Definition of eigenvalues and eigenvectors of a matrix

> Let $\mathbf{A}$ be any square matrix. A non-zero vector $\mathbf{v}$ is an **eigenvector** of $\mathbf{A}$ if
>
> $$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$
>
> for some number λ, called the corresponding **eigenvalue**.

NOTE: The German word "*eigen*" roughly translates as "own" or "belonging to". Eigenvalues and eigenvectors correspond to each other (are paired) for any particular matrix $\mathbf{A}$.

The solved examples below give some insight into what these concepts mean. First, a summary of what we're going to do:

How to find the eigenvalues and eigenvectors of a 2x2 matrix

1. Set up the **characteristic equation**, using $|\mathbf{A} - \lambda\mathbf{I}| = 0$
2. **Solve** the characteristic equation, giving us the **eigenvalues** (2 eigenvalues for a 2x2 system)
3. **Substitute** the eigenvalues into the two equations given by $\mathbf{A} - \lambda\mathbf{I}$
4. Choose a convenient value for x_1, then find x_2
5. The resulting values form the corresponding **eigenvectors** of **A** (2 eigenvectors for a 2x2 system)

There is no single **eigenvector formula** as such - it's more of a set of steps that we need to go through to find the eigenvalues and eigenvectors.

Example 1

We start with a system of two equations, as follows:

$$y_1 = -5x_1 + 2x_2$$

$$y_2 = -9x_1 + 6x_2$$

We can write those equations in matrix form as:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In general we can write the above matrices as:

$$\mathbf{y} = \mathbf{Av}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}, \text{ and}$$

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Step 1. Set up the characteristic equation, using $|\mathbf{A} - \lambda\mathbf{I}| = 0$

Our task is to find the **eigenvalues** λ, and **eigenvectors** $\mathbf{v}$, such that:

$$\mathbf{y} = \lambda\mathbf{v}$$

We are looking for **scalar values** λ (numbers, not matrices) that can replace the matrix $\mathbf{A}$ in the expression $\mathbf{y} = \mathbf{Av}$.

That is, we want to find λ such that :

$$-5x_1 + 2x_2 = \lambda x_1$$

$$-9x_1 + 6x_2 = \lambda x_2$$

Rearranging gives:

$$\begin{aligned} -(5-\lambda)x_1 + 2x_2 &= 0 \\ -9x_1 + (6-\lambda)x_2 &= 0 \end{aligned} \qquad (1)$$

This can be written using matrix notation with the identity matrix $\mathbf{I}$ as:

$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$, that is:

$$\left(\mathbf{A} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\mathbf{v} = 0$$

$$\left(\mathbf{A} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right)\mathbf{v} = 0$$

Step 2. Solve the characteristic equation, giving us the eigenvalues (2 eigenvalues for a 2x2 system)

In this example, the coefficient determinant from equations (1) is:

$$\begin{aligned}|\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix}-5-\lambda & 2\\-9 & 6-\lambda\end{vmatrix}\\ &= (-5-\lambda)(6-\lambda) - (-9)(2)\\ &= -30 - \lambda + \lambda^2 + 18\\ &= \lambda^2 - \lambda - 12\\ &= (\lambda+3)(\lambda-4)\end{aligned}$$

Now this equals 0 when:

$$(\lambda+3)(\lambda-4) = 0$$

That is, when:

$$\lambda = -3 \quad \text{or} \quad 4.$$

These two values are the **eigenvalues** for this particular matrix $\mathbf{A}$.

Step 3. Substitute the eigenvalues into the two equations given by $\mathrm{A} - \lambda\mathrm{I}$

Case 1: $\lambda_1 = -3$

When $\lambda = \lambda_1 = -3$, equations (1) become:

$$[-5-(-3)]x_1 + 2x_2 = 0$$

$$-9x_1 + [6-(-3)]x_2 = 0$$

That is:

$$\begin{aligned} -2x_1 + 2x_2 &= 0 \\ -9x_1 + 9x_2 &= 0 \end{aligned} \qquad (2)$$

Dividing the first line of Equations (2) by -2 and the second line by -9 (not really necessary, but helps us see what is happening) gives us the identical equations:

$$x_1 - x_2 = 0$$

$$x_1 - x_2 = 0$$

Step 4. Choose a convenient value for x_1, then find x_2

There are infinite solutions of course, where $x_1 = x_2$. We choose a convenient value for x_1 of, say 1, giving $x_2 = 1$.

Step 5. The resulting values form the corresponding eigenvectors of A (2 eigenvectors for a 2x2 system)

So the corresponding eigenvector is:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

NOTE: We could have easily chosen $x_1 = 3$, $x_2 = 3$, or for that matter, $x_1 = -100$, $x_2 = -100$. These values will still "work" in the matrix equation.

In general, we could have written our answer as "$x_1 = t$, $x_2 = t$, for any value t", however it's usually more meaningful to choose a convenient starting value (usually for x_1), and then derive the resulting remaining value(s).

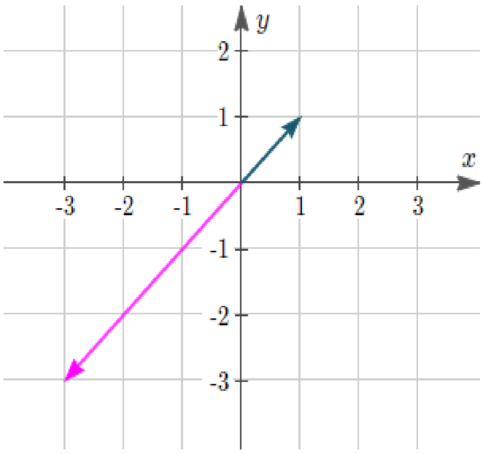
Is it correct?

We can check by substituting:

$$\begin{aligned}\mathbf{A}\mathbf{v}_1 &= \begin{bmatrix}-5 & 2\\-9 & 6\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\\ &= \begin{bmatrix}-3\\-3\end{bmatrix}\\ &= -3\begin{bmatrix}1\\1\end{bmatrix}\\ &= \lambda_1\mathbf{v}_1\end{aligned}$$

We have found an **eigenvalue** $\lambda_1 = -3$ and an **eigenvector** $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ for the matrix $\mathbf{A} = \begin{bmatrix}-5 & 2\\-9 & 6\end{bmatrix}$ such that $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$.

Graphically, we can see that matrix $\mathbf{A} = \begin{bmatrix}-5 & 2\\-9 & 6\end{bmatrix}$ acting on vector $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ is equivalent to multiplying $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ by the scalar $\lambda_1 = -3$. The result is applying a scale of -3.



Graph indicating the transform $\mathbf{y}_1 = \mathbf{A}\mathbf{v}_1$

Case 2: $\lambda_2 = 4$

When $\lambda = \lambda_2 = 4$, equations (1) become:

$$(-5-(4))x_1 + 2x_2 = 0$$

$$-9x_1 + (6-(4))x_2 = 0$$

That is:

$$-9x_1 + 2x_2 = 0$$

$$-9x_1 + 2x_2 = 0$$

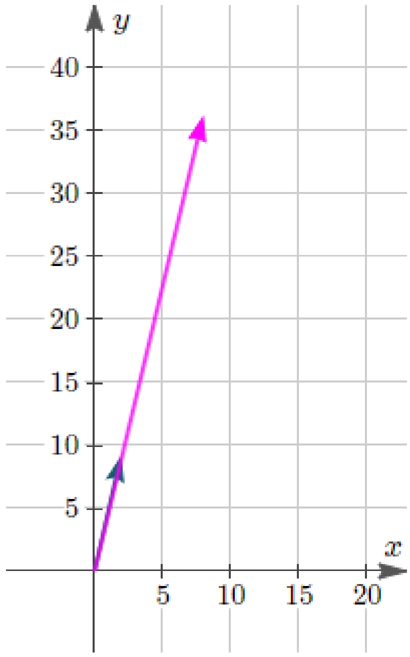
We choose a convenient value for x_1 of 2, giving $x_2 = 9$. So the corresponding eigenvector is:

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

We could check this by multiplying and concluding $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 9 \end{bmatrix}$, that is $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.

We have found an **eigenvalue** $\lambda_2 = 4$ and an **eigenvector** $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$ for the matrix $\mathbf{A} = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ such that $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.

Graphically, we can see that matrix $\mathbf{A} = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ acting on vector $\mathbf{v_2} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$ is equivalent to multiplying $\mathbf{v_2} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$ by the scalar $\lambda_2 = 4$. The result is applying a scale of 4.



Graph indicating the transform $\mathbf{y_2} = \mathbf{A}\mathbf{v_2} = \boldsymbol{\lambda_2}\mathbf{x_2}$