

INTRODUCTION TO HOMOLOGICAL INVARIANTS IN LOW-DIMENSIONAL TOPOLOGY

ARTEM KOTELSKIY

There is a common main idea in various constructions of low-dimensional topological invariants. One takes a topological object, associates a geometric construction to it, which involves some choices, and then gets an algebraic object out of it. The choices one makes in the geometric construction translate into chain homotopy in the algebraic world. Thus the chain homotopy type of the algebraic object becomes a topological invariant (in case of chain complexes their chain homotopy type is captured by homology).

Those auxiliary geometric constructions usually come from either symplectic geometry, or gauge theory. We focus on those which come from symplectic geometry.

1. SYMPLECTIC GEOMETRY

Here I will outline important constructions in symplectic geometry, which are used in low-dimensional topology.

Omitting a large amount of details and certain technical conditions [7], [3], Lagrangian Floer homology $HF_*(L_1, L_2)$ is a homological invariant of two Lagrangians inside a symplectic manifold $L_1, L_2 \rightarrow M^{2n}$, which is invariant under Hamiltonian isotopies of each Lagrangian. The underlying chain complex is generated by intersections: $CF_*(L_1, L_2) = \langle L_1 \cap L_2 \rangle_{\mathbb{F}_2}$. After picking an almost complex structure, the differential $d : CF_*(L_1, L_2) \rightarrow CF_*(L_1, L_2)$ comes from counting rigid pseudo-holomorphic discs between the intersection points, with Lagrangian boundary conditions on L_1 and L_2 :

 L_1 contributes 1 into coefficient c_{xy} in $d(x) = \sum_y c_{xy} \cdot y$.

Another invariant, which is relevant to my research, is Fukaya category of a symplectic manifold $\mathcal{F}(M^{2n})$ [9], [29], [3]. It is a unified structure, which captures how all Lagrangians intersect with each other. The objects in this category are Lagrangians L_i , and morphism spaces are Lagrangian Floer complexes $CF_*(L_i, L_j)$. The product is given by counting pseudo-holomorphic triangles: $\Delta : CF_*(L_1, L_2) \otimes CF_*(L_2, L_3) \rightarrow CF_*(L_1, L_3)$. It is not associative, and only is associative up to homotopy, given by counting pseudo-holomorphic rectangles. Thus $\mathcal{F}(M^{2n})$ is not a regular category, but rather is an A_∞ category (higher operations are given by counting pseudo-holomorphic polygons).

In the case $2n = 2$, where Lagrangians are curves on the surface, counting rigid pseudo-holomorphic discs is equivalent to counting immersed discs with convex angles at intersections. The Fukaya category in this case is similar to a curve complex, only it captures more information: minimal intersection numbers between the curves, and also all the immersed polygons with boundary on multiple curves.

2. HEEGAARD FLOER THEORY

One of the main 3-manifold techniques in the field is called Heegaard Floer homology, developed by Ozsváth and Szabó in [19], [20]. The construction works as follows: first one picks a Heegaard splitting of a 3-manifold along a surface: $Y^3 = U_1 \cup_{\Sigma_g} U_2$. Second, one constructs a symplectic manifold: $Sym^g(\Sigma_g) = (\Sigma_g)^g / S_g$. Then, having picked a Heegaard diagram with a basepoint for the splitting, one obtains two Lagrangian tori: $\mathbb{T}(U_1), \mathbb{T}(U_2) \subset Sym^g(\Sigma_g)$. At last, Heegaard Floer homology of Y^3 is Lagrangian Floer homology of these two tori $HF_*(\mathbb{T}(U_1), \mathbb{T}(U_2))$. There are different versions of Heegaard

Floer homology depending on how one incorporates the basepoint in the diagram: they are denoted by $\widehat{HF}(Y)$, $HF^-(Y)$, $HF^+(Y)$, $HF^\infty(Y)$. Among various applications, this theory can be used to effectively study surgery problems (for example, one can reprove property P [21]), homology cobordism group [22], contact structures [23]. One can also obtain such foundational 4-dimensional results as Donaldson's diagonalizability theorem and symplectic Thom conjecture for $\mathbb{C}P^2$ [22].

An analogous theory for knots, discovered independently in [25] and [27], is called knot Floer homology $\widehat{HFK}(K)$. It categorifies Alexander polynomial in the same way Khovanov homology categorifies Jones polynomial. $\widehat{HFK}(K)$ also detects genus of the knot [21], detects if the knot is fibered [10], [11], and provides lower bounds for 4-dimensional genus of the knot [26]. Another feature of knot Floer homology is that it contains essential information for studying Heegaard Floer homology $\widehat{HF}(S_{p/q}(K))$ of surgeries on a knot [24], [17].

One of the advantages of Heegaard Floer homology is that it can be computed for a large number of 3-manifolds (with small Heegaard diagrams) directly from its definition and properties. However, the construction of Heegaard Floer homology involves a choice of almost complex structure and subsequent counting of pseudo-holomorphic discs, and therefore a difficult and important question is how computable these invariants are in general. The first general algorithm to compute $\widehat{HF}(Y)$ was discovered by Sarkar and Wang in [28]. An effective algorithm to compute $\widehat{HF}(Y)$ was developed in [16]. Other versions of Heegaard Floer homology are theoretically algorithmically computable [18], but the methods are not practical.

3. GAUGE THEORY

There exists a parallel 3-manifold theory called monopole Floer homology. It is constructed using gauge theory by Kronheimer and Mrowka in [12], and is isomorphic to Heegaard Floer theory [13], [4]. This means that Heegaard Floer homology, at its core, allows one to solve in a geometric way Seiberg-Witten differential equations on a 3-manifold.

There is another pair of parallel Floer homology theories for 3-manifolds. The one coming from gauge theory is instanton homology $I(Y^3)$, developed in [8] by Floer using anti-self-dual Yang-Mills equations. The following is an original formulation of Atiyah-Floer conjecture, which describes what should be the parallel symplectic geometric theory. Having a Heegaard splitting $Y^3 = U_1 \cup_{\Sigma_g} U_2$, associate to U_i and Σ_g their $SU(2)$ representation varieties $R(U_i)$, $R(\Sigma_g)$. One then has maps $R(U_i) \rightarrow R(\Sigma_g)$. It was conjectured in [1] that instanton Floer homology $I(Y^3)$ should be equal to Lagrangian Floer homology $HF_*(R(U_1), R(U_2))$. Spaces $R(U_1)$, $R(U_2)$, $R(\Sigma_g)$ are singular, and thus symplectic instanton Floer homology $HF_*(R(U_1), R(U_2))$ was not possible to define at that moment. The symplectic side of the isomorphism, as well as the proof of Atiyah-Floer conjecture, are still under development [5], [6].

4. TQFT STRUCTURE

Importantly, Heegaard Floer homology fits into 3+1 TQFT framework. It means that for 3-manifolds it assigns vector spaces, and for 4-dimensional cobordisms it assigns maps between the vector spaces, which satisfy the composition law. Moreover, it was extended down to 2+1 TQFT by bordered Heegaard Floer homology theory, which was developed by Lipshitz, Ozsváth and Thurston in [14], [15]. This theory assigns a dg -algebra \mathcal{A} to a

parameterized surface, and an A_∞ module $\widehat{CFA}(Y_1)_\mathcal{A}$ or a D-structure ${}^1\widehat{ACFD}(Y_1)$ to a 3-manifold Y_1 with parameterized boundary ². A gluing theorem, needed for TQFT structure, states that if a $Y^3 = Y_1 \cup_{\Sigma_g} Y_2$, then $\widehat{HF}(Y^3) \cong \widehat{CFA}(Y_1)_\mathcal{A} \boxtimes^\mathcal{A} \widehat{CFD}(Y_2)$, where \boxtimes is an appropriate version of a tensor product. An alternative approach is to associate to a surface Σ_g a Fukaya category of $Sym^g(\Sigma_g \setminus 1pt)$, and, to a 3-manifold with $\partial Y = \Sigma_g$, associate an object in that category. An equivalence of these theories was addressed in [2].

REFERENCES

- [1] M. Atiyah. *New invariants of 3- and 4-dimensional manifolds*. The mathematical heritage of Hermann Weyl (Durham, NC, 1987), Proc. Sympos. Pure Math., vol. 48, Amer. Math. Soc., Providence, RI, 1988, pp. 285–299. MR 974342.
- [2] D. Auroux. *Fukaya categories of symmetric products and bordered Heegaard-Floer homology*. J. Gökova Geom. Topol. 4, 2010, 1–54. arXiv:1001.4323
- [3] D. Auroux. *A beginner’s introduction to Fukaya categories*. In Contact and symplectic topology, vol. 26 of Bolyai Soc. Math. Stud., Budapest, 2014, pp. 85–136. arXiv:1301.7056
- [4] V. Colin, P. Ghiggini, K. Honda. *The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions I*. Preprint, 2012. arXiv:1208.1074
- [5] A. Daemi, K. Fukaya. *Atiyah-Floer Conjecture: a Formulation, a Strategy to Prove and Generalizations*. Preprint, 2017. arXiv:1707.03924
- [6] S. Dostoglou, D. A. Salamon. *Self-dual instantons and holomorphic curves*. Ann. of Math. (2) 139, 1994, 581–640.
- [7] A. Floer. *Morse theory for Lagrangian intersections*. J. Differential Geom. 28 (1988), 513–547.
- [8] A. Floer. *An instanton-invariant for 3-manifolds*. Comm. Math. Phys. 118 (1988), no. 2, 215–240.
- [9] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction I and II*. AMS/IP Studies in Advanced Math., vol. 46, Amer. Math. Soc. and International Press, 2009.
- [10] P. Ghiggini. *Knot Floer homology detects genus-one fibred knots*. Amer. J. Math. 130 (2008), no. 5, 1151–1169. arXiv:0603445.
- [11] Y. Ni. *Knot Floer homology detects fibred knots*. Invent. Math. 170 (2007), no. 3, 577–608. arXiv:0607156.
- [12] P.B. Kronheimer, T.S. Mrowka. *Monopoles and three-manifolds*. Cambridge, University Press, 2008.
- [13] Ç. Kutluhan, Y.-J. Lee, C. Taubes. *HF = HM I: Heegaard Floer homology and Seiberg-Witten Floer homology*. Preprint, 2008. arXiv:1007.1979
- [14] R. Lipshitz, P. Ozsváth, D. Thurston. *Bordered Heegaard Floer homology: invariance and pairing*. Preprint, 2008. arXiv:0810.0687
- [15] R. Lipshitz, P. Ozsváth, D. Thurston. *Bimodules in bordered Heegaard Floer homology*. Geometry & Topology 19, 2015, 525–724. arXiv:1003.0598
- [16] R. Lipshitz, P. Ozsváth, D. Thurston. *Computing \widehat{HF} by factoring mapping classes*. Geom. Topol., 18 (5), 2547–2681, 2014. arXiv:1010.2550
- [17] C. Manolescu, P. Ozsváth. *Heegaard Floer homology and integer surgeries on links*. Preprint, 2010. 1011.1317
- [18] C. Manolescu, P. Ozsváth, D. Thurston. *Grid diagrams and Heegaard Floer invariants*. Preprint, 2009. 0910.0078
- [19] P. Ozsváth, Z. Szabó. *Holomorphic disks and topological invariants for closed three-manifolds*. Ann. of Math. 159 (2004), no. 3, 1027–1158. arXiv:0101206
- [20] P. Ozsváth, Z. Szabó. *Holomorphic disks and 3-manifold invariants: properties and applications*. Ann. of Math. 159 (2004), no. 3, 1159–1245. arXiv:0105202
- [21] P. Ozsváth, Z. Szabó. *Holomorphic disks and genus bounds*. Geometry & Topology, v.8 (2004), 311–334. arXiv:0311496
- [22] P. Ozsváth, Z. Szabó. *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*. Adv. Mat. 173 (2003), no. 2, 179–261. arXiv:0110170
- [23] P. Ozsváth, Z. Szabó. *Heegaard Floer homology and contact structures*. Duke Math. J. 129 (2005), 39–61. arXiv:0210127
- [24] P. Ozsváth, Z. Szabó. *Knot Floer homology and rational surgeries*. Algebraic & Geometric Topology 11 (2011) 1–68. arXiv:0504404
- [25] P. Ozsváth, Z. Szabó. *Holomorphic disks and knot invariants*. Adv. Math. 186 (2004), no. 1, 58–116. arXiv:0209056

¹ D-structure is essentially the same object as twisted complex, see [15, Remark 2.2.37].

² Appropriate objects called bimodules are assigned to 3-dimensional cobordisms between the surfaces, and the corresponding gluing theorems hold true.

- [26] P. Ozsváth, Z. Szabó. *Knot Floer homology and the four-ball genus*. Geom. Topol. 7 (2003), 615–639. arXiv:0301149
- [27] J. Rasmussen. *Floer homology and knot complements*. PhD Thesis, Harvard University, 2003. arXiv:0306378
- [28] S. Sarkar and J. Wang. *An algorithm for computing some Heegaard Floer homologies*. Ann. of Math. 171 (2010), no. 2, 1213–1236. arXiv:0607777
- [29] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lect. in Adv. Math., European Math. Soc., Zürich, 2008.

PRINCETON UNIVERSITY

E-mail address: `artemk@math.princeton.edu`