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# 1 Introduction

## 2 question 1

We want to solve the following equation for  $F(t) = F_0 \cos(\omega t)$ :

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \quad (1)$$

Equation 1 is a second order linear differential equation. According to theorem 3.5.2 in Boyce [1]:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \quad (2)$$

With  $Y(t)$ , any solution to the nonhomogeneous differential equation and with  $y_1(t)$  and  $y_2(t)$  that form a fundamental set of solutions to the homogeneous differential equation.

$$\ddot{y}(t) + \frac{\omega}{Q}\dot{y}(t) + \omega^2 y(t) = 0 \quad (3)$$

We first solve  $y(t)$  by seeing that, for constant values of  $\omega, m$  and  $Q$ , a solution for equation 3 is  $y(t) = e^{r t}$ . Applying to equation 3 yields:

$$(r^2 + \frac{\omega}{Q}r + \omega^2)y(t) = 0 \Rightarrow r^2 + \frac{\omega}{Q}r + \omega^2 = 0 \quad (4)$$

We find two solutions for  $r$ :

$$r_1 = \frac{-1}{2} \left( \frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2} \right), \quad r_2 = \frac{1}{2} \left( -\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2} \right)$$

We use this to define:

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

$r_1$  and  $r_2$  will be complex numbers for  $|Q| > \frac{1}{2}$  meaning that differential equation 1 will lead to a (damped) oscillator. Just as expected.

The method to find  $Y(t)$  is the method of undetermined coefficients described in section 3.5 of Boyce [1].

We assume the  $Y(t)$  is of the shape:

$$Y(t) = a_1 \cos(\omega t) + a_2 \sin(\omega t)$$

We find  $a_1 = 0$  and  $a_2 = \frac{F_0 Q}{\omega^2 m}$  (see appendix). Therefore we find:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{F_0 Q}{\omega^2 m} \sin(\omega t)$$

By imposing the initial conditions on the latter result (see appendix) we obtain:

$$c_1 = \frac{1}{r_2 - r_1} \left( -\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0(2r_2 - r_1) \right)$$

$$c_2 = \frac{1}{r_2 - r_1} \left( \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2 \right)$$

### 3 question 2

## 4 question 3

We want to solve equation 5 when a modulated external force is applied, this force follows from equation ??.

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2x(t) = F(t) \quad (5)$$

$$F(t) \quad (6)$$

## 5 conclusion

## References

- [1] W. E. Boyce, R. C. Dirpima, and D. B. Meade, *Elementary Differential Equations and Boundary Value Problems*. John Wiley and Sons, Inc., 11 ed., 2017.



## 6 Appendix

### 6.1 Question 1

#### 6.1.1 Solving $Y(t)$

In order to solve the particular solution to the differential equation we assume  $Y(t)$  is of the shape:

$$Y(t) = a_1 \cos(\omega t) + a_2 \sin(\omega t)$$

Differentiating with respect to  $t$  once and twice yields:

$$\dot{Y}(t) = -a_1 \omega \sin(\omega t) + a_2 \omega \cos(\omega t)$$

$$\ddot{Y}(t) = -a_1 \omega^2 \cos(\omega t) - a_2 \omega^2 \sin(\omega t)$$

If we substitute this for  $x(t)$  in equation 1 and with  $F(t) = F_0 \cos(\omega t)$  we find:

$$F(t)m = \ddot{x}(t) + m \frac{\omega}{Q} \dot{x}(t) + m \omega^2 x(t)$$

$$F_0 \cos(\omega t) = m [-a_1 \omega^2 \cos(\omega t) - a_2 \omega^2 \sin(\omega t)] + m \frac{\omega}{Q} [-a_1 \omega \sin(\omega t) + a_2 \omega \cos(\omega t)] + m \omega^2 [a_1 \cos(\omega t) + a_2 \sin(\omega t)]$$

$$F_0 \cos(\omega t) = \omega^2 m \cos(\omega t) \left( a_1 + \frac{a_2}{Q} - a_1 \right) + \omega^2 m \sin(\omega t) \left( a_2 - \frac{a_1}{Q} - a_2 \right)$$

$$F_0 \cos(\omega t) = a_2 \frac{\omega^2 m}{Q} \cos(\omega t) - a_1 \frac{\omega^2 m}{Q} \sin(\omega t)$$

From this follows:

$$-a_1 \frac{\omega^2 m}{Q} = 0$$

$$a_1 = 0$$

$$a_2 \frac{\omega^2 m}{Q} = F_0$$

$$a_2 = \frac{F_0 Q}{\omega^2 m}$$

Therefore the solution to differential equation 1 is:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \frac{F_0 Q}{\omega^2 m} \sin(\omega t)$$

We can find  $c_1$  and  $c_2$  by imposing the initial conditions on the latter equation:

$$x(0) = x_0 = c_1 + c_2$$

$$c_1 = x_0 - c_2$$

$$\dot{x}(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} + \frac{F_0 Q}{\omega m} \cos(\omega t)$$

$$\dot{x}(0) = \dot{x}_0 = c_1 r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}$$

$$\dot{x}_0 = [x_0 - c_2] r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}$$

$$c_2(r_2 - r_1) = \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1$$

$$c_2 = \frac{1}{r_2 - r_1} \left( \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2 \right)$$

We substitute this back into the equation for  $c_1$ :

$$\begin{aligned}c_1 &= x_0 - c_2 \\&= x_0 - \frac{1}{r_2 - r_1} \left( \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2 \right) \\&= \frac{1}{r_2 - r_1} \left( -\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0 (2 r_2 - r_1) \right)\end{aligned}$$