RP Propagation of Light - Amar Ramdas & Azat Arica

Research Practicum (Technische Universiteit Delft)

Propagation of light

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Index

Abstract

The goal of this research practicum was to analyse and apply Fermat's principles to several situations, such as a homogeneous medium, Snell-Descartes' law, a mirage and an optical fibre.

A mathematical explanation of Fermat's theorem is given, showing path light takes with the least amount of time.

Snell-Descartes' law was analytically backed using Fermat's theorem and it was applied to a homogeneous medium. Fermat's theorem was used to describe the phenomena of a mirage, and showed the light path taken by the mirage.

Lastly, it was used to describe the way light behaves inside an optical fibre. An equation was found that describes this behaviour, as well as a numerical approximation, which seemed to be valid as it complied with said equation.

1. Introduction

The goal of this report is to describe the propagation of light in a medium with a nonhomogeneous index of refraction. This is done using the Principle of Fermat, Snell-Descartes law and Euler-Lagrange equations.

Light plays an essential part in a major part of the processes that take place in our world. Furthermore, mankind has integrated the usage of photons in many of our technologies, including for example the fibre optics used in data transmission. This is why it's very important to understand the fundamentals and behaviour of photons in various mediums.

A large part of modern day understandings of the ways photons interact with different mediums is based upon the works of Fermat and Snell. Fermat stated that light travels between two points on the path that takes the shortest time, instead of the shortest distance as previously thought. [1]

In this paper, the principle of Fermat is first introduced and elaborated on and then applied to a homogeneous medium. It is then used to prove the law of Snell-Descartes and to explain the behaviour of light in a mirage. Finally it is used to analytically create an equation to describe the way light particles travel in an optical fibre, a very promising modern way to transmit data.

This experiment is part of the Research Practicum at the study Applied Physics at TU Delft.

2. Assignments

2.1 Principle of Fermat

Considering a medium with where $\vec{r}$ [1] are spatial points which form the ray path of a certain curve. For every point in space an index of refraction can be calculated with the function $n(\vec{r})$. Integrating this function as stated in equation [2] will result in the optical path. With ds being an infinitesimally small decimal of the optical path can be said that it equals the infinitesimally small length of the $\vec{r}$ as stated in equation [3] and the time derivative in equation [4].

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

[1]

$$L_{AB} = \int_A^B n(\vec{r})ds$$

[2]

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = d\|\vec{r}\|$$

[3]

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

[4]

Fermat's principle stats that light will always travel the shortest distance. Which means that the integral of equation [1] is stationary. Another approach of minimizing the optical path length is by integrating the Lagrangian. The optical Lagrangian is stated at equation [5]. Meaning the time integral of the optical Lagrangian will also result in the optical path as shown in equation [6]. It is important to remark that the variables x', y' and z' are the derivatives of the components of $\vec{r}$.

$$\mathcal{L}[x, y, z, x', y', z'] = n(x, y, z)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

[5]

$$L_{AB} = \int_{t_A}^{t_B} \mathcal{L}[x, y, z, x', y', z']t = \int_{t_A}^{t_B} n(x, y, z)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

[6]

In the manual can be found that the integral of equation [6] can be stated as three differential equations which are show in equations [7]. The sum of these differential equations can be found at equation [8].

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial x'}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial z'}\right) - \frac{\partial \mathcal{L}}{\partial z} = 0$$

[7]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial x'} + \frac{\partial \mathcal{L}}{\partial y'} + \frac{\partial \mathcal{L}}{\partial z'}\right) = \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial y} + \frac{\partial \mathcal{L}}{\partial z}$$

[8]

One way to solve the differential equation stated at equation [8] is by simply calculating the partial derivatives of the optical Lagrangians which are defined at equation [5]. The set of equations where this is done can be found at equation [9]

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}\left(n\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\right) = \frac{\partial n}{\partial x}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{ds}{dt}\left(\frac{\partial n}{\partial x}\right)$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial y}\left(n\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\right) = \frac{\partial n}{\partial y}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{ds}{dt}\left(\frac{\partial n}{\partial y}\right)$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{\partial}{\partial z}\left(n\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\right) = \frac{\partial n}{\partial z}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{ds}{dt}\left(\frac{\partial n}{\partial z}\right)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial x'}\right) = \frac{\frac{dx}{dt}n}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} = \frac{\frac{dx}{dt}n}{\frac{ds}{dt}} = \frac{dx}{ds}n$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = \frac{\frac{dy}{dt}n}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} = \frac{\frac{dy}{dt}n}{\frac{ds}{dt}} = \frac{dy}{ds}n$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial z'}\right) = \frac{\frac{dz}{dt}n}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} = \frac{\frac{dz}{dt}n}{\frac{ds}{dt}} = \frac{dz}{ds}n$$

[9]

At this point the results of equation [9] needs to be filled in equation [8]. After some algebraic manipulations found in equation [10] it results in the eikonal equation found in equation [11].

$$\frac{d}{dt}n\left(\frac{dx}{ds} + \frac{dy}{ds} + \frac{dz}{ds}\right) = \frac{ds}{dt}\left(\frac{\partial n}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial n}{\partial z}\right)$$

$$\frac{d}{dt}\left(n\frac{d}{ds}\vec{r}\right) = \frac{ds}{dt}\left(\vec{\nabla}n\right)$$

[10]

$$\frac{d}{ds}\left(n\frac{d}{ds}\vec{r}\right) = \vec{\nabla}n$$

[11]

2.2 Application

2.2.1 Homogeneous medium

Having light travel in a homogeneous medium means that every spatial point $\vec{r}$ encounters the same index of refraction. Since Fermat's principle stats that light travels the shortest distance between two points in space, an obvious ray path $\vec{r}$ would be linear.

To demonstrate this with the eikonal equation [11], it would mean that since the index of refraction is the same for every spatial point in space it can considered as a constant. The divergence of a constant value always results in the zero vector as seen in equation [12]. Since the $\partial x, \partial y$ and ∂z are infinitesimally small elements of the $\vec{r}$ vector, there are no unit vectors used.

$$\frac{d}{ds}\left(n\frac{d}{ds}\vec{r}\right) = \vec{\nabla}n = \frac{\partial n}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial n}{\partial z} = \vec{0} \tag{12}$$

This results in a second order homogeneous differential equation. Integrating both sides of equation [12] Results in equations [13] and [14]. Since an indefinite integral is used the integral of the zero part results in an sum of the integration constant C.

$$\int \frac{d}{ds}\left(n\frac{d}{ds}\vec{r}\right) ds = \int 0 ds \tag{13}$$

$$n\frac{d}{ds}\vec{r} = \vec{0} + C \tag{14}$$

Looking at the left side of equation [14] it states that the derivative of the ray path $\vec{r}$ over the optical path length s should always stay constant, which is only possible with a linear function. Another way to look at it is by formulating the derivative of equation [14] by using equation [15].

$$\frac{\vec{r}}{\|\vec{r}\|} = \hat{r} \tag{15}$$

$$n\frac{d}{ds}\vec{r} = n\frac{d\vec{r}}{d\|\vec{r}\|} = n\hat{r} \tag{16}$$

Equation [16] states that the derivative results in the unit vector $\hat{r}$. Meaning that the ray path can only travel in one direction, which makes it linearly.

2.2.2 Snell-Descartes Law

In a homogeneous medium, the path light takes will always be the shortest and at the same time the fastest route it can take. This can be seen in figure 1, where an observer looks at a picture of Willebrord Snellius in a system filled with just air divided in two arbitrary vertical sections:

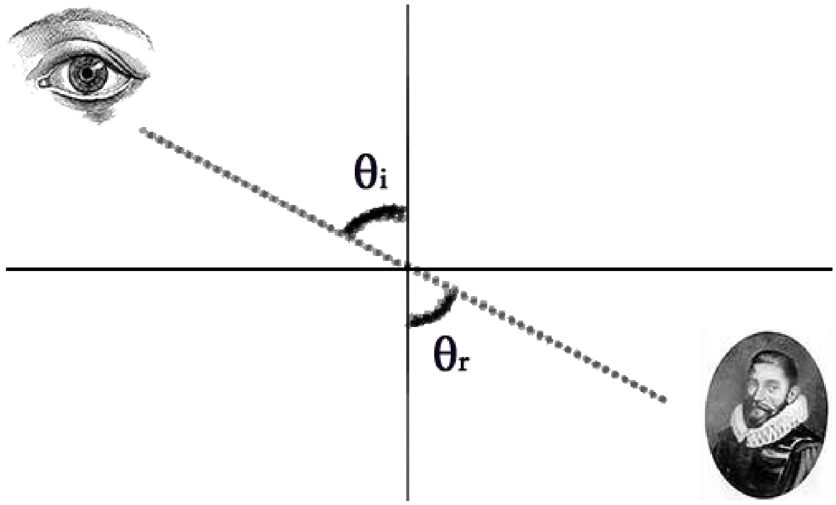

Figure 1

It is clear that the light takes a straight path, with the angle of impact and the angle of refraction being the same, or $\theta_i = \theta_r$.

However, with an inhomogeneous system, this is not always the case. If light travels through the border of two types of matter with different respective speeds of light, the shortest path is not always the fastest path. Take a look at figure 2 for example, where an observer looks at a picture of Snellius submerged in water:

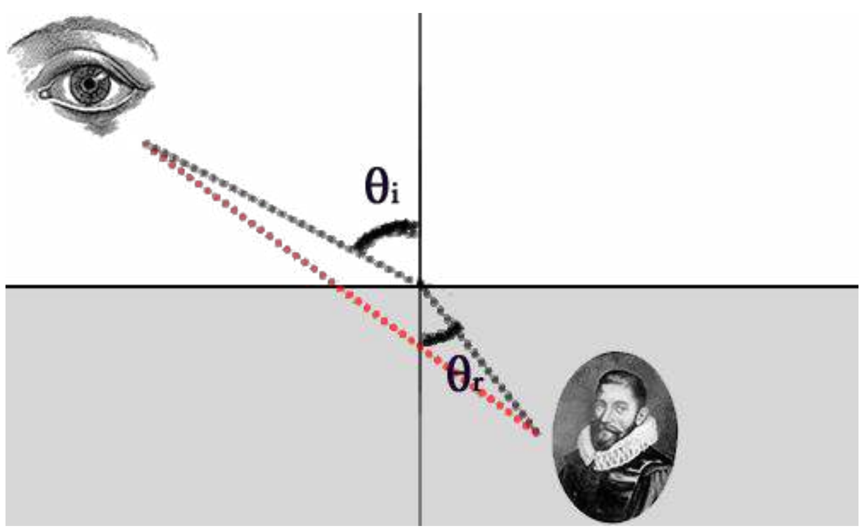

Figure 2

The shortest trajectory between the observer and Snellius is given in red, whereas the actual trajectory of light is given in black. The actual path the light takes is not the shortest one.

Fermat supposed this happens due to light taking the fastest path, opposed to the shortest one. This means that in figure 2, the light will bend towards the axis of refraction to accommodate a shorter path inside the medium with a slower speed of light, in this case water.

The mathematical backing of this theorem is a little bit more complex. First, to minimize the total time, equation [17] is used:

$$t = \frac{x_{air}}{v_i} + \frac{x_{water}}{v_r}$$

[17]

with x_{air} being the distance travelled in air, x_{water} the distance travelled in water, v_i the velocity of light in air and v_r the velocity of light in water. These distances, as well distance A, B, C, D and the Origin, are shown in figure 3.

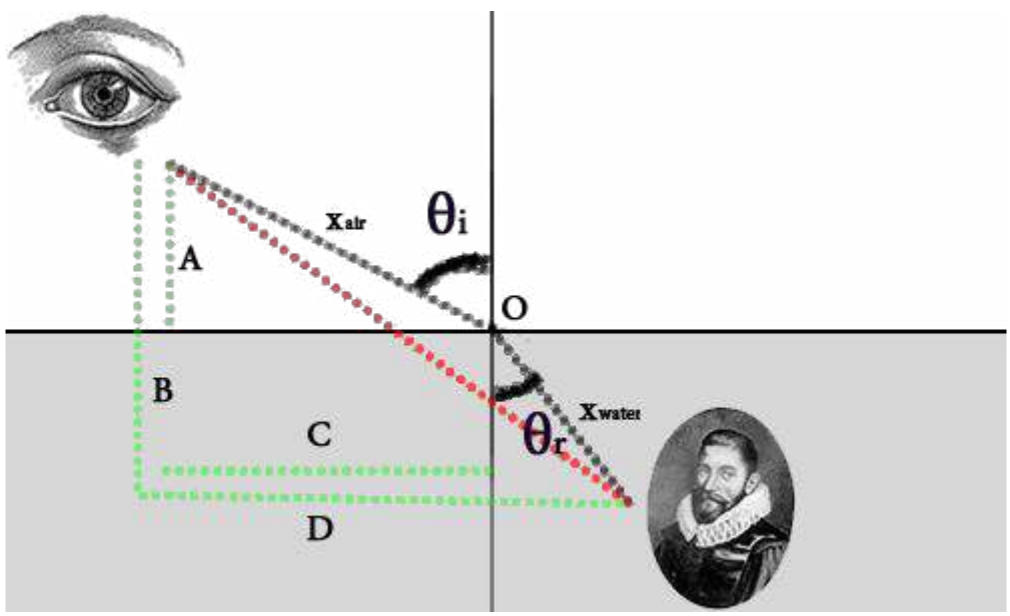

Figure 3

Applying Pythagoras' theorem to figure Z, equation [18] can be rewritten as:

$$t = \frac{\sqrt{A^2 + C^2}}{v_i} + \frac{\sqrt{B^2 + (D - C)^2}}{v_r}$$

[18]

The fastest path can be found by finding the minimum of the time function by differentiating over C. This can be done by solving the equation:

$$\frac{dt}{dx} = \frac{C}{v_i\sqrt{A^2 + C^2}} - \frac{D - C}{v_r\sqrt{B^2 + (D - C)^2}} = 0$$

This in turn can be rewritten into the equation:

$$\frac{dt}{dx} = \frac{sin\theta_i}{v_i} - \frac{sin\theta_r}{v_r} = 0$$

by introducing the angles of impact and refraction. This equation can also be rewritten to

$$v_t sin\theta_i = v_r sin\theta_r$$

Which in turn can be written into Snell's law using the fact that $v = \frac{c}{n}$, resulting in the equation:

$$n_i sin\theta_i = n_r sin\theta_r$$

2.2.3 Mirage

To know how mirages come to existence it is important to understand how refraction caused by different mediums on a visible level and quantum level. When a photon meets an atom, the energy cared by the photon is stored as potential energy in the atom and almost at the same time released. This causes a very short time delay on the propagation of the photon.

Eventually the wave character of the photon causes the wave front to change directions, which on visible level is translated as refraction. The Huygens-Fresnel principle illustrated that a photon that meets with an atom propagates as a point source. The time delay on several parts of the wave front causes the wave to change direction as seen in (figure). In a simplified situation it means that when a photon enters a different medium perpendicular to the surface of the medium, it will not be refracted at all.

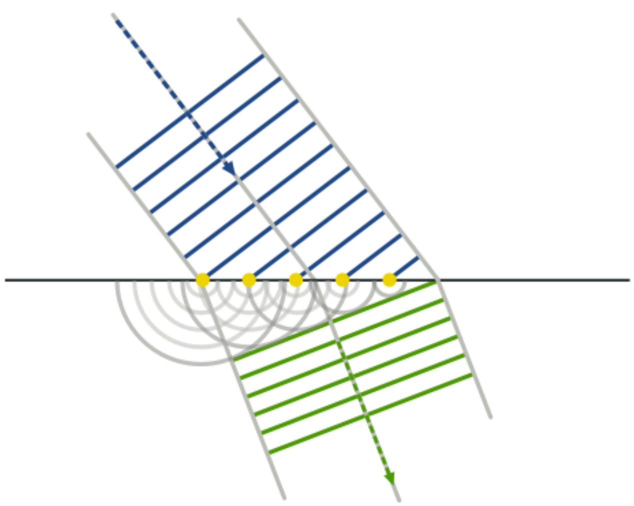

Figure 4: An illustrative representation of the Huygens-Fresnel principle

In the case of a mirage there are several factors which can cause the bending of light. For example, temperature, pressure of the atmosphere, concentrations of various atoms in air and the wavelength of visible light. When the air contains more density, mainly shorter waved light will be observed during sunset. This could cause the light bend even more and create a mirage at shorted distances.

The bending light in the case of a mirage still follows Huygens-Fresnel principle, but there is no clear passage between the mediums or a known orientation of surface. Which means that even when light propagates horizontally it will still bend, since the orientation of the surface is dynamic. Horizontal rays will be less effected by temperature effects, since the altitude does not change, but they can still bend because of the composition difference of the gasses for every infinitesimal slice of the medium they pass.

Every infinitesimal slice in the medium has a different index of refraction. Within the slices the index of refraction also varies depending on the temperature which is related to the height. Assuming the relation between temperature-height and temperature-refraction index are linear, it gives following equations that can be found at equation [19]

$$T(z) = T_0 - \alpha z$$

$$n(T) = n_0 + \beta T$$

$$n(T(z)) = n_0 + \beta(T_0 - \alpha z) = n_0 + \beta T_0 - \beta\alpha z$$

$$n(z) = a + bz$$

[19]

Where α and β are constants which can influence the slope of the linear relation. Some combination of constants can be eventually reduced to certain constants called a and b. At this point equation [11] comes into play. Calculating the gradient gives the following result that can be found at equation [20]

$$\frac{d}{ds}\left(n\frac{d}{ds}\vec{r}\right) = \vec{\nabla}n(z) = \frac{\partial(a+bz)}{\partial x}\hat{x} + \frac{\partial(a+bz)}{\partial y}\hat{y} + \frac{\partial(a+bz)}{\partial z}\hat{z} = b\hat{z}$$

[20]

Which means that the rate change of the position vector $\vec{r}$ only happens in the z-plane. Which is obvious since the light path in the case of a mirage only undergoes change rates in the height (which is in the z-direction). Since the second derivative is a constant, it can be concluded that the solution is most likely a quadratic function. The x-component of the position vector $\vec{r}$ increases with time, but does this linear unlike the z-component. Which means that the second derivative of the $\vec{r}$ vector in the x-direction will be the zero vector, or rather the change in the x-component of the $\vec{r}$ vector will be zero. These steps can be found at equation [21]

$$\frac{d}{ds}\left(n\frac{d}{ds}\vec{r}\right)\hat{x} = 0$$

$$\frac{d}{ds}\left(n\frac{dx}{ds}\right) = 0$$

[21]

Since this problem will be evaluated in two dimensions the events in the y-plane can be ignored. By using equation [3] ds can be rewritten for a two dimensional problem. Integration both sides of equation [21] will leave the left side equal to the integrating constant. This constant will be called γ. The following set of algebraic manipulations found in equation [22] should give the solution.

$$n\frac{dx}{ds} = n\frac{dx}{\sqrt{dx^2 + dz^2}} = \gamma$$

$$\left(n\frac{dx}{\sqrt{dx^2 + dz^2}}\right)^2 = \gamma^2 = n^2\frac{dx^2}{dx^2 + dz^2}$$

$$n^2dx^2 = \gamma^2dx^2 + \gamma^2dz^2$$

$$n^2dx^2 - \gamma^2dx^2 = \gamma^2dz^2 = dx^2(n^2 - \gamma^2)$$

$$\sqrt{\frac{\gamma^2}{n^2 - \gamma^2}dz^2} = \sqrt{dx^2} = dx = \sqrt{\frac{\gamma^2}{n^2 - \gamma^2}}dz$$

$$\int dx = \int \sqrt{\frac{\gamma^2}{(a + bz)^2 - \gamma^2}}dz$$

$$x = \gamma \int \frac{1}{\sqrt{(a + bz)^2 - \gamma^2}}dz = \gamma \int \frac{1}{\sqrt{b^2z^2 + 2abz + a^2 - \gamma^2}}dz$$

[22]

A general integral will be used where the values a is equal to b^2, b is equal to $2ab$ and c is equal to $a^2 - \gamma^2$. Equation [23] shows the general integral and the steps that continue with finally a solution for $z(x)$.

$$\int \frac{1}{\sqrt{ax^2 + bx + c}}dx = \frac{1}{\sqrt{-a}}arcsin\left(\frac{-2ax - b}{\sqrt{b^2 - 4ac}}\right)$$

$$x = \frac{1}{\sqrt{-b^2}}arcsin\left(\frac{-2b^2z - 2ab}{\sqrt{4a^2b^2 - 4b^2(a^2 - \gamma^2)}}\right)$$

$$z(x) = \frac{2ab + sin(x\sqrt{-b^2})\sqrt{4a^2b^2 - 4b^2(a^2 - \gamma^2)}}{-2b^2}$$

[23]

In figure 5 the solution of equation [23] is plotted in Matlab. The script of this plot can be found at the end of the report.

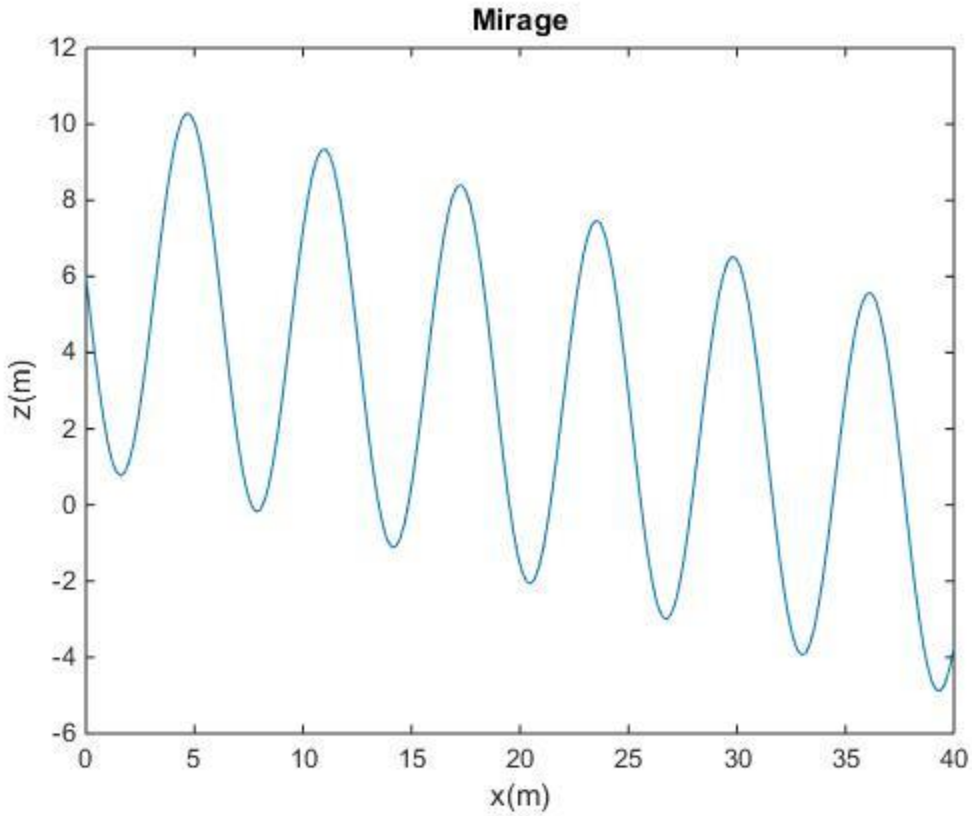

Figure 5: The solution found in equation [23] plotted in Matlab, with both 'z' and 'x' in meters.

2.3 Optical fiber

Optical fibers rely on the phenomena that light that goes into it on one end, and keeps getting refracted in such a way that it does not leave the fiber until it reaches the other end. To understand this principle more thoroughly, a closer look upon the Euler-Lagrange equations was taken. To apply this to optical fibers, the problem is simplified to a two dimensional problem, by only looking at the optical axis, in this case the optical fiber, and the perpendicular component used to determine the angle of refraction.

Doing so, the Euler Lagrange equations from before can be simplified to the equation:

$$\frac{\partial L}{\partial x} = \sqrt{x'^2 + y + z'^2}\,\frac{\partial n}{\partial x} = 0$$

[24]

as *n* does not rely on *x.* Using this, the following equation can be created:

$$\frac{d}{dt}\left(n(y)\frac{dx}{\sqrt{dx^2 + dy^2}}\right) = 0$$

[25]

In this equation, $\frac{dx}{\sqrt{dx^2+dy^2}}$ can be rewritten as the sinus of the angle between the incoming ray and the axis perpendicular to the optical axis using basic geometry: $sin\ i$. This results in the equation:

$$\frac{d}{dt}(n(y)\ sin\ i) = 0 \qquad [26]$$

As its derivative is 0, $n(y)\ sin\ \theta$ must be equal to a constant, in this case called *a,* resulting in formula (2.6):

$$n\ sin\ i = a \qquad [27]$$

Solving this equation for $n(r)\ =\ n_0\sqrt{1 - \alpha^2 r^2}$ with $\alpha < \frac{1}{r}$ a constant and *r* the radius of the fiber results in the equation:

$$n_0\sqrt{1 - \alpha r^2}\ sin\ i = a \qquad [28]$$

Note the distinction between alpha (α) and the constant *a. Sin i* can then be transformed back into $\frac{dx}{\sqrt{dx^2+dr^2}}$ resulting in the equation:

$$\frac{{n_0}^2\,(1-a^2r^2)}{dx^2+dr^2}(dx^2) = a^2 \qquad [29]$$

which was squared to separate *dx* , by first rewriting the equation into:

$$\frac{{n_0}^2\,(1-\alpha^2r^2)-a^2}{dr^2}(dx^2) = a^2 \qquad [30]$$

which in turn can be rewritten into the equation:

$$dx = \frac{a}{\sqrt{{n_0}^2\,(1-\alpha^2r^2)-a^2}}dr \qquad [31]$$

To find an analytical solution to this problem, some tricks are used. First, the function is rewritten as

$$dx = \frac{a}{\sqrt{{n_0}^2+a^2}}\frac{1}{\sqrt{1+\beta}}dr \qquad [32]$$

with $\beta = \frac{n_0\alpha^2}{n_0-a^2}$being a constant. This is done because the primitive of $\frac{1}{\sqrt{1+\beta}}$is $\frac{arcsin\,(\sqrt{\beta}\,r)}{\sqrt{1+\beta}}$

So after integrating both sides of formula [32] the equation becomes:

$$x = \frac{a}{n_0\alpha}\,arcsin\left(\frac{n_o\,\alpha}{\sqrt{{n_o}^2-a^2}}r\right) + C \qquad [33]$$

with *C* being an integration constant, which is determined by the height at which the light ray enters the fiber. This equation can then be rewritten to separate *r* like so:

$$y = \frac{\sqrt{{n_o}^2-a^2}}{n_0\alpha}\,arcsin\left(\frac{n_o\,\alpha}{a^2}x\right) + C \qquad [34]$$

From the RP Manual it was not very clear what formula was to be used for the numerical approach, so it was assumed formula (3.10) is supposed to be used (as it was referred to as "eq.˜refEq:6"). Using some basic Matlab code to loop through every x position like so:

```
for i=1:N
    r(i)=sqrt(n0^2-a^2)/(n0*alpha)*sin((x(i))*n0*alpha/a);
end
```

Resulting in figure 6, as seen below.

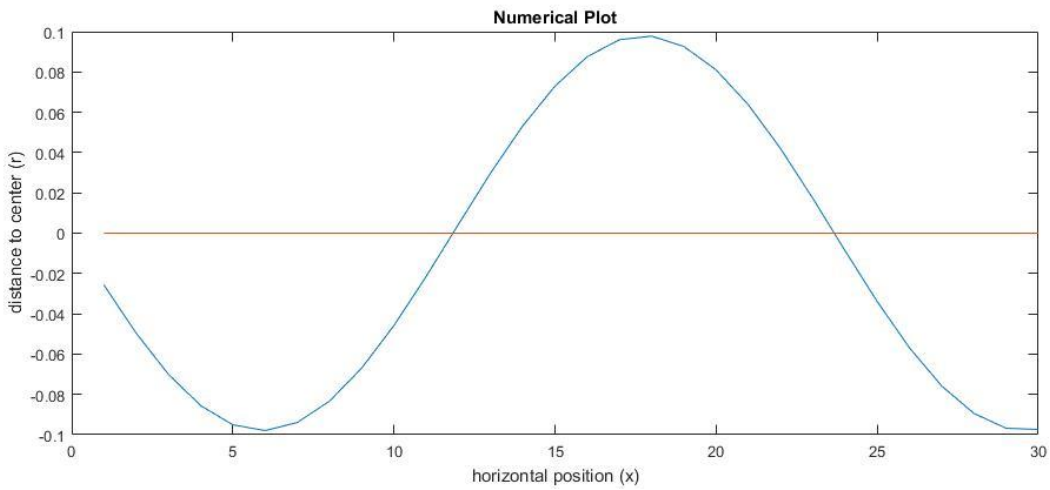

Figure 6. A numerical approximation to formula (3.10) the red line being the optical axis.

This analytical approximation seems to be correct as it has a sinusoidal form, oscillating around the centre of the fibre, resulting in a light ray that will stay inside said fibre. This makes the numerical calculation seem valid.

The advantage of using this fibre, with a dynamic index of refraction opposed to a fibre with a constant index is that this does not solemnly rely on the light rays coming in on an angle that causes it to be completely reflected back into the fibre. It relies on the index of refraction changing in such a way that the ray of light makes an oscillating movement, causing it to never leave the fibre.

3. Conclusion

In this paper, Fermat's principle was analysed and applied to a homogenous medium, Snell-Descartes' law, a mirage and an optical fibre. It was successfully used to analytically prove the law of reflection and transmission of light at the interface of two media with different indices of refraction.

It was used to describe the phenomena of a mirage, and showed the light path taken by the mirage.

Lastly, it was used to describe the way light behaves inside an optical fibre. An equation was found that describes this behaviour, as well as a numerical approximation, which seemed to be valid as it complied with said equation.

List of References

[1] En.wikipedia.org. (2017). *Pierre de Fermat*. [online] Available at: https://en.wikipedia.org/wiki/Pierre_de_Fermat#Work [Accessed 29 Jan. 2017].

Appendix – code used for mirage

```
clc;close all;clear all;
%stating values
xmin         = 0; xmax = 40; steps = 200;
z0           = 8; zperson = 2; zslope = linspace(z0,zperson,steps);
x0           = xmax; gamma = -5; beta = 1; alpha = 1; T0= 1;
a            = beta*T0; b = -beta*alpha;
%variables
x            = linspace(xmin,xmax,steps);
for i        = 1:steps
Solved(i)    = 2*a*b+sin(x(i)*sqrt((-b)^2))*sqrt((4*(a^2)*(b^2))-
(4*(b^2)*((a^2)-(gamma^2))))/-2*(b^2);
end
z            = Solved +zslope; plot(x,z);
xlabel('x(m)');ylabel('z(m)');title('Mirage')
```