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Delft, March 30th 2020
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1 Introduction

2 question 1

We want to solve the following equation for $F(t) = F_0 cos(\omega t)$:

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \tag{1}$$

Equation 1 is a second order linear differential equation. According to theorem 3.5.2 in Boyce [1]:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \tag{2}$$

With $Y(t)$, any solution to the nonhomogeneous differential equation and with $y_1(t)$ and $y_2(t)$ that form a fundamental set of solutions to the homogeneous differential equation.

$$\ddot{y}(t) + \frac{\omega}{Q}\dot{y}(t) + \omega^2 y(t) = 0 \tag{3}$$

We first solve $y(t)$ by seeing that, for constant values of ω,m and Q, a solution for equation 3 is $y(t) = e^{r\,t}$. Applying to equation 3 yields:

$$(r^2 + \frac{\omega}{Q}r + \omega^2)y(t) = 0 \quad \Rightarrow \quad r^2 + \frac{\omega}{Q}r + \omega^2 = 0 \tag{4}$$

We find two solutions for r:

$$r_1 = -\frac{1}{2}\left(\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right) \quad , \quad r_2 = \frac{1}{2}\left(-\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right)$$

We use this to define:

$$y_1(t) = e^{r_1 t} \quad , \quad y_2(t) = e^{r_2 t}$$

r_1 and r_2 will be complex numbers for $\mid Q \mid > \frac{1}{2}$ meaning that differential equation 1 will lead to a (damped) oscillator. Just as expected.

The method to find $Y(t)$ is the method of undetermined coefficients described in section 3.5 of Boyce [1].
We assume the $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

We find $a_1 = 0$ and $a_2 = \frac{F_0 Q}{\omega^2 m}$ (see appendix). Therefore we find:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

By imposing the initial conditions on the latter result (see appendix) we obtain:

$$c_1 = \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0 r_2\right)$$

$$c_2 = \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\right)$$

We can plot x as function of t for the three discussed values of Q (figure 2. For the sake of simplicity $\omega = 1$, $F_0 = 1$ and $m = 1$ in this plot. A scaled version of $F(t)$ is also plotted in the same axis in order to study the response of the oscillator with respect to driving force.

There are several conclusions that can be drawn from the graphs. For $Q = 1$, the oscillator will, after a period of amplification, oscillate with the same frequency as the driving force. However, with a phase shift. What's more, the oscillator seems to reach a steady state in approximately 1.5 oscillations where the amplitude stays constant. This steady state resembles a sine. The steady state is reached when the work

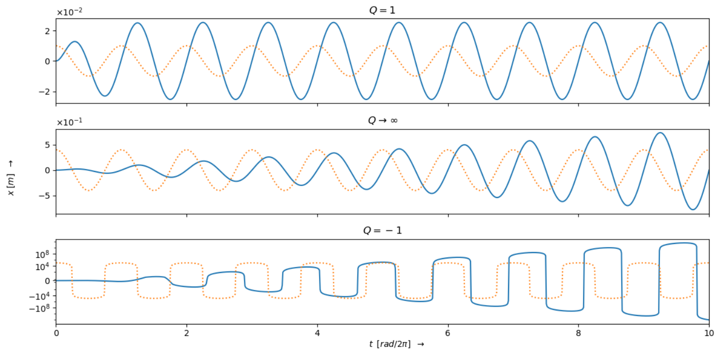

Figure 1: Graphs of $x(t)$ and $F(t)$ as function of t for different values of Q. The solid blue lines correspond with $x(t)$ and the orange dashed lines correspond to a scaled version of $F(t)$.

done by the driving force on the system equals the energy dissipation by the friction.
For $Q \to \infty$ we find that the oscillator will also oscillate at the same rate as $F(t)$ but with a phase shift. However, the amplitude does not stabilize and seems to increase linearly with time. This makes sense since, considering that there is no friction ($Q \to \infty$), there is no energy dissipation in the system, only work done on the system by the driving force. The system has the same frequency as the driving force so the energy gain by the system should be equal per oscillation. Therefore the oscillation amplitude, being directly proportional with the total energy in the system, should increase with the same amount each oscillation.
For $Q = -1$ we find that the oscillator does not reach the same oscillating frequency as the $F(t)$. It seems to be lagging behind the driving force as a result of the 'negative friction' exerting a force in the same direction as the velocity. We also see that the amplitude of the oscillation increases linearly on a logarithmic y-scale. Therefore, the amplitude must increase exponentially with respect to time. This can be explained by the total energy in the system that gets amplified as a result of the 'negative friction'.

3 question 2

For the three values of Q we now consider a situation with a finite excitation time such that $F(t) = F_0 cos(\omega t)$ for $t < 0$ and $F(t) = 0$ for t =.

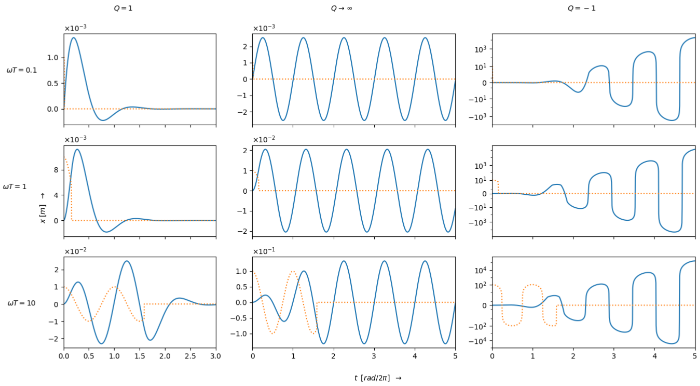

Figure 2: Graphs of $x(t)$ and $F(t)$ as function of t for different values of Q. The solid blue lines correspond with $x(t)$ and the orange dashed lines correspond to a scaled version of $F(t)$.

4 question 3

4.1 Analytical method

We want to solve equation 5 when a amplitude modulated external force is applied, this force follows from equation 6.

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \tag{5}$$

$$F(t) = F_0 t\frac{T-t}{T^2} = F_0\frac{t}{T} - F_0\frac{t^2}{T^2} \tag{6}$$

Since this equation 6 is relatively simple, it can be solved using the method of undetermined coefficients as outlined in paragraph 3.5 of Boyce [1].
The homogenous form has already been solved in the previous questions, the resulting roots are shown below in equation 7 and the solution in 8.

$$r_1 = -1/2\left(\frac{\omega}{Q} + \sqrt{\frac{\omega^2}{Q^2} - 4\omega^2}\right), r_2 = 1/2\left(-\frac{\omega}{Q} + \sqrt{\frac{\omega^2}{Q^2} - 4\omega^2}\right) \tag{7}$$

$$y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t} \tag{8}$$

Now for the particular solution $y_p(t)$ we use the aforementioned method, the derivation is show below. We start by assuming that $y_p(t)$ is of the shape:

$$y_p(t) = c_1 + c_2 \cdot t + c_3 \cdot t^2$$

If we then differentiate $y_p(t)$ two times and substitute the result into equation 5 we get the following:

$$y_p'(t) = c_2 + 2c_3 \cdot t, y_p"(t) = 2c_3$$

$$my"(t) + \frac{m\omega}{Q}y'(t) + m\omega^2 y(t) = F_0\frac{t}{T} - F_0\frac{t^2}{T^2}$$

$$m(2a_3) + \frac{m\omega}{Q}(a_2 + 2a_3 t) + m\omega^2(a_1 + a_2 t + a_3 t^2) = F_0\frac{t}{T} - F_0\frac{t^2}{T^2}$$

If we then equate the terms in front of the functions and it's derivatives we get the following:

$$m\omega^2 a_1 + \frac{m\omega a_2}{Q} + 2ma_3 = 0$$

$$m\omega^2 a_2 + \frac{2m\omega a_3}{Q} = \frac{F_0}{T}$$

$$m\omega^2 a_3 = -\frac{F_0}{T^2}$$

Solving the system of equations and substituting back into $y_p(t)$ yields:

$$a_1 = \frac{F_0(2Q^2 - QT\omega - 2)}{mQ^2T^2\omega^4}$$

$$a_2 = \frac{F_0(QT\omega + 2)}{mQT^2\omega^3}$$

$$a_3 = \frac{-F_0}{mT^2\omega^2}$$

$$y_p(t) = \frac{F_0 t}{mT\omega^2}(1 - t/T) + \frac{F_0}{mQT\omega^3}\left[\frac{2(t+Q)}{T} - \frac{2}{QT\omega} - 1\right] \tag{9}$$

4.2 Numerical method

To solve equation 5 numerically we first have to split the second order differential equations into a system of two first order equations. We do this by substituting two new time dependant functions for y, namely $u(t)$ and $v(t)$ equal to $y(t)$ and $y'(t)$ respectively. We can then derive the following system:

$$\begin{cases} \text{u(t) = y(t)} \\ \text{v(t) = y'(t)} \end{cases}$$

So that their derivatives become:

$$\begin{cases} \text{u'(t) = y'(t) = v(t)} \\ \text{v'(t) = y"(t)} \end{cases}$$

If we then substitute in these equations into the second order differential equation we get the following system:

$$u'(t) = y'(t) = v(t)$$
$$v'(t) = y''(t)$$
$$F(t) = my''(t) + m\frac{\omega}{Q}y'(t) + m\omega^2 y(t)$$

$$v'(t) = y''(t) = 1/m \cdot F(t) - \omega/Q \cdot y'(t) - \omega^2 \cdot y(t)$$
$$v'(t) = 1/m \cdot F(t) - \omega/Q \cdot v(t) - \omega^2 \cdot u(t)$$

So that we now have the following system of of first-order linear differential equations:

$$v'(t) = 1/m \cdot F(t) - \omega/Q \cdot v(t) - \omega^2 \cdot u(t) \tag{10}$$
$$u'(t) = v(t) \tag{11}$$

This system can be easily solved numerically using the following python code:

```
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp as solve

"""
|                    code in between lines                    |
"""

#Defining constants
m = 1            #Mass in kilogramme
Q = 1            #Quality factor
F_naught = 1     #force in newton
freqT = 0.1      #wT dimensionless
tmax = 10        #Maximum time to elapse

#Variables
T = 1
```

```python
freq = freqT/T

#Defining our timepoints
time = np.linspace(0,tmax,100)

def force(t):
    if t < T:
        return F_naught*t*(T-t)/(T**2)
    else:
        return 0


def system(t , func_array):
    u_prime = func_array[1]
    v_prime = 1/m * force(t) - freq/Q*func_array[1] -freq**2 *func_array[0]
    return [u_prime, v_prime]
    #function that takes in a vector [u, v]^T and returns [u' , v']^T

solution = solve(system, (0,tmax), [0,0], t_eval=time)
"""                 ^       ^       ^     \ Timepoints to evaluate at.
                    |       |        \ The initial values for u(t) and v(t)
                    |        \ The timespace in which system needs evaluating.
                     \ The system defining our system of first order ODE's
"""
plt.plot(solution.t,solution.y[0])
plt.show()
```

We can then use a similar program to plot the nine different scenarios, these are shown in figure 3.

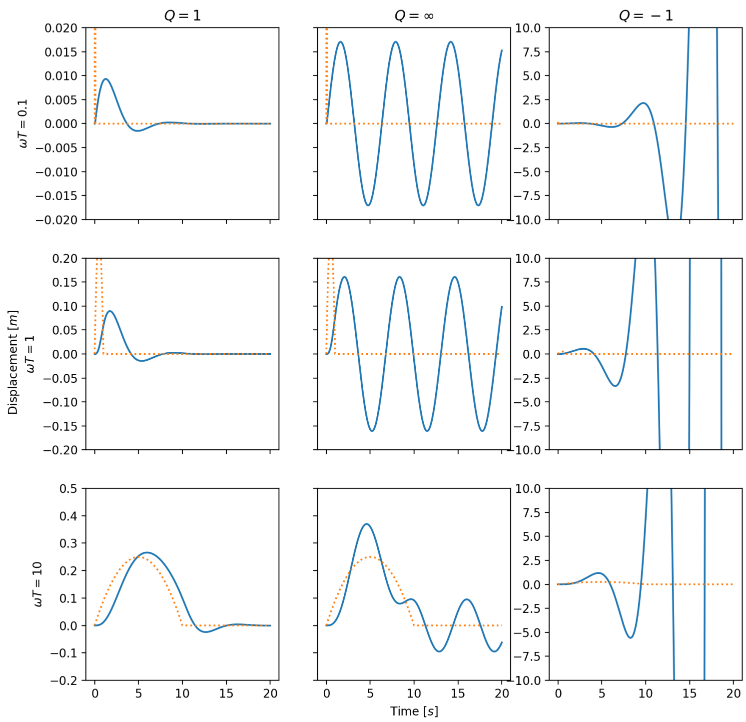

Figure 3: The resulting oscillations for a parabolic force modulation.

5 conclusion

References

[1] W. E. Boyce, R. C. Dirpima, and D. B. Meade, *Elementary Differential Equations and Boundary Value Problems*. John Wiley and Sons, Inc., 11 ed., 2017.

6 Appendix

6.1 Question 1

6.1.1 Solving $Y(t)$

In order to solve the particular solution to the differential equation we assume $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

Differentiating with respect to t once and twice yields:

$$\dot{Y}(t) = -a_1\omega \; sin(\omega t) + a_2\omega \; cos(\omega t)$$
$$\ddot{Y}(t) = -a_1\omega^2 \; cos(\omega t) - a_2\omega^2 \; sin(\omega t)$$

If we substitute this for $x(t)$ in equation 1 and with $F(t) = F_0 \; cos(\omega t)$ we find:

$$F(t)m = \ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t)$$
$$F_0 \; cos(\omega t) = m\left[-a_1\omega^2 \; cos(\omega t) - a_2\omega^2 \; sin(\omega t)\right] + m\frac{\omega}{Q}\left[-a_1\omega \; sin(\omega t) + a_2\omega \; cos(\omega t)\right] + m\omega^2\left[a_1 \; cos(\omega t) + a_2 \; sin(\omega t)\right]$$
$$F_0 \; cos(\omega t) = omega^2 m \; cos(\omega t)\left(a_1 + \frac{a_2}{Q} - a_1\right) + \omega^2 m \; sin(\omega t)\left(a_2 - \frac{a_1}{Q} - a_2\right)$$
$$F_0 \; cos(\omega t) = a_2\frac{\omega^2 m}{Q} \; cos(\omega t) - a_1\frac{\omega^2 m}{Q} \; sin(\omega t)$$

From this follows:

$$-a_1\frac{\omega^2 m}{Q} = 0 \qquad\qquad a_2\frac{\omega^2 m}{Q} = F_0$$
$$a_1 = 0 \qquad\qquad a_2 = \frac{F_0 Q}{\omega^2 m}$$

Therefore the solution to differential equation 1 is:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

We can find c_1 and c_2 by imposing the initial conditions on the latter equation:

$$x(0) = x_0 = c_1 + c_2$$
$$c_1 = x_0 - c_2$$

$$\dot{x}(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} + \frac{F_0 Q}{\omega m} cos(\omega t)$$
$$\dot{x}(0) = \dot{x}_0 = c_1 r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}$$
$$\dot{x}_0 = [x_0 - c_2]r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}$$
$$c_2(r_2 - r_1) = \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1$$
$$c_2 = \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\right)$$

We substitute this back into the equation for c_1:

$$\begin{aligned}
c_1 &= x_0 - c_2 \\
&= x_0 - \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\right) \\
&= \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0 r_2\right)
\end{aligned}$$