Baez and Muniain - Gauge Fields, Knots and Gravity

Part I: Electromagnetism

5 Rewriting Maxwell's Equations

Exercise 50. Consider a manifold $\mathbb{R} \times S$. Let E be a 1-form, and B a 2-form on S, such that, near $p \in S$,

$$E = E_i dx^i$$
$$B = \frac{1}{2} B_{ij} dx^i \wedge dx^j$$

Then, near $(t, p) \in \mathbb{R} \times S$, consider

$$B + E \wedge dt = \frac{1}{2} B_{ij} dx^i \wedge dx^j + E_i dx^i \wedge dt$$

But $\{dx^i \wedge dx^j, dx^i \wedge dt\}$ spans $\bigwedge^2 T^*_{(t,p)}(\mathbb{R} \times S)$, so any 2-form F on $\mathbb{R} \times S$ can be written in this way, locally. If F is globally defined, by thinking about its behaviour on the overlap between the chart we are working on and all intersecting charts, we see that also E and B must be, and we have this globally.
To show that writing F like this is unique, we just have to notice that $\{dx^i \wedge dx^j, dx^i \wedge dt\}$ is linearly independent. Then locally,

$$\frac{1}{2} B'_{ij} dx^i \wedge dx^j + E'_i dx^i \wedge dt = \frac{1}{2} B_{ij} dx^i \wedge dx^j + E_i dx^i \wedge dt$$

only if $B'_{ij} = B_{ij}$ and $E'_i = E_i$. This clearly extends to the whole manifold.

Exercise 51. Let $\omega \in \Omega(\mathbb{R} \times S)$. Locally,

$$\begin{aligned}
d\omega &= d(\omega_I dx^I) \\
&= \partial_\mu \omega_I dx^\mu \wedge dx^I \\
&= dx^\mu \wedge \partial_\mu \omega_I \wedge dx^I \\
&= dx^0 \wedge \partial_0 \omega_I dx^I + dx^i \wedge \partial_i \omega_I dx^I \\
&= \partial_0 \omega_I dx^0 \wedge dx^I + \partial^i \omega_I dx^i \wedge dx^I \\
&= dt + \partial_t \omega + d_S \omega
\end{aligned}$$

Exercise 52. Given $v \in V$, regard $g(v, \cdot)$ as a map $V \to \mathbb{R}$. Then it can be identified as an element of V^*, and hence we have a map $f : V \to V^*; v \mapsto g(v, \cdot)$. Then $\ker f = \{0\}$ by non-degeneracy, so this is injective. Then since $V \cong V^*$, injectivity implies surjectivity and hence this is a bijection, an isomorphism of vector spaces.

Exercise 53. Let the vector field $v = v^\mu e_\mu$ correspond to the 1-form $v_\nu f^\nu$ by $g(v, w) = v_\nu f^\nu(w)$ for all vector fields w. Then

$$\begin{aligned} v^\mu w^\nu g_{\mu\nu} &= v_\mu f^\mu(w^\nu e_\nu) \\ &= v_\mu w^\mu \\ \Rightarrow v_\mu &= g_{\mu\nu} v^\nu \end{aligned}$$

Exercise 54. Let the 1-form $\omega = \omega_\mu f^\mu$ correspond to the vector field $\omega^\nu e_\nu$ by $g(\omega^\nu e_\nu, v) = \omega(v)$ for all vector fields v. We could proceed as in the previous exercise, but since we already know that this correspondence is a bijection, we can just reverse that result by multiplying with the inverse metric:

$$\begin{aligned} g^{\mu\nu}\omega_\nu &= g^{\mu\nu} g_{\nu\lambda}\omega^\lambda \\ \Rightarrow \omega^\mu &= g^{\mu\nu}\omega_\nu \end{aligned}$$

Exercise 55. The Minkowski metric has signature (3,1), so $\eta_{00} = \eta(e_0, e_0) = -1$ and $\eta_{ii} = \eta(e_i, e_i) = 1$ for $i = 1, 2, 3$. Then in this basis

$$(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$$

Exercise 56. $g^\mu_\nu = g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$ since the metric with raised indices is the inverse metric.

Exercise 57. Let $(e^1, ..., e^n)$ be an orthonormal basis with $g(e^i, e^i) = \epsilon(i)$, where $\epsilon(i) = \pm 1$. We know that

$$\{e^{i_1} \wedge ... \wedge e^{i_p} \mid 1 \le i_1 < ... < i_p \le n\}$$

gives a basis of p-forms. To see it is orthonormal, first consider

$$\left\langle e^{i_1} \wedge ... \wedge e^{i_p}, e^{j_1} \wedge ... \wedge e^{j_p} \right\rangle$$

If the two arguments are distinct, there is at least one i_k which is distinct from all the j_ks. Therefore there is at least one row of $(g(e^i, e^j))$ which is full of zeroes, and hence the determinant vanishes. So this basis is orthogonal. Next consider

$$\begin{aligned}\left\langle e^{i_1} \wedge ... \wedge e^{i_p}, e^{i_1} \wedge ... \wedge e^{i_p} \right\rangle &= \det\bigl(g(e^{i_j}, e^{i_k})\bigr) \\ &= \prod_j g(e^{i_j}, e^{i_j}) \\ &= \prod_j \epsilon(j)\end{aligned}$$

This indeed has absolute value 1. So the basis is orthonormal. Now, the inner product is degenerate if for some $\omega \neq 0$, $\langle \omega, \eta \rangle = 0$ for all η. But for any $\omega = \alpha e^{i_1} \wedge ... \wedge e^{i_p}$, we can always choose $\eta = e^{i_1} \wedge ... \wedge e^{i_p}$, in which case

$$\langle \omega, \eta \rangle = \pm\alpha$$

which is non-zero if ω is. So the inner product is non-degenerate.

Exercise 58. Consider $\mathbb{R}^3$ with the Euclidean metric, and the 1-form

$$E = E_x dx + E_y dy + E_z dz$$

We have $\left\langle dx^i, dx^j \right\rangle = \delta^{ij}$, so

$$\langle E, E \rangle = E_x^2 + E_y^2 + E_z^2$$

Now consider the 2-form

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

Since the metric signature is strictly positive,

$$\langle dy \wedge dz, dy \wedge dz \rangle = 1$$

and so on, but since we need at least two non-zero components of $g(e^i, e^j)$ for the determinant not to vanish, all the other inner products must be zero. Thus

$$\langle B, B \rangle = B_x^2 + B_y^2 + B_z^2$$

Exercise 59. Consider $\mathbb{R}^4$ with the Minkowski metric, and the 2-form $F = B + E \wedge dt$.

$$\langle F, F \rangle = \langle B, B \rangle + 2 \langle B, E \wedge dt \rangle + \langle E \wedge dt, E \wedge dt \rangle$$

Now, the crossed term vanishes since at least two components of the corresponding matrix need to be non-zero to have a vanishing determinant. The matrix corresponding to the last term is block diagonal, so we can write

$$\langle E \wedge dt, E \wedge dt \rangle = \langle E, E \rangle \langle dt, dt \rangle = - \langle E, E \rangle$$

Thus

$$-\frac{1}{2} \langle F, F \rangle = \frac{1}{2} \left(\langle E, E \rangle - \langle B, B \rangle \right)$$

Exercise 60. Permuting elements of the basis of p-forms by $\sigma \in S_p$ just introduces an additional factor $\operatorname{sgn} \sigma$ to the determinant, so even permutations preserve orientation, and odd ones flip it.

Exercise 61. Let M be oriented. Then there exists a nowhere-vanishing volume form ω on M. Let (U_α, ϕ_α) be a chart on M. In this chart, for some nowhere-zero function f, we can write

$$\omega = \phi_\alpha^* (f dx^1 \wedge ... \wedge dx^n)$$

Either f is positive-definite or it is negative definite. If it is negative definite, define a new coordinate chart ϕ'_α on the same patch such that, if $\phi_\alpha(p) = (x_1, ..., x_n)$, $\phi'_\alpha(p) = (x_1, ..., x_{n-1}, -x_n)$. Then the corresponding function f' will be positive-definite. Then, we can construct an atlas using whichever of ϕ_α or ϕ'_α gives a positive-definite f by repeating this everywhere, which is possible since M is orientable.

Exercise 62. We say a diffeomorphism is orientation-preserving if the pullback it induces of a basis has the same orientation as that basis. Let $\{(U_\alpha, \phi_\alpha)\}$ be a chart on a manifold M, such that all the $\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ are orientation-preserving. We have the standard volume form $dx^1 \wedge ... \wedge dx^n$ on $\mathbb{R}^n$, and pullbacks $\phi_\alpha^*(dx^1 \wedge ... \wedge dx^n)$. Since all the transition functions are orientation-preserving, we can choose all the charts to be either orientation-preserving or orientation-reversing. Similarly to in the previous exercise, we can assume WLOG that they are all orientation-preserving. Then $\phi_\alpha^*(dx^1 \wedge ... \wedge dx^n)$ and $\phi_\beta^*(dx^1 \wedge ... \wedge dx^n)$ are volume forms on M with the same orientation, agreeing on the overlap $U_\alpha \cap U_\beta$. So by the smoothness of transition functions, we have a volume form everywhere on M. So M is orientable.

Exercise 63. Let M be oriented, and $\{e^\mu\}$ a positively-oriented basis of $T_p^* M$. Write $e^\mu = T^\mu_\nu dx^\nu$, and regard T as a linear transformation. This transformation must be invertible, and

$$dx^1 \wedge ... \wedge dx^n = (T^{-1})^1_{\nu_1} ... (T^{-1})^n_{\nu_n} e^{\nu_1} \wedge ... \wedge e^{\nu_n}$$
$$= \det T^{-1} e^1 \wedge ... \wedge e^n$$

But $g_{\mu\nu}$ is calculated using two factors of ∂_μ, so $\det g$ transforms in the opposite way, as $\det T^2$. That is,

$$\mathrm{vol}_p = \sqrt{|\det(g_{\mu\nu})_p|} dx^1 \wedge ... \wedge dx^n = \frac{|\det T|}{\det T} e^1 \wedge ... \wedge e^n$$

So if T is orientation preserving, i.e. if the e^i basis is of the orientation defined by the volume form, then

$$\mathrm{vol}_p = \sqrt{|\det(g_{\mu\nu})_p|} dx^1 \wedge ... \wedge dx^n = e^1 \wedge ... \wedge e^n$$

Exercise 64. By linearity of the wedge product and Hodge star, it is sufficient to consider the p-forms

$$\omega = e^{\mu_1} \wedge ... \wedge e^{\mu_p}$$
$$\mu = e^{\nu_1} \wedge ... \wedge e^{\nu_p}$$

From the definition of the Hodge star, we have

$$\omega \wedge *\mu = e^{\mu_1} \wedge ... \wedge e^{\mu_p} \wedge (\pm 1) e^{\nu_{p+1}} \wedge ... \wedge e^{\nu_n}$$

for some sign. If $\mu_i = \nu_j$ for some $1 \leq i \leq p$ and $p+1 \leq j \leq n$, this is zero, but so is $\langle \omega, \mu \rangle$, so certainly in that case $\omega \wedge *\mu = \langle \omega, \mu \rangle \, \mathrm{vol}$. So we assume that this is not the case, so the μ_i and ν_j are distinct and we can replace $\mu_i \to \nu_i$ for $1 \leq i \leq p$ without any ambiguity. Then we have

$$\omega \wedge *\mu = \pm e^{\nu_1} \wedge ... \wedge e^{\nu_n}$$

By the previous exercise, this is

$$\omega \wedge *\mu = \pm \, \mathrm{vol}$$

Now, the sign is

$$\mathrm{sgn}(\sigma_{i_1 ... i_n}) \epsilon(i_1) ... \epsilon(i_n) = \mathrm{sgn}(\sigma_{i_1 ... i_n}) g(e^{i_1}, e^{i_1}) ... g(e^{i_n}, e^{i_n})$$

The metric factors together give the sign of the determinant when we calculate $\langle \omega, \mu \rangle$ in the basis $\{e^{i_1} ... e^{i_n}\}$, so including the permutation sign, this entire factor is the sign of the determinant when we calculate $\langle \omega, \mu \rangle$ in the basis $\{e^1, ..., e^n\}$. That is,

$$\mathrm{sgn}(\sigma_{i_1 ... i_n}) g(e^{i_1}, e^{i_1}) ... g(e^{i_n}, e^{i_n}) = \det(g(e^{\mu_i}, e^{\nu_j})) = \langle \omega, \mu \rangle$$

Therefore we have

$$\omega \wedge *\mu = \langle \omega, \mu \rangle \,\mathrm{vol}$$

as desired.

Exercise 65. Let $\omega \in \Omega^1(\mathbb{R}^3)$ be given by $\omega = \omega_i dx^i$. Then

$$\begin{aligned} d\omega &= \partial_i \omega_j dx^i \wedge dx^j \\ *d\omega &= \partial_i \omega_j \varepsilon_{ijk} dx^k \end{aligned}$$

Explicitly,

$$*d\omega = \left(\frac{\partial \omega_y}{\partial z} - \frac{\partial \omega_z}{\partial y} \right) dx + \left(\frac{\partial \omega_z}{\partial x} - \frac{\partial \omega_x}{\partial z} \right) dy + \left(\frac{\partial \omega_x}{\partial y} - \frac{\partial \omega_y}{\partial x} \right) dz$$

(cf. $\boldsymbol{\nabla} \times \boldsymbol{\omega}$)

Exercise 66. Let $\omega \in \Omega^1(\mathbb{R}^3)$ be given by $\omega = \omega_i dx^i$. Then

$$\begin{aligned} *\omega &= \omega_i \varepsilon_{ijk} dx^j \wedge dx^k \\ d * \omega &= \partial_i \omega_i dx \wedge dy \wedge dz \\ *d * \omega &= \partial_i \omega_i \\ &= \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} \end{aligned}$$

(cf. $\boldsymbol{\nabla} \cdot \boldsymbol{\omega}$)

Exercise 67. Consider $\mathbb{R}^4$ with the Minkowski metric, and positively-oriented 1-form basis (dt, dx, dy, dz). On 1-forms,

$$\begin{aligned} *dt &= -dx \wedge dy \wedge dz \\ *dx &= -dt \wedge dy \wedge dz \\ *dy &= dt \wedge dz \wedge dx \\ *dz &= -dt \wedge dx \wedge dy \end{aligned}$$

On 2-forms,

$$\begin{aligned} *dt \wedge dx &= -dy \wedge dz \\ *dt \wedge dy &= dx \wedge dz \\ *dt \wedge dz &= -dz \wedge dy \\ *dx \wedge dy &= dt \wedge dz \\ *dy \wedge dz &= dt \wedge dx \\ *dz \wedge dx &= dt \wedge dy \end{aligned}$$

On 3-forms,

$$\begin{aligned} *dt \wedge dx \wedge dy &= -dz \\ *dt \wedge dy \wedge dz &= -dx \\ *dt \wedge dz \wedge dx &= -dy \\ *dx \wedge dy \wedge dz &= -dt \end{aligned}$$

And on 4-forms,

$$*dt \wedge dx \wedge dy \wedge dz = 1$$

It is easy to check that here

$$*^2 = (-1)^{p(4-p)+1}$$

Exercise 68. Now we want to generalise this observation to manifolds M with signature $(s, n-s)$. We will use the result of Exercise 69 for simplicity. Let ω be a p-form, in local coordinates

$$\omega = \frac{1}{p!}\omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

Then we have

$$\begin{aligned} *\omega &= \frac{1}{p!}\omega_{i_1 \dots i_p}\varepsilon^{i_1 \dots i_p}{}_{j_1 \dots j_{n-p}} e^{j_1} \wedge \dots \wedge e^{j_{n-p}} \\ *^2\omega &= \frac{1}{p!}\omega_{i_1 \dots i_p}\varepsilon^{i_1 \dots i_p}{}_{j_1 \dots j_{n-p}}\varepsilon^{j_1 \dots j_{n-p}}{}_{k_1 \dots k_p} e^{k_1} \wedge \dots \wedge e^{k_p} \\ &= \frac{1}{p!}\omega_{i_1 \dots i_p} g^{i_1 i_1'} \dots g^{i_p i_p'} g_{k_1 k_1'} \dots g_{k_p k_p'} \varepsilon_{i_1' \dots i_p' j_1 \dots j_{n-p}} \varepsilon^{j_1 \dots j_{n-p} k_1' \dots k_p'} e^{k_1} \wedge \dots \wedge e^{k_p} \\ &= \frac{1}{p!}\omega_{i_1 \dots i_p} g^{i_1 i_1'} \dots g^{i_p i_p'} g_{k_1 k_1'} \dots g_{k_p k_p'} (-1)^{(n-p)p} \varepsilon_{j_1 \dots j_{n-p} i_1' \dots i_p'} \varepsilon^{j_1 \dots j_{n-p} k_1' \dots k_p'} e^{k_1} \wedge \dots \wedge e^{k_p} \\ &= \frac{1}{p!}\omega_{i_1 \dots i_p} g^{i_1 i_1'} \dots g^{i_p i_p'} g_{k_1 k_1'} \dots g_{k_p k_p'} (-1)^{(n-p)p} \delta^{k_1'}_{i_1'} \dots \delta^{k_p'}_{i_p'} e^{k_1} \wedge \dots \wedge e^{k_p} \\ &= (-1)^{(n-p)p}\frac{1}{p!}\omega_{i_1 \dots i_p}(-1)^s \delta^{i_1}_{k_1} \dots \delta^{i_p}_{k_p} e^{k_1} \wedge \dots \wedge e^{k_p} \\ &= (-1)^{(n-p)p+s}\omega \end{aligned}$$

Thus

$$*^2 = (-1)^{(n-p)p+s}$$

Exercise 69. Let

$$\omega = \frac{1}{p!}\omega_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p}$$

Then

$$
\begin{aligned}
*\omega &= \frac{1}{p!}\omega_{i_1\dots i_p} * (e^{i_1} \wedge \dots \wedge e^{i_p}) \\
&= \frac{1}{p!}\omega_{i_1\dots i_p} (\quad)^{i_1\dots i_p}_{j_1\dots j_{n-p}} e^{j_1} \wedge \dots \wedge e^{j_{n-p}}
\end{aligned}
$$

The missing factor has given indices, and is the sign of the permutation

$$\{1, \dots, n\} \mapsto \{i_1, \dots, i_p, j_1, \dots, j_{n-p}\}$$

multiplied by $\epsilon(i_1)\dots\epsilon(i_p)$. Therefore for fixed indices what we are looking for is the number

$$\epsilon(i_1)\dots\epsilon(i_p)\varepsilon_{i_1\dots i_p j_1\dots j_{n-p}}$$

but the indices aren't quite right here. However, note that

$$
\begin{aligned}
\varepsilon^{i_1\dots i_p}{}_{j_1\dots j_{n-p}} &= g^{i_1 k_1}\dots g^{i_p k_p}\varepsilon_{k_1\dots k_p j_1\dots j_{n-p}} \\
&= \epsilon(i_1)\dots\epsilon(i_p)\varepsilon_{k_1\dots k_p j_1\dots j_{n-p}}
\end{aligned}
$$

Thus what we want is just

$$\varepsilon^{i_1\dots i_p}{}_{j_1\dots j_{n-p}}$$

So

$$*\omega = \frac{1}{p!}\omega_{i_1\dots i_p}\varepsilon^{i_1\dots i_p}{}_{j_1\dots j_{n-p}} e^{j_1} \wedge \dots \wedge e^{j_{n-p}}$$

Exercise 70. Recall from Exercise 66 that if ω is a 1-form on $\mathbb{R}^3$, $*d*\omega$ can be identified as the curl of the corresponding vector. So

$$*_s d_s *_s E = \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$$

Now consider the 2-form B. We have

$$
\begin{aligned}
B &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \\
*_s B &= B_x dx + B_y dy + B_z dz \\
d_s *_s B &= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) dz \wedge dx + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) dx \wedge dy \\
*_s d_s *_s B &= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) dx + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) dy + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) dz
\end{aligned}
$$

So

$$-\partial_t E + *_s d_s *_s B = j$$

is equivalent to

$$-\partial_t \boldsymbol{E} \cdot d\boldsymbol{x} + (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot d\boldsymbol{x} = \boldsymbol{j} \cdot d\boldsymbol{x}$$

i.e.

$$-\partial_t \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j}$$

Exercise 71. We have

$$\begin{aligned}
*E \wedge dt &= *(E_i dx^i \wedge dt) \\
&= E_x dy \wedge dz + E_y dz \wedge dx + E_z dy \wedge dz \\
&= *_s E
\end{aligned}$$

Then

$$\begin{aligned}
*B &= *(B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy) \\
&= B_x dt \wedge dx + B_y dt \wedge dy + B_z dt \wedge dz
\end{aligned}$$

and

$$*_s B = B_x dx + B_y dy + B_z dz$$

so

$$*B = - *_s B \wedge dt$$

Thus we have

$$*F = *_s E - *_s B \wedge dt$$

Then

$$\begin{aligned}
d *_s E &= \partial_t dt \wedge *_s E + d_s *_s E \\
&= *_s E \wedge dt + d_s *_s E
\end{aligned}$$

since $*_s E$ is a 2-form, and

$$d *_s B \wedge dt = d_s *_s B \wedge dt$$

Thus we have

$$d * F = *_s \partial_t E \wedge dt + d_s *_s E - d_s *_s B \wedge dt$$

Next, we have

$$\begin{aligned}
*(*_s \partial_t E \wedge dt) &= \partial_t * (E_x dy \wedge dz \wedge dt + E_y dz \wedge dx \wedge dt + E_z dx \wedge dy \wedge dt) \\
&= \partial_t (-E_x dx - E_y dy - E_z dz) \\
&= -\partial_t E
\end{aligned}$$

Then

$$\begin{aligned}
d_s *_s E &= d_s (E_x dy \wedge dz + E_y dz \wedge dz + E_z dz \wedge dy) \\
&= (\partial_x E_x + \partial_y E_y + \partial_z E_z) dx \wedge dy \wedge dz
\end{aligned}$$

So

$$*d_s *_s E = -(\partial_x E_x + \partial_y E_y + \partial_z E_z) dt$$

but

$$*_s d_s *_s E = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

So

$$*d_s *_s E = - *_s d_s *_s E \wedge dt$$

Lastly, we have

$$\begin{aligned} d_s *_s B \wedge dt &= -d_s(B_x dt \wedge dx + B_y dt \wedge dy + B_z dt \wedge dz) \\ &= (\partial_x B_y - \partial_y B_x) dt \wedge dx \wedge dy + (\partial_y B_z - \partial_z B_y) dt \wedge dy \wedge dz + (\partial_z B_x - \partial_x B_z) dt \wedge dz \wedge dx \end{aligned}$$

So

$$*d_s *_s B \wedge dt = -(\partial_x B_y - \partial_y B_x) dz - (\partial_y B_z - \partial_z B_y) dx - (\partial_z B_x - \partial_x B_z) dy$$

On the other hand,

$$\begin{aligned} *_s B &= B_x dx + B_y dy + B_z dz \\ d_s *_s B &= (\partial_x B_y - \partial_y B_x) dx \wedge dy + (\partial_y B_z - \partial_z B_y) dy \wedge dz + (\partial_z B_x - \partial_x B_z) dz \wedge dx \\ *_s d_s *_s B &= (\partial_x B_y - \partial_y B_x) dz + (\partial_y B_z - \partial_z B_y) dx + (\partial_z B_x - \partial_x B_z) dy \end{aligned}$$

Thus,

$$*d_s *_s B \wedge dt = - *_s d_s *_s B$$

So finally we have

$$*d * F = -\partial_t E - *_s d_s *_s E \wedge dt + *_s d_s *_s B$$

Then setting $*d * F = J$, we have the two equations

$$\begin{aligned} -\partial_t E + *_s d_s *_s B &= j \\ *_s d_s *_s E &= \rho \end{aligned}$$

as desired.

Exercise 72. Take $F_\pm = \frac{1}{2}(F \pm *F)$. Then certainly $F = F_+ + F_-$, and

$$\begin{aligned} *F_\pm &= \frac{1}{2}(*F \pm *^2 F) \\ &= \frac{1}{2}(*F \pm F) \\ &= \pm F_\pm \end{aligned}$$

Exercise 73. Take $F_\pm = \frac{1}{2}(iF \pm *F)$. Then certainly $F = F_+ + F_-$, and

$$\begin{aligned} *F_\pm &= \frac{1}{2}(i * F \mp F) \\ &= \frac{1}{2}i(*F \pm iF) \\ &= \pm iF_\pm \end{aligned}$$

Exercise 74. On 1-forms $*_s^2 = (-1)^{(3-1)} = 1$, so

$$\begin{aligned} *_s E = iB \quad &\Rightarrow \quad E = i *_s B \\ & \qquad *_s B = -iE \end{aligned}$$

So all we need to show that F is self-dual is $*_s E = iB$. In order for this to be the case, we need

$$*_s(E_x dx + E_y dy + E_z dz) = i(B_z dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy)$$

i.e. $E_i = iB_i$, so

$$B = -i(E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy)$$

Then obviously $*_s B = -iE$ follows.

Exercise 75. The second Maxwell equation is

$$\partial_t B + d_s E = 0$$

We have

$$\partial_t B = ik_0 B$$

and

$$d_s E = iE \wedge k_j dx^j$$

So

$$\begin{aligned} k_0 B + E \wedge k_j dx^j &= 0 \\ k_0 \boldsymbol{B} + \boldsymbol{E} \wedge k_j dx^j &= 0 \\ k_j dx^k j \wedge \boldsymbol{E} &= k_0 \boldsymbol{B} \end{aligned}$$

i.e. $\boldsymbol{k} \times \boldsymbol{E} \propto \boldsymbol{B}$. Equivalently,

$$k_j dx^j \wedge \boldsymbol{E} = -ik_0 *_s \boldsymbol{E}$$

Exercise 76. We have

$$\begin{aligned}
*_s\boldsymbol{E} &= e^{ik_\mu x^\mu} *_s (E_x dx + E_y dy + E_z dz) \\
&= e^{ik_\mu x^\mu}(E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy)
\end{aligned}$$

so

$$k_j dx^j \wedge (E_x dx + E_y dy + E_z dz) = -ik_0(E_x dy \wedge dz + E_y dz \wedge dx + E_z dz \wedge dy)$$

This can be written in the form $K\boldsymbol{E} = 0$:

$$\begin{pmatrix} ik_0 & -k_z & k_y \\ k_z & ik_0 & -k_x \\ -k_y & k_x & ik_0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

Therefore

$$\det K = ik_0(-k_0^2 + k_x^2 + k_z^2 + k_y^2) = 0$$

so

$$k_\mu k^\mu = 0$$

Exercise 77. The electric 1-form is

$$E = (dy - idz)e^{i(t-x)}$$

so the associated electric 3-vector field is

$$\boldsymbol{E} = \left(0, e^{i(t-x)}, -ie^{i(t-x)}\right)$$

Then

$$\begin{aligned}
B &= -i *_s (dy - idz)e^{i(t-x)} \\
&= -i(dz \wedge dx - idx \wedge dy)e^{i(t-x)}
\end{aligned}$$

so the associated magnetic 3-vector field is

$$\boldsymbol{B} = \left(0, -ie^{i(t-x)}, -e^{i(t-x)}\right)$$

Exercise 78. WLOG, take ${}^3k$ to be in the x-direction. We have already done the self-dual case, seeing how the solution is left-circularly polarised. Now consider the anti-self-dual case, where $B = i *_s E$. The first equation, $d_s B = 0$, does not change from the self-dual case. The second equation, $\partial_t B + d_s E = 0$, again gives us

$${}^3k \wedge E = k_0 \boldsymbol{B}$$

but now this is

$$^3k \wedge E = ik_0 *_s E$$

Therefore the determinant we have to set to zero is slightly different:

$$\begin{vmatrix} -ik_0 & -k_z & k_y \\ k_z & -ik_0 & -k_x \\ -k_y & k_z & -ik_0 \end{vmatrix} = 0$$

However, this again yields

$$ik_0(k_0^2 - k_x^2 - k_z^2 - k_y^2) = 0$$

i.e. $k_\mu k^\mu = 0$. Then we can take, WLOG,

$$k = dt - dx$$

Now, by $d_s B = 0$, $\boldsymbol{B} \wedge^3 k = 0$, so

$$\left\langle \boldsymbol{E}, ^3 k \right\rangle = 0$$

as before, and hence we can write $E = ady + bdz$ for some a, b. But then the second equation, $^3k \wedge E = ik_0 *_s E$, reads

$$\begin{aligned} (dt - dx) \wedge (ady + bdz) &= i *_s (ady + bdz) \\ &= i(adz \wedge dx + bdx \wedge dy) \end{aligned}$$

Therefore $b = ia$, so $\boldsymbol{E} \propto (0, 1, i)$, and hence the solution is right-circularly polarised.

Exercise 79. The parity transformation is $P : \mathbb{R}^4 \to \mathbb{R}^4; (t, x, y, z) \mapsto (t, -x, -y, -z)$. Recall that $F = B + E \wedge dT$ is self-dual if

$$*_s E = iB, \quad *_s B = -iE$$

or anti-self-dual if

$$*_s E = -iB, \quad *_s B = iE$$

Now, we have $P^* B = B$ and $P^* E = -E$, so if F is self-dual,

$$\begin{aligned} *_s P^* B &= *_s B \\ &= -iE \\ &= iP^* E \end{aligned}$$

and

$$\begin{aligned} *_s P^* E &= i *_s E \\ &= -iB \\ &= -iP^* B \end{aligned}$$

So if (E, B) constitute a self-dual solution, $(P^* E, P^* B)$ constitute an anti-self-dual solution. The converse follows from $P^2 = \text{id}$.