CS230 Assignment-3

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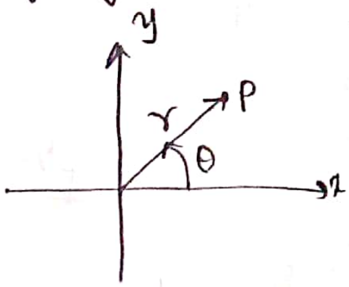
I. Continuous Distributions:

1) Given; $I = \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx$

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2/2} dx \int_{-\infty}^{+\infty} e^{-y^2/2} dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{(x^2+y^2)}{2}}\, dx\, dy$$

Evaluating Double Integral I^2 using Change of variables to polar coordinates.



$x = r\cos\theta$, $y = r\sin\theta$

$-\infty < x < -\infty$
$-\infty < y < -\infty$
$0 \leq \theta \leq 2\pi$
$0 \leq r < \infty$

$$\therefore I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} |J(r,\theta)|\, d\theta\, dr$$

$|J(r,\theta)|$:- Jacobian deter

$$|J(r,\theta)| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} \times r\, d\theta\, dr$$

$$I^2 = \int_0^{\infty} e^{-r^2/2}\, r\, dr \int_0^{2\pi} d\theta$$

$\Downarrow$ $\quad$ $\Downarrow$

$r^2/2 = a$ $\quad$ 2π.

$\Rightarrow r^2 = 2a$

$\Rightarrow \cancel{2}r = \cancel{2}\frac{da}{dr} \Rightarrow r\,dr = da$.

$$I^2 = (2\pi)\int_0^{\infty} e^{-a} da = 2\pi\left[\frac{e^{-a}}{-1}\right]_0^{\infty}$$

$$I^2 = 2\pi\left[0-(-1)\right] = 2\pi$$

$\therefore$ We get $I = \int_{-\infty}^{+\infty} e^{-x^2/2} dx = \sqrt{2\pi}$

Hence Proved.

(2) Given;

$$\text{Life time of a Computer chip } (T) \sim \underbrace{N}_{\text{Normal}}(\mu, \sigma^2)$$

where

Mean $(\mu) = 4.4 \times 10^6$ hours.

Standard Dev. $(\sigma) = 3 \times 10^5$ hours

→ Client requires that atleast 90% of Procured chips has a life time of 4×10^6 hours.

* So Given some large amount of chips; client will be wishing that; If he chose a chip; It should have $T \geq 4 \times 10^6$. 90% of the times he drew a chip from bunch.

So, let us evaluate $P(T \geq 4 \times 10^6)$

$$\therefore P(T \geq 4 \times 10^6) = 1 - P(T < 4 \times 10^6)$$

$$P(T < 4 \times 10^6) = P(T \leq 4 \times 10^6) - \cancelto{0}{P(T = 4 \times 10^6)}$$

$$\therefore P(T < 4 \times 10^6) = P(T \leq 4 \times 10^6)$$

($\because$ Prob. of a point in continuous dist $= 0$)

$$P(T < 4 \times 10^6) = P\left(\frac{T-\mu}{\sigma} \leq \frac{4 \times 10^6 - \mu}{\sigma}\right)$$

We know; if $X \sim N(\mu, \sigma^2)$

$$\underbrace{\boxed{\frac{X-\mu}{\sigma}}}_{\text{Standard Normal}} \sim N(0, 1)$$

$\therefore$ let $Z = \frac{T-\mu}{\sigma}$

2)

Now;

$$P(T < 4\times10^6) = P\left(Z \le \frac{4\times10^6 - \mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{4\times10^6 - 4.4\times10^6}{3\times10^5}\right)$$

$$= P\left(Z \le \frac{-0.4\times10}{3}\right)$$

$$P(T < 4\times10^6) = \underbrace{P\left(Z \le \frac{-4}{3}\right)}_{\text{CDF of Standard Normal Distribution}}$$

$$= \phi\left(\frac{-4}{3}\right)$$

$$P(T < 4\times10^6) = 0.0912 \quad \text{(From Table)}$$

$$\therefore P(T \ge 4\times10^6) = 1 - 0.0912 = 0.9088$$

So, we saw that; In random; A chip taken will have a life time greater than 4×10^6 hrs with probability 0.9088. Hence; On an average we can expect that atleast 90% of Total chips procured has a life time of 4×10^6 hrs.

Hence; Particular Client <u>should procure</u> from the chip manufacturer.

③ In a batch of 100 chips; Probability that there are atleast 4 chips whose lifetimes are less than 3.8×10^6 hrs:-

First; Let us evaluate;

Probability that a given chip has lifetime less than 3.8×10^6

ie; $P(T < 3.8 \times 10^6) = P(T \leq 3.8 \times 10^6) - P(T = 3.8 \times 10^6)$

$$\Rightarrow P(T < 3.8 \times 10^6) = P(T \leq 3.8 \times 10^6)$$

$$= P\left(\frac{T-\mu}{\sigma} \leq \frac{3.8 \times 10^6 - \mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{3.8 \times 10^6 - 4.4 \times 10^6}{3 \times 10^5}\right)$$

$$= P\left(Z \leq -\frac{0.6 \times 10}{3}\right)$$

$$= P(Z \leq -2)$$

We Know;

$$Z = \frac{T-\mu}{\sigma} \sim N(0,1) \longrightarrow \text{Standard Normal Dist.}$$

$$\therefore P(T < 3.8 \times 10^6) = P(Z \leq -2) \longrightarrow \text{CDF}$$

$$= \Phi(-2)$$

$$= 0.02275$$

Let us denote $P(T < 3.8 \times 10^6)$ for a Given Chip by 'p'

$\therefore P = 0.02275$ (success)

Now;

P(Out of 100 chips atleast 4 chips has lifetimes less than 3.8×10^6 hrs)

= P(out of 100 chips atleast 4 chips has success (P))

= 1 − P (out of 100 chips; less than 4 chips has success)

$$= 1 - \left[{}^{100}C_0 (P)^0 (1-P)^{100} + {}^{100}C_1 P^1 (1-P)^{99} + {}^{100}C_2\, P^2 (1-P)^{98} + {}^{100}C_3\, P^3 (1-P)^{97} \right]$$

where $\boxed{P = 0.02275}$

This is the prob. that a batch of 100 chips will contain atleast 4 chips whose lifetimes are less than 3.8×10^6 hrs.

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4) Given;

System can function for a random amount of time X.

& $f_X(x) = \begin{cases} Cxe^{-x/2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

↙ PDF

We need; Prob. that System functions for atleast 5 months!

$$P(X \geq 5) = \int_5^\infty f_X(x)\,dx$$

Since $5 < x < \infty$ $\Rightarrow f_X(x) = Cxe^{-x/2}$

$$P(X \geq 5) = \int_5^\infty C \times x e^{-x/2}\,dx$$

$$= C \int_5^\infty \underset{u}{x}\,\underset{v}{e^{-x/2}}\,dx \qquad \text{ILATE}$$

$$\Rightarrow C\left[x\int_5^\infty e^{-x/2}dx - \int_5^\infty\left(1 \times \int e^{-x/2}dx\right)dx\right]$$

$$\Rightarrow C\left[x \times \left(-2e^{-x/2}\right)\Big|_5^\infty + 2\int_5^\infty e^{-x/2}\,dx\right]$$

$$\Rightarrow C\left[-2xe^{-x/2}\Big|_5^\infty + 2 \times \left(-2e^{-x/2}\right)\Big|_5^\infty\right]$$

$$\Rightarrow P(X \geq 5) = c\left[-2xe^{-x/2}\Big|_5^\infty + -4e^{-x/2}\Big|_5^\infty \right]$$

$$= c\left[\left(\cancel{-2xe^{-x/2}\Big|_\infty} \to 0\right) - \left(-2\times5\times e^{-2.5}\right) + \cancel{-4e^{-x/2}\Big|_\infty} \to 0 - \left(-4e^{-2.5}\right)\right]$$

$$P(X\geq 5) = c\left[10e^{-2.5} + 4e^{-2.5}\right]$$

$$\boxed{P(X\geq 5) = c\times 14e^{-2.5}}$$

Let us find value of 'c'.

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_0^\infty cxe^{-x/2}\,dx = 1$$

$$\Rightarrow c\left[\int_0^\infty xe^{-x/2}\,dx\right] = 1$$

$$\Rightarrow c\left[\cancel{-2xe^{-x/2}\Big|_0^\infty} \to 0 + -4e^{-x/2}\Big|_0^\infty\right] = 1$$

$$\Rightarrow c\left[\cancel{-4e^{-x/2}\Big|_\infty} \to 0 - \left(-4e^{-x/2}\Big|_0\right)\right] = 1$$

$$\Rightarrow c\left[4e^{-0}\right] = 1 \Rightarrow \boxed{c = \frac{1}{4}}$$

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$\therefore P(X \geq 5) = \frac{\cancel{14}^{7}}{\cancel{4}_{2}} e^{-2.5} = 3.5 e^{-2.5}$

$\therefore$ Probability that system functions for atleast 5 months is $\boxed{3.5e^{-2.5}}$

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(5) Let 'X' be Discrete/Continuous R.V :-

we need to prove that :-

$$E(X) = \int_0^\infty P(X > x)\,dx - \int_0^\infty P(X < -x)\,dx$$

<u>Take the R.H.S part :-</u>

$$\text{RHS}:- \underbrace{\int_0^\infty P(X > x)\,dx}_{\alpha} - \underbrace{\int_0^\infty P(X < -x)\,dx}_{\beta}$$

$$\alpha = \int_0^\infty P(X > x)\,dx$$

$$\alpha = \int_0^\infty \left(\int_x^\infty f_X(z)\,dz \right) dx$$

Here ;

$x < z < \infty$

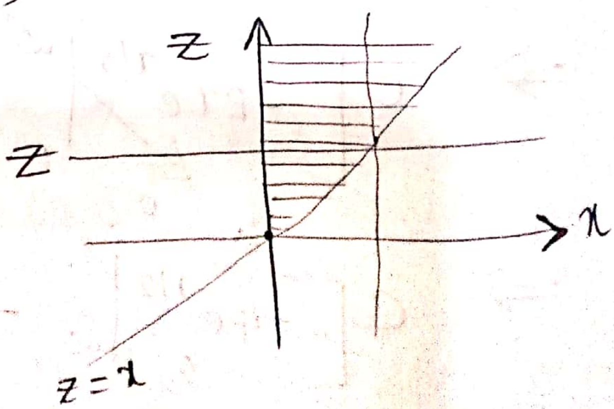
$0 < x < \infty$

So; By using Fubini's theorem;

For Interchanging the limits;

Now; We can write ;

$$\boxed{\begin{array}{l} 0 < z < \infty \\ 0 < x < z \end{array}}$$



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$$\therefore \alpha = \int_0^\infty \left(\int_x^\infty f_X(z)\,dz \right) dx = \int_0^\infty \left(\int_0^z f_X(z)\,dx \right) dz$$

$$\alpha = \int_0^\infty \left(f_X(z) \cdot \int_0^z dx \right) dz = \int_0^\infty f_X(z) \cdot z\,dz$$

$$\boxed{\therefore \alpha = \int_0^\infty z \cdot f_X(z)\,dz}$$

Now; $\beta = \int_0^\infty P(X<-x)\,dx$ Let $-x = t$

$-dx = dt$

$$\beta = \int_{-\infty}^0 P(X<t)(-dt)$$

$$\beta = -\int_{-\infty}^0 P(X<t)\,dt$$

Now;
$$\beta = \int_0^\infty P(X<-x)\,dx$$

Let $-x = t$ $\quad 0 < x < \infty$

$-dx = dt$ $\quad 0 < t < -\infty$

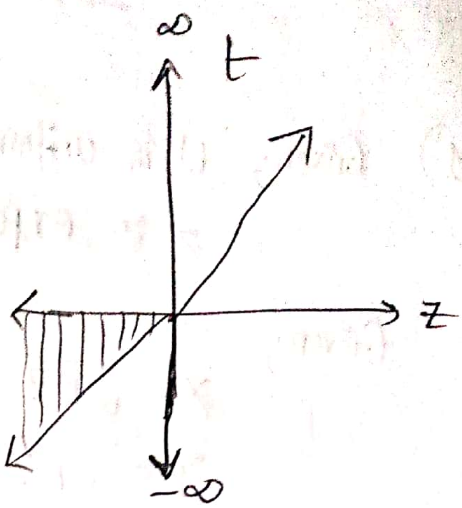
$$\therefore \beta = \int_0^{-\infty} P(X<t)\,(-dt) = -\int_0^{-\infty} P(X<t)\,dt$$

$$\beta = \int_{-\infty}^0 P(X<t)\,dt$$

$$\beta = \int_{-\infty}^0 \int_{-\infty}^t f_X(z)\,dz\,dt$$

(5) We have;

$$\beta = \int_{-\infty}^{0}\int_{-\infty}^{t} f_X(z)\,dz\,dt$$



Again; By using Fubini's theorem,

$-\infty < z < t$

$-\infty < t < 0$

$$\beta = \int_{-\infty}^{0}\int_{-\infty}^{t} f_X(z)\,dz\,dt = \int_{-\infty}^{0}\left(\int_{z}^{0} f_X(z)\,dt\right)dz$$

$$\beta = \int_{-\infty}^{0}\left(f_X(z)\int_{z}^{0} dt\right)dz = \int_{-\infty}^{0} f_X(z)(-z)\,dz$$

$$\boxed{\beta = -\int_{-\infty}^{0} z\, f_X(z)\,dz}$$

Now; R.H.S :- $\alpha - \beta$

$$\Rightarrow \int^{\infty} z\cdot f_X(z)\,dz + \int_{-\infty} z\cdot f_X(z)\,dz$$

$$\Rightarrow \int_{-\infty}^{0} z\cdot f_X(z)\,dz + \int_{0}^{\infty} z\cdot f_X(z)\,dz$$

$$\Rightarrow \int_{-\infty}^{\infty} z\cdot f_X(z)\,dz = \int_{-\infty}^{\infty} x\cdot f_X(x)\,dz$$

$$\Rightarrow E[X] \Rightarrow \boxed{RHS = LHS}$$

Hence;

$$\boxed{E[X] = \int_0^{\infty} P(X>x)\,dx - \int_0^{\infty} P(X<-x)\,dx}$$

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(6.) The PDF of X (the life time in hrs of elect. Device)

$$f_X(x) = \begin{cases} 10/x^2 & \text{if } x > 10 \\ 0 & \text{if } x \leq 10 \end{cases}$$

a) $P(X > 20) = \int_{20}^{\infty} f_X(x)\,dx$ $\quad \because 20 < x < \infty$

$= \int_{20}^{\infty} \frac{10}{x^2}\,dx$

$= 10 \int_{20}^{\infty} \frac{dx}{x^2} = 10 \int_{20}^{\infty} x^{-2}\,dx$ $\quad \int x^n dx = \frac{x^{n+1}}{n+1}$

$= 10 \left[\frac{x^{-2+1}}{-2+1}\right]_{20}^{\infty} = 10 \left[\frac{x^{-1}}{-1}\right]_{20}^{\infty}$

$= -10 \left[\frac{1}{x}\right]_{20}^{\infty}$

$= -10 \left[\frac{1}{\infty} - \frac{1}{20}\right] \Rightarrow \boxed{\frac{1}{2}}$

$\boxed{\therefore P(X > 20) = 1/2}$

b) We know; CDF of 'X' is defined as

$F_X(x) = P(X \leq x)$

$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

if $x \leq 10$; $\boxed{-\infty < t < x \leq 10}$

$\Rightarrow \boxed{t \leq 10}$ $\quad \therefore f_X(t) = 0$

Hence; if $x \leq 10$; $F_X(x) = 0$

if $x > 10$;

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$\Downarrow$$

$$= \int_{10}^{x} f_X(t)\,dt$$

$$F_X(x) = \int_{10}^{x} \frac{10}{t^2}\,dt$$

$$F_X(x) = 10\left[\frac{-1}{t}\right]_{10}^{x} = -10\left[\frac{1}{x} - \frac{1}{10}\right]$$

$$\boxed{F_X(x) = 1 - \frac{10}{x}}$$

$$\therefore \text{CDF of } X = \begin{cases} 0 & \text{if } x \leq 10 \\ 1 - \frac{10}{x} & \text{if } x > 10 \end{cases}$$

c) Let us assume that '6' devices given are independent of each other.

P(out of 6, atleast 3 will func^n for at least 15 hours)

= P(out of 6, 3 func^{n}s for atleast 15 hrs)

+ P(" " ", 4 func^{n}s " " " ")

+ P(" ", 5 " " " ")

+ P (" ", 6 " " " ")

Let us find First;

$$P(X \geq 15) = 1 - P(X < 15)$$

$$= 1 - P(X \leq 15)$$

$$= 1 - F_X(15)$$

$$= 1 - \left(1 - \frac{10}{15}\right)$$

$$P(X \geq 15) = \frac{\cancel{10}\,2}{\cancel{15}\,3} = 2/3.$$

$$\therefore P_{req} = {}^6C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3$$

$$+ {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$$

$$+ {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)$$

$$+ {}^6C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0$$

Probability that of '6' devices; atleast 3 will func^n for atleast 15 hrs.

$$\boxed{{}^6C_3 \times \frac{8}{36} + {}^6C_2 \frac{16}{36} + 6 \times \frac{32}{36} + \frac{64}{36}} \Longrightarrow \boxed{0.89986}$$

* We took an assumption that devices are Independent of each other

(7) $X \sim N(\mu, \sigma^2)$ $\quad \mu = 10$, $\sigma^2 = 36$

we know; from $\boxed{68\text{-}95\text{-}99 \text{ Rule}}$;

$$P(10-6 \le X \le 10+6) = P(4 \le X \le 16) = 0.68$$

$$P(10-2\times6 \le X \le 10+2\times6) = P(-2 \le X \le 22) = 0.95$$

$$P(10-3\times6 \le X \le 10+3\times6) = P(-8 \le X \le 28) = 0.99$$

a) $$P(X>5) = 1 - P(X \le 5)$$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \le \frac{5-\mu}{\sigma}\right)$$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \le \frac{5-10}{6}\right)$$ We have $\mu = 10$, $\sigma = 6$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \le \frac{-5}{6}\right)$$

We know; $\frac{X-\mu}{\sigma} = Z \rightarrow$ is a Standard Normal. $\because X \sim N(\mu, \sigma^2)$

$$\boxed{P(X>5) = 1 - \phi\left(\frac{-5}{6}\right)}$$

b) $P(4<X<16) = P(4 \le X \le 16)$ ($\because$ Prob. of single point in cont. R.V $= 0$)

$= 0.68$

(from 68-95-99 Rule)

c) $P(X<8) = P(X \le 8)$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{8-\mu}{\sigma}\right)$$

$$= P\left(\frac{X-10}{6} \le \frac{8-10}{6}\right)$$

$$= P\left(Z \le -\frac{\cancel{2}}{\cancel{6}3}\right) \qquad Z \sim N(0,1)$$

$$= \phi(-1/3) = \boxed{\phi(-1/3)}$$

d) $P(X<20) = P\left(\frac{X-10}{6} \le \frac{20-10}{6}\right)$

$$= P\left(\frac{X-10}{6} \le \frac{10}{6}\right)$$

$$= P\left(Z \le \frac{\cancel{10}5}{\cancel{6}3}\right) \qquad Z \sim N(0,1)$$

$$= \boxed{\phi(5/3)} \Rightarrow \phi(1.66)$$

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b) $P(4<X<16) = P(X<16) - P(X<4)$

$$= P(X \le 16) - P(X \le 4)$$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{16-10}{6}\right) - P\left(\frac{X-\mu}{\sigma} < \frac{4-10}{6}\right)$$

$$= P(Z<1) - P(Z<-1)$$

$$= \boxed{\phi(1) - \phi(-1)}$$

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II) Derived Distributions :-

1) Given; 'U' is uniform on $[0, 2\pi]$ 'U' & 'Z' are indep.
'Z' is expon. with $\lambda = 1$

Given;

$$X = \sqrt{2Z}\cos U$$
$$Y = \sqrt{2Z}\sin U$$

We know;
since, 'U' is uniform on $[0, 2\pi]$; $f_U(u) = \boxed{\frac{1}{2\pi}}$

'Z' is exp with $\lambda = 1$;

$$f_Z(z) = \lambda e^{-\lambda z} = \boxed{e^{-z}}$$

Since; 'U' & 'Z' are Independent R.V's;

$$f_{U,Z}(u,z) = f_U(u)\cdot f_Z(z)$$
$$= \frac{e^{-z}}{2\pi}$$

we had; Derived Distribution's PDF in multivariable case.

$$f_{Y_1 Y_2 Y_3 Y_4 \cdots Y_n} = f_{X_1 X_2 \cdots X_n}\left(h_1(y_1,y_2,\ldots y_n), h_2(\ ), \ldots h_n(\ )\right) \times \left|\det\left(J_n(y_1,y_2,\ldots y_n)\right)\right|$$

where;

$$x_1 = h_1(y_1, y_2, \ldots y_n)$$
$$x_2 = h_2(y_1, y_2, \ldots y_n)$$
$$\vdots$$
$$x_n = h_n(y_1, y_2, \ldots y_n)$$

$$\begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \cdots & \frac{\partial h_2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial h_n}{\partial y_1} & \frac{\partial h_n}{\partial y_2} & \cdots & \frac{\partial h_n}{\partial y_n} \end{vmatrix}$$

Using that formulal in our case;

$$f_{U,Z}(u,z) = f_{X,Y}(x,y)\,\left|\det(J_2(u,z))\right|$$

We had; $X = \sqrt{2Z}\cos U$

$Y = \sqrt{2Z}\sin U$

Now;

$$\det(J_2(u,z)) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\sqrt{2z}\sin u & \cancel{\sqrt{2}} \times \frac{1}{\cancel{2}\sqrt{z}} \cos u \\ \sqrt{2z}\cos u & \cancel{\sqrt{2}} \times \frac{1}{\cancel{2}\sqrt{z}} \sin u \end{vmatrix}$$

$$\det(J_2(u,z)) = -\sin^2 u - \cos^2 u = -1$$

Now; $f_{U,Z}(u,z) = f_{X,Y}(x,y)\,|-1|$

Now; $f_{X,Y}(x,y) = f_{U,Z}(u,z)$

$$f_{X,Y}(x,y) = \frac{e^{-z}}{2\pi}$$

Now; $x = \sqrt{2z}\cos u \longrightarrow x^2 = 2z\cos^2 u$ $\qquad x^2+y^2 = 2z$

$y = \sqrt{2z}\sin u \longrightarrow y^2 = 2z\sin^2 u$

$$f_{X,Y}(x,y) = \frac{e^{-\frac{(x^2+y^2)}{2}}}{2\pi}$$

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Now;

$$f_{X,Y}(x,y) = \frac{e^{-\frac{(x^2+y^2)}{2}}}{2\pi}$$

$$f_{X,Y}(x,y) = \left(\frac{e^{-x^2/2}}{2\pi}\right)\cdot\left(\frac{e^{-y^2/2}}{2\pi}\right)$$

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$$

where

$$f_X(x) = \frac{e^{-x^2/2}}{2\pi} \quad \& \quad f_Y(y) = \frac{e^{-y^2/2}}{2\pi}$$

are normal distributions with $\mu = 0$, $\sigma^2 = 1$

ie; $f_X(x)$, $f_Y(y) \sim N(0, 1)$

Standard Normal Distributions.

Also;
since $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$;

$\therefore$ We can say that X, Y are <u>Independent</u>

<u>Standard</u> Normal variables

(2) Given ;

'X' is said to have lognormal dist if $\log X \sim N(\mu, \sigma^2)$

If 'X' is lognormal with $E[\log X] = \mu$ &

$\text{var}(\log X] = \sigma^2$ find $f_X(x)$.

$\therefore \log X \sim N(\mu, \sigma^2)$

Let us call '$\log X$' as 'Y'

$\therefore Y \sim N(\mu, \sigma^2)$

We have; $Y = \log_e X$

$\Rightarrow \boxed{X = e^Y}$

We can calculate CDF :-

$F_X(x) = P(X \leq x)$

$= P(e^Y \leq x)$

$F_X(x) = P(Y \leq \log x) = F_Y(\log x)$

We know; $\boxed{f_X(x) = \frac{d}{dx}(F_X(x))}$

Applying diff on both sides;

$f_X(x) = f_Y(\log x) \frac{d(\log x)}{dx}$

$f_X(x) = f_Y(\log x) \frac{1}{x}$

We know; $\because Y \sim N(\mu, \sigma^2)$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$$

$$\therefore f_X(x) = f_Y(\log x) \times \frac{1}{x}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(\log x-\mu)^2}{2\sigma^2}} \times \frac{1}{x}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\frac{(\log x-\mu)^2}{2\sigma^2}\right\}$$

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(3) Given; 'U' is uniform R.V with $[0,1]$

$$f_U(u) = 1$$

$$Y = \log(U)$$

Now; $F_Y(y) = P(Y \leq y) = P(\log U \leq y)$

$$= P(U \leq e^y)$$

$$F_Y(y) = F_U(e^y)$$

Diff on both sides;

$$f_Y(y) = f_U(e^y) \cdot \frac{d(e^y)}{dy}$$

$$f_Y(y) = f_U(e^y) \cdot e^y$$

$$f_Y(y) = 1 \cdot e^y = e^y.$$

$\therefore$ PDF of Y is $\boxed{e^y}$

(iii) Other topics :-

(1) σ_x, σ_y are stand. deviations of R.V's X & Y.

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

a) Given;

$$Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) \geq 0$$

we know; $Var(Z) = E\left((Z - E[Z])^2\right)$

$$\therefore \quad E\left(\left(\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) - E\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right)\right)^2\right) \geq 0$$

we know; $\boxed{E[aX + bY] = aE[X] + bE[Y]} \longrightarrow \text{I}$

$$\therefore \quad E\left(\left(\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) - \left(\frac{E[X]}{\sigma_X} + \frac{E[Y]}{\sigma_Y}\right)\right)^2\right) \geq 0$$

$$E\left(\left(\left(\frac{X - E[X]}{\sigma_X}\right) + \left(\frac{Y - E[Y]}{\sigma_Y}\right)\right)^2\right) \geq 0$$

$$E\left[\left(\frac{X - E[X]}{\sigma_X}\right)^2 + \left(\frac{Y - E[Y]}{\sigma_Y}\right)^2 + 2\,\frac{(X - E[X])}{\sigma_X}\frac{(Y - E[Y])}{\sigma_Y}\right] \geq 0$$

Again using eq. I;

$$\left[\left(\frac{1}{\sigma_X}\right)^2 E\left((X - E[X])^2\right) + \left(\frac{1}{\sigma_Y}\right)^2 E\left((Y - E[Y])^2\right) + \frac{2}{\sigma_X \sigma_Y} E\left((X - E[X])(Y - E[Y])\right)\right] \geq 0$$

We know;

$$\sigma_Z = E[(\cancel{Z} - E[Z])^2]$$

$$\Rightarrow \frac{1}{\cancel{\sigma_X^2}} \times \cancel{\sigma_X^2} + \frac{1}{\cancel{\sigma_Y^2}} \times \cancel{\sigma_Y^2} + \frac{2}{\sigma_X \sigma_Y} Cov(X,Y) \geq 0$$

$$\Rightarrow 2 + \frac{2 Cov(X,Y)}{\sigma_X \sigma_Y} \geq 0$$

$$\Rightarrow 1 + Corr(X,Y) \geq 0$$

$$\Rightarrow \boxed{Corr(X,Y) \geq -1}$$

b) we need; $Corr(X,Y) \leq 1$

We know; $Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$

$$= E[XY] - E[X] \cdot E[Y]$$

Upon expanding the RHS of equation & using the concept of Max & Min;

we will get $Cov(X,Y)$ has maxima when $\boxed{X = Y}$

$$\therefore Cov(X,X) = E[X^2] - (E[X])^2$$

$$= Var(X)$$

$$Cov(X,X) = \sigma_X^2$$

$\therefore \text{Corr}(X,X) = \dfrac{\text{Cov}(X,X)}{\sigma_X^2} = \dfrac{\sigma_X^2}{\sigma_X^2} = 1$

$\therefore$ upper limit of $\text{Corr}(X,Y)$ is 1

$\therefore \quad \text{Corr}(X,Y) \le 1$

from a) part :- $\boxed{-1 \le \text{Corr}(X,Y) \le 1}$

Hence Proved.

c) σ_{X+Y} is the std. dev of $X+Y$;

$$\text{Var}(X+Y) = E\left[\left((X+Y) - E[X+Y]\right)^2\right]$$

$$= E\left[\left((X-E[X]) + (Y-E[Y])\right)^2\right]$$

$$= E\left[(X-E[X])^2 + (Y-E[Y])^2 + 2(X-E[X])(Y-E[Y])\right]$$

$$= E\left[(X-E[X])^2\right] + E\left[(Y-E[Y])^2\right] + 2E\left[(X-E[X])\cdot(Y-E[Y])\right]$$

$$= \sigma_X^2 + \sigma_Y^2 + 2\text{Cov}(X,Y)$$

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\text{Cov}(X,Y)$$

we had (from b)) $\quad \text{Corr}(X,Y) \le 1$

$$\Rightarrow \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} \le 1$$

$$\Rightarrow \boxed{Cov(X,Y) \leq \sigma_x \sigma_y}$$

$$\sigma^2_{X+Y} = \sigma_X^2 + \sigma_Y^2 + 2Cov(X,Y)$$

$$\sigma^2_{X+Y} \leq \sigma_X^2 + \sigma_Y^2 + 2\sigma_X \sigma_Y$$

$$\sigma^2_{X+Y} \leq (\sigma_X + \sigma_Y)^2$$

$$\Rightarrow \boxed{\sigma_{X+Y} \leq \sigma_X + \sigma_Y}$$

Hence Proved:

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② X & Y are Independent Poisson Random Variables with rate λ_1 & λ_2

Given; $Z = X + Y$

We need;

$$P_X(K) = \frac{e^{-\lambda_1}\lambda_1^K}{K!}$$

$$P_Z(z) = P(Z = z)$$

* Using result of Convolution in Discrete Random Variable case :-

$$P_Y(K) = \frac{e^{-\lambda_2}\lambda_2^K}{K!}$$

$$P_Z(z) = P(Z = z)$$

$$= \sum_{i=0}^{z} P(X = i \text{ and } Y = z - i)$$

$$= \sum_{i=0}^{z} P(X = i) \cdot P(Y = z - i)$$

$$= \sum_{i=0}^{z} \frac{e^{-\lambda_1}\lambda_1^i}{i!} \frac{e^{-\lambda_2}\lambda_2^{(z-i)}}{(z-i)!}$$

$$= \sum_{i=0}^{z} \frac{1}{i!(z-i)!} e^{-(\lambda_1+\lambda_2)} \cdot \lambda_1^i \lambda_2^{(z-i)}$$

$$= \sum_{i=0}^{z} \frac{z!}{i!(z-i)!} \frac{e^{-(\lambda_1+\lambda_2)} \cdot \lambda_1^i \lambda_2^{(z-i)}}{z!}$$

$$P_Z(z) = \sum_{i=0}^{z} \binom{z}{i} \lambda_1^{i} \lambda_2^{z-i} \frac{e^{-(\lambda_1+\lambda_2)}}{z!}$$

$$P_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \sum_{i=0}^{z} \binom{z}{i} \lambda_1^{i} \lambda_2^{(z-i)}$$

$$P_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)}}{z!} (\lambda_1+\lambda_2)^{z}$$

$\therefore$ $Z = X+Y$ is also a poisson Random variable with rate $\boxed{\lambda_1+\lambda_2}$

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(3) Given; X, Y & Z are Independent uniform Random variables in $[0,1]$.

& $W = X+Y+Z$

To find PDF of W;

Let us split this;

$$A = X+Y$$

$$W = A+Z$$

Since; $f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

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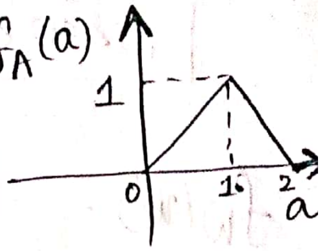
We know;

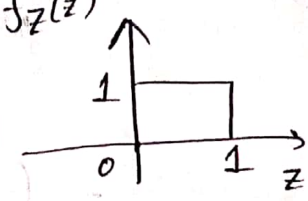
How $A = X + Y$ looks using Convolution.

$$f_A(a) = \begin{cases} a, & 0 < a < 1 \\ 2-a, & 1 < a < 2 \\ 0, & \text{else} \end{cases}$$



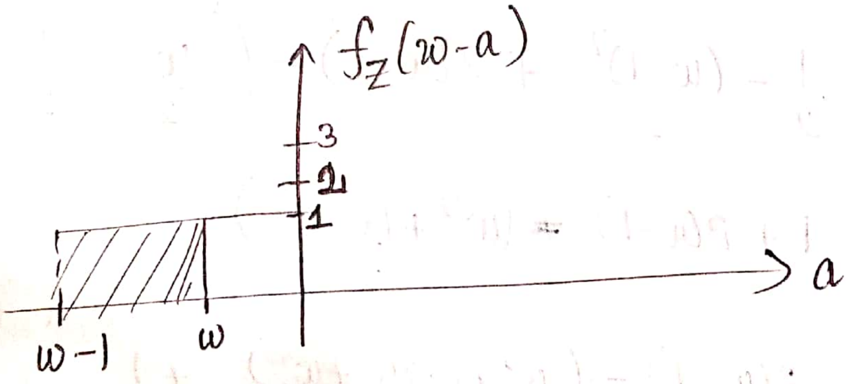
we also have;

$$f_Z(z) = \begin{cases} z, & 0 \le z \le 1 \\ 0, & \text{else} \end{cases}$$



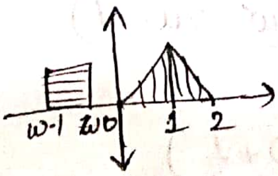
We need $f_W(w)$ where $W = A + Z$

$$f_W(w) = \int_{-\infty}^{+\infty} f_A(a) \cdot f_Z(w-a)\, da$$



Case-1:-

* $\boxed{w < 0}$



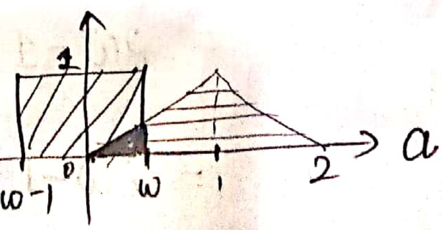
$$f_W(w) = \int 0 \cdot da$$

$$= 0$$

(No Overlap)

Case:-2

* $\boxed{0 < w < 1}$



$$f_W(w) = \int_0^w 1 \times a \times da$$

$$f_W(w) = \frac{w^2}{2}$$

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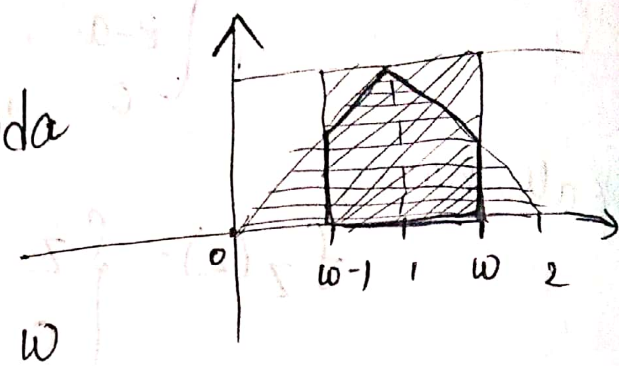
Case-3:-

$$\boxed{1 < w < 2}$$



$$f_W(w) = \int_{w-1}^{w} f_A(a) f_Z(w-a)\,da$$

$$= \int_{w-1}^{1} a(1)\,da + \int_{1}^{w} (2-a)\,da$$

$$= \left.\frac{a^2}{2}\right|_{w-1}^{1} + 2(w-1) - \left.\frac{a^2}{2}\right|_{1}^{w}$$

$$= \cancel{\frac{1}{2}} - \frac{(w-1)^2}{2} + 2(w-1) - \left(\frac{w^2}{2} - \cancel{\frac{1}{2}}\right)$$

$$= 1 + 2(w-1) - \frac{(w^2 + (w-1)^2)}{2}$$

$$= 2(w-1) - \frac{(w^2 + 1 - 2w + w^2)}{2} + 1$$

$$= 2(w) - \cancel{2} + \cancel{1} - \frac{(2w^2 - 2w + 1)}{2}$$

$$= 2w - 1 - \left(w^2 - w + \frac{1}{2}\right)$$

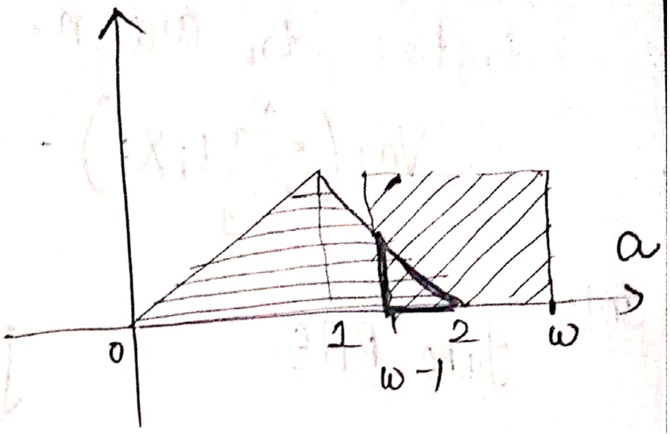
$$= 2w - 1 - w^2 + w - 1/2$$

$$= 3w - w^2 - 3/2$$

④ Case :- 4

$\boxed{w>2}$

$$f_W(w) = \int_{w-1}^{2} (1)(2-a)\,da$$



$$f_W(w) = \int_{w-1}^{2} (2-a)\,da$$

$$= 2\int_{w-1}^{2} da - \int_{w-1}^{2} a\,da$$

$$= 2(2-(w-1)) - \left(\frac{a^2}{2}\right)\Big|_{w-1}^{2}$$

$$f_W(w) = 2[3-w] - \frac{1}{2}\left[4-(w-1)^2\right]$$

$$f_W(w) = \cancel{6} - 2w - \cancel{2} + \frac{(w-1)^2}{2} = 4 - 2w + \frac{(w-1)^2}{2}$$

∴ Probability density funcn of $W = X+Y+Z$ is

$$f_W(w) = \begin{cases} 0 & ; \quad w<0 \\ \dfrac{w^2}{2} & ; \quad 0<w<1 \\ 3w - w^2 - 1.5 & ; \quad 1<w<2 \\ 4-2w+\dfrac{(w-1)^2}{2} & ; \quad w>2 \end{cases}$$

④ Given;

$X_1, X_2, \ldots X_n$ are n- Random Variables.

$t_1, t_2, \ldots t_n$ are n- Real numbers.

$$Var\left(\sum_{i=1}^{n} t_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j \, Cov(X_i, X_j)$$

Sol:-

Take RHS.

$$Var\left(\sum_{i=1}^{n} t_i X_i\right) = \underbrace{E\left[\left(\sum_{i=1}^{n} t_i X_i\right)^2\right]}_{A} - \underbrace{\left(E\left[\sum_{i=1}^{n} t_i X_i\right]\right)^2}_{B}$$

$$A = E\left[\left(\sum_{i=1}^{n} t_i X_i\right)^2\right] = E\left[\left(\sum_{i=1}^{n} t_i X_i\right)\left(\sum_{j=1}^{n} t_j X_j\right)\right]$$

$$A = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j X_i X_j\right]$$

$$A = \sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j \, E[X_i X_j]$$

linearity of Expectations

Now;

$$B = \left(E\left[\sum_{i=1}^{n} t_i X_i\right]\right)^2$$

$$B = \left(E\left[\sum_{i=1}^{n} t_i X_i\right]\right) \cdot \left(E\left[\sum_{i=1}^{n} t_i X_i\right]\right)$$

$$B = \left(\sum_{i=1}^{n} t_i \, E[X_i]\right) \cdot \left(\sum_{j=1}^{n} t_j \, E[X_j]\right)$$

$$B = \sum_{i=1}^{n} \sum_{j=1}^{n} t_i t_j E[X_i] E[X_j]$$

Now;

$$Var\left(\sum_{i=1}^{n} t_i X_i\right) = A - B$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} t_i t_j \left(E[X_i X_j] - E[X_i] E[X_j]\right)$$

$$Var\left(\sum_{i=1}^{n} t_i X_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} t_i t_j \, Cov(X_i, X_j)$$

Hence Proved.

---

(5) Given; $X = [X_1, X_2, X_3]^T$ is a random vector with means $\mu = [\mu_1, \mu_2, \mu_3]^T$.

Also;

'Σ' is Covariance matrix of 'X' Such that

$$\Sigma_{ij} = Cov(X_i, X_j)$$

We need to show;

$$\Sigma = E\left((X-\mu)(X-\mu)^T\right)$$

(5) Since we have X, μ :-

$$X-\mu = \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \\ X_3-\mu_3 \end{bmatrix}$$

$$(X-\mu)^T = [X_1-\mu_1, X_2-\mu_2, X_3-\mu_3]$$

$$(X-\mu)(X-\mu)^T = \sum_{i=1}^{3}\sum_{j=1}^{3}(X_i-\mu_i)(X_j-\mu_j)$$

$$E[(X-\mu)(X-\mu)^T] = E\left[\sum_{i=1}^{3}\sum_{j=1}^{3}(X_i-\mu_i)(X_j-\mu_j)\right]$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}E[(X_i-\mu_i)(X_j-\mu_j)]$$

Using lineartiy of Expectation

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}Cov(X_i, X_j)$$

We are also given that;

$$Cov(X_i, X_j) = \Sigma_{i,j}$$

So;

$$\boxed{\Sigma = E[(X-\mu)(X-\mu)^T]}$$

Hence Proved.