Elasticity Assignment 01

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-: Exercise 1st Chapter :-

Problem 1-1 compute the following:-

a) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

→ $a_{ii} = a_{11} + a_{22} + a_{33} = 1+4+1 = 6$ (scalar)

→ $a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$
$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 1 = 25$ (scalar)

→ $a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix}$ matrix

→ $a_{ij}b_j = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (vector)

→ $a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1$
$+ a_{32}b_3b_2 + a_{33}b_3b_3 = 1+0+2+0+0+0+0+0+4 = 7$ (scal

→ $b_ib_j = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \cancel{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ matrix

→ $b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1 + 0 + 4 = 5$ (scalar)

b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}$, $b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

→ $a_{ii} = a_{11} + a_{22} + a_{33} = 1+2+0 = 3$ (Scaler)

→ $a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + \cdots\cdots a_{33}a_{33} = 1+4+4+1+16+16 = 30$

→ $a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix}$ (matrix)

→ $a_{ij}b_j = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ (vector)

$\rightarrow$ $a_{ij}\, b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + \cdots\; a_{33} b_3 b_3 = 4+4+0+0+2+1+4+2+0 = 17$

$\rightarrow$ $b_i b_j = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (matrix)

$\rightarrow$ $b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3 = 4+1+1 = 6$ (scalar)

c) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\rightarrow$ $a_{ii} = a_{11} + a_{22} + a_{33} = 1+0+4 = 5$ (scalar)

$\rightarrow$ $a_{ij} a_{ij} = a_{11} a_{11} + a_{12} a_{12} + a_{13} a_{13} + \cdots\; a_{33} a_{33} = 1+1+\cancel{1+1}\,1+0+4+1+16 = 25$

$\rightarrow$ $a_{ij} a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix}$

$\rightarrow$ $a_{ij} b_j = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (vector)

$\rightarrow$ $a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + \cdots + a_{33} b_3 b_3 = 1+1+0+0+0+0+0+0+0 = 3$

$\rightarrow$ $b_i b_j = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow$ $b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3 = 1+1+0 = 2$ (scalar)

Problem 1-2 use decomposition result to express a_{ij} from above to express symetric & antisymetric matrices.

By definition:-

a) $$a_{ij} = \underbrace{\frac{1}{2}(a_{ij} + a_{ji})}_{\text{Symets}} + \underbrace{\frac{1}{2}(a_{ij} - a_{ji})}_{\text{Antisym}}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

By calculating it, will give a_{ij}

b) $a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$

$$= \underbrace{\frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}}_{\text{Symmetric}} + \underbrace{\frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}}_{\text{Anti-Symmetric}}$$

c) $a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$

$$= \underbrace{\frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}}_{\text{Symmetric}} + \underbrace{\frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}}_{\text{Anti-Symmetric}}$$

Problem 1-3 If a_{ij} is symmetric & b_{ij} is antisymmetric, Prove $a_{ij} b_{ij} = 0$

Generally, $a_{ij} b_{ij} = -a_{ji} b_{ji}$

$= -a_{ij} b_{ij}$

$+2a_{ij} b_{ij} = 0$

$\boxed{a_{ij} b_{ij} = 0}$

a) $a_{(ij)} a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T \right) = 0$

b) $a_{(ij)} a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T \right) = 0$

c) $a_{(ij)} a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T \right) = 0$

Problem 1-4 Explicitly verify following

i) $\delta_{ij}a_j = a_i$, ii) $\delta_{ij}a_{jk} = a_{ik}$

a) $\delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = \begin{bmatrix} \delta_{11}a_1 & \delta_{12}a_2 & \delta_{13}a_3 \\ \delta_{21}a_1 & \delta_{22}a_2 & \delta_{23}a_3 \\ \delta_{31}a_1 & \delta_{32}a_2 & \delta_{33}a_3 \end{bmatrix}$

$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$

b) $\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \end{bmatrix}$

$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$

Problem 1-5 Formally expand & proof the det$[a_{ij}]$ expression

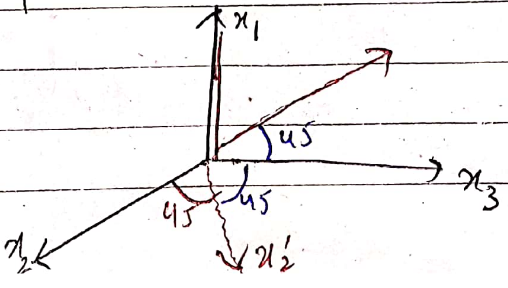
$\det(a_{ij}) = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{231}a_{12}a_{23}a_{31} + \varepsilon_{312}a_{13}a_{21}a_{32} + \varepsilon_{321}a_{13}a_{22}a_{31} + \varepsilon_{132}a_{11}a_{23}a_{32} + \varepsilon_{213}a_{12}a_{21}a_{33}$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

$= a_{13}(a_{21}a_{32} - a_{22}a_{31}) + a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$

$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Problem 1-6 New coordinate system is rotated through an angle of 45° about x_1-axis determine components of vector & matrix.



$$Q_{ij} = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{bmatrix}$$

$$= \begin{bmatrix} \cos(0) & \cos(90^\circ) & \cos(90^\circ) \\ \cos(90^\circ) & \cos(45^\circ) & \cos 45^\circ \\ \cos(90^\circ) & \cos(45^\circ) & \cos 45^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

From 1-1(a) $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

$$a_{ij}' = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4 & -1.5 \\ 0 & -2 & 1 \end{bmatrix}$$

From 1-1 (b) $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$

$$a_{ij}' = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

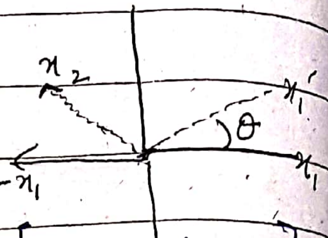
From 1-1 (c) $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$a_{ij}' = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 3.5 & 2.5 \\ -1/\sqrt{2} & 1.5 & 0.5 \end{bmatrix}$$

Problem 1-7 2D cordinate transformal in Polar cordinates

$$Q_{ij} = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') \\ \cos(x_2, x_1') & \cos(x_2, x_2') \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix}$$

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



$$b_{ij}' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{iq} Q_{jp} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$a_{ij}' = \begin{bmatrix} a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta & a_{12}\cos^2\theta - a_{21}\sin^2\theta - (a_{11}-a_{22}) \\ a_{21}\cos^2\theta - a_{12}\sin^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta & a_{11}\sin^2\theta\, a_{22}\cos^2\theta - (a_{12}+a_{21}) \end{bmatrix}$$

Problem 1-8 Show that second order retain its form of $a\delta_{ij}$ under transfor.

As $a_{ij}'\delta_{ij}' = Q_{ip} Q_{jq}\, a\delta_{pq} = a Q_{ip} Q_{jp} = a\delta_{ij}$

Problem 1-8 verify this remains same under transformation

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$\alpha'\delta_{ij}'\delta_{kl}' + \beta'\delta_{ik}'\delta_{jl}' + \gamma'\delta_{il}'\delta_{jk}' = Q_{im} Q_{jn} Q_{kp} Q_{lq} (x)$$

where $x = (\alpha\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np})$

$$= \alpha Q_{im} Q_{jm} Q_{lp} + \beta Q_{im} Q_{jn} Q_{km} Q_{ln} + \gamma Q_{im} Q_{jn} Q_{kn} Q_{lm}$$

$$= \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

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Problem 1-10 show that if $\beta = \gamma$ the

$$C_{ijkl} = C_{klij}$$

$$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}$$

$$= \alpha \delta_{ij}\delta_{kl} + \beta (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$= \alpha \delta_{kl}\delta_{ij} + \beta (\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li})$$

$$= C_{klij}$$

Q1-11

Show that the fundamental invariants ....

Solution:

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$

$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix}$

$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3.$

$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3.$

Q 1-12

Determine variants and principal value and direction of following matrices. Use determined ....

(a) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = a_{ij}$

$I_a = a_{jj} = a_{11} + a_{22} + a_{33} =$

$= -1 - 1 + 1$

$= -1$

$II_a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

$= \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$

$= (1-1) + (-1+0) + (-1+0) = -2.$

$III_a = \det[a_{ij}]$

$$= \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

$$= -1(-1-0) + 1(0-1) + 0(0-0).$$

$$= 1 - 1 = 0.$$

Thus characteristic equation is:

$$0 = -\lambda^3 + I_a\lambda^2 - II_a\lambda + III_a.$$

$$0 = -\lambda^3 + (-1)\lambda^2 - (-2)\lambda + 0.$$

$$0 = -\lambda^3 - \lambda^2 + 2\lambda.$$

$$\lambda(\lambda^2 + \lambda - 2) = 0.$$

$$\lambda(\lambda+2)(\lambda-1) = 0.$$

Roots:

$$\lambda_2 = 0, \lambda_1 = -2, \lambda_3 = 1$$

Case 1:

When $\lambda_1 = -2$...

using $(a_{ij} - \lambda\delta_{ij})n_j = 0.$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$n_1^{(1)} + n_2^{(1)} = 0.$$

$$n_3^{(1)} = 0 \Rightarrow n_1 = -n_2 = \pm\sqrt{2}/2$$

$$n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1.$$

$$\Rightarrow n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0)$$

Case 2:

When $\lambda_2 = 0$, using $(a_{ij} - \lambda\delta_{ij})n_j = 0.$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$.

$n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1$.

For $\lambda_3 = 1$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0.$$

$-2n_1^{(3)} + n_2^{(3)} = 0$

$n_1^{(3)} - 2n_2^{3} = 0 \Rightarrow n_1 = n_2 = 0, n_3 = 1.$

$n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1.$

$\Rightarrow n^{(3)} = \pm(0,0,1).$

Rotation matrix is given by:

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

and $a_{ij}' = Q_{ip}Q_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & +1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$I_a = -2 - 2 + 0 = -4$

$II_a = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} + 0 + 0 = 4 - 1 = 3.$

$III_a = -2(0) - 1(0) + 0 = 0.$

$\Rightarrow$ characteristic eq'n is:

$0 = -\lambda^3 + I_a\lambda^2 - II_a\lambda + III_a.$

$-\lambda^3 - 4\lambda^2 - 3\lambda = 0.$

$\lambda(\lambda^2 + 4\lambda + 3) = 0$

$\lambda(\lambda+3)(\lambda+1) = 0.$

Roots are

$$\lambda_1 = -3,\ \lambda_2 = -1,\ \lambda_3 = 0.$$

Case I:

$\lambda_1 = -3.$, by $(a_{ij} - \lambda\delta_{ij})n_j = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0.$$

$$n_1^{(1)} + n_2^{(1)} = 0.$$

$$n_3 = 0 \quad \Rightarrow \quad n_1^{(1)} = -n_2^{(1)} = +\sqrt{2}/2.$$

$$n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1$$

$$\Rightarrow \quad n^{(1)} = \pm\frac{\sqrt{2}}{2}(-1, 1, 0).$$

Case II:

when $\lambda_2 = -1$,

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0.$$

$$-n_1^{(2)} + n_2^{(2)} = 0.$$

$$n_3^{(2)} = 0. \quad \Rightarrow \quad n_1 = n_2 = \pm\sqrt{2}/2.$$

$$n_1^{(2)^2} + n_2^{(2)^2} + n_3^{(2)^2} = 1.$$

$$\Rightarrow n^{(2)} = \pm\frac{\sqrt{2}}{2}(1, 1, 0).$$

Case III;

when $\lambda_3 = 0$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$-2n_1^{(3)} + n_2^{(3)} = 0.$$

$$n_1^{(3)} - 2n_2^{(3)} = 0 \quad \Rightarrow \quad n_1 = n_2 = 0, \quad n_3^{(3)} = 1$$

$$n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1.$$

$$\Rightarrow \quad n^{(3)} = \pm(0, 0, 1).$$

The rotation matrix is given by $Q_{ij} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$.

$$a'_{ij} = Q_{ip}Q_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$I_a = -1-1 = -2$; $II_a = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 0 + 0 = 0$; $III_3 = -1(0) - 1(0) + 0 = 0$.

characteristic eq'n is:

$-\lambda^3 - 2\lambda^2 = 0$.

$\lambda^2(\lambda+2) = 0$.

Roots:

$\lambda_1 = -2,\ \lambda_2 = \lambda_3 = 0$.

Case I:

when $\lambda_1 = -2$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$\Rightarrow n_1^{(1)} + n_2^{(2)} = 0$

$n_2^{(1)} = 0 \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2$

$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$

$\Rightarrow n^{(1)} = \pm\frac{\sqrt{2}}{2}(-1, 1, 0)$.

Q1-15:

The dual vector a_i of an anti symmetric 2nd order tensor a_{ij} = defined as $a_i = -1/2\,\varepsilon_{ijk} a_{jk}$. Show....

Solution:

$$a_i = -1/2\,\varepsilon_{ijk} a_{jk}.$$

$$\varepsilon_{imn} a_i = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn} a_{jk}.$$

$$= -\frac{1}{2}\begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk}.$$

$$= -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}.$$

$$= -\frac{1}{2}(a_{mn} - a_{nm}) = -\frac{1}{2}(a_{mn} + a_{mn}) = -a_{mn}$$

$$\therefore\ a_{jk} = -\varepsilon_{ijk} a_i$$

Q1-16:

Using index notation, explicity verify

(a) $(1.8.5)_{1,2,3}$

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi.$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k}$$

$$= \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}.$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\nabla\psi.$$

$$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k} u_k.$$

$$= \nabla\phi\cdot u + \phi(\nabla u).$$

(b) $(1.8.5)_{4,5,6,7}$

$$\nabla\times(\phi U) = \varepsilon_{ijk}(\phi U_k)_{,j} = \varepsilon_{ijk}(\phi U_{k,j} + \phi_{,j} U_k)$$
$$= \varepsilon_{ijk}\phi_{,j}U_k + \phi\,\varepsilon_{ijk}U_{k} = \nabla\phi\times U + \phi(\nabla\times U)$$

$$\nabla\cdot(U\times V) = (\varepsilon_{ijk}U_jV_k)_{,i} = \varepsilon_{ijk}(U_jV_{k,i} + U_{j,i}V_k)$$
$$= V_k\,\varepsilon_{ijk}U_{j,i} + U_j\,\varepsilon_{ijk}V_{k,i} = V\cdot(\nabla\times U) - U\cdot(\nabla\times U)$$

$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0$ (symmetry & antisymmetry in jk)

$\nabla\cdot\nabla\phi = \nabla^2\phi.$

(c) $(1.8.5)_{8,9,10}$

$$\nabla\cdot(\nabla\times U) = (\varepsilon_{ijk}U_{k,j})_{,i} = \varepsilon_{ijk}U_{k,ji} = 0.$$

because of symmetry and antisymmetry in ij

$$\nabla\times(\nabla\times U) = \varepsilon_{mni}(\varepsilon_{ijk}U_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}U_{k,jn}$$
$$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})U_{k,jn}$$
$$= U_{n,nm} - U_{m,nn}$$
$$= \nabla(\nabla\cdot U) - \nabla^2 U.$$

$$U\times(\nabla\times U) = \varepsilon_{ijk}U_j(\varepsilon_{kmn}U_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}U_jU_{n,m}$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})U_jU_{n,m} = U_nU_{n,i} - U_mU_{i,m}$$
$$= \frac{1}{2}\nabla(U\cdot U) - U\nabla U.$$

Q 1-17: Extend the results found and determine the forms of ∇f, $\nabla \cdot u$, $\nabla^2 f$ and $\nabla \times u$ for a 3D cylindrical coordinate system.

Cylindrical coordinates:

$\xi^1 = r,\ \xi^2 = \theta,\ \xi^3 = z.$

$(ds)^2 = (dr)^2 + (r d\theta)^2 + (dz)^2.$

$\Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1.$

$\hat{e}_r = \cos\theta\, e_1 + \sin\theta\, e_2,$

$\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2,$

$\hat{e}_z = e_3.$

$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta,\quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$

$\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e}_z}{\partial z} = 0.$

$\nabla = \hat{e}_r\, \partial/\partial r + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z\, \partial/\partial z.$

$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r} \partial f/\partial\theta + \hat{e}_z\, \partial f/\partial z.$

$\nabla \cdot u = \frac{1}{r} \partial/\partial r\, (r u_r) + \frac{1}{r} \partial u_\theta/\partial\theta + \partial u_z/\partial z.$

$\nabla^2 f = \frac{1}{r} \partial/\partial r\, (r\, \partial f/\partial r) + \frac{1}{r^2} \partial^2 f/\partial\theta^2 + \partial^2 f/\partial z^2.$

$\nabla \times u = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta$

$+ \frac{1}{r}\left(\partial/\partial r\, (r u_\theta) - \partial u_r/\partial\theta\right)\hat{e}_z$

Q 1-18

Show that for spherical coordinates (R, ϕ, θ) $h_1 = 1,\ h_2 = R,\ h_3 = R\sin\phi$

Solution:

spherical coordinate: $\xi^1 = R,\ \xi^2 = \phi,\ \xi^3 = \theta.$

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3,\quad x^2 = \xi^1 \sin\xi^2 \sin\xi^3,\quad x^3 = \xi^1 \cos\xi^2.$

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Scale factors:

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1} \cdot \frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2} \frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R.$$

$$(h_3)^3 = \frac{\partial x^k}{\partial \xi^3} \frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi.$$

Unit vectors:

$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3.$

$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3.$

$\hat{e}_\theta = -\sin\phi\, e_1 + \cos\theta\, e_2$

$\partial\hat{e}_R/\partial R = 0, \quad \partial\hat{e}_R/\partial\phi = \hat{e}_\phi, \quad \partial\hat{e}_R/\partial\theta = \sin\phi\,\hat{e}_\theta$

$\partial\hat{e}_\phi/\partial R = 0, \quad \partial\hat{e}_\phi/\partial\phi = -\hat{e}_r, \quad \partial\hat{e}_\phi/\partial\theta = \cos\phi\,\hat{e}_\theta.$

$\partial\hat{e}_\theta/\partial R = 0, \quad \partial\hat{e}_\theta/\partial\phi = 0, \quad \partial\hat{e}_\theta/\partial\theta = \cos\phi\,\hat{e}_\theta$

$\nabla = \hat{e}_R\, \partial/\partial R + \hat{e}_\phi\, 1/R\, \partial/\partial\phi + \hat{e}_\theta\, 1/R\sin\phi\, \partial/\partial\theta.$

$\nabla f = \hat{e}_R\, \partial f/\partial R + \hat{e}_\phi\, 1/R\, \partial f/\partial\phi + \hat{e}_\theta\, 1/R\sin\theta\, \partial f/\partial\theta.$

$$\nabla\cdot u = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi\, u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}(R\sin\phi\, u_\phi) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}(R u_\theta).$$

$$= 1/R^2\, \partial/\partial R\,(R^2 u_R) + 1/R\sin\phi\, \partial/\partial\phi\,(\sin\phi\, u_\phi) + 1/R\sin\phi\, \partial/\partial\theta\,(u_\theta).$$

$$\nabla^2 f = 1/R^2\, \partial/\partial R\,(R^2\, \partial f/\partial R) + 1/R^2\sin\phi\, \frac{\partial}{\partial\phi}(\sin\phi\, \partial f/\partial\phi) + 1/R^2\sin^2\phi\, \partial^2 f/\partial\phi^2.$$

$$\nabla\times u = \left(1/R^2\sin\phi\left(\partial/\partial\phi\,(R\sin\phi\, u_\theta) - \partial/\partial\theta\,(R u_\phi)\right]\right)\hat{e}_R + \left(1/R\sin\phi\left(\partial/\partial\theta\,(u_R) - \partial/\partial R\,(R\sin\phi\, u_\theta)\right]\right)\hat{e}_\phi + \left(1/R\sin\phi - \partial/\partial\phi\,(u_R)\right)\hat{e}_\theta + \left(1/R\, \partial/\partial R\left[(R u_\phi) - \right.\right.$$

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$$= \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial \phi}(\sin\phi U_\theta) - \frac{\partial u}{\partial\theta} \right) \right] \hat{e}_R + \left[\frac{1}{R\sin\phi} \frac{\partial u_R}{\partial\theta} - \frac{1}{R}\frac{\partial}{\partial R}(R U_\theta) \right] \hat{e}_\phi + \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(R U_\phi) - \frac{\partial u_R}{\partial\phi} \right) \right] \hat{e}_\theta$$

Question : Transform strain displacement relation from cartesian to cylindrical and spherical coordinates.

(1) cylindrical co-ordinates.

$u_x = u_r\cos\theta - u_\theta\sin\theta$.

$u_y = u_r\sin\theta + u_\theta\cos\theta$.

$u_z = u_z$

Derivatives of $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ where $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$ is given by:

$$\partial/\partial x = \partial r/\partial x \, \partial/\partial r + \partial\theta/\partial x \, \partial/\partial\theta = \cos\theta \, \partial/\partial r - \frac{\sin\theta}{r}\partial/\partial\theta.$$

$$\partial/\partial y = \partial r/\partial y \, \partial/\partial r + \partial\theta/\partial y \, \partial/\partial\theta = \sin\theta \, \partial/\partial r + \frac{\cos\theta}{r}\partial/\partial\theta$$

It follows that.

$$\partial^2/\partial x^2 = \left(\cos\theta \, \partial/\partial r - \frac{\sin\theta}{r}\partial/\partial\theta\right)\left(\cos\theta \, \partial/\partial r - \frac{\sin\theta}{r}\partial/\partial\theta\right)$$

$$= \cos^2\theta \, \partial^2/\partial r^2 + \frac{\sin^2\theta}{r^2}\partial^2/\partial\theta^2 - \cos\theta\sin\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\partial/\partial\theta\right)$$

$$= \cos^2\theta \, \partial^2/\partial r^2 + \sin^2\theta/r^2 \, \partial^2/\partial\theta^2 - \cos\theta\sin\theta\left[-\frac{1}{r^2}\partial/\partial\theta + \frac{1}{r}\partial^2/\partial r\partial\theta\right] + \sin^2\theta/r \, \partial/\partial r - \sin\theta\cos\theta/r \, \partial^2/\partial\theta\partial r$$

$$= \cos^2\theta \, \partial^2/\partial r^2 + \sin^2\theta\left[\frac{1}{r}\partial/\partial r + \frac{1}{r^2}\partial^2/\partial\theta^2\right] + 2\sin\theta\cos\theta\left(\frac{1}{r^2}\partial/\partial\theta - \frac{1}{r}\partial^2/\partial r\partial\theta\right).$$

Likewise:

$$\partial^2/\partial y^2 = \sin^2\theta \, \partial^2/\partial r^2 + \cos^2\theta\left(\frac{1}{r}\partial/\partial r + \frac{1}{r^2}\partial r^2/\partial\theta^2\right) - 2\sin\theta\cos\theta\left(\frac{1}{r^2}\partial/\partial\theta - \frac{1}{r}\partial^2/\partial r\partial\theta\right).$$

$$e_{xx} = \partial u_x/\partial x = \cos\theta \, \partial/\partial r\,(u_r\cos\theta - u_\theta\sin\theta) - \sin\theta/r\,(\partial/\partial\theta(u_r\cos\theta - u_\theta\sin\theta))$$

$$= \partial u_r/\partial r \cos^2\theta + \left(u_\theta/r - \partial u_\theta/\partial r - 1/r \, \partial u_r/\partial \theta \right) \sin\theta\cos\theta$$
$$+ \left(u_r/r - 1/r \, \partial u_\theta/\partial \theta \right) \sin^2\theta.$$

$$e_{yy} = \partial u_y/\partial y = \sin\theta \, \partial/\partial r \left(u_r \sin\theta + u_\theta \cos\theta \right) + \cos\theta/r \, \partial/\partial\theta \left(u_r \sin\theta + u_\theta \cos\theta \right)$$

$$e_{xy} = 2\left(\partial x/\partial y + \partial y/\partial x \right).$$

thus

$$e_{rr} = \partial u_r/\partial r, \quad e_{\theta\theta} = 1/r \left(u_r + \partial u_\theta/\partial\theta \right),$$
$$e_{r\theta} = 1/2 \left(1/r \, \partial u_r/\partial\theta + 2 u_\theta/\partial r - u_\theta/r \right).$$

~~and~~

$$e_{zz} = \partial u_z/\partial z.$$