Multivariable Calculus

★ Variables, constants and Constant

$$F = ma + dv^2$$

Force F generated by a cars engine to its mass 'm', acceleration 'a', aerodynamic drag d and velocity v

→ If you are driving a car then you can change the speed and acceleration by pressing the accelerator pedal to adjust the force.

→ But here the mass and drag are fixed features just as the cars design.

→ Here force F is an independent variable which we can control it directly.

→ Speed and acceleration depends on the variables as they are consequences of your applied force.

→ Mass and drag coefficients are constants.

→ However, if you are a car designer (looking to design each engine size)

In this case your speed and acceleration are constant whereas 'mass' and drag become variables. (which we can adjust by redesigning the car).

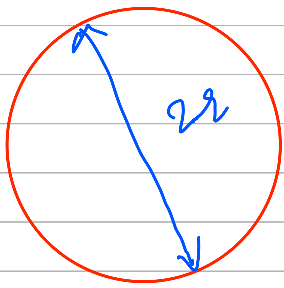
→ We refer these variable design constants as parameters.

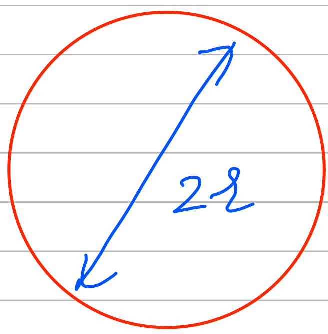
Example :-

Manufacturing of metal can

⇒ we need to understand the relationship between various key parameters

⇒ For approximation we can consider cans empty mass 'M' by breaking the area down into pieces.





thickness density.

$$m = \underbrace{2\pi r^2}_{\text{area of two circles}} t \rho + \underbrace{2\pi r h}_{\text{area of cylinder}} t \rho$$

what are the variables. ?

$$\frac{\partial m}{\partial h} = 2\pi r t \rho$$

$$\frac{\partial m}{\partial r} = 4\pi r t \rho + 2\pi h t \rho$$

$$\frac{\partial m}{\partial t} = 2\pi r^2 \rho + 2\pi r h \rho$$

$$\frac{\partial m}{\partial \rho} = 2\pi r^2 t + 2\pi r h t$$

→ This is what is called partial differentiation.

→ Differentiate w.r.t. anything.

$$f(x,y,z) = \sin(x)\, e^{yz^2}$$

$$\frac{\partial f}{\partial x} = \cos(x)\, e^{yz^2}$$

$$\frac{\partial f}{\partial y} = \sin(x)\, e^{yz^2}\, z^2$$

$$\frac{\partial f}{\partial z} = \sin(x)\, e^{yz^2}\, 2yz$$

Total derivative.

Let for the above function.

$$x = t - 1 \;; \quad y = t^2 \,, \quad z = \frac{1}{t}$$

we can substitute the value of t to the function and the get the derivative as.

$$f(t) = \sin(t-1)\, e^{t^2\left(\frac{1}{t}\right)^2}$$

$$f(t) = \sin(t-1)\, e$$

$$\frac{d f(t)}{dt} = \cos(t-1)\, e \,.$$

we use chain rule to calculate this to reduce the complexity.

⇒ derivative of the function w.r.t 't' is the sum of the chains of the other three variables as:

$$\frac{d f(x,y,z)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

we know these values.

Now lets find the derivative of each variable w.r.t. 't'

$x = t-1$ $\quad$ $y = t^2$ $\quad$ $z = \frac{1}{t}$

$\frac{dx}{dt} = 1$ $\quad$ $\frac{dy}{dt} = 2t$ $\quad$ $\frac{dz}{dt} = \frac{-1}{t^2}$

$\Rightarrow -t^{-2}$

$$\frac{df(x,y,z)}{dt} = \cos(x)\, e^{yz^2} \times 1$$
$$+ z^2 \sin(x)\, e^{yz^2} \times 2t$$
$$+ 2yz \sin(x)\, e^{yz^2} \times (-t^{-2})$$

put the values of x, y, z in terms of 't' and evaluate

$$\frac{df(x,y,z)}{dt} = \cos(t-1)\, e + 2t^{-1} \sin(t-1)\, e$$
$$- 2t^{-1} \sin(t-1)\, e$$

$$= \cos(t-1)\, e.$$

The Jacobian.

→ Jacobian of single function with many variables.

$$f(x_1, x_2, x_3, \ldots\ldots)$$

$$J = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \ldots\ldots \right]$$

This is the row vector

example :-

$$f(x, y, z) = x^2 y + 3z$$

To build jacobian we find each of the partial derivative of the function one by one.

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial z} = 3.$$

$$\frac{\partial f}{\partial y} = x^2$$

$$J = [2xy \quad x^2 \quad 3]$$

when we have specific x, y, z co-ordinates, we get the vector pointing in the direction of steepest slope of this function.

at the point $[0 \quad 0 \quad 0]$

$$J[0 \quad 0 \quad 0] = [0 \quad 0 \quad 3]$$

It is the vector of length 3 pointing directly in the z direction.

-> we can calculate the Jacobians for 100 of dimention.

Let consider the function.

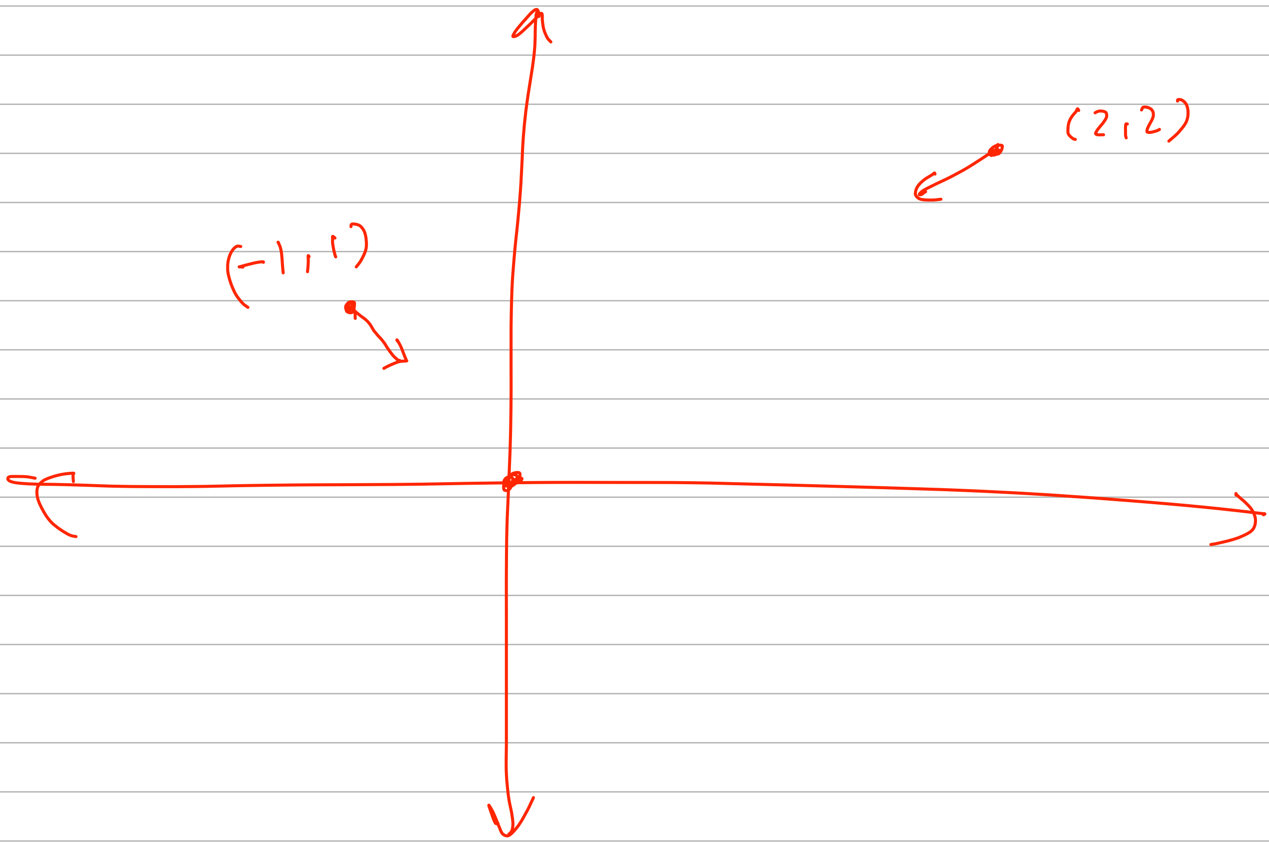
$$f(x, y) = e^{-(x^2+y^2)}$$

$$J = \left[-2x e^{-(x^2+y^2)} , -2y e^{-(x^2+y^2)} \right]$$

$J(-1, 1) =$

$J(2, 2) =$

$J(0, 0) =$



$\underline{J(0,0)}$ function is flat at this point.

This suggest that

→ a max

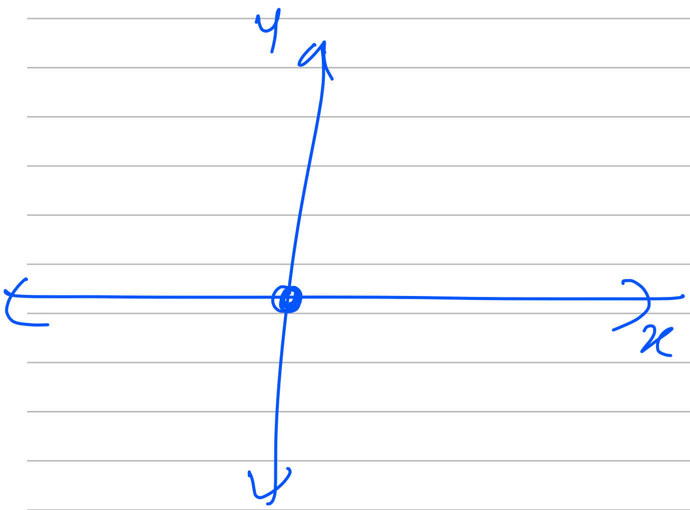
→ a min

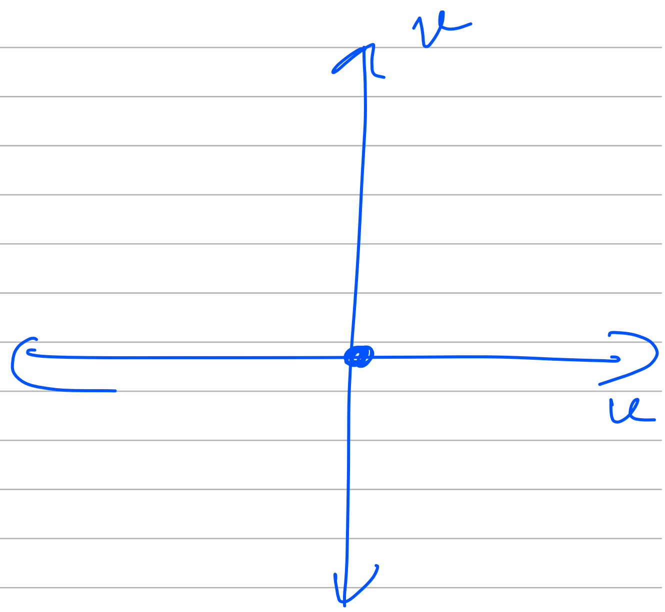
→ saddle.

i.e. origin must be the max. of this function.

Now lets consider the following ex.

$$u(x,y) = x - 2y$$
$$v(x,y) = 3y - 2x$$





Each point in (x,y) has corresponding location in (u,v)

$$J_u = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$$

$$J_v = \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Example.

$u(x,y) = x - 2y$

$v(x,y) = 3y - 2x$

$$J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

u and v are linear function of x and y.

if we want to apply the x, y vector $[2, 3]$

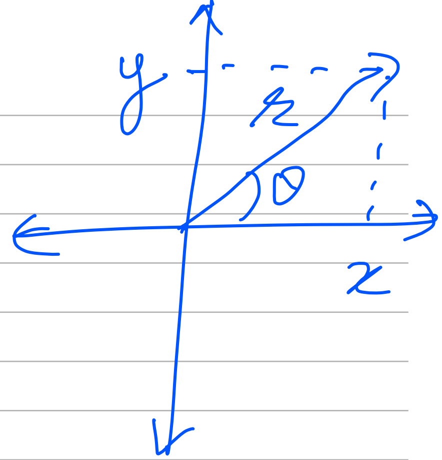
Then:

$$\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

Solve :-

$$x(r, \theta) = r \cos(\theta)$$

$$y(r, \theta) = r \sin(\theta)$$



Hessian:

$$f(x,y,z) = x^2 y z.$$