① Find the no. of integers b/w 1 and 720 both inclusive that are not divisible by any of the integers 2, 3 and 5.

→ No. of integers b/w 1 to 720 divisible by 2 is:-
$\frac{720}{2} = 360 = n(A)$

No. of integers b/w 1 to 720 divisible by 3 is:-
$\frac{720}{3} = 240 = n(B)$

No. of integers b/w 1 to 720 divisible by 5 is:-
$\frac{720}{5} = 145 = n(C)$

$n(A \cap B) = \frac{720}{6} = 120$

$n(A \cap C) = \frac{720}{10} = 72$

$n(B \cap C) = \frac{720}{15} = 48$

$n(A \cap B \cap C) = \frac{720}{30} = 24$

So, the req. no. of integers is:-
~~So~~ $n(A) + n(B) + n(C) - [n(A \cap B) + n(A \cap C) + n(B \cap C) + n(A \cap B \cap C)]$
$= 360 + 240 + 145 - (120 + 72 + 48 + 24)$
$= 745 - 264 = 481$

② Find the no. of natural no. not greater than 1000 which are not divisible by 3, 5 or 7.
Let A → set of N divisible by 3
B → " " " " " 5
C → " " " " " 7

$3 \overline{)1000}(333$
18
$\overline{}$
09
9
$\overline{}$
1

Then, $n(A) = 333$; $n(B) = 200$; $n(C) = 142$

$n(A \cap B) = \frac{1000}{15} = 66$

$n(A \cap B \cap C) = \frac{1000}{105} = 9$

$n(B \cap C) = \frac{1000}{35} = 25$

$n(C \cap A) = \frac{1000}{21} = 47$

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Hence: n(AUB
= n(A)+n(B)+n
= 546.
∴ The no. of
1000 − 546 = 45

③ Smoking habi
28% → A & B
What % do
→ n(A) = 55
n(B) 50
n(C) 28
n(AUBUC) =
∴ 100 − 97

④ If A = {
then show
→ AUB = {a,
(AUB)∩C = {

⑤ P.T. (A
Let (x,
Then (x,
⇒ (x,
⇒ (x∈
⇒ (x∈
⇒ x
⇒ (x,
∴ (A×B)
Again,
(A
Then,
⇒ (
⇒ (
⇒ (
⇒

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Hence: $n(A \cup B \cup C)$

$= n(A)+n(B)+n(C) - [n(A\cap B)+n(B\cap C)+n(C\cap A)+n(A\cap B\cap C)$

$= 546.$

$\therefore$ The no. of N not divisible by 3, 5 or 7 is

$1000 - 546 = 454$.

(3) Smoking habits :- $55\% \to A$; $50\% \to B$; $42\% \to C$;
$28\% \to A \,\&\, B$; $20\% \to A \,\&\, C$; $12\% \to B \,\&\, C$; $10\% \to A, B, C$.
What % don't smoke?

$\to$ $n(A) = 55$ $\quad$ $n(A\cap B) = 28$ $\quad$ $n(A\cap B\cap C) = 10$

$n(B) 50$ $\quad$ $n(A\cap C) = 20$

$n(C)\ 28$ $\quad$ $n(B\cap C) = 12$

$n(A\cup B\cup C) = 55 + 50 + 28 - (28 + 20 + 12 + 10) = 97$

$\therefore$ $100 - 97 = 3\%$. Students don't smoke.

(4) If $A = \{a,b,c,d,e\}$, $B = \{c,a,e,g\}$ & $C = \{b,e,f,g\}$,
then Show that $(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$.

$\to$ $A\cup B = \{a,b,c,d,e,g\}$ $\quad$ $(A\cap C) = \{b,e\}$; $(B\cap C) = \{e,g\}$

$(A\cup B)\cap C = \{b,e,g\}$ $\quad$ $(A\cap C)\cup(B\cap C) = \{b,e,g\}$

Proved.

(5) P.T. $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$.

Let (x,y) be an arbitrary element of set $(A\times B)\cap(C\times D)$

Then $(x,y) \in (A\times B)\cap(C\times D)$

$\Rightarrow (x,y)\in(A\times B)$ and $(x,y)\in(C\times D)$

$\Rightarrow (x\in A$ and $y\in B)$ and $(x\in C$ and $y\in D)$

$\Rightarrow (x\in A$ and $x\in C)$ and $(y\in B$ and $y\in D)$

$\Rightarrow x\in(A\cap C)$ and $y\in(B\cap D)$

$\Rightarrow (x,y)\in(A\cap C)\times(B\cap D)$

$\therefore (A\times B)\cap(C\times D) \subseteq (A\cap C)\times(B\cap D)$.

Again, let (a,b) be an arbitrary element of set $(A\cap C)\times(B\cap D)$.

Then, $(a,b)\in(A\cap C)\times(B\cap D)$

$\Rightarrow (a,b)\in(A\cap C)$ and $\cancel{(B\cap D)}$ $(a,b)\in(B\cap D)$

$\Rightarrow$ $\cancel{(a,b)\in}$ $(a\in A$ and $b\in C)$ and $(a\in B$ and $b\in D)$

$\Rightarrow (a\in A$ and $a\in B)$ and $(b\in C$ and $b\in D)$

$\Rightarrow a\in(A\times B)$ and $b\in(C\times D) \Rightarrow (a,b)\in(A\times B)\cap(C\times D)$

$\therefore (A\cap C)\times(B\cap D) \subseteq (A\times B)\cap(C\times D)$.

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(6) For any 3 sets :- A, B, C. Show that $A-(B-C) = (A-B) \cup (A \cap C)$

LHS :- $A-(B-C) = A \cap (B-C)^c$

$= A \cap (B \cap C^c)^c$

$= A \cap (B^c \cup C)$ [De-Morgan's rule]

$= (A \cap B^c) \cup (A \cap C)$

$= (A-B) \cup (A \cap C)$ proved.

(7) Let $A = \{1,2,3,4\}$ and $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(3,2)\}$. Is R is equivalence relation?

→ Reflexive → R is said to be reflexive relation if $(a,a) \in R \quad \forall \, a \in A$.

$A = \{1,2,3,4\}$ $\quad \{(1,1),(2,2),(3,3),(4,4)\}$ are present in R

∴ R is reflexive.

→ Symmetric → R is said to be symm. relⁿ if $(a,b) \in R \Rightarrow (b,a) \in R$.

$A = \{1,2,3,4\}$ $\quad \{(1,2),(1,3)\}$ are present in R but $\{(2,1),(3,1)\}$ are not present

∴ R is not symmetric.

→ Transitive → R is said to be transitive relⁿ if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$.

$A = \{1,2,3,4\}$ $\quad \{(1,1)$ and $(2,2)\}$ present in R

$\Rightarrow \{(1,2)\}$ also present.

∴ R is transitive.

∵ R is reflexive & transitive but not symm.

Hence, R is not equivalence relation.

(8) a) If $A = \{1,2,3\}$ and $B = \{x,y\}$, list all members of $A \times B$.

$A = \{1,2,3\}$ & $B = \{x,y\}$

$A \times B = \{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}$

b) If $A = \{2,4,6\}$ and $B = \{1,3,5,7\}$, find $A \cap B$.

→ ~~$A = \{2,4,6\}$~~ ϕ.

c) Z is a set of all integers and $f(x) = |x|$ as $x \in Z$. Show that f is not one to one.

→ $x_1, x_2 \in Z$

Now, $f(x_1) = f(x_2) \Rightarrow |x_1| = |x_2|$

or $x_1 = \pm x_2$

$\therefore$ f is not one to one.

d) If $A = \{a,b,c,d\}$ $B = \{b,c,p,q\}$, then find out $A \times B$, $B \times A$ and $A \Delta B$.

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

$= \{(a,b),(a,c),(a,p),(a,q),(b,b),(b,c),(b,p),(b,q),(c,b),(c,c),(c,p),(c,q),(d,b),(d,c),(d,p),(d,q)\}$

$B \times A = \{(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(p,a),(p,b),(p,c),(p,d),(q,a),(q,b),(q,c),(q,d)\}$

$A \Delta B = (A \cup B) - (A \cap B)$

$= \{a,b,c,d,p,q\} - \{b,c\} = \{a,d,p,q\}$.

e) Define power set. Find the power set of $\{a,b,c\}$

→ A set formed of all the sub-sets of the set as its elements is called power set of S and is symbolically denoted by $P(S)$

Let, $S = \{a,b,c\}$

$P(S) = \{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\phi\}$.

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MOD-2

① Find the remainder when the sum $1^5+2^5+3^5+\ldots+100^5$ is divided by 5.

→ Given expression is:

$1^5+2^5+\ldots+100^5 = (1^5-1)+(2^5-2)+\ldots+(100^5-100)+(1+2+\ldots+100)$

$= 5K_1 + 5K_2 + \ldots + 5K_{100} + \frac{100\times 101}{2}$ [Fermat's theorem]

$= 5\{K_1+K_2+\ldots+K_{100}+1010\}$

Hence, $1^5+2^5+\ldots+100^5$ leaves remainder 0 when divided by 5.

② Find the remainder when the sum $1!+2!+3!+\ldots+100!$ is divided by 5.

→ Clearly, from 5! onwards each term is divisible by 5.

So, we consider only $1!+2!+3!+4!$

By Wilson's theory $4! \equiv -1 \pmod 5$, 5 being a prime no. i.e., $4!+1$ is divisible by 5.

So, we are to consider only $2!+3!$ i.e., 8 which clearly leaves 3 as remainder when divided by 5.

So, the remainder asked for is 3.

③ Find gcd (595, 252) and express it in the form $252m+595n$.

By division algorithm:-

$595 = 2\times 252 + 91$

$252 = 2\times 91 + 70$

$91 = 1\times 70 + 21$

$70 = 3\times 21 + 7$

$21 = 3\times 7 + 0$

∵ the last non-zero remainder is 7

gcd (595, 252) = 7.

Now, $7 = 70 - 3\times 21$

$= 70 - 3(91 - 1\times 70) = 70 - 3(91-70)$

$= 4\times 70 - 3\times 91$

$= 4(252-2\times 91) - 3\times 91$

$= 4\times 252 - 11\times 91 = 4\times 252 - 11(595 - 2\times 252)$

$= 26\times 252 - 11\times 595.$

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④ Find the remainder ...

$1!+2!+3!+\ldots$

$6! = 720 \equiv 0 \pmod{\ldots}$

When $n \geq 6$, ...

∴ $n!$ ...

∴ $(1!+2!+3\ldots$

$\equiv (1+2+6+24+\ldots$

$\equiv 153 \pmod{\ldots}$

$\equiv 17.$

⑤ Find integers ...

⇒ $512 = 1\times 32\ldots$

$320 = 1\times 1\ldots$

$192 = 1\times 1\ldots$

$128 = 2\times\ldots$

Thus, 64 ...

Hence, the ...

$m512\ldots$

Now, from ...

⑥ a) If ...

Let ...

then ...

∴ a...

As ...

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4) Find the remainder when the sum $1! + 2! + 3! + \dots + 100!$ is divisible by 18.

$6! = 720 \equiv 0 \pmod{18}$

When $n \geq 6$, $n! = 18k$; k is a non-zero integer.

$\therefore n! \equiv 0 \pmod{18}$; $n \geq 6$

$\therefore (1! + 2! + 3! + 4! + 5! + \dots + 100!) \pmod{18}$

$\equiv (1 + 2 + 6 + 24 + 120) \pmod{18}$

$\equiv 153 \pmod{18}$

$\equiv 17.$

5) Find integers m and n such that $512m + 320n = 64$.

$512 = 1 \times 320 + 192$ — (i)

$\Rightarrow 320 = 1 \times 192 + 128$ — (ii)

$192 = 1 \times 128 + 64$ — (iii)

$128 = 2 \times 64 + 0$ — (iv)

Thus, 64 is the gcd of 512 and 320.

Hence, there exist integers, m and n such that

$m512 + n320 = 64$

Now, from (iii), we have,

$$
\begin{aligned}
64 &= 192 - 1 \times 128 \\
&= 192 - (320 - 192) \\
&= 2 \times 192 - 320 \\
&= 2(512 - 320) - 320 \\
&= 2 \times 512 - 2 \times 320 - 320 \\
&= 2 \times 512 - 3 \times 320
\end{aligned}
$$

$\therefore m = 2$, $n = -3$.

6) a) If $\gcd(a, b) = 1$, P.T. $\gcd(a^2, b^2) = 1$.

Let $\gcd(a^2, b^2) = d$

Then, $a^2 = dk_1$, $b^2 = dk_2$, $k_1, k_2 \in \mathbb{N}$

$\therefore a = \sqrt{dk_1}$, $b = \sqrt{dk_2}$

As $\gcd(a, b) = 1$ we get $\gcd(\sqrt{d}\sqrt{k_1}, \sqrt{d}\sqrt{k_2}) = 1$

So, $\sqrt{d} = 1$ or $d = 1$

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b) Find 2 integers u and v satisfying $63u + 55v = 1$

$\Rightarrow$ $\because$ gcd $(63, 55) = 1$

$\therefore$ eqⁿ $63u + 55v = 1$ has integral solutions.

Now, $63 = 1 \times 55 + 8$

$55 = 6 \times 8 + 7$

$8 = 1 \times 7 + 1$

$\therefore 1 = 8 - 7 \times 1 = (63 - 55) - (55 - 6 \times 8)$

$= 63 - 2 \times 55 + 6(63 - 55)$

$= 7 \times 63 - 8 \times 55$

So, $u = 7$, $v = -8$

(7) State Division algorithm. Show that every sq. integer is of the form $5k$, $5k \pm 1$ for some integer k.

$\rightarrow$ Division of one integer by another plays an imp. role in the study of integers.

If $b \neq 0$ and a are integers, then we can divide a by b.

If q is the quotient and r is the remainder, then we can say a is completely divisible by b if $r = 0$. indeed we can state that :-

If $a, b \in \mathbb{Z}$, $b \neq 0$, then there exist $q, r \in \mathbb{Z}$ such that $a = bq + r$, $0 \leq r < |b|$ for ex,

if $a = 37$, $b = 5$ then $q = 7$, $r = 2$.

2nd part: Let x be any +ve integer :-

Then $x = 5q$ or $x = 5q+1$ or $x = 5q+4$ for integer x.

If $x = 5q$, $x^2 = (5q)^2 = 25q^2 = 5(5q^2) = 5n$ $\quad (n = 5q^2)$

If $x = 5q+1$, $x^2 = (5q+1)^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1 = 5n + 1$

If $x = 5q+4$, $x^2 = (5q+4)^2 = 25q^2 + 40q + 16 = 5(5q^2 + 8q + 3) + 1 = 5n + 1$

$\therefore$ In each ~~term~~ of 3 cases x^2 is either of the form $5q$ or $5q+1$ or $5q+4$ and for integer q.

(8) a) P.T. $6^{n+2} + 7^{2n+1}$ is divisible by 43, for each natural no. n.

+55V=1.

ns.

integer is
n K.
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integer x.
$n = 5q^2$)
$+1 = 5n+1$
$)+1 = 5n+1$
orm

each natural

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Let $P(n) = 6^{n+2} + 7^{2n+1}$

Clearly, $P(0)$ is true as $P(0) = 6^2 + 7 = 36 + 7 = 43$.

Also, $P(1)$ is true as $P(1) = 6^3 + 7^3 = 559$, divisible by 43.

So assume $P(m)$ is divisible by 43, i.e.,

$P(m) = 6^{m+2} + 7^{2m+1} = 43k, \quad k \in \mathbb{Z}$

Now, $P(m+1) = 6^{m+3} + 7^{2m+3} = \cancel{6 \cdot 6^{m+2}}\ 6 \cdot 6^{m+2} + 49 \cdot 7^{2m+1}$

$= 6(6^{m+2} + 7^{2m+1}) + 43 \cdot 7^{2m+1}$

$= 6 \times 43k + 43 \cdot 7^{2m+1}$

$= 43(6k + 7^{2m+1}) = 43k' \quad (k' \in \mathbb{Z})$

Thus, $P(m+1)$ is divisible by 43.

Hence, by induction principle, $6^{n+2} + 7^{2n+1}$ is divisible by 43.

b) If $\gcd(a,b) = 1$, show that $\gcd(a+b,\ a^2 - ab + b^2) = 1$ or 3.

$\rightarrow \gcd(a,b) = 1$

Let $d = \gcd(a+b,\ a^2 - ab + b^2)$

Then $d \mid a+b$

So, $d \mid (a+b)^2$

Again, $d \mid (a^2 - ab + b^2)$

So, $d \mid \{(a+b)^2 - (a^2 - ab + b^2)\}$ i.e., $d \mid 3ab$

$\because$ 3 is a prime, i.e., $d = 3$ or $d \mid ab$

But as $\gcd(a,b) = 1$, $d = 1$

Thus d is 1 or 3.

(9) Define GCD of 2 integers a and b. ~~then an integer~~
~~When a and b are (non-zero) integer,~~

~~d(70) is~~

The greatest common divisor (gcd) of two integers a and b is the largest +ve integer d that divides both a and b without leaving a remainder.

$d = \gcd(a,b)$ if $d \mid a$ and $d \mid b$, and for any integer e such that $e \mid a$ and $e \mid b$, we have $e \leq d$.

Use Eucledian algo. to find integers u and v such that $\gcd(72, 120) = 72u + 120v$.

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gcd (72, 120) = 72u + 120v

We know: 120 = 1×72 + 48

72 = 1×48 + 24

48 = 2×24 + 0

Now, 24 = 72 − 1×48

= 72 − 1(120 − 1×72)

= 3×72 − 1×120

⇒ u = 3, v = −1.

State and prove fundamental theory of arithmetic.

→ Euclidean theory

Every +ve integer > 1 can be uniquely factored into prime numbers. This means that for any integer $n > 1$, there exist a unique set of prime numbers $p_1, p_2, \ldots, p_k$ and +ve integers $e_1, e_2, \ldots, e_k$ such that $n = p_1^{e_1}, p_2^{e_2}, p_k^{e_k}$

where, $p_1 < p_2 < \cdots < p_k$ are prime nos. and $e_1, e_2, \ldots, e_k$ are +ve integers.

Proof → : Existence
: Uniqueness.

Existence

1. Base case → Smallest +ve integer > 1 is 2, which is a prime no. Hence, it's already a product of primes

2. Inductive step → Assume that every integer from 2 to n can be factored into primes. We need to show that n+1 can also be factored into primes.

- If n+1 is a prime no., then it is already product of primes (itself)

- If n+1 is not prime no., then it can be written as product of 2 integers a & b such that $2 \leq a \leq b$ and $a \cdot b = n + 1$.

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Uniqueness → Suppose that a +ve integer n can be factored into primes in 2 diff. ways :-

$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$ { p_i and q_j are prime nos. }

$n = q_1^{f_1} \cdot q_2^{f_2} \cdot \ldots \cdot q_m^{f_m}$

Suppose p_1 → Smallest prime factor in the 1st factoriz^n.

∵ p_1 divides n, it must also divide the product of $q_1^{f_1} \cdot q_2^{f_2} \cdot \ldots \cdot q_m^{f_m}$.

If $a|b$ and $a|c$ then P.T. $a|(bx+cy)$ for arbitrary integers x and y.

∵ $a|b$, we have $a|xb$, where x is an integer

Again $a|c$ then $a|yc$, where y is an integer.

∴ $a|(xb+yc)$ for 2 integers x and y.

If $a|bc$ and $\gcd(a,b)=1$ then P.T. $a|c$.

Let $\gcd(a,b)=1$.

∴ ∃ 2 integers m and n such that

$ma+nb = 1 = \gcd(a,b)$ —(i)

(i) × c, we get,

$mac + nbc = c$

Now, $a|bc \Rightarrow a|nbc$

∴ $a|mac \Rightarrow a|c$ (proved.)

Find all possible value of x, for $345x \equiv 18 \pmod{912}$.

∵ ~~gcd~~ $(345, 912) = 3$

So, $345x \equiv 18 \pmod{912}$ —(i)

$115x \equiv 6 \pmod{104}$ —(ii)

We 1st solve : $\gcd(115, 104) = 1$

$$
\begin{array}{l}
104)\,115\,(1 \\
\quad\underline{104} \\
\quad\; 11)\,104\,(9 \\
\qquad\;\; \underline{99} \\
\qquad\;\;\; 5)\,11\,(2 \\
\qquad\qquad \underline{10} \\
\qquad\qquad\; 1
\end{array}
$$

$1 = 11 - 5\times 2$

$= 11 - (104 - 9\times 11)$

$= 10\times 11 - 104$

$= 10\times (115 - 10{\cancel{4}}\times 10.4) - 104$

$115 \times 10 \equiv 1 \pmod{104}$

⇒

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(10) Let m, n be integers not both zero. P.T.
$\gcd(km, kn) = k \cdot \gcd(m,n)$ for any +ve integer k.

→ Let $d = \gcd(m,n)$
Then, $ma + nb = d$ when a and b are integers.
$\therefore a(km) + b(kn) = kd$
i.e., $\gcd(km, kn) = kd = k \gcd(m,n)$
If k is any integer, then result becomes
$\gcd(km, kn) = k \gcd(m,n)$.

(11) a) State and prove the recursion th. of gcd.
GCD Algorithm 2: Euclid's Algorithm
for $m \geq n > 0$,
$$\gcd(m,n) = \begin{cases} n & \text{if } n \text{ divides } m \text{ with no remainder} \\ \gcd(n, \text{remainder of } \frac{m}{n}) & \text{otherwise} \end{cases}$$

We can rewrite m as follows:-
$$m = n\left[\frac{m}{n}\right] + \text{remainder of } \frac{m}{n}.$$

Proof:-
Division Algorithm → Acc. to division algo., for any integers a and b (with $b \neq 0$), there exist unique integers q (quotient) and r (remainder) such that:-
$a = bq + r$ where $0 \leq r < b$
Here, $r = a \mod b$
Common divisors →
If d is a c.d. of a and b, then:
$d|a$ and $d|b$
By definition of divisibility, if $d|a$ & $d|b$, then
$d \mid (a - bq)$
$\because a = bq + r$, we can rewrite this
$d | r$
$\therefore d$ is also a divisor of r.

Thus, $\gcd(a,b) = \gcd(b, a \bmod b)$.

b) i) P.T. $\gcd(a,b) = \gcd(a, a-b)$

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Let $p = \gcd(a, \ldots$
If $h|a$, $h|(a-b \ldots$
But $p = \text{gc}\ldots$
Hence, $p = \ldots$

(12) State the pr...
→ States that ...
a least elem...
S of +ve i...
an element ...
This principle ...

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Let $p = \gcd(a, b)$ Then, $p|a$, $p|b$ so, $p|(a-b)$

If $h|a$, $h|(a-b)$ then, $h|a-(a-b)$ i.e., $h|b$

But $p = \gcd(a, ~~a~~ b)$. So, $h|p$

Hence, $p = \gcd(a, a-b)$

(12) State the ~~pr~~ principle of well ordering.

→ States that every non-empty set of +ve integers contains a least element. In other words, if you have a set S of +ve integers & S is not empty, then there exists an element m in S such that $m \leq n$ for all n in S.

This principle is fundamental in no. theory.

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① If $A \cup B = A \cup C$ and $A \cap B = A \cap C$. P.T. $B = C$.

→ Absorptive property :- $B = B \cap (B \cup A)$

Commutative property :- $B = B \cap (A \cup B)$

" $B = B \cap (A \cup C)$ {∵ $A \cup B = A \cup C$ given}

Distributive property :- $B = (B \cap A) \cup (B \cap C)$

$= (A \cap B) \cup (B \cap C)$

$= (A \cap C) \cup (B \cap C)$ {∵ $A \cap B = A \cap C$ given}

$= (C \cap A) \cup (C \cap B)$

$= C \cap (A \cup B)$

$= C \cap (A \cup C)$

$= C$ (absorptive property)

Hence, $B = C$ (proved)

② If A and B are 2 sets then P.T. $A - B$, $A \cap B$ and $B \cap A$ are pairwise disjoint.

→ We have to prove $(A-B) \cap (A \cap B) = \phi$

if possible, let $x \in (A-B) \cap (A \cap B)$

$\Rightarrow x \in (A-B)$ and $x \in (A \cap B)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \in B)$

So this is not possible bcz there is no element which satisfy $x \notin B$ and $x \in B$.

So, $x \notin (A-B) \cap (A \cap B)$

So, $(A-B) \cap (A \cap B) = \phi$

again,

to prove $(A-B) \cap (B \cap A) = \phi$

if possible, let $x \in (A-B) \cap (B \cap A)$

$\Rightarrow x \in (A-B)$ and $x \in (B \cap A)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in B$ and $x \in A)$

So this is not possible that $x \notin B$ and $x \in B$

So, $x \notin (A-B) \cap (B \cap A)$

So, $(A-B) \cap (B \cap A) = \phi$

proved.

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③ Show that the re[...]

$R = \{(a,b) : a, b$ [...]

→ Reflexive :-

$(a^2 + a^2) = 2a^2$

So, $(a, a) \in$ [...]

Symmetric :-

$(a^2 + b^2)$ is mu[...]

So, $(b^2 + a^2)$ is al[...]

Transitive :-

$(a^2 + b^2)$ is mult[...]

$(a^2 + b^2) + (b^2 + c^2$ [...]

$(a^2 + c^2) + 2b^2$

$\Rightarrow (a^2 + c^2)$ is [...]

So, $(a, c) \in$ [...]

Hence, R is [...]

④ Let z be the [...]

such a way [...]

as $b = a^r$, [...]

(z, ρ) is [...]

→ For Reflexive: [...]

For Anti [...]

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$B = C$.

$\because A \cup B = A \cup C$ given}

$\because A \cap B = A \cap C$ given}

perty)

$A - B$, $A \cap B$

$x \in B$)

no element

$x \in A$)

$x \in B$

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(3) Show that the relⁿ. R on Z is an equivalence relⁿ.
$R = \{(a,b) : a, b \in Z$ and $a^2 + b^2$ is a multiple of 2}

→ Reflexive :-
$(a^2 + a^2) = 2a^2$ is a multiple of 2
So, $(a, a) \in R \Rightarrow$ reflexive.

Symmetric :-
$(a^2 + b^2)$ is multiple of 2 $\Rightarrow (a,b) \in R$
So, $(b^2 + a^2)$ is also " " $\Rightarrow (b,a) \in R$ } ⇒ Symmetric.

Transitive :-
$(a^2 + b^2)$ is multiple of 2 and $(b^2 + c^2)$ is multiple of 2
$(a^2 + b^2) + (b^2 + c^2)$ is also multiple of 2
$(a^2 + c^2) + 2b^2$ is multiple of 2.
$\Rightarrow (a^2 + c^2)$ is also multiple of 2
So, $(a, c) \in R \Rightarrow$ Transitive relⁿ.
Hence, R is an equivalence relⁿ.

(4) Let z be the set of all integer. Define a f in such a way that $a f b$ holds if b can be expressed as $b = a^{r}$, for some +ve integer r, so that, (z, f) is PO set.

→ For Reflexive :- $a = a^{1}$, when $r = 1$
$a f a \Rightarrow f$ is reflexive.

For Anti Symm. :- $a f b$ and $b f c$
$b = a^{r}$, $c = b^{r_2}$
$c = b^{r_2}$
$\Rightarrow c = a^{r_1 r_2}$ [$\because r_1, r_2$ are +ve integers then, $r_1 \times r_2$ is also +ve]

$\therefore a f c$
$\therefore f$ is transitive.

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(5) Show that $f: R \to R$ defined by $f(x) = x^2$ is not bijective.
(where $x \in R$)

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

here, $f(x_1) = f(x_2)$ doesn't imply $x_1 = x_2$

$\therefore$ f is not injective.

$y = x^2$

$\Rightarrow x = \sqrt{y}$

Let y be any arbitrary in the co-domain R,

$\therefore y = x^2 \Rightarrow x = \pm\sqrt{y} \notin R$, for -ve real no. y in R.

So, each element in co-domain set R has no pre-image under f.

Hence, f is not surjective.

$\therefore$ f is not bijective.

(6) Find domain of $f(x)$ where $f(x) = \sqrt{\log \frac{4x - x^2}{3}}$, where $f(x)$ and x are real.

$\rightarrow$ let $f(x) \rightarrow$ real (R), if $\log \frac{4x - x^2}{3} \geq 0$

$\Rightarrow \frac{4x - x^2}{3} \geq e^0 = 1$

$\Rightarrow 4x - x^2 \geq 3$

$\Rightarrow x^2 - 4x + 3 \leq 0$

$\Rightarrow x^2 - (3+1)x + 3 \; \cancel{(x-3)} \leq 0$

$\Rightarrow x(x-3) - 1(x-3) \leq 0$

$\Rightarrow (x-3)(x-1) \leq 0$

either $x - 1 \leq 0$ & $x - 3 \geq 0$

$\Rightarrow x \leq 1$ & $x \leq 3$

$\therefore x \in [1, 3]$

Hence, domain of $f(x)$: $x \in [1, 3]$. Ans.

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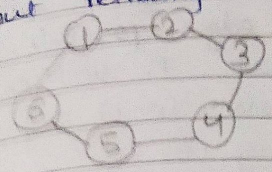
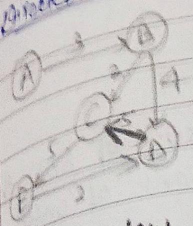
i) What is a cycle in a graph?
→ Starting from a particular vertex, if you can traverse a series of edges that eventually lead you back to the starting vertex without retracing any edge, you have formed a cycle.

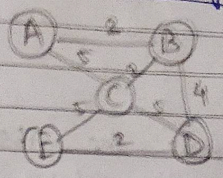


Cycle in a graph

Directed graph



Undirected graph



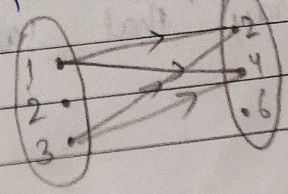
ii) Which law of inference states that "if A implies B, and B implies C, then A implies C"?
→ Transitive law.

iii) What is symm. diff. of set A and B?
→ Set which contains the elements which are either in set A or in set B but not in both.
$A \Delta B = (A-B) \cup (B-A)$

iv) How many reflexive rel^n. are possible on a set with n elements?
→ $N = 2n(n-1)$

v) What do you mean by a relation?
→ A set is a collection of well defined objects of particular kind. Rel^n is always studied b/w 2 sets.
Relation → defined as an collection of ordered pairs (a, b) where a belongs to the elements from set A and b from set B and the ~~fun~~ rel^n. is from A to B but not vice-versa.

ex:-



$x) : x^2$ is not bijective

$x_1 : x_2$

co-domain R,

-ve real no.

set R has no

$= \sqrt{\log \frac{4x - x^2}{3}}$,

$x^2 \geq 0$

]. Ans.

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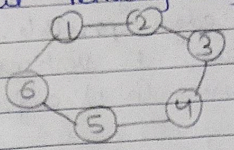
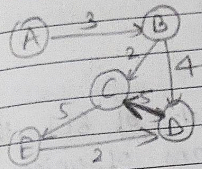
(1) i) What is a cycle in a graph?

→ Starting from a particular vertex, if you can traverse a series of edges that eventually lead you back to the starting vertex without retracing any edge, you have formed a cycle.

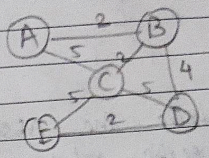


Cycle in a graph.

Directed graph



Undirected graph



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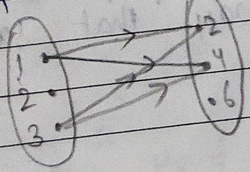
ex:-



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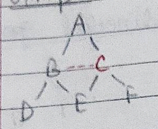
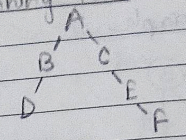
v) What is a Spanning tree in a graph?

→ It is a subgraph that includes all the vertices of the original graph & is a tree.
It connects all the vertices together without any cycles, using the min. no. of edges possible.
If original graph has n vertices
Spanning tree will have (n-1) edges.

Graph



Spanning tree



vii) Write the Absorption laws in respect of SET theory.

→ Specific relationship b/w the operations of union & intersection.

1. Union - Intersection Absorption Law :- $A \cup (A \cap B) = A$
2. Intersection - Union Absorption Law :- $A \cap (A \cup B) = A$

viii) Write the contrapositive of $\sim p \rightarrow \sim q$.

→ Reverse the hypothesis →
$\sim p$ and the conclusion $\sim q$, making it $\sim q \rightarrow \sim p$.

→ Negate both parts →
The negation of $\sim q$ is q, and negation of $\sim p$ is p.
Thus, contrapositive of $\sim p \rightarrow \sim q$ is $\boxed{q \rightarrow p}$.

ix) What is the inverse of $p \rightarrow \sim q$?

→ $\sim p \rightarrow q$.

x) K6 is a planar graph. T or F

→ True. Planar graph is a graph that can be drawn on a plane without any edges crossing each other.
∵ K6 is complete graph on 6 vertices, it can be drawn in a way that satisfies the conditions of planarity.

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xi) What is cantors ...
It is a mathema...
set of real no. ...
a greater cardinal...

xii) De. Morgan's la...
It relates the ...
sets and vice...
1. Complement of ...
intersection of ...
A ...
2. Complement o...
complements :- ...

(2) What is an ...
of a function ...
→ An inverse fu...
the original ...
an input x ...
as f^{-1}, ma...
If f(...
then f^{-1}...
To find in...
1. Start with...
2. Swap the...
3. Solve for...
4. Rename...

Example
Let...

Step :- 1.
Step :- 2.
Step :- 3.
Step :- 4 :

So, in...

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xi) What is cantor's diagonal argument?

It is a mathematical proof used to demonstrate that the set of real no. is uncountably infinite, meaning it has a greater cardinality than the set of natural no.

xii) De. Morgan's law of set operations.

It relates the complement of a union & intersection of sets and vice-versa.

1. Complement of the union of 2 sets is eq. to the intersection of their complements :-

$$(A \cup B)' = A' \cap B'$$

2. Complement of intersection of 2 sets = union of their complements :- $(A \cap B)' = A' \cup B'$

(2) What is an inverse function? How do you obtain inverse of a function? Explain with an example.

→ An inverse function is a function that reverses the effect of the original function. If you have a func^n f that maps an input x to an output y, the inverse func^n, denoted as f^{-1}, maps y back to x.

If $f(x) = y$

then $f^{-1}(y) = x$.

To find inverse of a function :-

1. Start with the eq^n of the original func^n. $y = f(x)$
2. Swap the variables x and y. Means :- $x = f(y)$
3. Solve for y to express y in terms of x.
4. Rename y as $f^{-1}(x)$ to denote the inverse func^n.

Example

Let $f(x) = 2x + 3$

Step :-1. $y = 2x + 3$

Step :-2. Swap x and y : $x = 2y + 3$

Step :-3. Solve for y : $x = 2y + 3 \Rightarrow x - 3 = 2y$

$\Rightarrow y = \dfrac{x-3}{2}$

Step :-4. Write the inverse func^n.

$$f^{-1}(x) = \frac{x-3}{2}$$

So, inverse of $f(x) = 2x+3$ is $f^{-1}(x) = \dfrac{x-3}{2}$. Ans

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(3) Five speakers A, B, C, D, E speak in a meeting one after the other. Find the prob. that A speaks before B.

→ As there are 5 speakers & A will always speaks before B.

∴ keep A in 1^{st} posn.

A - - - -

Remaining 4 will be arranged in $^4P_4$ ways.

$^4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4!}{1} = 4 \times 3 \times 2 \times 1 = 24.$

Probability $= 1/5$.

(4) Prove by induction: n^2+n is even where n is a +ve integer.

→ $P(n) = n^2+n$

$P(1) = 1^2+1 = 2 = 0 \pmod 2$, true for $P(1)$

Inductive step for $P(n+1)$:

$P(n+1) = (n+1)^2 + (n+1) = n^2+2n+1+n+1 = n^2+n+2(n+1)$

Hence for n^2+n is an even integer

$\forall\ n \in N$ (proved)

(5) There are 5 white balls, 4 red balls and 3 blue balls in a box. If you draw 2 balls at random from the box, what is the prob. that both are either white or red?

→ ~~P(drawing 2 white balls) $= \frac{5}{8} \times \frac{4}{7}$~~

Box contains 5 white balls, 4 red balls, 3 blue balls. Total = 12 balls.

No. of ways to choose 2 balls out of 12 is given by:- $^{12}C_2 = \frac{12!}{2!(12-2)!} = \frac{12 \times 11 \times 10!}{2 \times 10!} = 66$

- No. of ways to choose 2 white balls from 5 white balls

$^5C_2 = \frac{5!}{2!(3!)} = \frac{5 \times 4}{2} = 10$

- No. of ways to choose 2 red balls from 4 red balls

$^4C_2 = \frac{4!}{2!(2!)} = \frac{4 \times 3}{2} = 6.$

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No. of ways to ... $^3C_2 = \frac{3 ...}{2! ...}$

No. of ways ... balls & 1 ... $5C_1 \times {}^4C_1$

Total no. ...

∴ Req. P...

(6) In how m... 5 friends ...

→ 1st consider ... they are in... invited.

For each ... (invite or ...

Thus, ... all 5 ...

∵ We n... one frien...

∴ Req. ...

(7a) Define ... set th... perform... In class... of dist... for ex - ... be re... Set ... 1. Union (... element... • Ex:- ...

No. of ways to choose 2 blue balls from 3 blue balls
$^3C_2 = \frac{3!}{2!(1!)} = \frac{3\times2}{2} = 3$.

No. of ways to choose 1 white ball from 5 white balls & 1 red ball from 4 red balls:-
$^5C_1 \times {}^4C_1 = 5\times4 = 20$
Total no. of favourable outcomes $= 10+6+20 = 36$.
$\therefore$ Req. $P = \frac{\text{no. of fav. outcomes}}{\text{total no. of outcomes}} = \frac{36}{66} = \frac{6}{11}$ Ans.

(6) In how many way you can invite one/more of your 5 friends to your birthday party?
→ 1st consider that each friend has 2 possibilities: either they are invited to the ~~tea~~ B'day party or they are not invited.
For each of the 5 friends, this gives us 2 choices (invite or don't invite)
Thus, total no. of ways to make choice for all 5 friends :-
$2^5 = 32$

$\because$ We need to count only the scenarios where atleast one friend is invited, we subtract this one scenario:-
$32-1 = 31$
$\therefore$ Req. no. of ways $= \boxed{31}$.

(7a) Define a set with proper ex. acc. to the classical set theory. What are the diff. set operations usually performed - explain with ex.
In classical set theory, a set is defined as a collection of distinct objects, considered as an object in its own right.
For ex - the set of vowels in the English alphabet can be represented as $V = \{a,e,i,o,u\}$.
Set Operations :-
1. Union ($\cup$) → Union of 2 sets: A and B is the set of elements that are in either A or B or in both.
· Ex:- Let $A = \{1,2,3\}$ Let $B = \{3,4,5\}$
Then, $A\cup B = \{1,2,3,4,5\}$

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2. Intersection($\cap$) - Intersection of 2 sets A and B is the set of elements that are in both A and B.
ex:- let $A = \{1, 2, 3\}$ let $B = \{3, 4, 5\}$
Then, $A \cap B = \{3\}$

3. Difference - The diff. of sets A and B (also known as the complement of B in A) is the set of elements that are in A but not in B.
ex:- let $A = \{1, 2, 3\}$ let $B = \{3, 4, 5\}$
$A - B = \{1, 2\}$ and $B - A = \{4, 5\}$

4. Symmetric Difference (Δ) - The symm. diff. of two sets A and B is the set of elements that are in either of the sets A or B but not in their intersection.
let $A = \{1, 2, 3\}$ $B = \{3, 4, 5\}$
Then, $A \Delta B = \{1, 2, 4, 5\}$

5. Complement (A^c) - The complement of a set A is the set of all elements in the universal set U that are not in A.
Ex:- $U = \{1, 2, 3, 4, 5\}$ let $A = \{1, 2, 3\}$
then, $A^c = \{4, 5\}$

6. Cartesian product ($\times$) - Cartesian product of two sets A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
Ex:- Let $A = \{1, 2\}$ let $B = \{x, y\}$
Then, $A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$

b) What do you mean by finite & infinite set? Also, explain the concepts of finite countable sets, infinite countable sets and infinite uncountable sets with proper example.

→ Finite set → set that contains a specific no. of elements.
→ no. of elements can be counted & is a finite no.

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ex:- Set of vowels in English alphabet : $\{a, e, i, o, u\}$. This set has 5 elements.

Infinite sets → set that has an unlimited no. of elements. These sets don't have a finite no. of elements & can't be completely listed.

Finite countable sets → have limited no. of elements, we have count them.

ex → Set of N less than 10. :- $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Infinite countable sets → Infinite set where the elements can be put into a one-to-one correspondence with the N. We can count the elements even though there are infinitely many of them.

ex:- Set of all N :- $\{1, 2, 3, 4, \dots\}$

Set of all Z :- $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Infinite uncountable sets → An infinite set where the elements can't be put into a one-to-one correspondence with the N. There are too many elements to count, even in principle.

ex. Set of all real no. b/w 0 and 1. This set ~~indicates~~ includes all possible decimal expansions b/w 0 and 1, which is too vast to be matched one-to-one with the N.

(8a) How many non-negative integral solⁿ. are there of the eqⁿ. $x_1 + x_2 + x_3 + x_4 = 20$?

→

b) Show that no. of prime no. is infinite.

→ Let there are only finitely many prime numbers.
Let the finite set of all prime numbers be $\{p_1, p_2, p_3, \ldots, p_n\}$

consider the number P defined as the product of all these prime no. + 1.

$$P = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$$

By construction, P is > 1 and is not divisible by any of the prime no. in the set $\{p_1, p_2, p_3, \ldots p_n\}$

This is bcoz, if you divide P by any p_i (where $1 \leq i \leq n$), the remainder will be 1.

→ If P is prime, then ~~it~~ is a new prime no. not in our original finite list of primes.

→ If P is composite, then it must have a prime divisor.

∵ Our assumption leads to a contradiction, the no of prime numbers must be infinite.

(9a) Show that universal quantifiers distributes over conjunction and existential quantifier distributes over disjunction.

→ Universal quantifier distributes over conjunction:-

To show that :- $\forall x (P(x) \land Q(x)) \equiv (\forall x P(x)) \land (\forall x Q(x))$:

Left to Right ($\Rightarrow$):

Suppose, $\forall x (P(x) \land Q(x))$ is true.

This means, for every x, $P(x) \land Q(x)$ is true.

Then, both $P(x)$ & $Q(x)$ must be true for every x.

∴ $\forall x P(x)$ is true, and $\forall x Q(x)$ is true.

hence, $(\forall x P(x)) \land (\forall x Q(x))$ is true.

Right to left ($\Leftarrow$):

Suppose, $(\forall x P(x)) \land (\forall x Q(x))$ is true.

This means, $\forall x P(x)$ is true and $\forall x Q(x)$ is true.

If $\forall x P(x)$ is true, $P(x)$ is true for every x.

If $\forall x Q(x)$ is true, then $Q(x)$ is true for every x.

∴ for every x, both $P(x)$ and $Q(x)$ are true, i.e., $P(x) \land Q(x)$ is true.

Hence, $\forall x (P(x) \land Q(x))$ is true.

Existential Quantifier Distributes Over Disjunction:-
~~Left~~ To Show that: $\exists x(P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$:
Left to Right ($\Rightarrow$):
Suppose $\exists x(P(x) \vee Q(x))$ is true.
This means there exists some x such that $P(x) \vee Q(x)$ is true.

If $P(x) \vee Q(x)$ is true for some x, then either $P(x)$ is true or $Q(x)$ is true for that particular x.
$\therefore$ either $\exists x P(x)$ is true or $\exists x Q(x)$ is true.
Hence, $(\exists x P(x)) \vee (\exists x Q(x))$ is true.
Right to Left ($\Leftarrow$):
Suppose, $(\exists x P(x)) \vee (\exists x Q(x))$ is true.
This means, either $\exists x P(x)$ is true or $\exists x Q(x)$ is true.
If $\exists x P(x)$ is true, then $P(x)$ is true.
If $\exists x Q(x)$ is true, then $Q(x)$ is true.
In either case, there exists some x such that $P(x) \vee Q(x)$ is true.
Hence, $\exists x(P(x) \vee Q(x))$ is true.

Thus we proved:- (i) $\forall x(P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$
(ii) $\exists x(P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$

b) Income tax evasion:-
Brown: Jones is guilty & Smith is innocent.
Jones: If Brown is guilty, then so is Smith.
Smith: I am innocent, but at least one of others is guilty.
Assuming everyone told the truth. Who is/are guilty/innocent
$\rightarrow$

(10) a) What do you mean by Eulerian and Hamiltonian walks?

→ Eulerian & Hamiltonian walks are concepts from graph theory used to model pairwise relations b/w objects.

1. <u>Eulerian walk</u>:

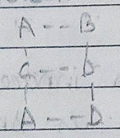
→ a walk in a graph that visits every edge exactly once.

→ if the walk starts & ends at the same vertex, it is called an Eulerian circuit or Eulerian cycle.

→ A connected graph has an Eulerian circuit if and only if every vertex has an even degree.

It has an Eulerian path if and only if exactly 0 or 2 vertices have an odd degree.

Eulerian circuit



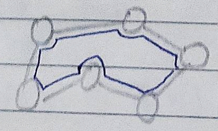
A → B → D → C → A → D → C → A

2. <u>Hamiltonian walks</u>:-

→ a walk that visits every vertex ~~at~~ exactly once.

→ ~~op~~ ~~a~~ the walk starts & ends at the same vertex, it is called Hamiltonian circuit / cycle.

→ There is no simple characterization like Eulerean paths for Hamiltonian paths, and determining whether a Hamiltonian path or cycle exists in a graph is generally more complex (NP-complete)



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b) What is vertex colouring and colouring of edges in graph theo?

→ Vertex colouring & edge colouring are methods of assigning colors to elements of a graph subject to certain contraints.

Vertex colouring →

Assignment of colours to the vertices of a graph such that no 2 adjacent vertices share the same color.

Goal is to use the min. no. of colors needed to achieve this condition, known as chromatic no. of the graph.

Given a graph $G = (V, E)$, a vertex colouring is a function $f: V \to C$ where C is set of colors, such that, $f(u) \neq f(v)$ for every edge $(u, v) \in E$.

Ex:-

For a simple Δ graph, the chromatic no. is 3, as each vertex must be a diff. color to ensure no 2 adjacent vertices share the same color.

Ex:. ν assignment in mobile networks

Edge colouring →

Assignment of colors to the edges of a graph such that no 2 edges that share the common vertex have the same color. Goal here is to use the min. no. of colors, known as chromatic index or edge chromatic no. of graph.

Given a graph $G = (V, E)$, an edge coloring is a function $g: E \to C$ where C is set of colors, such that $g(e_1) \neq g(e_2)$ for every pair of edges e_1 and e_2 that share a common vertex.

Ex:-

For a simple cycle graph with 4 vertices, the chromatic index is 2, as 2 colors are sufficient to color the edges so that no 2 edges sharing a vertex have the same color.

Ex:. assignment of ν to radio links, traffic light phases in

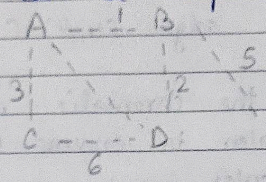
c) Define Minimal Spanning tree (MST) with an example.

→ A Minimal Spanning Tree (MST) is a subset of the edges in a connected, undirected graph that connects all the vertices together without any cycles and with

the min. possible total edge weight.
MST is imp. in various applications like network design, circuit design and clustering.
Ex:- consider a graph with 4 vertices (A,B,C,D) and the following edges with their respective weights

- A-B : 1
- A-C : 3
- A-D : 4
- B-C : 2
- B-D : 5
- C-D : 6

Graph can be represented as:-



(11) a) What is a bipartite graph? How do you determine if a graph is bipartite or not?
→ A bipartite graph is a type of graph where the set of vertices can be divided into 2 distinct & independent sets U and V such that every edge connects a vertex in U to a vertex in V.

A graph $G = (V, E)$ is bipartite if there exists a partition of the vertex set V into 2 disjoint sets U and V such that every edge $e \in E$ has one endpoint in U & other endpoint in V.

Determining → 2-coloring method
→ BFS or DFS.

- BFS or DFS:-

→ Start from any vertex and assign it the 1st color.
→ Traverse its neighbours, assigning them 2nd color.
→ Continue traversing, alternating colors for each level
→ If you encounter a vertex that needs to be colored the same as an adjacent vertex, the graph is not bipartite.

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Ex:- A — B — C
| | |
D — E — F

- Start with vertex A, color it with color 1.
- Color B with color 2, D with color 2
- Move to D, color E with color 1
- Move to E, color F with color 2
- No conflicts arise, so the graph is bipartite.

b) Show that trees are bipartite graphs.

→ We need to demonstrate that for any tree $T = (V, E)$, we can partition its vertex set V into 2 disjoint sets V_1 and V_2 such that no two vertices within the same set are adjacent.

- A tree is a connected, acyclic graph.
- A graph is bipartite if its vertices can be divided into 2 disjoint sets such that no 2 vertices within the same set share an edge.

BFS layering

- Perform a BFS starting from the root r. This process will layer the tree based on the dist. from the root.
  - Layer 0: vertex r
  - Layer 1: vertices directly connected to r
  - Layer 2: vertices connected to layer 1 vertices, & so on

Verification - consider any edge (u, v) in the tree T. By the nature of BFS, u and v must be in adjacent layers. Thus, if u is in an even-numbered layer, v will be in an odd-numbered layer, and vice-versa. This ensures that u and v are in diff. sets (V_1 and V_2).

c) Determine the chromatic polynomial of K_n.

→ The chromatic polynomial $P(K_n, k)$ of a complete graph K_n with n vertices is the function that counts the no. of ways to color the graph using k colors such that no 2 adjacent vertices have the same color.

For a complete graph K_n, each vertex is connected to every other vertex. To find the chromatic polynomial, we start by noting that the 1st vertex can be colored in k ways. The 2nd vertex must be colored in diff. color than the 1st, so it has $k-1$ choices. Similarly, the 3rd vertex has $k-2$ choices, and so on.

Thus, the chromatic polynomial $P(K_n, k)$ is given by the product of the choices:-

$$P(K_n, k) = k(k-1)(k-2)\cdots(k-n+1)$$

Or $P(K_n, k) = \prod_{i=0}^{n-1} (k-i)$

- For $n=1$:

  $P(K_1, k) = k$

- For $n=2$:

  $P(K_2, k) = k(k-1)$

- For $n=3$:

  $P(K_3, k) = k(k-1)(k-2)$

- For $n=4$:

  $P(K_4, k) = k(k-1)(k-2)(k-3)$