Tutorial - 6.

1) Expected payoff of player playing S

$= a(1-x) + bx$

where x fraction of population use strategy T.

Similarly, expected payoff of player playing T $= c(1-x) + dx$

S is evolutionary stable if for all sufficiently small values of $x > 0$,

$$a(1-x) + bx > c(1-x) + dx$$

As $x \to 0$,

if $a > c$, L.H.S > R.H.S

if $a < c$; L.H.S < R.H.S

if $a = c$, L.H.S > R.H.S when $b > d$

So, S is evolutionary stable when

i) $a > c$ or

ii) $a = c$, $b > d$.

2) $\begin{pmatrix} 2,2 & 1,2 \\ 2,1 & 2,2 \end{pmatrix}$; $A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$

$B = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$

NE →

(e_1, e_1)

(e_2, e_2)

Now check whether such strategies are evolutionary stable.

a strategy $(\hat{x}, \hat{y})$ is an ESS if for all $\varepsilon > 0$ and all $(x, y) \in (\Delta_A \setminus \{\hat{x}\}) \times (\Delta_B \setminus \{\hat{y}\})$

$$x \cdot A(\varepsilon y + (1-\varepsilon)\hat{y}) < \hat{x} \cdot A(\varepsilon y + (1-\varepsilon)\hat{y})$$

$$(\varepsilon x + (1-\varepsilon)\hat{x}) \cdot By < (\varepsilon x + (1-\varepsilon)\hat{x}) \cdot B\hat{y}$$

check for $(\hat{x}, \hat{y}) = (e_1, e_1)$

choose a point $\left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right)$,

$$x \cdot A\left(\varepsilon y + (1-\varepsilon)\hat{y}\right)$$

$$= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 - \varepsilon/2 \\ \varepsilon/2 \end{pmatrix}$$

$$= 2 - \frac{\varepsilon}{4} > 2 - \varepsilon/2$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 - \varepsilon/2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 - \varepsilon/2 \\ \varepsilon/2 \end{pmatrix}$$

$$= \hat{x} \cdot A\left(\varepsilon y + (1-\varepsilon)\hat{y}\right)$$

$\therefore$ (e_1, e_1) is not an ESS.

on the other hand for $(\hat{x}, \hat{y}) = (e_2, e_2)$

$$x \cdot A\left(\varepsilon y + (1-\varepsilon)\hat{y}\right) = \begin{pmatrix} x_1 \\ 1 - x_1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \varepsilon y_1 \\ 1 - \varepsilon y_1 \end{pmatrix}$$

$$= x_1(1 + \varepsilon y_1) + 2(1 - x_1)$$

$$< 2x_1 + 2(1 - x_1) = 2$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 + \varepsilon y_1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \varepsilon y_1 \\ 1 - \varepsilon y_1 \end{pmatrix} = \hat{x} \cdot A \left(\varepsilon y + (1-\varepsilon\right.$$

and

$$(\varepsilon x + (1-\varepsilon)\hat{x}).By =$$

$$\begin{pmatrix} \varepsilon x_1 \\ 1-\varepsilon x_1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ 1-y_1 \end{pmatrix}$$

$$= 2\varepsilon x_1 + (2-y_1)(1-\varepsilon x_1)$$

$$< 2\varepsilon x_1 + 2(1-\varepsilon x_1) = 2$$

$$= \begin{pmatrix} \varepsilon x_1 \\ 1-\varepsilon x_1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \hat{y}.B(\varepsilon x + (1-\varepsilon)\hat{x}).$$

$\therefore (e_2, e_2)$ is an ESS.

3 $$\begin{pmatrix} 1,-1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix}$$

Replicator eqn :-

$$\dot{x}_i = x_i((Ay)_i - x.Ay)$$

$$= x_i((-1)^{i+1} y_i - x_1 y_1 + x_2 y_2)$$

$$\dot{y}_j = y_j(((x^T B)_j - x.By)$$

$$= y_j((-1)^j x_j + x_1 y_1 - x_2 y_2)$$

$x_2 = 1 - x_1, \quad y_2 = 1 - x_1$

$\dot{x}_1 = x_1(1 - x_1)$

$\dot{y}_1 = -y_1(1 - y_1)$

$(0,1) \to$ sink

$(0,0), (1,1) \to$ saddle

$(1,0) \to$ sink

4)

	L	R
U	0, 0	(2, 2)
D	(1, 5)	(1, 5

Shares of U, D be $x, 1-x$

Shares of L, R be $y, 1-y$

Expected payoff of a U type individual $=$

$0 \cdot y + 2 \cdot (1-y)$

$= 2 - 2y$

" " " D " " $=$

$1 \cdot y + 1 \cdot (1-y)$

$= 1$

" " " L " " $= 5 - 5x$

" " " R " " $= 2x + 5(1-x)$

$= 5 - 3x$

Average $= x \cdot 2(1-y) + (1-x) \cdot 1$

Replicator eqns

$$\frac{dx}{dt} = x\left[2(1-y) - x\left[2(1-y)\right] - (1-x)\right]$$

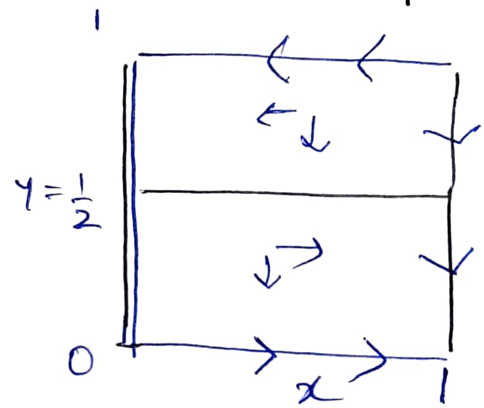
$$= x(1-x)(1-2y)$$

$$\frac{dy}{dt} = y(1-y)(-2x)$$

$y = \frac{1}{2}$

Black lines $\rightarrow \frac{dx}{dt} = 0$

Blue line $\rightarrow \frac{dy}{dt} = 0$



Rest points :

$$\{(0,y) \mid 0 \le y \le 1\} \cup \{(1,0)\} \cup \{(1,1)\}$$

if $1 > y \geq \frac{1}{2}$, $0 < x < 1 \Rightarrow \frac{dx}{dt}, \frac{dy}{dt} < 0$

$(1,0) \rightarrow$ stable rest point

(U, R)

observe that (U, R) is a NE as well.

$\rightarrow$ The observation that a stable rest point of the replicator dynamics is a strict NE (This is an iff condition for 2×2 bimatrix game)

$\rightarrow$ Try to prove.

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$$\begin{pmatrix} 0 & 10 & 1 \\ 10 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

NE :- $e_3, \left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Case 1:- e_3 is an ESS or not ?

for e_3 to be an ESS,

$$x \cdot A(\varepsilon x + (1-\varepsilon)e_3) < e_3 \cdot A(\varepsilon x + (1-\varepsilon)e_3)$$

Now since $x \cdot A e_3 = 1$ for all $x \in \Delta$,

$$\Rightarrow \quad x \cdot Ax < e_3 \cdot Ax$$

$$x \cdot Ax = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 10x_2 + x_3 \\ 10x_1 + x_3 \\ 1 \end{pmatrix}$$

$$= 20x_1x_2 + (1 + x_1 + x_2)x_3$$

$$e_3 \cdot Ax = 1$$

Choose $\tilde{x} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \in \Delta \setminus \{e_3\}$

then $\tilde{x} \cdot A\tilde{x} = 5 > 1 = e_3 \cdot A\tilde{x}$

$\therefore$ e_3 is not an ESS.

Case 2:- For $\tilde{x} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ to be an ESS,

$y \cdot Ay < \tilde{x} \cdot Ay$ holds for all $y \neq \tilde{x}$ in the neighbourhood of $\tilde{x}$.

let $y = \left(\frac{1}{2} + \delta_1, \frac{1}{2} + \delta_2, -\delta_1 - \delta_2\right)$, where $\delta_1 + \delta_2 \leq 0$, (not both 0 at a time)

Then,

$$y \cdot Ay = \left(\tfrac{1}{2}+\delta_1\right)\left(5+10\delta_2-\delta_1-\delta_2\right)+\left(\tfrac{1}{2}+\delta_2\right)\left(5+10\delta_1-\delta_1-\delta_2\right)-\delta_1-\delta_2$$

$$= \left(\tfrac{1}{2}+\delta_1\right)\left(5+9\delta_2-\delta_1\right)+\left(\tfrac{1}{2}+\delta_2\right)\left(5+9\delta_1-\delta_2\right)-\delta_1-\delta_2$$

$$= 5+8\delta_1+8\delta_2+18\delta_1\delta_2-\delta_1^2-\delta_2^2$$

Similarly,

$$\tilde{x} \cdot Ay = 5+5\delta_1+5\delta_2-\delta_1-\delta_2$$
$$= 5+4(\delta_1+\delta_2)$$

$$\Rightarrow \quad 4(\delta_1+\delta_2)+18\delta_1\delta_2-\delta_1^2-\delta_2^2<0 \rightarrow (1)$$

w.l.g assume $|\delta_2| \leq |\delta_1|$ and $\delta_2 = \lambda\delta_1$ with $|\lambda| \leq 1$

$$\Rightarrow \quad (\lambda+1)\delta_1 \leq 0$$

$\therefore$ either $\delta_1 < 0$ and $\lambda \in (-1,1]$

or $\delta_1 > 0$ and $\lambda = -1$.

So (1) becomes,

$$4(\lambda+1)\delta_1+18\lambda\delta_1^2-(\lambda^2+1)\delta_1^2<0$$

with $\delta_1<0,\ \lambda\in(-1,1]$ or $\delta_1>0,\ \lambda=-1$.

If $\delta_1<0,\ \lambda\in(-1,1]$ then (2) becomes,

$$4(\lambda+1) + 18\lambda\delta_1 - (\lambda^2+1)\delta_1 > 0$$

which obviously holds for $|\delta_1|$ small.

If $\delta_1 > 0$, $\lambda = -1$ then ② is equivalent to $-18\delta_1 - 2\delta_1 < 0$ which again holds.

$\therefore \tilde{x}$ is an ESS.

6). $B = A + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\dot{x}_i = x_i\left((Bx)_i - x\cdot Bx\right)$$

$$= x_i\left(\left(Ax + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x\right) - x\cdot Ax - x\cdot\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x\right)$$

$$= x_i\left(\left(Ax + \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix}\right) - x\cdot Ax - x_1(x_1+x_2+x_3)\right)$$

$$= x_i\left((Ax_i) + x_1 - x\cdot Ax - x_1\right)$$

$$= x_i\left((Ax)_i - x\cdot Ax\right).$$

The replicator dynamics of B is same as replicator dynamics of A.