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③ Newton-Raphson method

Q) $x^3 - 4x - 9$

Sol^n:- $f(0) = (0)^3 - 4(0) - 9 = -9$
$f(1) = (1)^3 - 4(1) - 9 = -12$
$f(2) = (2)^3 - 4(2) - 9 = -9$ -ve
$f(3) = (3)^3 - 4(3) - 9 = 6$ +ve
Stop

Interval:- $[2, 3]$
$a = 2$, $b = 3$

$f(a) = -9 \qquad f(b) = 6$

Initial approximation:-
$x_0 = 3$

Find x_1:- $\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$

$x_0 = 3$
$f(x_0) = x^3 - 4x - 9$
$f'(x_0) = 3x^2 - 4$

$$= x_0 - \frac{(x^3 - 4x - 9)}{3x^2 - 4}$$

$$x_1 = 3 - \frac{(3^3 - 4\times 3 - 9)}{3(3)^2 - 4} = 2.739$$

$$x_2 = 2.739 - \left[\frac{(2.739)^3 - 4(2.739) - 9}{3(2.739)^2 - 4}\right]$$

$$= 2.706$$

$$x_3 = 2.706 - \frac{(2.706)^3 - 4(2.706) - 9}{3(2.706)^2 - 4}$$

$= 2.706$

$f(x_2) = (2.693)^3 - \cancel{8}4(2.693) - 9$
$= -0.2416$ (-ve)

$\because$ -ve, $a = 2.693$

$x_3 = \dfrac{(2.693)6 - 3(-0.2416)}{6 + 0.2416} = 2.704$

$f(x_3) = (2.704)^3 - 4(2.704) - 4 = -0.045$ (-ve)

$\because$ -ve, $a = -0.045$

$x_4 = \dfrac{(2.204)6 - 3(-0.045)}{6 + 0.045} = \underline{2.706}$

$f(x_4) = (2.706)^3 - 4(2.706) - 4$
$= -0.009 \approx 0$

Hence:- $x = 2.706$.

(2) Regula-Falsi method

Q> Find root of $x^3 - 4x - 9 = 0$

$f(x) = x^3 - 4x - 9$
$f(0) = 0^3 - 4.0 - 9 = -9$
$f(1) = 1^3 - 4(1) - 9 = -12$
$f(2) = 2^3 - 4(2) - 9 = -9$ (-ve)
$f(3) = 3^3 - 4(3) - 9 = 6$ (+ve) Stop

$\therefore$ Interval $[2, 3]$ → $a = 2$, $b = 3$

$f(a) = -9$ $\quad$ $f(b) = 6$

formula:- $\boxed{x_i = \dfrac{af(b) - bf(a)}{f(b) - f(a)}}$

$x_1 = \dfrac{2(6) - 3(-9)}{6 - (-9)} = \dfrac{12 + 27}{15} = \dfrac{39}{15}$

$= 2.6$

$f(x_1) = (2.6)^3 - 4(2.6) - 9 = -1.824$
(-ve)

-ve value so, $a = 2.6$, $f(a) = -1.824$
$b = 3$ (remain same)

$x_2 = \dfrac{2.6(6) - [3(-1.824)]}{6 - (-1.824)} = 2.693$

MOD - V

Numerical solution of Algebraic Equation

1) Bisection method.
2) Regula - Falsi method
3) Newton - Raphson method.

Bisection - method :

Find a real root of $x^3 - x = 1$ by using bisection method

Sol:

$$x^3 - x = 1$$

or, $x^3 - x - 1 = 0$

$f(0) = 0^3 - 0 - 1 = -1$

$f(1) = 1^3 - 1 - 1 = -1 \rightarrow f(a)$

$f(2) = 2^3 - 2 - 1 = 5 \rightarrow f(b)$

$\therefore a = 1$ & $b = 2$

$-ve$

$+ve$

concept

i) $f(x) = -ve$
$x = a$ & $f(x) = f(a)$

ii) $f(x) = +ve$
$x = b$ & $f(x) = f(b)$

$(1.5)^3 - (1.5) - 1$

(1.25)

a	b	f(x)	f(b)	$x = \frac{a+b}{2}$	f(x)
				1.5	0.875
1	2	-1	5	1.25	~~[illegible]~~ -0.2968
1	1.5	-1	0.875		
1.25	1.5	-0.2968	0.875	1.375	0.2246
1.25	1.375	-0.2968	0.2246	1.3125	-0.0515
1.3125	1.375	-0.0575	0.2246	1.3437	0.0823
				1.3246	

4th Iteration

$$x = \frac{19 - (1.000) + 5(1.000)}{25} = 1$$

$$y = \frac{-18 - (1) - (1.000)}{20} = -1$$

$$z = \frac{7 - 3(1) - 4(-1)}{8} = 1$$

~~Where Values~~

From 3^{rd} & 4^{th} Iteration, the value of x, y, z are equal

So,

$x = 1$

$y = -1$

$z = 1$ Ans.

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Initially :- $x=0,\ y=0,\ z=0$

1st Iteration :- $x = \frac{19-0+5\times0}{25} = \frac{19}{25} = 0.76$

$y = \frac{(-18-0.76-0)}{20} = \frac{-18-(0.76)}{20}$

$= -0.938$

$z = \frac{7-3(0.76)-4(-0.938)}{8}$

$= 1.059.$

2nd Iteration :-

$x = \frac{19-(-0.938)+5(1.059)}{25} = 1.009$

$y = \frac{-18-(1.009)-(1.059)}{20} = -1.003$

$z = \frac{7-3(1.009)-4(-1.003)}{8} = 0.998$

3rd Iteration :-

$x = \frac{19-(-1.003)+5(0.998)}{25} = 1.000$

$y = \frac{-18-(1.000)-(0.998)}{20} = -1.000$

$z = \frac{7-3(1.000)-4(-1.000)}{8} = 1.000$

$Ux = y$

$$\begin{bmatrix} 2 & -6 & 8 \\ 0 & 19 & -23 \\ 0 & 0 & 40/19 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ -58 \\ 200/19 \end{bmatrix}$$

solve

(4) Gauss - Seidel Iterative Method

$$3x + 4y + 8z = 7$$
$$x + 20y + z = -18$$
$$25x + y - 5z = 19$$

Rewrite → diagonally dominant

$$25x + y - 5z = 19$$
$$x + 20y + z = -18$$
$$3x + 4y + 8z = 7$$

$25x = 19 - y + 5z$

$x =$

$25x$

$$x = \frac{19 - y + 5z}{25}$$

$$y = \frac{-18 - x - z}{20}$$

$$z = \frac{7 - 3x - 4y}{8}$$

$$\begin{bmatrix} y_1 \times 1 + 0 \times y_2 + 0 \times y_3 \\ \frac{5}{2} y_1 + y_2 + 0 \\ \frac{3}{2} y_1 + \frac{10}{19} y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

$\Rightarrow \quad y_1 = 24$ — (i)

$\frac{5}{2} y_1 + y_2 = 2$

$\Rightarrow \frac{5}{\cancel{2}} \times \cancel{24}^{12} + y_2 = 2$

$\Rightarrow \quad 60 + y_2 = 2 \Rightarrow y_2 = -58$ — (ii)

$\frac{3}{\cancel{2}} \times \cancel{24}^{12} + \frac{10}{19} \times (-58) + y_3 = 16$

$$\begin{array}{r} 19 \\ \times 2 \\ \hline 38 \end{array}$$

$\Rightarrow 36 - \frac{580}{19} + y_3 = 16$

$\Rightarrow \quad y_3 + \frac{684 - 580}{19} = 16$

$\Rightarrow \quad y_3 + \frac{104}{19} = 16$

$\Rightarrow \quad y_3 = \frac{200}{19}$

$16 - \frac{104}{19}$

$\frac{304 - 104}{19}$

$= \frac{200}{19}$

$y_1 = 24$

$y_2 = -58$

$y_3 = 200/19$

$U_{11} = 2$ $U_{12} = -6$ $U_{13} = 8$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 5/2 & 1 & 0 \\ 3/2 & 10/19 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -6 & 8 \\ 0 & 19 & -23 \\ 0 & 0 & 40/19 \end{bmatrix}$$

$AX = B$

$LUX = B$

Let $UX = Y$

$LY = B$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\underset{3\times3}{\begin{bmatrix} 1 & 0 & 0 \\ 5/2 & 1 & 0 \\ 3/2 & 10/19 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

$$\underset{3\times1}{\begin{bmatrix} y_1 + 0 + 0 \\ \frac{5}{2}y_2 + y_2 + 0 \\ \frac{3}{2}y_3 + \frac{10}{19}y_3 + y_3 \end{bmatrix}} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

$\Rightarrow$ $y_1 = 24$ —— (i)

$\frac{7y_2}{2} = 2$ $\Rightarrow y_2 = \frac{4}{7}$ —— (ii)

3) LU factorization Method

$2x - 6y + 8z = 24$

$5x + 4y - 3z = 2$

$3x + y + z = 16$

$$A = \begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11}+0+0 & u_{12}+0+0 & u_{13}+0+0 \\ l_{21}u_{11} & u_{12}l_{21}+u_{22} & u_{13}l_{21}+u_{23} \\ u_{11}l_{31} & u_{12}l_{31}+l_{32}u_{22} & u_{13}l_{31}+u_{23}l_{32}+u_{33} \end{bmatrix}$$

$\boxed{AX = B}$

$LUX = B$

$\boxed{A = LU}$

$u_{11} = 2$

$u_{12} = -6$

$u_{13} = 8$

$l_{21}u_{11} = 5$

$\Rightarrow l_{21} = 5/2$

$3-3/2$ 3

$2-\frac{3}{2}$ $1/2$ 2

$$\xrightarrow[R_3-R_1]{R_2-3R_1} \begin{array}{ccc:ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 3/2 & & & \\ 0 & & & & & \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -3 & -9/2 & -1/2 \\ 0 & 1 & 0 & 12 & 25/2 & 3/2 \\ 0 & 0 & 1 & -5 & -7/2 & -1/2 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & -9/2 & -1/2 \\ 12 & 25/2 & 3/2 \\ -5 & -7/2 & -1/2 \end{bmatrix}$$

$\boxed{AX = B}$

$\Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{3\times 3}{\begin{bmatrix} -3 & -9/2 & -1/2 \\ 12 & 25/2 & 3/2 \\ -5 & -7/2 & -1/2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}}$$

$-\frac{9}{2}\times \cancel{18}^{9}$ $-\frac{1}{2}\times\cancel{16}^{8}$ $\frac{25}{2}\times\cancel{18}^{9}$ $\frac{3}{2}\times\cancel{16}^{8}$ $-\frac{7}{2}\times\cancel{18}^{9}$ $-\frac{1}{2}\times\cancel{16}^{8}$

$$= \underset{3\times 1}{\begin{bmatrix} (-30)+(-81)+(-8) \\ (120)+(225)+(24) \\ (-50)+(-56)+(-8) \end{bmatrix}}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -119 \\ 369 \\ -114 \end{bmatrix}$$

$\therefore x = -119$
$y = 369$
$z = -114$.

Ans

② Matrix Inversion.

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

$$AX = B.$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

Identity Matrix, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(A : I)$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

rules
$A = I$

$$R_1/2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$[A : B]$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$2R_3 \longrightarrow \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -2 & 2 & 4 \end{array}\right]$$

$$R_3 - R_2 \longrightarrow \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$0x + 0y + z = 1$

$\Rightarrow z = 1$

$-2y + z = 3$

$\Rightarrow -2y + 1 = 3$

$\Rightarrow -2y = 2$

$\Rightarrow y = -1$

$x + 2y + z = 0$

$\Rightarrow x + (-2) + 1 = 0$

$\Rightarrow x - 2 + 1 = 0$

$\Rightarrow x - 1 = 0$

$\Rightarrow x = 1$

$\therefore x = 1$

$y = -1$

$z = 1.$

So, $X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

Ans

MOD-4

Numerical sol^n of a System of Linear Equation:-

1) Gauss elimination Method
2) Matrix Inversion
3) LU factorization method
4) Gauss-Seidel iterative method.

① Gauss Elimination Method

Q) $x + 2y + z = 0$
$2x + 2y + 3z = 3$
$-x - 3y = 2$

$\boxed{Ax = B}$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

augmented form

Step ① ↳ $[A : B]$

$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -1 & 0 & 2 \end{array}\right]$

Step ② Lower triangle (left) = 0.

Apply $R_2 \to R_2 - 2R_1$

$\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ -1 & -1 & 0 & 2 \end{array}$

$R_3 \to R_3 + R_1$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$

$2+(-2)$

$2+1$

$R_3 \to 2R_3 + R_2$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 7 \end{array}\right]$