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CS230 - Assignment-1

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Method-1

1) Given;

A, B are events in a probability space such that $A \subseteq B$.

Let us assume; Our Sample Space is 'Ω'

Let Cardinality of Ω be 'N'

(i.e; Total no. of possible outcomes in our sample space is N) $N \in \mathbb{N}$ (Natural No)

* Given; $A \subseteq B$

{We know; If two sets X & Y are such that X is a subset of Y ($X \subseteq Y$) then; Cardinality of $X \leq Y$}

→ Let cardinality of 'A' is a (ie; no of ways in which event 'A' occurs is 'a')

cardinality of 'B' is b

So, $\boxed{a \leq b}$ (where $a, b \in \mathbb{N}$) (Natural numbers)

We know; Probability of occurrence of an event $P(E) = \dfrac{\text{no. of favourable Outcomes}}{\text{Total no of Outcomes}}$

Here; $P(A) = \frac{a}{N}$

$P(B) = \frac{b}{N}$.

Since;

$$a \leq b$$

$$\frac{a}{N} \leq \frac{b}{N}$$

$$\Rightarrow \boxed{P(A) \leq P(B)}$$

Hence proved.

1)

$\boxed{\text{Method-2}}$

Given; $A \subseteq B$

Now; let us solve only using axioms.

Given two events A, B; let us construct an event C such that;

$$\boxed{C = A^c \cap B}$$

Now;

$$A \cap C = A \cap (A^c \cap B)$$

$$A \cap C = (A \cap A^c) \cap B = \phi \cap B = \phi$$

Hence; Events A & C are Exclusive events.

also; sets A & C are disjoint.

Now; using Finite additivity Axiom;

$$\boxed{P(A \cup C) = P(A) + P(C)} \quad \text{①} \quad (\because A \cap C = \phi)$$

Now;

$$A \cup C = A \cup (A^C \cap B)$$

$$A \cup C = (A \cup A^C) \cap (A \cup B)$$

$$A \cup C = A \cup B$$

Since $A \subseteq B \Rightarrow \boxed{A \cup C = B}$ ②

Putting ② in ①

$$P(B) = P(A) + P(C).$$

$$\Rightarrow P(C) = P(B) - P(A)$$

By non-negativity axiom ; $P(C) \geq 0$

$$\Rightarrow \boxed{P(B) \geq P(A)}$$

Hence Proved.

2) Let us assume our Sample Space is 'Ω'
with cardinality of 'N'.

* Given; an event 'A' whose cardinality is 'a'

We Know;

Complement of A i.e; 'A^c' is the event
where possible outcomes are 'any outcome
except those which are in A'

$\therefore$ Cardinality of A^c = Total outcomes (or) (Cardinality of Ω) − Cardinality of A

$$\boxed{\text{Cardinality of } A^c = N - a}$$

*(Note that Cardinality means no. of elements in a Set.
Here we are representing an event with set of all possible outcomes)

Since; Probability of occurrence of an event E, $P(E) = \frac{\text{No. of favourable outcomes}}{\text{Total possible outcomes}}$

$$\therefore P(A) = \frac{a}{N}$$

$$P(A^c) = \frac{N-a}{N}$$

$$\Rightarrow 1 - P(A^c) = 1 - \left(\frac{N-a}{N}\right)$$

$$\Rightarrow 1 - P(A^c) = 1 - \left(1 - \frac{a}{N}\right)$$

$$\Rightarrow 1 - P(A^c) = \cancel{1} - \cancel{1} + \frac{a}{N}$$

$$\Rightarrow \boxed{1 - P(A^c) = P(A)}$$

Hence Proved.

Method-2

2) We know; If sample space is Ω;

$\boxed{P(\Omega) = 1}$ (from Normalization axiom)

Let us take an event 'A' of Ω.

It is clear that;

Set Ω can be formed by A & A^C also.

So; $\boxed{\Omega = A \cup A^C}$

We know;

$$A \cap A^C = \phi$$

So; we can use Finite additivity axiom.

$$P(A \cup A^C) = P(A) + P(A^C)$$

$$P(\Omega) = P(A) + P(A^C)$$

$$1 = P(A) + P(A^C)$$

$$\therefore \boxed{P(A) = 1 - P(A^C)}$$

Hence Proved.

3) Given;

$A_1, A_2, A_3, \ldots A_n$ are set of finite events which are not neccessarily mutually Disjoint.

we need to show;

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

First, let us prove an Identity ie;

For any events A_1, A_2 of Sample Space Ω

$$\mathrm{I} \leftarrow \boxed{P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)}$$

Note that; $A_1, A_2 - A_1$ are exclusive events

and $(A_1 \cup (A_2 - A_1)) = (A_1 \cup A_2)$

We can also say that;

$$A_1 \cap (A_2 - A_1) = \phi.$$

Now; $P(A_1 \cup A_2) = P(A_1 \cup (A_2 - A_1))$

* using Finite Additivity of Prob. Axiom; RHS becomes:-

$$\boxed{P(A_1 \cup A_2) = P(A_1) + P(A_2 - A_1)} \rightarrow 1$$

Note that; $(A_2 - A_1)$ & $(A_2 \cap A_1)$ are also exclusive

and $(A_2 - A_1) \cup (A_2 \cap A_1) = A_2$

also; It implies; $(A_2 - A_1)$ & $(A_2 \cap A_1)$ are disjoint

* Again; Using Finite Additivity axiom on below eqn.

$$P(A_2) = P\left((A_2 - A_1) \cup (A_2 \cap A_1)\right)$$

$$P(A_2) = P(A_2 - A_1) + P(A_2 \cap A_1)$$

$$\Rightarrow \boxed{P(A_2) = P(A_2 - A_1) + P(A_1 \cap A_2)} \rightarrow 2$$

The difference of 1 & 2 yields:-

$$\mathrm{I} \leftarrow \boxed{P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)}$$

Now; Lets prove $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$ using Mathematical Induction.

<u>For n=1 Case;</u>

$$P\left(\bigcup_{i=1}^{1} A_i\right) \leq \sum_{i=1}^{1} P(A_i)$$

$$\Rightarrow P(A_1) \leq P(A_1) \longrightarrow \text{True}.$$

Let us assume the stat. is True for n=K case

$$\therefore P\left(\bigcup_{i=1}^{K} A_i\right) \leq \sum_{i=1}^{K} P(A_i)$$

Now, we need check for n=K+1 case.

$$P\left(\bigcup_{i=1}^{K+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{K} A_i\right) \cup A_{K+1}\right)$$

using equation I on RHS :-

$$P\left(\bigcup_{i=1}^{K+1} A_i\right) = P\left(\bigcup_{i=1}^{K} A_i\right) + P\left(A_{K+1}\right) - P\left(\bigcup_{i=1}^{K} A_i \cap A_{K+1}\right)$$

Now,

$$P\left(\bigcup_{i=1}^{K} A_i \cap A_{K+1}\right) = P\left(\bigcup_{i=1}^{K} A_i\right) + P\left(A_{K+1}\right) - P\left(\bigcup_{i=1}^{K+1} A_i\right)$$

II.

By the nonnegativity Axiom of probability ;

$$P\left(\bigcup_{i=1}^{K} A_i \cap A_{K+1}\right) \geq 0$$

$$\therefore \; P\left(\bigcup_{i=1}^{K} A_i\right) + P\left(A_{K+1}\right) - P\left(\bigcup_{i=1}^{K+1} A_i\right) \geq 0$$

$$\Rightarrow \; P\left(\bigcup_{i=1}^{K+1} A_i\right) \leq P\left(\bigcup_{i=1}^{K} A_i\right) + P\left(A_{K+1}\right)$$

using non negativity Axiom repeatedly yields :-

$$\Rightarrow \; P\left(\bigcup_{i=1}^{K+1} A_i\right) \leq \sum_{i=1}^{K} P\left(A_i\right) + P\left(A_{K+1}\right)$$

$$\Rightarrow \; \boxed{P\left(\bigcup_{i=1}^{K+1} A_i\right) \leq \sum_{i=1}^{K+1} P\left(A_i\right)}$$

$\because$ Assuming the eqn is true for $n=K$ implied it is true for $n=K+1$.

$\therefore$ By Mathematical Induction; eqn is True $\forall n$.

$$\therefore \; \boxed{P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P\left(A_i\right)}$$

Hence Proved

4. A Partition of Sample Space (Ω) is a collection of disjoint events $S_1, S_2, \dots S_n$ such that $\boxed{\Omega = \bigcup_{i=1}^{n} S_i}$

a) We Need to show;

For any event 'A'; we have $P(A) = \sum_{i=1}^{n} P(A \cap S_i)$

Sol:-

As $\Omega = \bigcup_{i=1}^{n} S_i$ we can say;

$$\boxed{A = \bigcup_{i=1}^{n} (A \cap S_i)}$$

i.e, $A = (A \cap S_1) \cup (A \cap S_2) \cup (A \cap S_3) \dots \cup (A \cap S_n)$

* We are given; $S_1, S_2, \dots S_n$ are disjoint sets.

So; $(A \cap S_1), (A \cap S_2), (A \cap S_3) \dots (A \cap S_n)$ are also disjoint sets.

i.e, $\forall i$ '$A \cap S_i$' sets are disjount.

Now;

$$P(A) = P\left(\bigcup_{i=1}^{n} (A \cap S_i)\right) \longrightarrow I$$

Using the Finite Additivity Axiom;

$$P\left(\bigcup_{i=1}^{n} (A \cap S_i)\right) = P(A \cap S_1) + P(A \cap S_2) + \cdots + P(A \cap S_n)$$

Substitute in eq. I;

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \cdots + P(A \cap S_n)$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(A \cap S_i)}$$

Hence Proved.

5) We have; Events A & $A_1, A_2, \ldots$ such that,

$A_1 \subset A_2 \subset A_3 \cdots$

$$A = \bigcup_{i \geq 1} A_i$$

$\boxed{\text{Prove:} \rightarrow P(A) = \lim_{n\to\infty} P(A_n)}$

Sol:- From Given Info;

$$A_j \subset A_{j+1}$$

So; $A_j \cap A_{j+1} = A_j$

Let us define $\boxed{C_{j+1} = A_{j+1} \cap A_j^c}$

Assuming $A_0 = \phi$ will make:-

$$C_{1+0} = A_{1+0} \cap \phi^c$$

$$C_1 = A_1 \cap \phi^c$$

$$\boxed{C_1 = A_1}$$

NOTE:-

$C_1 \cup C_2 = A_1 \cup (A_2 \cap A_1^c)$

$C_1 \cup C_2 = (A_1 \cup A_2) \cap \Omega$

$\boxed{C_1 \cup C_2 = A_1 \cup A_2}$

$= A_2$

We have;

$\boxed{A_1 \subset A_2}$

Now; Let us find C_2.

$$\boxed{C_2 = A_2 \cap A_1^c}$$

Let us check

$$C_1 \cap C_2 = A_1 \cap (A_2 \cap A_1^c)$$

$$C_1 \cap C_2 = A_1 \cap (A_1^c \cap A_2)$$

$$C_1 \cap C_2 = (A_1 \cap A_1^c) \cap A_2 = \phi$$

Let us check for general C_K, C_{K+1}

$$C_K = A_K \cap A_{K-1}^c$$

$$C_{K+1} = A_{K+1} \cap A_K^c$$

Now;

$$C_K \cap C_{K+1} = \left(A_K \cap (A_{K-1}^c)\right) \cap \left(A_{K+1} \cap A_K^c\right)$$

$$C_K \cap C_{K+1} = \left(A_{K-1}^c \cap A_K\right) \cap \left(A_K^c \cap A_{K+1}\right)$$

$$C_K \cap C_{K+1} = A_{K-1}^c \cap \left(A_K \cap A_K^c\right) \cap \left(A_{K+1}\right)$$

$$\boxed{C_K \cap C_{K+1} = \phi}$$

$\therefore$ $C_1, C_2, \ldots$ are disjoint with

$$\bigcup_{j=1}^{n} C_j = A_n \longrightarrow (2)$$

and

$$\bigcup_{j=1}^{\infty} C_j = \bigcup_{j=1}^{\infty} A_j = A \longrightarrow (3)$$

Now; $P(A) = P\left(\bigcup_{j=1}^{\infty} C_j\right) \longrightarrow (1)$

Using Countable Additivity Axiom ;

$$P(A) = P\left(\bigcup_{j=1}^{\infty} C_j\right) = \sum_{j=1}^{\infty} P(C_j)$$

$$P(A) = \lim_{n \to \infty} \sum_{j=1}^{n} P(C_j)$$

Using Finite Additivity Axiom

$$\cancel{P(A) = \lim_{n \to \infty} P\left[\bigcup_{j=1}^{n} C_j\right]}$$

$$P(A) = \lim_{n \to \infty} P\left(\bigcup_{j=1}^{n} C_j\right)$$

from eq. (2)

$$\boxed{P(A) = \lim_{n \to \infty} P(A_n)}$$

Hence Proved.

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⑥ Given;
We have an event 'A' and some events $A_1, A_2, \ldots$
Such that :-

$$A_1 \supseteq A_2 \supseteq A_3 \ldots \text{ and}$$

$$A = \bigcap_{i \geq 1} A_i$$

To use our axioms and also result from q.⑤
Let us convert above scenario using complements.

$$A_1^c \subseteq A_2^c \subseteq A_3^c \ldots$$

$$A^c = \left(\bigcap_{i \geq 1} A_i\right)^c$$

(using Demorgans Law;

$$(A \cap B)^c = A^c \cup B^c$$

$$\boxed{A^c = \bigcup_{i \geq 1} A_i^c}$$

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We have proved in Q-5 that;

$$A_1 \subset A_2 \subset A_3 \cdots \quad \& \quad A = \bigcup_{i \geq 1} A_i$$

Implied $P(A) = \lim_{n\to\infty} P(A_n)$

Now; We have; (using analogy)

$$A_1^c \subseteq A_2^c \subseteq A_3^c \cdots \quad \& \quad A^c = \bigcup_{i \geq 1} A_i^c$$

It Gives; $\boxed{P(A^c) = \lim_{n\to\infty} P(A_n^c)}$ ②

We Know; $\boxed{P(A) + P(A^c) = 1}$ (Proved in Q-2) ①

Substituting ① in eq. ②

$$1 - P(A) = \lim_{n\to\infty} \left(1 - P(A_n)\right)$$

$$\Rightarrow \cancel{1} - \cancel{P(A)} = \cancel{1} - \lim_{n\to\infty} P(A_n)$$

$$\Rightarrow \boxed{P(A) = \lim_{n\to\infty} P(A_n)}$$

Hence Proved