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1 Introduction

From quantum effects to pendulums and from data exchange to clocks, there are many phenomena and devices that are governed by harmonic oscillators. Since harmonic oscillators can be used to describe any system with a smooth potential near a stable equilibrium and many corresponding applications, it is a popular topic in physics. The harmonic oscillator is defined as a system in subject to a restoring force that is proportional to the displacement from an equilibrium position. As a result of this restoring force, harmonic oscillators will describe an oscillatory motion.
The aim of this study is to give insights in the behaviour of different types of harmonic oscillators subject to driving forces. This could, for example, be useful for data exchange in quantum physics.
For this experiment we will take a look at a basic pendulum subject to a driving force and a frictional force such as in the schematic drawing in figure 1. With x, the angle of excitation, m the mass of the pendulum, $F(t)$ the driving force, F_f the friction force and F_g the gravitational force on the mass.
We can solve $x(t)$ using Newtons second law of motion:

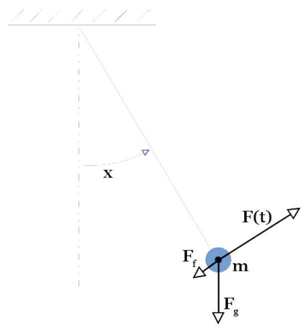

Figure 1: Schematic drawing of basic pendulum subject to a driving, gravitational and a frictional force corresponding to the symbols $F(t)$, F_g and F_f respectively. x corresponds to the excitation of the pendulum.

$$m\,a = \sum F$$
$$m\ddot{x} = -F_{g,x} - F_f + F(t)$$

To solve x as function of t, we suppose F_f is directly proportional to the velocity and note that the gravitational force in the x direction is given by $F_{g,x} = -F_g sin(x) = -m\,g\,sin(x)$, with g the gravitational constant.

$$m\ddot{x} = -m\;g\;sin(x) - \alpha\;\dot{x} + F(t)$$

For small values of x we can use a Taylor expansion for $sin(x)$ which yields $sin(x) \approx x$. Therefore, for small values of x we get the following differential equation:

$$m\ddot{x} + \alpha\;\dot{x} + m\;g\;x = F(t)$$

When there would be no frictional or driving force we would get:

$$m\ddot{x} + m\;g\;x = 0$$

This is straight forward to solve and gives:

$$x(t) = \beta_1 e^{i\omega t} + \beta_2 e^{-i\omega t}$$

With β_1, β_2 two arbitrary constants that depend on the initial conditions and $\omega = \sqrt{g}$ the natural frequency of the pendulum. We also define $alpha = \frac{m\,\omega}{Q}$, with Q the quality factor of the system. This would mean that for higher values of Q, the frictional force would be less and there would be fewer energy dissipation by the system. The m and ω terms are necessary in this definition of α in order to make Q dimensionless. Finally we find the following differential equation:

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \tag{1}$$

In this report, the behaviour of a pendulum subject by three types of driving forces will be studied. This will be done for a pendulum with a conventional frictional force, one without friction and one with 'inverse' friction. This corresponds to the values of $Q = 1$, $Q \to \infty$ and $Q = -1$ respectively. In the first part of this report, we will study the behaviour of the driven oscillator for an infinite driving force of $F(t) = F_0 \cos \omega t$. This is solved analytically . In the second part, a finite excitation $F(t) = F_0 cos(\omega t)$ for $0 < t < T$ and $F(t) = 0$ for $t \geq T$ with three different values $\omega T = 0.1$, $\omega T = 1$ and $\omega T = 10$ are solved analytically and studied. In the final part a finite driving force of $F(t) = F_0 t \frac{T-t}{T^2}$ for the same values of T as in the previous section.

2 Infinite excitation

We want to solve equation 1 for $F(t) = F_0 cos(\omega t)$. Equation 1 is a second order linear differential equation. According to theorem 3.5.2 in Boyce [**?**]:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \tag{2}$$

With $Y(t)$, any solution to the nonhomogeneous differential equation and with $y_1(t)$ and $y_2(t)$ that form a fundamental set of solutions to the homogeneous differential equation.

$$\ddot{y}(t) + \frac{\omega}{Q}\dot{y}(t) + \omega^2 y(t) = 0 \tag{3}$$

We first solve $y(t)$ by seeing that, for constant values of ω,m and Q, a solution for equation 3 is $y(t) = e^{r\,t}$. Applying to equation 3 yields:

$$(r^2 + \frac{\omega}{Q} r + \omega^2) y(t) = 0 \quad \Rightarrow \quad r^2 + \frac{\omega}{Q} r + \omega^2 = 0 \tag{4}$$

We find two solutions for r:

$$r_1 = -\frac{1}{2}\left(\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right) \quad , \quad r_2 = \frac{1}{2}\left(-\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right)$$

We use this to define:

$$y_1(t) = e^{r_1 t} \quad , \quad y_2(t) = e^{r_2 t}$$

r_1 and r_2 will be complex numbers for $\mid Q \mid > \frac{1}{2}$ meaning that differential equation 1 will lead to a (damped) oscillator. Just as expected.

The method to find $Y(t)$ is the method of undetermined coefficients described in section 3.5 of Boyce [**?**].
We assume the $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

We find $a_1 = 0$ and $a_2 = \frac{F_0 Q}{\omega^2 m}$ (see appendix). Therefore we find:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

By imposing the initial conditions on the latter result (see appendix) we obtain:

$$c_1 = \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0 r_2\right)$$

$$c_2 = \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\right)$$

We can plot x as function of t for the three discussed values of Q (figure 3. For the sake of simplicity $\omega = 1$, $F_0 = 1$ and $m = 1$ in this plot. A scaled version of $F(t)$ is also plotted in the same axis in order to study the response of the oscillator with respect to driving force.

We can observe several things from the graphs. For $Q = 1$, the oscillator will, after a period of amplification, oscillate with the same frequency as the driving force. However, with a phase shift. What's more, the oscillator seems to reach a steady state in approximately 1.5 oscillations where the amplitude stays constant. This steady state resembles a sine. From this it can be concluded that $Q = 1$ corresponds to a damped oscillator for which a steady state is reached when the work done by the driving force on the system equals the energy dissipation by friction.
For $Q \to \infty$ we find that the oscillator will also oscillate at the same rate as $F(t)$ but with a phase shift.

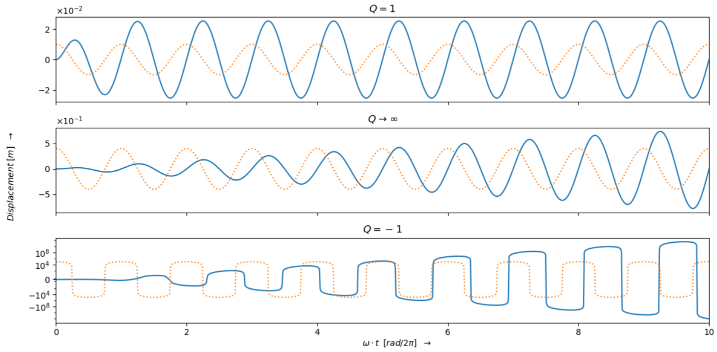

Figure 2: Graphs of $x(t)$ and $F(t)$ as function of t for different values of Q. The solid blue lines correspond with $x(t)$ and the orange dashed lines correspond to a scaled version of $F(t)$.

However, the amplitude does not stabilize and seems to increase linearly with time. Therefore, $Q \to \infty$, could correspond to a frictionless driven oscillator. For this situation, there is no energy dissipation in the system, only work done on the system by the driving force. The system has the same frequency as the driving force so the energy gain by the system should be equal per oscillation. Therefore the oscillation amplitude, being directly proportional with the total energy in the system, should increase with the same amount each oscillation.
For $Q = -1$ we find that the oscillator does not reach the same oscillating frequency as the $F(t)$. It seems to be lagging behind the driving force. We also see that the amplitude of the oscillation increases linearly on a logarithmic y-scale. Therefore, the amplitude increases exponentially with respect to time. The behaviour of the oscillation can be explained by a 'negative' friction exerting a force on the system in the direction of the motion instead of in the opposite direction for 'regular' friction. Therefore, the systems energy is amplified every oscillation.

3 Finite excitation

For the three values of Q we now consider a situation with a finite excitation time such that $F(t) = F_0 cos(\omega t)$ for $0 < t < T$ and $F(t) = 0$ for $t \geq T$. We can solve $x(t)$ analytically, using the result for a driving force of $F(t) = F_0 cos(\omega t)$ for $t > 0$. Using this, we can find $x(T)$ and $\dot{x}(T)$. Subsequently, We can use this as the initial conditions in the same function $x(t)$, however with $F_0 = 0$. We also need to shift the function $t = T$ to future.

$$x(t)(t) = \begin{cases} x_1(t) & , 0 < t < T \\ x_2(t-T) & , t \geq T \end{cases}$$

With:

$$x_1(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$
$$x_2(t) = d_1 y_1(t) + d_2 y_2(t)$$

c_1, c_2, y_1 and y_2 are defined as in the previous section and for d_1 and d_2:

$$d_1 = \frac{1}{r_2 - r_1}(-\dot{x}_1(T) + x_1(T) r_2)$$
$$d_2 = \frac{1}{r_2 - r_1}(\dot{x}_1(T) - x_1(T) r_1)$$

We can plot graphs for the values $\omega T = 0.1$, $\omega T = 1$ and $\omega T = 10$ for the three discussed values of Q. A scaled version of $F(t)$ is also plotted in the same axes in order to study the system with respect to the driving force:

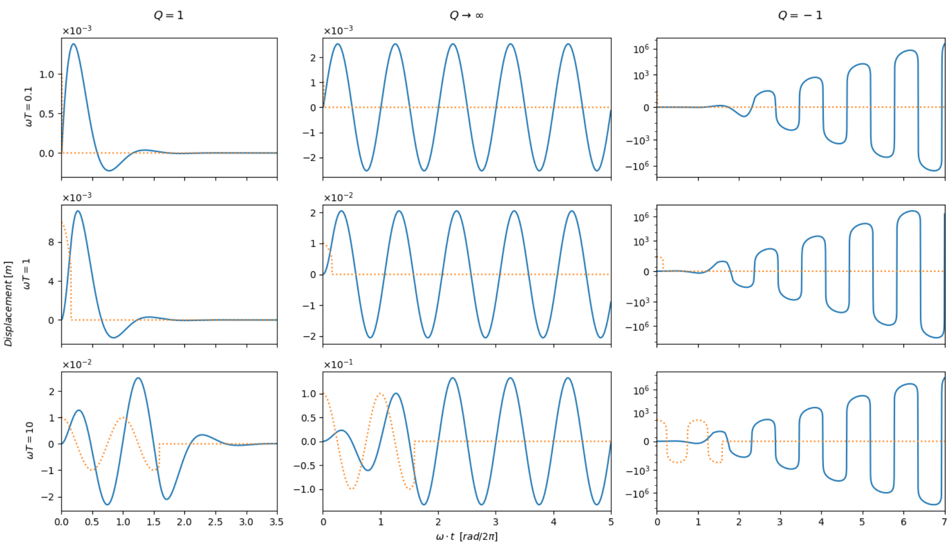

Figure 3: Graphs of $x(t)$ and $F(t)$ as function of t for different values of Q and T. The solid blue lines correspond with $x(t)$ and the orange dashed lines correspond to a scaled version of $F(t)$.

We can make a number of observations from the graphs. Considering $Q = 1$, we find that , when there is no driving force, $x(t)$ will converge to zero. This is logical, since for $t > T$, there is only energy dissipation in the system by the friction. We also see that the graphs for $\omega T = 0.1$ and $\omega T = 1$, the graphs are practically of identical shape, only differing by the amplitude and a slight time shift. The graph for $\omega T = 10$ shows that the shape of $x(t)$ is very different when T is in a different phase of the oscillation.

For $Q \to \infty$ we find that for $t = T$, the oscillator will reach a steady state. This is logical since there is no energy dissipation or work done on the system. Therefore, the total energy in the system will remain constant. The amplitude of the steady state is the amplitude that corresponds with the energy in the system at $t = T$. We also see that the peaks of the three graphs are relatively close to each other. We can conclude from the graphs that the value of T does not affect the shape of $x(t)$ and hardly affects the phase for an oscillator without friction. The amplitude is related to the excitation time. For $Q = -1$ we see that the three graphs are very similar of shape, phase. Considering the logarithmic scale on the graphs, relative amplitude between $\omega T = 10$ and $\omega T = 0.1$ is more than a factor ten. The graphs do show some differences in shape in the long term. However, after about 2 oscillations, these differences become indivisible. What's more is the amplification of the amplitude which seems equally for the three graphs and similar to the case with infinite excitation time in the previous section. It seems as for this system with 'negative' friction. therefore, it seems as if the excitation time does not make a considerable difference to the system in the long term.

4 Amplitude modulation

4.1 Analytical method

We want to solve equation 5 when a amplitude modulated external force is applied, this force follows from equation 6.

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \tag{5}$$

$$F(t) = F_0 t\frac{T-t}{T^2} = F_0\frac{t}{T} - F_0\frac{t^2}{T^2} \tag{6}$$

Since this equation 6 is relatively simple, it can be solved using the method of undetermined coefficients as outlined in paragraph 3.5 of Boyce [**?**].
The homogenous form has already been solved in the previous questions, the resulting roots are shown below in equation 7 and the solution in 8.

$$r_1 = -1/2\left(\frac{\omega}{Q} + \sqrt{\frac{\omega^2}{Q^2} - 4\omega^2}\right), r_2 = 1/2\left(-\frac{\omega}{Q} + \sqrt{\frac{\omega^2}{Q^2} - 4\omega^2}\right) \tag{7}$$

$$y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t} \tag{8}$$

Now for the particular solution $y_p(t)$ we use the aforementioned method, the derivation is show below. We start by assuming that $y_p(t)$ is of the shape:

$$y_p(t) = c_1 + c_2 \cdot t + c_3 \cdot t^2$$

If we then differentiate $y_p(t)$ two times and substitute the result into equation 5 we get the following:

$$y_p'(t) = c_2 + 2c_3 \cdot t, y_p"(t) = 2c_3$$

$$my"(t) + \frac{m\omega}{Q}y'(t) + m\omega^2 y(t) = F_0\frac{t}{T} - F_0\frac{t^2}{T^2}$$
$$m(2a_3) + \frac{m\omega}{Q}(a_2 + 2a_3 t) + m\omega^2(a_1 + a_2 t + a_3 t^2) = F_0\frac{t}{T} - F_0\frac{t^2}{T^2}$$

If we then equate the terms in front of the functions and it's derivatives we get the following:

$$m\omega^2 a_1 + \frac{m\omega a_2}{Q} + 2ma_3 = 0$$
$$m\omega^2 a_2 + \frac{2m\omega a_3}{Q} = \frac{F_0}{T}$$
$$m\omega^2 a_3 = -\frac{F_0}{T^2}$$

Solving the system of equations and substituting back into $y_p(t)$ yields:

$$a_1 = \frac{F_0(2Q^2 - QT\omega - 2)}{mQ^2T^2\omega^4}$$
$$a_2 = \frac{F_0(QT\omega + 2)}{mQT^2\omega^3}$$
$$a_3 = \frac{-F_0}{mT^2\omega^2}$$

$$y_p(t) = \frac{F_0 t}{mT\omega^2}(1 - t/T) + \frac{F_0}{mQT\omega^3}\left[\frac{2(t+Q)}{T} - \frac{2}{QT\omega} - 1\right] \tag{9}$$

4.2 Numerical method

To solve equation 5 numerically we first have to split the second order differential equations into a system of two first order equations. We do this by substituting two new time dependant functions for y, namely $u(t)$ and $v(t)$ equal to $y(t)$ and $y'(t)$ respectively. We can then derive the following system:

$$\begin{cases} \text{u(t) = y(t)} \\ \text{v(t) = y'(t)} \end{cases}$$

So that their derivatives become:

$$\begin{cases} \text{u'(t) = y'(t) = v(t)} \\ \text{v'(t) = y"(t)} \end{cases}$$

If we then substitute in these equations into the second order differential equation we get the following system:

$$\begin{aligned} u'(t) &= y'(t) = v(t) \\ v'(t) &= y''(t) \\ F(t) &= my''(t) + m\frac{\omega}{Q}y'(t) + m\omega^2 y(t) \end{aligned}$$

$$\begin{aligned} v'(t) &= y''(t) = 1/m \cdot F(t) - \omega/Q \cdot y'(t) - \omega^2 \cdot y(t) \\ v'(t) &= 1/m \cdot F(t) - \omega/Q \cdot v(t) - \omega^2 \cdot u(t) \end{aligned}$$

So that we now have the following system of of first-order linear differential equations:

$$v'(t) = 1/m \cdot F(t) - \omega/Q \cdot v(t) - \omega^2 \cdot u(t) \tag{10}$$

$$u'(t) = v(t) \tag{11}$$

This system can be easily solved numerically using the python code shown in the appendix section 6.2. We can then use a similar program to plot the nine different scenarios, these are shown in figure 4.

When looking at the graph we can see the different values of ωT plotted in rows and the different values of Q in columns, all plots share the same time domain but only plots on the same row and in the first and second column share displacement axis.
The different values of ωT make noticeable differences in the graphs, the smallest value $\omega T = 0.1$ acts a lot like an impulse function, applying a short but strong force to set de oscillator into motion. We can see that for different quality factors the system reacts differently, for a normal friction force the system gradually returns to the equilibrium position, for $Q = \infty$ there is not friction force and the system start oscillating forever and looks like it had a starting velocity due to the short impulse force, for the quality factor of $Q = -1$ the friction force actually puts more energy into the system causing it to spiral out of control. For the higher values of ωT we can see that the force is no longer representative of an impulse, meaning that the acceleration is not instantaneous, in the graph this can be seen by the small curve at the start of the plots where the oscillating mass needs to gain energy first. The different quality factors have the same effect as before. The other noticeable effect is that in the bottom middle graph for $Q = \infty$ and $\omega T = 10$ we can see that the duration of the force applied is to long, at about 7 seconds the force starts to work against the oscillation thus having the adverse to the intended effect of a driving force. This effect can also be seen though less noticeably so with the graph to the left where the force is also pushing the mass away from equilibrium while the mass is already moving towards the equilibrium position.

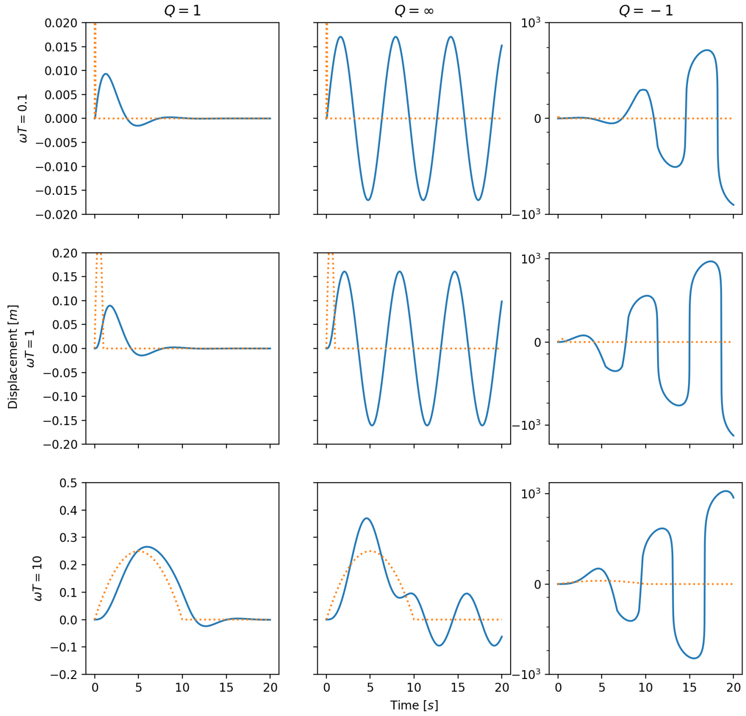

Figure 4: The resulting oscillations for a parabolic force modulation.

5 conclusion

After having worked through all the question we have learned a new method for solving differential equations, we have gained new intuition on how different systems react to different damping and driving forces. In question 3 we realised that it is easier and faster to solve differential equations numerically than analytically.

6 Appendix

6.1 Question 1

6.1.1 Solving $Y(t)$

In order to solve the particular solution to the differential equation we assume $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

Differentiating with respect to t once and twice yields:

$$\dot{Y}(t) = -a_1\omega\, sin(\omega t) + a_2\omega\, cos(\omega t)$$
$$\ddot{Y}(t) = -a_1\omega^2\, cos(\omega t) - a_2\omega^2\, sin(\omega t)$$

If we substitute this for $x(t)$ in equation 1 and with $F(t) = F_0\, cos(\omega t)$ we find:

$$
\begin{aligned}
F(t)m &= \ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t)\\
F_0\, cos(\omega t) &= m\left[-a_1\omega^2\, cos(\omega t) - a_2\omega^2\, sin(\omega t)\right] + m\frac{\omega}{Q}\left[-a_1\omega\, sin(\omega t) + a_2\omega\, cos(\omega t)\right] + m\omega^2\left[a_1\, cos(\omega t) + a_2\, sin(\omega t)\right]\\
F_0\, cos(\omega t) &= omega^2 m\, cos(\omega t)\left(a_1 + \frac{a_2}{Q} - a_1\right) + \omega^2 m\, sin(\omega t)\left(a_2 - \frac{a_1}{Q} - a_2\right)\\
F_0\, cos(\omega t) &= a_2\frac{\omega^2 m}{Q}\, cos(\omega t) - a_1\frac{\omega^2 m}{Q}\, sin(\omega t)
\end{aligned}
$$

From this follows:

$$-a_1\frac{\omega^2 m}{Q} = 0 \qquad\qquad a_2\frac{\omega^2 m}{Q} = F_0$$
$$a_1 = 0 \qquad\qquad a_2 = \frac{F_0 Q}{\omega^2 m}$$

Therefore the solution to differential equation 1 is:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

We can find c_1 and c_2 by imposing the initial conditions on the latter equation:

$$
\begin{aligned}
x(0) = x_0 &= c_1 + c_2\\
c_1 &= x_0 - c_2
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}(t) &= c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} + \frac{F_0 Q}{\omega m} cos(\omega t)\\
\dot{x}(0) = \dot{x}_0 &= c_1 r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}\\
\dot{x}_0 &= [x_0 - c_2]r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}\\
c_2(r_2 - r_1) &= \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\\
c_2 &= \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\right)
\end{aligned}
$$

We substitute this back into the equation for c_1:

$$\begin{aligned} c_1 &= x_0 - c_2 \\ &= x_0 - \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\right) \\ &= \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0 r_2\right) \end{aligned}$$

6.2 Question 3

```
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp as solve

"""
|                         code in between lines                         |
"""

#Defining constants
m = 1              #Mass in kilogramme
Q = 1              #Quality factor
F_naught = 1       #force in newton
freqT = 0.1        #wT dimensionless
tmax = 10          #Maximum time to elapse

#Variables
T = 1
freq = freqT/T

#Defining our timepoints
time = np.linspace(0,tmax,100)

def force(t):
    if t < T:
        return F_naught*t*(T-t)/(T**2)
    else:
        return 0


def system(t , func_array):
    u_prime = func_array[1]
    v_prime = 1/m * force(t) - freq/Q*func_array[1] -freq**2 *func_array[0]
    return [u_prime, v_prime]
    #function that takes in a vector [u, v]^T and returns [u' , v']^T

solution = solve(system, (0,tmax), [0,0], t_eval=time)
"""                  ^       ^       ^     \ Timepoints to evaluate at.
                     |       |        \ The initial values for u(t) and v(t)
                     |        \ The timespace in which system needs evaluating.
                      \ The system defining our system of first order ODE's
"""
plt.plot(solution.t,solution.y[0])
plt.show()
```