Baez and Muniain - Gauge Fields, Knots and Gravity

Part 2: Gauge Fields

(I have not done Chapter 2.1 (Symmetry) since most of the content I have covered at least once before.)

2 Bundles and Connections

Exercise 55. Let $\phi_\alpha : U_\alpha : \mathbb{R}^n$ be charts on M forming an atlas. Define

$$V_\alpha = \{v \in TM \mid \pi(v) \in U_\alpha\}$$

Then clearly $\{V_\alpha\}$ is an open cover of TM. Further define

$$\begin{aligned} \psi_\alpha : V_\alpha &\to \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n} \\ v &\mapsto (\phi_\alpha(\pi(v)), (\phi_\alpha)_* v) \end{aligned}$$

where we regard $(\phi_\alpha)_* v$ as an element of $\mathbb{R}^n$ by identifying $\mathbb{R}^n \cong T_{\phi_\alpha(p)}\mathbb{R}^n$. with the obvious topology on TM the ψ_α map open subsets of TM to open subsets of $\mathbb{R}^{2n}$ homeomorphically, and $\psi_\alpha \circ \psi_\beta^{-1}$ is smooth with smooth inverse since $\phi_\alpha \circ \phi_\beta^{-1}$ is. Thus TM is a $2n$-manifold. Clearly $\pi : TM \to M$ is smooth since locally $TM = U_\alpha \times \mathbb{R}^n$, and then π is just the identity restricted to the first factor.

Exercise 56. Let $\pi : E \to M$ and $\pi' : E' \to M'$ be bundles, with $\psi : E \to E'$ and $\phi : M \to M'$. Let $u \in E$ and $\pi(u) = p \in M$. Then if ψ is a bundle morphism and ϕ the induced map on base spaces, we require

$$\begin{aligned} \pi' \circ \psi(u) &= \phi(p) \\ &= \phi \circ \pi(u) \end{aligned}$$

That is, $\pi' \circ \psi = \phi \circ \pi$.
Given ψ, we can construct ϕ uniquely: if $\pi(u) = p$ and $\pi' \circ \psi(u) = q$, we must have $\phi(p) = q$. In this way we know everything about ϕ.

Exercise 57. Let $\phi : M \to M'$. This induces the pushforward $\phi_* : TM \to TM'$. In local coordinates x^μ on M and y^α on M', define

$$V = V^\mu(p)\frac{\partial}{\partial x^\mu} \in TM$$

Then the pushforward is

$$\phi_* V = V^\mu(p)\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial}{\partial y^\alpha}$$

which is smooth since TM and TM' are smooth manifolds.

Exercise 58. Let $\phi : M \to M'$ be a diffeomorhpism. Then ϕ_* is also a diffeomorphism since it is a $C^\infty(M)$-linear map

$$\frac{\partial}{\partial x^\mu} \mapsto \frac{\partial \phi^\alpha(x)}{\partial x^\mu}\frac{\partial}{\partial y^\alpha}$$

Therefore ϕ_* is a bundle isomorphism.

Exercise 59. Locally, i.e. within a coordinate patch U_α, we can regard $p \in M$ as some x^μ and $v \in T_pM$ as some

$$v = v^\mu \frac{\partial}{\partial x^\mu}$$

This tangent space basis may be used over all of U_α, so TM locally looks like $U_\alpha \times \mathbb{R}^n$, which is trivial.

Exercise 60. Take two copies of $\mathbb{R}$, U and V, and glue them on the strictly positive half-lines by the identity. Given the trivial bundle on each line, glue $U \times \mathbb{R}$ to $V \times \mathbb{R}$ by, say, $(x, y) \sim (x, xy)$. Then this is not locally trivial, since any attempt at a local trivialisation $\phi(x, y) = (x, y)$ fails to be well-defined.

Exercise 61. Clearly fibres of TM are $\mathbb{R}^n$, and also TM is locally trivial. So it is a vector bundle.

Exercise 62. The Möbius band is a bundle over S^1 with fibres $\mathbb{R}$. Use two charts, U_N

and U_S, to cover S^1. Then we can have the local trivialisations

$$\begin{aligned}\phi_N : E\mid_{U_N} &\to U_N \times \mathbb{R} \\ u &\mapsto (\theta, t) \\ \phi_S : E\mid_{U_S} &\to U_S \times \mathbb{R} \\ u &\mapsto (\theta, -t)\end{aligned}$$

Therefore this is a real line bundle.

Exercise 63. Let $\psi : E \to E'$ be a diffeomorphic vector bundle morphism. Then it has a diffeomorphic inverse $\psi^{-1} : E' \to E$. Furthermore, this is a bundle morphism since if $\pi'(u_1') = \pi'(u_2')$ then u_1' and u_2' are mapped to by the same fibre restriction $\psi\mid_{E_p}$. Finally, it is a vector bundle morphism since the inverse of a linear map is linear.

Exercise 64. Clearly a section in $\Gamma(TM)$ smoothly assigns tangent vectors v to points $p \in M$, making it a vector field. Since this identification is pointwise, the required properties of vector fields (linearity and Leibniz) follow from the same properties of tangent vectors.

Exercise 65. Let $s, s' in \Gamma(E)$, $f, g \in C^\infty(M)$ and $p \in$M. Then

(i) We have

$$\begin{aligned}(f(s+s'))(p) &= f(p)(s+s')(p) \\ &= f(p)(s(p)+s'(p))\end{aligned}$$

and

$$(f(s))(p) = f(p)s(p)$$

So

$$f(s+s') = fs + fs'$$

(ii) We have

$$\begin{aligned}(f+g)(s)(p) &= (f+g)(p)s(p) \\ &= (f(p)+g(p))s(p)\end{aligned}$$

So

$$(f+g)s = fs + gs$$

(iii) We have

$$
\begin{aligned}
(fg)(s)(p) &= (fg)(p)s(p) \\
&= f(p)g(p)s(p)
\end{aligned}
$$

and

$$
\begin{aligned}
f(gs)(p) &= f(p)(gs)(p) \\
&= f(p)g(p)s(p)
\end{aligned}
$$

So

$$(fg)s = f(gs)$$

(iv) Finally,

$$
\begin{aligned}
(1(s))(p) &= 1(p)s(p) \\
&= s(p)
\end{aligned}
$$

So

$$1s = s$$

These results make $\Gamma(E)$ a module over $C^\infty(M)$.

Exercise 66. On the Möbius bundle we have local trivialisations

$$
\begin{aligned}
\phi_N(u) &= (\theta, t) \\
\phi_S(u) &= (\theta, -t)
\end{aligned}
$$

for the usual atlas $\{U_N, U_S\}$ of S^1. Take $u = s(\theta)$ for a generic section s. Then

$$
\begin{aligned}
\phi_N(s(\theta)) &= (\theta, t) \\
\phi_S(s(\theta)) &= (\theta, -t)
\end{aligned}
$$

Therefore there must be a zero on $U_N \cap U_S$. However, global triviality requires the existence of a basis of sections, and basis elements can never be zero, so it requires n non-vanishing sections (here $n = 1$). But there are none, and hence the Möbius bundle is non-trivial.

Exercise 67. Let $\pi : E \to M$ be a vector bundle with fibres E_p. Then an atlas on M naturally induces one on E just as in the case of $E = TM$. Now, take the dual fibres E_p^* and construct the dual bundle

$$E^* = \bigcup_{p \in M} E_p^*$$

with the obvious projection π^*. Make this into a manifold by mapping

$$v \mapsto (\phi_\alpha(\pi^*(v)), (\phi_\alpha)_* \theta(v))$$

where $\theta : E_p \to E_p^*$ is the dual vector space isomorphism. The local trivialisation on E makes it fibrewise-linear; clearly this is inherited here.
Given a basis of sections $e_i \in \Gamma(E)$, we have induced bases $\{e_i(p)\}$ on each E_p, and hence induced dual bases $\{e^j(p)\}$ on each E_p^*, where $e_i(p)e^j(p) = \delta_i^j$. To extend the $\{e^j(p)\}$ to a basis of sections, there is a unique construction to get smooth sections, by not relabelling basis elements between E_p and E_q for nearby p, q. This is possible precisely since the bases $\{e_i(p)\}$ have been induced from smooth sections $e_i \in \Gamma(E)$.

Exercise 68. let $s \in \Gamma(E)$ be a section of a vector bundle over M, and $\lambda \in \Gamma(E^*)$. Then certainly

$$p \mapsto \lambda(p)(s(p))$$

is a map $M \to \mathbb{R}$, and moreover a smooth map, since λ and s are smooth sections. It is furthermore linear in λ and s since the dual vector space maps $\lambda(p) : s(p) \to \mathbb{R}$ are, and indeed this is $C^\infty(M)$-linearity since the function is smooth and defined by the dual actions $\lambda(p) : s(p) \to \mathbb{R}$ pointwise. That is, $\lambda(s) \in C^\infty(M)$, defined in this way.

Exercise 69. A 1-form is a smooth $C^\infty(M)$-linear map $\mathrm{Vect}(M) \to \mathbb{R}$. Identifying $\mathrm{Vect}(M) = \Gamma(TM)$, we see by the previous exercise that 1-forms are sections of $(TM)^* = T^*M$.

Exercise 70. Given vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ over M with fibres E_p and E'_p and local trivialisations $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$, construct

(i) $E \oplus E'$, having fibres $E_p \oplus E'_p$. Define local trivialisations $\{\chi_{\alpha\beta}\}$ on $E \oplus E'$ by

$$\chi_{\alpha\beta}(u) = \begin{cases} \phi_\alpha(u) & u \in E_p \subset E_p \oplus E'_p \\ \psi_\beta(u) & u \in E'_p \subset E_p \oplus E'_p \end{cases}$$

where $p = \pi(u)$. Then $E \oplus E'$ is a vector bundle.

(ii) $E \otimes E'$, having fibres $E_p \otimes E'_p$. Define local trivialisations

$$\chi_{\alpha\beta} = \phi_\alpha \otimes \psi_\beta$$

Exercise 71. Let E, E' be vector bundles over M, and $s \in \Gamma(E)$ and $s' \in \Gamma(E')$. Define $(s, s') \in \Gamma(E \oplus E')$ by

$$(s, s')(p) = (s(p), s'(p))$$

Clearly this exists, and is unique since the decomposition of $E \oplus E'$ into E and E' is. Define $s \otimes s' \in \Gamma(E \otimes E')$ by

$$(s \otimes s')(p) = s(p) \otimes s'(p)$$

Again, this clearly exists, and is by construction unique.

Exercise 72. Let E and E' have local bases of sections e_i and e'_j. This induces bases $\{e_i(p)\}$ of E_p and $\{e'_j(p)\}$ of E'_p. Then $\{e_i(p) \otimes e'_j(p)\}$ is a basis of $E_p \otimes E'_p$. This can be extended, at least in some neighbourhood, so that locally, a section of $E \otimes E'$ is of the form

$$s = s^{ij} e_i \otimes e'_j$$

i.e.

$$s(p) = s^{ij}(p) e_i(p) \otimes e_j(p)$$

Then we can choose coverings of E and E' and subordinate partitions of unity to make this global (this construction will remain locally finite).

Exercise 73. Let $E \to M$ be a vector bundle. Then we have

$$\bigwedge E = \bigcup_{p \in M} \bigwedge E_p$$

Define the projection $\pi : \bigwedge E \to M$ by

$$\pi(u) = p$$

where $u \in \bigwedge E_p$. Now, let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. Define

$$V_\alpha = \{u \in \bigwedge E \mid \pi(u) \in U_\alpha\}$$

So $\{V_\alpha\}$ covers $\bigwedge E$. Now, let $\{\tilde{\phi}_\alpha\}$ be a local trivialisation associated with this atlas on M, and write

$$\tilde{\phi}_\alpha(u) = (p, \omega_\alpha)$$

We want to define a homeomorphism $\psi_\alpha : V_\alpha \to \mathbb{R}^{n+2^k}$ which maps the p factor to $\phi_\alpha(p)$, and the ω_α factor to some open subset of $\mathbb{R}^{2^k}$ (recall that $\dim \bigwedge E_p = 2^k$. To do this, take ω (suppressing the chart-labelling subscript) and expand it in the local coordinates provided by ϕ_α:

$$\omega = \omega^{(0)} + \sum_{m=1}^{n} \omega^{(m)}_{\mu_1 \ldots \mu_m} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m}$$

where we demand that the μ_i be strictly increasing. Then we can map

$$\omega \mapsto (\omega^{(0)}, \omega^{(1)}_1, \omega^{(1)}_2, \ldots, \omega^{(2)}_{12}, \omega^{(2)}_{13}, \ldots)$$

In this way we can build an atlas on $\bigwedge E$, making it an $(n+2^k)$-manifold. Then $\pi : \bigwedge E \to M$ is locally trivial with fibres $\bigwedge E_p$, and hence a vector bundle.

Exercise 74. From the atlas constructed in the previous exercise it is clear that

$$\bigwedge E = \bigoplus_{i=0}^{n} \bigwedge^{i} E$$

If E has fibres E_p, the bundle $\bigwedge^0 E$ has fibres $\bigwedge^0 E_p \cong \mathbb{R}$. Therefore a section of $\bigwedge^0 E$ assigns real numbers to points in M smoothly, and hence can be identified in a one-to-one way with smooth functions on M. Similarly, the bundle $\bigwedge^1 E$ has fibres $\bigwedge^1 E_p \cong E_p$, with a canonical isomorphism. Therefore we can identify sections of $\bigwedge^1 E$ with sections of E in a one-to-one way.

Exercise 75. That given any $\omega, \mu \in \Gamma(\bigwedge E)$ there exists some $(\omega \wedge \mu) \in \Gamma(\bigwedge E)$ basically follows from Exercise 73. Now, we know that $\Gamma(\bigwedge E)$ is a vector space. In addition, for $\omega, \mu, \eta \in \Gamma(\bigwedge E)$ and $f, g \in C^\infty(M)$ we trivially have

$$\begin{aligned}
\omega + \mu &= \mu + \omega \\
(\omega + \mu) + \eta &= \omega + (\mu + \eta) \\
\omega \wedge (\mu \wedge \eta) &= (\omega \wedge \mu) \wedge \eta \\
\omega \wedge (\mu + \eta) &= \omega \wedge \mu + \omega \wedge \eta \\
(\omega + \mu) \wedge \eta &= \omega \wedge \eta + \mu \wedge \eta \\
1\omega &= \omega \\
f(g\omega) &= (fg)\omega \\
f(\omega + \mu) &= f\omega + f\mu \\
(f + g)\omega &= f\omega + g\omega
\end{aligned}$$

This makes $\Gamma(\bigwedge E)$ with $\wedge$ into an algebra over $C^\infty(M)$.
Clearly if $\omega, \mu \in \Gamma(\bigwedge^i E)$, $\omega + \mu \in \Gamma(\bigwedge^i E)$ and $f\omega \in \Gamma(\bigwedge^i E)$. Also $0 \in \Gamma(\bigwedge^i E)$ (locally

$0 = dx^1 \wedge ... \wedge dx^1$ with i factors, for instance). So $\Gamma(\bigwedge^i E)$ is a linear subspace of $\Gamma(\bigwedge E)$. A section $\omega \in \Gamma(\bigwedge E)$ is a locally finite sum of wedge products of $v \in \Gamma(E)$, which just follows from the construction of tensor bundles.

Exercise 76. A section of $\bigwedge^i T^*M = (\bigwedge^i TM)^*$ takes an antisymmetric collection of i vector fields $C^\infty(M)$-linearly to $\mathbb{R}$. This is precisely what i-forms do.

Exercise 77. We require

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

on $U_\alpha \cap U_\beta \cap U_\gamma$, so in particular,

$$g_{\alpha\beta}g_{\beta\alpha}g_{\alpha\alpha} = 1$$

But since we also impose $g_{\alpha\alpha} = 1$, this is

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

Consider sequences $(\alpha_1, ..., \alpha_n)$ and $(\beta_1, ..., \beta_m)$. Using the cocycle rule,

$$\begin{aligned} g_{\alpha_1\alpha_2}g_{\alpha_2\alpha_3}\cdots g_{\alpha_{n-1}\alpha_n} &= g_{\alpha_1\alpha_3}g_{\alpha_3\alpha_4}\cdots g_{\alpha_{n-1}\alpha_n} \\ &= ... \\ &= g_{\alpha_1\alpha_n} \end{aligned}$$

Similarly

$$g_{\beta_1\beta_2}\cdots g_{\beta_{m-1}\beta_m} = g_{\beta_1\beta_m}$$

So if $\alpha_1 = \beta_1$ and $\alpha_n = \beta_m$,

$$g_{\alpha_1\alpha_2}\cdots g_{\alpha_{n-1}\alpha_n} = g_{\beta_1\beta_2}\cdots g_{\beta_{m-1}\beta_m}$$

Exercise 78. Begin with

$$E = \bigcup_{p\in M} (U_\alpha \times V)/G$$

and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, with $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$. Firstly, we have vector space fibres since G acts only on the second factor of the $U_\alpha \times V$, i.e. it never glues (p, v) and (q, v') if $q \neq p$. The content of the conditions on the transition functions is that E is locally trivial, since there is no gluing within each $U_\alpha \times V$. To see this, first consider any $p \in U_\alpha$ that is not in an overlap region. The only gluings are $(p, v) \sim (p, g_{\alpha\alpha}v) = (p, v)$, i.e. we are only quotienting by the identity. Now consider any $p \in U_\alpha \cap U_\beta$. We have

$$(p, v) \sim (p, g_{\beta\alpha}v) \sim (p, g_{\alpha\beta}g_{\beta\alpha}v)$$

(p, v) and $(p, g_{\alpha\beta} g_{\beta\alpha} v)$ are in $U_\alpha \times V$ (whereas $(p, g_{\beta\alpha} v)$ is in $U_\beta \times V$), so $g_{\alpha\beta} g_{\beta\alpha} = 1$ ensures no such non-trivial gluing. Now consider $p \in U_\alpha \cap U_\beta \cap U_\gamma$. On $U_\alpha \times V$ we have the glued points

$$(p, v) \sim (p, g_{\alpha\gamma} g_{\gamma\beta} g_{\beta\alpha} v)$$

Again our condition ensures no such non-trivial gluing. Now that these three results have been established it is easy to see that they also imply the same for intersections of four or more coordinate patches. Thus local triviality survives, and we have a vector bundle.

Exercise 79. Let $p \in U_\alpha$ and T be a transformation

$$T : [p, v]_\alpha \mapsto [p, d\rho(x) v]_\alpha$$

living in $\mathfrak{g}$, where $d\rho = \rho_*$ is the pushforward of ρ. Then

$$\begin{aligned} [p, v]_\alpha &= [p, g_{\beta\alpha} v]_\beta \\ &= [p, v']_\beta \end{aligned}$$

where $v' = g_{\beta\alpha} v$. On the RHS

$$\begin{aligned} [p, d\rho(x) v]_\alpha &= [p, g_{\beta\alpha} d\rho(x) v]_\beta \\ &= [p, g_{\beta\alpha} d\rho(x) g_{\alpha\beta} v']_\beta \\ &= [p, g_{\beta\alpha} d\rho(x) g_{\beta\alpha}^{-1} v']_\beta \\ &= [p, \mathrm{Ad}_{g_{\beta\alpha}} d\rho(x) v']_\beta \\ &= [p, d\rho(x') v']_\beta \end{aligned}$$

for some $x' \in \mathfrak{g}$.

Exercise 80. Let $\phi : \mathbb{R}^4 \to \mathbb{C}^3$ be a pion field transforming under the spin-1 representation of $SU(2)$. Consider the equation

$$(\partial^\mu \partial_\mu + m^2 + \lambda \phi^i \phi_i)\phi = 0$$

where $\phi^i \phi_i$ is the $\mathbb{C}^3$ inner product. Under action of $SU(2)$, this is clearly invariant, and therefore the LHS becomes

$$(\partial^\mu \partial_\mu + m^2 + \lambda \phi^i \phi_i) U_1(g) \phi$$

where U_1 is the spin-1 representation of $SU(2)$ and $g \in SU(2)$. But $g \neq g(x)$, so this is indeed

$$U_1(g)(\partial^\mu \partial_\mu + m^2 + \lambda \phi^i \phi_i)\phi = 0$$

Therefore the action of $SU(2)$ maps solutions to solutions.

Exercise 81. Let $T : \Gamma(E) \to \Gamma(E)$ be a C^∞-linear map. Let $s \in \Gamma(E)$ and $\{f_\alpha\}$ be a partition of unity. Then

$$s = \sum_\alpha f_\alpha s$$

Furthermore, if $\{U_\alpha\}$ is an open cover of M to which $\{f_\alpha\}$ is subordinate, we can say that each $f_\alpha s$ is zero outside U_α, and we can work locally on $U_\alpha \times V$. But on a trivial bundle sections are the same as functions, i.e. $\Gamma(U_\alpha \times V) \cong C^\infty(U_\alpha, V)$. Then locally we have transformations $T \in \mathrm{End}(C^\infty(U_\alpha, V))$ acting on $f_\alpha s$. That is, we can regard any $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(E)$ as defined by a sum

$$T(s) = \sum_\alpha \tilde{T}_\alpha(f_\alpha s)$$

for appropriately chosen $\tilde{T}_\alpha \in \mathrm{End}(C^\infty(U_\alpha, V))$. Then T corresponds to $\tilde{T}_\alpha$ just as $s \in \Gamma(E)$ corresponds to $s_\alpha \in C^\infty(U_\alpha, V)$, and it is possible to find an appropriate set of $\tilde{T}_\alpha$ such that taking the sum we get a smooth section:

$$T \in \Gamma(\mathrm{End}(E))$$

Exercise 82. The set of gauge transformations forms a group, by

$$(gh)(p) = g(p)h(p)$$
$$g^{-1}(p) = g(p)^{-1}$$

To see this, notice that if $g(p), h(p) \in G$ for all $p \in M$, and they depend smoothly on p, then also $g(p)h(p) \in G$ is a gauge transformation. Similarly so is $g(p)^{-1}$. Furthermore, $e(p) = \mathrm{id}_G$ for all $p \in M$ is clearly a gauge transformation.

Exercise 83. By Exercise 72, we can write the $\mathrm{End}(E)$-valued 1-form

$$A = \sum_i T_i \otimes \omega_i$$

where $T_i \in \Gamma(\mathrm{End}(E))$ and $\omega_i \in \Omega^1(M)$. In general this will not be unique; suppose we also have

$$A = \sum_i \tilde{T}_i \otimes \tilde{\omega}_i$$

That is,

$$\sum_i T_i \otimes \omega_i = \sum_i \tilde{T}_i \otimes \tilde{\omega}_i$$

Working locally (which we are doing anyway whenever we introduce A) this is

$$\sum_i T^k_{ij}\omega_{i\mu}e^j \otimes e_k \otimes dx^\mu = \sum_i \tilde{T}^k_{ij}\tilde{\omega}_{i\mu}e^j \otimes e_k \otimes dx^\mu$$
$$\sum_i T^k_{ij}\omega_{i\mu} = \sum_i \tilde{T}^k_{ij}\tilde{\omega}_{i\mu}$$

for each j, k, μ. Now, using the first form, we have

$$\begin{aligned} A(v) &= \sum_i \omega_i(v)T_i \\ &= \sum_i v^\mu \omega_{i\mu}T^k_{ij}e^j \otimes e_k \end{aligned}$$

But using the second form, we have

$$\begin{aligned} A(v) &= \sum_i \tilde{\omega}_i(v)\tilde{T}_i \\ &= \sum_i v^\mu \tilde{\omega}_{i\mu}\tilde{T}^k_{ij}e^j \otimes e_k \end{aligned}$$

For $A(v)$ to be well-defined, these two must be equal. That is, we must have

$$\sum_i v^\mu \omega_{i\mu}T^k_{ij}e^j \otimes e_k = \sum_i v^\mu \tilde{\omega}_{i\mu}\tilde{T}^k_{ij}e^j \otimes e_k$$
$$\sum_i \omega_{i\mu}T^k_{ij} = \sum_i \tilde{\omega}_{i\mu}\tilde{T}^k_{ij}$$

for each j, k, μ. But this is precisely what we just checked was required of $\tilde{T}$ and $\tilde{\omega}$ given T and ω. So $A(v)$ is well-defined.

Exercise 84. We want to show that, given the standard flat connection D^0 on a local trivialisation, and a vector potential A, $D = D^0 + A$ is a connection. That is,

$$D_v s = D^0_v s + A(v)s$$

is a connection. Furthermore, we want to show that all connections are related to D^0 in this way for some vector potential A.

First we check that $D = D^0 + A$ is a connection. We have

$$
\begin{aligned}
D_v(\alpha s) &= D^0_v(\alpha s) + A(v)(\alpha s) \\
&= \alpha D^0_v s + \alpha A(v) s \\
D_v(s+t) &= D^0_v(s+t) + A(v)(s+t) \\
&= D^0_v s + A(v)s + D^0_v t + A(v)t \\
D_v(fs) &= D^0_v(fs) + A(v)(fs) \\
&= v(f)s + f D^0_v s + f A(v) s \\
&= v(f)s + f D_v s \\
D_{v+w} s &= D^0_{v+w} s + A(v+w)s \\
&= D^0_v s + A(v)s + D^0_w s + A(w)s \\
D_{fv} s &= D^0_{fv} s + A(fv)s \\
&= f D^0_v s + f A(v) s
\end{aligned}
$$

We see in checking this that what makes $D^0 + A$ a connection is the $C^\infty(M)$-linearity of A. Now we further want to check that all connections D can be written in this way for appropriate choice of gauge potential A. To see this we just need to check that $D - D^0$ is in fact a vector potential, i.e. $C^\infty(M)$-linear. Indeed,

$$
\begin{aligned}
D_{fv} s - D^0_{fv} s &= f D_v s - f D^0_v s \\
&= f(D_v s - f D^0_v s) \\
D_v(fs) - D^0_v(fs) &= v(f)s + f D_v s - v(f)s - f D^0_v s \\
&= f(D_v s - D^0_v s)
\end{aligned}
$$

Therefore A is indeed a vector potential, and we are done.

Exercise 85. First we want to show that D' is a connection. We have

$$D'_v s = g D_v(g^{-1} s)$$

Then

$$\begin{aligned}
D'_v(\alpha s) &= dD_v(g^{-1}\alpha s)\\
&= \alpha g D_v(g^{-1}s)\\
D'_v(s+t) &= gD_v(g^{-1}(s+t))\\
&= gD_v(g^{-1}s) + gD_v(g^{-1}t)\\
D'_v(fs) &= gD_v(g^{-1}fs)\\
&= gv(f)D_v(g^{-1}s) + gfD_v(g^{-1}s)\\
&= v(f)gD_v(g^{-1}s) + fgD_v(g^{-1}s)\\
D'_{v+w}s &= gD_{v+w}(g^{-1}s)\\
&= gD_v(g^{-1}s) + gD_w(g^{-1}s)\\
D'_{fv}s &= gD_{fv}(g^{-1}s)\\
&= fgD_v(g^{-1}s)
\end{aligned}$$

Thus D' is indeed a connection.

Exercise 86. Introduce a local trivialisation $\phi_\alpha : E\mid_{U_\alpha} \to U_\alpha \times V$, and the associated flat connection D^0, satisfying $D^0_v s = v(s^i)e_i$. We can write $D = D^0 + A$. Then

$$\begin{aligned}
D'_v s &= gD_v(g^{-1}s)\\
&= gD^0_v(g^{-1}s) + gA(v)(g^{-1}s)
\end{aligned}$$

Locally and explicitly, we have

$$\begin{gathered}
g = \rho(g)^j_k e_k \otimes e^k\\
s = s^i e_i\\
A = A^j_{\mu i} e^i \otimes e_j \otimes dx^\mu
\end{gathered}$$

Thus,

$$\begin{aligned}
g^{-1}s &= \rho(g^{-1})^j_i s^i e_j\\
D^0_v(g^{-1}s) &= D^0_v(\rho(g^{-1})^j_i s^i e_j)\\
&= v(\rho(g^{-1})^j_i s^i)e_j\\
&= v^\mu \partial_\mu(\rho(g^{-1})^j_i s^i)e_j\\
gD^0_v(g^{-1}s) &= \rho(g)^i_j v^\mu \partial_\mu(\rho(g^{-1})^j_k s^k)e_i\\
&= \rho(g)^i_j v^\mu \partial_\mu \rho(g^{-1})^j_k s^k e_i + v^\mu \partial_\mu s^i e_i
\end{aligned}$$

And

$$gA(v)(g^{-1}s) = \rho(g)^i_j e_i \otimes e^j (A^k_{\mu l} v^\mu \rho(g^{-1})^l_m s^m e_k)$$
$$= \rho(g)^i_j A^j_{\mu l} v^\mu \rho(g^{-1})^l_m s^m e_i$$

We therefore have

$$D'_v s = v^\mu \partial_\mu s^i e_i + \rho(g)^i_j v^\mu (\partial_\mu \rho(g^{-1})^j_k s^k + A^j_{\mu l} \rho(g^{-1})^l_m s^m) e_i$$

However,

$$D^0_v s = v(s^i) e_i = v^\mu \partial_\mu s^i e_i$$

So we see that

$$D'_v s = D^0_v s = \rho(g)^i_j v^\mu (\partial_\mu \rho(g^{-1})^j_k s^k + A^j_{\mu l} \rho(g^{-1})^l_m s^m) e_i$$

That is,

$$A'(v)(s) = \rho(g)^i_j v^\mu s^k (\partial_\mu \rho(g^{-1})^j_k + A^j_{\mu l} \rho(g^{-1})^l_k) e_i$$

Now, writing

$$A' = (A')^j_{\mu i} e^i \otimes e_j \otimes dx^\mu$$

we have

$$A'(v)(s) = (A')^i_{\mu k} v^\mu s^k e_i$$

So we have found that

$$(A')^i_{\mu k} = \rho(g)^i_j (\partial_\mu \rho(g^{-1})^j_k + A^j_{\mu l} \rho(g^{-1})^l_k)$$

That is,

$$A'_\mu = g A_\mu g^{-1} + g \partial_\mu g^{-1}$$

Now, A_μ lives in $\mathfrak{g}$ since D is a G-connection. Then

$$g A_\mu g^{-1} = \mathrm{Ad}_g A_\mu$$

so this also lives in $\mathfrak{g}$. Consider the other term: explicitly, this is

$$\rho(g)^i_j \partial_\mu \rho(g^{-1})^j_k$$

We can write

$$\rho(g^{-1})^j_k = \exp\bigl(-\lambda_i(x^\mu) T^i\bigr)$$

where the T^i span $\mathfrak{g}$ for some functions λ_i. Then

$$\partial_\mu \rho(g^{-1})^j_k = -\partial_\mu \lambda_l T^l \rho(g^{-1})^j_k$$

Therefore this term is

$$-\partial_\mu \lambda_l \rho(g^{-1})^i_j T^l \rho(g^{-1})^j_k$$

This also lives in $\mathfrak{g}$. Therefore A'_μ lives in $\mathfrak{g}$. Thus D' is a G-connection.

Exercise 87. Consider a generic connection $D = D^0 + A$ in some local trivialisation. We want to show that this is always gauge-equivalent to a temporal gauge connection $D' = D^0 + A'$, i.e. with $A'_0 = 0$. A' and A must be related by

$$A'_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1}$$

That is, we want to choose g such that

$$gA_0 g^{-1} + g\partial_0 g^{-1} = 0$$

i.e.

$$\partial_0 g^{-1} = A_0 g^{-1}$$

We can write

$$g = \exp\{-x(t,p)\}$$

where x is a $\mathfrak{g}$-valued function. Then

$$\partial_0 g^{-1} = \partial_t x g^{-1}$$

Therefore we want to choose x such that

$$\partial_t x = A_0$$

This is always possible: set

$$x(t,p) = \int_0^t A_0(s,p)ds$$

Then we are done.

Exercise 88. We have the covariant derivative, in some local trivialisation,

$$D_{\gamma'(t)}u(t) = \frac{d}{dt}u(t) + A(\gamma'(t))u(t)$$

where A is the vector potential of D in this local trivialisation. Now,

$$\gamma'(t) = \frac{\partial\gamma^\mu(t)}{\partial t}\frac{\partial}{\partial x^\mu}$$

(where by γ^μ we really mean $\phi \circ \gamma$ where ϕ is the local coordinate chart). Then

$$A(\gamma'(t)) = A_\mu \frac{\partial\gamma^\mu(t)}{\partial t}$$

In some other local trivialisation,

$$\gamma'(t) = \frac{\partial \gamma^\alpha(t)}{\partial t}\frac{\partial}{\partial y^\alpha}$$

and $A = \tilde{A}_\alpha \otimes dy^\alpha$, where

$$\tilde{A}_\alpha = A_\mu \frac{\partial x^\mu}{\partial y^\alpha}$$

Therefore here,

$$\begin{aligned} A(\gamma'(t)) &= \tilde{A}_\alpha \frac{\partial \gamma^\alpha(t)}{\partial t} \\ &= A_\mu \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial \gamma^\nu(t)}{\partial t}\frac{\partial y^\alpha}{\partial x^\nu} \\ &= A_\mu \frac{\partial \gamma^\mu(t)}{\partial t} \end{aligned}$$

So $A(\gamma'(t))$ is indeed independent of local trivialisation. Therefore the operator

$$D_{\gamma'(t)} = \frac{d}{dt} + A(\gamma'(t))$$

is. (Indeed, the equation

$$D_{\gamma'(t)}u(t) = \frac{d}{dt}u(t) + A(\gamma'(t))u(t)$$

under a change of local trivialisation with section basis change $e_i \to \Lambda_i^j e_j$ also picks up a factor of Λ everywhere, and this must by definition be invertible, so drops straight out.)

Exercise 89. Put a norm $|| \; ||$ on V and define the induced norm on $\mathrm{End}(V)$ by

$$||T|| = \sup\{||Tu|| \mid ||u|| = 1\}$$

Define

$$K = \sup\{||A(\gamma'(t))|| \mid t \in [0,T]\}$$

Now consider the n^{th} term in the sum for $u(t)$:

$$(-1)^n \int_{t \geq t_1 \geq \ldots \geq t_n \geq 0} A(\gamma'(t_1))\ldots A(\gamma'(t_n))dt_n \ldots dt_1 \; u$$

When we take the norm obviously the $(-1)^n$ disappears, and we also have a factor $||u||$. For the integral,

$$\int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} A(\gamma'(t_1))\ldots A(\gamma'(t_n))dt_n \ldots dt_1$$

where $t \geq t_1 \geq ... \geq t_{n-1}$, the norm must be at most

$$||A(\gamma'(t))||^n \frac{t^n}{n!} = t^n K^n \frac{1}{n!}$$

Thus, the n^{th} term has norm at most

$$t^n K^n ||u|| \frac{1}{n!}$$

Therefore the sum converges.
We also have

$$\frac{d}{dt} u(t) = \sum_{n=1}^{\infty} (-1)^n A(\gamma'(t)) \int_{t \geq t_2 ... \geq t_n} A(\gamma'(t_2))...A(\gamma'(t_n)) dt_n...dt_2$$

It is not too hard to see that this must be convergent in a similar way. So du/dt is finite, i.e. $u(t)$ is differentiable. In fact, we can clearly take t-derivatives arbitrarily many times and have similar looking sums which will all converge in a similar way, so $u(t)$ is C^∞ in t. Finally, though we already know this, as a consistency check we can confirm

$$\begin{aligned}
\frac{d}{dt} u(t) &= A(\gamma'(t)) \sum_{n=1}^{\infty} (-1)^n \int_{t \geq t_2 \geq ... \geq t_n} A(\gamma'(t_2))...A(\gamma'(t_n)) dt_n...dt_2 \\
&= A(\gamma'(t))(-1) \sum_{m=0} (-1)^m \int_{t \geq t_1 \geq ... \geq t_m} A(\gamma'(t_1))...A(\gamma'(t_m)) dt_m...dt_1 \\
&= -A(\gamma'(t)) u(t)
\end{aligned}$$

where in the second line we have relabelled not only the summation index but also the integration variables.

Exercise 90. Assume γ is contained in a neighbourhood over which we can locally trivialise. $H(\gamma, D)$ is then defined by an equation

$$\frac{d}{dt} u(t) + A(\gamma'(t)) u(t) = 0$$

where t parameterises γ. If we perform a reparameterisation $t \to s$, both terms just pick up a factor dt/ds, which clearly drops out. Therefore this equation is invariant under reparameterisation, and hence $H(\gamma, D)$ is invariant of parameterisation of γ.

Exercise 91. If α and β are composable into $\beta\alpha$, then clearly parallel transporting some u along α then along β is equivalent to parallel transporting it along $\beta\alpha$. Thus

$$H(\beta\alpha, D) = H(\beta, D) H(\alpha, D)$$

Then, given an inverse path

$$\alpha^{-1}(t) = \alpha(T - t)$$

we must have

$$H(\alpha^{-1}, D)H(\alpha, D)u = u$$

i.e.

$$H(\alpha^{-1}, D) = H(\alpha, D)^{-1}$$

Then, trivially

$$H(1_p, D)u = u$$

where 1_p is the constant loop at p, i.e.

$$H(1_p, D) = 1$$

Then it follows that

$$\begin{aligned} H(1_q\alpha, D) &= H(1_q, D)H(\alpha, D) \\ &= H(\alpha, D) \end{aligned}$$

and

$$\begin{aligned} H(\alpha 1_p, D) &= H(\alpha, D)H(1_p, D) \\ &= H(\alpha, D) \end{aligned}$$

Exercise 92. For generality, suppose γ is not confined to a locally trivial neighbourhood, and rewrite it as a product of paths that are

$$\gamma = \gamma_n ... \gamma_1$$

where $\gamma_i : [t_i, t_{i+1}] \to M$ with $t_1 = 0$ and $t_{n+1} = T$. Then

$$\begin{aligned} H(\gamma, D') &= H(\gamma_n, D')...H(\gamma_1, D') \\ &= g(\gamma(T))H(\gamma_n, D)g(\gamma(t_n))...g(\gamma(t_1))H(\gamma_1, D)g(\gamma(0))^{-1} \\ &= g(\gamma(T))H(\gamma_n, D)...H(\gamma_1, D)g(\gamma(0))^{-1} \\ &= g(\gamma(T))H(\gamma, D)g(\gamma(0))^{-1} \end{aligned}$$

Therefore this result holds anyway.

Exercise 93. Assume γ is confined within a local trivialisation. The map $H(\gamma, D) : E_p \to E_q$ is then defined by

$$\frac{d}{dt}u(t) = -A(\gamma'(t))u(t)$$

and solved by

$$u(t) = Pe^{-\int_0^t A(\gamma'(s))ds}u(0)$$

A lives in $\mathfrak{g}$, so the path-ordered exponential lives in G. Therefore $H(\gamma, D)$ does. This will remain true even if we need to generalise to $\gamma = \gamma_n \ldots \gamma_1$.