Baez and Muniain - Gauge Fields, Knots and Gravity

Part 2: Gauge Fields

3 Curvature and the Yang-Mills Equation

Exercise 94. Let $\gamma : [0,1] \to M$ be a square of infinitesimal side length ε lying in the x^μ-x^ν plane in local coordinates. If $\gamma(0) = \gamma(1) = p$, let $v \in E_p$, and denote by $v(t)$ the parallel transport of v around γ. Then $v' = v(1) \in E_p$, and

$$v' = H(\gamma, D)v$$

Now, specifically, the result of parallel transport is given by

$$v(t) = Pe^{-\int_0^t A(\gamma'(s))ds}v$$

That is,

$$H(\gamma, D) = Pe^{-\int_0^1 A(\gamma'(s))ds}$$

(Since γ is infinitesimal we can obviously work within a local trivialisation and hence use A.) Going to second order in ε,

$$\begin{aligned} H(\gamma, D) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P \left(\int_0^1 A(\gamma'(s))ds \right)^n \\ &= 1 - \int_0^1 A(\gamma'(s))ds + \frac{1}{2} P \left(\int_0^1 A(\gamma'(s))ds \right)^2 + \mathcal{O}(\varepsilon^3) \end{aligned}$$

Now, we can parameterise γ explicitly in the x^μ-x^ν plane by

$$\gamma(s) = \begin{cases} (4s\varepsilon, 0) & 0 \le s \le \frac{1}{4} \\ (\varepsilon, \varepsilon(4s-1)) & \frac{1}{4} \le s \le \frac{1}{2} \\ (\varepsilon(3-4s), \varepsilon) & \frac{1}{2} \le s \le \frac{3}{4} \\ (0, 4\varepsilon(1-s)) & \frac{3}{4} \le s \le 1 \end{cases}$$

(although recall that holonomy is independent of parameterisation). Then

$$\gamma'(s) = \begin{cases} 4\varepsilon\partial_\mu & 0 \leq s \leq \frac{1}{4} \\ 4\varepsilon\partial_\nu & \frac{1}{4} \leq s \leq \frac{1}{2} \\ -4\varepsilon\partial_\mu & \frac{1}{2} \leq s \leq \frac{3}{4} \\ -4\varepsilon\partial_\nu & \frac{3}{4} \leq s \leq 1 \end{cases}$$

First compute the first order integral. We have

$$\begin{aligned} \int_0^1 A(\gamma'(s))ds = \int_0^{1/4} 4\varepsilon A_\mu(4\varepsilon s, 0)ds + \int_{1/4}^{1/2} 4\varepsilon A_\nu(\varepsilon, \varepsilon(4s-1))ds \\ - \int_{1/2}^{3/4} 4\varepsilon A_\mu(\varepsilon(3-4s), \varepsilon)ds - \int_{3/4}^{1} 4\varepsilon A_\nu(0, 4\varepsilon(1-s))ds \end{aligned}$$

We can expand the vector potential components around $(0,0)$:

$$\begin{aligned} A_\mu(4\varepsilon s, 0) &= A_\mu(0,0) + \varepsilon \frac{d}{d\varepsilon} A_\mu(4s\varepsilon, 0) \mid_{\varepsilon=0} + \mathcal{O}(\varepsilon^2) \\ &= A_\mu(0,0) + 4\varepsilon s \partial_\mu A_\mu(0,0) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Similarly, to first order in ε, and suppressing $(0,0)$ coordinates,

$$\begin{aligned} A_\nu(\varepsilon, \varepsilon(4s-1)) &= A_\nu + \varepsilon\partial_\mu A_\nu + \varepsilon(4s-1)\partial_\nu A_\nu \\ A_\mu(\varepsilon(3-4s), \varepsilon) &= A_\mu + \varepsilon(3-4s)\partial_\mu A_\mu + \varepsilon\partial_\nu A_\mu \\ A_\nu(0, 4\varepsilon(1-s)) &= A_\nu + 4\varepsilon(1-s)\partial_\nu A_\nu \end{aligned}$$

Now, we have

$$\begin{aligned} \int_0^{1/4} 4\varepsilon A_\mu(4\varepsilon s, 0)ds &= 4\varepsilon_0^{1/4}(A_\mu + 4\varepsilon s\partial_\mu A_\mu)ds \\ &= \varepsilon A_\mu + \frac{1}{2}\varepsilon^2 \partial_\mu A_\mu \end{aligned}$$

$$\begin{aligned} \int_{1/4}^{1/2} 4\varepsilon A_\nu(\varepsilon, \varepsilon(4s-1))ds &= 4\varepsilon \int_{1/4}^{1/2} (A_\nu + \varepsilon\partial_\nu A_\nu + \varepsilon(4s-1)\partial_\nu A_\nu)ds \\ &= \varepsilon A_\nu + \varepsilon^2(\partial_\mu A_\nu - \partial_\nu A_\nu) + \frac{3}{2}\varepsilon^2 \partial_\nu A_\nu \end{aligned}$$

$$\begin{aligned} \int_{1/2}^{3/4} 4\varepsilon A_\mu(\varepsilon(3-4s), \varepsilon)ds &= 4\varepsilon \int_{1/2}^{3/4} (A_\mu + \varepsilon(3-4s)\partial_\mu A_\mu + \varepsilon\partial_\nu A_\mu)ds \\ &= \varepsilon A_\mu + 3\varepsilon^2 \partial_\mu A_\mu + \varepsilon^2 \partial_\nu A_\mu - \frac{7}{2}\varepsilon^2 \partial_\mu A_\mu \end{aligned}$$

$$\int_{3/4}^{1} 4\varepsilon A_\nu(0, 4\varepsilon(1-s))ds = 4\varepsilon \int_{3/4}^{1} (A_\nu + 4\varepsilon(1-s)\partial_\nu A_\nu)ds$$
$$= \varepsilon A_\nu + 4\varepsilon^2 \partial_\nu A_\nu - \frac{7}{2}\varepsilon^2 \partial_\nu A_\nu$$

Putting this all together, we have

$$\int_0^1 A(\gamma'(s))ds = \varepsilon^2(\partial_\mu A_\nu - \partial_\nu A_\mu) + \mathcal{O}(\varepsilon^3)$$

Now, consider the second integral.

$$\frac{1}{2}P\left(\int_0^1 A(\gamma'(s))ds\right)^2 = \int_{1\geq s_1 \geq s_2 \geq 0} A(\gamma'(s_1))A(\gamma'(s_2))ds_2 ds_1$$

We will only need to go to zeroeth order to get $\mathcal{O}(\varepsilon^2)$ terms. So we can regard A as a constant. Then the first integral factor is

$$\int_0^{s_1} A(\gamma'(s_2))ds_2 = 4\varepsilon \begin{cases} s_1 A_\mu & 0 \leq s_1 \leq \frac{1}{4} \\ \frac{1}{4}A_\mu + \left(s_1 - \frac{1}{4}\right) A_\nu & \frac{1}{4} \leq s_1 \leq \frac{1}{2} \\ \left(\frac{3}{4} - s_1\right) A_\mu + \frac{1}{4}A_\nu & \frac{1}{2} \leq s_1 \leq \frac{3}{4} \\ (1-s_1)A_\nu & \frac{3}{4} \leq s_1 \leq 1 \end{cases}$$

Doing the second integral factor in the same way, we have

$$\frac{1}{2}P\left(\int_0^1 A(\gamma'(s))ds\right)^2 = 16\varepsilon^2\Bigg[\int_0^{1/4} sA\mu A_\mu ds + \int_{1/4}^{1/2} A_\nu \left(\frac{1}{4} - \left(s - \frac{1}{4}\right)A_\nu\right)ds$$
$$+ \int_{1/2}^{3/4} A_\mu \left(\left(\frac{3}{4} - s\right)A_\mu + \frac{1}{4}A_\nu\right)ds + \int_{3/4}^{1} (1-s)A_\nu A_\nu ds\Bigg]$$
$$= 16\varepsilon^2\Big[\frac{1}{32}A_\mu A_\mu + \frac{1}{16}A_\nu(A_\mu - A_\nu) + \frac{3}{32}A_\nu A_\nu + \frac{1}{8}A_\mu A_\mu$$
$$+ \frac{1}{16}A_\mu A_\nu - \frac{5}{16}A_\mu A_\mu + \frac{1}{4}A_\nu A_\nu - \frac{7}{32}A_\nu A_\nu\Big]$$
$$= \varepsilon^2(A_\nu A_\mu - A_\mu A_\nu)$$

Finally we have

$$H(\gamma, D) = 1 - \varepsilon^2(\partial_\mu A_\nu - \partial_\nu A_\mu - A_\nu A_\mu + A_\mu A_\nu) + \mathcal{O}(\varepsilon^3)$$
$$= 1 - \varepsilon^2 F_{\mu\nu} + \mathcal{O}(\varepsilon^3)$$

Exercise 95. Let γ_0 and γ_1 be homotopic loops from p to q. Then $\gamma_0^{-1}\gamma_1$ is a loop at p. If γ_0 and γ_1 are infinitesimal with length scale ε, we have by the previous exercise

$$H(\gamma_0^{-1}\gamma_1, D) \sim 1 - \varepsilon^\mu \varepsilon^\nu F_{\mu\nu}$$

So if D is a flat connection,

$$H(\gamma_0^{-1}\gamma_1, D) = 1$$

So

$$H(\gamma_0, D) = H(\gamma_1, D)$$

If on the other hand γ_0 and γ_1 are not infinitesimal, we can break them up into products of infinitesimal paths and then perform infinitesimal homotopies on these infinitesimal factors (possible since γ_0 and γ_1 are homotopic). By the above result the holonomy will never change under an individual homotopy, and since holonomy of products is the product of holonomies, we will obtain the very same result.

Exercise 96. Let M be 1-dimensional and consider $F(u, v)s$. In local coordinates,

$$F(u, v)s = u^\mu v^\nu F(\partial_\mu, \partial_\nu)s$$

But in one dimension $\mu = \nu$ necessarily. So by the antisymmetry of F this is identically zero. So any connection on a bundle over a 1-manifold is flat.

Exercise 97. By Exercise 72 a section $\tilde{s} \in \Gamma(E \otimes \bigwedge^p T^*M)$ can be written, not necessarily uniquely, as

$$\tilde{s} = s \otimes \omega$$

where $s \in \Gamma(E)$ and $\omega \in \Gamma(\bigwedge^p T^*M) = \Omega^p(M)$.

Exercise 98. Writing $\tilde{s} = s \otimes \omega$ is not necessarily unique. However, in local coordinates, if $\{dx^I\}$ is a basis of p-forms, where I ranges over multi-indices, we can write $\omega = \omega_I dx^I$, and then $\tilde{s}$ can be written uniquely (that is, given this local coordinate patch) as $\tilde{s} = s_I \otimes dx^I$, where we have defined $s_I = s\omega_I$. Then for an ordinary form μ we define the wedge product $\tilde{s} \wedge \mu$ by

$$(s \otimes \omega) = s \otimes (\omega \wedge \mu)$$

The RHS is

$$s_I \otimes (dx^I \wedge \mu)$$

which is unambiguous. Therefore this is uniquely defined in these coordinates. It is also clear that this definition is coordinate free, as changing coordinates will just give a Jacobian factor on each side.

Exercise 99. Locally, let $v = v^\mu \partial_\mu$, and s be an arbitrary section of E. We have

$$\begin{aligned} D_v s &= v^\mu D_\mu s \\ &= D_\mu s \otimes dx^\mu(v) \end{aligned}$$

since $dx^\mu(v) = v^\mu$. Therefore we can write

$$d_D s = D_\mu s \otimes dx^\mu$$

Exercise 100. Locally, we can write an End(E)-valued form uniquely as $T_I \otimes dx^I$, and an E-valued form uniquely as $s_I \otimes dx^I$. Then we have

$$(T_I \otimes dx^I) \wedge (s_J \otimes dx^J) = T_I(s_J) \otimes (dx^I \wedge dx^J)$$

Thus we have a unique formula for the wedge product of an End(E)-valued form with an E-valued form. Further, if $f \in C^\infty(M)$,

$$(fT_I \otimes dx^I) \wedge (s_J \otimes dx^J) = fT_I(s_J) \otimes (dx^I \wedge dx^J)$$

and

$$\begin{aligned} (T_I \otimes dx^I) \wedge (fs_J \otimes dx^J) &= T_I(fs_J) \otimes (dx^I \wedge dx^J) \\ &= fT_I(s_J) \otimes (dx^I \wedge dx^J) \end{aligned}$$

since $T_I \in \Gamma(\mathrm{End}(E))$ is $C^\infty(M)$-linear. Therefore this wedge product is $C^\infty(M)$-linear in both arguments.

Exercise 101. See Exercise 24 of Section 1.

Exercise 102. To check D^* is a connection, we have

$$\begin{aligned} (D_v^*(\alpha\lambda))(s) &= v(\alpha(\lambda(s)) - \alpha\lambda D_v(S)) \\ &= \alpha(v(\lambda(s)) - \lambda D_v(s)) \\ &= \alpha(D_v^*\lambda)(s) \end{aligned}$$

Then

$$\begin{aligned} (D_v^*(\lambda + \mu))(s) &= v((\lambda + \mu)(s)) - (\lambda + \mu)D_v(s) \\ &= v(\lambda(s)) + v(\mu(s)) - \lambda D_v(s) - \mu D_v(s) \\ &= (D_v^*\lambda)(s) + (D_v^*\mu)(s) \end{aligned}$$

Next,

$$\begin{aligned}(D_v^*(f\lambda))(s) &= v(f\lambda(s)) - f\lambda D_v(s)\\ &= v(f)\lambda(s) + fv(\lambda(s)) - f\lambda D_v(s)\\ &= v(f)\lambda(s) + (D_v^*\lambda)(s)\end{aligned}$$

And

$$\begin{aligned}(D_{v+w}^*\lambda)(s) &= (v+w)(\lambda(s)) - \lambda D_{v+w}(s)\\ &= v(\lambda(s)) + w(\lambda(s)) - \lambda D_v(s) - \lambda D_w(s)\\ &= (D_v^*\lambda)(s) + (D_w^*\lambda)(s)\end{aligned}$$

Finally,

$$\begin{aligned}(D_{fv}^*\lambda)(s) &= fv(\lambda(s)) - \lambda D_{fv}(s)\\ &= fv(\lambda(s)) - f\lambda D_v(s)\\ &= f(D_v^*\lambda)(s)\end{aligned}$$

for all $v, w \in \Gamma(TM)$, $\lambda \in \Gamma(E^*)$, $s \in \Gamma(E)$, $\alpha \in \mathcal{F}$, $f \in C^\infty(M)$. Therefore D^* is indeed a connection on E^*.

Exercise 103. To check $D \oplus D'$ is a connection, we have

$$\begin{aligned}(D \oplus D')_v(\alpha s, \alpha s') &= (D_v(\alpha s), D'_v(\alpha s'))\\ &= \alpha(D_v s, D'_v s')\end{aligned}$$

Then

$$\begin{aligned}(D \oplus D')_v(s+t, s'+t') &= (D_v(s+t), D'_v(s'+t'))\\ &= (D_v s, D'_v s) + (D_v t, D'_v t')\end{aligned}$$

Next,

$$\begin{aligned}(D \oplus D')_v(fs, fs') &= (D_v(fs), D'_v(fs'))\\ &= (v(f)(s) + fD_v s, v(f)(s') + fD'_v s')\\ &= v(f)(s, s') + f(D_v s, D'_v s')\end{aligned}$$

And

$$\begin{aligned}(D \oplus D')_{v+w}(s, s') &= (D_{v+w} s, D'_{v+w} s')\\ &= (D_v s, D'_v s') + (D_w s, D'_w s')\end{aligned}$$

Finally,

$$\begin{aligned}(D \oplus D')_{fv}(s, s') &= (D_{fv}s, D'_{fv}s') \\ &= f(D_v s, D'_v s')\end{aligned}$$

for all $v, w \in \Gamma(TM)$, $s, t \in \Gamma(E)$, $s', t' \in \Gamma(E')$, $\alpha \in \mathcal{F}$, $f \in C^\infty(M)$. Therefore $D \oplus D'$ is indeed a connection on $E \oplus E'$.

Exercise 104. To check $D \otimes D'$ is a connection, we have

$$\begin{aligned}(D \otimes D')_v(\alpha s \otimes s') &= D_v(\alpha s) \otimes s' + \alpha s \otimes D'_v s' \\ &= \alpha(D_v s \otimes s' + s \otimes D'_v s') \\ &= \alpha(D_\otimes D')_v(s \otimes s')\end{aligned}$$

For the addition axiom we first need to confirm the Leibniz one:

$$\begin{aligned}(D \otimes D')_v(fs \otimes s') &= D_v(fs) \otimes s' + fs \otimes D'_v s' \\ &= v(f)(s) \otimes s' + fD_v(s) \otimes s' + fs \otimes D'_v s' \\ &= v(F)s \otimes s' + f((D \otimes D')_v(s \otimes s'))\end{aligned}$$

Returning to the addition axiom, consider

$$(D \otimes D')_v(s \otimes s' + t \otimes t')$$

Locally, we can write $s \otimes s' = s^{ia} e_i \otimes e'_a$ and $t \otimes t' = t^{ia} e_i \otimes e'_a$, where s^{ia} and t^{ia} are just functions. Then

$$\begin{aligned}(D \otimes D')_v(s \otimes s' + t \otimes t') &= (D \otimes D')_v((s^{ia} + t^{ia})e_i \otimes e'_a) \\ &= v(s^{ia} + t^{ia})(e_i \otimes e'_a) + (s^{ia} + t^{ia})((D_v e_i) \otimes e'_a + e_i \otimes D_v e'_a)\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}(D \otimes D')_v(s \otimes s') &= (D \otimes D')_v(s^{ia} e_i \otimes e'_a) \\ &= v(s^{ia})(e_i \otimes e'_a) + s^{ia}(D \otimes D')_v(e_i \otimes e'_a) \\ &= v(s^{ia})(e_i \otimes e'_a) + s^{ia}((D_v e_i) \otimes e'_a + e_i \otimes D'_v e'_a)\end{aligned}$$

and hence also

$$(D \otimes D')_v(t \otimes t') = v(t^{ia})(e_i \otimes e'_a) + t^{ia}((D_v e_i) \otimes e'_a + e_i \otimes D'_v e'_a)$$

Thus we see that in fact

$$(D \otimes D')_v(s \otimes s' + t \otimes t') = (D \otimes D')_v(s \otimes s') + (D \otimes D')_v(t \otimes t')$$

Next,

$$\begin{aligned}(D \otimes D')_{v+w}(s \otimes s') &= D_{v+w}s \otimes s' + s \otimes D'_{v+w}s' \\ &= D_v s \otimes s' + D_w s \otimes s' + s \otimes D'_v s' + s \otimes D'_w s' \\ &= (D \otimes D')_v(s \otimes s') + (D \otimes D')_w(s \otimes s')\end{aligned}$$

Finally,

$$\begin{aligned}(D \otimes D')_{fv}(s \otimes s') &= D_{fv}s \otimes s' + s \otimes D'_{fv}s' \\ &= fD_v s \otimes s' + s \otimes fD'_v s' \\ &= f(D \otimes D')_v(s \otimes s')\end{aligned}$$

Therefore $D \otimes D'$ is indeed a connection on $E \otimes E'$.

Exercise 105. $\mathrm{End}(E) \cong E \otimes E^*$, and hence a connection D on E induces a connection $\tilde{D} = D \otimes D^*$ on $\mathrm{End}(E)$. We have

$$\begin{aligned}(D^*_v\lambda)(t) &= v(\lambda(t)) - \lambda D_v t \\ (D \otimes D^*)_v(s \otimes \lambda(t)) &= (D_v s) \otimes \lambda(t) + s \otimes (D^*_v\lambda)(t) \\ &= \lambda(t)D_v s + s \otimes (v(\lambda(t)) - \lambda(D_v t)) \\ &= \lambda(t)D_v s + v(\lambda(t))s - s \otimes \lambda(D_v t)\end{aligned}$$

Consider

$$D_v((s \otimes \lambda)(t)) - (s \otimes \lambda)(D_v t)$$

Working in local coordinates, we confirm that

$$\begin{aligned}(s \otimes \lambda)(t) &= s^i\lambda_j t^k(e_i \otimes e^j)(e_k) \\ &= s^i\lambda_j t^j e_j \\ &= \lambda(t)s\end{aligned}$$

Thus

$$\begin{aligned}D_v((s \otimes \lambda)(t)) &= D_v(\lambda(t)s) \\ &= v(\lambda(t))s + \lambda(t)D_v s\end{aligned}$$

and hence we confirm that

$$D_v((s \otimes \lambda)(t)) - (s \otimes \lambda)(D_v t) = (D \otimes D^*)_v(s \otimes \lambda)(t)$$

Now, writing $T = s \otimes \lambda$ and replacing t with s, we have that

$$(D \otimes D^*)_v(T)(s) = D_v(Ts) - T(D_v s)$$

Using D to mean the induced connection on $\mathrm{End}(E)$ as well as the original connection on E, this is

$$(D_v T)(s) = D_v(Ts) - T(D_v s)$$

Exercise 106. Let D be a connection on E, $\omega \in \Gamma(\mathrm{End}(E) \otimes \bigwedge^p T^*M)$ and $\mu \in \Gamma(E \otimes \bigwedge T^*M)$. Then consider the covariant exterior derivative on $\mathrm{End}(E)$:

$$d_D(\omega \wedge \mu) = D_\mu(\omega \wedge \mu) \otimes dx^\mu$$

Locally, write $\omega = \omega_I \otimes dx^I$ and $\mu = \mu_J \otimes dx^J$, where $\omega_I \in \Gamma(\mathrm{End}(E))$ and $\mu_J \in \Gamma(E)$. Then

$$\begin{aligned} \omega \wedge \mu &= (\omega_I \otimes dx^I) \wedge (\mu_J \otimes dx^J) \\ &= \omega_I(\mu_J) \otimes (dx^I \wedge dx^J) \end{aligned}$$

Then

$$\begin{aligned} d_D(\omega \wedge \mu) &= d_D(\omega_I(\mu_J)) \wedge dx^I \wedge dx^J \\ &= D_\mu(\omega_I(\mu_J)) \otimes dx^\mu \wedge dx^I \wedge dx^J \\ &= ((D_\mu \omega_I)(\mu_J) + \omega_I(D_\mu \mu_J)) \otimes dx^\mu \wedge dx^I \wedge dx^J \end{aligned}$$

On the other hand,

$$\begin{aligned} d_D\omega &= d_D\omega_I \wedge dx^I \\ &= (D_\mu \omega_I) \otimes (dx^\mu \wedge dx^I) \\ d_D \wedge \mu &= ((D_\mu \omega_I) \otimes (dx^\mu \wedge dx^I)) \wedge (\mu_J \otimes dx^J) \\ &= (D_\mu \omega_I)(\mu_J) \otimes (dx^\mu \wedge dx^I \wedge dx^J) \end{aligned}$$

And

$$\begin{aligned} d_D\mu &= d_D\omega_J \wedge dx^J \\ &= (D_\mu \omega_J) \otimes (dx^\mu \wedge dx^J) \\ \omega \wedge d_D\mu &= (\omega_I \otimes dx^I) \wedge ((D_\mu \mu_J) \otimes (dx^\mu \wedge dx^J)) \\ &= \omega_I(D_\mu \mu_J) \otimes (dx^I \wedge dx^\mu \wedge dx^J) \\ &= (-1)^p \omega_I(D_\mu \mu_J) \otimes (dx^\mu \wedge dx^I \wedge dx^J) \end{aligned}$$

Thus we have found that

$$d_D(\omega \wedge \mu) = d_D\omega \wedge \mu + (-1)^p \omega \wedge d_D\mu$$

Exercise 107. This is too tedious to bear.

Exercise 108. Let $\tilde{S}, \tilde{T} \in \Gamma(\mathrm{End}(E) \otimes \bigwedge T^*M)$. If we write $\tilde{S} = S \otimes \omega$ and $\tilde{T} = T \otimes \mu$, define their wedge product by

$$(S \otimes \omega) \wedge (T \otimes \mu) = ST \otimes (\omega \wedge \mu)$$

To check this is well-defined we can use the unique local forms $\tilde{S} = S_I \otimes dx^I$ and $\tilde{T} = T_I \otimes dx^I$. Then define

$$\tilde{S} \wedge \tilde{T} = S_I T_J \otimes (dx^I \wedge dx^J)$$

This is unique and concrete, we just have to make sure it is $C^\infty(M)$-linear, and hence reproduces the above definition. We have

$$(f\tilde{S}) \wedge \tilde{T} = fS_I T_J \otimes (dx^I \wedge dx^J)$$

and

$$\begin{aligned} \tilde{S} \wedge (f\tilde{T}) &= S_I(fT_J) \otimes (dx^I \wedge dx^J) \\ &= fS_I T_J \otimes (dx^I \wedge dx^J) \end{aligned}$$

since S_I is $C^\infty(M)$-linear. Therefore this definition is $C^\infty(M)$-linear, and works as desired.

Exercise 109. Let $\omega \in \Gamma(\mathrm{End}(E) \otimes \bigwedge^p T^*M)$ and $\mu \in \Gamma(\mathrm{End}(E) \otimes \bigwedge^q T^*M)$. Then define their **graded commutator** $[\omega, \mu]$ by

$$[\omega, \mu] = \omega \wedge \mu - (-1)^{pq} \mu \wedge \omega$$

Then

$$\begin{aligned} -(-1)^{pq}[\mu, \omega] &= -(-1)^{pq} \mu \wedge \omega - (-1)^{2pq+1} \omega \wedge \mu \\ &= \omega \wedge \mu - (-1)^{pq} \mu \wedge \omega \\ &= [\omega, \mu] \end{aligned}$$

If also $\eta \in \Gamma(\mathrm{End}(E) \otimes \bigwedge^r T^*M)$, consider

$$\begin{aligned} [\omega, [\mu, \eta]] &= [\omega, \mu \wedge \eta - (-1)^{qr} \eta \wedge \mu] \\ &= \omega \wedge \mu \wedge \eta - (-1)^{p(q+r)} \mu \wedge \eta \wedge \omega - (-1)^{qr} \omega \wedge \eta \wedge \mu + (-1)^{qr+p(q+r)} \eta \wedge \mu \wedge \omega \end{aligned}$$

Then similarly

$$\begin{aligned}(-1)^{p(q+r)}[\mu,[\eta,\omega]] &= (-1)^{p(q+r)}(\mu\wedge\eta\wedge\omega - (-1)^{q(r+p)}\eta\wedge\omega\wedge\mu \\ &\quad - (-1)^{rp}\mu\wedge\omega\wedge\eta + (-1)^{rp+q(r+p)}\omega\wedge\eta\wedge\mu) \\ &= (-1)^{p(q+r)}\mu\wedge\eta\wedge\omega - (-1)^{r(p+q)}\eta\wedge\omega\wedge\mu \\ &\quad - (-1)^{pq}\mu\wedge\omega\wedge\eta + (-1)^{qr}\omega\wedge\eta\wedge\mu\end{aligned}$$

So

$$\begin{aligned}[\omega,[\mu,\eta]] + (-1)^{p(q+r)}[\mu,[\eta,\omega]] &= \omega\wedge\mu\wedge\eta - (-1)^{pq}\mu\wedge\omega\wedge\eta \\ &\quad - (-1)^{r(p+q)}\eta\wedge\omega\wedge\mu + (-1)^{qr+p(q+r)}\eta\wedge\mu\wedge\omega\end{aligned}$$

Also,

$$\begin{aligned}(-1)^{r(p+q)}[\eta,[\omega,\mu]] &= (-1)^{r(p+q)}(\eta\wedge\omega\wedge\mu - (-1)^{r(p+q)}\omega\wedge\mu\wedge\eta \\ &\quad - (-1)^{pq}\eta\wedge\mu\wedge\omega + (-1)^{pq+r(p+q)}\mu\wedge\omega\wedge\eta) \\ &= (-1)^{r(p+q)}\eta\wedge\omega\wedge\mu - \omega\wedge\mu\wedge\eta \\ &\quad - (-1)^{r(p+q)+pq}\eta\wedge\mu\wedge\omega + (-1)^{pq}\mu\wedge\omega\wedge\eta\end{aligned}$$

Thus we finally have

$$[\omega,[\mu,\eta]] + (-1)^{p(q+r)}[\mu,[\eta,\omega]] + (-1)^{r(p+q)}[\eta,[\omega,\mu]] = 0$$

This is the **graded Jacobi identity**.
Now, let $A \in \Gamma(\mathrm{End}(E)\otimes\bigwedge T^*M)$. Locally we can write $A = A_I \otimes dx^I$. Then

$$A\wedge A = A_I(A_J)\otimes(dx^I\wedge dx^J)$$

is not necessarily zero. Consider $[A,A]$. The commutator is linear in both arguments, so assume WLOG that $A \in \Gamma(\mathrm{End}(E)\otimes\bigwedge^p T^*M)$. Then

$$[A,A] = -(-1)^{p^2}[A,A]$$

So if p is even $[A,A]=0$. If p is odd this should not necessarily be the case?? Finally, $A\wedge A \in \Gamma(\mathrm{End}(E)\otimes\bigwedge^{2p} T^*M)$, so

$$\begin{aligned}[A,A\wedge A] &= A\wedge A\wedge A - (-1)^{2p^2}A\wedge A\wedge A \\ &= 0\end{aligned}$$

Exercise 110. Under a gauge transformation g, we get a new connection related to the old one by

$$D'_v(s) = gD_v(g^{-1}s)$$

Let $\eta \in \Gamma(E \otimes \bigwedge T^*M)$. Then write $\eta = \eta_I \otimes dx^I$. We have

$$\begin{aligned}
d_{D'}\eta &= d_{D'}\eta_I \wedge dx^I \\
&= D'_\nu \eta_I \otimes dx^\nu \wedge dx^I \\
&= gD_\nu(g^{-1}\eta_I) \otimes dx^\nu \wedge dx^I \\
&= gd_D(g^{-1}\eta)
\end{aligned}$$

Exercise 111. Let $T \in \Gamma(\mathrm{End}(E))$ and $s \in \Gamma(E)$. Then we have

$$\begin{aligned}
(D'_\nu T)(s) &= D'_\nu(Ts) - T(D'_\nu s) \\
&= gD_\nu(g^{-1}Ts) - (Tg)D_\nu(g^{-1}s) \\
&= g(D_\nu(g^{-1}Tgg^{-1}s) - (g^{-1}Tg)D_\nu(g^{-1}s)) \\
&= gD_\nu(g^{-1}Tg)(g^{-1}s) \\
&= gD_\nu(\mathrm{Ad}(g^{-1})T)g^{-1}(s) \\
&= \mathrm{Ad}(g)D_\nu(\mathrm{Ad}(g^{-1})T)(s)
\end{aligned}$$

So

$$D'_\nu T = \mathrm{Ad}(g)D_\nu(\mathrm{Ad}(g^{-1})T)$$

Exercise 112. Let $\eta \in \Gamma(\mathrm{End}(E) \otimes \bigwedge T^*M)$. Then

$$\begin{aligned}
d_{D'}\eta &= d_{D'}\eta_I \wedge dx^I \\
&= D'_\nu \eta_I \otimes dx^\nu \wedge dx^I \\
&= \mathrm{Ad}(g)d_D(\mathrm{Ad}(g^{-1})\eta)
\end{aligned}$$