Ans to the Q No 1 (a)

Given

$$z_1 = x_1 \quad \text{and} \quad z_2 = x_3$$

$$\Rightarrow \dot{z}_1 = \dot{x}_1 = u_1 \qquad \dot{z}_2 = \dot{x}_3 = x_2 u_1$$

$$\dot{x}_3 = x_2 u_1$$

$$\Rightarrow x_2 = \frac{\dot{x}_3}{u_1} = \frac{\dot{z}_2}{u_1} = \frac{\dot{z}_2}{\dot{z}_1}$$

$$\Rightarrow \dot{x}_2 = \frac{d}{dt}\left(\frac{\dot{z}_2}{\dot{z}_1}\right)$$

$$= \frac{1}{\dot{z}_1}\ddot{z}_2 + \dot{z}_2\left(-\dot{z}_1^{-2}\right) \cdot \ddot{z}_1$$

$$u_1 = \dot{x}_1 = \dot{z}_1$$

$$u_2 = \dot{x}_2 = \frac{1}{\dot{z}_1}\ddot{z}_2 - \frac{\dot{z}_2 \cdot \ddot{z}_1}{\dot{z}_1^2}$$

So the system is differentially flat.

Ans to the Q 1 (b)

$$z_1(t) = \alpha_{11}\psi_1 + \alpha_{12}\psi_2 + \alpha_{13}\psi_3 + \alpha_{14}\psi_4$$

$$\Rightarrow z_1(t) = \alpha_{11} \times 1 + \alpha_{12} t + \alpha_{13} t^2 + \alpha_{14} t^3$$

$$\Rightarrow \dot{z}_1(t) = 0 + \alpha_{12} + 2\alpha_{13} t + 3\alpha_{14} t^2$$

$$z_2(t) = \alpha_{21}\psi_1 + \alpha_{22}\psi_2 + \alpha_{23}\psi_3 + \alpha_{24}\psi_4$$

$$\Rightarrow z_2(t) = \alpha_{21} \times 1 + \alpha_{22} t + \alpha_{23} t^2 + \alpha_{24} t^3$$

$$\Rightarrow \dot{z}_2(t) = 0 + \alpha_{22} + 2\alpha_{23} t + 3\alpha_{24} t^2$$

$$x = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{24} \end{bmatrix} \qquad B = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix}$$

$$A x = B$$

$z_1(0) = \alpha_{11}$ $\quad \dot{z}_1(0) = \alpha_{12}$ $\quad z_2(0) = \alpha_{21}$ $\quad \dot{z}_2(0) = \alpha_{22}$

$$z_1(T) = \alpha_{11} + \alpha_{12}T + \alpha_{13}T^2 + \alpha_{14}T^3$$

$$\dot{z}_1(T) = \alpha_{12} + 2\alpha_{13}T + 3\alpha_{14}T^2$$

$$z_2(T) = \alpha_{21} + \alpha_{22}T + \alpha_{23}T^2 + \alpha_{24}T^3$$

$$\dot{z}_2(T) = \alpha_{22} + 2\alpha_{23}T + 3\alpha_{24}T^2$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & T & T^2 & T^3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T & T^2 & T^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2T & 3T^2 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix} = \begin{array}{c} y(0) \\ \begin{bmatrix} 1 \\ 1 \\ -3 \\ 0 \\ 5 \\ 1 \\ 5 \\ 5 \end{bmatrix} \end{array} = \begin{array}{c} y(\pi) \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 10 \\ 1 \\ 5 \\ 16 \end{bmatrix} \end{array}$$

Ans to the Q No 1. (a)

$$z_1(t) = \alpha_{11}\psi_1 + \alpha_{12}\psi_2 + \alpha_{13}\psi_3 + \alpha_{14}\psi_4 + \alpha_{15}\psi_5 + \alpha_{16}\psi_6$$

$$\Rightarrow z_1(t) = \alpha_{11} + \alpha_{12}t + \alpha_{13}t^2 + \alpha_{14}t^3 + \alpha_{15}t^4 + \alpha_{16}t^5$$

$$\Rightarrow \dot{z}_1(t) = \alpha_{12} + 2\alpha_{13}t + 3\alpha_{14}t^2 + 4\alpha_{15}t^3 + 5\alpha_{16}t^4$$

$$z_2(t) = \alpha_{21} + \alpha_{22}t + \alpha_{23}t^2 + \alpha_{24}t^3 + \alpha_{25}t^4 + \alpha_{26}t^5$$

$$\dot{z}_1(t) = 0 + \alpha_{22} + 2\alpha_{23}t + 3\alpha_{24}t^2 + 4\alpha_{25}t^3 + 5\alpha_{26}t^4$$

$$Ax = B$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & T & T^2 & T^3 & T^4 & T^5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 & 4T^3 & 5T^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & T & T^2 & T^3 & T^4 & T^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2T & 3T^2 & 4T^3 & 5T^4 \end{bmatrix}}_{A,\ \cancel{12\times 8}\ 8\times 12} \underbrace{\begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{15} \\ \alpha_{16} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{24} \\ \alpha_{25} \\ \alpha_{26} \end{bmatrix}}_{x,\ 12\times 1} = \underbrace{\begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix}}_{B,\ 8\times 1}$$

Ans to the Q No 2 (a)

Given, $z_1 = x$ $\quad$ $z_2 = y$

$\dot{z}_1 = \dot{x}$ $\quad$ $\dot{z}_2 = \dot{y}$

$= v(t)\cos\theta(t)$ — (i) $\quad$ $= v(t)\sin\theta(t)$ — (ii)

(i) / (ii) $\rightarrow$

$$\cot\theta(t) = \frac{\dot{z}_1}{\dot{z}_2}$$

$$\theta(t) = \cot^{-1}\left[\frac{\dot{z}_1}{\dot{z}_2}\right]$$

$$\dot{z}_1 = v(t)\cos\theta(t)$$

$$v(t) = \frac{\dot{z}_1}{\cos\theta(t)}$$

$$= \frac{\dot{z}_1}{\cos\left[\cot^{-1}\left(\frac{\dot{z}_1}{\dot{z}_2}\right)\right]}$$

So the system is differentially flat.

Ans to the Q No 2 (b)

$$z_1(t) = \alpha_{11}\psi_1 + \alpha_{12}\psi_2 + \alpha_{13}\psi_3 + \alpha_{14}\psi_4$$

$$\Rightarrow z_1(t) = \alpha_{11} + \alpha_{12}t + \alpha_{13}t^2 + \alpha_{14}t^3$$

$$z_2(t) = \alpha_{21}\psi_1 + \alpha_{22}\psi_2 + \alpha_{23}\psi_3 + \alpha_{24}\psi_4$$

$$= \alpha_{21} + \alpha_{22}t + \alpha_{23}t^2 + \alpha_{24}t^3$$

$$\dot{z}_2(t) = \alpha_{22} + \alpha_{23}2t + \alpha_{24}3t^2$$

$$\dot{z}_1(t) = \alpha_{12} + \alpha_{13}2t + \alpha_{14}3t^2$$

$$\ddot{z}_1(t) = 2\alpha_{13} + 6\alpha_{14}t$$

$Ax = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & T & T^2 & T^3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T & T^2 & T^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & T & T^2 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{24} \end{bmatrix} = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ z_2(0) \\ \dot{z}_2(0) \\ z_1(T) \\ \dot{z}_1(T) \\ z_2(T) \\ \dot{z}_2(T) \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.5 \\ 5 \\ 0 \\ 5 \\ -0.5 \end{bmatrix}$$

Ans to the Q No 2(c)

$\ddot{z}_1(t) = 2\alpha_{13} + 6\alpha_{14}(t)$

$\ddot{z}_1(0) = 2\alpha_{13} = \ddot{x}_1(0)$

$\ddot{z}_2(t) = 2\alpha_{23} + 6\alpha_{24}t$

$\ddot{z}_2(0) = 2\alpha_{23} = \ddot{x}_2(0)$

$\dot{z}_1(0) = \alpha_{12}$

$\dot{z}_2(0) = \alpha_{22}$

$\theta(0) = -\pi/2$

$v(0) = 0.5$

$$\therefore \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos(-\pi/2) & -0.5\sin(-\pi/2) \\ \sin(-\pi/2) & 0.5\cos-(\pi/2) \end{bmatrix} \begin{bmatrix} a(t) \\ \omega(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\alpha_{13} \\ 2\alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a(0) \\ \omega(0) \end{bmatrix}$$

// $Ax = b$

$x = A^{-1}b$

$$\Rightarrow \begin{bmatrix} a(0) \\ \omega(0) \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2\alpha_{13} \\ 2\alpha_{23} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a(0) \\ \omega(0) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2\alpha_{13} \\ 2\alpha_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 2\alpha_{23} \\ 4\alpha_{13} \end{bmatrix}$$

Afterward, using $a(0)$, $w(0)$ calculate $v(0.1)$ and $\theta(0.1)$ also $x(0.1)$, $y(0.1)$ then we can also calculate $a(0.1)$, $w(0.1)$ from those values.

Rest are shown in the codes.

code link : https://colab.research.google.com/drive/1-TZ7SkObRmmGNmCy6x24_8QNYplUPB8D?usp=sharing