Assignment 3
CS 230 /CS 561: Probability and Statistics for CS
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INSTRUCTIONS: Solve all problems. Please submit your solutions for assignment in google classroom. You may refer to standard normal tables if required.

I Continuous distributions

1. Let $I = \int_{-\infty}^{\infty} e^{-x^2/2} dx$. Show that $I = \sqrt{2\pi}$ by writing

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2} dx \int_{-\infty}^{\infty} e^{-y^2/2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy$$

   and the evaluating the double integral by means of a change of variables to polar coordinates.

2. The lifetime of a computer chip produced by a certain firm are normally distributed with mean 4.4×10^6 hours with a standard deviation of 3×10^5 hours. If a client wishes to procure a large batch of chips and has a requirement that at least 90% of procured chips has a lifetime of 4×10^6 hours. Should this particular client procure from the chip manufacturer?

3. In the above problem, what is the probability that a batch of 100 chips will contain at least 4 chips whose lifetimes are less than 3.8×10^6 hours?

4. A system can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

   what is the probability that the system functions for atleast 5 months?

5. Show that the expected value of a discrete or continuous random variable X satisfies

$$E(X) = \int_0^{\infty} P(X > x) dx - \int_0^{\infty} P(X < -x) dx$$

6. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & \text{if } x > 10 \\ 0 & \text{if } x \leq 10 \end{cases}$$

   (a) Find $P(X > 20)$

(b) What is the cumulative distribution function of X?

(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumption is required to solve this?

7. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute (you can state the answers in terms of $\phi(.)$ the standard normal cdf)

(a) $P(X > 5)$

(b) $P(4 < X < 16)$

(c) $P(X < 8)$

(d) $P(X < 20)$

II Derived distributions

1. If U is uniform on $(0, 2\pi)$ and Z, independent of U, is exponential with rate 1, show that X and Y defined by

$$X = \sqrt{2Z}\cos U$$

$$Y = \sqrt{2Z}\sin U$$

are independent standard normal variables.

2. A random variable X is said to have a lognormal distribution if $\log X$ is normally distributed. If X is lognormal with $E[\log X] = \mu$ and $\mathrm{Var}(\log X) = \sigma^2$ determine the probability density function of X.

3. Let U be uniform random variable with range $[0, 1]$. Let $Y = \log(U)$. Find the pdf of Y?

III Other topics

1. The standard deviation of a random variable is the positive square root of its variance. Letting σ_X and σ_Y denote the standard deviations of the random variables X and Y, we define the correlation of X and Y by

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

(a) Starting with the inequality $Var(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}) \geq 0$, show that $-1 \leq Corr(X, Y)$

(b) Prove the inequality

$$-1 \leq Corr(X, Y) \leq 1$$

(c) If σ_{X+Y} is the standard deviation of $X + Y$, show that

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y$$

2. If X and Y are independent Poisson random variables with rate λ_1 and λ_2. Find the probability density function of $Z = X + Y$ using convolution?

3. If X, Y and Z are independent uniform random variables in the range $[0, 1]$. Find the probability density function of $W = X + Y + Z$ using convolution?

4. Let $X_1, X_2 \ldots X_n$ are n random variables and let $t_1, t_2 \ldots t_n$ are n real numbers. Prove that

$$Var\left(\sum_{i=1}^{n} t_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j Cov(X_i, X_j)$$

5. Let $\mathbf{X} = [X_1, X_2, X_3]^T$ be a random vector with means $\boldsymbol{\mu} = [\mu_1, \mu_2, \mu_3]^T$. Let Σ denote the covariance matrix of X, such that $\Sigma_{i,j} = Cov(X_i, X_j)$ (the entry in the ith row and jth column). Prove that

$$\Sigma = \mathbb{E}((\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T)$$