14-10-20

CS230-Assignment-2

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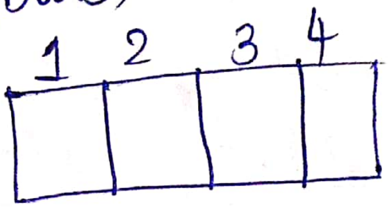
CSE

I) CONDITIONAL PROBABILITY:

1) Analysing Monty Hall Problem if there are 4 Gates (Doors)



* Let us denote Goats in 3 doors as G_1, G_2, G_3.
* Treasure as T.

→ Let us assume the case where HOST opens only 1 Door.

→ Sample Space (Ω)

$$\Omega = \{ (G_1, G_2), (G_1, G_3), (G_2, G_1), (G_2, G_3), (G_3, G_1), (G_3, G_2), (T, G_1), (T, G_2), (T, G_3) \}$$

In our Sample Space (a, b)

First One represents the option chosen by Contestant

Second one represents the option chosen by HOST.

Let us find Probabilities of each element in Ω.

$$P((G_1, G_2)) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

Contestant can chose any 1 Door from 4.

Once contest chooses Door with Goat; Host has to chose from rem. two.

Similarly for

$$P((G_1, G_3)) = \frac{1}{8}$$

$$P((G_2, G_1)) = \frac{1}{8}$$

$$P((G_2, G_3)) = \frac{1}{8}$$

$$P((G_3, G_1)) = \frac{1}{8}$$

$$P((G_3, G_2)) = \frac{1}{8}$$

Now;

$$P((T, G_1)) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

Once Contest chooses door with treasure; he has 3 options

Similarly;

$$P((T, G_2)) = \frac{1}{12}$$

$$P((T, G_3)) = \frac{1}{12}$$

Now;

* Probability of Winning when Person Sticks to his Initial choice $= P((T,G_1)) + P((T,G_2)) + P((T,G_3))$

$$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{12}$$

$$= \frac{1}{4}$$

* Probability of Winning when Person choses to Switch from Initial choice =

$$= P((G_1,G_2)) + P((G_1,G_3)) + P((G_2,G_3)) + P((G_2,G_1)) + P((G_3,G_2)) + P((G_3,G_1))$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$\Rightarrow \frac{6}{8} = \frac{3}{4}$$

∴ The probability that person wins is more when person chooses to SWITCH from Initial Choice $\left(\frac{3}{4} > \frac{1}{4}\right)$

Switch Not switch.

NOTE:- There is other case where HOST opens 2 doors out of 4.
I have simulated this case in Question 2.

③ **AIM:-** Pairwise Independence for Several events is not sufficient for claiming Full Independence.

We know;

Three events X, Y, Z are said to be $\boxed{\text{INDEPENDENT}}$ if it satisfies the below 4 eqns:-

1) $P(X \cap Y \cap Z) = P(X)\,P(Y)\,P(Z)$
2) $P(X \cap Y) = P(X) \cdot P(Y)$
3) $P(Y \cap Z) = P(Y) \cdot P(Z)$
4) $P(X \cap Z) = P(Z) \cdot P(X)$

If only $\boxed{2, 3, 4}$ hold & $\boxed{1}$ doesnot hold we say X, Y, Z are only PAIRWISE INDEPENDENT.

Let's take an example :-

* Two independent fair coin tosses

Event X: 1st toss shows Head

Event Y: 2nd toss shows Head

Event Z: 1st & 2nd tosses give same result.

$\Omega = \{HH, HT, TH, TT\}$

$P(X) = \frac{2}{4} = \frac{1}{2}$

$P(Y) = \frac{2}{4} = \frac{1}{2}$

$P(Z) = \frac{2}{4} = \frac{1}{2}$

(3) Now;

$$P(X \cap Y) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(X) \cdot P(Y)$$

$$P(X \cap Z) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(X) \cdot P(Z)$$

$$P(Y \cap Z) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(Y) \cdot P(Z)$$

$\therefore$ Here; X, Y, Z are Pairwise Independent.

Now; Let us Check;

$$P(X \cap Y \cap Z) = \frac{1}{4}$$

$$P(X) \cdot P(Y) \cdot P(Z) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$\therefore P(X \cap Y \cap Z) \neq P(X) \cdot P(Y) \cdot P(Z)$$

Hence; X, Y, Z are not INDEPENDENT.

Here; Pairwise Independence of events X, Y, Z is not Implying complete Independence

$\therefore$ Pairwise Independence is not Sufficient to claim full Independence.

④ AIM:-

$P(E_1, E_2, \ldots E_n) = \prod_{i=1}^{n} P(E_i)$ is not enough to claim pairwise independence

Lets take an example:-

Consider two independent rolls of a fair six sided dice

Event A: First roll is 1/2/3.

Event B: ~~First~~ roll is 3/4/5.

Event C: Sum of two rolls is 9.

$$\Omega = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & \underline{(3,3)} & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

$$P(A) = \frac{6+6+6}{36} = \frac{18}{36} = \frac{1}{2}$$

$$P(B) = \frac{6+6+6}{36} = \frac{18}{36} = \frac{1}{2}$$

$$P(C) = P(\{(3,6), (6,3), (5,4), (4,5)\})$$

$$P(C) = \frac{4}{36} = \frac{1}{9}$$

$$P(A) \cdot P(B) \cdot P(C) = \frac{1}{36}$$

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$P(A \cap B \cap C) = P((3,6)) = \frac{1}{36}$

$$\boxed{\therefore P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)}$$

Now, Lets check whether it is implying Pairwise Independence.

$P(A \cap B) = \frac{6}{36} = \frac{1}{6}$

$P(A \cap C) = \frac{1}{36}$

$P(B \cap C) = \frac{3}{36}$

$P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(A) \cdot P(C) = \frac{1}{2} \times \frac{1}{9} = \frac{1}{18}$

$P(B) \cdot P(C) = \frac{1}{2} \times \frac{1}{9} = \frac{1}{18}$

It is clear from above that;

$P(A \cap B) \neq P(A) \cdot P(B)$

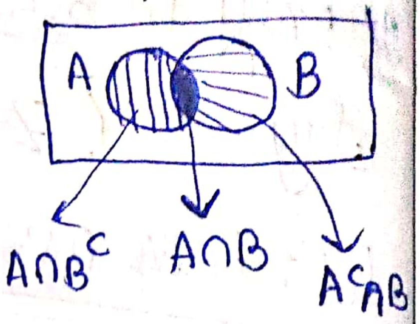
$P(A \cap C) \neq P(A) \cdot P(C)$

$P(B \cap C) \neq P(B) \cdot P(C)$

Hence; This Example shows that the condition; $P(E_1, E_2, \ldots E_n) = \prod_{i=1}^{n} P(E_i)$ is not enough to claim pairwise Independence.

⑤ Given ; A & B are Independent ;

$\Rightarrow \boxed{P(A \cap B) = P(A) \cdot P(B)}$



a) $P(A \cap B^c) = P(A) - P(A \cap B)$

$= P(A) - P(A) \cdot P(B)$

$= P(A)(1 - P(B))$

$$\boxed{P(A \cap B^c) = P(A) P(B^c)}$$

Hence A & B^c are Independent.

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b) $P(A^c \cap B) = P(B) - P(A \cap B)$

$$= P(B) - P(A)P(B)$$

$$= P(B)\left(1 - P(A)\right)$$

$$\boxed{P(A^c \cap B) = P(A^c) \cdot P(B)}$$

Hence, A^c and B are also Independent.

---

c) $P(A^c \cap B^c) = 1 - P(A \cup B)$

$$= 1 - \left(P(A) + P(B) - P(A \cap B)\right)$$

$$= \left(1 - P(A)\right) + \left(-P(B)\right) + P(A \cap B)$$

$$= \left(1 - P(A)\right) + P(A)P(B) - P(B)$$

$$= \left(1 - P(A)\right) + P(B)\left(P(A) - 1\right)$$

$$= \left(1 - P(A)\right)\left(1 - P(B)\right)$$

$$\boxed{P(A^c \cap B^c) = P(A^c) \cdot P(B^c)}$$

Hence; A^c & B^c are also Independent.

(6) Let us assume an event A. Conditioned on other event B.

From definition of Conditional Probability;

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{with } P(B) \geq 0$$

I) NON NEGATIVITY AXIOM.

We know; If 'S' is an event $P(S) \geq 0$

$\therefore P(A \cap B) \geq 0$

We also know $P(B) \geq 0$

$\therefore P(A|B) \geq 0$ Hence 1st axiom is proved

II) NORMALIZATION

$P(\Omega) = 1$

Now; $P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

$$\boxed{\therefore P(\Omega|B) = 1}$$

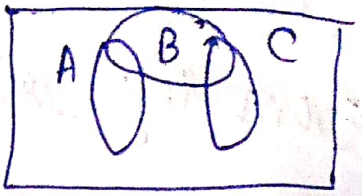
Also; $P(\phi) = 0$

Now; $P(\phi|B) = \frac{P(\phi \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$.

$$\boxed{\therefore P(\phi|B) = 1}$$

Hence 2nd axiom is proved.

(iii) Let A & C be two mutually exclusive events.
'B' is a conditional event



$$P(A\cup C|B) = \frac{P((A\cup C)\cap B)}{P(B)}$$

$$P(A\cup C|B) = \frac{P((A\cap B)\cup(C\cap B))}{P(B)}$$

Since; A & C are mutually exclusive events;
$A\cap B$ & $C\cap B$ are also mutually exclusive.

$$\because (A\cap B)\cap(C\cap B)$$
$$\Rightarrow (\underbrace{A\cap C}_{\phi})\cap B\cap B = \phi$$

$\therefore$ By Probability axiom;

$$P((A\cap B)\cup(C\cap B)) = P(A\cap B) + P(C\cap B)$$

$$\therefore P(A\cup C|B) = \frac{P(A\cap B) + P(C\cap B)}{P(B)}$$

$$P(A\cup C|B) = \frac{P(A\cap B)}{P(B)} + \frac{P(C\cap B)}{P(B)}$$

$$\boxed{P(A\cup C|B) = P(A|B) + P(C|B)}$$

This proves Axiom-3

* Hence; Conditional Probability obeys all the axioms of Probability.

7) **AIM:-** UNCONDITIONAL INDEPENDENCE doesnot necessarily imply CONDITIONAL INDEPENDENCE.

Lets consider an example :-

Two independent tosses of a fair coin:

Event A: 1st toss shows H

Event B: 2nd toss shows H

Event C: Both tosses show same result.

$\Omega = \{HH, HT, TH, TT\}$

$P(A) = \frac{2}{4} = \frac{1}{2}$ | $P(B) = \frac{2}{4} = \frac{1}{2}$ | $P(C) = \frac{2}{4} = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A) \cdot P(B)$

$\therefore$ A & B are UNCONDITIONALLY INDEPENDENT.

Now; Assume 'C' happened :-

$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{1/2} = 1/2$

$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{1/4}{1/2} = 1/2$

$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{1/4}{1/2} = 1/2$

$\therefore P(A|C) \cdot P(B|C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\therefore P(A \cap B|C) \neq P(A|C) \cdot P(B|C)$

$\therefore$ A & B are CONDITIONALLY DEPENDENT.

This proves the fact that;
Unconditional Independence doesnot necessarily imply Conditional Indep.

8) AIM:- CONDITIONAL INDEPENDENCE doesnot necessarily imply UNCONDITIONAL INDEPENDENCE.

Ex:- Assume we have two coins.

1st coin is regular & unbiased : $P(H) = \frac{1}{2}$

2nd coin is irregular (Head) : $P(H) = 1$

First we will choose a coin & then toss it two times.

A: 1st coin toss results H

B: 2nd coin toss results H

C: Coin 1 (perfect) is selected.

$P(A|C) = \frac{1}{2}$ $P(B|C) = \frac{1}{2}$

$P(A \cap B|C) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A|C) \cdot P(B|C)$

$\therefore$ A & B are conditionally independent.

Now;
$$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c)$$
$$= \frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{2} = 3/4$$

$$P(B) = P(B|C)P(C) + P(B|C^c)P(C^c)$$
$$= \frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{2} = 3/4$$

$$P(A \cap B) = P(A \cap B|C)P(C) + P(A \cap B|C^c) \cdot P(C^c)$$
$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + 1 \times 1 \times \frac{1}{2} = 5/8$$

$$P(A) \cdot P(B) = \frac{9}{16} \neq \frac{5}{8} = P(A \cap B)$$

Hence; It proves the fact that;

Conditional Independence doesnt necessarily imply unconditional Independence.

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II DISCRETE RANDOM VARIABLES:-

(1) $$\lim_{n\to\infty}\lim_{m\to\infty} 2^{m-n} \neq \lim_{m\to\infty}\lim_{n\to\infty}\left(2^{m-n}\right)$$

L.H.S:- $$\lim_{n\to\infty}\lim_{m\to\infty} 2^{m-n} = \lim_{n\to\infty}\left(\lim_{m\to\infty} 2^{m-n}\right)$$

$$= \lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{2^m}{2^n}\right)$$

$$= \lim_{n\to\infty}(\infty)$$

$$= \infty$$

$$\boxed{\therefore \lim_{n\to\infty}\lim_{m\to\infty} 2^{m-n} \to \infty} \longrightarrow (1)$$

R.H.S:-

$$\lim_{m\to\infty}\lim_{n\to\infty} 2^{m-n} = \lim_{m\to\infty}\left(\lim_{n\to\infty} 2^{m-n}\right)$$

$$= \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{2^m}{2^n}\right)$$

$$= \lim_{m\to\infty} 0$$

$$= 0$$

$$\boxed{\therefore \lim_{m\to\infty}\lim_{n\to\infty} 2^{m-n} = 0} \longrightarrow (2)$$

from eq. (1) & (2) ; it can be proved that

$$\lim_{n\to\infty}\lim_{m\to\infty}\left(2^{m-n}\right) \neq \lim_{m\to\infty}\lim_{n\to\infty}\left(2^{m-n}\right)$$

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③

a) Bernoulli Random Variable:-

We know;

A random variable 'X' is called Bernoulli Random Variable, if

$$P_X(0) = 1-P$$
$$P_X(1) = P$$

where $0 \le P \le 1$ is the probability that the trial is a 'success'

★

$$E[X] = \sum_x x\, P_X(x)$$
$$= 0\times(1-P) + 1\times P$$
$$\boxed{E[X] = P}$$

★ $Var[X] = E[(X - E(X))^2]$

$$Var[X] = E[(X-P)^2]$$
$$= \sum_x (x-P)^2 P_X(x)$$
$$= (0-P)^2\times(1-P) + (1-P)^2\times P$$
$$= (1-P)P^2 + (1-P)^2 P$$

$$Var[X] = (1-P)P(\cancel{P} + 1 - \cancel{P}) = \boxed{P(1-P)}$$

b) Binomial Random Variable :-

Let 'p' is Probability of Success.

'X' is a Binomial RV which is equal to no. of successes in 'n' independent trails.

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

Let us define some Random Variables :-

$$X_i = \begin{cases} 1; & \text{if success in trail } i, \\ 0; & \text{otherwise} \end{cases}$$

$$E[X_i] = 1 \cdot P + 0 \cdot (1-P) = P.$$

We can write $X = \sum_i X_i$

We know; $E[X] = E\left[\sum_i X_i\right]$

$$= \sum_i E[X_i] = \sum_i P$$

$$\boxed{E[X] = nP}$$

Now;

$$Var(X_i) = E[X_i^2] - (E[X_i])^2$$
$$= E[X_i] - (E[X_i])^2$$
$$= p - p^2 = p(1-p)$$

$$\boxed{\therefore Var(X_i) = p(1-p)}$$

From our definition of X_i's ; It is clear that X_i's are independent of each other.

$$\therefore Var(X) = Var\left(\sum_i X_i\right)$$
$$= \sum_i Var(X_i)$$
$$= \sum_i p(1-p)$$

$$\boxed{Var(X) = np(1-p)}$$

c) Geometric Random Variable :-

Let 'p' is the probability of success.

X be the number of trials required until the first success; then 'X' is said to be a Geometric R.V.

$$P_X(x) = \begin{cases} (1-p)^{x-1} p \; ; x = 1,2,3 \ldots \\ 0 \; ; \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$\boxed{\text{Let } (1-p) = q}$

$$= \sum_{x=1}^{\infty} x(q)^{x-1} p.$$

$$= p \sum_{x=1}^{\infty} x(q^{x-1})$$

$$= p \sum_{x=1}^{\infty} \frac{dq^x}{dq}$$

$$= p \sum_{x=1}^{\infty} \frac{d}{dq}(q^x)$$

(Exchanging Summation to ∞ & derivative is similar to exch- of Limits)

In General Interchanging limits is not a correct way As proven in Q-1. But in some cases; it will yield correct result.

$$\therefore E[X] = p \frac{d}{dq}\left(\sum_{x=1}^{\infty} q^x\right)$$

$$= p \frac{d}{dq}\left(\frac{q}{1-q}\right)$$

$$= p \times \left(\frac{(1-q)\times 1 - q(-1)}{(1-q)^2}\right)$$

$$= \frac{p(1-\cancel{q}+\cancel{q})}{(1-q)^2} = \frac{p}{(1-q)^2}$$

Since $1-p=q \Rightarrow \boxed{1-q=p}$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$\boxed{\therefore E[X] = \frac{1}{p}}$$

Now;

$$Var[X] = E[X^2] - (E[X])^2$$

Now; $E[X^2] = \sum_{x=1}^{\infty} x^2 P_X(x)$

$$= \sum_{x=1}^{\infty} x^2 p(1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} \quad 1-p=q$$

$$E[X^2] = p\sum_{x=1}^{\infty} x^2 q^{x-1}$$

$\boxed{q = 1-p}$

We Know; $$\sum_{x=1}^{\infty} x^2 q^{x-1} = \frac{1+q}{(1-q)^3}$$ $\boxed{\because q<1}$

$$\therefore \quad E[X^2] = \frac{p(1+q)}{(1-q)^3}$$ $\because \boxed{1-q=p}$

$$= \frac{p(1+1-p)}{p^3}$$

$$E[X^2] = \frac{p(2-p)}{p^3} = \frac{2-p}{p^2}$$

We had $E[X] = \frac{1}{p}$

$$\therefore \quad Var(X) = \left(\frac{2-p}{p^2}\right) - \left(\frac{1}{p}\right)^2$$

$$\boxed{Var(X) = \frac{1-p}{p^2}}$$

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d) POISSON RANDOM VARIABLE :-

A Random Variable X, Taking on one of the Values 0,1,2 is said to be a Poisson RV with λ for

$$P_X(K) = P(X=K) = \frac{e^{-\lambda}\lambda^K}{K!} \qquad \lambda>0 \rightarrow \text{Rate Parameter}$$

$$X \in \{0,1,2,\ldots\}$$

Now;

$$E[X] = \sum_x x P_X(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} = \frac{e^{-\lambda}\times 0\times\lambda^0}{0!} + \sum_{x=1}^{\infty}\frac{x\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda}\sum_{x=\cancel{0}1}^{\infty}\frac{\lambda^x\ \lambda}{(x-1)!\ \lambda}$$

$$= e^{-\lambda}\lambda\sum_{x=\cancel{0}1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$E[X] = \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

Now; let $y = x - 1$

$$E[X] = \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

We know;

$$\boxed{e^a = \sum_{y=0}^{\infty} \frac{a^y}{y!}}$$

$$E[X] = \left(\lambda e^{-\lambda}\right)\left(e^{\lambda}\right)$$

$$E[X] = \lambda e^{(\lambda-\lambda)}$$

$$\boxed{\therefore E[X] = \lambda}$$

Now; $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \sum_x x^2 \frac{\lambda^x e^{-\lambda}}{x!} \quad ?$$

First letus find;

$$E[X(X-1)] = \sum_x x(x-1) \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x(x-1)\lambda^x}{x!}$$

(when $x=0$; $x=1$ terms becom 0)

$$= e^{-\lambda} \sum_{x=2}^{\infty} \frac{\cancel{x(x-1)}\lambda^x}{\cancel{x(x-1)}(x-2)!}$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

II. 3) d) $E[X(X-1)] = e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$

Let $y = x-2$

$$E[X(X-1)] = \lambda^2 e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

$$E[X(X-1)] = \lambda^2 e^{-\lambda} e^{\lambda} \quad \left(\because \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{\lambda}\right)$$

$$E[X(X-1)] = \lambda^2$$

$$E[X^2 - X] = \lambda^2$$

$$E[X^2] - E[X] = \lambda^2$$

$$E[X^2] = \lambda^2 + E[X] = \lambda^2 + \lambda$$

Now; $Var(X) = E[X^2] - (E[X])^2$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{Var(X) = \lambda}$$

Assignment - 2

II.

5) $\boxed{Ch-2;\ Ex:-6}$

Given; 5 fair coins are tossed.

E: Event that all coins land heads

$$I_E = \begin{cases} 1; & \text{if } E \text{ occurs} \\ 0; & \text{if } E^c \text{ occurs} \end{cases}$$

$\Omega = \{ (H,H,H,H,H),$
$(H,H,H,H,T), (H,H,H,T,T) \ldots\ldots,$
(H,H,H,T,H)
$\vdots \quad \vdots$

$\Omega = \{ 0H, 1H, 2H, 3H, 4H, 5H \}$

$0H \to$ 1 way $({}^5C_0)$; $1H \to {}^5C_1$ ways; $2H \to {}^5C_2$ ways; $3H \to {}^5C_3$ ways; $4H \to {}^5C_4$ ways; $5H \to$ 1 way $({}^5C_5)$

1) $I_E = 1$ if the outcome is 5 Heads
i.e; (H,H,H,H,H)

2) $$P\{I_E = 1\} = \frac{1}{({}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5)}$$

$$P\{I_E = 1\} = \frac{1}{(1+5+10+10+5+1)} = \frac{1}{32}$$

6) $\boxed{\text{Ch-2; Ex:-8}}$

Distribution func^n of 'X' is

$$F(b) = \begin{cases} 0; & b < 0 \\ \frac{1}{2}; & 0 \le b < 1 \\ 1; & 1 \le b < \infty \end{cases}$$

We know;

$$F(b) = P(X \le b)$$

$$= \sum_{x \le b} P_X(x) \quad \left(\text{For Discrete R.V} \right)$$

$$F(b) = P(X = b) + P(X < b)$$

We know,

$$P(a) = \boxed{P(x = a) = F(x = a) - F(x < a)}$$

Let $x = b$;
$a = 0$

$$* \quad P(b=0) = F(b=0) - F(b<0)$$

$$= \frac{1}{2} - 0$$

$$\boxed{P(b=0) = \frac{1}{2}}$$

$$* \quad P(b=1) = F(b=1) - F(b<1)$$

$$= 1 - 1/2$$

$$= 1/2$$

$$\therefore \boxed{P(b=1) = \frac{1}{2}}$$

$\boxed{P(b) = 0 \text{ for all other 'b'.}}$

$$\therefore P_X(x) = \begin{cases} \frac{1}{2}; & b = 0 \\ \frac{1}{2}; & b = 1 \\ 0; & \text{else} \end{cases}$$

7) $\boxed{\text{Ch:2} \quad \text{Ex:-9}}$

Given; Distribution func^n of X; F is

$$F(b) = \begin{cases} 0, & b<0 \\ \frac{1}{2}; & 0 \le b < 1 \\ \frac{3}{5}; & 1 \le b < 2 \\ 4/5; & 2 \le b < 3 \\ 9/10; & 3 \le b < 3.5 \\ 1; & b \ge 3.5 \end{cases}$$

Now;

we know; $F(b) = P(X \le b)$

$$\boxed{P(x=a) = F(x=a) - F(x<a)}$$

Now;

* $P(x=0) = F(x=0) - F(x<0)$

$P(x=0) = \frac{1}{2} - 0 = \frac{1}{2}$

* $P(x=1) = F(x=1) - F(x<1)$

$P(x=1) = \frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10}$

* $P(x=2) = F(x=2) - F(x<2)$

$P(x=2) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

* $P(x=3) = F(x=3) - F(x<3)$

$P(x=3) = \frac{9}{10} - \frac{4}{5} = \frac{9-8}{10} = \frac{1}{10}$

7) $P(x=3.5) = F(x=3.5) - F(x<3.5)$

$P(x=3.5) = 1 - \frac{9}{10} = 1 - 0.9 = \frac{1}{10}$

$\therefore$ The Probability mass funcⁿ of X

$$P_X(x) = \begin{cases} \frac{1}{2} \; ; & x=0 \\ \frac{1}{10} \; ; & x=1 \\ \frac{1}{5} \; ; & x=2 \\ \frac{1}{10} \; ; & x=3 \\ \frac{1}{10} \; ; & x=3.5 \end{cases}$$

8) $\boxed{\text{Ch:2 Ex:-13}}$

A fair coin is flipped 10 times.
Let 'X' is a R.V which has the value of no of coins the person correctly predicted.

Given;
Person claims to have ESP $\Rightarrow \boxed{X = 7}$

It is so clear that the Probability that the individual would have done at least this well if he had no ESP is $P(X=7) + P(X=8) + P(X=9) + P(X=10)$

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(or) In short it is $P(X \geq 7)$.

Note that;

Our defined Random Variable 'X';

obeys Binomial Distibution

ie; $P_X(x) = \binom{n}{x} \left(\frac{1}{2}\right)^{n-x} \left(\frac{1}{2}\right)^{x}$

Here; $n = 10$

Since coin is fair;

$P(H) = P(T) = \frac{1}{2}$

P(correctly Predicting) = P(not correct) $= \frac{1}{2}$

$\therefore P_X(x) = \binom{10}{x} \left(\frac{1}{2}\right)^{10}$

$\boxed{{}^{n}C_{r} = \frac{n!}{r!(n-r)!}}$

Now;

$P(X \geq 7) = \sum_{x=7}^{10} \binom{10}{x} \left(\frac{1}{2}\right)^{10}$

$\frac{\cancel{7!} \times 10 \times \cancel{8}^{4} \times \cancel{9}^{3}}{\cancel{7!}\, \cancel{3!}\, \cancel{6}\,\cancel{7}}$

$= \left(\frac{1}{2^{10}}\right)\left({}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right)$

${}^{10}C_7 \Downarrow {}^{10}C_3 \downarrow 120$

${}^{10}C_8 \Downarrow {}^{10}C_2 \downarrow \frac{\cancel{10}^{5} \times 9 \times \cancel{8!}}{\cancel{2!}\,\cancel{8!}}$

${}^{10}C_9 \Downarrow {}^{10}C_1 \Downarrow \frac{10!}{9!\,1!} = 10$

${}^{10}C_{10} \downarrow 1$

$P(X \geq 7) = \frac{(120 + 45 + 10 + 1)}{2^{10}} = 0.171875$

9) $\boxed{\text{Ch:2 EX:17}}$

Given; an $\boxed{\text{experiment}}$ can result in $\boxed{\text{one}}$ of r possible outcomes with ith outcome having 'P_i' Prob.

$$\sum_{i=1}^{r} P_i = 1$$

* If 'n' of these experiments are performed & if outcome of any one of the 'n' does not affect outcome of the other $n-1$ experiments;

⇒ Probability that 1st outcome appears x_1 times
$\boxed{P}$ 2nd " " x_2 "
rth " " x_r " is

Now; probability that 1st outcome appears x_1 times among 'n' experiments;
$P(1)$

$$P(1) = {}^{n}C_{x_1} (P_1)^{x_1}$$

Similarly;
Probability that 2nd outcome appears x_2 times among $(n-x_1)$ experiments is

$$P(2) = {}^{(n-x_1)}C_{x_2} (P_2)^{x_2}$$

Similarly for rth outcome.

Since 1st, 2nd, ... rth Outcomes are Independent of each other;

Probability that 1st outcome appears x_1 times,
(P) 2nd " " x_2 times
⋮
rth " " x_r times if

$$P = \left[{}^{n}C_{x_1}(P_1)^{x_1}\right]\left[{}^{n-x_1}C_{x_2}(P_2)^{x_2}\right]\left[{}^{n-x_1-x_2}C_{x_3}(P_3)^{x_3}\right] \cdots \left[{}^{n-(x_1+x_2+\cdots+x_{r-1})}C_{x_r}(P_r)^{x_r}\right]$$

$$P = \left({}^{n}C_{x_1}\,{}^{n-x_1}C_{x_2}\,{}^{n-(x_1+x_2)}C_{x_3}\cdots\cdots{}^{n-(x_1+x_2+\cdots x_{r-1})}C_{x_r}\right) \cdot \left((P_1)^{x_1}(P_2)^{x_2}(P_3)^{x_3}\cdots(P_r)^{x_r}\right)$$

We also have; $\boxed{x_1+x_2+\cdots x_{r-1}+x_r = n}$

$$P = \left({}^{n}C_{x_1}\,{}^{n-x_1}C_{x_2}\,{}^{n-(x_1+x_2)}C_{x_3}\cdots\cdots{}^{x_r}C_{x_r}\right) \cdot \left(P_1^{x_1}P_2^{x_2}\cdots P_r^{x_r}\right)$$

we have; $\boxed{{}^{n}C_r = \dfrac{n!}{r!\,(n-r)!}}$

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$$\therefore P = \left(\frac{n!}{x_1!\,\cancel{(n-x_1)!}} \cdot \frac{\cancel{(n-x_1)!}}{x_2!\,\cancel{(n-x_1-x_2)!}} \cdot \frac{\cancel{(n-x_1-x_2)!}}{x_3!\,\cancel{(n-x_1-x_2-x_3)!}} \cdots \frac{\cancel{x_r!}}{x_r!\,0!} \right) \cdot \left(p_1^{x_1} \cdot p_2^{x_2} \cdots p_r^{x_r} \right)$$

$$P = \frac{n!}{x_1!\,x_2!\,x_3!\cdots x_r!}\, p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$$

Hence Proved.

---

10) Let X_i denote the number of times the i^{th} type outcome occurs, $i = 1, 2, \ldots r$

for $\boxed{0 \le j \le n;}$

$$P\left(X_i = x_i,\ i = 1, 2, \ldots r-1 \mid X_r = j \right)$$

We know; from definition of conditional probability;

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Here;

$$P(X_i = x_i;\ i = 1, 2, \ldots r-1 \mid X_r = j) = \frac{P\left(X_r = j \text{ and } X_i = x_i\right)}{P(X_r = j)}$$

$i = 1, 2, \ldots r-1$

We also know that

Events ; $(X_i = x_i \,;\, i = 1, 2, \dots r-1) : A$

and $(X_r = j) : B$ are independent of each other.

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} = P(A)$$

$$\therefore P(A|B) = P(A)$$

$$\Rightarrow P(X_i = x_i \text{ for } i = 1, 2, \dots r-1 \mid X_r = j)$$

$$= P(X_i = x_i, i = 1, 2, \dots r-1)$$

$$= P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \dots \cdot P(X_{r-1} = x_{r-1})$$

1st outcome occurs x_1 times.

$(r-1)^{th}$ outcome occurs x_{r-1} times.

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$$= p_1^{x_1} p_2^{x_2} \cdots p_{r-1}^{x_{r-1}}$$

Hence;

$$P(X_i = x_i, i = 1, 2, \cdots r-1 \mid X_r = j)$$

$$= p_1^{x_1} p_2^{x_2} \cdots p_{r-1}^{x_{r-1}}$$

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11) $\boxed{\text{Ch: 2 , Ex: 23}}$

→ P(Head) = P

→ Coin is flipped successively until the r^{th} head appears

→ X = No. of flips required

for $\boxed{n \geq r}$,

$$P(X=n) = \binom{n-1}{r-1} p^{r} (1-p)^{n-r}$$

We know that;

In order for 'X' to be equal to 'n';

The first (n-1) flips must have (r-1) heads

and in the nth flip we must have head

So that we stop our experiment at n flips.

Let event; A: first (n-1) flips must had (r-1) heads

B: nth flip resulted in Head.

Our task is done if both A & B happens;

P(A ∩ B) = P(A) · P(B) (Since A & B are independent)

$$\therefore P(X=n) = \underbrace{\binom{n-1}{r-1} p^{r-1} (1-p)^{(n-1-(r-1))}}_{A} \times \underbrace{P}_{B}$$

$$\boxed{P(X=n) = \binom{n-1}{r-1} p^{r} (1-p)^{n-r} \quad , n \geq r}$$

Hence Proved

12) $\boxed{\text{Ch: 2 , Ex: 24}}$

Given; Probability Mass function of 'X' is given by:

$$P(K) = \binom{r+K-1}{r-1} p^r (1-p)^K$$

$K = 0, 1, ---$

$$P(K) = \binom{r+K-1}{r-1} p^{r-1} . p . (1-p)^{(r+K-1)-(r-1)}$$

$$P_X(K) = \binom{r+K-1}{r-1} p^{r-1} . (1-p)^{(r+K-1)-(r-1)} \times p$$

Let us assume we have a coin

* $P(\text{Heads}) = p$.
* coin is successively flipped until the $\underline{r^{th}}$ head appear.

Now; A Random Variable 'X' be the (K) number that must be added to 'r' So that the number of flips required will be $(r+K)$. [r+X in general]

$$\therefore P(X=K) = \binom{r+K-1}{r-1} p^{r-1} . (1-p)^{(r+K-1-r+1)} \times p$$

Implies $(r+K)$ flips are req.

So; Among '$r+K$' flips;
We must have $(r-1)$ Heads in $(r+K-1)$ flips & the last flip must be head.

$$\therefore P(X=K) = \binom{r+K-1}{r-1} p^{r-1} (1-p)^{(r+K-1)-(r-1)} \times p$$

$$P(X=K) = \binom{r+K-1}{r-1} p^{r} (1-p)^{K}. \quad K=0,1,\ldots$$

Hence an Interpretation of 'X' is done!!!

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