Khovanov homology via immersed curves.

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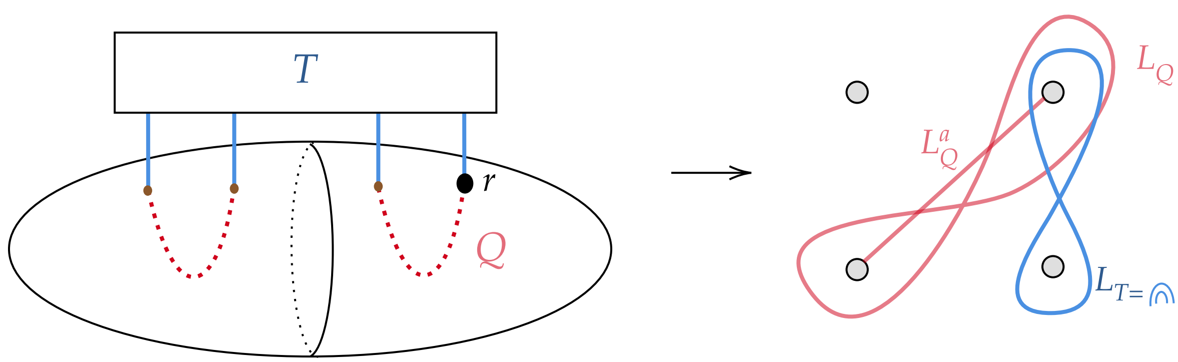
1. Statement.

Theorem 1

Suppose $(S^3, K) = (D^3, Q) \cup_{(S^2, 4pt)} (D^3, T)$ is a decomposition of a knot into two 4-ended tangles. Then reduced and unreduced Khovanov homology is isomorphic to Lagrangian Floer homology of immersed curves inside $S^2 \setminus 4pt$:

$$Kh_r(K; F_2) \cong HF_*\left(L_Q^a, L_T\right),$$
$$Kh(K; F_2) \cong HF_*(L_Q, L_T),$$

where L_Q^a, L_Q are immersed curves in $S^2 \setminus 4pt$ depicted below, and L_T is a tangle invariant taking form of another immersed curve, possibly with a local system, in $S^2 \setminus 4pt$.



The main idea for the proof is to first study bordered Khovanov invariants for tangles, and then translate them into geometric objects inside the Fukaya category $Fuk\left(S^2 \setminus 4pt\right)$. The proof can be split up into following steps:

$$CKh_r(K) \overset{Step\ 1}{\simeq} \widehat{A}(Q)_{B_r} \boxtimes {}^{B_r}\widehat{D}(T) \overset{Step\ 3}{\simeq} M\left(L_Q^a\right)_A \boxtimes {}^{A}N(L_T) \overset{Step\ 2}{\simeq} CF_*\left(L_Q^a, L_T\right).$$

Remark. These notes are based on the talk I gave at the conference "Perspectives on bordered Heegaard Floer theory" in Montreal, May 2018. In the forthcoming paper [KWZ] we substantially improve the results presented here. Among other things we remove the assumption that Q is a trivial tangle, obtain Bar-Natan's deformation of Khovanov homology via wrapped Floer homology, and along the way classify objects in the fully wrapped Fukaya category of the 3-punctured 2-disc.

Acknowledgements. I would like to thank Paul Kirk for helpful discussions and particularly for explaining the results of [HHHK].

2. Step 1. Bordered Khovanov theory.

In [Kh] Khovanov developed invariants for tangles, taking values in homotopy equivalence classes of dg-modules over a certain arc algebra. In [Man1] Manion rephrased that construction in the language of bordered theories analogous to [LOT1], [LOT2]. Here we review this construction. We will use 4-ended tangles as examples.

Note that this construction is almost equivalent to Bar-Natan's dotted cobordism theory [BN], see [Man2] for a precise comparison.

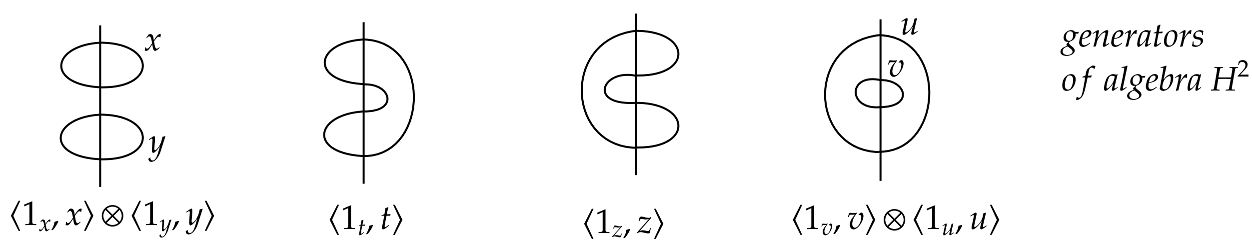
We will use the language of A_∞ (or dg) algebras, A modules, D structures, and AA/DA/DD bimodules. See [LOT1] and [LOT2] for an introduction to these algebraic structures.

2.1. Arc algebra H^n.

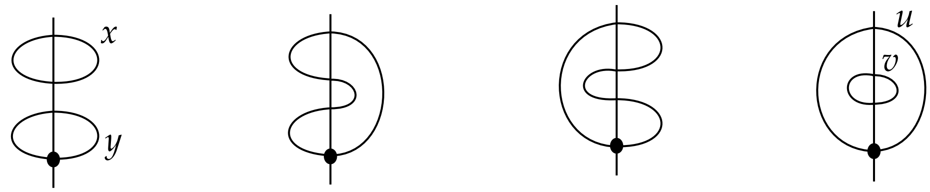
2.1.1. Generators.

Consider a plane $\mathbb{R}\times\mathbb{R}$, and a set of $2n$ points on a vertical line $0\times\mathbb{R}$. Consider crossingless arc diagrams in the left half plane $\mathbb{R}^-\times\mathbb{R}$, consisting of n arcs $\{a_1,..., a_n\}$ with boundary on those $2n$ points, up to isotopy. The number of such arc diagrams is the n-th Catalan number C_n. Analogously consider such arc diagrams in $\mathbb{R}^+\times\mathbb{R}$, to the right of the vertical line. The generators over F_2 of the arc algebra H^n are Khovanov complexes of all possible closures of left and right arc diagrams, viewed as in 3 dimensional space (later projections of our tangles and knots will be drawn on this $\mathbb{R}\times\mathbb{R}$ plane). We will call these generators "distinguished generators", or sometimes just "generators".

Example (generators of H^2).
For Khovanov generators we denote v^- as a variable x and v^+ as 1_x. We have $2\cdot 2+2\cdot 2+1+1 = 12$ generator in H^2:



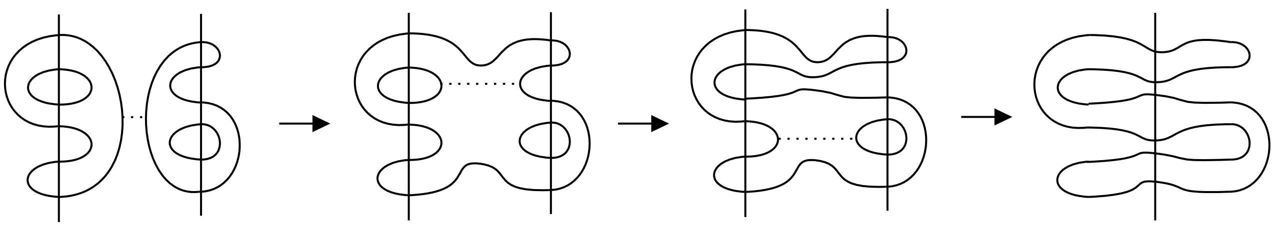
Because we will be working with reduced Khovanov homology, we will need the reduced version of H^2, which we denote by B_r. The reduced version is very similar, we only ignore v^- on the components which are reduced, and thus algebra B_r will have 6 generators:



$\langle 1_x, x\rangle \otimes \langle 1_y\rangle$ $\quad$ $\langle 1_t\rangle$ $\quad$ $\langle 1_z\rangle$ $\quad$ $\langle 1_v, v\rangle \otimes \langle 1_u\rangle$ $\quad$ *generators of algebra B_r*

2.1.2. Multiplication.

Multiplication in algebra H^n is defined via merge/split maps in Khovanov homology theory. On can multiply element a and b only if the right arc diagram of a coincides with the mirror of the left arc diagram of b (otherwise the multiplication is 0). The slit/merge maps come from a natural sequence of cobordism connecting outermost arcs in the middle, which we explain by an example picture below:

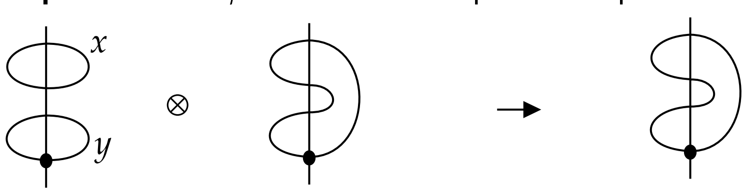


$$a \otimes b \longmapsto a \cdot b$$

If we are working with reduced arc algebra, then if one gets v^- on the reduced component as a result of a multiplication, then that term in the multiplication is set to be 0.

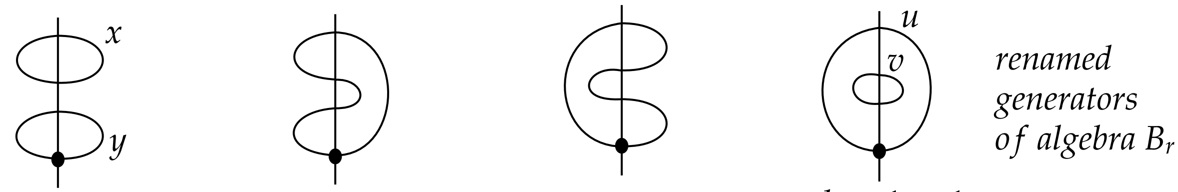
Notice, that elements of the type $1_x \otimes 1_y \otimes \cdots \otimes 1_z$ are idempotents in the algebra H^2. Notice that each distinguished generator gen of H^n has a unique "left" idempotent l_{gen} of the form $1_x \otimes 1_y \otimes \cdots \otimes 1_z$, such that $l_{gen} \cdot gen \neq 0$. The notion of "right" idempotent for distinguished generators is analogous. We denote the idempotent subring by k.

Multiplication in B_r. Here is an example of multiplication in B_r, which is reduced H^2 arc algebra.



$$(1_x \otimes 1_y) \otimes 1_t \longmapsto \langle 1_t \rangle$$
$$(x \otimes 1_y) \otimes 1_t \longmapsto 0 \quad \text{(because we are in the reduced case)}$$

Let us relabel the elements of B_r this way:



$l_0 = 1_x \otimes 1_y$ $\quad$ $p = 1_t$ $\quad$ $q = 1_z$ $\quad$ $l_1 = 1_v \otimes 1_u$

$c_0 = x \otimes 1_y$ $\quad$ $c_1 = v \otimes 1_u$

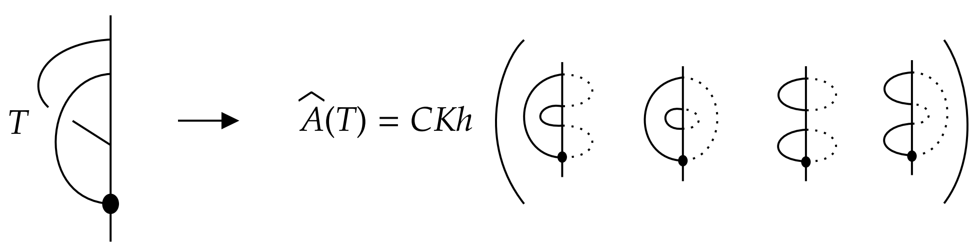
Then we have the following "path algebra of a quiver" representation for B_r, which we will often use later on.

$$B_r = \begin{tikzcd} l_0 \arrow[r, bend left, "p"] & l_1 \arrow[l, bend left, "q"] \end{tikzcd} \Big/ \begin{array}{l} pqp = 0 \\ qpq = 0 \end{array}$$

2.2. A modules associated to tangles.

Suppose we are given $2n$-ended tangle T, such that its projection on xy-plane has its ends on the vertical line $0 \times \mathbb{R}$, and the rest of the projection of the tangle is to the left of this line. Then the diagram of the projection will produce a right A module $\widehat{A}(T)_{H^n}$ over H^n, where:

1) over F_2 the generators of $\widehat{A}(T)_{H^n}$ are the generators of Khovanov complexes of all possible resolutions of the tangle together with all possible closures of it from the right by arc diagrams:



2) the differential $m_{1|0} : \widehat{A}(T) \to \widehat{A}(T)$ is given by merge/split maps at the places of crossings.

3) the action $m_{1|1} : \widehat{A}(T) \otimes H^n \to \widehat{A}(T)$ is given by cobordism maps completely analogous to the way multiplication $H^n \otimes H^n \to H^n$ was defined in Section 2.1.2 above. Notice that as a module over the idempotent subalgebra k the module $\widehat{A}(T)$ has the following structure: each generator g of $\widehat{A}(T)$ has a unique distinguished idempotent r_g in k which preserves it: $g \cdot r_g = g$. All other distinguished idempotents annihilate g.

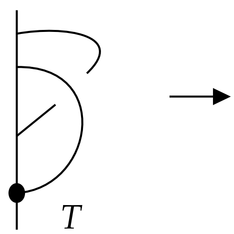
4) higher actions $m_{1|k} : \widehat{A}(T) \otimes \left(H^n\right)^{\otimes k} \to \widehat{A}(T)$ for $k \geqslant 2$ are equal to 0.

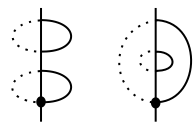
See [Man, Section 3.2] or [Kh] for a precise definition. In [Kh] it was proved that the homotopy type of $\widehat{A}(T)$ as a dg-module is an invariant of the tangle T.

2.3. D structures associated to tangles.

Suppose we are given $2n$-ended tangle T, such that its projection on xy-plane has is ends on the vertical line $0 \times \mathbb{R}$, and the rest of the projection of the tangle is to the right of this line. Then this diagram will produce a left D structure ${}^{H^n}\widehat{D}(T)$ over H^n, where:

1) over F_2 the generators of ${}^{H^n}\widehat{D}(T)$ are the generators of Khovanov complexes of all possible resolutions of the tangle closed by a specific arc diagram from the left: the one which mirrors the arc diagram on the right side. Moreover, on those circles which go through the line $0 \times \mathbb{R}$ we only pick up the 1_x type of generators:

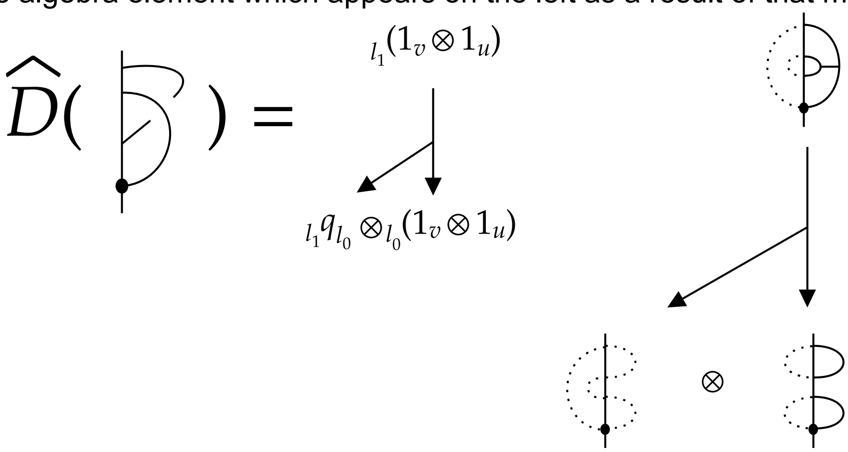




$$\widehat{D}(T) = \langle 1_x \otimes 1_y \,, 1_v \otimes 1_u \rangle_{F_2} = (F_2)^2$$

2) the left action by the idempotent subalgebra $k \otimes \widehat{D}(T) \to \widehat{D}(T)$ is determined by the arc diagram from the left: as usual each generator has its own unique left idempotent which preserves it. In the example on the picture we have: $\widehat{D}(T) = \langle {}_{l_0}(1_x \otimes 1_y) \,, {}_{l_1}(1_v \otimes 1_u) \rangle_{F_2}$, where the subscript indicates the unique idempotent.

3) the differential $\delta^1 : \widehat{D}(T) \to H^n \otimes_k \widehat{D}(T)$ is determined by the merge/split maps at the places of crossings + the algebra element which appears on the left as a result of that merge/split map:



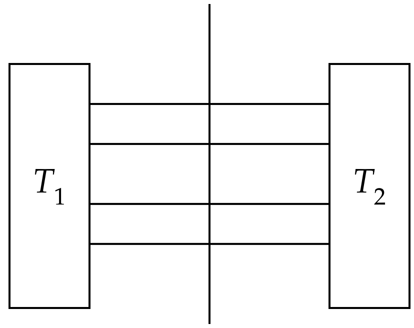
See [Man, Section 3.1] for a precise definition. Analogously to the invariance of Khovanov homology $Kh(K)$ and A module $\widehat{A}(T)$, it can be proved that the homotopy type of $\widehat{D}(T)$ is an invariant of the tangle T.

2.4. Gluing theorem for tangles.

Because the algebraic objects above were constructed specifically for the this purpose, one gets:

Theorem 2
Having two tangles T_1 and T_2 as on the picture below, one has the following homotopy equivalences:

$$CKh(T_1 \cup T_2) \simeq \widehat{A}(T_1)_{H^n} \boxtimes {}^{H^n}\widehat{D}(T_2),$$
$$\widehat{A}(T_1)_{H^n} \simeq \overline{\widehat{D}(m(T_1))}^{H^n} \boxtimes {}_{H^n}H^n_{H^n},$$
$$CKh(T_1 \cup T_2) \simeq Mor\left({}^{H^n}\widehat{D}(m(T_1)), {}^{H^n}\widehat{D}(T_2)\right) \cong \overline{\widehat{D}(m(T_1))}^{H^n} \boxtimes {}_{H^n}H^n_{H^n} \boxtimes {}^{H^n}\widehat{D}(T_2).$$

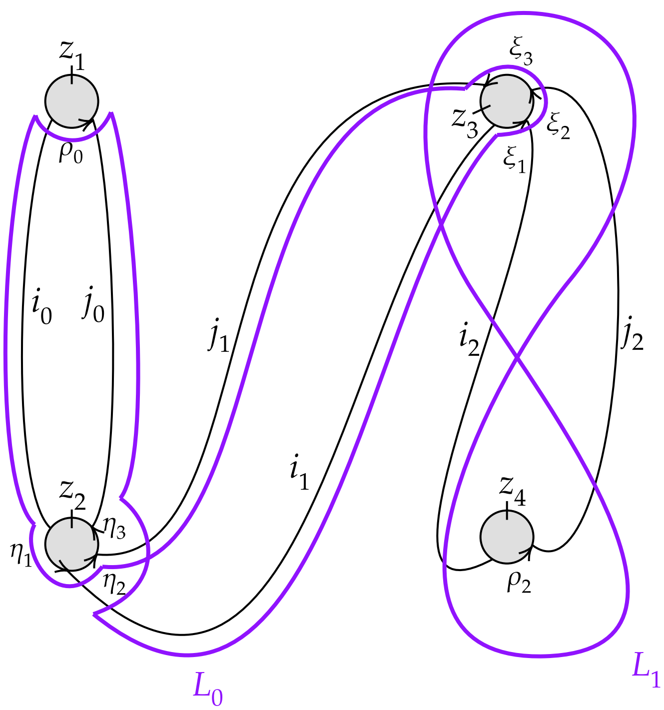


In the reduced case everything works analogously, one only needs to work over reduced arc algebra.

The algebraic structures with the bar above them are the duals, see [LOT3, Definition 2.5]

3. Step 2. The partially wrapped Fukaya category of $S^2\backslash 4pt$, and classification of its objects as immersed curves with local systems.

We will use $\overline{P} = S^2\backslash\{4\ discs\}$ with basepoints (stops) $\{z_1, z_2, z_4, z_4\}$ on the boundaries as a model for $S^2\backslash 4pt$, see the picture below. Moreover, we will parameterize it by arcs $\{i_0, i_1, i_2, j_0, j_1, j_2\}$ (which we will sometimes denote by $\{arc_0, \dots, arc_5\}$). Such a parameterization defines a chord quiver algebra A with relations, over the idempotent ring $I = \langle i_0, i_1, i_2, j_0, j_1, j_2\rangle_{F_2}$, see [HKK], [Kot1] (in [Kot1] the orientation conventions are slightly different). Algebra A is going to be our basic algebraic structure, so we describe its explicit quiver representation below:



$$A = \begin{tikzcd} i_0 \arrow[r, "\eta_1"] \arrow[d, "\rho_0"'] & i_1 \arrow[r, "\xi_1"] \arrow[d, "\eta_2"] & i_2 \arrow[d, "\xi_2"'] \arrow[d, bend left, "\rho_2"] \\ j_0 & j_1 \arrow[l, "\eta_3"] & j_2 \arrow[l, "\xi_3"] \end{tikzcd} \Big/ \begin{array}{l} \xi_1\rho_2 = 0 \\ \rho_2\xi_3 = 0 \\ \xi_3\eta_3 = 0 \\ \eta_1\xi_1 = 0 \end{array}$$

$$^A N(L_0) = \begin{tikzcd} i_0 \arrow[r, "\eta_{12}"] \arrow[d, "\rho_0"'] & j_1 \\ j_0 & i_1 \arrow[l, "\eta_{23}"] \arrow[u, "\xi_{123}"'] \end{tikzcd}$$

$$^A N(L_1) = \begin{tikzcd} i_2 \arrow[r, "\xi_{23}"] \arrow[d, "\rho_2"'] & j_1 \\ j_2 & i_1 \arrow[l, "\xi_{12}"] \arrow[u, "\eta_2"'] \end{tikzcd}$$

Denote the category of up-to-homotopy-bounded left D structures (or, equivalently, twisted complexes) over the algebra A by $^A Mod$. Denote the category of A modules (which is a different name for A_∞ modules) over A by $_A Mod$. These two categories are quasi-equivalent, see [LOT2, Section 2.3.3].

In [HKK] the Fukaya category of a marked surface $Fuk\left(\overline{P}, \{z_1, z_2, z_4, z_4\}\right)$ is defined to be ${}^{A}Mod$. It makes sense to define it like this, because the partially wrapped Fukaya category $\mathcal{F}_{pw}\left(\overline{P}, \{z_1, z_2, z_4, z_4\}\right)$ is generated by 6 objects $\{i_0, i_1, i_2, j_0, j_1, j_2\}$ (see [Aur] for this statement and the definition of $\mathcal{F}_{pw}$), and one has quasi-isomorphism (or even isomorphism, if one uses the simplest perturbations) $\bigoplus_{i,j} hom_{pw}(arc_i, arc_j) \cong A$.

In papers [HKK] and [HRW] it was proved that homotopy equivalence classes of objects in ${}^{A}Mod$ are in 1-1 correspondence with immersed curves, possibly having a local system:

$$\left\{Objects\ in\ Fuk\left(\overline{P}, \{z_1, z_2, z_4, z_4\}\right)\ up\ to\ homotopy\right\}$$

$$1to1\ (by\ def) \updownarrow$$

$$\left\{Objects\ in\ {}^{A}Mod\ up\ to\ homotopy\right\} \quad \underset{\underset{{}^{A}bar_r(A)^{A}\boxtimes\cdot}{\longleftarrow}}{\overset{\overset{{}_{A}A_{A}\boxtimes\cdot}{\longrightarrow}}{1to1}} \quad \left\{Objects\ in\ {}_{A}Mod\ up\ to\ homotopy\right\}$$

$$1to1\ (see\ [HKK]\ and\ [HRW]) \updownarrow$$

$$\left\{Immersed\ curves\ with\ local\ systems\ in\ \overline{P}\ up\ to\ homotopy\right\}$$

Having an immersed curve (a circle $L: S^1 \to \overline{P}$ or an arc $L: ([0,1], \{0,1\}) \to \left(\overline{P}, \partial\overline{P} \setminus \{z_1, z_2, z_4, z_4\}\right)$, the corresponding D structure ${}^{A}N(L)$ is defined in [HKK, Section 4.1], see the picture above for two examples ${}^{A}N(L_0), {}^{A}N(L_1)$. We drew the curve L_0 in a specific way which suggests how the D structure ${}^{A}N(L_0)$ is written down.

If the curve L is an immersed circle, then there is an A module assigned to it, which is defined by

$$M(L)_A := \bigoplus_k CF_*(L, ark_k),$$

see [Kot1] for a precise definition. If the curve is an arc, the corresponding A module is also $M(L)_A := \bigoplus_k CF_*\left(L, \widetilde{ark_k}\right)$, one just needs to perturb the arcs towards the stops as it is done in the partially wrapped context.

An important type DD structure ${}^{A}bar_r(A)^{A}$ is defined as follows:

$$\begin{gathered}{}^{A}bar_r(A)^{A} := \langle \{b(e_1, e_2, \dots, e_l) |\ e_k \in A, e_k \otimes_I e_{k+1} \neq 0, e_k e_{k+1} = 0, \} \rangle_{F_2},\\ \delta^1 : {}^{A}bar_r(A)^{A} \to A \otimes_I {}^{A}bar_r(A)^{A} \otimes_I A,\\ \delta^1(b(e_1, e_2, \dots, e_l)) := e_1 \otimes b(e_2, \dots, e_l)) \otimes 1 + 1 \otimes b(e_1, e_2, \dots, e_{l-1})) \otimes e_l,\end{gathered}$$

where $1 \in A$ is the unit which is equal to the sum of the idempotents. See [Kot1] for the geometric interpretation of this type DD structure (technically, there we described its dual ${}^{A}\overline{bar_r(A)}^{A}$). Notice that ${}^{A}bar_r(A)^{A} \boxtimes {}_{A}A_{A} = {}^{A}[I]_{A}$, identity DA bimodule (see [LOT2] for the definition of the identity bimodule).

Below we list important relationships between these objects, as well as for how to recover Lagrangian Floer homology of curves. See [Kot1] for a precise definition of Lagrangian Floer homoloy CF_*, and for the proof of 4. The proof of statements 2,5,6 is analogous to the proof of 4 in [Kot1], 3 follows from 4 and 6, and 7 is easy to see from the definition. (below while taking $CF_*(L_0, L_1)$ we always assume admissibility condition for the pair (L_0, L_1), i.e. that the partially wrapped perturbation for L_1 near the boundary was done, and there are no immersed annuli with boundaries on $L_0 \cup L_1$, see [Kot1] for the precise definition of admissibility.)

1. ${}^{A}N(L) \overset{[HKK],\,[HRW]}{\longleftrightarrow} \left[immersed\ curve\ L\ in\ \left(\overline{P}, \partial\overline{P} \setminus \{z_1, z_2, z_4, z_4\}\right)\right] \rightarrow M(L)_A \overset{def}{\simeq} \bigoplus_k CF_*(L, arc_k)$
2. $CF_*(L_0, L_1) \simeq Mor\left({}^{A}N(L_0), {}^{A}N(L_1)\right) \simeq \overline{N(L_0)}^{A} \boxtimes {}_{A}A_{A} \boxtimes {}^{A}N(L_1)$
3. $CF_*(L_0, L_1) \simeq M(L_0)_A \boxtimes {}^{A}N(L_1)$
4. $CF_*(L_0, L_1) \simeq Mor(M(L_1)_A, M(L_0)_A) \simeq M(L_0)_A \boxtimes {}^{A}\overline{bar_r(A)}^{A} \boxtimes {}_{A}\overline{M(L_1)}$
5. $M(L_0)_A \simeq \overline{N(L_0)}^{A} \boxtimes {}_{A}A_{A}$
6. ${}^{A}N(L_1) \simeq {}^{A}\overline{bar_r(A)}^{A} \boxtimes {}_{A}\overline{M(L_1)}$
7. $\bigoplus_k CF_*(arc_k, L_0) \simeq {}_{A}\overline{M(L_0)}$

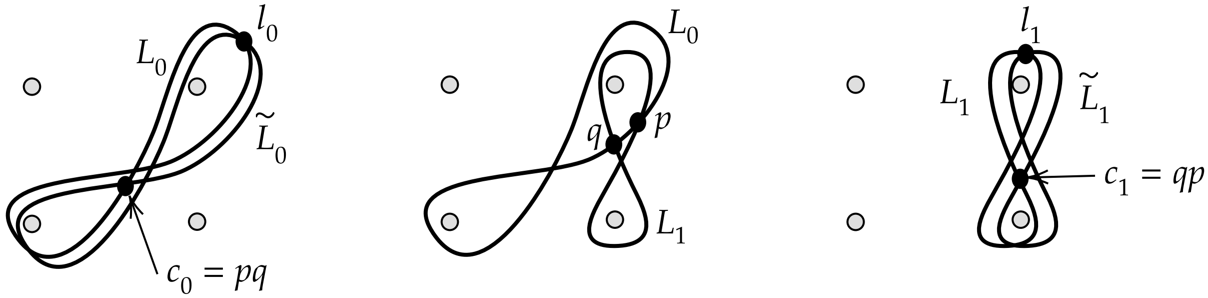
4. Step 3. Going from algebra B_r to algebra A.

4.1. Khovanov arc algebra is contained inside $Fuk(S^2\setminus 4pt)$.

Let us start first from the statement which inspired the construction that follows.

Theorem 3 *([HHHK])*
Suppose L_0, L_1 are the curves in $S^2\setminus 4pt$ depicted below. Then
1) There are perturbations associated to these curves, such that reduced Khovanov arc algebra on 4 strands B_r embeds into $Fuk(L_0, L_1)$ computed with those perturbations. The correspondence between algebra elements and intersections is depicted below. This embedding is an A_∞ homomorphism.

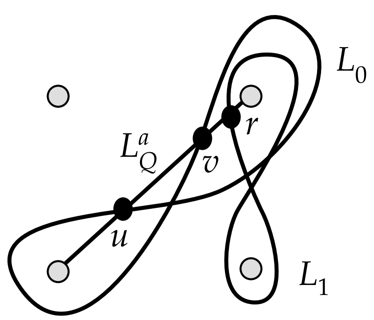
$$
B_r = \begin{tikzcd} l_0 \arrow[r, bend left, "p"] & l_1 \arrow[l, bend left, "q"] \end{tikzcd} \Big/ \begin{matrix} pqp = 0 \\ qpq = 0 \end{matrix} \quad \hookrightarrow \quad Fuk(L_0, L_1) = \bigoplus_{0 \leqslant i,j \leqslant 1} CF_*(L_i, L_j)
$$

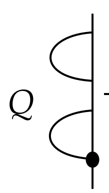


2) There is a following isomorphism of A_∞ modules:

$$
\left(CF_*\left(L_Q^a, L_0\right) \oplus CF_*\left(L_Q^a, L_1\right)\right)_{B_r} \cong \widehat{A}(Q)_{B_r},
$$

where L_Q^a is an arc and $\widehat{A}(Q)_{B_r}$ is an A module associated to the tangle Q depicted below:



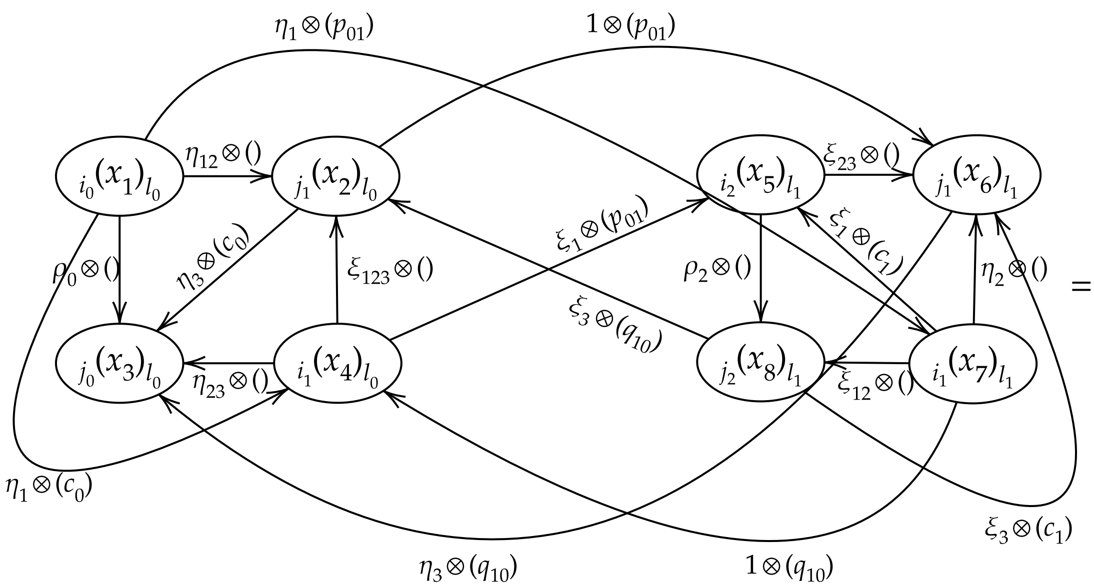


$$
\longrightarrow \quad \widehat{A}(Q)_{B_r} = \begin{tikzcd} (u)_{l_0} & & (v)_{l_0} \arrow[ll, "c_0"'] \arrow[dl, "p"] \\ & (r)_{l_1} \arrow[ul, "q"] & \end{tikzcd} = \left(\overline{\widehat{D}(Q)}^{B_r}\right) \boxtimes {}_{B_r}B_{r_{B_r}}
$$

$$
\overline{\widehat{D}(Q)}^{B_r} = \langle y_{l_0} \rangle_{F_2}
$$

4.2. The bimodule ${}^A DA_{B_r}$.

Suppose we are given a D structure over B_r, for example ${}^{B_r}\widehat{D}(T)$, coming from some tangle T via Khovanov bordered theory. Using the theorem above, the authors in [HHHK] reinterpret ${}^{B_r}\widehat{D}(T)$ as a twisted complex, i.e. iterated mapping cone, inside the $Fuk\left(S^2 \backslash 4pt\right)$ with the basic objects L_0 and L_1. Given that we use $Fuk\left(\overline{P}, \{z_1, z_2, z_4, z_4\}\right) = {}^A Mod$ as a model for $Fuk\left(S^2 \backslash 4pt\right)$, we want to translate everything into D structures over A. It turns out that the following DA bimodule, which we denote by ${}^A DA_{B_r}$, does the job for us:



$= {}^{A}DA_{B_r}$

The definition of the immersed curve associated to a tangle is then as follows:

$$\text{tangle } T \xrightarrow[{[BN]}]{[Kh],} {}^{B_r}\widehat{D}(T) \xrightarrow{{}^{A}DA_{B_r}\boxtimes\cdot} {}^{A}DA_{B_r} \boxtimes {}^{B_r}\widehat{D}(T) = {}^{A}N(L_T) \xrightarrow{[HKK],\,[HRW]} \left[\text{immersed curve } L_T \text{ in } \left(\overline{P}, \partial\overline{P} \setminus \{z_1, z_2, z_4, z_4\}\right)\right]$$

Of course, one wants to still be able to prove the pairing result. With that in mind, what properties do we want ${}^{A}DA_{B_r}$ to satisfy? It turns out that the only property we really need is the following homotopy equivalence:

$$M\left(L_Q^a\right)_A \boxtimes {}^{A}DA_{B_r} \simeq \widehat{A}(Q)_{B_r}, \tag{1}$$

which corresponds to the 2nd statement of the above theorem. The left hand-side can be computed using formula 5. above Section 4:

$$M\left(L_Q^a\right)_A = \langle x_{i_1}\rangle^A \boxtimes {}_A A_A = \begin{tikzcd} & x \arrow[r, "\xi_1"] \arrow[dl, "\eta_{23}"'] \arrow[d, "\eta_2"] \arrow[dr, "\xi_{12}"] \arrow[rrd, bend left, "\xi_{123}"] & x \otimes \xi_1 \arrow[d, "\xi_2"] \arrow[dr, "\xi_{23}"] & \\ x \otimes \eta_{23} & x \otimes \eta_2 \arrow[l, "\eta_3"'] & x \otimes \xi_{12} \arrow[r, "\xi_3"] & x \otimes \xi_{123} \end{tikzcd}$$

The right hand-side we know: $$\widehat{A}(Q)_{B_r} = \begin{tikzcd} (u)_{l_0} & (v)_{l_0} \arrow[l, "c_0"'] \arrow[dl, "p"] \\ (r)_{l_1} \arrow[u, "q"] & \end{tikzcd}$$

A computer assisted computation using a package [Kot2] then shows that (1) is true.

4.3. The construction of ${}^{A}DA_{B_r}$.

This subsection can be skipped, as it serves only as an explanation of how we arrived to the bimodule

${}^{A}DA_{B_r}$. Suppose hypothetically that ${}^{B_r}\widehat{D}(T)$ is a twisted complex representing some curve L_T inside $\mathcal{F}_{pw}\left(\overline{P}, \{z_1, z_2, z_4, z_4\}\right)$. Then we would have

$$ {}_{A}\overline{M(L_T)} \simeq \bigoplus_k CF_*(arc_k, L_T) \simeq {}_{A}AA_{B_r} \boxtimes {}^{B_r}\widehat{D}(T)\ , $$

where ${}_{A}AA_{B_r} \stackrel{\text{def}}{=} \bigoplus_{k=0,\dots,5;j=0,1} CF_*(arc_k, L_j)$. Thus we would have

$$ {}^{A}N(L_T) \simeq {}^{A}\overline{bar_r(A)}^{A} \boxtimes {}_{A}\overline{M(L_T)} \simeq {}^{A}\overline{bar_r(A)}^{A} \boxtimes {}_{A}AA_{B_r} \boxtimes {}^{B_r}\widehat{D}(T). $$

The bimodule ${}^{A}DA_{B_r}$ was written down as a box tensor product ${}^{A}\overline{bar_r(A)}^{A} \boxtimes {}_{A}AA_{B_r}$. The bimodule ${}_{A}AA_{B_r}$ was written down with computer assistance [Kot2] in 3 steps:

(1) Computing the dg-algebra $C = \bigoplus_{k=0,\dots,5;j=0,1} Mor\left({}^{A}N(arc_k), {}^{A}N(L_j)\right)$.

(2) Canceling differentials in C via homological perturbation lemma (see [Seidel_book]), thus obtaining an A_∞ algebra $\mathcal{F}_{pw}(arc_k, L_j)$ with $\mu_1 = 0$.

(3) Writing down the bimodule ${}_{A}AA_{B_r} = \bigoplus_{k=0,\dots,5;j=0,1} CF_*(arc_k, L_j)$ based on $\mathcal{F}_{pw}(arc_k, L_j)$.

Remark. On step (2) one can only cancel differentials in summands of type $Mor\left({}^{A}N(L_j), {}^{A}N(L_i)\right)$ and $Mor\left({}^{A}N(arc_k), {}^{A}N(arc_j)\right)$, and then in step (3) simply do the rest of the cancellations on the AA bimodule side. This reduces the complexity of the computations. The resulting bimodule is written below.

The bimodule ${}_{A}AA_{B_r} \stackrel{\text{def}}{=} \bigoplus_{k=0,\dots,5;j=0,1} CF_*(arc_k, L_j)$:

Generators with their left and	AA type actions:	((et2,), x8, (), x3)
right idempotents:	((), x1, (c0,), x3)	((et2,), x9, (), x1)
j0___x7___l0	((), x1, (p01,), x2)	((et23,), x7, (), x3)
i2___x5___l0	((), x10, (q10,), x8)	((et3,), x7, (), x8)
i1___x1___l0	((), x11, (c1,), x13)	((ks1,), x4, (), x2)
i1___x3___l0	((), x11, (q10,), x12)	((ks1,), x5, (), x3)
j2___x11___l1	((), x12, (p01,), x13)	((ks12,), x11, (), x2)
i2___x4___l1	((), x2, (q10,), x3)	((ks12,), x12, (), x3)
j2___x13___l1	((), x4, (c1,), x6)	((ks123,), x9, (), x3)
i0___x0___l0	((), x4, (q10,), x5)	((ks2,), x11, (), x4)
j1___x8___l0	((), x5, (p01,), x6)	((ks2,), x12, (), x5)
j2___x12___l0	((), x9, (c0,), x8)	((ks2,), x13, (), x6)
i2___x6___l1	((), x9, (p01,), x10)	((ks23,), x10, (), x6)
i1___x2___l1	((et1,), x1, (), x0)	((ks23,), x9, (), x5)
j1___x10___l1	((et12,), x9, (), x0)	((ks3,), x10, (), x13)
j1___x9___l0	((et2,), x10, (), x2)	((ks3,), x9, (), x12)
	((r2,), x11, (), x6)	((r0,), x7, (), x0)

5. Assembling the proof.

The proof of Theorem 1 is given by the following sequence of homotopy equivalences:

$$CKh_r(K) \overset{(a)}{\simeq} \widehat{A}(Q)_{B_r} \boxtimes {}^{B_r}\widehat{D}(T) \overset{(b)}{\simeq} \left(M\left(L_Q^a\right)_A \boxtimes {}^{A}DA_{B_r}\right) \boxtimes {}^{B_r}\widehat{D}(T) \overset{(c)}{\simeq}$$

$$\simeq M\left(L_Q^a\right)_A \boxtimes \left({}^{A}DA_{B_r} \boxtimes {}^{B_r}\widehat{D}(T)\right) \overset{(d)}{\simeq} M\left(L_Q^a\right)_A \boxtimes {}^{A}N(L_T) \overset{(e)}{\simeq} CF_*\left(L_Q^a, L_T\right),$$

where

(a) is bordered Khovanov theory explained in Section 2.4.

(b) follows from (1) in Section 4.2.

(c) follows from the fact that box tensor product is associative up to homotopy, see [LOT2, Proposition 2.3.15].

(d) is a definition of D structure ${}^{A}N(L_T)$, and immersed curve L_T, see Section 4.2.

(e) is formula 3. above Section 4. It can be proved analogously to the pairing theorem in [Kot1].

Remark. Differential in the box tensor product can be also understood as a differential in the morphism space in the category of twisted complexes. Namely, if one takes [Seidel_book, Formula 3.20], and takes a special case of it where $d = 1$, and X_1 is a twisted complex with $\delta_{X_1} = 0$, then the resulting formula $\mu^1_{TwA}(a_1) = \sum \mu^k_A(a_1, \delta_{X_0}, \ldots, \delta_{X_0})$ is identical to the formula of the differential in the box tensor product.

Also, because we have $M(L_0)_A \boxtimes {}^{A}DA_{B_r} \simeq (F_2)^2 \otimes \widehat{A}(Q)_{B_r}$ (this statement is analogous to formula (1) in Section 4.2), the full unreduced $CKh(K)$ is:

$$CKh(K) \simeq (F_2)^2 \otimes Kh_r(K) \simeq (F_2)^2 \otimes \left[\widehat{A}(Q)_{B_r} \boxtimes {}^{B_r}\widehat{D}(T)\right] \simeq \left(M(L_0)_A \boxtimes {}^{A}DA_{B_r}\right) \boxtimes {}^{B_r}\widehat{D}(T) \simeq$$

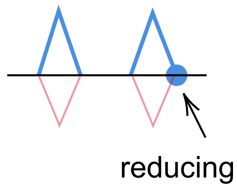
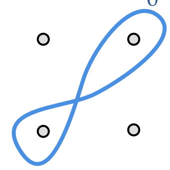
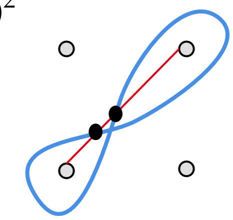
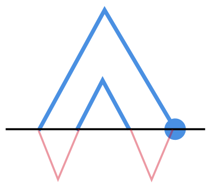
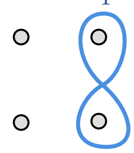
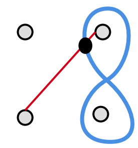
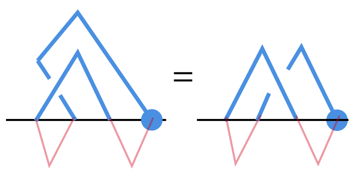
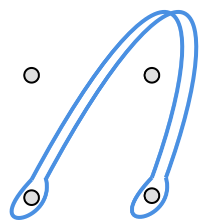
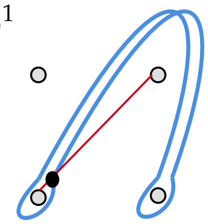
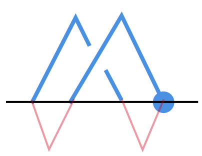
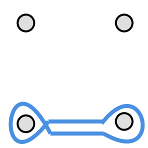
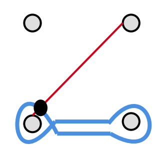
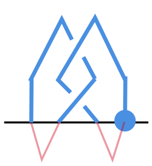
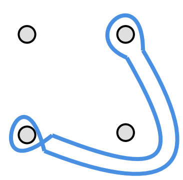
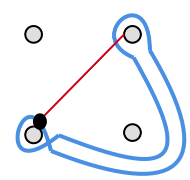
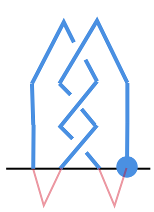
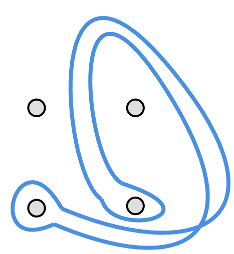
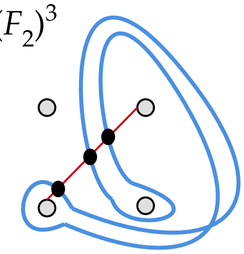
$$\simeq M(L_0)_A \boxtimes \left({}^{A}DA_{B_r} \boxtimes {}^{B_r}\widehat{D}(T)\right) \simeq M(L_0)_A \boxtimes {}^{A}N(L_T) \simeq CF_*(L_Q, L_T),$$

where the first homotopy equivalence is true because we are working over F_2, and the last one follows from $L_0 = L_Q$.

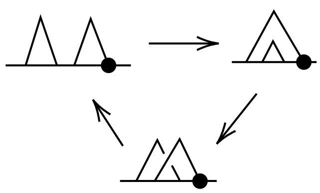
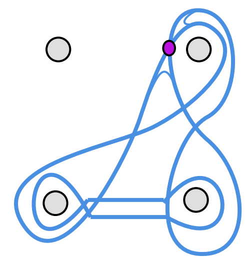
6. Examples.

The following are examples of tangles, their associated D structures, the corresponding curves, and the ways those curves give Khovanov homology when paired with the red arc.

Tangle T	Left D structure ${}^{B_r}\widehat{D}(T)$	Curve $L_{Kh}(T)$, obtained from D structure ${}^{A}DA_{B_r} \boxtimes {}^{B_r}\widehat{D}(T)$	Khovanov homology of the tangle T glued with $\overline{\cup\ \cup}$

	l_0	L_0 	$Kh_r(OO) = (F_2)^2$ 
	l_1	L_1 	$Kh_r(O) = (F_2)^1$ 
	$l_1 \xrightarrow{q_{10}} l_0$		$Kh_r(O) = (F_2)^1$ 
	$l_0 \xrightarrow{p_{01}} l_1$		$Kh_r(O) = (F_2)^1$ 
	$l_0 \xrightarrow{p_{01}} l_1 \xrightarrow{c_1} l_1$		$Kh_r(Hopf) = (F_2)^1$ 
	$\begin{array}{ccc} l_0 & & l_2 \\ \downarrow p_{01} & & \uparrow c_1 \\ l_1 & \xrightarrow{c_1} & l_1 \end{array}$		$Kh_r(RHT) = (F_2)^3$ 

Here are 3 manifestations of exact triangle for Khovanov invariants of tangles. The first one is a picture of three tangles, where the smoothing of the middle one was resolved in two ways (left and right). The philosophy of Khovanov theory is that any Khovanov invariant associated to tangles should form an exact triangle for such tangles. And indeed, the D structures in the second column form an exact triangle, or equivalently: the mapping cone of the horizontal arrow is homotopy equivalent to the middle D structure. In our setting the corresponding fact is that the curves associated to those three tangles form an exact triangle. This follows from the fact that resolving the purple point on the picture gives the third curve, see [Abo, Lemma 5.4].

Exact triangle for Khovanov homology	Exact triangle for D structures	Exact triangle for curves
	$\begin{tikzcd} l_0 \arrow[rr] & & l_1 \arrow[dl] \\ & l_0 \xrightarrow{p_{01}} l_1 \arrow[ul] & \end{tikzcd}$	

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