1 Principle of Fermat

Question: Proof that equation 1 and 2 can be reduced to equation 3.

$$L[x, y, z, \dot{X}, \dot{y}, \dot{z}] = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \tag{2}$$

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{3}$$

Answer:

First noting that ds is a small element of distance travelled. Therefore taking into account the x, y and z direction, ds is given by:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \tag{4}$$

A small distance travelled in a trivial direction, lets say dx, can be approximated by as $dx = dt \cdot \dot{x}$. Therefore ds can be rewritten as:

$$ds = dt\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{5}$$

Rewriting gives:

$$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{dt} \tag{6}$$

If we combine the equations in equation 2 in vector notation we get:

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} L - \nabla L = 0 \tag{7}$$

Rewriting and filling in equation 1 gives:

$$\frac{d}{dt}\left[\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \left[n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] \tag{8}$$

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \vec{\nabla}n \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{9}$$

Using equation 6 to to replace the $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and rewriting the $\dot{\vec{r}}$ vector gives the following:

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\frac{ds}{dt}} = \vec{\nabla}n \frac{ds}{dt} \tag{10}$$

$$\frac{d}{dt}\frac{n \cdot \frac{d}{dt}\vec{r}}{\frac{ds}{dt}} = \vec{\nabla}n \frac{ds}{dt} \tag{11}$$

Rewriting yields the equation that was to be proved:

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{12}$$

2 Application

2.1 Homogeneous medium

Question: Using equation 3, show how light is travelling in a homogeneous medium.

Answer:

Equation 3 can be rewritten using the chain rule:

$$\frac{d\vec{r}}{ds}\frac{d}{ds}n + n\frac{d^2\vec{r}}{ds^2} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} n \tag{13}$$

Note that for a homogeneous medium, the index of refraction, n, is constant. Therefore $\frac{dn}{ds} = 0$, $\frac{\partial n}{\partial x} = 0$, $\frac{\partial n}{\partial y} = 0$ and $\frac{\partial n}{\partial z} = 0$. Using this in the previous equation yields:

$$n\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{14}$$

$$\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{15}$$

This implies that the direction and velocity of the light is not changed as the light travels through the medium. Therefore, it travels in a straight line with a constant velocity of $v = c/n$.

2.2 Snell-Descartes Law

Question: Express first geometrically and then analytically Snell and Descartes law of reflection and transmission of the light at the interface between two media of different index of refraction n_1 and n_2, using the Principle of Fermat and equation 3.

Answer:

2.2.1 Geometrical

The speed of light in a medium is inversely proportional to the refractive index. Therefore, the shortest path (in distance) between two points in materials with different refractive indices is not always the fastest (in time). This phenomenon is nicely described by a 2-dimensional analogy of a beach (see figure 1). The maximum speed on foot on beach is significantly higher than the maximum swimming speed in the water. So if somebody would need to get from a point A on the beach to a point B in the water, the direct route from A to B (dashed line in figure 1) would intuitively be slower than the path with a shorter swimming distance (solid line in figure 1).

It is possible to calculate the fastest route between point A and B If we add the parameters v_1, v_2, θ_1, θ_2, a, b, c and d which corresponds respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in figure 2.

The time it takes to travel from point A to B, t, can easily be found dividing the path on the beach and the water, respectively l_{beach} and l_{water} by the corresponding speed:

$$t = l_{beach}/v_1 + l_{water}/v_2 \tag{16}$$

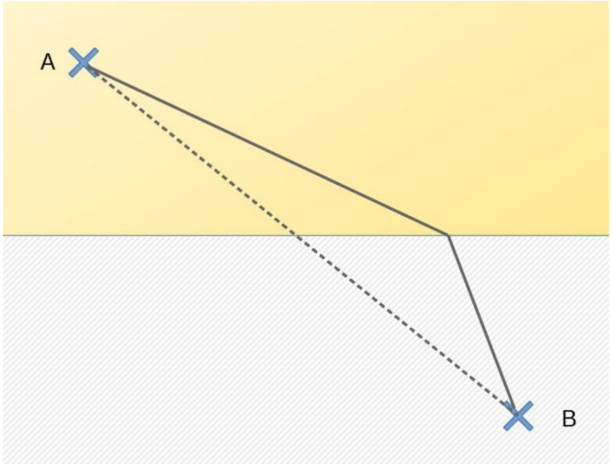


Figure 1: Diagram of a 2-dimensional beach analogy of the interface between two media with different index of refraction. The upper-half corresponds to the beach and the lower-half to the sea. The dashed line corresponds to the direct route between point A and B with the shortest distance. The solid line corresponds to a route that is intuitively faster than the direct route.

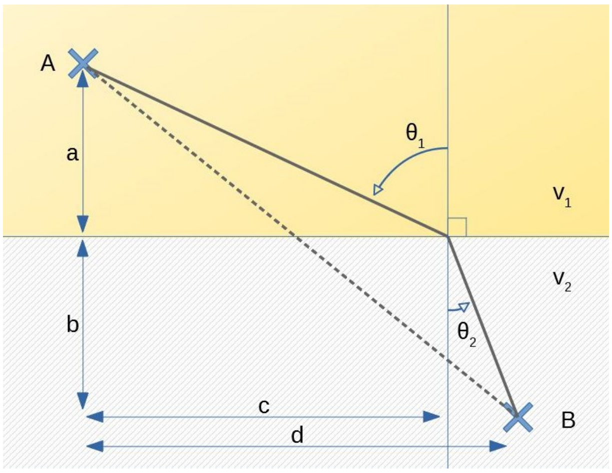


Figure 2: Diagram of beach analogy parameters v_1, v_2, θ_1, θ_2, a, b, c and d. These correspond respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in the diagram.

Using the pythagoras theorem we find:

$$t = \sqrt{a^2 + c^2}/v_1 + \sqrt{b^2 + (d-c)^2}/v_2 \tag{17}$$

If there is a fastest path, there should be an optimum value for c for which $dt/dc = 0$. Therefore, applying the principle of Fermat to equation 17 leads to the following:

$$0 = \frac{c}{v_1\sqrt{a^2 + c^2}} + \frac{c-d}{v_2\sqrt{b^2 + (d-c)^2}} \tag{18}$$

Using the trigonometric identity $sin(\theta) = (adjacentside)/(diagonalside)$ for right-angled triangle we obtain:

$$0 = sin(\theta_1)/v_1 - sin(\theta_2)/v_2 \tag{19}$$

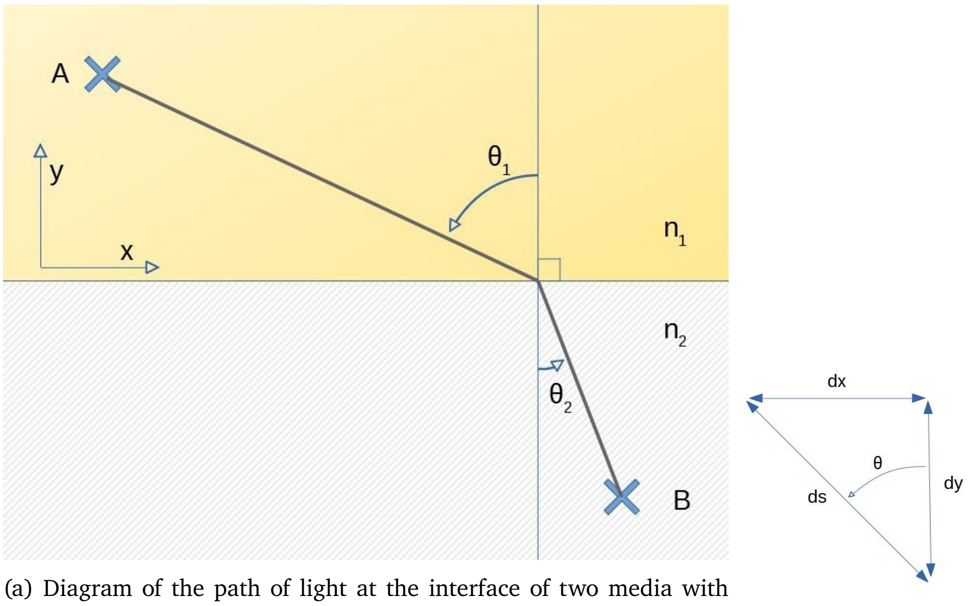
If we rewrite this and use the fact that the speed of light in a medium is given by $v = c/n$ we obtain Snell-Descartes law:

$$n_1\ sin(\theta_1) = n_2\ sin(\theta_2) \tag{20}$$

This equation basically tells us, that for a interface to a higher refractive index, so where the light slows down, the light bends to the normal.

2.2.2 Analytical

For the analytical derivation of Snell-Descartes law we will use a similar diagram as in the geometrical derivation with a coordinate system added as in figure 3a.



(a) Diagram of the path of light at the interface of two media with different refractive indices n_1 and n_2. θ_1 and θ_2 correspond to the angle with the normal.

(b) ds in relation to dx and dy.

Figure 3

If we write equation 3 for only the x-component and use the fact that n is independent of x in our diagram, we get the following:

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = \frac{dn}{dx} \tag{21}$$

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = 0 \tag{22}$$

The ds in the latter equation is defined as in figure 3b. If we keep a fixed ds for both media we obtain the following equality:

$$\frac{d}{ds}\left[n_1\frac{dx_1}{ds}\right] = \frac{d}{ds}\left[n_2\frac{dx_2}{ds}\right] \tag{23}$$

Rewriting and using the trigonometric identity, $sin(\theta) = dx/ds$, yields the Snell-Descartes law:

$$n_1\frac{dx_1}{ds}] = n_2\frac{dx_2}{ds} \tag{24}$$

$$n_1\ sin(\theta_1) = n_2\ sin(\theta_2) \tag{25}$$

2.3 Mirage

The mirage is a common phenomenon when the ground is very warm and the temperature of the air decreases with altitude; in this case the density then increases as well as its index of refraction.
Sketch what is happening.

In figure 4 a situation in which a mirage occurs has been sketched. In this figure a few parameters were introduced, an x- and z-distance respectivly denoting the horizontal and vertical distance from the feet of the observer and an angle Θ which is the angle between the horizontal and the unbent light path from the observer. A gradient effect is also applied to the image, where the colour is darker the index of refraction is higher.
Since the index of refraction of the air varies with the temperature and the temperature increases as the z-coordinate increases, the path of light rays will be bent. Thus there will be a light ray coming from the sky bending in such a way that it lands in the eye of an observer. This is the mirage effect where light follows a different path than one might expect.

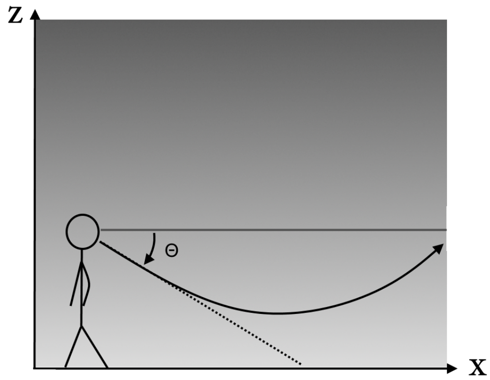


Figure 4: A sketch of situation where a mirage effect occurs.

Find the relation between the index of refraction with the altitude if we assume that the gradient of the temperature changes linearily with the altituder.
In equation 26 the temperature at a height z is defined using a ground temperature T_0 and a lapse rate c $[K/m]$ so that it is linearly decreasing. To relate the index of refraction n to the temperature at height $T(z)$, the Gladstone-Dale relation is used. In equation 27 this relation is shown with a proportionality constant K_{air}. Finally using the equation of state 28 it is possible to relate the index of refraction n to height z. Finding the relation for the pressure at height $P(z)$ is show below in the derivation and the result is displayed in equation 29.

$$T(z) = T_0 - c \cdot z \tag{26}$$

$$n - 1 \propto K_{air}\rho \tag{27}$$

$$\rho = \frac{1}{R_{sp,air}} \frac{P(z)}{T(z)} \tag{28}$$

$$dT/dz \equiv -c$$

$$dP = -g_0 \rho dz$$

$$
\begin{aligned}
dP &= g_0 \frac{\rho}{c} dT \\
\frac{dP}{P} &= \frac{g_0}{c \cdot R_{sp,air}} \frac{dT}{T} \\
\int_{P0}^{P} dP &= \int_{T_0}^{T} \frac{g_0}{c \cdot R_{sp,air}} \frac{dT}{T} \\
\ln P - \ln P_0 &= \big[\ln T - \ln T_0 \big] \cdot \frac{g_0}{c \cdot R_{sp,air}} \\
\frac{P}{P_0} &= (\frac{T}{T_0})^{\frac{g_0}{c \cdot R_{sp,air}}} \\
P &= P_0 \cdot (\frac{T}{T_0})^{\frac{g_0}{c \cdot R_{sp,air}}} \qquad (29)
\end{aligned}
$$

After deriving equation 29 it is know possible to use both equation 26 and equation 29 to derive the air density ρ with equation 28. The result is show in equation 30, which will be combined with the Gladstone-Dale relation for a equation that equates the index of refraction n with the height z. Shown in equation 31.

$$
\rho = \frac{P_0}{R_{sp,air} \cdot T(z)} \cdot (\frac{T(z)}{T_0})^{\frac{g_0}{c \cdot R_{sp,air}}}
$$

$$
\begin{aligned}
\rho &= \frac{P_0}{R_{sp,air}} \cdot \frac{T(z)^{\frac{g_0}{cR_{sp,air}}}}{T(z)} \cdot (T_0)^{-\frac{g_0}{cR_{sp,air}}} \\
\rho &= \Big[T(z) \Big]^{\frac{g_0}{cR_{sp,air}} - 1} \frac{P_0}{R_{sp,air}} \cdot T_0^{-\frac{g_0}{cR_{sp,air}}} \\
\rho &= \Big[T_0 - c \cdot z \Big]^{\frac{g_0}{cR_{sp,air}} - 1} \frac{P_0}{R_{sp,air}} \cdot T_0^{-\frac{g_0}{cR_{sp,air}}} \\
\rho &= \frac{P_0}{T_0 R_{sp,air}} \Big[1 - c \cdot z T_0^{-\frac{g_0}{cR_{sp,air}}} \Big] \\
\rho &= \frac{P_0}{T_0 R_{sp,air}} \Big[1 - c \cdot z T_0^{-\frac{g_0}{cR_{sp,air}}} \Big] \qquad (30)
\end{aligned}
$$

$$
n(z) = K_{air} \rho + 1 = 1 + \frac{P_0 K_{air}}{T_0 R_{sp,air}} \Big[1 - c \cdot z T_0^{-\frac{g_0}{cR_{sp,air}}} \Big] \qquad (31)
$$

Express analytically the trajectory of the light in this situation.
For an analytical solution it is easier to linearise equation 31.

$$
\begin{aligned}
n_l(z) &= n_0 + \alpha \cdot z \\
n_l(z) &= n(0) + z \cdot \frac{d}{dz} n(z)|_{z=0}
\end{aligned}
$$

To get to an analytical solution; equation 3 needs to be solved. This is done below.

$$
\vec{\nabla} \cdot n(z) = \frac{\partial}{\partial z} n(z) = \alpha \vec{e}_z
$$

$$
\begin{aligned}
\frac{d}{ds} \Big[n(z) \frac{d\vec{r}}{ds} \Big] &= \vec{\nabla} \cdot n(z) = \alpha \vec{e}_z \\
\frac{d}{ds} \Big[n(z) \frac{d\vec{x} + d\vec{z}}{\sqrt{dx^2 + dz^2}} \Big] &= \alpha \vec{e}_z \\
\frac{d}{ds} \Big[n(z) \frac{dx}{\sqrt{dx^2 + dz^2}} \Big] &= 0 \\
\frac{d}{ds} \Big[n(z) \frac{dz}{\sqrt{dx^2 + dz^2}} \Big] &= \alpha
\end{aligned}
$$

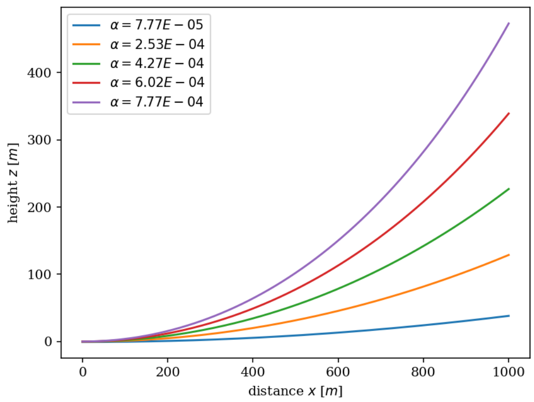
$$n(z) \cdot \frac{dx}{\sqrt{dx^2 + dy^2}} = c$$
$$n(z)^2 \cdot \frac{dx^2}{dx^2 + dy^2} = c^2$$
$$n(z)^2 \cdot dx^2 = c^2(dx^2 + dy^2)$$
$$\frac{dz}{dx} = \frac{\sqrt{n(z)^2 - c^2}}{c} = \frac{\sqrt{(n_0 + \alpha \cdot z)^2 - c^2}}{c}$$

The above first-order nonlinear ordinary differential equation has been solved using wolfram and is displayed below in 32. In this equation k is an arbitrary constant and c is the total path length. α is the defined linearisation constant.

$$z(x) = -\frac{c^2 \exp{(\alpha x + \alpha k)/c} + \exp{-(\alpha x + \alpha k)/c} - 2n_0}{2\alpha} \tag{32}$$

<u>Plot</u> several of these trajectories for different relevant cases.

Below in figure **??** a few trajectories for different α-values are plotted. These values range from $\alpha = 7.77 \cdot 10^{-7}$ the linearisation constant equal to $d/dz \cdot n(z)|_{z=0}$ to the same constant a magnitude larger.



<u>Find the closest distance</u> from an observer where a mirage can be visible.
In the figure above an α-value was used, this value was equal to the derivate of our index of refraction function $n(z)$ evealuated at $x = 0$. This value was dependant on the ground temperature T_0, the ground pressure P_0 and the lapse rate of the temperature c. For these the following values were used: $T_0 = 323.15$ K, $P_0 = 101325$ Pa, a lapse rate of $c = 1$ K/m and an initial angle of looking down $\theta = 15$ degrees. With these conditions you would see the mirage about 500 m in front of you.

<u>Forsee</u> what could happen on the north pole when a warm wind is present.
On the North Pole with a warm wind blowing an inverse effect as described before would be seen, since the ground is cold and the air heats up as the height increases. The index of refraction n would then decrease the higher off the ground. Light would then bend downward instead of up allowing you to see the ground when you look at the sky.