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We want to solve the following equation for $F(t) = F_0 cos(\omega t)$:

$$m\ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t) = F(t) \tag{1}$$

Equation 1 is a second order linear differential equation. According to theorem 3.5.2 in Boyce [?]:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \tag{2}$$

With $Y(t)$, any solution to the nonhomogeneous differential equation and with $y_1(t)$ and $y_2(t)$ that form a fundamental set of solutions to the homogeneous differential equation.

$$\ddot{y}(t) + \frac{\omega}{Q}\dot{y}(t) + \omega^2 y(t) = 0 \tag{3}$$

We first solve $y(t)$ by seeing that, for constant values of ω,m and Q, a solution for equation 3 is $y(t) = e^{r\,t}$. Applying to equation 3 yields:

$$(r^2 + \frac{\omega}{Q}r + \omega^2)y(t) = 0 \quad \Rightarrow \quad r^2 + \frac{\omega}{Q}r + \omega^2 = 0 \tag{4}$$

We find two solutions for r:

$$r_1 = \frac{-1}{2}\left(\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right) \quad , \quad r_2 = \frac{1}{2}\left(-\frac{\omega}{Q} + \sqrt{\left(\frac{\omega}{Q}\right)^2 - 4\omega^2}\right)$$

We use this to define:

$$y_1(t) = e^{r_1 t} \quad , \quad y_2(t) = e^{r_2 t}$$

r_1 and r_2 will be complex numbers for $\mid Q \mid > \frac{1}{2}$ meaning that differential equation 1 will lead to a (damped) oscillator. Just as expected.

The method to find $Y(t)$ is the method of undetermined coefficients described in section 3.5 of Boyce [?].
We assume the $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

We find $a_1 = 0$ and $a_2 = \frac{F_0 Q}{\omega^2 m}$ (see appendix). Therefore we find:

$$x(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

By imposing the initial conditions on the latter result (see appendix) we obtain:

$$c_1 = \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0(2\,r_2 - r_1)\right)$$

$$c_2 = \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2\right)$$

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5 conclusion

6 Appendix

6.1 Question 1

6.1.1 Solving $Y(t)$

In order to solve the particular solution to the differential equation we assume $Y(t)$ is of the shape:

$$Y(t) = a_1 cos(\omega t) + a_2 sin(\omega t)$$

Differentiating with respect to t once and twice yields:

$$\dot{Y}(t) = -a_1\omega\ sin(\omega t) + a_2\omega\ cos(\omega t)$$
$$\ddot{Y}(t) = -a_1\omega^2\ cos(\omega t) - a_2\omega^2\ sin(\omega t)$$

If we substitute this for $x(t)$ in equation 1 and with $F(t) = F_0\ cos(\omega t)$ we find:

$$\begin{aligned}
F(t)m &= \ddot{x}(t) + m\frac{\omega}{Q}\dot{x}(t) + m\omega^2 x(t)\\
F_0\ cos(\omega t) &= m\left[-a_1\omega^2\ cos(\omega t) - a_2\omega^2\ sin(\omega t)\right] + m\frac{\omega}{Q}\left[-a_1\omega\ sin(\omega t) + a_2\omega\ cos(\omega t)\right] + m\omega^2\left[a_1\ cos(\omega t) + a_2\ sin(\omega t)\right]\\
F_0\ cos(\omega t) &= omega^2 m\ cos(\omega t)\left(a_1 + \frac{a_2}{Q} - a_1\right) + \omega^2 m\ sin(\omega t)\left(a_2 - \frac{a_1}{Q} - a_2\right)\\
F_0\ cos(\omega t) &= a_2\frac{\omega^2 m}{Q}\ cos(\omega t) - a_1\frac{\omega^2 m}{Q}\ sin(\omega t)
\end{aligned}$$

From this follows:

$$-a_1\frac{\omega^2 m}{Q} = 0 \qquad\qquad a_2\frac{\omega^2 m}{Q} = F_0$$
$$a_1 = 0 \qquad\qquad a_2 = \frac{F_0 Q}{\omega^2 m}$$

Therefore the solution to differential equation 1 is:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \frac{F_0 Q}{\omega^2 m} sin(\omega t)$$

We can find c_1 and c_2 by imposing the initial conditions on the latter equation:

$$\begin{aligned}
x(0) = x_0 &= c_1 + c_2\\
c_1 &= x_0 - c_2
\end{aligned}$$

$$\begin{aligned}
\dot{x}(t) &= c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} + \frac{F_0 Q}{\omega m} cos(\omega t)\\
\dot{x}(0) = \dot{x}_0 &= c_1 r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}\\
\dot{x}_0 &= [x_0 - c_2]r_1 + c_2 r_2 + \frac{F_0 Q}{\omega m}\\
c_2(r_2 - r_1) &= \dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_1\\
c_2 &= \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2\right)
\end{aligned}$$

We substitute this back into the equation for c_1:

$$\begin{aligned} c_1 &= x_0 - c_2 \\ &= x_0 - \frac{1}{r_2 - r_1}\left(\dot{x}_0 - \frac{F_0 Q}{\omega m} - x_0 r_2\right) \\ &= \frac{1}{r_2 - r_1}\left(-\dot{x}_0 + \frac{F_0 Q}{\omega m} + x_0(2\,r_2 - r_1)\right) \end{aligned}$$