CHAPTER 2

Fuzzy Sets and Fuzzy Logic

2. 1 INTRODUCTION

Fuzzy sets were introduced by L.A Zadeh in 1965 to represent/manipulate data and information possessing nonstatistical uncertainties. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems.

Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning. The conventional approaches to knowledge representation lack the means for representing the meaning of fuzzy concepts. As a consequence, the approaches based on first order logic and classical probability theory do not provide an appropriate conceptual framework for dealing with the representation of commonsense knowledge, since such knowledge is by its nature both lexically imprecise and noncategorical. The development of fuzzy logic was motivated in large measure by the need for a conceptual frame work which can address the issue of uncertainty and lexical imprecision.

2.2 WHAT IS FUZZY LOGIC?

Fuzzy logic is all about the relative importance of precision: How important is it to be exactly right when a rough answer will do? All books on fuzzy logic begin with a few good quotes on this very topic, and this is no exception. Here is what some clever people have said in the past:

Precision is not truth.

—Henri Matisse

Sometimes the more measurable drives out the most important.

—Rene Dubos

Vagueness is no more to be done away with in the world of logic than friction in mechanics.

—Charles Sanders Peirce

I believe that nothing is unconditionally true, and hence I am opposed to every statement of positive truth and every man who makes it.

—H. L. Mencken

So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.

—Albert Einstein

As complexity rises, precise statements lose meaning and meaningful statements lose precision.

—L. A Zadeh

Some pearls of folk wisdom also echo these thoughts:

Don't lose sight of the forest for the trees.

Don't be penny wise and pound foolish.

Fuzzy logic is a fascinating area of research because it does a good job of trading off between significance and precision - something that humans have been managing for a very long time (Fig. 2.1). Fuzzy logic sometimes appears exotic or intimidating to those unfamiliar with it, but once you become acquainted with it, it seems almost surprising that no one attempted it sooner. In this sense fuzzy logic is both old and new because, although the modern and methodical science of fuzzy logic is still young, the concepts of fuzzy logic reach right down to our bones.

Fuzzy logic is a convenient way to map an input space to an output space. This is the starting point for everything else, and the great emphasis here is on the word "convenient." What do I mean by mapping input space to output space? Here are a few examples: You tell me how good your service was at a restaurant, and I'll tell you what the tip should be. You tell me how hot you want the water, and I'll adjust the faucet valve to the right setting. You tell me how far away the subject of your photograph is, and I'll focus the lens for you. You tell me how fast the car is going and how hard the motor is working, and I'll shift the gears for you.

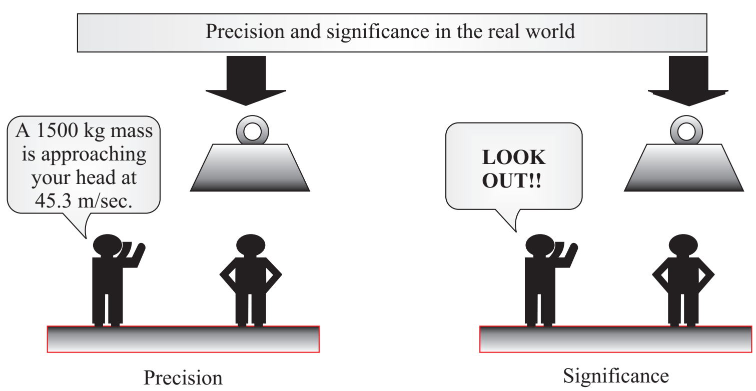

Fig. 2.1 Precision and significance.

2.3 HISTORICAL BACKGROUND

Almost forty years have passed since the publication of first paper on fuzzy sets. Where do we stand today? In viewing the evolution of fuzzy logic, three principal phases may be discerned.

The first phase, from 1965 to 1973, was concerned in the main with fuzzification, that is, with generalization of the concept of a set, with two-valued characteristic function generalized to a membership function taking values in the unit interval or, more generally, in a lattice. The basic issues and applications which were addressed were, for the most part, set-theoretic in nature, and logic and reasoning were not at the center of the stage.

The second phase, 1973-1999, two key concepts were introduced in this paper: (a) the concept of a linguistic variable; and (b) the concept of a fuzzy if-then rule. Today, almost all applications of fuzzy set theory and fuzzy logic involve the use of these concepts.

The term ***fuzzy logic*** was used for the first time in 1974. Today, ***fuzzy logic*** is used in two different senses: (a) a narrow sense, in which fuzzy logic, abbreviated as FLn, is a logical system which is a generalization of multivalued logic; and (b) a wide sense, in which fuzzy logic, abbreviated as FL, is a union of FLn, fuzzy set theory, possibility theory, calculus of fuzzy if-then rules, fuzzy arithmetic, calculus of fuzzy quantifiers and related concepts and calculi. The distinguishing characteristic of FL is that in FL everything is, or is allowed to be, a matter of degree. Today, the term ***fuzzy logic*** is used, for the most part, in its wide sense.

Perhaps the most striking development during the second phase of the evolution was the naissance and rapid growth of fuzzy control, alongside the boom in fuzzy logic applications, especially in Japan. There were many other major developments in fuzzy-logic-related basic and applied theories, among them the genesis of possibility theory and possibilistic logic, knowledge representation, decision analysis, cluster analysis, pattern recognition, fuzzy arithmetic; fuzzy mathematical programming, fuzzy topology and, more generally, fuzzy mathematics. Fuzzy control applications proliferated but their dominance in the literature became less pronounced.

Soft computing came into existence in 1981, with the launching of BISC (Berkeley Initiative in Soft Computing) at UC Berkeley. Basically, soft computing is a coalition of methodologies which collectively provide a foundation for conception, design and utilization of intelligent systems. The principal members of the coalition are: fuzzy logic, neurocomputing, evolutionary computing, probabilistic computing, chaotic computing, rough set theory and machine learning. The basic tenet of soft computing is that, in general, better results can be obtained through the use of constituent methodologies of soft computing in combination rather than in a stand-alone mode. A combination which has attained wide visibility and importance is that of neuro-fuzzy systems. Other combinations, e.g., neuro-fuzzy-genetic systems, are appearing, and the impact of soft computing is growing on both theoretical and applied levels.

An important development in the evolution of fuzzy logic, marking the beginning of the third phase, 1996 — is the genesis of computing with words and the computational theory of perceptions. Basically, development of computing with words and perceptions brings together earlier strands of fuzzy logic and suggests that scientific theories should be based on fuzzy logic rather than on Aristotelian, bivalent logic, as they are at present. A key component of computing with words is the concept of Precisiated Natural Language (PNL). PNL opens the door to a major enlargement of the role of natural languages in scientific theories. It may well turn out to be the case that, in coming years, one of the most important application-areas of fuzzy logic, and especially PNL, will be the Internet, centering on the conception and design of search engines and question-answering systems.

From its inception, fuzzy logic has been (and to some degree still is) an object of skepticism and controversy. In part, skepticism about fuzzy logic is a reflection of the fact that, in English, the word ***fuzzy*** is usually used in a pejorative sense. But, more importantly, for some fuzzy logic is hard to accept because by abandoning bivalence it breaks with centuries-old tradition of basing scientific theories on bivalent logic. It may take some time for this to happen, but eventually abandonment of bivalence will be viewed as a logical development in the evolution of science and human thought.

2.4 CHARACTERISTICS OF FUZZY LOGIC

Some of the essential characteristics of fuzzy logic relate to the following:

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
- In fuzzy logic, everything is a matter of degree.
- In fuzzy logic, knowledge is interpreted a collection of elastic or, equivalently, fuzzy constraint on a collection of variables.
- Inference is viewed as a process of propagation of elastic constraints.
- Any logical system can be fuzzified.

2.5 CHARACTERISTICS OF FUZZY SYSTEMS

There are two main characteristics of fuzzy systems that give them better performance for specific applications:

- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the system with a mathematical model that is difficult to derive.
- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

2.6 FUZZY SETS

2.6.1 Fuzzy Set

Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function (Fig. 2.2).

$$\mu_A : X \to [0, 1] \qquad \text{...(2.1)}$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

It is clear that A is completely determined by the set of tuples

$$A = \{(u, \mu_A(u)) \mid u \in X\} \qquad \text{...(2.2)}$$

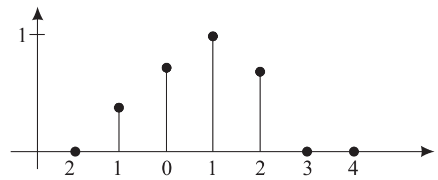

Fig. 2.2 A discrete membership function for "x is close to 1".

Frequently we will write $A(x)$ instead of $\mu A(x)$. The family of all fuzzy sets in X is denoted by $F(X)$. If $X = \{x_1, ..., x_n\}$ is a finite set and A is a fuzzy set in X then we often use the notation

$$A = \mu_1/x_1 + ... + \mu_n/x_n \quad ...(2.3)$$

where the term μ_i/x_i, $i = 1,...,n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

Example 2.1: The membership function (Fig. 2.3) of the fuzzy set of real numbers "close to 1", can be defined as

$$A(t) = \exp(-\beta(t-1)^2)$$

where β is a positive real number.

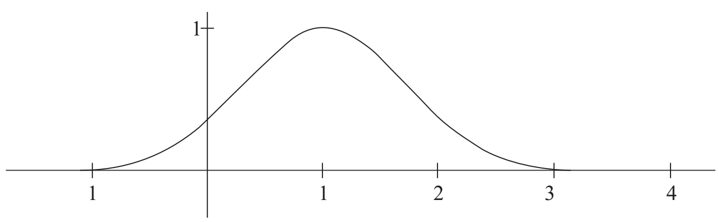

Fig. 2.3 A membership function for "x is close to 1".

Example 2.2: Assume someone wants to buy a cheap car. Cheap can be represented as a fuzzy set on a universe of prices, and depends on his purse (Fig. 2.4). For instance, from the Figure cheap is roughly interpreted as follows:

- Below Rs. 300000 cars are considered as cheap, and prices make no real difference to buyer's eyes.

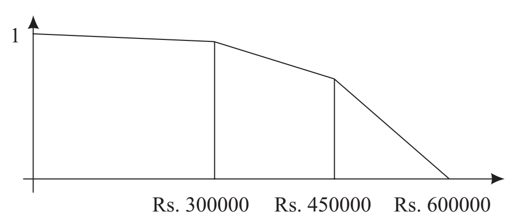

Fig. 2.4 Membership function of "cheap".

- Between Rs. 300000 and Rs. 450000, a variation in the price induces a weak preference in favor of the cheapest car.
- Between Rs. 450000and Rs. 600000, a small variation in the price induces a clear preference in favor of the cheapest car.
- Beyond Rs. 600000 the costs are too high (out of consideration).

2.6.2 Support

Let A be a fuzzy subset of X; the support of A, denoted supp (A), is the crisp subset of X whose elements all have nonzero membership grades in A.

$$\text{supp}(A) = \{x \in X \mid A(x) > 0\}. \qquad ...(2.5)$$

2.6.3 Normal Fuzzy Set

A fuzzy subset A of a classical set X is called normal if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal.

2.6.4 α-Cut

An a-level set of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^\alpha$ and is defined by

$$[A]^\alpha = \begin{cases} \{t \,\varepsilon\, X | A(t) \geq \alpha\} & \text{if} \quad \alpha > 0 \\ cl(\text{supp}\, A) & \text{if} \quad \alpha = 0 \end{cases} \qquad ...(2.6)$$

where cl(supp A) denotes the closure of the support of A.

Example 2.3: Assume

$$X = \{-2, -1, 0, 1, 2, 3, 4\} \text{ and}$$

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4.$$

In this case

$$[A]^\alpha = \begin{cases} \{-1, 0, 1, 2, 3\} & \text{if} \quad 0 \leq \alpha \leq 0.3 \\ \{0, 1, 2\} & \text{if} \quad 0.3 < \alpha \leq 0.6 \\ \{1\} & \text{if} \quad 0.6 < \alpha \leq 1 \end{cases}$$

2.6.5 Convex Fuzzy Set

A fuzzy set A of X is called convex if $[A]^\alpha$ is a convex subset of $X \forall \alpha \in [0, 1]$. An α-cut of a triangular fuzzy number is shown in Fig. 2.5.

In many situations people are only able to characterize numeric information imprecisely. For example, people use terms such as, about 5000, near zero, or essentially bigger than 5000. These are examples of what are called fuzzy numbers. Using the theory of fuzzy subsets we can represent these fuzzy numbers as fuzzy subsets of the set of real numbers. More exactly,

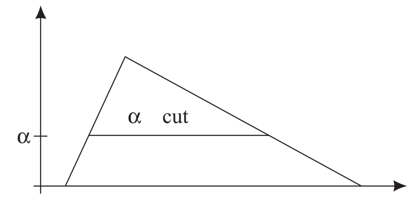

Fig. 2.5 An α-cut of a triangular fuzzy number.

2.6.6 Fuzzy Number

A fuzzy number (Fig. 2.6) A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by F .

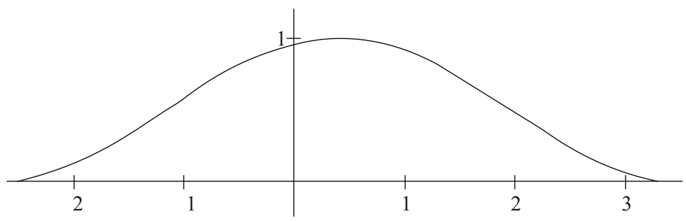

Fig. 2.6 Fuzzy number.

2.6.7 Quasi Fuzzy Number

A quasi-fuzzy number A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions

$$\lim_{1 \to \infty} A(t) = 0 \qquad ...(2.7)$$

Let Abe a fuzzy number. Then $[A]^\gamma$ is a closed convex (compact) subset of $\Re$ for all $\gamma \in [0,1]$. Let us introduce the notations

$$a_1(\gamma) = \min [A]^\gamma, \quad a_2(\gamma) \max [A]^\gamma \qquad ...(2.8)$$

In other words, a_1 (γ) denotes the left-hand side and a_2 (γ) denotes the right-hand side of the γ - cut. It is easy to see that

if $$\alpha \leq \beta \text{ then } [A]^\alpha \supset [A]^\beta \qquad ...(2.9)$$

Furthermore, the left-hand side function

$$a_1 : [0, 1] \to \Re \qquad ...(2.10)$$

is monotone increasing and lower semicontinuous, and the right-hand side function

$$a_2 : [0, 1] \to \Re \qquad ...(2.11)$$

is monotone decreasing and upper semicontinuous.

We shall use the notation

$$[A]^\gamma = [a_1(\gamma),\ a_2(\gamma)] \qquad \text{...(2.12)}$$

The support of A is the open interval $[a_1(0), a_2(0)]$ and it is illustrated in Fig. 2.7.

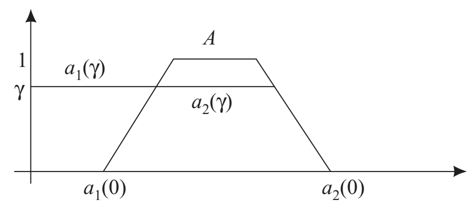

Fig. 2.7 The support of A is $[a_1(0), a_2(0)]$.

If A is not a fuzzy number then there exists an $\gamma \in [0, 1]$ such that $[A]^\gamma$ is not a convex subset of R. The not fuzzy number is shown in Fig. 2.8.

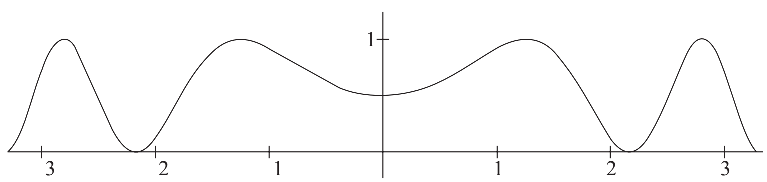

Fig. 2.8 Not fuzzy number.

2.6.8 Triangular Fuzzy Number

A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \le t \le a \\ 1 - \dfrac{a-t}{\beta} & \text{if } a \le t \le a + \beta \\ 0 & \text{otherwise} \end{cases} \qquad \text{...(2.13)}$$

and we use the notation $A = (\alpha, \alpha, \beta)$. It can easily be verified that

$$[A]^\gamma = [a - (1-\gamma)\alpha,\ a + (1-\gamma)\beta],\ \forall_\gamma \in [0,1] \qquad \text{...(2.14)}$$

The support of A is $(a - \alpha, b + \beta)$.

A triangular fuzzy number (Fig. 2.9) with center a may be seen as a fuzzy quantity "x is approximately equal to a".

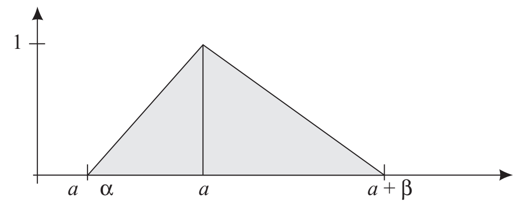

Fig. 2.9 Triangular fuzzy number.

2.6.9 Trapezoidal Fuzzy Number

A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width and right width β if its membership function has the following form.

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \le t \le a \\ 1 & \text{if } a \le t \le b \\ 1 - \dfrac{t-b}{\beta} & \text{if } a \le t \le b + \beta \\ 0 & \text{otherwise} \end{cases} \qquad \text{...(2.15)}$$

and we use the notation $A = (a, b, \alpha, \beta)$. It can easily be shown that

$$[A]^{\gamma} = [a - (1-\gamma)\alpha,\ b + (1-\gamma)\beta],\ \forall_{\gamma} \in [0, 1] \qquad \text{...(2.16)}$$

The support of is $(a - \alpha, b + \beta)$.

A trapezoidal fuzzy number (Fig. 2.10) may be seen as a fuzzy quantity "x is approximately in the interval $[a, b]$".

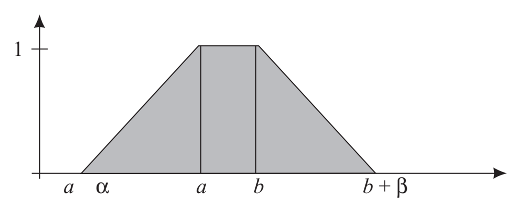

Fig. 2.10 Trapezoidal fuzzy number.

2.6.10 Subsethood

Let A and B are fuzzy subsets of a classical set X. We say that A is a subset of B if $A(t) \le B(t)$, $\forall t \in X$. The subsethood is illustrated in Fig. 2.11.

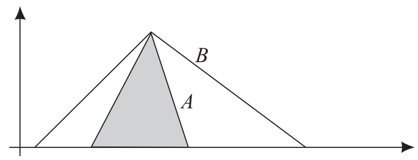

Fig. 2.11 *A* is a subset of *B*.

2.6.11 Equality of Fuzzy Sets

Let A and B are fuzzy subsets of a classical set X. A and B are said to be equal, denoted $A = B$, if $A \subset B$ and $B \subset A$. We note that $A = B$ if and only if $A(x) = B(x)$ for $x \in X$.

2.6.12 Empty Fuzzy Set

The empty fuzzy subset of X is defined as the fuzzy subset Ø of X such that $Ø(x) = 0$ for each $x \in X$. It is easy to see that $Ø \subset A$ holds for any fuzzy subset A of X.

2.6.13 Universal Fuzzy Set

The largest fuzzy set in X, called universal fuzzy set (Fig. 2.12) in X, denoted by 1_X, is defined by $1_X(t) = 1$, $\forall t \in X$. It is easy to see that $A \subset 1_X$ holds for any fuzzy subset A of X.

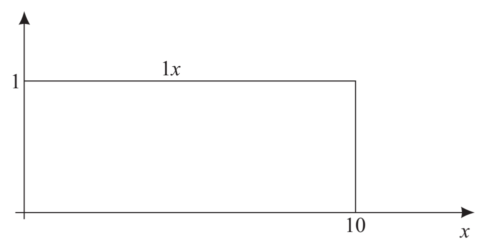

Fig. 2.12 The graph of the universal fuzzy subset in $X = [0, 10]$.

2.6.14 Fuzzy Point

Let A be a fuzzy number. If supp $(A) = \{x_0\}$, then A is called a fuzzy point (Fig. 2.13) and we use the notation $A = \bar{x}_0$.

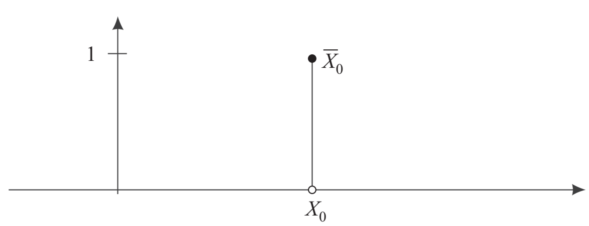

Fig. 2.13 Fuzzy point.

Let $A = \bar{x}_0$ be a fuzzy point. It is easy to see that

$$[A]^{\gamma} = [x_0, x_0] = \{x_0\}, \forall_{\gamma} \in [0, 1] \quad ...(2.17)$$

2.7 OPERATIONS ON FUZZY SETS

We extend the classical set theoretic operations from ordinary set theory to fuzzy sets. We note that all those operations which are extensions of crisp concepts reduce to their usual meaning when the fuzzy subsets have membership degrees that are drawn from {0, 1}. For this reason, when extending operations to fuzzy sets we use the same symbol as in set theory.

Let A and B are fuzzy subsets of a nonempty (crisp) set X.

2.7.1 Intersection

The intersection of A and B is defined as

$$(A \cap B)(t) = \min \{A(t), B(t)\} = A(t) \wedge B(t) \text{ for all } t \in X \quad ...(2.18)$$

The intersection of A and B is shown in Fig. 2.14.

2.7.2 Union

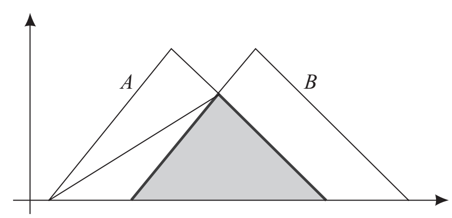

Fig. 2.14 Intersection of two triangular fuzzy numbers.

The union of A and B is defined as

$$(A \cup B)(t) = \max \{A(t), \quad B(t)\} = A(t) \vee B(t) \text{ for all } t \in X \quad ...(2.19)$$

The union of two triangular numbers is shown in Fig. 2.15.

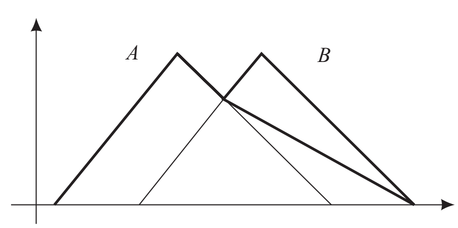

Fig. 2.15 Union of two triangular fuzzy numbers.

2.7.3 Complement

The complement of a fuzzy set A is defined as

$$(\neg A)(t) = 1 - A(t) \qquad \text{...(2.20)}$$

A closely related pair of properties which hold in ordinary set theory are the law of excluded middle

$$A \vee \neg A = X \qquad \text{...(2.21)}$$

and the law of non-contradiction principle

$$A \wedge \neg A = \phi \qquad \text{...(2.22)}$$

It is clear that $\neg 1_X = \phi$ and $\neg\phi = 1_X$, however, the laws of excluded middle and noncontradiction are not satisfied in fuzzy logic.

Lemma–2.1: The law of excluded middle is not valid. Let $A(t)$ =1/2, $\forall t \in R$, then it is easy to see that

$$\begin{aligned}(\neg A \vee A)(t) &= \max\{\neg A(t), A(t)\}\\ &= \max\{1 - 1/2, 1/2\}\\ &= 1/2 \neq 1\end{aligned}$$

Lemma–2.2: The law of non-contradiction is not valid. Let $A(t)$ =1/2, $\forall t \in R$, then it is easy to see that

$$\begin{aligned}(\neg A \vee A)(t) &= \text{mix}\{\neg A(t), A(t)\}\\ &= \text{mix}\{1 - 1/2, 1/2\}\\ &= 1/2 \neq 0\end{aligned}$$

However, fuzzy logic does satisfy De Morgan's laws

$$\neg(A \wedge B) = \neg A \vee \neg B, \quad \neg(A \vee B = \neg A \wedge \neg B)$$