Relative error :-

$$\left(\frac{0.1}{0.2} + \frac{2.0}{3.0} \right) = 0.1167$$

Absolute error :- $6.0 \times 0.1167 = \underline{0.7002}$

Result :- 6.0 ± 0.7002

→ analyze how errors pass through mathematical operations.

If I/P has a small error, O/P might have a larger error after multiple steps.

Q) Evaluate :- e^x at $x = 1$ using
$\boxed{e^x \approx 1 + x + \frac{x^2}{2!}}$

Sol^n :- $e^1 \approx 1 + 1 + \frac{1}{2} = \underline{2.5}$

Actual value :- $e^1 = 2.71828$

Error = $2.71828 - 2.5 = 0.21828$

Q) Round to 3 decimal
$x = 3.14159265$

∵ 4th digit is 5
∴ $x = 3.142$

Rounding error = $3.142 - 3.14159265$
$= 0.00040735$

Q) $A = 2.0 \pm 0.1$ → e_1 Propagat^n of error.
$B = 3.0 \pm 0.2$ → e_2
(a = 2.0, b = 3.0)

a) Add^n :- $A + B = (2.0 + 3.0) = 5.0$
Result :- 5.0 ± 0

b) Multip^n :- $A \times B = 2.0 \times 3.0 = 6.0$

3.142
3.14
3.149 → 3.15

MOD-1

→ Approximation in Numerical computation.

1> Truncation & Rounding Errors.
2> fixed & floating-point arithmetic
3> Propagation of errors.

→ Truncation error → when an infinite process (Taylor series) is cut off at a finite no. of terms.
(truncate)
↓
cut

Ex:- using only 2 terms of a series instead of full infinite terms.

→ Rounding error → when no. are rounded to fit the comp's limited precision
Ex:- 3.141592 → 3.14 in calculator.

→ Fixed ~~error~~ point → No. are stored with a fixed no. of digits before & after the decimal.
used when precision & range are small & known

→ Floating-point → No. are stored in arithmetic notation (1.23×10^5)
IEEE 754 Standard

→ Propagation of Error:-
Small errors in I/P → can grow + affect final result

Solve system of equations:-

$$\boxed{\begin{aligned} y_2 - 1.9375\,y_1 &= 0 \\ y_3 + y_1 - 1.9375\,y_2 &= 0 \\ y_2 - 1.9375\,y_3 &= 0. \end{aligned}}$$

Ans. $\rightarrow y_1 = y_2 = y_3 = 0.$

Given: $\frac{d^2y}{dx^2} = -y$

$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = -y_i$

$\Rightarrow$ (Multiplying both sides by h^2)

$\Rightarrow (y_{i+1} - 2y_i + y_{i-1}) = -y_i (h^2)$

$\Rightarrow \boxed{y_{i+1} - 2y_i + y_{i-1} + h^2 y_i = 0}$ — (a)

$h = 0.25$

$\Rightarrow h^2 = (0.25)^2 = 0.0625$

Put h^2 in eq^n (i)

$\cancel{y_{i+1} + y_{i-1} - 2}$

$y_{i+1} + y_{i-1} - y_i (2 - h^2) = 0$

$\Rightarrow y_{i+1} + y_{i-1} - (2 - 0.0625) y_i = 0$

$\Rightarrow \boxed{y_{i+1} + y_{i-1} - 1.9375 y_i = 0}$

At $i = 1$:- $y_2 + \cancel{y_0}^{0} - 1.9375 y_1 = 0$

$\Rightarrow y_2 - 1.9375 y_1 = 0$ — (i)

At $i = 2$:- $y_3 + y_1 - 1.9375 y_2 = 0$ — (ii)

At $i = 3$:- $\cancel{y_4}^{0} + y_2 - 1.9375 y_3 = 0$ — (iii)

Q> $\frac{d^2y}{dx^2} = -y$, on interval $[0, 1]$

with boundary conditions:-

$y(0) = 0$ → initial value

$y(1) = 0$ → final value.

Step-size, $h = 0.25$.

Sol^n:- i> Discretize the interval:-

From $x=0$ to $x=1$, Step size, $h=0.25$

$x_0 = 0$, ~~$x_1 = 0.25$~~

$x_1 = 0 + h = 0 + 0.25 = 0.25$

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$x_3 = 0.5 + 0.25 = 0.75$

$x_4 = 0.75 + 0.25 = 1$.

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$y_0 = 0$, y_1, y_2, y_3

$y_4 = 0$.

ii> Replace 2nd derivate using finite difference

$$\boxed{\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}}$$ → Taylor's series.

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(4) Finite Difference Method

→ to solve diff. eqⁿ. by replacing derivatives with differences.

→ diff. eqⁿ. → into set of algebraic eqⁿ.

→ Solving ODEs
Solving PDEs (heat flow, wave motion, fluid dynamics)

→ Calculus (derivatives) ✗
approximate nearby values ✓✓

→ called forward difference rul.

→ Types of Difference :-

i) Forward diff.	$\frac{y_{i+1} - y_i}{h}$	Current to next point
ii) Backward diff.	$\frac{y_i - y_{i-1}}{h}$	Current to previous point.
iii) Central diff.	$\frac{y_{i+1} - y_{i-1}}{2h}$	Avg. slope b/w neighbours

h = Step size
y_i = function value at x_i

ii) $f(x_n, y_n) = \frac{dy}{dx}$

$\Rightarrow \cancel{f(0, 1) = 0+1}$

$x = x_0 + h = 0 + 0.1 = 0.1$

$y = 1.1$

$f(x_n, y_n) = f(0.1, 1.1)$ $\qquad \frac{dy}{dx} = x + y$

$f(0.1, 1.1) = 0.1 + 1.1 = 1.2$

$y_1^{(correct)} = 1 + \frac{0.1}{2}(1 + 1.2) = 1.11$

$\therefore \ y(0.1) \approx 1.11$ Ans

(3) Predictor corrector Method

→ 2-step numerical method:

→ Solve ODEs.

→ 1st → predicts next value using approx. method (Euler's or Runge-Kutta)

→ Then → corrects value using a better formula (Trapezoidal Rule)

→ General ODE form:- $\boxed{\frac{dy}{dx} = f(x, y)}$, $\boxed{y(x_0) = y_0}$

→ Working?

Step (1) Predictor Step (estimate next y)

Ex:- Euler's method:-

$$\boxed{y_{n+1}^{(predict)} = y_n + h \cdot f(x_n, y_n)}$$

Step (2) Corrector Step (improve guess)

Trapezoidal rule:-

$$y_{n+1}^{(predict)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(predict)})\right]$$

Q> $\frac{dy}{dx} = x + y$, $y(0) = 1$, $h = 0.1$

Sol:- i> $\cancel{f(0,1)} \Rightarrow f(x_n, y_n) = \frac{dy}{dx}$

$$f(0, 1) = 0 + 1 = 1$$

$$y_1^{(predict)} = y_n + h \cdot f(x_n, y_n)$$

$$= 1 + 0.1(1) = 1.1$$

$k_1 = 0.1$

$k_2 = 0.11$

$k_3 = 0.1105$

$k_4 = 0.12105$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\Rightarrow y(0.1) = 1 + \frac{1}{6}(0.1 + 2(0.11) + 2(0.1105) + 0.12105)$$

$$\approx 1.1103.$$

Hence, $y(0.1) \approx 1.1103$.

$y(x) \quad x+y = \frac{dy}{dx} = f(x, y)$

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Q) $\frac{dy}{dx} = x+y$, $\quad y(0) = 1$, $\quad h = 0.1$

$x = 0$
$y = 1$

calculate $y(0.1)$

Sol :- $K_1 = h \cdot f(x_n, y_n)$

$= 0.1 f(0, 1) = 0.1(0+1) = 0.1$

$K_2 = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$

$= 0.1 f(0.05, 1.05)$

$= 0.1(0.05 + 1.05) = 0.11$

$K_3 = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right)$

$= 0.1 f(0.05, 1.055)$

$= 0.1(0.05 + 1.055) = 0.1105$

$K_4 = h \cdot f(x_n + h, y_n + K_3)$

$= 0.1 f(0 + 0.1, 1 + 0.1105)$

$= 0.1(0.1 + 1.1105) = 0.12105$

2) Runge - Kutta Method

- → improves Euler's method
- → calculating Slop at multiple points
- → taking weighted avg. of those slopes.
- → much more accurate results.
- → General ODE form:-

$$\boxed{\frac{dy}{dx} = f(x, y)} \quad , \ y(x_0) = y_0$$

- → we need to find y at some point x using Small steps of size h.

- → Runge- Kutta 4th order formula.

$$K_1 = h \cdot f(x_n, y_n)$$

$$K_2 = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

$$K_3 = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$$

$$K_4 = h \cdot f(x_n + h, y_n + K_3)$$

$$\boxed{y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)}$$

→ Steps:- Start Initial point (x_0, y_0)

Calculate K_1, K_2, K_3, K_4.

Avg. them using formula.

Get next y-vale.

Repeat for next Step.

→ approximates solⁿ of 1st order differential eqⁿ.

$$\frac{dy}{dx} = f(x, y)$$

→ initial condition: $y(x_0) = y_0$

→ Euler's formula: $y_n + h \cdot f(x_n, y_n)$

h = step size
x_n, y_n = current x & y values
$f(x_n, y_n)$ = Slop at current point
y_{n+1} = next y value
$x_{n+1} = x_n + h$

→ Ex:- Given $\frac{dy}{dx} = x + y$; $y(0) = 1$

use $h = 0.1$, find approx. value at $x = 0.1$

Solution :- $f(x_0, y_0) = f(0, 1) = 0 + 1 = 1$
$y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 0.1.1 = 1.1$

So, approx. value ≈ 1.1

$x_1 = x_0 + h \Rightarrow 0 + 0.1 \Rightarrow 0.1$
$x_2 = x_1 + h \Rightarrow 0.1 + 0.1 \Rightarrow 0.2$
$x_3 = x_2 + h \Rightarrow 0.2 + 0.1 \Rightarrow 0.3$
$x_4 = x_3 + h \Rightarrow 0.3 + 0.1 \Rightarrow 0.4$

equn (2)

$$\boxed{y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})}$$

$\frac{dy}{dx} = x + 2y$

$0 + 0 = 0$

$0.1 + 0 \Rightarrow 0.1$

$0.2 + (2 \times 0.01) \Rightarrow 0.22$

$y_1 = y_0 + 0.1 f(x_0, y_0) \Rightarrow 0 + 0.1 \times 0 \Rightarrow 0$

$y_2 = y_1 + 0.1 f(x_1, y_1) \Rightarrow 0 + 0.1 \times 0.1 = 0.01$

$y_3 = y_2 + 0.1 f(x_2, y_2) \Rightarrow 0.01 + 0.1 \times 0.22 = 0.032$

$y_4 = y_3 + 0.1 f(x_3, y_3) \Rightarrow 0.032 + 0.1 \times 0.364 = 0.0684$

$\therefore$ $x = 0.4$, $y = 0.0684$ Ans

~~Runge - Kutta method~~

$= \boxed{2.706}$ Stop.

MOD-6

Numerical Solⁿ of Ordinary differential eqⁿ.

1) Euler's method
2) Runge - Kutta methods
3) Predictor - corrector methods.
4) Finite Difference method.

Euler's method

Numerical method $\xrightarrow{\text{finding}}$ solution $\xrightarrow{\text{of}}$ differential equⁿ.

$\frac{dy}{dx}$ → ind, → dep

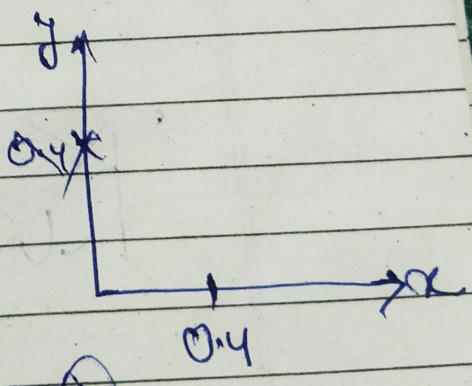
initial condition $x=0, y=0$

Stepsize

(Q.) Given $\frac{dy}{dx} = x+2y$, $y(0) = 0$; $h = 0.1$

find $y(0.4)$

$x = 0.4$, $y = ?$



equⁿ (1)

$$\boxed{X_n = X_{n-1} + h}$$

equⁿ (2)

$$\boxed{Y_n = Y_{n-1} + hf(x_{n-1}, y_{n-1})}$$