_0\ cos(\omega t) = omega^2 m\ cos(\omega t)\left(a_1 + \frac{a_2}{Q} - a_1\right) + \omega^2 m\ sin(\omega t)\left(a_2 - \frac{a_1}{Q} - a_2\right)$$
$$F_0\ cos(\omega t) = a_2\frac{\omega^2 m}{Q}\ cos(\omega t) - a_1\frac{\omega^2 m}{Q}\ sin(\omega t)$$

From this follows:

$$-a_1\frac{\omega^2 m}{Q} = 0 \qquad\qquad a_2\frac{\omega^2 m}{Q} = F_0$$
$$a_1 = 0 \qquad\qquad a_2 = \frac{F_0 Q}{\omega^2 m}$$

Therefore the solution to differential equation 1 is:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

We can find c_1 and c_2 by imposing the initial conditions on the latter equation:

$$x(0) = x_0 = c_1 + c_2$$
$$c_1 = x_0 - c_2$$

$$\dot{x}(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} + \frac{F_0 Q}{\omega m} cos(\omega t)$$
$$\dot{x}(0) = \dot{x}_0 = c_1 r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}$$
$$\dot{x}_0 = [x_0 - c_2]r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}$$
$$c_2(r_2 - r_1) = \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1$$
$$c_2 = \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2\right)$$

We substitute this back into the equation for c_1:

$$\begin{aligned} c_1 &= x_0 - c_2 \\ &= x_0 - \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2\right) \\ &= \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0(2\, r_2 - r_1)\right) \end{aligned}$$