1 Principle of Fermat

Question: Proof that 1 and 2 can be reduced to equation 3.

$$L[x,y,z,\dot{X},\dot{y},\dot{z}] = n(x,y,z)\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \tag{2}$$

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{3}$$

Answer:

First noting that ds is a small element of distance travelled. Therefore taking into account the x, y and z direction, ds is given by:

$$ds = \sqrt{dx^2+dy^2+dz^2} \tag{4}$$

A small distance travelled in a trivial direction, lets say dx, can be approximated by as $dx = dt \cdot \dot{x}$. Therefore ds can be rewritten as:

$$ds = dt\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \tag{5}$$

Rewriting gives:

$$\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} = \frac{ds}{dt} \tag{6}$$

If we combine the equations in equation 2 in vector notation we get:

$$\frac{d}{dt}\begin{pmatrix}\frac{\partial}{\partial \dot{x}}\\ \frac{\partial}{\partial \dot{y}}\\ \frac{\partial}{\partial \dot{z}}\end{pmatrix} L - \nabla L = 0 \tag{7}$$

Rewriting and filling in equation 1 gives:

$$\frac{d}{dt}\left[\begin{pmatrix}\frac{\partial}{\partial \dot{x}}\\ \frac{\partial}{\partial \dot{y}}\\ \frac{\partial}{\partial \dot{z}}\end{pmatrix} n(x,y,z)\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\right] = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\left[n(x,y,z)\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\right] \tag{8}$$

$$\frac{d}{dt}\frac{n\cdot\dot{\vec{r}}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \vec{\nabla}n\,\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \tag{9}$$

Using equation 6 to to replace the $\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$ and rewriting the $\dot{\vec{r}}$ vector gives the following:

$$\frac{d}{dt}\frac{n\cdot\dot{\vec{r}}}{\frac{ds}{dt}} = \vec{\nabla}n\,\frac{ds}{dt} \tag{10}$$

$$\frac{d}{dt}\frac{n\cdot\frac{d}{dt}\vec{r}}{\frac{ds}{dt}} = \vec{\nabla}n\,\frac{ds}{dt} \tag{11}$$

Rewriting yields the equation that was to be proved:

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{12}$$

2 Application

2.1 Homogeneous medium

Question: Using equation 3, show how light is travelling in a homogeneous medium.

Answer:

Equation 3 can be rewritten using the chain rule:

$$\frac{d\vec{r}}{ds}\frac{d}{ds}n + n\frac{d^2\vec{r}}{ds^2} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} n \tag{13}$$

Note that for a homogeneous medium, the index of refraction, n, is constant. Therefore $\frac{dn}{ds} = 0$, $\frac{\partial n}{\partial x} = 0$, $\frac{\partial n}{\partial y} = 0$ and $\frac{\partial n}{\partial z} = 0$. Using this in the previous equation yields:

$$n\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{14}$$

$$\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{15}$$

This implies that the direction and velocity of the light is not changed as the light travels through the medium. Therefore, it travels in a straight line with a constant velocity of $v = c/n$.