# Baez and Muniain - Gauge Fields, Knots and Gravity

Part I: Electromagnetism

## 6 De Rham Theory in Electromagnetism

**Exercise 80.** On $\mathbb{R}^2$, consider

$$E = \frac{xdy - ydx}{x^2 + y^2}$$

We have

$$\begin{aligned} dE &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy \wedge dx \\ &= \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) dx \wedge dy \\ &= \frac{1}{(x^2+y^2)^2}(x^2+y^2-2x^2+x^2+y^2-2y^2) dx \wedge dy \\ &= 0 \end{aligned}$$

Let $\gamma_0$ be the path

$$\gamma_0(t) = (-\cos t, \sin t)$$

Then $dx = \sin t dt$ and $dy = \cos t dt$. So

$$E|_{\gamma_0} = \frac{-\cos^2 t - \sin^2 t}{\cos^2 t + \sin^2 t} dt$$
$$= -dt$$

So

$$\begin{aligned} \int_{\gamma_0} E &= \int_0^\pi (-dt) \\ &= -\pi \end{aligned}$$

On the other hand, on

$$\gamma_1(t) = (-\cos t, -\sin t)$$

there is an extra minus sign everywhere, so we get

$$\int_{\gamma_1} E = \pi$$

**Exercise 81.** Let $p, q \in \mathbb{R}^n$, and $\gamma_0$ and $\gamma_1$ two smooth curves from $p$ to $q$. Then define the family $\gamma_s$ of curves by

$$\gamma_s(t) = (1-s)\gamma_0 + s\gamma_1$$

Clearly this exists, is smooth and agrees with $\gamma_0$ and $\gamma_1$. Any two paths between given points in $\mathbb{R}^n$ are homotopic.

**Exercise 82.** Suppose $E$ is exact and write $E = -d\phi$. Then if $\gamma(0) = \gamma(1) = p$,

$$\begin{aligned}\int_\gamma E &= -\int_\gamma d\phi \\ &= -\phi(\gamma(1)) + \phi(\gamma(0)) \\ &= -\phi(p) + \phi(p) \\ &= 0\end{aligned}$$

On the other hand, suppose $\int_\gamma E = 0$ for all loops $\gamma$ at $p$. Now, if $\gamma(1/2) = q$, we can define the paths from $p$ to $q$

$$\begin{aligned}\gamma_0(t) &= \gamma\left(\frac{1}{2}t\right) \\ \gamma_1(t) &= \gamma\left(\frac{1}{2}(1-t)\right)\end{aligned}$$

By choosing $\gamma$ appropriately, $\gamma_0$ and $\gamma_1$ thus defined can be any pair of paths between any two points (we presuppose connectedness). But then

$$\int_\gamma E = \int_{\gamma_0} E - \int_{\gamma_1} E$$

so

$$\int_{\gamma_0} E = \int_{\gamma_1} E$$

and hence $E$ is exact.
Therefore $E$ is exact iff $\int_\gamma E = 0$ for all loops $\gamma$.

**Exercise 83.** We can put coordinates $(\theta, x)$ on $S^1 \times M$. By the previous exercise, the 1-form $d\theta$ on $S^1$ is not exact since

$$\int_0^{2\pi} d\theta = 2\pi \neq 0$$

Then $d\theta$ as a 1-form on $S^1 \times M$ is not exact either (the loop purely in the $S^1$ direction is still available). On the other hand, it is obviously closed. Therefore $H^1(S^1 \times M)$ is not trivial.

**Exercise 84.** Then $n$-disk is

$$D^n = \{(x_1, ..., x_n) \mid x_1^2 + ... + x_n^2 \leq 1\}$$

Define open sets

$$\begin{aligned} U_i^+ &= \{x \in D^n \mid x_i > 0\} \\ U_i^- &= \{x \in D^n \mid x_i < 0\} \end{aligned}$$

and maps

$$\begin{aligned} \phi_i^+ &: U_i^+ \to \mathbb{H}^n \\ (x_1, ..., x_n) &\mapsto \frac{|x|}{x_i}(x_1, ..., x_n, ..., x_i) - (0, ..., 0, 1) \\ \phi_i^- &: U_i^- \to \mathbb{H}^n \\ (x_1, ..., x_n) &\mapsto \frac{|x|}{x_i}(x_1, ..., x_n, ..., x_i) - (0, ..., 0, 1) \end{aligned}$$

Then the union of all of these open sets is $D^n$ itself. Clearly the transition functions are defined and smooth on their domains. Furthermore we note that a point in $D^n$ is mapped to $\partial\mathbb{H}^n$ only if $|x| = 1$, so as expected $\partial D^n = S^{n-1}$.

**Exercise 85.** Let $M$ be a manifold with boundary and $p \in \partial M$. Then $p$ gets mapped to $\partial\mathbb{H}^n$, its $n^{\text{th}}$ component $x_n > 0$. The other directions are clearly alright - we just need to check the derivative in the $n^{\text{th}}$ direction is well-defined. But smoothness of maps involving $\mathbb{H}^n$ is defined by extending the $n^{\text{th}}$ direction to some $-\varepsilon$, in which case derivatives can clearly be defined. But coordinate charts are smooth maps to $\mathbb{H}^n$, so we are fine.

**Exercise 86.** We have already seen that our definition of integration is coordinate-independent (up to a sign depending on orientation - we assume that two different coordinate systems have the same orientation). Now, if $\{f_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$, and $\{g_\beta\}$ another, subordinate to $\{V_\beta\}$, then

$$\begin{aligned}\sum_\alpha \int_{U_\alpha} f_\alpha \omega &= \sum_\alpha \int_{U_\alpha} (\sum_\beta g_\beta)\omega \\ &= \sum_\alpha \sum_\beta \int_{U_\alpha} f_\alpha g_\beta \omega \\ &= \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} f_\alpha g_\beta \omega\end{aligned}$$

where we have used in the first line that partitions of unity sum to 1, in the second that only finitely many of the $g_\beta$ are non-zero anywhere to move the summation over $\beta$ outside the integral, and in the third that the support of $g_\beta$ is contained in $V_\beta$ to restrict the domain. Similarly

$$\sum_\beta \int_{V_\beta} g_\beta \omega = \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} f_\alpha g_\beta \omega$$

So,

$$\sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\beta \int_{V_\beta} g_\beta \omega$$

Thus our definition of integration is independent of the partition of unity we choose.

**Exercise 87.** We saw already in Exercise 84 that $\partial D^n = S^{n-1}$.

**Exercise 88.** Let $M = [0,1]$. Then an exact top-form on $M$ is some $df = f'(x)dx$ in local coordinates. Stokes' theorem reads

$$\int_M f'(x)dx = \int_{\partial M} f$$

and $\partial M = \{0, 1\}$. Now, orientation is defined by a choice of an equivalence class of top-forms; if we take $dx$ to be a representative of this class, it defines orientation in the sense that increasing $x$ is outward-facing at 1 and inward facing at 0. Thus

$$\int_\partial Mf = f(1) - f(0)$$

and hence

$$\int_0^1 f'(x)dx = f(1) - f(0)$$

**Exercise 89.** Let $M = [0, \infty)$. Stokes' theorem gives us

$$\begin{aligned} \int_M f' dx &= \int_{\partial M} f \\ &= -f(0) \end{aligned}$$

where the sign follows as in the previous exercise. However, if $f$ is not compactly supported, $\int_M f' dx$ may be divergent, whereas $f(0)$ is defined. Therefore Stokes' theorem cannot hold on non-compact manifolds without the assumption that the forms involved are compactly supported.

**Exercise 90.** Let $S \subset M$ be a $k$-dimensional submanifold. Put an atlas $\{U_i, \phi_i)\}$ on $M$. For each $p \in S$, choose $i$ such that $p \in U_i$. Then $\phi_i(U_i \cap S) \subset \mathbb{R}^k$. So define $V_i = U_i \cap S$ and $\psi_i = \phi_i|_{V_i}$ for all $i$. Then $\{(V_i, \psi_i)\}$ is an atlas on $S$, since smoothness of transition functions is clearly inherited from the atlas on $M$.

**Exercise 91.** Clearly $S^{n-1} \subset \mathbb{R}^n$ is both closed and bounded, so by Heine-Borel it is compact.

**Exercise 92.** Let $M$ be a manifold and $S \subset M$ open. Let $\{(U_i, \phi_i)\}$ be an atlas on $M$. Then all the $U_i \cap S$ are open in $M$, so define $V_i = U_i \cap S$ and $\psi_i = \phi_i \mid_{U_i \cap S}$. Then $\{(V_i, \psi_i)\}$ is an atlas on $S$, so $S$ is a manifold.

**Exercise 93.** If $S$ is a $k$-dimensional submanifold with boundary of $M$, then $S$ is a manifold with boundary, just as in Exercise 90 (with $\mathbb{R}^k$ replaced by $\mathbb{H}^k$).
Then, if $\{(U_i, \phi_i) \mid i \in I\}$ is an atlas on $S$, consider the restriction to $J \subset I$, which includes exactly those $i$ such that $\phi_i(U_i) \cap \partial\mathbb{H}^k \neq \emptyset$. Then define $V_j = U_j \cap \partial S$ and $\psi_j = \phi_j \mid_{V_j}$. Then $\{(V_j, \psi_j) \mid j \in J\}$ is an atlas on $\partial S$ such that $\psi_j(V_j) \subset \partial\mathbb{H}^k$. Then $\psi_j$ can be regarded as having for its codomain $\mathbb{R}^{k-1}$. So $\partial S$ is a $(k-1)$-dimensional manifold. Clearly $\partial S$ is a submanifold of $S$, and clearly the relation 'is a submanifold of' is transitive, so $\partial S$ is a submanifold of $M$.

**Exercise 94.** Didn't we do this in Exercise 84?

**Exercise 95.** Let $S \subset \mathbb{R}^2$ be a 2-dimensional compact orientable submanifold with boundary. Let $\omega \in \Omega^1(S)$. Stokes' theorem gives us

$$\int_S d\omega = \int_{\partial S} \omega$$

Write $\omega = \omega_x dx + \omega_y dy$. Then

$$d\omega = (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy$$

so we have

$$\int_S \left( \frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dx \wedge dy = \int_{\partial S} (\omega_x dx + \omega_y dy)$$

This is just Green's theorem.

**Exercise 96.** Let $S \subset \mathbb{R}^3$ be a 2-dimensional orientable submanifold with boundary. Partition $S$ up into a finite number of smaller such submanifolds $S_i$, so that $S = \bigcup_i S_i$, and $S_i \cap S_j = \partial S_i \cap \partial S_j$ which is either empty or a 1-dimensional submanifold. Now, if we integrate something over $\partial S$ it will be just the same as integrating over all the $\partial S_i$ and summing, since orientations on coinciding boundaries will be opposite, cancelling everything except those $\partial S_i$ which are in $\partial S$. If $\omega \in \Omega^1(S)$, then Stokes' theorem on $S_i$ says

$$\int_{S_i} d\omega = \int_{\partial S_i} \omega$$

We can choose our partition of $S$ such that $S_i$ is small enough that in some coordinates it lies only in the $xy$-plane. Then WLOG we can also write $\omega = \omega_x dx + \omega_y dy$. Then

$$d\omega = (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy$$

So

$$\int_{S_i} (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy = \int_{\partial S_i} \omega$$

Define $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z) = (\omega_x, \omega_y, 0)$. Then

$$\boldsymbol{\nabla} \times \boldsymbol{\omega} = (0, 0, \partial_x \omega_y - \partial_y \omega_x)$$

so

$$\int_{S_i} (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \cdot d\boldsymbol{S} = \int_{S_i} (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy$$

Also,

$$\begin{aligned} \int_{\partial S_i} \boldsymbol{\omega} \cdot d\boldsymbol{r} &= \int_{\partial S_i} (\omega_x, \omega_y, 0) \cdot (dx, dy, 0) \\ &= \int_{\partial S_i} \omega \end{aligned}$$

So we have

$$\int_{S_i} (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \cdot d\boldsymbol{S} = \int_{\partial S_i} \boldsymbol{\omega} \cdot d\boldsymbol{r}$$

and hence

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{\omega} \cdot d\boldsymbol{r}$$

This is the classical Stokes' theorem.

**Exercise 97.** Let $S \subset \mathbb{R}^3$ be a 3-dimensional compact orientable submanifold with boundary, and $\omega \in \Omega^2(S)$. Write

$$\omega = \omega_{xy} dx \wedge dy + \omega_{yz} dy \wedge dz + \omega_{zx} dz \wedge dz$$

Then

$$d\omega = (\partial_z \omega_{xy} + \partial_x \omega_{yz} + \partial_y \omega_{zx}) dx \wedge dy \wedge dz$$

Define $\boldsymbol{\omega} = (\omega_{yz}, \omega_{zx}, \omega_{xy})$. Then

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) dx \wedge dy \wedge dz = d\omega$$

Also,

$$\omega = \boldsymbol{\omega} \cdot d\boldsymbol{S}$$

Thus, Stokes' theorem reads

$$\int_S (\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) dV = \int_{\partial S} \boldsymbol{\omega} \cdot d\boldsymbol{S}$$

This is Gauss' theorem.

**Exercise 98.** Let $\omega \in \Omega^p(N)$ be closed, and $\phi : M \to N$. Then

$$d\phi^* \omega = \phi^* d\omega = 0$$

so $\phi^* \omega \in \Omega^p(M)$ is closed.
Then if $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(N)$,

$$\phi^* \omega = \phi^* d\eta = d\phi^* \eta$$

so $\phi^* \omega \in \Omega^p(M)$ is exact.

**Exercise 99.** $\phi : M \to M'$ induces the pullback on forms $\phi^* : \Omega^p(M') \to \Omega^p(M)$, and hence the pullback on cohomology $\phi^* : H^p(M') \to H^p(M)$ given by

$$\phi^*([\omega]) = [\phi^* \omega]$$

We have to check this is well-defined. Let $[\omega] = [\mu]$ in $H^p(M')$. Then for some $\alpha \in \Omega^{p-1}(M')$ we have $\omega = \mu + d\alpha$. Then

$$\begin{aligned}
\phi^*([\omega]) = [\phi^*\omega] &= [\phi^*(\mu + d\alpha)] \\
&= [\phi^*\mu + \phi^* d\alpha] \\
&= [\phi^*\mu + d\phi^*\alpha] \\
&= [\phi^*\mu] \\
&= \phi^*([\mu])
\end{aligned}$$

Thus $\phi^*$ is well-defined on cohomology. Since pullbacks on forms can be composed contravariantly, the same is immediately true for pullbacks on cohomology.

**Exercise 100.** We have

$$\begin{aligned}
*dz &= dx \wedge dy \\
&= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= r\cos^2\theta dr \wedge d\theta - r\sin^2\theta d\theta \wedge dr \\
&= r dr \wedge d\theta \\
\Rightarrow *j &= f(r) r dr \wedge d\theta
\end{aligned}$$

**Exercise 101.** We have

$$\begin{aligned}
d\theta &= \frac{xdy - ydx}{r^2} \\
*d\theta &= \frac{1}{r^2}(xdz \wedge dz - ydy \wedge dz) \\
&= \frac{1}{r^2}(xdz \wedge (\cos\theta dr - r\sin\theta d\theta) - y(\sin\theta dr + r\cos\theta d\theta) \wedge dz) \\
&= \frac{1}{r^2}(r\cos^2\theta dz \wedge dr - r^2\sin\theta\cos\theta dz \wedge d\theta - r\sin^2\theta dr \wedge dz - r^2\sin\theta\cos\theta d\theta \wedge dz) \\
&= \frac{1}{r^2} r dz \wedge dr \\
&= \frac{1}{r} dz \wedge dr
\end{aligned}$$

**Exercise 102.** We have

$$\begin{aligned}
d * B &= d(g(r)d\theta) \\
&= g'(r) dr \wedge d\theta
\end{aligned}$$

So if $d * B = *j$,

$$g'(r) = rf(r)$$

**Exercise 103.** Consider the $n$-torus, $T^n = S^1 \times ... \times S^1$. Put coordinates $(\theta_1, ..., \theta_n)$ on $T^n$. Then define $n$ projection maps $p_i : T^n \to S^1$ by

$$p_i(\theta_1, ..., \theta_n) \mapsto \theta_i$$

Now, $d\theta$ is a closed but not exact 1-form on $S^1$. Consider the pullback, $p_i^* d\theta = d\theta_i$. Clearly this is also closed. Furthermore it cannot be exact, since $\theta_i$ is not actually a function on $T^n$. We have $n$ such linearly independent forms, so $H^1(T^n)$ is at least $n$-dimensional.

**Exercise 104.** Let $E = e(r)dr$ be a 1-form on $\mathbb{R} \times S^2$ with the given metric. Then clearly $dE = 0$ since $e$ is only a function of $r$. Now, we have

$$|g| = f(r)^4 \sin^2 \phi$$
$$\sqrt{|g|} = f(r)^2 \sin \phi$$

So

$$\begin{aligned} *dr &= \frac{f(r)^2 \sin \phi}{2!} \varepsilon^r{}_{ij} dx^i \wedge dx^j \\ &= f(r)^2 \sin \phi d\theta \wedge d\phi \end{aligned}$$

Thus

$$*E = e(r) f(r)^2 \sin \phi d\theta \wedge d\phi$$

Then,

$$d * E = \frac{\partial}{\partial r}(e(r) f(r)^2) \sin \phi dr \wedge d\theta \wedge d\phi$$

Clearly then $d * E = 0$ is satisfied if

$$e(r) = \frac{q}{4\pi f(r)^2}$$

**Exercise 105.** $\mathbb{R} \times S^2$ is simply connected, so since $E$ is closed we can find some $\varphi$ (terrible notation used in the book!) such that $E = -d\varphi$. We then will have

$$\frac{q}{4\pi f(r)^2} dr = -d\varphi$$

We can take

$$\varphi(r) = -\int_0^r \frac{q}{4\pi f(s)^2} ds$$

(Note that $f(r) > 0$ for all $r$ so this is defined.)

**Exercise 106.** To integrate, we need to pick an orientation. Take the standard one defined by $+r^2 \sin\theta d\theta \wedge d\phi$. Now consider the integral

$$\int_{S^2} *E = \int_{S^2} \frac{q}{4\pi} \sin\phi d\theta \wedge d\phi$$

With our choice of orientation, $d\theta \wedge d\phi$ just becomes $d\theta d\phi$. Then we have

$$\begin{aligned}\int_{S^2} *E &= \frac{q}{2} \int_0^{\pi} \sin\phi d\phi \\ &= q\end{aligned}$$

**Exercise 107.** In the previous exercise we took the standard volume form $+r^2 \sin\theta d\theta \wedge d\phi$ to define the orientation. Namely, our orientation is along increasing $r$ for all $r$. However, if we are in the $r < 0$ universe, this is not the appropriate orientation, since the normal to $S^2$ here should be in the direction of decreasing $r$. Therefore when we calculate the charge of the wormhole in that universe we should replace $d\theta \wedge d\phi$ with $-d\theta d\phi$. Therefore we end up with an overall minus sign, i.e.

$$\int_{S^2} *E = -q$$

Thus each the ends of the wormhole appear to have opposite charges.

**Exercise 108.** In $n$ dimensions, $E$ is a 1-form, but $*E$ is an $(n-1)$-form. We have 'charge without charge' if there exists an $(n-1)$-surface which we can integrate $*E$ over to get something non-zero. If we can do this, $*E$ will be a closed but not exact $(n-1)$-form, and hence $H^{n-1}$ of our space must be non-trivial.

**Exercise 109.** We have

$$\begin{aligned}d\omega &= \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} dy \wedge dz + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} dz \wedge dx + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} dx \wedge dy \\ &= \frac{1}{(x^2+y^2+z^2)^{5/2}}(3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2) dx \wedge dy \wedge dz \\ &= 0\end{aligned}$$

**Exercise 110.** On $\mathbb{R}^n \backslash \{0\}$, define the $(n-1)$-form

$$\omega = \frac{1}{(x_1^2 + ... + x_n^2)^{n/2}} \sum_{i=1}^{n} (-1)^{i-1} x_i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

We have

$$\begin{aligned}
d\omega &= \sum_{i=1}^{n}(-1)^{i-1}\partial_i \frac{x_i}{(x_1^2+...+x_n^2)^{n/2}}(-1)^{i-1}dx^1\wedge...\wedge dx^n \\
&= \sum_{i=1}^{n}\left(\frac{1}{(x_1^2+...+x_n^2)^{n/2}} - \frac{n}{2}\frac{2x_i^2}{(x_1^2+...+x_n^2)^{n/2+1}}\right)dx^1\wedge...\wedge dx^n \\
&= \frac{1}{(x_1^2+...+x_n^2)^{n/2+1}}\sum_{i=1}^{n}(x_1^2+...+x_n^2-nx_i^2)dx^1\wedge...\wedge dx^n \\
&= 0
\end{aligned}$$

On the other hand, if we integrate over the unit $n$-sphere,

$$\begin{aligned}
\int_{S^{n-1}}\omega &= \sum_{i=1}^{n}(-1)^{i-1}\int_{S^{n-1}}\frac{x_i dx^1\wedge...\wedge\widehat{dx^i}\wedge...\wedge dx^n}{(x_1^2+...+x_n)^2} \\
&= \sum_{i=1}^{n}(-1)^{i-1}\int_{S^{n-1}}\sqrt{1-\sum_{j\neq i}x_j^2}dx^1\wedge...\wedge\widehat{dx^i}\wedge...\wedge dx^n
\end{aligned}$$

If we choose the standard orientation, we replace

$$dx^1\wedge...\wedge\widehat{dx^i}\wedge...\wedge dx^n \to (-1)^{i-1}dx^1...\widehat{dx^i}...dx^n$$

So we have

$$\int_{S^{n-1}}\omega = \sum_{i=1}^{n}\int_{S^{n-1}}\sqrt{1-\sum_{j\neq i}x_j^2}dx^1...\widehat{dx^i}...dx^n$$

which we can convince ourselves is non-zero. Therefore $\omega$ cannot be exact, and so $H^{n-1}(\mathbb{R}^{n-1}\backslash\{0\}) \neq 0$. In fact it is always $\mathbb{R}$.

**Exercise 111.** The integral in $\mathbb{R}^3\backslash\{0\}$ works just as the integral of $*E$ in Exercise 106. The integral in $\mathbb{R}^3$ is clearly zero, since in this space any closed form is exact, and $S^2$ has no boundary.