Delft University of Technology

Research Practicum Differential Equations

Propagation of the Light: Euler-Lagrange Equations

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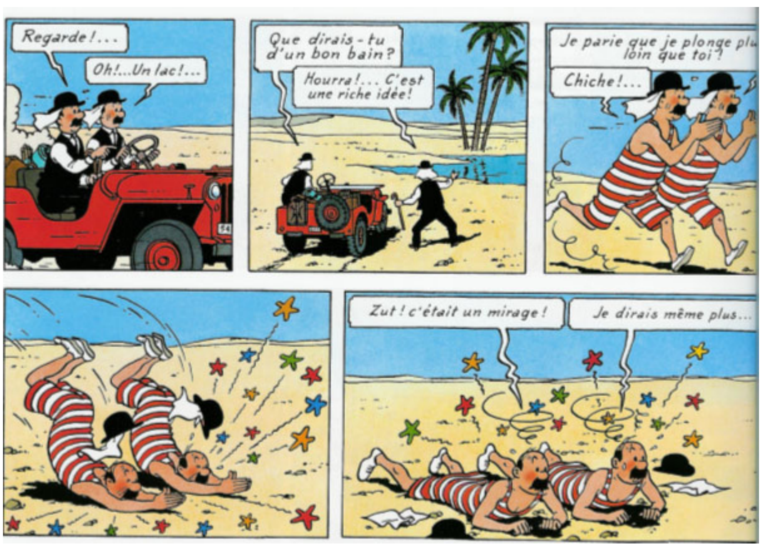
We are interested to study how light is propagating in a medium with non-homogenous index of refraction. To do so, we will use the **Principle of Fermat** (1601-1665) and the **Snell-Descartes law** (1621 & 1637). We will describe the propagation of light, in term of differential equations and solve it for different cases such as for the mirage and the optical fiber.



Your full assignment is to write a report in which you treat the propagation of light in a non-homogeous index of refraction media and derive the answers to the assignments below.

1 Principle of Fermat

Let's consider an inhomogenous medium, defined by an index of refraction $n(\vec{r})$ which depends on the position $\vec{r}$ in space and a curve (C) between two points A and B. The optical path along (C) is defined as:

$$L_{AB} = \int_A^B n(\vec{r})ds \tag{1.1}$$

First show by introducing the time t that Eq. (1.1) can be rewritten as:

$$L_{AB} = \int_{t_A}^{t_B} \mathcal{L}\left[x, y, z, \dot{x}, \dot{y}, \dot{z}\right] dt \tag{1.2}$$

with $\mathcal{L}$ the Lagrangian (named after Lagrange, 1736-1813) defined as:

$$\mathcal{L}\left[x, y, z, \dot{x}, \dot{y}, \dot{z}\right] = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1.3}$$

and $\dot{x}$, $\dot{y}$ and $\dot{z}$ the temporal derivative of the components x, y, z of the position $\vec{r}$.

Fermat demonstrated that the light will follow the path between A and B which is the shortest, i.e. that makes the integral of Eq. (1.1) stationary (meaning minimum). This is equivalent to solve the Euler-Lagrange equations present in the theory of the *Calculus of variations*:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0, \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = 0, \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} - \frac{\partial \mathcal{L}}{\partial z} = 0 \tag{1.4}$$

$$\tag{1.5}$$

Prove that these equations can be reduced to:

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla} n \tag{1.6}$$

where s is the distance travelled by the light. This is known as the differential equation of light propagation. In the next chapter, we will only use this Eq. (1.6) to solve our problems.

2 Application

2.1 Homogenous medium

Using Eq. (1.6), show how light is traveling in a homogenous medium. Be aware that the demonstration is not trivial, even the answer is. Does the result correspond to your expectation?

2.2 Snell-Descartes Law

Snell and Descartes have at the same time (re-)discovered the law of reflection and transmission of the light at the interface between two media of different index of refraction n_1 and n_2.
Express first geometrically (using only a drawing with a ruler) and then analytically their findings using the Principle of Fermat and Eq. (1.6).

2.3 Mirage

The mirage is a common phenomenon when the ground is very warm and the temperature of the air decreases with altitude; in this case the density then increases as well as its index of refraction.

- ▷ Sketch what is happening, introduce the necessary input parameters.
- ▷ Find the relation of the index of refraction with the altitude if we assume that the gradient of temperature changes linearly with the altitude. You will need to introduce new input parameters.
- ▷ Express analytically (with a function) the trajectory of the light in this situation. This function should depends of the input parameters you have introduced.
- ▷ Plot several of these trajectories for different relevant cases.
- ▷ Using your own approximation (temperature, height, etc..), find the closest distance from an observer where a mirage can be visible. Does it correspond to your own experience of a mirage?
- ▷ Could you foresee what will happen in the North Pole when a warm wind is present?

2.4 Optical fiber

An *optical fiber* is one way to guide light efficiently from one point to an other. It is currently used for data communication: it offers low loss and very high bandwidth, ideal for the requirements of the internet. Generally, we can describe an optical fiber as a medium with an index of refraction n which depends only on the distance from its axis for example $\mathcal{O}_x$. We assume an incoming beam in the plane $\mathcal{O}_{xy}$ at start crossing $\mathcal{O}_x$ with an angle θ_0.

- ▷ Show that the light beam will stay in this plane and that, by solving the Euler-Lagrange equation (1.6), the equation for the beam path can be written as:

$$n \sin i = a \tag{2.1}$$

  with a a constant to determine and i the angle that the ray makes with the normal.

- ▷ Solve Eq. (2.1) for $n(r) = n_0\sqrt{1-\alpha^2 r^2}$ with $\alpha < 1/R$ a constant and R the radius of the fiber. You should end up with an analytical function.

Usually solution of differential equations are not necessary analytical and one needs to solve the problem with a numerical approach for example defining a space with a finite grid where numerical "value" will be computed *step by step*. In our case, it would be a grid where we define by which nodes the beam of light passes.

- ▷ Using your own (matlab) algorithm applied to the Eq. (2.1) for example, plot a numerical solution of the problem.
- ▷ Compare the numerical solutions with the analytical functions found earlier. Discuss the validity of the numerical calculation.
- ▷ What is advantage of such a fiber compared to a fiber with a constant index?