1 Principle of Fermat

Question: Proof that equation 1 and 2 can be reduced to equation 3.

$$L[x, y, z, \dot{X}, \dot{y}, \dot{z}] = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \tag{2}$$

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{3}$$

Answer:

First noting that ds is a small element of distance travelled. Therefore taking into account the x, y and z direction, ds is given by:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \tag{4}$$

A small distance travelled in a trivial direction, lets say dx, can be approximated by as $dx = dt \cdot \dot{x}$. Therefore ds can be rewritten as:

$$ds = dt\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{5}$$

Rewriting gives:

$$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{dt} \tag{6}$$

If we combine the equations in equation 2 in vector notation we get:

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} L - \nabla L = 0 \tag{7}$$

Rewriting and filling in equation 1 gives:

$$\frac{d}{dt}\left[\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\left[n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] \tag{8}$$

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \vec{\nabla}n\ \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{9}$$

Using equation 6 to to replace the $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and rewriting the $\dot{\vec{r}}$ vector gives the following:

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\frac{ds}{dt}} = \vec{\nabla}n\ \frac{ds}{dt} \tag{10}$$

$$\frac{d}{dt}\frac{n \cdot \frac{d}{dt}\vec{r}}{\frac{ds}{dt}} = \vec{\nabla}n\ \frac{ds}{dt} \tag{11}$$

Rewriting yields the equation that was to be proved:

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{12}$$

2 Application

2.1 Homogeneous medium

Question: Using equation 3, show how light is travelling in a homogeneous medium.

Answer:

Equation 3 can be rewritten using the chain rule:

$$\frac{d\vec{r}}{ds}\frac{d}{ds}n + n\frac{d^2\vec{r}}{ds^2} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} n \tag{13}$$

Note that for a homogeneous medium, the index of refraction, n, is constant. Therefore $\frac{dn}{ds} = 0$, $\frac{\partial n}{\partial x} = 0$, $\frac{\partial n}{\partial y} = 0$ and $\frac{\partial n}{\partial z} = 0$. Using this in the previous equation yields:

$$n\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{14}$$

$$\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{15}$$

This implies that the direction and velocity of the light is not changed as the light travels through the medium. Therefore, it travels in a straight line with a constant velocity of $v = c/n$.

2.2 Snell-Descartes Law

Question: Express first geometrically and then analytically Snell and Descartes law of reflection and transmission of the light at the interface between two media of different index of refraction n_1 and n_2, using the Principle of Fermat and equation 3.

Answer:

2.2.1 Geometrical

The speed of light in a medium is inversely proportional to the refractive index. Therefore, the shortest path (in distance) between two points in materials with different refractive indices is not always the fastest (in time). This phenomenon is nicely described by a 2-dimensional analogy of a beach (see figure 1). The maximum speed on foot on beach is significantly higher than the maximum swimming speed in the water. So if somebody would need to get from a point A on the beach to a point B in the water, the direct route from A to B (dashed line in figure 1) would intuitively be slower than the path with a shorter swimming distance (solid line in figure 1).

It is possible to calculate the fastest route between point A and B If we add the parameters v_1, v_2, θ_1, θ_2, a, b, c and d which corresponds respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in figure 2.

The time it takes to travel from point A to B, t, can easily be found dividing the path on the beach and the water, respectively l_{beach} and l_{water} by the corresponding speed:

$$t = l_{beach}/v_1 + l_{water}/v_2 \tag{16}$$

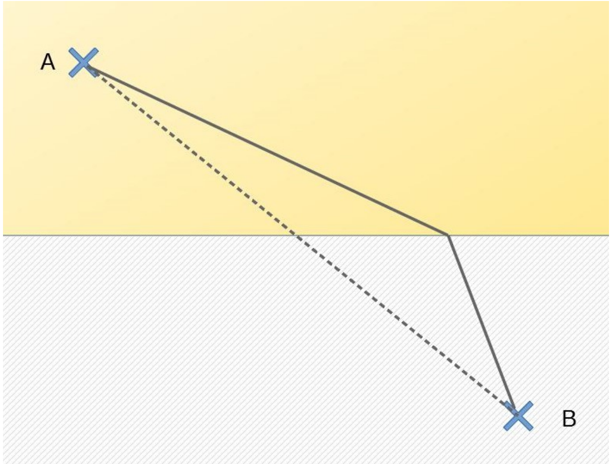


Figure 1: Diagram of a 2-dimensional beach analogy of the interface between two media with different index of refraction. The upper-half corresponds to the beach and the lower-half to the sea. The dashed line corresponds to the direct route between point A and B with the shortest distance. The solid line corresponds to a route that is intuitively faster than the direct route.

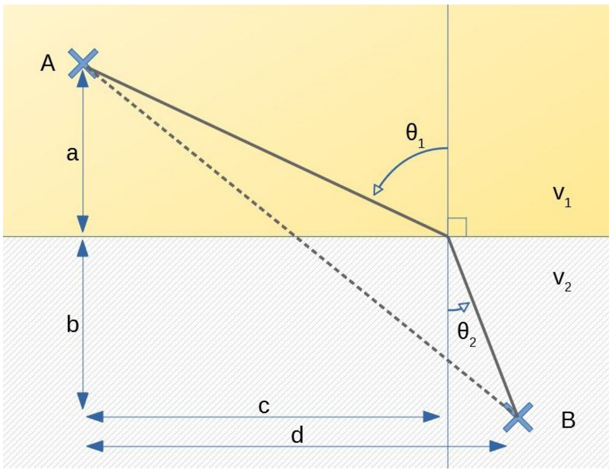


Figure 2: Diagram of beach analogy parameters v_1, v_2, θ_1, θ_2, a, b, c and d. These correspond respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in the diagram.

Using the pythagoras theorem we find:

$$t = \sqrt{a^2 + c^2}/v_1 + \sqrt{b^2 + (d - c)^2}/v_2 \tag{17}$$

If there is a fastest path, there should be an optimum value for c for which $dt/dc = 0$. Therefore, applying the principle of Fermat to equation 17 leads to the following:

$$0 = \frac{c}{v_1\sqrt{a^2 + c^2}} + \frac{c - d}{v_2\sqrt{b^2 + (d - c)^2}} \tag{18}$$

Using the trigonometric identity $sin(\theta) = (adjacentside)/(diagonalside)$ for right-angled triangle we obtain:

$$0 = sin(\theta_1)/v_1 - sin(\theta_2)/v_2 \tag{19}$$

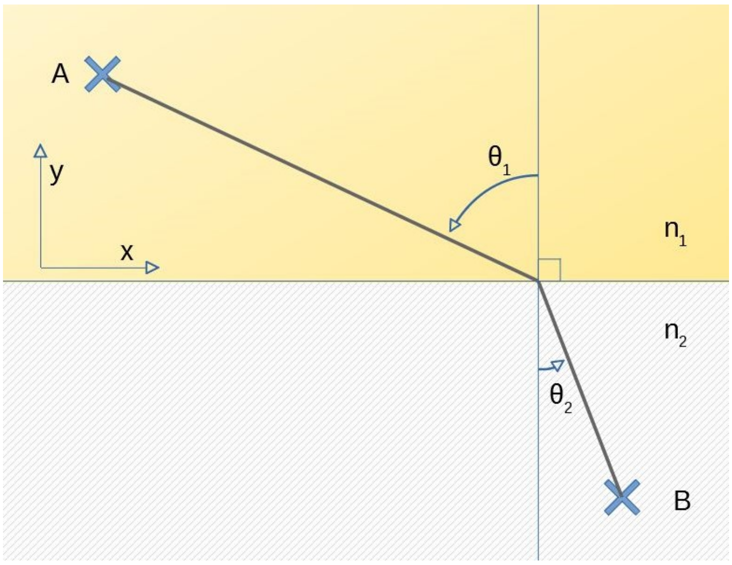
If we rewrite this and use the fact that the speed of light in a medium is given by $v = c/n$ we obtain Snell-Descartes law:

$$n_1 \; sin(\theta_1) = n_2 \; sin(\theta_2) \tag{20}$$

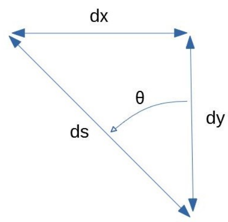
This equation basically tells us, that for a interface to a higher refractive index, so where the light slows down, the light bends to the normal.

2.2.2 Analytical

For the analytical derivation of Snell-Descartes law we will use a similar diagram as in the geometrical derivation with a coordinate system added as in figure 3a.



(a) Diagram of the path of light at the interface of two media with different refractive indices n_1 and n_2. θ_1 and θ_2 correspond to the angle with the normal.



(b) ds in relation to dx and dy.

Figure 3

If we write equation 3 for only the x-component and use the fact that n is independent of x in our diagram, we get the following:

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = \frac{dn}{dx} \tag{21}$$

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = 0 \tag{22}$$

The ds in the latter equation is defined as in figure 3b. If we keep a fixed ds for both media we obtain the following equality:

$$\frac{d}{ds}\left[n_1\frac{dx_1}{ds}\right] = \frac{d}{ds}\left[n_2\frac{dx_2}{ds}\right] \tag{23}$$

Rewriting and using the trigonometric identity, $sin(\theta) = dx/ds$, yields the Snell-Descartes law:

$$n_1\frac{dx_1}{ds}] = n_2\frac{dx_2}{ds} \tag{24}$$

$$n_1 \; sin(\theta_1) = n_2 \; sin(\theta_2) \tag{25}$$