UNIT 1

Null Space and Nullity of a Matrix

1. What is Null Space?

- The **null space** of a matrix A is the set of all vectors x such that: A·x=0
- This means when you multiply matrix AAA with some vector xxx, the result is the **zero vector**.
- It helps us find **linear relationships** among the columns of the matrix.

2. Size of the Null Space (Nullity)

- If A is a matrix of size m×n , then:
  - **Nullity of AAA =** dimension of the null space of AAA, i.e., the number of free variables in the solution of A·x = 0
  - It tells us how many **independent linear relations** exist among the columns.

3. Example:

Given matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{bmatrix}$$

We want to solve:

$$A \cdot x = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 4: Solving the system of equations

You start with the system of equations from the matrix multiplication:

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 2x_3 - x_4 = 0 \\ 2x_1 + 4x_2 + 0x_3 + 6x_4 = 0 \end{cases}$$

What we do next:

1. Use these equations to express some variables in terms of others.
2. From the first two equations, subtract one from the other to eliminate some variables:

$$(x_1 + 2x_2 + 2x_3 - x_4) - (x_1 + 2x_2 + x_3 + x_4) = 0 - 0$$

This simplifies to:

$$x_3 - 2x_4 = 0 \quad \Rightarrow \quad x_3 = 2x_4$$

3. Substitute $x_3 = 2x_4$ back into the first equation:

$$x_1 + 2x_2 + (2x_4) + x_4 = 0$$

Simplify:

$$x_1 + 2x_2 + 3x_4 = 0$$

4. Solve for x_1:

$$x_1 = -2x_2 - 3x_4$$

So, general solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

- The variables x_2 and x_4 are called **free variables** because you can choose any values for them.
- Once you pick values for x_2 and x_4, you can find x_1 and x_3 using the formulas we found earlier.
- So, every solution to the equation $A \cdot x = 0$ is made by mixing (adding and scaling) two special vectors — one for x_2 and one for x_4.
- This means the **null space** is made up of all possible combinations of these two vectors.

Psuedo Inverse for given matrix :

Pseudo inverse or Moore – Penrose inverse is the generalization of the matrix inverse

If the matrix is invertible then its inverse will be equal to pseudo inverse and denoted by A+.

It is used when:

- The matrix is **not square** or
- The matrix is **not invertible**

- If the columns of a matrix A are linearly independent, so $A^T \cdot A$ is invertible and we obtain with the following formula the pseudo inverse:

$$\mathbf{A^+ = (A^T \cdot A)^{-1} \cdot A^T}$$

- Here A^+ is a left inverse of A, what means: $A^+ \cdot A = E$.
- However, if the rows of the matrix are linearly independent, we obtain the pseudo inverse with the formula:

$$A^+ = A^T \cdot (A \cdot A^T)^{-1}$$

- This is a right inverse of A, what means: $A \cdot A^+ = E$.
- If both the columns and the rows of the matrix are linearly independent, then the matrix is invertible and the pseudo inverse is equal to the inverse of the matrix.

If A has rank deficient, then the Pseudo inverse of A id defined as

$$A^+ = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$$

$$\text{If } \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \text{ then } \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \end{bmatrix}$$

Problems:

$$1. \textit{Find the pseudo inverse of } A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Sol:

$$\textit{Given } A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$\textit{Here } \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5 \neq 0$$

$$rank(A) = 2$$

Then the pseudo inverse of A is $A^{+} = A^{T}(AA^{T})^{-1}$

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad A^{T} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$AA^{T} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 15 \\ 15 & 30 \end{bmatrix}$$

$$|AA^{T}| = 15(30-15) = 225$$

$$(AA^{T})^{-1} = \frac{1}{225}\begin{vmatrix} 30 & -15 \\ -15 & 15 \end{vmatrix} = \begin{bmatrix} 2/15 & -1/15 \\ -1/15 & 1/15 \end{bmatrix}$$

$$A^{+} = A^{T}(AA^{T})^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2/15 & -1/15 \\ -1/15 & 1/15 \end{bmatrix} = \begin{bmatrix} -2/15 & 3/15 \\ 1/15 & 1/15 \\ 0 & 1/15 \\ 5/15 & -2/15 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} -2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 5 & -2 \end{bmatrix}$$

$3.\ Find\ the\ pseudo\ inverse\ of\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

Sol:

$$Given\ \ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$Here\ all\ \min ors\ of\ order\ two\ are\ zero$

$\therefore rank(A) = 1$

Since for the given matrix the rank is not equal to the number of rows or columns therefore it has to be solved by singular value decomposition.

$Then\ the\ pseudo\ inverse\ of\ A\ u\sin g\ SVD\ is\ A^{+} = V\Sigma^{+}U^{T} = V\Sigma^{-1}U^{T}$

Compute $A^T A = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 45 \end{bmatrix}$

It has eigen values $\lambda_1 = 70, \lambda_2 = 0, \lambda_3 = 0$ and the corresponding

eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$

The singular values of A are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{70}, \sigma_2 = \sqrt{\lambda_1} = \sqrt{0}, \sigma_3 = \sqrt{\lambda_3} = \sqrt{0}$$

Normalized vectors are $v_1 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix},\ v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, v_2 = \begin{bmatrix} 3/\sqrt{70} \\ 6/\sqrt{70} \end{bmatrix}$

Normalized vectors are $v_1 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix},\ v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 3/\sqrt{70} \\ 6/\sqrt{70} \\ -5/\sqrt{70} \end{bmatrix}$

So Thus $V = \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & 3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & 6/\sqrt{70} \\ 3/\sqrt{14} & 0 & -5/\sqrt{70} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{70} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Also we can find $u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$$U = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\therefore A^{+} = V\Sigma^{+}U^{T}$$

$$= \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & 3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & 6/\sqrt{70} \\ 3/\sqrt{14} & 0 & -5/\sqrt{70} \end{bmatrix} \begin{bmatrix} 1/\sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

What Are Eigenvalues and Eigenvectors?

The mathematical expression is:

$$A\vec{x} = \lambda\vec{x}$$

- Here, A is a matrix, $\vec{x}$ is a **vector**, and λ is a **scalar**.
- λ is called an **eigenvalue** — it tells **how much the vector stretches or shrinks.**
- $\vec{x}$ is the corresponding **eigenvector** — the direction that stays the same after transformation by A.

Key Idea: When you multiply a matrix by its eigenvector, the result is just a **scaled version** of that vector (not rotated).

Step 1: Write the Characteristic Equation

For a 2×2 matrix, the **characteristic equation** is:

$$\lambda^2 - (\text{Trace}) \cdot \lambda + (\text{Determinant}) = 0$$

Purpose:
This equation helps find the values of λ, which are the eigenvalues.

Step 2: Find the Trace (Sum of Diagonal Elements)

The trace of a matrix is the sum of its main diagonal elements.

From the matrix A:

$$\text{Trace} = 1 + (-1) = 0$$

Purpose:
The trace is used in the characteristic equation to form the middle term.

Step 3: Find the Determinant of the Matrix

Use the formula:

$$\text{Determinant} = (1)(-1) - (1)(3) = -1 - 3 = -4$$

Purpose:
The determinant is used as the constant term in the characteristic equation.

Step 4: Form the Characteristic Equation

Substitute the values into:

$$\lambda^2 - (\text{Trace})\lambda + (\text{Determinant}) = 0$$

$$\lambda^2 - 0\lambda - 4 = 0 \Rightarrow \lambda^2 - 4 = 0$$

Step 5: Solve the Characteristic Equation

$$\lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

Final Eigenvalues:

$$\boxed{\lambda_1 = 2, \quad \lambda_2 = -2}$$

◆ Characteristic Equation Definition:

The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of matrix A.

◆ Key Properties:

1. **Eigenvalues:**
   - The characteristic equation of an $n \times n$ matrix will give n **roots.**
   - These roots are called **eigenvalues**, **characteristic roots**, or **latent roots** of the matrix.
2. **Eigenvectors:**
   - For each eigenvalue λ, the equation

     $$A\vec{x} = \lambda\vec{x}$$

     has a **non-zero solution** $\vec{x}$, which is called the **eigenvector.**
   - The eigenvector tells the **direction** that does not change under the transformation.

A **positive** eigenvalue λ means **stretching**, and a **negative** one means **flipping** and then stretching.

Imagine a line pointing in a direction (vector). After applying matrix A, if it still points in the same direction (or opposite), but is just longer or shorter, then that direction is an **eigenvector**, and the **scaling factor** is the **eigenvalue.**

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

We want to solve:

$$A\vec{x} = \lambda\vec{x}$$

Which means: When matrix A acts on a vector $\vec{x}$, it stretches (or shrinks) it by a factor λ. That vector $\vec{x}$ is called an **eigenvector**, and the value λ is called an **eigenvalue**.

Step 2: Write the Characteristic Equation

The eigenvalues of a matrix are found by solving:

$$\det(A - \lambda I) = 0$$

Where I is the identity matrix and λ is a scalar (eigenvalue).

Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}$$

Step 3: Compute the Determinant

Now, calculate:

$$\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - (3)(1)$$

Use the identity $(a-b)(a+b) = a^2 - b^2$ if helpful:

$$\begin{gathered} = (1-\lambda)(-1-\lambda) - 3 \\ = -[(1-\lambda)(1+\lambda)] - 3 \\ = -(1-\lambda^2) - 3 \\ = -1 + \lambda^2 - 3 = \lambda^2 - 4 \end{gathered}$$

So, the **characteristic equation** is:

$$\lambda^2 - 4 = 0$$

Step 4: Solve for Eigenvalues

$$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

✅ **Eigenvalues:**

- $\lambda_1 = 2$
- $\lambda_2 = -2$

Step 5: Find Eigenvectors

Now, for each eigenvalue, solve:

$$(A - \lambda I)\vec{x} = 0$$

Let $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

- **For $\lambda = 2$:**

$$A - 2I = \begin{pmatrix} 1-2 & 1 \\ 3 & -1-2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

Now solve:

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first row:

$$-x + y = 0 \Rightarrow y = x$$

So, eigenvector:

$$\vec{x}_1 = \begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Choose $x = 1$, we get:

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- **For $\lambda = -2$:**

$$A + 2I = \begin{pmatrix} 1+2 & 1 \\ 3 & -1+2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Now solve:

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first row:

$$3x + y = 0 \Rightarrow y = -3x$$

So, eigenvector:

$$\vec{x}_2 = \begin{pmatrix} x \\ -3x \end{pmatrix} = x \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Choose $x = 1$, we get:

$$\vec{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

- **Eigenvalues:**
- $\lambda_1 = 2$
- $\lambda_2 = -2$

- **Eigenvectors:**
- For $\lambda = 2$: $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- For $\lambda = -2$: $\vec{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$