CHAPTER 3

Fuzzy Relations

3.1 INTRODUCTION

A classical relation can be considered as a set of tuples, where a tuple is an ordered pair. A binary tuple is denoted by (u, v), an example of a ternary tuple is (u, v, w) and an example of n-ary tuple is $(X_1, ..., X_n)$.

Example 3.1: Let X be the domain of man {John, Charles, James} and Y the domain of women {Diana, Rita, Eva}, then the relation "married to" on $X \times Y$ is, for example

{(Charles, Diana), (John, Eva), (James, Rita)}

3.2 FUZZY RELATIONS

3.2.1 Classical N-Array Relation

Let $X_1, ..., X_n$ be classical sets. The subsets of the Cartesian product $X_1 \; x \; \ldots x \; X_n$ are called n-ary relations. If $X_1 = ... = X_n$ and $R \subset X^n$, then R is called an n-ary relation in X. Let R be a binary relation in $\Re$. Then the characteristic function of R is defined as

$$X_R(u, v) = \begin{cases} 1 & \text{if} \quad (u, v) \in R \\ 0 & \text{otherwise} \end{cases} \qquad ...(3.1)$$

Example 3.2 Consider the following relation

$$(u, v) \in R \Leftrightarrow u \in [a,\ b] \text{ and } \quad v \in [0,\ c] \qquad ...(3.2)$$

$$X_R(u, v) = \begin{cases} 1 & \text{if} \quad (u, v) \in [a, b] x [0, c] \\ 0 & \text{otherwise} \end{cases}$$

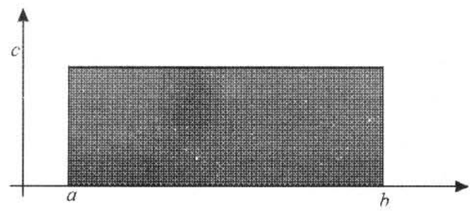

Fig. 3.1

3.2.2 Reflexivity

Let R be a binary relation in a classical set X. Then R is reflexive if $\forall u \in U$: $(u, v) \in R$.

3.2.3 Anti-Reflexivity

R is anti-reflexive if $\forall u \in U$: $(u, v) \notin R$.

3.2.4 Symmetricity

R is symmetric if from $(u, v) \in R \rightarrow (u, v) \in R$, $\forall u, v \in U$.

3.2.5 Anti-Symmetricity

R is anti-symmetric if $(u, v) \in R$ and $(v, u) \in R$ then $u = v$, $\forall u, v \in U$.

3.2.6 Transitivity

R is transitive if $(u, v) \in R$ and $(v, w) \in R$ then $(u, w) \in R$, $\forall u, v, w \in U$.

Example 3.3: Consider the classical inequality relations on the real line $\Re$. It is clear that $\leq$ is reflexive, anti-symmetric and transitive, $<$ is anti-reflexive, anti-symmetric and transitive.

Other important properties of binary relations are:

3.2.7 Equivalence

R is an equivalence relation if, R is reflexive, symmetric and transitive.

3.2.8 Partial Order

R is a partial order relation if it is reflexive, anti-symmetric and transitive.

3.2.9 Total Order

R is a total order relation if it is partial order and $\forall u, v \in R$, $(u, v) \in R$ or $(v, u) \in R$ hold.

Example 3.4: Let us consider the binary relation "subset of". It is clear that we have a partial order relation. The relation $\leq$ on natural numbers is a total order relation.

Consider the relation "mod 3" on natural numbers

$$\{(m, n) \mid (n - m) \bmod 3 \equiv 0\}$$

This is an equivalence relation.

3.2.10 Binary Fuzzy Relation

Let X and Y be nonempty sets. A fuzzy relation R is a fuzzy subset of $X \times Y$.
In other words, $R \in F(X \times Y)$.

If $X = Y$ then we say that R is a binary fuzzy relation in X.

Let R be a binary fuzzy relation on R. Then $R(u, v)$ is interpreted as the degree of membership of the ordered pair (u, v) in R.

Example 3.5: A simple example of a binary fuzzy relation on $U = \{1, 2, 3\}$, called "approximately equal" can be defined as

$$R(1, 1) = R(2, 2) = R(3, 3) = 1$$

$$R(1, 2) = R(2, 1) = R(2, 3) = R(3, 2) = 0.8$$

$$R(1, 3) = R(3, 1) = 0.3$$

The membership function of R is given by

$$R(u, v) = \begin{cases} 1 & \text{if} \quad u = v \\ 0.8 & \text{if} \quad |u - v| = 1 \\ 0.3 & \text{if} \quad |u - v| = 2 \end{cases}$$

In matrix notation it can be represented as

$$\begin{bmatrix} & 1 & 2 & 3 \\ 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{bmatrix}$$

3.3 OPERATIONS ON FUZZY RELATIONS

Fuzzy relations are very important because they can describe interactions between variables. Let R and S be two binary fuzzy relations on $X \times Y$.

3.3.1 Intersection

The intersection of R and S is defined by

$$(R \wedge S)(u, v) = \min\{R(u, v), S(u, v)\} \qquad ...(3.3)$$

Note that $R: X \times Y \rightarrow [0, 1]$, i.e. R the domain of R is the whole Cartesian product $X \times Y$.

3.3.2 Union

The union of R and S is defined by

$$(R \vee S)\,(u, v) = \min\{R(u, v), S(u, v)\} \qquad ...(3.4)$$

Example 3.6: Let us define two binary relations

R = "x is considerable larger than y"

$$\begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{array}$$

S = "x is very close to y"

$$\begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.4 & 0 & 0.9 & 0.6 \\ x_2 & 0.9 & 0.4 & 0.5 & 0.7 \\ x_3 & 0.3 & 0 & 0.8 & 0.5 \end{array}$$

The intersection of R and S means that "x is considerable larger than y" and "x is very close to y".

$$(R \wedge S)\,(x, y) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.4 & 0 & 0.1 & 0.6 \\ x_2 & 0 & 0.4 & 0 & 0 \\ x_3 & 0.3 & 0 & 0.7 & 0.5 \end{array}$$

The union of R and S means that "x is considerable larger than y" or "x is very close to y".

$$(R \vee S)\,(x, y) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.8 & 0 & 0.9 & 0.7 \\ x_2 & 0.9 & 0.8 & 0.5 & 0.7 \\ x_3 & 0.9 & 1 & 0.8 & 0.8 \end{array}$$

Consider a classical relation R on $\mathfrak{R}$.

$$R(u, v) = \begin{cases} 1 & \text{if} \quad (u, v) \in [a, b]\, x\, [0, c] \\ 0 & \text{otherwise} \end{cases} \qquad ...(3.5)$$

It is clear that the projection (or shadow) of R on the X-axis is the closed interval $[a, b]$ and its projection on the Y-axis is $[0, c]$.

If R is a classical relation in $X \times Y$, then

$$\prod_x = \{x \in X | \exists y \in Y(x, y) \in R\} \quad \text{...(3.6)}$$

$$\prod_y = \{y \in Y | \exists x \in X(x, y) \in R\} \quad \text{...(3.7)}$$

where $\prod_x$ denotes projection on X and $\prod_y$ denotes projection on Y.

3.3.3 Projection

Let R be a fuzzy binary fuzzy relation on $X \times Y$. The projection of R on X is defined as

$$\prod_x (x) = \sup \{R(x, y) \mid y \in Y\} \quad \text{...(3.8)}$$

and the projection of R on Y is defined as

$$\prod_y (y) = \sup \{R(x, y) \mid x \in X\} \quad \text{...(3.9)}$$

Example 3.7: Consider the relation

$$R = \text{“}x \text{ is considerable larger than } y\text{”} = \left[\begin{array}{ccccc} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0 \end{array}\right]$$

then the projection on X means that

- x_1 is assigned the highest membership degree from the tuples (x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), i.e. $\prod_x(x_1) = 1$, which is the maximum of the first row.
- x_2 is assigned the highest membership degree from the tuples (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), i.e. $\prod_x(x_2) = 0.8$, which is the maximum of the second row.
- x_3 is assigned the highest membership degree from the tuples (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), i.e. $\prod_x(x_3) = 1$, which is the maximum of the third row.

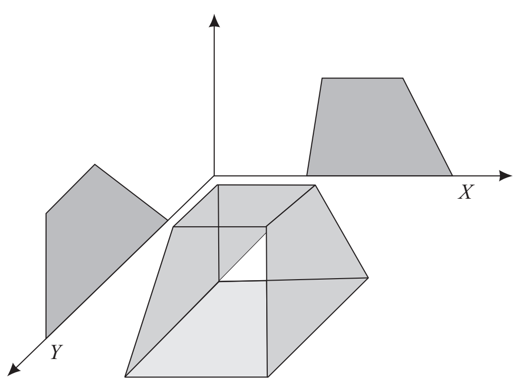

Fig. 3.2 Shadows of a fuzzy relation.

3.3.4 Cartesian Product of Two Fuzzy Sets

The Cartesian product of $A \in F(X)$ and $B \in F(Y)$ is defined as

$$(A \times B)(u, v) = \min\{A(u), B(v)\} \qquad ...(3.10)$$

for all $u \in X$ and $v \in Y$.

It is clear that the Cartesian product of two fuzzy sets (Fig. 3.3) is a fuzzy relation in $X \times Y$.

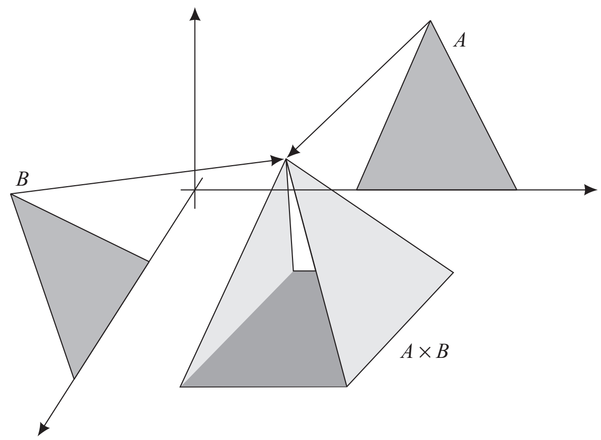

Fig. 3.3 Cartesian product of two fuzzy sets.

If A and B are normal, then $\prod_y (A \times B) = B$ and $\prod_x (A \times B) = A$.

Really,

$$\begin{aligned}\prod_x(x) &= \sup\{A \times B(x, y) \mid y\} \\ &= \sup\{A(x) \wedge B(y) \mid y\} \\ &= \min\{A(x), \sup\{B(y)\} \mid y\} \qquad ...(3.11) \\ &= \min\{A(x), 1\} \\ &= A(x)\end{aligned}$$

3.3.5 Shadow of Fuzzy Relation

The sup-min composition of a fuzzy set $C \in F(X)$ and a fuzzy relation $R \in F(X \times Y)$ is defined as

$$(C \circ R)(y) = \sup \min\{C(x), R(x, y)\} \qquad ...(3.12)$$

for all $x \in X$ and $y \in Y$.

The composition of a fuzzy set C and a fuzzy relation R can be considered as the shadow of the relation R on the fuzzy set C (Fig. 3.4).

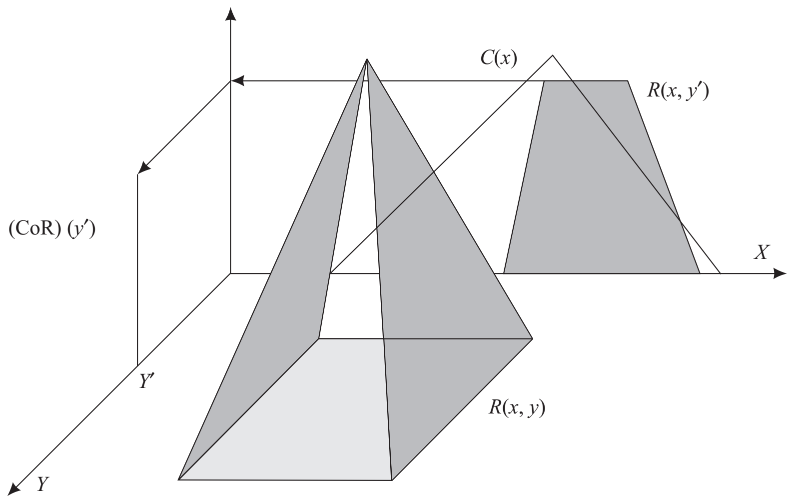

Fig. 3.4 Shadow of fuzzy relation R on the fuzzy set C.

Example 3.8: Let A and B be fuzzy numbers and let

$$R = A \times B$$

a fuzzy relation.

Observe the following property of composition

$$A \circ R = A \circ (A \times B) = A$$

$$B \circ R = B \circ (A \times B) = B$$

Example 3.9: Let C be a fuzzy set in the universe of discourse $\{1, 2, 3\}$ and let R be a binary fuzzy relation in $\{1, 2, 3\}$. Assume that $C = 0.2/1 + 1/2 + 0.2/3$ and

$$R = \begin{bmatrix} & 1 & 2 & 3 \\ 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{bmatrix}$$

Using the definition of sup-min composition we get

$$C \circ R = (0.2/1 + 1/2 + 0.2/3) \circ \begin{bmatrix} & 1 & 2 & 3 \\ 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{bmatrix} = 0.8/1 + ½ + 0.8/3$$

Example 3.10: Let C be a fuzzy set in the universe of discourse [0, 1] and let R be a binary fuzzy relation in [0, 1]. Assume that $C(x) = x$ and $R(x, y) = 1 - |x - y|$.

Using the definition of sup-min composition we get

$$C \text{ o } R(y) = \sup \min\{x, 1 - |x - y|\} = \frac{1 + y}{2}$$

for all $x \in [0, 1]$ and $y \in [0, 1]$

3.3.6 Sup-Min Composition of Fuzzy Relations

Let $R \in F(X \times Y)$ and $S \in F(Y \times Z)$. The sup-min composition of R and S, denoted by $R \text{ o } S$ is defined as

$$(R \text{ o } S)\,(u, w) = \sup \min\{R(u, v), S(v, w)\} \qquad \text{...(3.13)}$$

for $v \in Y$

It is clear that $R \text{ o } S$ is a binary fuzzy relation in $X \times Z$.

Example 3.11: Consider two fuzzy relations

$$R = \text{“}x \text{ is considerable larger than } y\text{”} = \begin{bmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{bmatrix}$$

$$S = \text{“ } y \text{ is very close to } z\text{”} = \begin{bmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{bmatrix}$$

Then their composition is

$$\text{R o S} = \begin{bmatrix} & z_1 & z_2 & z_3 \\ x_1 & 0.6 & 0.8 & 0.5 \\ x_2 & 0 & 0.4 & 0 \\ x_3 & 0.7 & 0.9 & 0.7 \end{bmatrix}$$

formally,

$$\begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{array} \circ \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{array} = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.6 & 0.8 & 0.5 \\ x_2 & 0 & 0.4 & 0 \\ x_3 & 0.7 & 0.9 & 0.7 \end{array}$$

i.e., the composition of R and S is nothing else, but the classical product of the matrices R and S with the difference that instead of addition we use maximum and instead of multiplication we use minimum operator.