Aluffi - Algebra: Chapter 0

Chapter 1: Preliminaries: Set Theory and Categories

August 11, 2018

1 Naive Set Theory

Exercise 1.1. Russell's paradox demonstrates a contradiction in naive set theory. Any definable collection is a set. Let R be the collection of all sets which are not members of themselves, i.e. $R = \{A \mid A \notin A\}$. Then R is a set. But then if $R \in R$, $R \notin R$. We have a contradiction.

Exercise 1.2. Let $\sim$ be a relation on a set S. By reflexivity, every equivalence class $[a] = \{b \in S \mid a \sim b\}$ contains at least one element, $a \in [a]$. Similarly, every element of S is in an equivalence class. Then, if $a \in [b]$ and $a \in [c]$, $a \sim b$ and $a \sim c$. By transitivity, $[b] = [c]$. Therefore distinct equivalence classes are disjoint. Thus an equivalence relation defines a partition, $\mathcal{P}_\sim$, of S.

Exercise 1.3. Let $\mathcal{P}$ be a partition of a set S, and define a relation $\sim$ on S by: $a \sim b$ iff $a, b \in A$ for some $A \in \mathcal{P}$. Then we immediately have reflexivity, symmetry, and transitivity, so $\sim$ is an equivalence relation.

Exercise 1.4. Given the set $\{1,2,3\}$, we have four partitions: $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{1,2,3\}\}$. Since equivalence relations are in one-to-one correspondence with partitions, there are therefore four distinct equivalence relations on $\{1,2,3\}$.

Exercise 1.5. If a relation is reflexive and symmetric but not transitive, the associated equivalence classes are not disjoint, so the relation does not define a partition. Consider $S = \{1,2,3\}$, and define a relation R, thought of as a subset $R \subset S \times S$, by

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}$$

Since we have $(x, x) \in R$ for all $x \in S$, we have reflexivity. Since we have $(x, y) \in R$ for every $(y, x) \in R$, we have symmetry. However, while we have $(1, 2)$ and $(1, 3)$, we do not have $(2, 3)$, so R is not transitive.

Exercise 1.6. Define a relation $\sim$ on $\mathbb{R}$ by $a \sim b$ iff $b - a \in \mathbb{Z}$. Then $a - a = 0 \in \mathbb{Z}$ (reflexivity), $a - b \in \mathbb{Z} \Rightarrow b - a = -(a - b) \in \mathbb{Z}$ (symmetry), and $a - b \in \mathbb{Z}$, $b - c \in \mathbb{Z} \Rightarrow a - c = (a - b) + (b - c) in \mathbb{Z}$ (transitivity). Therefore $\sim$ is an equivalence relation. Then the quotient $R/\sim$ is the real line with all numbers separated by integers identified, i.e. the circle S^1 with radius $1/2\pi$.
Similarly, define $\tilde{\sim}$ on $\mathbb{R}^2$ by $(a_1, a_2) \tilde{\sim} (b_1, b_2)$ iff $a_1 \sim b_1$ and $a_2 \sim b_2$. By the same logic this is also an equivalence relation. Similarly, it is the torus T^2, where both S^1 factors have radius $1/2\pi$.

2 Functions between Sets

Exercise 2.1. Suppose S has n elements. A bijection $f : S \to S$ is just a permutation of these n elements, so there are $n!$ different such bijections.

Exercise 2.2. Let $A \neq \emptyset$, and $f : A \to B$ be a function.
If f has a right inverse, there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. Then for every $b \in B$, $f(g(b)) = b$. So f must be surjective.
If f is surjective, for every $b \in B$ there is some $a \in A$ such that $f(a) = b$. To make sure g is well-defined, let $g(b) = a$, for *one* such a. We are allowed to do this by the axiom of choice. Then $f(g(b)) = b$, so $f \circ g = \mathrm{id}_B$, and g is the right-inverse of f.
Therefore f has a right-inverse iff it is surjective.

Exercise 2.3. Let $f : A \to B$ be a bijection.

(i) f has an inverse $f^{-1} : B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. Then by Corollary 1.3 f^{-1} is trivially itself a bijection.

(ii) Now let $g : B \to C$ be a bijection as well, and consider $g \circ f$. We have that $f(a) = f(a')$ iff $a = a'$, and $g(f(a)) = g(f(a'))$ iff $f(a) = f(a')$, so $g(f(a)) = g(f(a'))$ iff $a = a'$. So $g \circ f$ is injective. Next, for all $c \in C$, there is some $b \in B$ such that $g(b) = c$, and for all $b \in B$ there is some $a \in A$ such that $f(a) = b$. Therefore for all $c \in C$ there is some $a \in A$ such that $g(f(a)) = c$. So $g \circ f$ is surjective. Therefore it is a bijection, i.e. compositions of bijections are bijections.

Exercise 2.4. Let S be a set of sets, and define a relation $\sim$ on S by $A \sim B$ iff $A \cong B$. Then, reflexivity follows from the existence of the identity (which is a bijection), symmetry follows from the first part of the previous exercise, and transitivity from the second part. Therefore $\sim$ is an equivalence relation.

Exercise 2.5.
Definition. A function $f : A \to B$ is an **epimorphism** if: for all sets Z and functions $\alpha', \alpha'' : B \to Z$,

$$\alpha' \circ f = \alpha'' \circ f \Rightarrow \alpha'\alpha''$$

Proposition. A function is surjective iff it is an epimorphism.

Proof. ($\Rightarrow$) Let $f : A \to B$ be surjective. Then it has a right-inverse $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. Then if $\alpha' \circ f = \alpha'' \circ f$,

$$\begin{aligned} (\alpha' \circ f) \circ g &= (\alpha'' \circ f) \circ g \\ \alpha' \circ (f \circ g) &= \alpha'' \circ (f \circ g) \\ \alpha' &= \alpha'' \end{aligned}$$

so f is an epimorphism.
($\Leftarrow$) Suppose f is an epimorphism, and define

$$\begin{aligned} \alpha' &: B \cup \{B\};\ b \mapsto b \\ \alpha'' &: B \cup \{B\};\ b \mapsto \begin{cases} b & b \in \operatorname{Im} f \\ B & b \notin \operatorname{Im} f \end{cases} \end{aligned}$$

Then $\alpha' \circ f = \alpha \circ f$, but $\alpha' = \alpha''$ only if $\operatorname{Im} f = \emptyset$, i.e. if f is surjective. Since f is an epimorphism, this must be the case. $\square$

Exercise 2.6. Given sets A, B, we have the projections

$$\begin{aligned} \pi_A &: A \times B \to A;\ (a, b) \mapsto a \\ \pi_B &: A \times B \to B;\ (a, b) \mapsto b \end{aligned}$$

A **section** s of π_A is a map $s : A \to A \times B$ such that $\pi_A \circ s = \mathrm{id}_A$. Consider the map $g : A \to A \times B$ given by $g = \mathrm{id}_A \times f$, i.e. $g(a) = (a, f(b))$, where $f : A \to B$. This is naturally induced by f. Then $\pi_A \circ s(a) = a$, so g is a section of π_A.

Exercise 2.7. Let $f : A \times B$ be a function. It maps every $a \in A$ to exactly one $b \in B$. The graph of f is

$$\Gamma_f = \{(a, b) \in A \times B \mid b = f(a)\}$$

Since $f(a)$ is uniquely determined by a, every element of Γ_f can be uniquely specified by the corresponding element of A. Therefore $\Gamma_f \cong A$.

Exercise 2.8. Let $f : \mathbb{R} \to \mathbb{C}$ be defined by $f(r) = e^{2\pi ir}$. Define $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $f(x) = f(y)$. That is, $x \sim y$ iff $x - y \in \mathbb{Z}$ (cf. Exercise 1.6). Now let us decompose f as in Theorem 2.7. $\pi : \mathbb{R} \to \mathbb{R}/\sim$ is given by $\pi(r) = r \mod 1$. Next, $\tilde{f}([r]) = f(r) = e^{2\pi ir}$. $\iota : \mathbb{C} \hookrightarrow \mathbb{C}$ is obvious.

Exercise 2.9. Suppose $A' \cap B' = \emptyset$, $A'' \cap B'' = \emptyset$, and that we can define isomorphisms

$$f : A' \to A'';\ a' \mapsto a''$$
$$g : B' \to B'';\ b' \mapsto b''$$

Then define $h : A' \cup B' \to A'' \cup B''$ by

$$h(x) = \begin{cases} f(x) & x \in A' \\ g(x) & x \in B' \end{cases}$$

Since f and g are bijections, and $A' \cap B' = A'' \cap B'' = \emptyset$, h is also a bijection. That is, $A' \cup B' \cong A'' \cup B''$. Therefore, if we define the disjoint union $A \sqcup B$ by first mapping each A and B to new disjoint sets by isomorphisms, then take the usual union, this definition is only up to isomorphisms of the disjoint sets.

Exercise 2.10. Let A and B be finite sets. The set B^A is the set of all functions $f : A \to B$. f must be defined for all $a \in A$, of which there are $|A|$, and for each a it must pick exactly one $b \in B$, of which there are $|B|$. Therefore the number of such functions is $|B^A| = |B|^{|A|}$.

Exercise 2.11. The **power set** $\mathcal{P}(A)$ of A is the set of all subsets of A. Suppose A is finite. Then $\mathcal{P}(A)$ consists of: $\emptyset$; $|A|$ sets of one element; $\binom{|A|}{2}$ sets of two elements;...; A itself. Therefore

$$\begin{aligned} |\mathcal{P}(A)| &= \sum_{n=0}^{|A|} \binom{|A|}{n} \\ &= 2^{|A|} \end{aligned}$$

Therefore, if B is some set of two elements, by the previous exercise, B^A has $2^{|A|}$ elements, and therefore we can set up a bijection between it and $\mathcal{P}(A)$.

3 Categories

Exercise 3.1. Let C be a category. Define the **opposite category**, C^{op}, by

- $\mathrm{Obj}(\mathsf{C}^{op}) = \mathrm{Obj}(\mathsf{C})$
- for all $A, B \in \mathrm{Obj}(\mathsf{C}^{op})$, $\mathrm{Hom}_{\mathsf{C}^{op}}(A,B) = \mathrm{Hom}_{\mathsf{C}}(B,A)$

We have identities in C^{op} since we do in C, and the inverse of an identity is an identity. Suppose $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,B)$, $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B,C)$. Then $f \in \mathrm{Hom}_{\mathsf{C}}(B,A)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(C,B)$, so we have the composition $fg \in \mathrm{Hom}_{\mathsf{C}}(C,A)$. Therefore we have $fg \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,C)$, and this defines composition in C^{op}. Clearly it is associative, and identities inherited from C are indeed identities under this composition. Therefore C^{op} is a category. Diagrammatically, we take C to C^{op} by flipping all arrows.

Exercise 3.2. Let A be a finite set. Then $\mathrm{End}_{\mathsf{Set}}$ has $|A|^{|A|}$ elements, by Exercise 2.10.

Exercise 3.3. Suppose $\mathrm{Hom}_{\mathsf{C}}(a,b)$ and $\mathrm{Hom}_{\mathsf{C}}(b,c)$ are non-empty, i.e. $\mathrm{Hom}_{\mathsf{C}}(a,b) = \{(a,b)\}$, $\mathrm{Hom}_{\mathsf{C}}(b,c) = \{(b,c)\}$. Composition is defined by $(a,b)\circ(b,c) = (a,c) \in \mathrm{Hom}_{\mathsf{C}}(a,c)$, which exists since $\sim$ is transitive. For the case $c = b$, $\mathrm{Hom}_{\mathsf{C}}(b,b)$ is obviously not empty, since $\sim$ is reflexive, and if C is a category, it must include an identity element. Indeed, it is the singlet $\{(b,b)\}$, and $(a,b)\circ(b,b) = (a,b)$, so acts as the identity on the right for any morphism into b. Similarly for identities on the left.

Exercise 3.4. We cannot use $>$ as the relation in the construction of Example 3.3 since it is not reflexive, and hence we have no identities and C cannot be a category.

Exercise 3.5. Example 3.4 is a construction like that in Example 3.3, with the relation $\subseteq$ on the set $\mathcal{P}(S)$.

Exercise 3.6. Define a category V by

- $\mathrm{Obj}(\mathsf{V}) = \mathbb{N}$
- $\mathrm{Hom}_{\mathsf{V}}(n,m) = \mathbb{R}^{m\times n}$

Then define the composition of $f \in \mathrm{Hom}_{\mathsf{V}}(n,m)$ with $g \in \mathrm{Hom}_{\mathsf{V}}(m,p)$ in terms of normal matrix multiplication, so that $fg \in \mathrm{Hom}_{\mathsf{V}}(n,p)$. Clearly $1_n \in \mathrm{Hom}_{\mathsf{V}}(n,n)$ is the unit matrix, which is the identity under this notion of composition. We also know that matrix multiplication is associative. Notice that while this definition might seem to break down if

n or $m = 0$, in which case $\mathbb{R}^{m \times n} = \{0\}$, but we only need to compose such morphisms with others with n or $m = 0$, in which case we can trivially define composition by $0 \circ 0 = 0$.

Exercise 3.7. For a category C, and object $A \in \mathrm{Obj}(\mathsf{C})$, we have the coslice category C^A defined in the following way.

- $\mathrm{Obj}(\mathsf{C}^A) = \{f \in \mathrm{Hom}_{\mathsf{C}}(A, Z) \mid Z \in \mathrm{Obj}(\mathsf{C})\}$, i.e. objects of C^A are arrows

$$\begin{tikzcd} A \arrow[d, "f"] \\ Z \end{tikzcd}$$

- Given $f_1, f_2 \in \mathsf{C}^A$, we have the morphism $f_1 \to f_2$ given by the commuting diagram

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Z_1 \arrow[rr, "\sigma"] & & Z_2 \end{tikzcd}$$

  where $\sigma : Z_1 \to Z_2$ is a morphism of C.

Compositions are given by the diagram

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\ Z_1 \arrow[r, "\sigma_1"] & Z_2 \arrow[r, "\sigma_2"] & Z_3 \end{tikzcd}$$

which reduces to

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_3"] & \\ Z_1 \arrow[rr, "\sigma_2\sigma_1"] & & Z_3 \end{tikzcd}$$

Exercise 3.8. Define a category C by

- $\mathrm{Obj}(\mathsf{C})$ is the class of all infinite sets
- $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is the set of all set-functions $f : A \to B$

Then clearly $\mathrm{Obj}(\mathsf{C})$ is a subcollection of the collection of all sets, and $\mathrm{Hom}_{\mathsf{C}}(A, B) = \mathrm{Hom}_{\mathsf{Set}}(A, B)$. Composition, associativity, and identities are then inherited by virtue of the fact that Set is a category, so C is a category, and indeed it is a subcategory of Set.

Exercise 3.9. We can arrive at a notion of multisets by considering normal sets endowed with equivalence relations. For instance, given the set $\{a, b, c, d\}$, with the equivalence relation $\sim$ such that $a \sim d$, is equivalent to a multiset $\{a, a, b, c\}$. Define the category MSet by

- $\mathrm{Obj}(\mathsf{MSet})$ are such sets together with equivalence relations
- $\mathrm{Hom}_{\mathsf{MSet}}(A, B)$ are set-functions $f : A \to B$ such that $f(a') \sim_B f(a'')$ if $a' \sim_A a''$

The identity is the usual identity map $1_A : A \to A$, and indeed if $a' \sim a''$, $1_A(a') \sim 1_A(a'')$. Let $f : A \to B$ and $g : B \to C$. We can define the composition $h = g \circ f$ in the usual way, since the equivalence of elements is preserved by both f and g, and similarly associativity. Thus MSet is a category. Clearly Set is a subcategory, with the associated subcollection of objects those 'enhanced sets' whose equivalence relations are trivial. In fact, since they are trivial, there are no restrictions on the set-functions allowed to be morphisms, so in fact Set is a full subcategory of MSet.

Exercise 3.10. Sometimes it will make sense to talk about subobjects, such as when the objects of a category are sets. Given any object A of a category C, its subobjects are in one-to-one correspondence with morphisms $A \to \Omega$ for a particular object Ω, which is called a **subobject classifier**. For the case of Set, we know from Exercise 2.11 that the set of all subsets of a set A, $\mathcal{P}(A)$, can be put into one-to-one correspondence with B^A, where $|B| = 2$, i.e. the set of all functions $f : A \to B$. Thus we have that Ω is any set of order 2.

Exercise 3.11. $\mathsf{C}^{A,B}$ is constructed just as $C_{A,B}$, but with all arrows reverse (except $\sigma : Z \to Z$ in the definition of composition). Similarly for $\mathsf{C}^{\alpha,\beta}$ from $C_{\alpha,\beta}$.

4 Morphisms

Exercise 4.1. For three morphisms f, g, h in a category C such that

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have, by definition of a category, $(fg)h = f(gh)$. Now consider

$$Z_1 \xrightarrow{f_1} ... \xrightarrow{f_n} Z_n$$

for $n \geq 4$, and suppose we know that all parenthesisations of $(n-1)$ morphisms are equivalent. Denote an arbitrary parenthesisation of $f_1, ..., f_n$ by f, and write $f = hg$, where

g is some parenthesisation of $f_1, ..., f_i$ for some $1 \leq i \leq n$ and h is some parenthesisation of $f_{i+1}, ..., f_n$. Then we can write

$$\begin{aligned} g &= \Big(f_i\big(f_{i-1}(...(f_2 f_1))\big)\Big) \\ &= f_i g' \end{aligned}$$

which defines g', and

$$h = \Big(\big(...((f_n f_{n-1}) f_{n-2})...\big) f_{i+1}\Big)$$

Then we have

$$f = hg = h(f_i g') = (h f_i) g'$$

which brings any initial parenthesisation f into the canonical form

$$f = \Big(\big(...((f_n f_{n-1}) f_{n-2})...\big) f_1\Big)$$

Thus by induction on n all parenthesisations of a composition of any number of morphisms is equivalent.

Exercise 4.2. Define the category C as in Example 3.3, i.e. construct it from a set S and a relation $\sim$ on S which is reflexive and transitive. If $\sim$ is also symmetric, then for each morphism (a, b), we also have the morphism (b, a). Indeed, $(a, b) \circ (b, a) = (a, a)$, and $(b, a) \circ (a, b) = (b, b)$, so (b, a) is the inverse of (a, b). Therefore every morphism in the category has an inverse, and is therefore an isomorphism. Thus this category is a groupoid.

Exercise 4.3. Let A and B be objects of a category C, and $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$.

(i) Suppose f has a right-inverse, g. Then suppose that, for any object Z and morphisms $\beta', \beta'' \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$,

$$\beta' \circ f = \beta'' \circ f$$

Then

$$\begin{aligned} (\beta' \circ f) \circ g &= (\beta'' \circ f) \circ g \\ \beta' \circ (f \circ g) &= \beta'' \circ (f \circ g) \\ \beta' &= \beta'' \end{aligned}$$

so f is epic.

(ii) Consider the category constructed as in Example 3.3 from $\mathbb{Z}$ and $\leq$. As in Example 4.10, every morphism in this category is epic. However, a morphism $f : n \to m$, where $m < n$, cannot have a right-inverse. This demonstrates hat it is not in general true that epimorphisms have right-inverses.

Exercise 4.4.

(i) Let $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ be monomorphisms, and suppose that for any $Z \in \mathrm{Obj}(\mathsf{C})$ and $\alpha', \alpha'' \in \mathrm{Hom}_{\mathsf{C}}(Z, A)$,

$$(g \circ f) \circ \alpha' = (g \circ f) \circ \alpha''$$

Then

$$g \circ (f \circ \alpha') = g \circ (f \circ \alpha'')$$

Since g is monic,

$$f \circ \alpha' = f \circ \alpha''$$

and since f is monic,

$$\alpha' = \alpha''$$

Therefore $g \circ f$ is monic.

(ii) Then given a category C, we can define a subcategory C_{mono} by

- $\mathrm{Obj}(\mathsf{C}_{mono}) = \mathrm{Obj}(\mathsf{C})$
- $\mathrm{Hom}_{\mathsf{C}_{mono}}(A, B) = \{f \in \mathrm{Hom}_{\mathsf{C}}(A, B) \mid f \text{ monic}\}$

This is a category by virtue of the previous result and the fact that all identities are monic. However, the analogous structure defined by including only non-monic morphisms is not a subcategory, since it includes no identities.

(iii) Now let f, g be epic, and suppose that for all $Z \in \mathrm{Obj}(\mathsf{C})$ and $\beta', \beta'' \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$

$$\beta' \circ (g \circ f) = \beta'' \circ (g \circ f)$$

Then

$$(\beta' \circ f) \circ f = (\beta'' \circ g) \circ f$$

Since f is epic,

$$\beta' \circ g = \beta'' \circ g$$

and since g is epic

$$\beta' = \beta''$$

Therefore $g \circ f$ is epic.

(iv) Then we can construct C_{epic} due to this result, and the fact that identities are epic. Again, the analogous structure defined by taking all non-epic morphisms contains no identities and therefore is not a category.

Exercise 4.5. Consider the category MSet as constructed in Exercise 3.9. Let $z', z'' \in Z \in \mathrm{Obj}(\mathsf{C})$, and suppose $z' \sim z''$. Then, monomorphisms are naturally defined by

$$f \circ \alpha'(z) \sim_B f \circ \alpha''(z) \Rightarrow \alpha'(z) \circ \alpha''(z)$$

where $f : A \to B$, for any $\alpha', \alpha'' \in \mathrm{Hom}_{\mathsf{MSet}}(Z, A)$. Similarly, we naturally define epimorphisms by

$$\beta' \circ f(a') \sim \beta'' \circ f(a'') \Rightarrow f(a') \sim f(a'')$$

5 Universal Properties

Exercise 5.1. Let F be a final object in a category C. Then for all objects A in C, there exists a unique morphism $A \to F$ in C. The opposite category, C^{op}, is defined to have the same objects as C, but $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(B, A)$. Therefore F is an object in C^{op}, and there is a unique morphism $F \to A$ for every object $A \in \mathrm{Obj}(\mathsf{C}^{op}) = \mathrm{Obj}(\mathsf{C})$. Therefore F is initial in C^{op}.

Exercise 5.2. In the category Set, the empty set is initial, and is the only initial set. To see this, realise that if a set S is infinite there can be no unique morphism $S \to A$ for all sets A, and indeed the same is true if $|S| \geq 1$. This leaves exactly one set, $\emptyset$. On the other hand, every singleton is a final set.

Exercise 5.3. Let C be a category. If F is final in C, there is a unique morphism $F \to F$, which must be the identity. Suppose F_1 and F_2 are final. Then there are unique morphisms $f : F_1 \to F_2$ and $g : F_2 \to F_1$. Since F_1 is final, the composition $gf : F_1 \to F_1$ must be the identity 1_{F_1}, and since F_2 is final, the composition $fg : F_2 \to F_2$ must be the identity 1_{F_2}. Therefore $f : F_1 \to F_2$ is an isomorphism.

Exercise 5.4. A pointed set is an ordered pair (S, s), where S is a set and $s \in S$. The category of pointed sets, Set^*, has morphisms $f : (S, s) \to (T, t)$ such that $f(s) = t$. Suppose (S, s) is initial, i.e. that $(S, s) \to (T, t)$ is unique for any (T, t). If S has more than one element, say if $S = \{s, s'\}$, then there are many functions taking $f(s) = t$ (i.e. since $f(s')$ could be any element of T). Therefore S must be the singleton $\{s\}$. So initial objects in Set^* are $(\{s\}, s)$. Similarly final objects.

Exercise 5.5. Consider a set A with an equivalence relation $\sim$, and the category $\mathsf{Q}_\sim$, whose objects are set-functions $\phi : A \to Z$ such that if $a' \sim a''$ then $\phi(a') = \phi(a'')$. We use the notation for these objects (ϕ, Z). Suppose (ϕ, Z) is a final object of $\mathsf{Q}_\sim$. Then there is a unique commuting diagram

$$\begin{tikzcd} Z' \arrow[r, "\sigma"] & Z \\ & A \arrow[ul, "\phi'"] \arrow[u, "\phi"'] \end{tikzcd}$$

for any (ϕ', Z'). Knowing that the final objects of Set are singletons, we can see that $(\phi, Z) = (c, \{c\})$, where $c : A \to \{c\}$ is the constant function, which therefore trivially satisfies the condition in the definition of $\mathsf{Q}_\sim$. Then also σ is the constant function to c, and the diagram commutes. This is the unique choice, from the uniqueness of σ and the result for Set.

Exercise 5.6. Define the category C as in Example 3.3 with the set $\mathbb{Z}^+$ and the relation of divisibility, i.e. there exists a morphism $d \to m$ iff $m/d \in \mathbb{Z}^+$, and this morphism, if it exists, is unique.

(i) The category $\mathsf{C}_{d,m}$ has for objects diagrams

$$\begin{tikzcd} & d \\ n \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & m \end{tikzcd}$$

where $d/n, m/n \in \mathbb{Z}^+$. A morphism in $\mathsf{C}_{d,m}$ is a diagram

$$\begin{tikzcd} & & d \\ n' \arrow[r, "\sigma"] \arrow[urr, bend left, "f'"] \arrow[drr, bend right, "g'"'] & n \arrow[ur, "f"] \arrow[dr, "g"] & \\ & & m \end{tikzcd}$$

where $d/n', m/n', n/n' \in \mathbb{Z}^+$. This is unique if n is the highest common factor of d and m. Therefore we take the product in C to be $n = d \times m = \text{hcf}(d, m)$.

(ii) The category $\mathsf{C}^{d,m}$ has for objects diagrams

$$\begin{tikzcd} d \arrow[dr, "f"] & \\ & n \\ m \arrow[ur, "g"] & \end{tikzcd}$$

where $d/n, m/n \in \mathbb{Z}^+$. A morphism in $\mathsf{C}^{d,m}$ is a diagram

$$\begin{tikzcd}
d \arrow[dr, "f'"] \arrow[drr, bend left, "f"] & & \\
& n' \arrow[r, "\sigma"] & n \\
m \arrow[ur, "g'"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

where $n'/d, n'/m, n/n' \in \mathbb{Z}^+$. This is unique if n is the lowest common multiple of d and m. Therefore we take the coproduct in C to be $n = d \sqcup m = \mathrm{lcd}(d, m)$.

Exercise 5.7. Consider sets $A' \cong A''$, $B' \cong B''$ such that $A' \cap B' = \emptyset$, $A'' \cap B'' = \emptyset$. Define inclusions $\iota_{A'} : A' \hookrightarrow A' \cup B'$, $\iota_{B'} : B' \hookrightarrow A' \cup B'$. Now, let Z be a set and define morphisms $f_{A'} : A' \to Z$ and $f_{B'} : B' \to Z$. Then define a morphism $\sigma : A' \sqcup B' \cong A' \cup B' \to Z$ by

$$\sigma(x) = \begin{cases} f_{A'}(x) & x \in A' \\ f_{B'}(x) & x \in B' \end{cases}$$

Then we have the commuting diagram

$$\begin{tikzcd}
A' \arrow[dr, hook, "\iota_{A'}"] \arrow[drr, bend left, "f_{A'}"] & & \\
& A' \sqcup B] \arrow[r, "\sigma"] & Z \\
B] \arrow[ur, hook, "\iota_{B'}"] \arrow[urr, bend right, "f_{B'}"] & &
\end{tikzcd}$$

This mirrors the proof that disjoint unions are coproducts, so $A' \sqcup B \cong A' \cup B'$ is a disjoint union. Now, let $f : A' \to A''$ and $g : B' \to B''$ be isomorphisms, and define inclusions $\iota_{A''} : A'' \hookrightarrow A'' \cup B''$ and $\iota_{B''} : B'' \hookrightarrow A'' \cup B''$. Then define $\sigma : A'' \sqcup B'' \cong A'' \cup B'' \to Z$ by

$$\sigma(x) = \begin{cases} f_{A'} \circ f^{-1} & x \in A'' \\ f_{B'} \circ g^{-1} & x \in B'' \end{cases}$$

Then we have the commuting diagram

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[drrr, bend left, "f_{A'}"] & A'' \arrow[dr, hook, "\iota_{A''}"] & & \\
& & A' \sqcup B \arrow[r, "\sigma"] & Z \\
B' \arrow[r, "g"] \arrow[urrr, bend right, "f_{B'}"] & B'' \arrow[ur, hook, "\iota_{B''}"] & &
\end{tikzcd}$$

Thus we have that $A'' \sqcup B'' \cong A'' \cup B''$ is a disjoint union. Then, by Proposition 5.4, $A' \sqcup B'$ and $A'' \sqcup B''$ are isomorphic.

Exercise 5.8. If $A \times B$ is a product in C, then

$$\begin{tikzcd} & A \\ A \times B \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$$

is a final object in $\mathsf{C}_{A,B}$. Similarly, if $B \times A$ is a product in C,

$$\begin{tikzcd} & B \\ B \times A \arrow[ur] \arrow[dr] & \\ & A \end{tikzcd}$$

is a final object in $\mathsf{C}_{B,A}$. But this is the same auxiliary category as $C_{A,B}$ by their definitions. So both satisfy the same universal property for the product of A and B, and hence $A \times B \cong B \times A$.

Exercise 5.9. Let C be a category, and A, B, C objects in C. Then we define the three-product $A \times B \times C \in \text{Obj}()$ of A, B, C by the universal property that: if π_A, π_B, π_C are the usual projections, and Z any object in C with morphisms $f_A : Z \to A$, $f_B : Z \to B$, $f_C : Z \to C$, then there is a unique morphism $\sigma : Z \to A \times B \times C$ such that

$$\begin{tikzcd} & & A & \\ Z \arrow[rr, bend left, "f_A"] \arrow[r, "\sigma"] \arrow[drr, "f_C"] \arrow[drrr, bend right, "f_B"'] & A \times B \times C \arrow[ur, "\pi_A"] \arrow[dr, "\pi_C"] \arrow[drr, bend left, "\pi_B"] & & \\ & & C & B \end{tikzcd}$$

commutes. Now, consider the products $A \times B$ and $(A \times B) \times C$. Using the same notation conventions, from the first, we have

$$\begin{tikzcd}
 & & A \\
Z \arrow[r, "\sigma'"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] & A \times B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_C"] & \\
 & & C
\end{tikzcd}$$

where σ is unique, and from the second

$$\begin{tikzcd}
 & & A \times B \\
Z \arrow[r, "\sigma'"] \arrow[urr, bend left, "f_{A\times B}"] \arrow[drr, bend right, "f_C"'] & (A \times B) \times C \arrow[ur, "\pi_{A\times B}"] \arrow[dr, "\pi_C"] & \\
 & & C
\end{tikzcd}$$

where σ' is unique. Clearly these can be combined to create a diagram for a three-product, where the morphism $\sigma : Z \to (A \times B) \times C$ is unique. So $(A \times B) \times C$ is a three-product, and similarly so is $A \times (B \times C)$. Then, since both of these satisfy the same universal property in C, by Proposition 5.4 they are isomorphic.

Exercise 5.10. Now suppose we have defined $(n-1)$ products in C, and consider $A_1 \times ... \times A_{n-1}$ and $(A_1 \times ... \times A_{n-1}) \times A_n$. In the same way as the previous exercise we get to a notion of the n-product in C, and therefore have all n-products by induction. Similarly, we see that all parenthesisations are isomorphic.

Exercise 5.11. Let A, B be sets, and $\sim_A, \sim_B$ be equivalence relations on them, respectively. Then define a relation $\sim$ on $A \times B$ by

$$(a_1, b_1) \sim (a_2, b_2) \iff a_1 \sim_A a_2,\ b_1 \sim_B b_2$$

Clearly this is an equivalence relation. Now, by the quotient universal property, we have that, for any set Z and function $\phi : A \times B \to Z$, there is a unique function $\bar{\phi} : (A \times B)/\sim \to Z$ which makes

$$\begin{tikzcd}
(A \times B)/\sim \arrow[rr, "\bar{\phi}"] & & Z \\
 & A \times B \arrow[ul, "\pi_\sim"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

commute. Therefore a unique function $\pi^A : (A \times B)/ \sim \to A/ \sim_A$ exists if a function $A \times B \to A/ \sim_A$ does. But this latter is just $\pi_{\sim_A} \circ \pi_A$, so π^A exists. Similarly π^B. Now consider the diagram

$$\begin{tikzcd}
 & & A/ \sim_A \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] & (A \times B)/ \sim \arrow[ur, "\pi^A"] \arrow[dr, "\pi^B"] & \\
 & & B/ \sim_B
\end{tikzcd}$$

where σ is yet to be defined. Define by

$$\begin{aligned} f : Z &\to A/ \sim_A \times B/ \sim_B \\ z &\mapsto (f_A(z), f_B(z)) \end{aligned}$$

which is clearly unique given f_A, f_B. Then, by the quotient universal property we have the diagram

$$\begin{tikzcd}
A/ \sim_A \arrow[rr, "\bar{1}_A"] & & A \\
 & A \arrow[ul, "\pi_{\sim_A}"] \arrow[ur, "1_A"'] &
\end{tikzcd}$$

so we have a unique function $\bar{1}_A : A/ \sim_A \to A$, and similarly a function $\bar{1}_B : B/ \sim_B \to B$. Then define $g : Z \to A \times B$ by

$$g(z) = \big(\bar{1}_A(f_A(z)), \bar{1}_B(f_B(z))\big)$$

which is also unique given f_A, f_B. Then finally define σ by $\sigma = \pi_\sim \circ g$. This is also unique, and hence $(A \times B)/ \sim$ satisfies the universal property for the product of $A/ \sim_A$ and $B/ \sim_B$. Finally, by Proposition 5.4,

$$(A/ \sim_A) \times (B/ \sim_B) \cong (A \times B)/ \sim$$

Exercise 5.12. Recall that if $A, B, C \in \mathrm{Obj}(\mathsf{C})$, and $\alpha : A \to C$ and $\beta : B \to C$ are morphisms in C, then $\mathsf{C}_{\alpha,\beta}$ is a category with objects

$$
\begin{tikzcd}
 & A \arrow[dr, "\alpha"] & \\
Z \arrow[ur, "f"] \arrow[dr, "g"'] & & C \\
 & B \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

where $Z \in \mathrm{Obj}(\mathsf{C})$ and f and g are morphisms in C. Morphisms in $\mathsf{C}_{\alpha,\beta}$ are commuting diagrams

$$
\begin{tikzcd}
 & & A \arrow[dr, "\alpha"] & \\
Z_1 \arrow[r, "\sigma"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "g_1"'] & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & & C \\
 & & B \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

We then define a **fibred product** $A \times_C B \in \mathrm{Obj}(\mathsf{C})$ as an object of C such that, together with morphisms $\pi_A : A \times_C B \to A$ and $\pi_B : A \times_C \to B$, it is a final object in $\mathsf{C}_{\alpha,\beta}$. That is,

$$
\begin{tikzcd}
 & A \arrow[dr, "\alpha"] & \\
A \times_C B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & & C \\
 & B \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

is final in $\mathsf{C}_{\alpha,\beta}$. Then the universal property is that $A \times_C B$ satisfies the diagram

$$
\begin{tikzcd}
 & & A \arrow[dr, "\alpha"] & \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times_C B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & & C \\
 & & B \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

for any Z, $f : Z \to A$, $g : Z \to B$, in which there is a unique σ making the diagram commute.
Similarly, the category $\mathsf{C}^{\alpha,\beta}$ has objects

$$
\begin{tikzcd}
& A \arrow[dr, "f"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & C \\
& B \arrow[ur, "g"'] &
\end{tikzcd}
$$

and morphisms

$$
\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \\
& B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}
$$

where σ is unique. So we define a **fibred coproduct** $A \sqcup_C B \in \mathrm{Obj}(\mathsf{C})$ as an object of C that, together with morphisms $\iota_A : A \hookrightarrow A \sqcup_C B$ and $\iota_B : B \to A \sqcup_C B$, is an initial object in $C^{\alpha,\beta}$. That is, the universal property is that $A \sqcup_C B$ satisfies the diagram

$$
\begin{tikzcd}
& A \arrow[dr, hook, "\iota_A"] \arrow[drr, bend left, "f"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & A \sqcup_C B \arrow[r, "\sigma"] & Z \\
& B \arrow[ur, hook, "\iota_B"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}
$$

for any Z, $f : A \to Z$, $g : B \to Z$, in which there is a unique σ making the diagram commute.