Baez and Muniain - Gauge Fields, Knots and Gravity

Part 2: Gauge Fields

4 Chern-Simons Theory

Exercise 116. Locally,

$$\begin{aligned}
F_{\mu\nu} &= [D_\mu, D_\nu] \\
&= [D^0_\mu + A_\mu, D^0_\nu + A_\nu] \\
&= [D^0_\mu, D^0_\nu] + [D^0_\mu, A_\nu] - [D^0_\nu, A_\mu] + [A_\mu, A_\nu] \\
&= (F_0)_{\mu\nu} + [D^0_\mu, A_\nu] - [D^0_\nu, A_\mu] + [A_\mu, A_\nu] \\
\Rightarrow F &= F_0 + \frac{1}{2}\left([D^0_\mu, A_\nu] - [D^0_\nu, A_\mu] + [A_\mu, A_\nu]\right) \otimes dx^\mu \wedge dx^\nu \\
&= F_0 + [D^0_\mu, A] \otimes dx^\mu + A \wedge A \\
&= F_0 + dA + A \wedge A
\end{aligned}$$

as expected, where we have used $[A, A] = A \wedge A + A \wedge A$.

Exercise 117. Let $\omega =\in \Gamma(\mathrm{End}(E) \otimes \bigwedge^p T^*M)$ and $\mu \in \Gamma(\mathrm{End}(E) \otimes \bigwedge^q T^*M)$, and write $\omega = \omega_I \otimes dx^I$ and $\mu = \mu_J \otimes dx^J$. Then

$$\omega \wedge \mu = \omega_I \mu_J \otimes dx^I \wedge dx^J$$

Similarly,

$$\begin{aligned}
\mu \wedge \omega &= \mu_J \omega_I \otimes dx^J \wedge dx^I \\
&= (-1)^{pq} \mu_J \omega_I \otimes dx^I \wedge dx^J
\end{aligned}$$

Then

$$\mathrm{tr}(\mu \wedge \omega) = (-1)^{pq}\, \mathrm{tr}(\omega_J \mu_I) dx^I \wedge dx^J$$

Now,

$$(\mu_J \omega_I)^i_j = \mu^i_{Jk} \omega^k_{Ij}$$

So

$$\begin{aligned}\operatorname{tr}(\mu_J\omega_I) &= \mu^i_{Jk}\omega^k_{Ij}\\ &= \operatorname{tr}(\omega_I\mu_J)\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{tr}(\mu\wedge\omega) &= (-1)^{pq}\operatorname{tr}(\omega_I\mu_J)dx^I\wedge dx^J\\ &= (-1)^{pq}\operatorname{tr}(\omega\wedge\mu)\end{aligned}$$

Therefore we say that the trace is graded cyclic on wedge products of $\mathrm{End}(E)$-valued forms. This implies that

$$\begin{aligned}\operatorname{tr}([\omega,\mu]) &= \operatorname{tr}(\omega\wedge\mu-(-1)^{pq}\mu\wedge\omega)\\ &= \operatorname{tr}(\omega\wedge\mu)-(-1)^{pq}\operatorname{tr}(\mu\wedge\omega)\\ &= \operatorname{tr}(\omega\wedge\mu)-\operatorname{tr}(\omega\wedge\mu)\\ &= 0\end{aligned}$$

Exercise 118. Let $\omega\in\Gamma(\mathrm{End}(E)\otimes\bigwedge^p T^*M)$, and write $\omega=\omega_I\otimes dx^I$. Then

$$\begin{aligned}d_D\omega &= d_D\omega_I\wedge dx^I\\ &= D_\mu\omega_I\otimes dx^\mu\wedge dx^I\\ \operatorname{tr}(d_D\omega) &= \operatorname{tr}(D_\mu\omega_I)\otimes dx^\mu\wedge dx^I\end{aligned}$$

Let $s\in\Gamma(E)$ be $s=s^ie_i$. Then

$$(D_\mu\omega_I)(s)=D_\mu(\omega_Is)-\omega_I(D_\mu s)$$

We have

$$\begin{aligned}\omega_Is &= \omega^i_{Ij}s^je_i\\ D_\mu(\omega_Is) &= D^0_\mu(\omega^i_{Ij}s^j)e_i+A^i_{\mu j}\omega^j_{Ik}s^ke_i\end{aligned}$$

and

$$\begin{aligned}D_\mu s &= D^0_\mu s^ie_i+A^i_{\mu j}s^je_i\\ \omega_I(D_\mu s) &= \omega^i_{Ij}(D^0_\mu s^je_i+A^j_{\mu k}s^ke_i)\end{aligned}$$

Therefore

$$\begin{aligned}
(D_\mu \omega_I)(s) &= [(D^0_\mu \omega^i_{Ij})s^j + A^i_{\mu j}\omega^j_{Ik}s^k - \omega^i_{Ij}D^0_\mu s^i - \omega^i_{Ij}A^j_{\mu k}s^l]e_i \\
(D_\mu \omega_I)^i_j s^j &= (D^0_\mu \omega^i_{Ij})s^j + (A^i_{\mu k}\omega^k_{Ij} - A^k_{\mu j}\omega^s_{Ik})s^j \\
(D_\mu \omega_I)^i_j &= D^0_\mu \omega^i_{Ij} + A^i_{\mu k}\omega^k_{Ij} - A^k_{\mu j}\omega^s_{Ik}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{tr}(D_\mu \omega_I) &= D^0_\mu \omega^i_{Ii} + A^i_{\mu k}\omega^k_{Ii} - A^k_{\mu i}\omega^i_{Ik} \\
&= D^0_\mu \operatorname{tr}(\omega_I)
\end{aligned}$$

So we have

$$\begin{aligned}
\operatorname{tr}(d_D \omega) &= D^0_\mu \operatorname{tr}(\omega_I) \otimes dx^\mu \wedge dx^I \\
&= D^0_\mu \operatorname{tr}(\omega) \otimes dx^\mu \\
&= d \operatorname{tr}(\omega)
\end{aligned}$$

Exercise 119. Let M be compact and oriented, and $\dim M = n$. Let $\omega \in \Gamma(\operatorname{End}(E) \otimes \bigwedge^p T^*M)$ and $\mu \in \Gamma(\operatorname{End}(E) \otimes \bigwedge^q T^*M)$, where $p + q = n - 1$. We have

$$d_D(\omega \wedge \mu) = d_D\omega \wedge \mu + (-1)^p \omega \wedge d_D\mu$$

Then by the previous exercise, taking the trace we have

$$d \operatorname{tr}(\omega \wedge \mu) = \operatorname{tr}(d_D\omega \wedge \mu) + (-1)^p \operatorname{tr}(\omega \wedge d_D\mu)$$

The LHS is an n-form, and M is compact, so we can integrate over M:

$$\int_M d \operatorname{tr}(\omega \wedge \mu) = \int_M \operatorname{tr}(d_D\omega \wedge \mu) + (-1)^p \int_M \operatorname{tr}(\omega \wedge d_D\mu)$$

So assuming $\partial M = \emptyset$, we have by Stokes' theorem

$$\int_M \operatorname{tr}(d_D\omega \wedge \mu) = (-1)^{p+1} \int_M \operatorname{tr}(\omega \wedge d_D\mu)$$

Now, put a metric on M, so that we have $*$, and let $q = p$ (I think there is an error in the question where it says to take $p + q = n$). We have

$$\omega \wedge *\mu = \omega_I \mu_J \otimes dx^I \wedge *dx^J$$

Earlier we found that

$$dx^I \wedge *dx^J = g^{IJ}\,\mathrm{vol}$$

where here by g^{IJ} we mean p metrics with the appropriate indices. That is,

$$\omega \wedge *\mu = \omega_I \mu^I\,\mathrm{vol}$$

Locally, this is

$$\omega \wedge *\mu = \omega^i_{Ij}\mu^{Ij}_k e_i \otimes e^k$$

So

$$\begin{aligned}\mathrm{tr}(\omega \wedge *\mu) &= \omega^i_{Ij}\mu^{Ij}_i \\ &= \mathrm{tr}(\mu \wedge *\omega)\end{aligned}$$

This is an n-form, so we can integrate over M to obtain

$$\int_M \mathrm{tr}(\omega \wedge *\mu) = \int_M \mathrm{tr}(\mu \wedge *\omega)$$

Exercise 120. Suppose M is not compact. Then in general we cannot define

$$S_{YM} = \frac{1}{2}\int_M \mathrm{tr}(F \wedge *F)$$

However, we can define

$$\delta S_{TM} = \frac{1}{2}\int_M \delta\,\mathrm{tr}(F \wedge *F)$$

if we restrict to variations δA with compact support. Then we have

$$\begin{aligned}\delta S_{YM} &= \frac{1}{2}\int_M \mathrm{tr}(\delta F \wedge *F + F \wedge *\delta F) \\ &= \int_M \mathrm{tr}(\delta F \wedge *F) \\ &= \int_M \mathrm{tr}(d_D \delta A \wedge *F)\end{aligned}$$

Now we are inclined to use

$$d\,\mathrm{tr}(\delta A \wedge *F) = \mathrm{tr}(d_D \delta A \wedge *F) - \mathrm{tr}(\delta A \wedge d_D * F)$$

but we ought to check this is legitimate. Indeed since δA has compact support so does each term here, so this is ok. Then again assuming $\delta M = \emptyset$, we have

$$\delta S_{YM} = \int_M \mathrm{tr}(\delta A \wedge d_D * F)$$

Thus $\delta S_{YM} = 0$ implies

$$d_D * F = 0$$

and hence the Yang-Mills equation

$$*d_D * F = 0$$

Exercise 121. Consider electromagnetism:

$$S(A) = -\frac{1}{2}\int_M F \wedge *F$$

where $F = dA$ and $A \in \Omega^1(M)$. Now, we can follow the previous derivations of equations of motion, but with just d instead of d_D to get

$$d * F = 0$$

and of course $dF = 0$ follows from $F = dA$.
Now we want to generalise beyond electromagnetism. On $M = \mathbb{R} \times S$, we can again write

$$F = B + E \wedge dt$$

but now B and E are End(E)-valued 2- and 1-forms. We can define an inner product on End(E)-valued p-forms by

$$\langle A, B\rangle = A_I B_J \left\langle dx^I, dx^J \right\rangle$$

where $\left\langle dx^I, dx^J \right\rangle$ is the usual inner product on $\Omega^p(M)$. However, it is not immediately obvious that this is symmetric. To check, write

$$\begin{aligned}\langle A, B\rangle &= A^i_{Ij} e_i \otimes e^j (B^k_{Jl} e_k \otimes e^l) \left\langle dx^I, dx^J \right\rangle \\ &= A^i_{Ij} B^{Ij}_i\end{aligned}$$

This is therefore a legitimate inner product. As before, we have

$$F \wedge *F = \langle F, F\rangle \,\mathrm{vol}$$

and

$$\langle F, F\rangle = \langle B, B\rangle - \langle E, E\rangle$$

Thus we can write

$$S_{YM}(A) = \frac{1}{2}\int_M (\langle E, E\rangle - \langle B, B\rangle)\,\mathrm{vol}$$

Exercise 122. Let E be a $U(1)$ bundle over M with standard fibre the fundamental representation. Since this is a $U(1)$ bundle the first Chern form is just $(i/2\pi)F$. Let Σ be an arbitrary 2d compact submanifold of M without boundary, and define a loop $\gamma : S^1 \to \Sigma$, which we can regard as defining two pieces Σ^+ and Σ^- of Σ (in the standard computation in the context of magnetic monopoles a la Dirac, we take γ to map out the equator of S^2 and talk about northern and southern hemispheres). Now, since Σ^+ is compact, we can choose a finite open cover $(U_i)_{i\in I}$, over which we can locally trivialise. Then define a collection $(\gamma_i)_{i\in I}$ of loops such that the image of each γ_i is only in U_i, and such that

$$\gamma = \prod_{i\in I} \gamma_i$$

(For instance, if we had a sphere, we could imagine these γ_i looked like lines of longitude and latitude covering a hemisphere up to the equator.) Now, put a connection D on E. Then we have

$$H(\gamma, D) = \prod_{i\in I} H(\gamma_i, D)$$

For each factor, we can work in a local trivialisation and write $D = D^0 + A$. Then, since this is a $U(1)$ bundle, the holonomy takes the simple form

$$H(\gamma_i, D) = e^{-\int_{\gamma_i} A}$$

Let Σ_i be the surface enclosed by γ_i, i.e. $\gamma_i = \partial\Sigma_i$. Then

$$H(\gamma_i, D) = e^{-\int_{\Sigma_i} F}$$

Then

$$\begin{aligned} H(\gamma, D) &= \prod_{i\in I} e^{-\int_{\Sigma_i} F} \\ &= e^{-\sum_{i\in I}\int_{\Sigma_i} F} \\ &= e^{-\int_{\Sigma^+} F} \end{aligned}$$

Now, we can do a similar thing with Σ^-. Since this inherits the opposite orientation to Σ^+, we have $\partial\Sigma^- = \gamma^{-1}$. Thus we calculate

$$H(\gamma^{-1}, D) = e^{-\int_{\Sigma^-} F}$$

We therefore have

$$\begin{aligned} H(\gamma, D)H(\gamma^{-1}, D) &= e^{-\int_{\Sigma^+} F} e^{-\int_{\Sigma^-} F} \\ 1 &= e^{-\int_{\Sigma} F} \end{aligned}$$

Thus we must have

$$F = 2n\pi i$$

for some $n \in \mathbb{Z}$, i.e.

$$\frac{i}{2\pi} F \in \mathbb{Z}$$

The first Chern class is integral.

Exercise 123. Let E be a trivial bundle over M and write $D = D^0 + A$. Define $A_s = A$ and

$$F_s = d_{D^s} A_s = sdA + s^2 A \wedge A$$

as before. The calculation of the k^{th} Chern-Simons form proceeds as in the case $k = 2$, except with F_s^{k-1} the second factor in the wedge product. That is, we obtain

$$\begin{aligned}
\operatorname{tr}(F \wedge F) &= kd \int_0^1 \operatorname{tr}\Big(A \wedge F_s^{k-1}\Big) ds \\
&= kd \int_0^1 \operatorname{tr}\Big(A \wedge (sdA + s^2 A \wedge A)^{k-1}\Big) ds
\end{aligned}$$

This is as explicit we can get for generic k: we would want to use the binomial theorem for F_s^{k-1}, but we cannot, since non-commutativity of dA and $A \wedge A$ prevents gathering terms. The trace helps somewhat (completely for $k = 3$), but not totally, since, for instance, I cannot use the cyclic property of the trace to gather $\operatorname{tr}\big(A^3 \wedge dA^2\big)$ with $\operatorname{tr}\big(A \wedge dA \wedge A^2 \wedge dA\big)$ (these both appear in $\operatorname{tr}\big(A \wedge F_s^3\big)$).

Exercise 124. We have

$$\begin{aligned}
\frac{d}{ds} S_{CS}(A_s)\Big|_{s=0} &= \frac{d}{ds} \int_S \operatorname{tr}\left(A_s \wedge dA_s + \frac{2}{3} A_s \wedge A_s \wedge A_s\right)\Big|_{s=0} \\
&= \int_S \operatorname{tr}\left(\frac{dA_s}{ds} \wedge dA_s + A_s \wedge \frac{d}{ds} dA_s + 2A_s \wedge A_s \wedge \frac{dA_s}{ds}\right)\Big|_{s=0}
\end{aligned}$$

where we have used graded cyclicity to gather terms to form the last here. Then, writing

$$d\left(A_s \wedge \frac{dA_s}{ds}\right) = dA_s \wedge \frac{dA_s}{ds} - A_s \wedge \frac{d}{ds} dA_s$$

and using $\partial S = \emptyset$, and using graded cyclicty further, we have

$$\begin{aligned}
\frac{d}{ds} S_{CS}(A_s)\Big|_{s=0} &= 2 \int_S \operatorname{tr}\left(\frac{dA_s}{ds} \wedge dA_s + A_s \wedge A_s \wedge \frac{dA_s}{ds}\right)\Big|_{s=0} \\
&= 2 \int_S \operatorname{tr}(([T, A] - dT) \wedge dA + A \wedge A \wedge ([T, A] - dT))
\end{aligned}$$

Then using

$$dT \wedge dA = d(T \wedge dA)$$

and $\partial S = \emptyset$ again, we obtain

$$\frac{d}{ds} S_{CS}(A_s)\Big|_{s=0} = 2 \int_S \mathrm{tr}([T, A] \wedge dA + A \wedge A \wedge ([T, A] - dT))$$