1 Principle of Fermat

Question: Proof that equation 1 and 2 can be reduced to equation 3.

$$L[x, y, z, \dot{X}, \dot{y}, \dot{z}] = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \tag{2}$$

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{3}$$

Answer:

First noting that ds is a small element of distance travelled. Therefore taking into account the x, y and z direction, ds is given by:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \tag{4}$$

A small distance travelled in a trivial direction, lets say dx, can be approximated by as $dx = dt \cdot \dot{x}$. Therefore ds can be rewritten as:

$$ds = dt\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{5}$$

Rewriting gives:

$$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{dt} \tag{6}$$

If we combine the equations in equation 2 in vector notation we get:

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} L - \nabla L = 0 \tag{7}$$

Rewriting and filling in equation 1 gives:

$$\frac{d}{dt}\left[\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\left[n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] \tag{8}$$

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \vec{\nabla}n\,\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{9}$$

Using equation 6 to to replace the $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and rewriting the $\dot{\vec{r}}$ vector gives the following:

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\frac{ds}{dt}} = \vec{\nabla}n\,\frac{ds}{dt} \tag{10}$$

$$\frac{d}{dt}\frac{n \cdot \frac{d}{dt}\vec{r}}{\frac{ds}{dt}} = \vec{\nabla}n\,\frac{ds}{dt} \tag{11}$$

Rewriting yields the equation that was to be proved:

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{12}$$

2 Application

2.1 Homogeneous medium

Question: Using equation 3, show how light is travelling in a homogeneous medium.

Answer:

Equation 3 can be rewritten using the chain rule:

$$\frac{d\vec{r}}{ds}\frac{d}{ds}n + n\frac{d^2\vec{r}}{ds^2} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} n \tag{13}$$

Note that for a homogeneous medium, the index of refraction, n, is constant. Therefore $\frac{dn}{ds} = 0$, $\frac{\partial n}{\partial x} = 0$, $\frac{\partial n}{\partial y} = 0$ and $\frac{\partial n}{\partial z} = 0$. Using this in the previous equation yields:

$$n\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{14}$$

$$\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{15}$$

This implies that the direction and velocity of the light is not changed as the light travels through the medium. Therefore, it travels in a straight line with a constant velocity.

2.2 Snell-Descartes Law

Question: Express first geometrically and then analytically Snell and Descartes law of reflection and transmission of the light at the interface between two media of different index of refraction n_1 and n_2, using the Principle of Fermat and equation 3.

Answer:

2.2.1 Geometrical

The speed of light in a medium is inversely proportional to the refractive index. Therefore, the shortest path (in distance) between two points in materials with different refractive indices is not always the fastest (in time). This phenomenon is nicely described by a 2-dimensional analogy of a beach (see figure 1). The maximum speed on foot on beach is significantly higher than the maximum swimming speed in the water. So if somebody would need to get from a point A on the beach to a point B in the water, the direct route from A to B (dashed line in figure 1) would intuitively be slower than the path with a shorter swimming distance (solid line in figure 1).

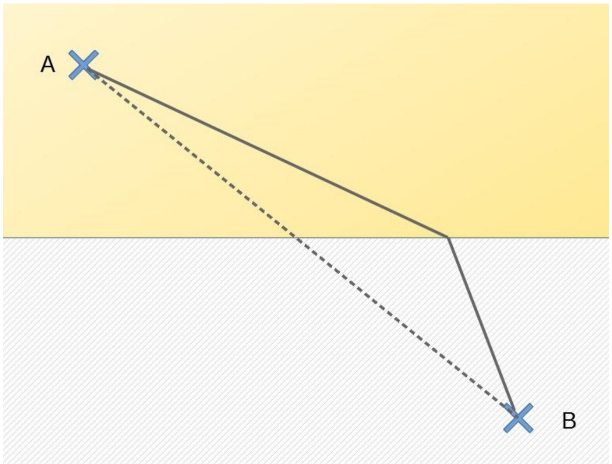


Figure 1: Diagram of a 2-dimensional beach analogy of the interface between two media with different index of refraction. The upper-half corresponds to the beach and the lower-half to the sea. The dashed line corresponds to the direct route between point A and B with the shortest distance. The solid line corresponds to a route that is intuitively faster than the direct route.

It is possible to calculate the fastest route between point A and B If we add the parameters v_1, v_2, θ_1, θ_2, a, b, c and d which corresponds respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in figure 2.

The time it takes to travel from point A to B, t, can easily be found dividing the path on the beach and the water, respectively l_{beach} and l_{water} by the corresponding speed:

$$t = l_{beach}/v_1 + l_{water}/v_2 \tag{16}$$

Using the pythagoras theorem we find:

$$t = \sqrt{a^2 + c^2}/v_1 + \sqrt{b^2 + (d-c)^2}/v_2 \tag{17}$$

If there is a fastest path, there should be an optimum value for c for which $dt/dc = 0$. Therefore, applying the principle of Fermat to equation 17 leads to the following:

$$0 = \frac{c}{v_1\sqrt{a^2 + c^2}} + \frac{c-d}{v_2\sqrt{b^2 + (d-c)^2}} \tag{18}$$

We now need the following trigonometric identity for right-angled triangles:

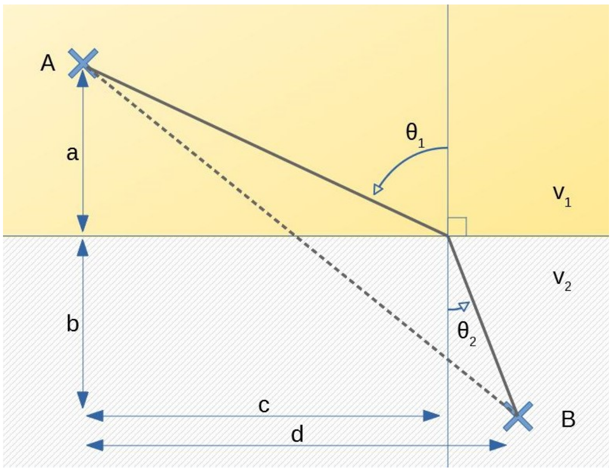


Figure 2: Diagram of beach analogy with parameters v_1, v_2, θ_1, θ_2, a, b, c and d. These correspond respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in the diagram.

$$sin(\theta) = (adjacent - side)/(diagonal - side) \tag{19}$$

Filling in this identity yields:

$$0 = sin(\theta_1)/v_1 - sin(\theta_2)/v_2 \tag{20}$$

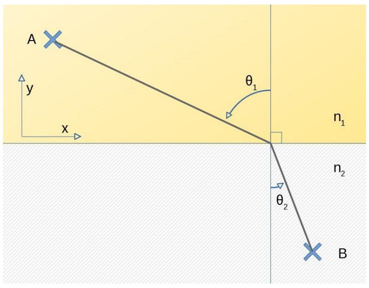
If we rewrite this and use the fact that the speed of light in a medium is given by $v = c/n$ we obtain Snell-Descartes law:

$$n_1\ sin(\theta_1) = n_2\ sin(\theta_2) \tag{21}$$

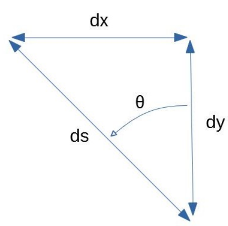
This equation basically tells us, that for a interface to a higher refractive index, so where the light slows down, the light bends to the normal.

2.2.2 Analytical

For the analytical derivation of Snell-Descartes law we will use a similar diagram as in the geometrical derivation with a coordinate system added as in figure 3a.



(a) Diagram of the path of light at the interface of two media with different refractive indices n_1 and n_2. θ_1 and θ_2 correspond to the angle with the normal.



(b) ds in relation to dx and dy.

Figure 3

If we write equation 3 for only the x-component and use the fact that n is independent of x in our diagram, we get the following:

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right]=\frac{dn}{dx} \tag{22}$$

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right]=0 \tag{23}$$

The ds in the latter equation is defined as in figure 3b. If we keep a fixed ds for both media we obtain the following equality:

$$\frac{d}{ds}\left[n_1\frac{dx_1}{ds}\right]=\frac{d}{ds}\left[n_2\frac{dx_2}{ds}\right] \tag{24}$$

Integrating both sides with respect to ds and using the trigonometric identity from equation 19 yields the Snell-Descartes law:

$$n_1\frac{dx_1}{ds}=n_2\frac{dx_2}{ds} \tag{25}$$

$$n_1\ sin(\theta_1)=n_2\ sin(\theta_2) \tag{26}$$

2.3 Mirage

Question: The mirage is a common phenomenon when the ground is very warm and the temperature of the air decreases with altitude; in this case the density then increases as well as its index of refraction.
Sketch what is happening.

Answer:
In figure 4 a situation in which a mirage occurs has been sketched. In this figure a few parameters were introduced, an x- and z-distance respectively denoting the horizontal and vertical distance from the feet of the observer and an angle Θ which is the angle between the horizontal and the unbent light path from the observer. A gradient effect is also applied to the image, where the colour is darker the index of refraction is higher.
Since the index of refraction of the air varies with the temperature and the temperature increases as the z-coordinate increases, the path of light rays will be bent. Thus there will be a light ray coming from the sky bending in such a way that it lands in the eye of an observer. This is the mirage effect where light follows a different path than one might expect.

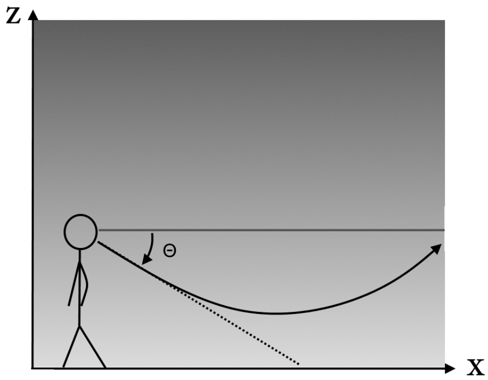


Figure 4: A sketch of situation where a mirage effect occurs.

Question: Find the relation between the index of refraction with the altitude if we assume that the gradient of the temperature changes linearly with the altitude.

Answer:
In equation 27 the temperature at a height z is defined using a ground temperature T_0 and a lapse rate c $[K/m]$ so that it is linearly decreasing. To relate the index of refraction n to the temperature at height $T(z)$, the Gladstone-Dale relation is used. In equation 28 this relation is shown with a proportionality constant K_{air}. Finally using the equation of state (equation 29), it is possible to relate the index of refraction n to height z. Finding the relation for the pressure at height $P(z)$ is shown below in the derivation and the result is displayed in equation 30.

$$T(z) = T_0 - c \cdot z \tag{27}$$

$$n - 1 \propto K_{air}\rho \tag{28}$$

$$\rho = \frac{1}{R_{sp,air}} \frac{P(z)}{T(z)} \tag{29}$$

$$dT/dz \equiv -c$$

$$dP = -g_0 \rho dz$$

$$
\begin{aligned}
dP &= g_0 \frac{\rho}{c} dT \\
\frac{dP}{P} &= \frac{g_0}{c \cdot R_{sp,air}} \frac{dT}{T} \\
\int_{P0}^{P} dP &= \int_{T_0}^{T} \frac{g_0}{c \cdot R_{sp,air}} \frac{dT}{T} \\
\ln P - \ln P_0 &= \big[\ln T - \ln T_0 \big] \cdot \frac{g_0}{c \cdot R_{sp,air}} \\
\frac{P}{P_0} &= (\frac{T}{T_0})^{\frac{g_0}{c \cdot R_{sp,air}}} \\
P &= P_0 \cdot (\frac{T}{T_0})^{\frac{g_0}{c \cdot R_{sp,air}}} \qquad (30)
\end{aligned}
$$

After deriving equation 30 it is know possible to use both equation 27 and equation 30 to derive the air density ρ with equation 29. The result is show in equation 31, which will be combined with the Gladstone-Dale relation for a equation that equates the index of refraction n with the height z. Shown in equation 32.

$$
\rho = \frac{P_0}{R_{sp,air} \cdot T(z)} \cdot (\frac{T(z)}{T_0})^{\frac{g_0}{c \cdot R_{sp,air}}}
$$

$$
\begin{aligned}
\rho &= \frac{P_0}{R_{sp,air}} \cdot \frac{T(z)^{\frac{g_0}{cR_{sp,air}}}}{T(z)} \cdot (T_0)^{-\frac{g_0}{cR_{sp,air}}} \\
\rho &= \Big[T(z)\Big]^{\frac{g_0}{cR_{sp,air}}-1} \frac{P_0}{R_{sp,air}} \cdot T_0^{-\frac{g_0}{cR_{sp,air}}} \\
\rho &= \Big[T_0 - c \cdot z\Big]^{\frac{g_0}{cR_{sp,air}}-1} \frac{P_0}{R_{sp,air}} \cdot T_0^{-\frac{g_0}{cR_{sp,air}}} \\
\rho &= \frac{P_0}{T_0 R_{sp,air}} \Big[1 - c \cdot z T_0^{-\frac{g_0}{cR_{sp,air}}}\Big] \\
\rho &= \frac{P_0}{T_0 R_{sp,air}} \Big[1 - c \cdot z T_0^{-\frac{g_0}{cR_{sp,air}}}\Big] \qquad (31) \\
n(z) = K_{air}\rho + 1 &= 1 + \frac{P_0 K_{air}}{T_0 R_{sp,air}} \Big[1 - c \cdot z T_0^{-\frac{g_0}{cR_{sp,air}}}\Big] \qquad (32)
\end{aligned}
$$

<u>Question:</u> *Express analytically the trajectory of the light in this situation.*

<u>Answer:</u>
For an analytical solution it is easier to linearise equation 32.

$$
\begin{aligned}
n_l(z) &= n_0 + \alpha \cdot z \\
n_l(z) &= n(0) + z \cdot \frac{d}{dz} n(z)|_{z=0}
\end{aligned}
$$

To get to an analytical solution; equation 3 needs to be solved. This is done below.

$$
\vec{\nabla} \cdot n(z) = \frac{\partial}{\partial z} n(z) = \alpha \vec{e}_z
$$

$$
\begin{aligned}
\frac{d}{ds}\Big[n(z)\frac{d\vec{r}}{ds}\Big] &= \vec{\nabla} \cdot n(z) = \alpha \vec{e}_z \\
\frac{d}{ds}\Big[n(z)\frac{d\vec{x} + d\vec{z}}{\sqrt{dx^2 + dz^2}}\Big] &= \alpha \vec{e}_z \\
\frac{d}{ds}\Big[n(z)\frac{dx}{\sqrt{dx^2 + dz^2}}\Big] &= 0 \\
\frac{d}{ds}\Big[n(z)\frac{dz}{\sqrt{dx^2 + dz^2}}\Big] &= \alpha
\end{aligned}
$$

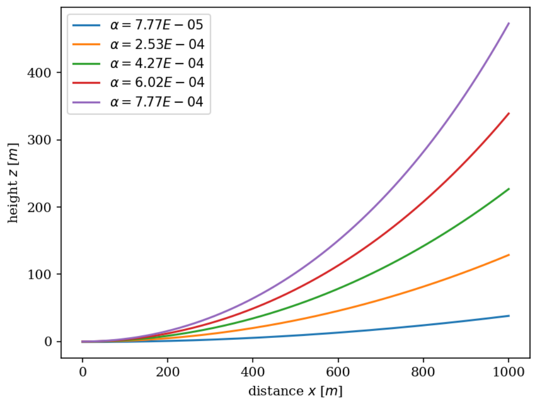
$$
\begin{aligned}
n(z) \cdot \frac{dx}{\sqrt{dx^2 + dy^2}} &= c \\
n(z)^2 \cdot \frac{dx^2}{dx^2 + dy^2} &= c^2 \\
n(z)^2 \cdot dx^2 &= c^2(dx^2 + dy^2) \\
\frac{dz}{dx} = \frac{\sqrt{n(z)^2 - c^2}}{c} &= \frac{\sqrt{(n_0 + \alpha \cdot z)^2 - c^2}}{c}
\end{aligned}
$$

The above first-order non-linear ordinary differential equation has been solved using wolfram and is displayed below in equation 33. In this equation, k is an arbitrary constant and c is the total path length. α is the defined linearisation constant.

$$
z(x) = -\frac{c^2 \exp{(\alpha x + \alpha k)/c} + \exp{-(\alpha x + \alpha k)/c} - 2n_0}{2\alpha} \tag{33}
$$

Question: *Plot several of these trajectories for different relevant cases.*

Answer:
Below in figure **??** a few trajectories for different α-values are plotted. These values range from $\alpha = 7.77 \cdot 10^{-7}$ the linearisation constant equal to $d/dz \cdot n(z)|_{z=0}$ to the same constant a magnitude larger.



Question: *Find the closest distance from an observer where a mirage can be visible.*

Answer:
In the figure above an α-value was used, this value was equal to the derivative of our index of refraction function $n(z)$ evaluated at $x = 0$. This value was dependant on the ground temperature T_0, the ground pressure P_0 and the lapse rate of the temperature c. For these the following values were used: $T_0 = 323.15$ K, $P_0 = 101325$ Pa, a lapse rate of $c = 1$ K/m and an initial angle of looking down $\theta = 15$ degrees. With these conditions you would see the mirage about 500 m in front of you.

Question: *Forsee what could happen on the north pole when a warm wind is present.*

Answer:
On the North Pole with a warm wind blowing an inverse effect as described before would be seen, since the ground is cold and the air heats up as the height increases. The index of refraction n would then decrease the higher off the ground. Light would then bend downward instead of up allowing you to see the ground when you look at the sky.

2.4 Optical fibre

2.4.1

Question: If we assume an incoming beam in the $\vartheta_{x,y}$ plane at start crossing ϑ_x with an angle θ_0. Show that the light beam will stay in this plane and that, by solving the Euler-Lagrange equation 3, the equation for the beam path can be written as:

$$n\ sin(i) = a \tag{34}$$

Answer:

For this question it is assumed that the optical fibre is cylindrical with its axis ϑ_x (see figure 5). The incident-surface will therefore be in the $\vartheta_{y,z}$ plane.

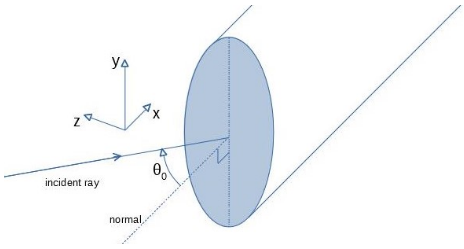


Figure 5: 3-dimensional diagram of fibre with incoming light beam making an angle θ_0 with the ϑ_x axis.

When the incoming beam is in the $\vartheta_{x,y}$ plane, the angle with the normal is θ_0 with ϑ_x in the ϑ_y direction and $\theta_z = 0$ in the ϑ_z direction. If we use Snell-Descartes law (equation 26) for θ_z with $\theta_{z,1} = 0$ we obtain $\theta_{z,2} = k \cdot \pi\ [rad]$, with $k = -1, 0, 1, 2, ...$ Therefore, the beam of light will stay in the $\vartheta_{x,y}$ plane at the interface, when entering the optical fibre.
It is the property of a cylinder that the outer surface is always perpendicular with the radius. For this light beam. Since the light beam is travelling in the $\vartheta_{x,y}$ direction, the outer surface it reaches will be in the $\vartheta_{x,z}$ plane. The angle with the surface normal in the ϑ_z axis is still zero. From the internal reflection it follows that the angle in the ϑ_z axis will remain zero. Therefore, for every reflection at the outer surface, the beam will stay in the $\vartheta_{x,y}$ plane.

Since we know that the beam will stay in the $\vartheta_{x,y}$ plane, the problem can be simplified to a 2-dimensional problem (see figure 6). The infinitely small segment ds is related to ds, dy, dx and i as in figure 6.

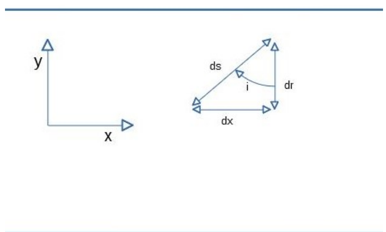


Figure 6: 2-dimensional fibre including the relation between ds, ds, dy, dx and i.

We can write equtaion 3 for only the x-component and use that n is independent of x:

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = \frac{dn}{dx} \tag{35}$$

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = 0 \tag{36}$$

If we now integrate with respect to ds on both sides we get the following:

$$n\frac{dx}{ds} = a \tag{37}$$

With a an arbitrary constant. From equation 19 it follows that $sin(i) = dx/ds$. Using this, we obtain the requested equation:

$$n\ sin(i) = a \tag{38}$$

2.4.2

Question: Solve equation 34 for $n(r) = n_0\sqrt{1-\alpha^2r^2}$ with $\alpha < 1/r$ a constant and r the radius of the fibre.You should end up with an analytical function.

Answer:
From figure 6 and equation 19 it follows that $sin(i) = dx/ds$. Using this and the Pythagoras theorem in equation 34 yields:

$$n\frac{dx}{\sqrt{dr^2+dx^2}} = a$$

Filling in $n(r)$ and some rewriting results in the following:

$$n_0\sqrt{1-\alpha^2r^2}\frac{dx}{\sqrt{dr^2+dx^2}} = a$$

$$n_0^2(1-\alpha^2r^2)\frac{dx^2}{dr^2+dx^2} = a^2$$

$$n_0^2(1-\alpha^2r^2)dx^2 = a^2(dr^2+dx^2)$$

$$(n_0^2 - n_0^2\alpha^2r^2 - a^2)dx^2 = a^2dr^2$$

$$dx^2 = \frac{a^2dr^2}{n_0^2 - n_0^2\alpha^2r^2 - a^2}$$

$$dx = \frac{a\ dr}{\sqrt{n_0^2 - n_0^2\alpha^2r^2 - a^2}}$$

If we now integrate both sides we will get:

$$\int dx = \int \frac{a}{\sqrt{n_0^2 - n_0^2\alpha^2r^2 - a^2}}dr$$

$$x = \frac{a}{\sqrt{n_0^2-a^2}}\int \frac{1}{1-\left(\frac{rn_0\alpha}{\sqrt{n_0^2-a^2}}\right)^2}dr$$

To solve this integral we need the following substitution:

$$\frac{r\ n_0\ \alpha}{\sqrt{n_0^2-a^2}} = sin(u)$$

$$dr = \frac{\sqrt{n_0^2-a^2}}{n_0\alpha}cos(u)du$$

$$u = arcsin\left(\frac{rn_0\alpha}{\sqrt{n_0^2-a^2}}\right)$$

Using the substitution we get:

$$x = \frac{a}{n_0 \ \alpha} \int \frac{\cos(u)}{\sqrt{1 - \sin^2(u)}} du$$

If we use that $1 - \sin^2(x) = \cos^2(x)$, we obtain:

$$x = \frac{a}{n_0 \ \alpha} \int \frac{\cos(u)}{\sqrt{\cos^2(u)}} du$$

$$x = \frac{a}{n_0 \ \alpha} \int du$$

$$x = \frac{a}{n_0 \ \alpha} u$$

Inverting the substitution and doing some rewriting yields the following equation:

$$x = \frac{a}{n_0 \ \alpha} arcsin \left(\frac{r \ n_0 \ \alpha}{\sqrt{n_0^2 - a^2}} \right)$$

$$\sin\left(\frac{n_0 \ \alpha \ x}{a}\right) = \frac{r n_0 \alpha}{\sqrt{n_0^2 - a^2}}$$

$$r(x) = \frac{\sqrt{n_0^2 - a^2}}{n_0 \alpha} \sin\left(\frac{n_0 \ \alpha \ x}{a}\right)$$

Since there are no discontinuities in $n(r)$ and $dn(r)/dr$, we would expect that there are also no discontinuities in $r(x)$ and $dr(x)/dx$. Since $r =\mid y \mid$ for the light beam in the $\vartheta_{x,y}$ plane. The only possible solution of $y(x)$ is the following:

$$y(x) = \pm \frac{\sqrt{n_0^2 - a^2}}{n_0 \alpha} \sin\left(\frac{n_0 \ \alpha \ x}{a}\right) \tag{39}$$