1 Principle of Fermat

Question: Proof that equation 1 and 2 can be reduced to equation 3.

$$L[x, y, z, \dot{X}, \dot{y}, \dot{z}] = n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0, \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \tag{2}$$

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{3}$$

Answer:

First noting that ds is a small element of distance travelled. Therefore taking into account the x, y and z direction, ds is given by:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \tag{4}$$

A small distance travelled in a trivial direction, lets say dx, can be approximated by as $dx = dt \cdot \dot{x}$. Therefore ds can be rewritten as:

$$ds = dt\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{5}$$

Rewriting gives:

$$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{dt} \tag{6}$$

If we combine the equations in equation 2 in vector notation we get:

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} L - \nabla L = 0 \tag{7}$$

Rewriting and filling in equation 1 gives:

$$\frac{d}{dt}\left[\begin{pmatrix} \frac{\partial}{\partial \dot{x}} \\ \frac{\partial}{\partial \dot{y}} \\ \frac{\partial}{\partial \dot{z}} \end{pmatrix} n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \left[n(x, y, z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right] \tag{8}$$

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \vec{\nabla}n\ \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{9}$$

Using equation 6 to to replace the $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and rewriting the $\dot{\vec{r}}$ vector gives the following:

$$\frac{d}{dt}\frac{n \cdot \dot{\vec{r}}}{\frac{ds}{dt}} = \vec{\nabla}n\,\frac{ds}{dt} \tag{10}$$

$$\frac{d}{dt}\frac{n \cdot \frac{d}{dt}\vec{r}}{\frac{ds}{dt}} = \vec{\nabla}n\,\frac{ds}{dt} \tag{11}$$

Rewriting yields the equation that was to be proved:

$$\frac{d}{ds}\left[n\frac{d\vec{r}}{ds}\right] = \vec{\nabla}n \tag{12}$$

2 Application

2.1 Homogeneous medium

Question: Using equation 3, show how light is travelling in a homogeneous medium.

Answer:

Equation 3 can be rewritten using the chain rule:

$$\frac{d\vec{r}}{ds}\frac{d}{ds}n + n\frac{d^2\vec{r}}{ds^2} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} n \tag{13}$$

Note that for a homogeneous medium, the index of refraction, n, is constant. Therefore $\frac{dn}{ds} = 0$, $\frac{\partial n}{\partial x} = 0$, $\frac{\partial n}{\partial y} = 0$ and $\frac{\partial n}{\partial z} = 0$. Using this in the previous equation yields:

$$n\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{14}$$

$$\frac{d^2\vec{r}}{ds^2} = \vec{0} \tag{15}$$

This implies that the direction and velocity of the light is not changed as the light travels through the medium. Therefore, it travels in a straight line with a constant velocity of $v = c/n$.

2.2 Snell-Descartes Law

Question: Express first geometrically and then analytically Snell and Descartes law of reflection and transmission of the light at the interface between two media of different index of refraction n_1 and n_2, using the Principle of Fermat and equation 3.

Answer:

2.2.1 Geometrical

The speed of light in a medium is inversely proportional to the refractive index. Therefore, the shortest path (in distance) between two points in materials with different refractive indices is not always the fastest (in time). This phenomenon is nicely described by a 2-dimensional analogy of a beach (see figure 1). The maximum speed on foot on beach is significantly higher than the maximum swimming speed in the water. So if somebody would need to get from a point A on the beach to a point B in the water, the direct route from A to B (dashed line in figure 1) would intuitively be slower than the path with a shorter swimming distance (solid line in figure 1).

It is possible to calculate the fastest route between point A and B If we add the parameters v_1, v_2, θ_1, θ_2, a, b, c and d which corresponds respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in figure 2.

The time it takes to travel from point A to B, t, can easily be found dividing the path on the beach and the water, respectively l_{beach} and l_{water} by the corresponding speed:

$$t = l_{beach}/v_1 + l_{water}/v_2 \tag{16}$$

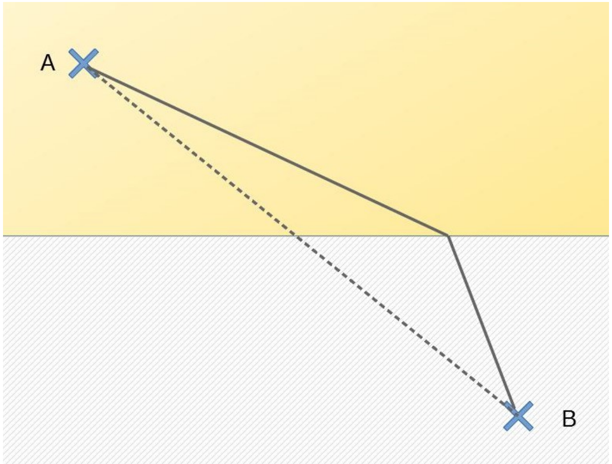


Figure 1: Diagram of a 2-dimensional beach analogy of the interface between two media with different index of refraction. The upper-half corresponds to the beach and the lower-half to the sea. The dashed line corresponds to the direct route between point A and B with the shortest distance. The solid line corresponds to a route that is intuitively faster than the direct route.

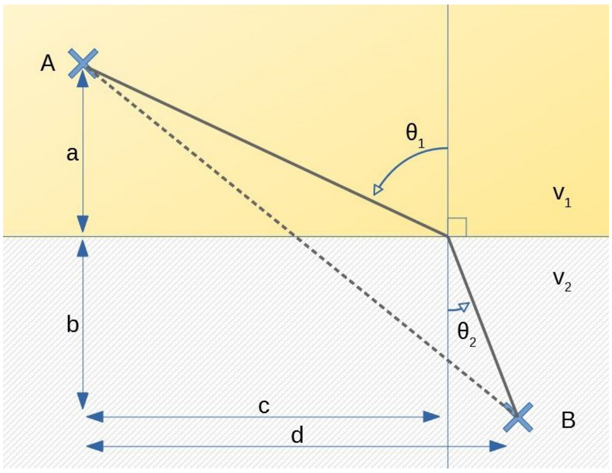


Figure 2: Diagram of beach analogy parameters v_1, v_2, θ_1, θ_2, a, b, c and d. These correspond respectively to the propagation speed on the beach, the propagation speed in the water, the angle of the path on the beach with the normal, the angle of the path in the water with the normal and distances which can be seen in the diagram.

Using the pythagoras theorem we find:

$$t = \sqrt{a^2 + c^2}/v_1 + \sqrt{b^2 + (d - c)^2}/v_2 \tag{17}$$

If there is a fastest path, there should be an optimum value for c for which $dt/dc = 0$. Therefore, applying the principle of Fermat to equation 17 leads to the following:

$$0 = \frac{c}{v_1\sqrt{a^2 + c^2}} + \frac{c - d}{v_2\sqrt{b^2 + (d - c)^2}} \tag{18}$$

Using the trigonometric identity $sin(\theta) = (adjacentside)/(diagonalside)$ for right-angled triangle we obtain:

$$0 = sin(\theta_1)/v_1 - sin(\theta_2)/v_2 \tag{19}$$

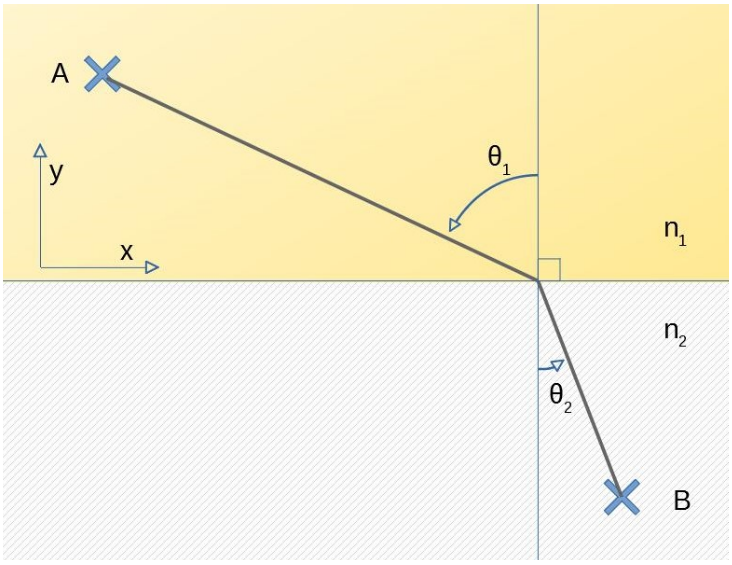
If we rewrite this and use the fact that the speed of light in a medium is given by $v = c/n$ we obtain Snell-Descartes law:

$$n_1 \ sin(\theta_1) = n_2 \ sin(\theta_2) \tag{20}$$

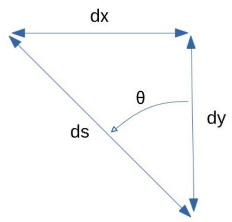
This equation basically tells us, that for a interface to a higher refractive index, so where the light slows down, the light bends to the normal.

2.2.2 Analytical

For the analytical derivation of Snell-Descartes law we will use a similar diagram as in the geometrical derivation with a coordinate system added as in figure 3a.



(a) Diagram of the path of light at the interface of two media with different refractive indices n_1 and n_2. θ_1 and θ_2 correspond to the angle with the normal.



(b) ds in relation to dx and dy.

Figure 3

If we write equation 3 for only the x-component and use the fact that n is independent of x in our diagram, we get the following:

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = \frac{dn}{dx} \tag{21}$$

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = 0 \tag{22}$$

The ds in the latter equation is defined as in figure 3b. If we keep a fixed ds for both media we obtain the following equality:

$$\frac{d}{ds}\left[n_1\frac{dx_1}{ds}\right] = \frac{d}{ds}\left[n_2\frac{dx_2}{ds}\right] \tag{23}$$

Integrating both sides with respect to ds and using the trigonometric identity, $sin(\theta) = dx/ds$, yields the Snell-Descartes law:

$$n_1\frac{dx_1}{ds} = n_2\frac{dx_2}{ds} \tag{24}$$

$$n_1 \ sin(\theta_1) = n_2 \ sin(\theta_2) \tag{25}$$

2.3 Optical fibre

2.3.1

Question: If we assume an incoming beam in the $\vartheta_{x,y}$ plane at start crossing ϑ_x with an angle θ_0. Show that the light beam will stay in this plane and that, by solving the Euler-Lagrange equation 3, the equation for the beam path can be written as:

$$n\ sin(i) = a \tag{26}$$

Answer:

For this question it is assumed that the optical fibre is cylindrical with its axis ϑ_x (see figure 4). The incident-surface will therefore be in the $\vartheta_{y,z}$ plane.

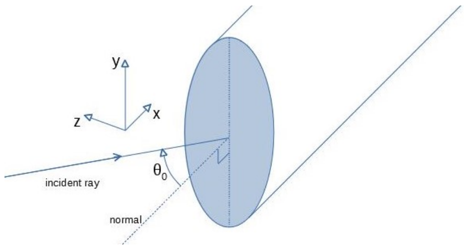


Figure 4: 3-dimensional diagram of fibre with incoming light beam making an angle θ_0 with the ϑ_x axis.

When the incoming beam is in the $\vartheta_{x,y}$ plane, the angle with the normal is θ_0 with ϑ_x in the ϑ_y direction and $\theta_z = 0$ in the ϑ_z direction. If we use Snell-Descartes law (equation 25) for θ_z with $\theta_{z,1} = 0$ we obtain $\theta_{z,2} = k \cdot \pi[rad]$ with $k = -1, 0, 1, 2, ...$. Therefore, the beam of light will stay in the $\vartheta_{x,y}$ plane at the interface when entering the optical fibre.
It is the property of a cylinder that the outer surface is always perpendicular with the radius. For this light beam. Since the light beam is travelling in the $\vartheta_{x,y}$ direction, the outer surface it reaches will be in the $\vartheta_{x,z}$ plane. The angle with the surface normal in the ϑ_z axis is still zero. From the internal reflection it follows that the angle in the ϑ_z axis will remain zero. Therefore, for every reflection at the outer surface, the beam will stay in the $\vartheta_{x,y}$ direction.

Since we know that the beam will stay in the $\vartheta_{x,y}$ plane, the problem can be simplified to a 2-dimensional problem (see figure 5). The infinitely small segment ds is related to ds, dy, dx and i as in figure 5.

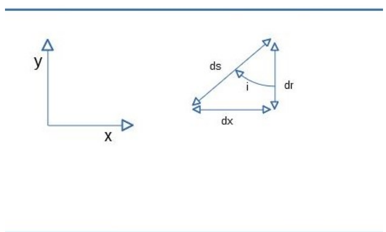


Figure 5: 2-dimensional fibre including the relation between ds, ds, dy, dx and i.

We can write equtaion 3 for only the x-component and use that n is independent of x:

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = \frac{dn}{dx} \tag{27}$$

$$\frac{d}{ds}\left[n\frac{dx}{ds}\right] = 0 \tag{28}$$

If we now integrate with respect to ds on both sides we get the following:

$$n\frac{dx}{ds} = a \tag{29}$$

With a an arbitrary constant. Using the trigonometric identity $sin(i) = dx/ds$ we obtain the requested equation:

$$n\ sin(i) = a \tag{30}$$

2.3.2

Question: Solve equation 26 for $n(r) = n_0\sqrt{1-\alpha^2r^2}$ with $\alpha < 1/r$ a constant and r the radius of the fibre.You should end up with an analytical function.
Answer:

From figure 5 and trigonometric identities it follows that $sin(i) = dx/ds$. Using the Pythagoras theorem and filling this in in equation 26 we get:

$$n\frac{dx}{\sqrt{dr^2+dx^2}} = a \tag{31}$$

Filling in $n(r)$ and some rewriting results in the following:

$$n_0\sqrt{1-\alpha^2r^2}\frac{dx}{\sqrt{dr^2+dx^2}} = a \tag{32}$$

$$n_0^2(1-\alpha^2r^2)\frac{dx^2}{dr^2+dx^2} = a^2 \tag{33}$$

$$n_0^2(1-\alpha^2r^2)dx^2 = a^2(dr^2+dx^2) \tag{34}$$

$$(n_0^2 - n_0^2\alpha^2r^2 - a^2)dx^2 = a^2dr^2 \tag{35}$$

$$dx^2 = \frac{a^2dr^2}{n_0^2 - n_0^2\alpha^2r^2 - a^2} \tag{36}$$

$$dx = \frac{a\ dr}{\sqrt{n_0^2 - n_0^2\alpha^2r^2 - a^2}} \tag{37}$$

If we now integrate both sides we will get:

$$\int dx = \int \frac{a}{\sqrt{n_0^2 - n_0^2\alpha^2r^2 - a^2}}dr \tag{38}$$

$$x = \frac{a}{\sqrt{n_0^2-a^2}}\int \frac{1}{1-\left(\frac{rn_0\alpha}{\sqrt{n_0^2-a^2}}\right)^2}dr \tag{39}$$

To solve this integral we need the following substitution:

$$\frac{rn_0\alpha}{\sqrt{n_0^2-a^2}} = sin(u) \tag{40}$$

$$dr = \frac{\sqrt{n_0^2-a^2}}{n_0\alpha}cos(u)du \tag{41}$$

$$u = arcsin\left(\frac{rn_0\alpha}{\sqrt{n_0^2-a^2}}\right) \tag{42}$$

Using the substitution we get:

$$x = \frac{a}{n_0\ \alpha} \int \frac{\cos(u)}{\sqrt{1 - \sin^2(u)}} du \tag{43}$$

If we use that $1 - \sin^2(x) = \cos^2(x)$, we obtain:

$$x = \frac{a}{n_0\ \alpha} \int \frac{\cos(u)}{\sqrt{\cos^2(u)}} du \tag{44}$$

$$x = \frac{a}{n_0\ \alpha} \int du \tag{45}$$

$$x = \frac{a}{n_0\ \alpha} u \tag{46}$$

Inverting the substitution and doing some rewriting yields the following equation:

$$x = \frac{a}{n_0\ \alpha} arcsin\left(\frac{r\ n_0\ \alpha}{\sqrt{n_0^2 - a^2}}\right) \tag{47}$$

$$\sin\left(\frac{n_0\ \alpha\ r}{a}\right) = \frac{r n_0 \alpha}{\sqrt{n_0^2 - a^2}} \tag{48}$$

$$r(x) = \frac{\sqrt{n_0^2 - a^2}}{n_0 \alpha} \sin\left(\frac{n_0\ \alpha\ r}{a}\right) \tag{49}$$

The Since there are no discontinuities in $n(r)$ and $dn(r)/dr$, we would expect that there are also no discontinuities in $r(x)$ and $dr(x)/dx$. Since $r = \mid y \mid$ for the light beam in the $\vartheta_{x,y}$ plane. The only possible solution of $y(x)$ is the following:

$$y(x) = \frac{\sqrt{n_0^2 - a^2}}{n_0 \alpha} \sin\left(\frac{n_0\ \alpha\ r}{a}\right) \tag{50}$$