UNIT-V
FUNCTIONS OF SEVERAL VARIABLES

Partial Differential Co-Efficient

- Let Z be a function in two or more variables, it can be differentiated with respect to each of the variable by assuming that it varies only with that variable and others treated as constants. These differential Co-efficients are Known as Partial differential Co-efficient. They are denoted by $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial u}{\partial t}$ etc..

Examples

- **If u $=e^x \sin y$,find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.**

$$\frac{\partial u}{\partial x} = e^x \sin y$$
$$\frac{\partial u}{\partial y} = e^x \cos y$$

- $\boldsymbol{u = \sin(x^2+y^2)}$

$$\frac{\partial u}{\partial x} = 2x\, \sin(x^2+y^2)$$

$$\frac{\partial u}{\partial y} = 2y\, \sin(x^2+y^2)$$

Chain Rule For Partial Differentiation

- $u = f(x,y)$ and $x = f(s), y = f(s)$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

- $\phi = f(u,v,w)$and $u = f(x), \mathrm{v} = f(x), w = f(x),$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}$$

Problems

- **If $\boldsymbol{\phi = f(y-z, z-x, x-y)}$, show that $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = 0$**

Given $\phi = f(y-z, z-x, x-y)$

Let $u = y - z$

$v = z - x$

$w = x - y$

By chain rule,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial\phi}{\partial u}(0) + \frac{\partial\phi}{\partial v}(-1) + \frac{\partial\phi}{\partial w}(1)$$

$$= \frac{\partial\phi}{\partial w} - \frac{\partial\phi}{\partial v} \quad \text{------------------------}(1)$$

Similarly,

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u} - \frac{\partial\phi}{\partial w} \quad \text{---------------------------}(2)$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial v} - \frac{\partial\phi}{\partial u} \quad \text{-------------------------}\ (3)$$

Adding (1) ,(2) and (3), we get,

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z} = 0$$

- **If u $=e^x \sin y$ where $x = st^2$ and y=s²t .Find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.**

Given that u $=e^x \sin y$, where $x = st^2$ and y =s²t

By chain rule,

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= e^x(t^2 \sin y + 2st \cos y)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= e^x(2st \sin y + s^2 \cos y).$$

- **If $v = (y - z)(z - x)(x - y)$, prove that $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0$.**

$$v = (y - z)(z - x)(x - y)$$

$$= (yz - z^2 - xy + xz)(x - y)$$

$$= xyz - xz^2 - x^2y + x^2z - y^2z + yz^2 + xy^2 - xyz$$

$$= xy^2 + yz^2 + x^2z - xz^2 - x^2y - y^2z$$

$$\frac{\partial v}{\partial x} = y^2 + 2xz - 2xy - z^2 \qquad \text{-------------}(1)$$

$$\frac{\partial v}{\partial y} = 2xy + z^2 - x^2 - 2yz \qquad \text{--------------}(2)$$

$$\frac{\partial v}{\partial z} = 2yz + x^2 - 2xz - y^2 \qquad \text{----------------}(3)$$

$Adding$ (1), (2) and (3)

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = y^2 + 2xz - 2xy - z^2 + 2xy + z^2 - x^2 - 2yz$$

$$+2yz + x^2 - 2xz - y^2 = 0$$

- **If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ (L6)**

Given $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

Let $u = f(p, q, r)$, where $p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right)$$

$$= \frac{1}{y}\frac{\partial u}{\partial p} - \frac{z}{x^2}\frac{\partial u}{\partial r} \qquad ------------------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial r}(0)$$

$$= \frac{1}{z}\frac{\partial u}{\partial q} - \frac{x}{y^2}\frac{\partial u}{\partial p} \qquad ------------------(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}\ (0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{\partial u}{\partial r} - \frac{y}{z^2}\frac{\partial u}{\partial q} \qquad ------------------ (3)$$

Therefore, from (1), (2) and (3), we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} + \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q}$$

$$= 0 = \text{RHS}$$

- **If $u = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$. (L6)**

Given $u = f(x, y)$, $x = r\cos\theta, y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\ (\cos\theta) + \frac{\partial u}{\partial y}\ (\sin\theta) \quad --(1)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}\ (r\cos\theta) \quad ------------------------------------(2)$$

Therefore, from (1) and (2), we get,

$$\text{RHS} = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\ (\cos\theta) + \frac{\partial u}{\partial y}\ (\sin\theta)\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}\ (r\cos\theta)\right)^2$$

$$= \left[\left(\frac{\partial u}{\partial x}\right)^2 \cos^2\theta + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\ \cos\theta\sin\theta + \sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{1}{r^2}\left[r^2\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\ r^2\cos\theta\sin\theta + r^2\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2 + sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 (cos^2\theta + sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 (sin^2\theta + cos^2\theta) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \text{LHS}$$

- **If $\boldsymbol{u = f(x-y, y-z, z-x)}$, prove that $\boldsymbol{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$**

Given $u = f(x-y, y-z, z-x)$

Let $u = f(p,q,r)$, where $p = x-y, q = y-z, r = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad ----------------------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$= \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \quad ----------------------(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \quad ----------------------(3)$$

Therefore, From (1), (2) and (3), we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} = 0 = \text{RHS}$$

- **If $\boldsymbol{u = f(x^2+2yz, y^2+2zx)}$,prove that**

$$\boldsymbol{(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0}$$

Given $u = f(x^2 + 2yz, y^2 + 2zx)$

Let $u = f(p, q)$, where $p = x^2 + 2yz, q = y^2 + 2zx$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}$$

$$= \frac{\partial u}{\partial p}(2x) + \frac{\partial u}{\partial q}(2z)$$

$$= 2x\frac{\partial u}{\partial p} + 2z\frac{\partial u}{\partial q} \quad ----------------------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y}$$

$$= \frac{\partial u}{\partial p}(2z) + \frac{\partial u}{\partial q}(2y)$$

$$= 2z\frac{\partial u}{\partial p} + 2y\frac{\partial u}{\partial q} \quad --------------------(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z}$$

$$= \frac{\partial u}{\partial p}(2y) + \frac{\partial u}{\partial q}(2x)$$

$$= 2y\frac{\partial u}{\partial p} + 2x\frac{\partial u}{\partial q} \quad ----------------------(3)$$

Therefore, From (1), (2) and (3), we get,

$$(y^2 - zx)\ \frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$$

$$= 2x\ \ (y^2 - zx)\frac{\partial u}{\partial p} + 2z\ \ (y^2 - zx)\frac{\partial u}{\partial q} + 2z(x^2 - yz)\frac{\partial u}{\partial p}$$

$$+ 2y(x^2 - yz)\frac{\partial u}{\partial q} + 2y(z^2 - xy)\frac{\partial u}{\partial p} + 2x(z^2 - xy)\frac{\partial u}{\partial q}$$

$$= \frac{\partial u}{\partial p}[2xy^2 - 2zx^2 + 2zx^2 - 2yz^2 + 2yz^2 - 2xy^2]$$

$$+ \frac{\partial u}{\partial q}[2zy^2 - 2xz^2 + 2yx^2 - 2zy^2 + 2xz^2 - 2yx^2]$$

$$= 0 = \text{RHS}$$

- **If $u = (x^2+y^2+z^2)^{1/2}$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$**

Given that $u = (x^2+y^2+z^2)^{1/2}$

By chain rule,

$$\frac{\partial u}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2x)$$

$$= \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}$$

By quotient rule,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\sqrt{(x^2 + y^2 + z^2)}\,(1) - x\,(\frac{1}{2})(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2x)}{(\sqrt{(x^2 + y^2 + z^2)})^2}$$

$$= \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \quad ------------------- (1)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \quad ------------------- (2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \qquad ------------------(3)$$

Adding (1)(2)and (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{x^2 + z^2 + y^2 + z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{2}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{2}{u}$$

- **If $\boldsymbol{z = f(u, v)}$ where $\boldsymbol{u = x + y}$ and $\boldsymbol{v = x - y}$,**

  show that $\boldsymbol{2\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}}$

Given $z = f(u, v)$

where $u = x + y, v = x - y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad ------------(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(-1)$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \qquad -------------(2)$$

Therefore, From (1) and (2), we get,

$$\text{RHS} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= 2\frac{\partial z}{\partial u} = \text{LHS} \ .$$

- **If $z = f(x, y)$, where $x = u + v, y = uv$, prove that $\mathbf{u}\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$.(L6)**

Given $z = f(x, y)$

where $x = u + v, y = uv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(v)$$

$$= \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} \quad ------------------------(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(u)$$

$$= \frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \quad -------------------(2)$$

Therefore, From (1) and (2), we get,

$$\text{LHS} = \mathrm{u}\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y}$$

$$= (u + v)\frac{\partial z}{\partial x} + (uv + uv)\frac{\partial z}{\partial y}$$

$$= (u + v)\frac{\partial z}{\partial x} + 2uv\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = \text{RHS}$$

- **Show that $\frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y} = 2logu$, where $logu = \frac{(x^3+y^3)}{(3x+4y)}$.**

Differentiating partially w.r.to x by quotient rule,

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{(3x+4y)(3x^2)-(x^3+y^3)(3)}{(3x+4y)^2}=\frac{6x^3+12x^2y-3y^3}{(3x+4y)^2}--(1)$$

Differentiating partially w.r.to y by quotient rule,

$$\frac{1}{u}\frac{\partial u}{\partial y}=\frac{(3x+4y)(3y^2)-(x^3+y^3)(4)}{(3x+4y)^2}=\frac{8y^3+9xy^2-4x^3}{(3x+4y)^2}--(2)$$

Multiplying (1) by x and (2) by y we get,

$$\frac{x}{u}\frac{\partial u}{\partial x}=\frac{6x^4+12x^3y-3xy^3}{(3x+4y)^2}\quad ---------------(3)$$

$$\frac{y}{u}\frac{\partial u}{\partial y}=\frac{8y^4+9xy^3-4x^3y}{(3x+4y)^2}-----------------(4)$$

Adding (4) and (5), we get,

$$\frac{x}{u}\frac{\partial u}{\partial x}+\frac{y}{u}\frac{\partial u}{\partial y}=\frac{6x^4+12x^3y-3xy^3+8y^4+9xy^3-4x^3y}{(3x+4y)^2}$$

$$=\frac{6x^4+8x^3y+6xy^3+8y^4}{(3x+4y)^2}$$

$$=\frac{6x(x^3+y^3)+8y(x^3+y^3)}{(3x+4y)^2}$$

$$=\frac{(6x+8y)(x^3+y^3)}{(3x+4y)^2}$$

$$=\frac{2(3x+4y)(x^3+y^3)}{(3x+4y)^2}$$

$$=\frac{2(x^3+y^3)}{(3x+4y)}$$

$$\frac{x}{u}\frac{\partial u}{\partial x}+\frac{y}{u}\frac{\partial u}{\partial y}=2logu$$

- **If $z = f(u,v)$, where $u = x^2-y^2$ and $v = 2xy$, prove that**

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right].$$

Given $z=f(u,v)$

where $u = x^2 - y^2, v = 2xy$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(2y)$$

$$= 2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v} \qquad ---------------(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x)$$

$$= -2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v} \qquad ---------------(2)$$

Therefore, From (1) and (2), Squaring and addding, we get,

$$\text{LHS} = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right)^2 + \left(-2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right)^2$$

$$= 4x^2\left(\frac{\partial z}{\partial u}\right)^2 + 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4y^2\left(\frac{\partial z}{\partial v}\right)^2 + 4y^2\left(\frac{\partial z}{\partial u}\right)^2 - 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4x^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$= \left(\frac{\partial z}{\partial u}\right)^2[4x^2 + 4y^2] + \left(\frac{\partial z}{\partial v}\right)^2[4x^2 + 4y^2]$$

$$= [4x^2 + 4y^2]\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \text{RHS}$$

<u>Total Differential Coefficient Of A Function</u>

Let Z be a function in two variables x and y.If Z is continuous, then the total differential coefficient of Z is given by $dz = \frac{\partial Z}{\partial x}\,dx + \frac{\partial Z}{\partial y}\,dy$

Examples

- **Find the total differential coefficient of the function $\boldsymbol{u = tan\,(3x - y + 2z)}$.**

Given, $u = tan\,(3x - y + 2z)$.

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$ ------------------(1)

$$\frac{\partial u}{\partial x} = 3sec^2(3x - y + 2z)$$

$$\frac{\partial u}{\partial y} = -sec^2(3x - y + 2z)$$

$$\frac{\partial u}{\partial z} = 2sec^2(3x - y + 2z)$$

Substituting in (1)

$$du = 3sec^2(3x - y + 2z)dx - sec^2(3x - y + 2z)dy + 2sec^2(3x - y + 2z)dz$$

$$du = sec^2(3x - y + 2z)(3dx - dy + 2dz)$$

- **Find $\boldsymbol{\frac{du}{dt}}$, if $\boldsymbol{u = log(x + y + z)}$, where $\boldsymbol{x = e^{-t}, y = sint, z = cost}$**

Given, $u = log(x + y + z)$,

where $x = e^{-t},\ y = sint,\ z = cost$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= \frac{1}{x+y+z}(-e^{-t}) + \frac{1}{x+y+z}(cost) + \frac{1}{x+y+z}(-sint)$$

$$= \frac{cost - sint - e^{-t}}{e^{-t} + sint + cost}$$

- **Find $\boldsymbol{\frac{du}{dt}}$, if $\boldsymbol{u = e^{xy}}$, where $\boldsymbol{x = (a^2 - t^2)^{1/2}, y = sin^3 t}$**

Given, $u = e^{xy}$, where $x = (a^2 - t^2)^{1/2}, y = sin^3 t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= ye^{xy}\frac{1}{2}(a^2 - t^2)^{\frac{1}{2}-1}(-2t) + xe^{xy}3sin^2tcost$$

$$= e^{xy}\left[\frac{-yt}{\sqrt{a^2-t^2}} + 3xsin^2tcost\right]$$

- **Find $\frac{du}{dt}$, if $u = x^3y^2 + x^2y^3$ $where\ x = at^2, y = 2at$.**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= (3x^2y^2 + 2xy^3)(2at) + (2x^3y + 3x^2y^2)(2a)$$

$$= (3a^2t^4 4a^2t^2 + 2at^2 8a^3t^3)(2at) + (2a^3t^6 2at + 3a^2t^4 4a^2t^2)(2a)$$

$$=4a^4t^5(3t + 4)(2at) + 4a^4t^6(t + 3)(2a)$$

$$= 8a^5t^6(3t + 4) + 8a^5t^6(t + 3)$$

$$= 8a^5t^6(3t + 4 + t + 3)$$

$$= 8a^5t^6(4t + 7)$$

- **Find $\frac{du}{dt}$, if u $= \frac{x}{y}$, where $x = e^t$, and $y = log\ t$. (L1)**

Given, $u = \frac{x}{y}$, where $x = e^t$, and $y = log\ t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{y}e^t + \left(\frac{-x}{y^2}\right)\frac{1}{t}$$

$$= \frac{1}{log\ t}\ e^t + \frac{-e^t}{(log\ t)^2}\frac{1}{t}$$

$$= \frac{e^t}{log\ t}\left(1 - \frac{1}{t\ log\ t}\right)$$

- **If $u = sin^{-1}(x - y)$, where $x = 3t$ and $y = 4t^3$. Show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.**

Given, $u = sin^{-1}(x - y)$

where $x = 3t$ and $y = 4t^3$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(3) - \frac{1}{\sqrt{1-(x-y)^2}}(12t^2) = \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

Now $1-(x-y)^2 = 1-(3t-4t^3)^2$

$$= 1 - t^2(3-4t^2)^2$$

$$= 1 - t^2(9-24t^2+16t^4)$$

$$= 1 - 9t^2 + 24t^4 - 16t^6$$

$$= 1 - t^2 - 8t^2 + 8t^4 + 16t^4 - 16t^6$$

$$= (1-t^2)(1-8t^2+16t^4)$$

$$= (1-t^2)(1-4t^2)^2$$

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

Implicit Function

A function of the form $f(x, y) = 0$ is called an implicit function.

e.g.1. $6x^3 + 12x^2y - 3y^3 = 0$

e.g.2. $x^3 + y^3 = 3ax^2y$

For an implicit function $f(x, y) = 0$,

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

- **Find $\frac{dy}{dx}$, when $x^3 + y^3 = 3ax^2y$**

Let $f(x, y) = x^3 + y^3 - 3ax^2y$.

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{\partial f}{\partial x} = 3x^2 - 6axy$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax^2$$

$$\frac{dy}{dx} = \frac{-(3x^2 - 6axy)}{3y^2 - 3ax^2}$$

$$= -\frac{3x(x - 6ay)}{3(y^2 - ax^2)} = \frac{-x(x - 6ay)}{(y^2 - ax^2)}$$

- **Find $\frac{dy}{dx}$, when $x^y + y^x = c$**

Let $u(x, y) = x^y + y^x - c$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$$\frac{\partial u}{\partial x} = yx^{y-1} + y^x logy$$

$$\frac{\partial u}{\partial y} = x^y logx + xy^{x-1}$$

$$\therefore \quad \frac{dy}{dt} = \frac{-(yx^{y-1} + y^x logy)}{x^y logx + xy^{x-1}}$$

Taylor's Theorem For A Function Of Two Variables.

If $f(x, y)$ and all its partial derivatives are finite and continuous at all points , then the Taylor series of $f(x, y)$ about the point (a, b) is given by

$$f(x, y) = f(a, b) + \frac{1}{1!}\left[(x - a)f_x(a, b) + (y - b)f_y(a, b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)\right.$$
$$\left.+(y-b)^2 f_{yy}(a,b)\right]+\cdots$$

- **Write the Taylor's series expansion of x^y near the point (1, 1) up to the second degree terms**

Taylor's series expansion of x^y near the point (1, 1) is given by

$$x^y = f(1,1)+\frac{1}{1!}\left[(x-1)\frac{\partial f(1,1)}{\partial x}+(y-1)\frac{\partial f(1,1)}{\partial y}\right]+$$
$$\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,1)}{\partial x^2}+2(x-1)(y-1)\frac{\partial^2 f(1,1)}{\partial x\partial y}+(y-1)^2\frac{\partial^2 f(1,1)}{\partial y^2}\right]$$
$$+\cdots$$

Function	Value at (1,1)
$f=x^y$	1
$f_x = yx^{y-1}$	1
$f_y = x^y logx$	0 $\quad [since\ log1 = 0]$
$f_{xx} = y(y-1)x^{y-2}$	0
$f_{xy} = yx^{y-1}logx + x^{y-1}$	1
$f_{yy} = x^y(logx)^2$	0

$$x^y = 1+\frac{1}{1!}[(x-1)1+(y-1)0]$$
$$+\frac{1}{2!}[(x-1)^2(0)+2(x-1)(y-1)+(y-1)^2(0)]$$
$$x^y = 1+\frac{1}{1!}[(x-1)]+\frac{1}{2!}[2(x-1)(y-1)]+\cdots$$

- **Write the Taylor series expansion of $e^x\log(1+y)$ in powers of x and y up to the terms of first degree.**

Taylor's series expansion of $e^x \log(1+y)$ near the point $(0,0)$ or Maclaurin's expansion is given by

$$e^x \log(1+y) = f(0,0) + \frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x} + (y-0)\frac{\partial f(0,0)}{\partial y}\right] + \cdots$$

Function	Value at (0,0)
$f = e^x \log(1+y)$	0 $[since \log 1 = 0]$
$f_x = e^x \log(1+y)$	0
$f_y = e^x \frac{1}{1+y}$	1

$$\therefore \quad e^x \log(1+y) = 0 + \frac{1}{1!}[(x)0 + (y)1] + \cdots = y$$

- **Expand x^2y+3y-2 in powers of $(x-1)$ and $(y+2)$ up to the third terms**

Taylor's series about the point (a, b)is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Taylor's series about the point $(1,-2)$is given by

$$x^2y + 3y - 2 = f(1,-2) + \frac{1}{1!}\left[(x-1)\frac{\partial f(1,-2)}{\partial x} + (y+2)\frac{\partial f(1,-2)}{\partial y}\right] +$$

$$\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,-2)}{\partial x^2} + 2(x-1)(y+2)\frac{\partial^2 f(1,-2)}{\partial x \partial y} + (y+2)^2\frac{\partial^2 f(1,-2)}{\partial y^2}\right] + \cdots$$

Function	Value at (1,-2)
$f = x^2y+3y-2$	$(1)^2(-2) + 3(-2) - 2 = -2 - 6 - 2 = -10$
$f_x = 2xy$	$2(1)(-2) = -4$
$f_y = x^2 + 3$	$(1)^2 + 3 = 4$

$f_{xx}=2y$	$2(-2)=-4$
$f_{xy}=2x$	$2(1)=2$
$f_{yy}=0$	0
$f_{xxx}=0$	0
$f_{xxy}=2$	2
$f_{xyy}=0$	0
$f_{yyy}=0$	0

Using the table values

$$x^2y+3y-2=-10+\frac{1}{1!}[(x-1)(-4)+(y+2)4]+\frac{1}{2!}[(x-1)^2(-4)+$$
$$2(x-1)(y+2)2+(y+2)^2 0\,]+\frac{1}{3!}[(x-1)^3(0)+3(x-1)^2(y+2)2+$$
$$+3(x-1)(y+2)^2(0)+(y+2)^3 0]+\cdots$$
$$=-10+\frac{1}{1!}[-4(x-1)+4(y+2)]+\frac{1}{2!}[-4(x-1)^2+4(x-1)(y+2)]$$
$$+\frac{1}{3!}[6(x-1)^2(y+2)]+\cdots$$
$$=-10-4[(x-1)-(y+2)]-2[(x-1)^2-(x-1)(y+2)\,]$$
$$+[(x-1)^2(y+2)]+\cdots$$

- **Expand $f(x,y)=x^2y+siny+e^x$ in Taylor's series about the point (1,π).(L2)**

Taylor's series about the point (a, b)is given by

$$f(x,y)=f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)\right.$$
$$\left.+(y-b)^2f_{yy}(a,b)\right]+\cdots$$

Taylor series about the point $(1,\pi)$ is

$$x^2y + siny + e^x = f(1,\pi) + \frac{1}{1!}\left[(x-1)\frac{\partial f(1,\pi)}{\partial x} + (y-\pi)\frac{\partial f(1,\pi)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,\pi)}{\partial x^2} + 2(x-1)(y-\pi)\frac{\partial^2 f(1,\pi)}{\partial x \partial y}\right.$$

$$\left. + (y-\pi)^2\frac{\partial^2 f(1,\pi)}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[(x-1)^3\frac{\partial^3 f(1,\pi)}{\partial x^3} + 3(x-1)^2(y-\pi)\frac{\partial^3 (1,\pi)}{\partial x^2 \partial y}\right.$$

$$\left. + \ 3(x-1)(y-\ \pi)^2\frac{\partial^3 f(1,\pi)}{\partial x \partial y^2} + \ (y-\pi)^3\frac{\partial^3 f(1,\pi)}{\partial y^3}\right] + \cdots$$

Function	$Value\ at\,(1,\pi)$
$f\ = x^2y + siny + e^x$	$f\ = \pi + e$
$f_x = 2xy + e^x$	$f_x = 2\pi + e$
$f_y = x^2 + cosy$	$f_y = 0$
$f_{xx} = 2y + e^x$	$f_{xx} = 2\pi + e$
$f_{xy} = 2x$	$f_{xy} = 2$
$f_{yy} = -siny$	$f_{yy} = 0$
$f_{xxx} = e^x$	$f_{xxx} = e$
$f_{xxy} = 2$	$f_{xxy} = 2$
$f_{xyy} = 0$	$f_{xyy} = 0$
$f_{yyy} = -cosy$	$f_{yyy} = 1$

$$x^2y + siny + e^x = \pi + e + \frac{1}{1!}[(x-1)(2\pi + e) + (y-\pi)(0)]$$

$$+\frac{1}{2!}[(x-1)^2(2\pi + e) + 2(x-1)(y-\pi)(2) + (y-\pi)^2(0)]$$

$$+\frac{1}{3!}[(x-1)^3e+3(x-1)^2(y-\pi)(2)+3(x-1)(y-\pi)^2(0)$$
$$+(y-\pi)^3(1)]+\cdots$$

$$\therefore x^2y+siny+e^x=\pi+e+\frac{1}{1!}[(x-1)(2\pi+e)]+$$
$$\frac{1}{2!}[(x-1)^2(2\pi+\ e)+4(x-1)(y-\pi)]$$
$$+\frac{1}{3!}[e(x-1)^3+6(x-1)^2(y-\pi)+(y-\pi)^3]+\cdots$$

- **Write the Taylor's series expansion of $e^x siny$ near the point $(-1,\pi/4)$ up to the third degree terms.**

Taylor's series about the point (a, b)is given by

$$f(x,y)=f(a,b)+\frac{1}{1!}\big[(x-a)f_x(a,b)+(y-b)f_y(a,b)\big]$$
$$+\frac{1}{2!}\big[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)$$
$$+(y-b)^2f_{yy}(a,b)\big]+\cdots$$

Taylor's series about the point $\left(-1,\frac{\pi}{4}\right)$ is given by

$$e^x siny=f\left(-1,\frac{\pi}{4}\right)+\frac{1}{1!}\left[(x+1)\frac{\partial f\left(-1,\frac{\pi}{4}\right)}{\partial x}+\left(y-\frac{\pi}{4}\right)\frac{\partial f\left(-1,\frac{\pi}{4}\right)}{\partial y}\right]$$
$$+\frac{1}{2!}\left[(x+1)^2\frac{\partial^2 f\left(-1,\frac{\pi}{4}\right)}{\partial x^2}+2(x+1)\left(y-\frac{\pi}{4}\right)\frac{\partial^2 f\left(-1,\frac{\pi}{4}\right)}{\partial x\partial y}+\left(y-\frac{\pi}{4}\right)^2\frac{\partial^2 f\left(-1,\frac{\pi}{4}\right)}{\partial y^2}\right]+\frac{1}{3!}\left[(x+1)^3\frac{\partial^3 f\left(-1,\frac{\pi}{4}\right)}{\partial x^3}+3(x+1)^2\left(y-\frac{\pi}{4}\right)\frac{\partial^3 f\left(-1,\frac{\pi}{4}\right)}{\partial x^2\partial y}+\right.$$
$$\left.3(x+1)\left(y-\frac{\pi}{4}\right)^2\frac{\partial^3 f\left(-1,\frac{\pi}{4}\right)}{\partial x\partial y^2}+(y-\frac{\pi}{4})^3\frac{\partial^3 f(-1,\frac{\pi}{4})}{\partial y^3}\right]+\cdots\ldots....$$

Function	Value at $\left(-1,\frac{\pi}{4}\right)$
$f = e^{x}siny$	$f = e^{-1}sin\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_x = e^{x}siny$	$f_x = e^{-1}sin\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_y = e^{x}cosy$	$f_y = e^{-1}cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xx} = e^{x}siny$	$f_{xx} = e^{-1}sin\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xy} = e^{x}cosy$	$f_{xy} = e^{-1}cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{yy} = -e^{x}siny$	$f_{yy} = -e^{-1}sin\frac{\pi}{4} = -\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xxx} = e^{x}siny$	$f_{xxx} = e^{-1}cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xxy} = e^{x}cosy$	$f_{xxy} = e^{-1}cos\frac{\pi}{4} = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{xyy} = -e^{x}siny$	$f_{xyy} = -e^{-1}sin\frac{\pi}{4} = -\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$
$f_{yyy} = -e^{x}cosy$	$f_{yyy} = -e^{-1}sin\frac{\pi}{4} = -\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)$

$$e^{x}siny = \frac{1}{e}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{1!}\left[(x+1)\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right) + \left(y-\frac{\pi}{4}\right)\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$+\frac{1}{2!}\left[(x+1)^{2}\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right) + 2(x+1)\left(y-\frac{\pi}{4}\right)\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right) + \left(y-\frac{\pi}{4}\right)^{2}\left(-\frac{1}{e}\frac{1}{\sqrt{2}}\right)\right]$$

$$+\frac{1}{3!}\left[(x+1)^3\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)+3(x+1)^2\left(y-\frac{\pi}{4}\right)\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)\right.$$

$$\left.+\ 3(x+1)\left(y-\frac{\pi}{4}\right)^2\left(-\frac{1}{e}\left(\frac{1}{\sqrt{2}}\right)\right)+\left(y-\frac{\pi}{4}\pi\right)^3\left(-\frac{1}{e}\frac{1}{\sqrt{2}}\right)\right]+\cdots$$

$$\Rightarrow\ e^x siny=\tfrac{1}{e}\left(\tfrac{1}{\sqrt{2}}\right)\left[1+(x+1)+\left(y-\tfrac{\pi}{4}\right)+\tfrac{1}{2!}\left\{(x+1)^2+\right.\right.$$

$$\left.2(x+1)\left(y-\tfrac{\pi}{4}\right)-\left(y-\tfrac{\pi}{4}\right)^2\right\}+\tfrac{1}{3!}\left\{(x+1)^3+\ 3(x+1)^2\left(y-\tfrac{\pi}{4}\right)-\right.$$

$$\left.\left.3(x+1)\left(y-\tfrac{\pi}{4}\right)^2-\left(y-\tfrac{\pi}{4}\right)^3\right\}+\cdots\right]$$

<u>Maclaurin's Expansion Of $f(x,y)$</u>

Taylor's series about the point (0,0) is known as Maclaurin's Expansion

Maclaurin's expansion of $f(x,y)$ is given by

$$f(x,y)=f(0,0)+\frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x}+(y-0)\frac{\partial f(0,0)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-0)^2\frac{\partial^2 f(0,0)}{\partial x^2}+2(x-0)(y-0)\frac{\partial^2 f(0,0)}{\partial x\partial y}+(y-0)^2\frac{\partial^2 f(0,0)}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[(x-0)^3\frac{\partial^3 f(0,0)}{\partial x^3}+3(x-0)^2(y-0)\frac{\partial^3 f(0,0)}{\partial x^2\partial y}\right.$$

$$\left.+\ 3(x-0)(y-0)^2\frac{\partial^3 f(0,0)}{\partial x\partial y^2}+(y-0)^3\frac{\partial^3 f(0,0)}{\partial y^3}\right]$$

- **Write down the Maclaurin's series for $sin(x+y)$.**

Maclaurin's expansion of $f(x,y)$ is given by

$$f(x,y)=f(0,0)+\frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x}+(y-0)\frac{\partial f(0,0)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-0)^2\frac{\partial^2 f(0,0)}{\partial x^2}+2(x-0)(y-0)\frac{\partial^2 f(0,0)}{\partial x\partial y}+(y-0)^2\frac{\partial^2 f(0,0)}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[(x-0)^3\frac{\partial^3 f(0,0)}{\partial x^3}+3(x-0)^2(y-0)\frac{\partial^3 f(0,0)}{\partial x^2\partial y}\right.$$

$$\left.+3(x-0)(y-0)^2\frac{\partial^3 f(0,0)}{\partial x\partial y^2}+(y-0)^3\frac{\partial^3 f(0,0)}{\partial y^3}\right]+\cdots$$

Function	Value at (0,0)
$f=\sin(x+y)$	0
$f_x=\cos(x+y)$	1
$f_y=\cos(x+y)$	1
$f_{xx}=-\sin(x+y)$	0
$f_{xy}=-\sin(x+y)$	0
$f_{yy}=-\sin(x+y)$	0
$f_{xxx}=-\cos(x+y)$	-1
$f_{xxy}=-\cos(x+y)$	-1
$f_{xyy}=-\cos(x+y)$	-1
$f_{yyy}=-\cos(x+y)$	-1

Substituting the table values,

$$\sin(x+y)=0+\tfrac{1}{1!}[(x-0)1+(y-0)1]\ +\tfrac{1}{2!}[(x-0)^2 0+$$
$$2(x-0)(y-0)0+(y-0)^2 0]+\tfrac{1}{3!}[(x-0)^3(-1)+$$
$$3(x-0)^2(y-0)(-1)+3(x-0)(y-0)^2(-1)+$$
$$(y-0)^3(-1)]+\cdots$$

$$\sin(x+y)=(x+y)-\tfrac{1}{3!}(x+y)^3+\cdots$$

- **Write down the Maclaurin's series for e^{x+y} .**

Maclaurin's expansion of e^{x+y} is given by

$$e^{x+y}\ =f(0,0)+\frac{1}{1!}\left[(x-0)\frac{\partial f(0,0)}{\partial x}+(y-0)\frac{\partial f(0,0)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-0)^2\frac{\partial^2 f(0,0)}{\partial x^2}+2(x-0)(y-0)\frac{\partial^2 f(0,0)}{\partial x\partial y}+(y-0)^2\frac{\partial^2 f(0,0)}{\partial y^2}\right.$$
$$\left.+\cdots\right]$$

Function	Value at (0,0)
$f=e^{x+y}$	1
$f_x=e^{x+y}$	1
$f_y=e^{x+y}$	1
$f_{xx}=e^{x+y}$	1
$f_{xy}=e^{x+y}$	1
$f_{yy}=e^{x+y}$	1

$$\therefore \quad e^{x+y}=1+\frac{1}{1!}[(x)1+(y)1]+\frac{1}{2!}[(x)^2 1+2(x)(y)1+(y)^2 1]+\cdots$$

$$\Longrightarrow \quad e^{x+y}=1+\frac{1}{1!}(x+y)+\frac{1}{2!}(x+y)^2+\cdots$$

Jacobian.

If $u(x,y)$ and $v(x,y)$ are functions in two variables x and y,then the Jacobian of u and v w.r.t x and y is given by the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$.

Properties Of Jacobian

1. $\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$
2. $\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}$
3. If $u(x,y)$ and $v(x,y)$ are functionally independent, then

   $$\frac{\partial(u,v)}{\partial(x,y)}=0$$

- **If $u = 2xy, v = x^2 - y^2, x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial(u,v)}{\partial(r,\theta)}$.**

By the property of jacobian,

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} \quad ------------(1)$$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$= -4y^2 - 4x^2$$

$$= -4(x^2 + y^2)$$

$$= -4(r^2cos^2\theta + r^2sin^2\theta)$$

$$= -4\,r^2 \quad ------------(2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= (rcos^2\theta + rsin^2\theta)$$

$$= r \quad ------------(3)$$

substituting (2)and (3) in (1)

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4\,r^2\,(r) = -4r^3$$

- **If $x = rcos\theta, y = rsin\theta$ and $z = \varphi$. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$.**

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} cos\theta & -rsin\theta & 0 \\ sin\theta & rcos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= cos\theta(rcos\theta - 0) + rsin\theta(sin\theta - 0) + 0$$

$$= rcos^2\theta + rsin^2\theta \;= r$$

- **If $x = \; u\,(1 + v), y \;=\; v\,(1 + u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.**

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (\,1 + v)\,(1 + u) - uv$$

$$= 1 + u + v + uv - uv$$

$$= 1 + u + v$$

- **If $x{=}e^r sec\theta, y = e^r\; tan\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$.**

$$Given,\; x{=}e^r\; sec\theta \quad \Longrightarrow \frac{\partial x}{\partial r} = e^r\; sec\theta; \quad \frac{\partial x}{\partial \theta} = e^r sec\theta\; tan\theta;$$

$$y = e^r\; tan\theta \quad \Longrightarrow \frac{\partial y}{\partial r} = e^r\; tan\theta; \quad \frac{\partial y}{\partial \theta} = e^r \sec^2\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$=\begin{vmatrix} e^r \sec\theta & e^r \sec\theta\tan\theta \\ e^r \tan\theta & e^r \sec^2\theta \end{vmatrix} = e^{2r}(\sec^3\theta) - e^{2r}(\sec\theta\tan^2\theta)$$

$$= e^{2r}\sec\theta(\sec^2\theta - \tan^2\theta) = e^{2r}\sec\theta$$

- **If $u = x^2, v = y^2$, prove that $\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} = 1$.**

If $u = x^2 \Rightarrow x = \sqrt{u}$ *and* $v = y^2 \Rightarrow y = \sqrt{v}$

$$\frac{\partial(x,y)}{\partial(u,v)}.\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{vmatrix} \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix}$$

$$= \frac{1}{4\sqrt{uv}}.4xy = \frac{1}{4\sqrt{x^2y^2}}.4xy = \frac{1}{4xy}.4xy = 1$$

- **If $u = xyz, v + xy + yz + zx, w = x + y + z.$ Find $\frac{\partial(u,v,w)}{\partial(x,y,w)}$**

$$\frac{\partial(u,v,w)}{\partial(x,y,w)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(x^2z + zxy - x^2y - xyz) - 1(xyz + y^2z - xy^2 - xyz) + 1(xyz + z^2y - xyz - xz^2)$$

$$= x^2(z-y) - y^2(z-x) + z^2(y-x)$$

- **If $x = uv$, $y = \frac{u}{v}$, show that $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$**

Given, $x = uv$ --------------------(1)

$y = \frac{u}{v}$ -------------------(2)

(1) multiplied with (2)

$xy = uv.\frac{u}{v} = u^2$ ------------(3)

(1) divided with (2)

$\frac{x}{y} = \frac{uv}{\frac{u}{v}} = v^2$ -----------------(4)

From (1)&(2) ,

$\frac{\partial x}{\partial u} = v$

$\frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = \frac{1}{v}$

$\frac{\partial y}{\partial v} = \frac{-u}{v^2}$

From (3)&(4)

$2u\frac{\partial u}{\partial x} = y$

$2u\frac{\partial u}{\partial y} = x$

$2v\frac{\partial v}{\partial x} = \frac{1}{y}$

$2v\frac{\partial v}{\partial y} = -\frac{x}{y^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} \begin{vmatrix} \frac{y}{2u} & \frac{x}{2u} \\ \frac{1}{2vy} & \frac{-x}{2vy^2} \end{vmatrix}$$

$$= \left[\left(v.\frac{-u}{v^2}\right) - \left(u.\frac{1}{v}\right)\right]\left[\left(\frac{y}{2u}.\frac{-x}{2vy^2}\right) - \left(\frac{x}{2u}.\frac{1}{2vy}\right)\right]$$

$$= \left[\frac{-u}{v} - \frac{u}{v}\right]\left[-\frac{xy}{4uvy^2} - \frac{x}{4uvy}\right] = \left[\frac{-2u}{v}\right]\left[-\frac{2x}{4uvy}\right]$$

$$= \frac{x}{yv^2} = \frac{x}{y\left(\frac{x}{y}\right)} = 1$$

- **If we transform from three dimensional Cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ). Show that the Jacobian of x, y, z with respect to r, θ, ϕ is $r^2 sin\theta$**

Spherical polar co-ordinates are ,

$$x = r\sin\theta cos\varphi$$

$$y = rsin\ \theta sin\varphi$$

$$z = rcos\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} sin\theta cos\varphi & rcos\theta cos\varphi & -rsin\theta sin\varphi \\ sin\theta sin\varphi & rcos\theta sin\varphi & rsin\theta cos\varphi \\ cos\theta & -rsin\theta & 0 \end{vmatrix}$$

$$= cos\theta[r^2 sin\theta cos\theta cos^2\varphi + r^2 sin\theta cos\theta sin^2\varphi]$$
$$+ rsin\theta[rsin^2\theta cos^2\varphi + rsin^2\theta sin^2\varphi]$$

$$= cos\theta r^2 sin\theta cos\theta[cos^2\varphi + sin^2\varphi] + rsin\theta rsin^2\theta[cos^2\varphi + sin^2\varphi]$$

$$= [cos^2\varphi + sin^2\varphi](r^2 sin\theta cos^2\theta + r^2 sin\theta sin^2\theta)$$

$$= r^2 sin\theta(cos^2\theta + sin^2\theta) = r^2 sin\theta$$

- **If $u = x + y + z, uv = y + z, uvw = z$, evaluate. $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.**

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Let $\quad u = x + y + z \qquad$ ---------------(1)

$uv = y + z$ ----------------(2)

$uvw = z$ ----------------(3)

Put (2) in (1) we get,

$$u = x + uv \quad \Rightarrow x = u - uv \quad \Rightarrow x = u(1-v)$$

Put (3) in (2) we get,

$$uv = y + uvw \quad \Rightarrow y = uv - uvw \quad \Rightarrow y = uv(1-w)$$

From (3) we get, $z = uvw$.

$$\therefore x = u(1-v) \quad \Rightarrow \frac{\partial x}{\partial u} = 1 - v, \frac{\partial x}{\partial v} = -u, \frac{\partial x}{\partial w} = 0$$

$$y = uv - uvw \quad \Rightarrow \frac{\partial y}{\partial u} = v(1-w); \ \frac{\partial y}{\partial v} = u(1-w); \ \frac{\partial y}{\partial w} = -uv$$

$$z = uvw \quad \Rightarrow \frac{\partial z}{\partial u} = vw; \quad \frac{\partial z}{\partial v} = uw; \quad \frac{\partial z}{\partial v} = uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} (1-v) & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$=(1-v)(u^2v - u^2vw + u^2vw) + u(uv^2 - uv^2w + uv^2w)$$

$$= u^2v - u^2v^2 + u^2v^2 = u^2v$$

Stationary Points.

Let $f(x,y)$ be a function in x and y. Then the points at which $\frac{\partial f}{\partial x} = 0 \ and \ \frac{\partial f}{\partial y} = 0$ are called stationary points. At these points the function takes an extreme value.

Maximum Value, Minimum Value And Extreme Value Of A Function Of Two Variables.

4. A function is said to have a maximum value at the point (a,b) if $f(a,b) > f(a+h, b+k)$ for all small values of h and k.
5. A function is said to have a minimum value at the point (a,b)

if $f(a,b) < f(a+h, b+k)$ for all small values of h and k.

6. A function is said to have an extreme value at the point (a,b) if it is either maximum or minimum at (a,b).

Define Saddle Point Of A Function $f(x,y)$.

Let $f(x,y)$ be a function in x and y.The point (a,b) is said to be a saddle point ,if the function is neither maximum nor minimum at that point

Working Rule To Find Maximum/ Minimum Value

- Find the stationary points (a,b)
- Find the values A= $\frac{\partial^2 f}{\partial x^2}$, B= $\frac{\partial^2 f}{\partial x \partial y}$, C= $\frac{\partial^2 f}{\partial y^2}$ and $\Delta = AC - B^2$ at all the stationary points.
- If $\Delta > 0$ and A or B> 0 at (a,b),Then the function has a minima at (a,b)
- If $\Delta > 0$ and A or B< 0 at (a,b),Then the function has a maxima at (a,b)
- If $\Delta < 0$,Then (a,b) is a saddle point.
- If Δ=0,Then the nothing can be decided.
- **Examine the stationary points of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ and also state their nature.**

Given that $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

To find stationary points

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow 3(x^2 - 1) = 0$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1 \text{ -----------------(1)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 12 = 0 \Rightarrow 3(y^2 - 4) = 0$$

$$\Rightarrow y^2 = 4$$

$$\Rightarrow y = \pm 2 \text{ -----------------(2))}$$

$\therefore$ The stationary points are $(-1,-2), (-1,2), (1,-2)$, and $(1,2)$

$A = \frac{\partial^2 f}{\partial x^2} = 6x$

B= $\frac{\partial^2 f}{\partial x \partial y} = 0$

C= $\frac{\partial^2 f}{\partial y^2} = 6y$

Points	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x \partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	$\Delta = AC - B^2$	Extremum
$(-1,-2)$	$-6 < 0$	0	$-12 < 0$	$72 > 0$	Maximum
$(-1,2)$	$-6 < 0$	0	12	$-72 < 0$	Saddle point
$(1,-2)$,	6	0	-12	$-72 < 0$	Saddle point
$(1,2)$	$6 > 0$	0	$12 > 0$	$72 > 0$	Minimum

The maximum value at $(-1,-2)$ is

$$f(x,y) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20 = 38$$

The minimum value at $(1,2)$ is

$$f(x,y) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$$
$$= 1 + 8 - 3 - 24 + 20 = 2$$

- **Examine $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values**

  To find stationary points :

  $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$

  $\Rightarrow x^2 + y^2 - 10x + 24 = 0$ --------------------------(1)

  $\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 30y = 0$

  $\Rightarrow \quad y(6x - 30) = 0$

$\Longrightarrow \quad y = 0 \text{ or } 6x - 30 = 0$

$\Longrightarrow \quad y = 0 \text{ or } x = 5$

Put $y = 0$ in (1)

$\Longrightarrow x^2 - 10x + 24 = 0$

$(x - 6)(x - 4) = 0 \Longrightarrow x = 4,6$

$\therefore$ For $y = 0$ *the points are* $(4,0)$ *and* $(6,0)$.

Let $x = 5$ in (1), we get, $25 + y^2 - 50 + 24 = 0$

$\Longrightarrow (y^2 - 1) = 0 \Longrightarrow y^2 = 1 \Longrightarrow y = \pm 1$

The points are $(5,1)$ $(5,-1)$

$\therefore$ The stationary points are $(4,0), (6,0), (5,1), (5,-1)$

$A = \frac{\partial^2 f}{\partial x^2} = 6x - 30$; $C = \frac{\partial^2 f}{\partial y^2} = 6x - 30$

$B = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$\Delta = AC - B^2$

POINTS	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x \partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	$\Delta = AC - B^2$	EXTREMUM
$(4,0)$	$-6<0$	0	$-6<0$	$36>0$	MAXIMA
$(6,0)$	$6>0$	0	$6>0$	$36>0$	MINIMA
$(5,-1)$	0	$-6<0$	0	$-36<0$	SADDLE POINT
$(5,1)$	0	$6>0$	0	$-36<0$	SADDLE POINT

- **Examine the function $f(x, y) = x^3y^2(12 - x - y)$ for extreme values**

Given that $f(x, y) = 12x^3y^2 - x^4y^2 - x^3y^3$

To find stationary points

$\frac{\partial f}{\partial x} = 0 \Longrightarrow 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Longrightarrow x^2y^2(36 - 4x - 3y) = 0$

$\Longrightarrow x = 0, y = 0 \text{ or } 36 - 4x - 3y = 0$

$\Longrightarrow 4x + 3y = 36$ -----------------(1)

$\frac{\partial f}{\partial y} = 0 \Longrightarrow 24x^3y - 2x^4y - 3x^3y^2 = 0 \Longrightarrow x^3y(24 - 2x - 3y) = 0$

$\Longrightarrow x = 0, y = 0 \text{ or } 24 - 2x - 3y = 0$

$\Longrightarrow 2x + 3y = 24$ ------------------(2)

Solving (1) and (2),

(2)x 2 $\Longrightarrow 4x + 6y = 48$ -------------------(3)

(1)$\Longrightarrow$ $4x + 3y = 36$ ------------------(4)

$(3) - (4) \quad \Longrightarrow \quad 3y = 12 \Longrightarrow y = 4 \quad and\ x = 6$

$x = 0\ in\ (1) \Longrightarrow y = 12 \Longrightarrow (0,12)$

$y = 0\ in\ (1) \Longrightarrow x = 9 \Longrightarrow (9,0)$

$x = 0\ in\ (2) \Longrightarrow y = 8 \Longrightarrow (0,8)$

$y = 0\ in\ (2) \Longrightarrow x = 12 \Longrightarrow (12,0)$

The stationary points are (0,0), (0,12), (9,0), (0,8),(12,0) *and* (6,4)

$$A = \frac{\partial^2 f}{\partial x^2} = 72xy^2 - 12x^2y^2 - 6xy^3$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 72x^2y - 8x^3y - 9x^2y^2$$

$$C = \frac{\partial^2 f}{\partial y^2} = 24x^3 - 2x^4 - 6x^3y$$

Points	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x \partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	$\Delta = AC - B^2$	Extremum
(0,0)	0	0	0	0	Nothing can be decided

(0,12)	0	0	0	0	Nothing can be decided
(9,0)	0	0	4374	0	Nothing can be decided
(0,8)	0	0	0	0	Nothing can be decided
(12,0)	0	0	0	0	Nothing can be decided
(6,4)	-2304	-1728	-2592	> 0	Maximum

The maximum value at (6,4) is

$f(x, y)=(6)^3(4)^2(12-6-4) = 6912$

__Lagrange's Method For Constrained Maxima And Minima__

Let $f(x, y, z)$ be the function whose maximum/ minimum to be found subject to the constraint $\varphi(x, y, z) = 0$.

By Lagrange's Method,

- Form the auxiliary function $F = f + \lambda\,\varphi$,where λ is the Lagrangian multiplier.
- Solve for (x, y, z) from the equations

$\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and $\varphi(x, y, z) = 0$ to find maximum/ minimum value of $f(x, y, z)$

- **Examine the minimum value of $x^2+y^2+z^2$, when $xyz = a^3$.**

Let $f(x,y,z) = x^2 + y^2 + z^2$ and $\varphi(x,y,z) = xyz - a^3$

By Lagrange's method $F = f + \lambda\varphi$

$$(i.e)\ F = (x^2 + y^2 + z^2) + \lambda(xyz - a^3)$$

$$\frac{\partial \mathrm{F}}{\partial x} = 0 \implies 2x + \lambda \mathrm{yz} = 0$$

$$\implies \lambda = -\frac{2x}{yz} \quad \text{-----------------------------}(1)$$

$$\frac{\partial \mathrm{F}}{\partial \mathrm{y}} = 0 \implies 2y + \lambda xz = 0$$

$$\implies \lambda = -\frac{2y}{xz} \quad \text{-----------------------------}(2)$$

$$\frac{\partial \mathrm{F}}{\partial \mathrm{z}} = 0 \implies 2\mathrm{z} + \lambda \mathrm{xy} = 0$$

$$\implies \lambda = -\frac{2z}{xy} \quad \text{-----------------------------}(3)$$

From (1) and (2), we get

$$-\frac{2x}{yz} = -\frac{2y}{xz}$$

$$\implies x^2 = y^2$$

$$\implies x = y \quad \text{-----------------------------}(4)$$

From (2) and (3) We get

$$-\frac{2y}{xz} = -\frac{2z}{xy}$$

$$\implies y^2 = z^2$$

$$\implies y = z \quad \text{-------------------------------}(5)$$

From (4) and (5), we get

$$x = y = z$$

Using this in $xyz = a^3$, we get,

$$x(x)(x) = a^3$$

$\Longrightarrow \; x^3 = a^3$

$\Longrightarrow \quad x = a$

$\therefore x = y = z = a$

$\therefore \; (a, a, a)$ *is a point of minima and* $f_{min} = a^2 + a^2 + a^2 = 3a^2$

- **The temperature at any point (x, y, z) in space is given by $T = kxyz^2$, where k is a constant. Determine the highest temperature on the surface of the sphere $x^2+y^2+z^2 = a^2$**

Given Temperature $T = kxyz^2$,

such that $x^2+y^2+z^2 = a^2$

$$f(x, y, z) = kxyz^2$$

$$\varphi(x, y, z) = x^2 + y^2 + z^2 - a^2$$

By Lagrange's Method,

Let $F = f + \lambda\varphi$, where λ is Lagrangian Multiplier.

$$\Longrightarrow \quad F = kxyz^2 + \lambda(x^2 + y^2 + z^2 - a^2)$$

$$\therefore \quad \frac{\partial F}{\partial x} = 0 \quad \Longrightarrow kyz^2 + 2\lambda x = 0,$$

$$\Longrightarrow \lambda = \frac{-kyz^2}{2x} \quad ------------------(1)$$

$$\frac{\partial F}{\partial y} = 0 \quad \Longrightarrow kxz^2 + 2\lambda y = 0,$$

$$\Longrightarrow \lambda = \frac{-kxz^2}{2y} \quad ----------------------(2)$$

$$\frac{\partial F}{\partial z} = 0 \quad \Longrightarrow 2kxyz + 2\lambda z = 0,$$

$$\Longrightarrow \lambda = \frac{-2kxyz}{2z} = -kxy \quad ------------- (3)$$

$From\ (1)\ and\ (2), we\ get,$

$$\frac{-kyz^2}{2x} = \frac{-kxz^2}{2y}$$

$\Rightarrow \quad y^2z^2 = x^2z^2$

$\Rightarrow \quad y^2 = x^2$

$\Rightarrow \quad x = y \quad -------------(4)$

$From\ (2)\ and\ (3), we\ get,$

$$\frac{-kxz^2}{2y} = -kxy$$

$\Rightarrow \quad xz^2 = 2xy^2$

$\Rightarrow \quad z^2 = 2y^2$

$\Rightarrow \quad \sqrt{2}\,y = z \quad ------------- (5)$

$From\ (4)\ and\ (5), we\ get, x = y\ \ \&\ z = \sqrt{2}\,y$

$Using\ this\ in\ x^2 + y^2 + z^2 - a^2 = 0, we\ get,$

$$y^2 + y^2 + \left(\sqrt{2}\,y\right)^2 - a^2 = 0,$$

$\Rightarrow \quad 2y^2 + 2y^2 - a^2 = 0,$

$\Rightarrow \quad 4y^2 - a^2 = 0$

$\Rightarrow \quad 4y^2 = a^2$

$\Rightarrow \quad y^2 = \frac{a^2}{4} \ \Rightarrow y = \frac{a}{2}$

$From\ (4), we\ get, \quad x = \frac{a}{2}$

$From\ (5), we\ get, \quad z = \sqrt{2}\,\frac{a}{2} = \frac{a}{\sqrt{2}}$

Therefore, the maximum temperature on the given surface is

$$T = k\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{\sqrt{2}}\right)^2 = k\,\frac{a^4}{8}$$

- **Determine the minimum value of $x^2 + y^2 + z^2$ when $x + y + z = 3a$.**

Let $f(x,y,z) = x^2 + y^2 + z^2$

and $\varphi(x,y,z) = x + y + z - 3a = 0$

By Lagrange's Method,

Let $F = f + \lambda\varphi$, where λ is Lagrangian Multiplier.

$F = (x^2 + y^2 + z^2) + \lambda(x + y + z - 3a)$

$$\frac{\partial F}{\partial x} = 0 \quad \Rightarrow 2x + \lambda = 0,$$

$$\Rightarrow \lambda = -2x \text{ - - - - - - - - - - - - - - - } (1)$$

$$\frac{\partial F}{\partial y} = 0 \quad \Rightarrow 2y + \lambda = 0,$$

$$\Rightarrow \lambda = -2y \text{ - - - - - - - - - - - - - - - } (2)$$

$$\frac{\partial F}{\partial z} = 0 \quad \Rightarrow 2z + \lambda = 0,$$

$$\Rightarrow \lambda = -2z \text{ - - - - - - - - - - - - - - - } (3)$$

From (1), (2) *and* (3), *we get*,

$-2x = -2y = -2z$

$\Rightarrow \quad x = y = z$

Using this in $x + y + z - 3a = 0$, we get

$3x - 3a = 0 \quad \Rightarrow x = a$

$$\Rightarrow \quad x = a = y = z$$

Therefore, Minimum value of $f(x,y,z) = a^2 + a^2 + a^2 = 3a^2$

- **Determine the volume of the largest rectangular solid which can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$**

Let the volume of the solid be xyz which is maximised in such a way

that it can be inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Let $f(x,y,z) = xyz$ and $\varphi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

By Lagrange's method $F = f + \lambda\,\varphi$

$(i.e)$ $F = xyz + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right]$

$$\therefore\ \frac{\partial F}{\partial x} = 0 \Rightarrow yz + \frac{2x\lambda}{a^2} = 0$$

$$\Rightarrow \lambda = \frac{-yza^2}{2x} ----------------(1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \frac{2y\lambda}{b^2} = 0$$

$$\Rightarrow \lambda = \frac{-xzb^2}{2y} ----------------(2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \frac{2z\lambda}{c^2} = 0$$

$$\Rightarrow \lambda = \frac{-yxc^2}{2z} ----------------(3)$$

From (1) &(2) we get

$$\frac{-yza^2}{2x} = \frac{-xzb^2}{2y} \Rightarrow \frac{ya^2}{x} = \frac{xb^2}{y}$$

$$\Rightarrow y^2a^2 = x^2b^2$$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} ----------------(4)$$

From (2) &(3) we get

$$\frac{-xzb^2}{2y} = \frac{-yxc^2}{2z} \Rightarrow \frac{zb^2}{y} = \frac{yc^2}{z}$$

$$\Rightarrow \frac{y}{b} = \frac{z}{c} ----------------(5)$$

From (4) &(5) we get

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \implies y = \frac{bx}{a}, z = \frac{cx}{a} \quad ------------(6)$$

We know that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\implies \quad \frac{x^2}{a^2} + \frac{b^2x^2}{a^2b^2} + \frac{c^2x^2}{a^2c^2} = 1$$

$$\implies 3\,\frac{x^2}{a^2} = 1$$

$$\implies x^2 = \frac{a^2}{3} \qquad \implies x = \frac{a}{\sqrt{3}}$$

$\therefore$ Equation (6) $\implies$ $y = \frac{bx}{a} = \frac{b(\frac{a}{\sqrt{3}})}{a} = \frac{b}{\sqrt{3}}$

$$and \quad z = \frac{cx}{a} = \frac{c(\frac{a}{\sqrt{3}})}{a} = \frac{c}{\sqrt{3}}$$

$\therefore$ The rectangular solid in a cube with dimensions are

$$x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}, \qquad z = \frac{c}{\sqrt{3}}$$

$\therefore$ Volume $= xyz = \frac{a}{\sqrt{3}} \times \frac{b}{\sqrt{3}} \times \frac{c}{\sqrt{3}} = \frac{abc}{3\sqrt{3}}$

- **Determine the minimum value of $x^m y^n z^p$ when $x + y + z = a$.**

Let $f(x, y, z) = x^m y^n z^p$ and $\varphi(x, y, z) = x + y + z - a$

By Lagrange's method $F = f + \lambda\,\varphi$

$(i.e)$ $F = x^m y^n z^p + \lambda(x + y + z - a)$

$\therefore$ $\frac{\partial F}{\partial x} = 0 \implies mx^{m-1}y^n z^p + \lambda = 0$

$$\implies \quad \lambda = -\,mx^{m-1}y^n z^p \quad ------------------(1)$$

$$\frac{\partial F}{\partial y}=0 \implies nx^m y^{n-1} z^p+\lambda=0$$

$$\implies \lambda=-nx^m y^{n-1} z^p \quad \text{-------------------(2)}$$

$$\frac{\partial F}{\partial z}=0 \implies p\,x^m y^n z^{p-1}+\lambda=0$$

$$\implies \lambda=-p\,x^m y^n z^{p-1} \quad \text{------------------(3)}$$

From (1) &(2), we get

$$-mx^{m-1}y^n z^p=-nx^m y^{n-1} z^p$$

$$\implies mx^{m-1}y^n=nx^m y^{n-1}$$

$$\implies my=nx$$

$$\implies \frac{y}{n}=\frac{x}{m} \quad -------------------(4)$$

From (2) &(3), we get

$$-nx^m y^{n-1} z^p=-p\,x^m y^n z^{p-1}$$

$$\implies nz=py$$

$$\implies \frac{y}{n}=\frac{z}{p} \quad -------------------(5)$$

From (4) &(5), we get

$$\frac{x}{m}=\frac{y}{n}=\frac{z}{p} \implies y=\frac{nx}{m}, z=\frac{px}{m}$$

Now consider , $x+y+z=a$

$$\implies x+\frac{nx}{m}+\frac{px}{m}=a$$

$$\implies mx+nx+px=am$$

$$\therefore \; x=\frac{am}{m+n+p}$$

$$y=\frac{n\left(\frac{am}{m+n+p}\right)}{m}=\frac{na}{m+n+p}$$

$$z = \frac{p\left(\frac{am}{m+n+p}\right)}{m} = \frac{pa}{m+n+p}$$

$\therefore$ *The minimum value of* $f(x, y, z)$

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{na}{m+n+p}\right)^n \left(\frac{pa}{m+n+p}\right)^p$$

$$= a^{m+n+p} \frac{m^m n^n p^p}{(m+n+p)^{m+n+p}}$$