Baez and Muniain - Gauge Fields, Knots and Gravity

Part I: Electromagnetism

1 Maxwell's Equations

Exercise 1. Let $\boldsymbol{k} \in \mathbb{R}^3$ and $\omega = |\boldsymbol{k}|$. Let $\boldsymbol{E} \in \mathbb{C}^3$ be fixed, and satisfy $\boldsymbol{k} \cdot \boldsymbol{E} = 0$ and $\boldsymbol{k} \times \boldsymbol{E} = -i\omega\boldsymbol{E}$. Define

$$\boldsymbol{\varepsilon}(t, \boldsymbol{x}) = \boldsymbol{E}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$$

Then

$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{E} &= i\boldsymbol{k} \cdot \boldsymbol{E}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \\ &= 0\end{aligned}$$

Then,

$$\begin{aligned}\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} &= i\boldsymbol{k} \times \boldsymbol{E}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \\ &= \omega\boldsymbol{E}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}\end{aligned}$$

and

$$\frac{\partial\boldsymbol{\varepsilon}}{\partial t} = -i\omega\boldsymbol{E}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$$

so indeed

$$\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} = -\frac{\partial\boldsymbol{\varepsilon}}{\partial t}$$

2 Manifolds

Exercise 2. Let X, Y be topological spaces. We say that $f : X \to Y$ is topological-continuous if for any open $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open.
On the other hand, we say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is ϵ-continuous if, for all $x \in \mathbb{R}^n$ and $\epsilon > 0$, there exists some $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| > \epsilon$.
Consider $f : \mathbb{R}^n \to \mathbb{R}^m$. Suppose it is topological-continuous, and consider the open ball $B(f(x), \epsilon)$. $f^{-1}(B(f(x), \epsilon))$ is then open in $\mathbb{R}^n$, so we can choose δ such that $x \in B(x, \delta) \subseteq$

$f^{-1}(B(f(x),\epsilon))$. That is, such that $|y-x|<\delta$ implies $|f(y)-f(x)|<\epsilon$. So f is ϵ-continuous.
Conversely, suppose f is ϵ-continuous. For any $U \subseteq \mathbb{R}^m$ (with non-empty intersection with the image of f) we can choose some $x \in \mathbb{R}^n$ and $\epsilon > 0$ such that $f(x) \in B(f(x)\epsilon) \subset U$. Then we can choose any δ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$, so $x \in B(x,\delta) \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is open. So f is topological-continuous.
Therefore, these two definitions are equivalent (on Euclidean spaces).

Exercise 3. Consider $S^n \subset \mathbb{R}^{n+1}$, with the induced topology. Define

$$U_N = S^n \backslash \{(0,...,0,-1)\} \quad \phi_N : U_N \to \mathbb{R}^n$$
$$\phi_N : \boldsymbol{x} \mapsto \frac{1}{1+x_{n+1}}(x_1,...,x_n)$$

and

$$U_S = S^n \backslash \{(0,...,0,1)\} \quad \phi_S : U_S \to \mathbb{R}^n$$
$$\phi_S : \boldsymbol{x} \mapsto \frac{1}{1-x_{n+1}}(x_1,...,x_n)$$

Then U_N and U_S are open, and $U_N \cup U_S = S^n$, so $\{U_N, U_S\}$ is an open cover of S^n, and $\phi_{N,S}$ are homeomorphisms to open sets in $\mathbb{R}^n$. Now, let $\boldsymbol{y} \in \phi_N(U_N) \cap \phi_S(U_S) \subseteq \mathbb{R}^n$. We have

$$\phi_N^{-1}(\boldsymbol{y}) = (1+y_{n+1})(y_1,...,y_n,1)$$

where we have defined

$$y_{n+1} = \sqrt{1-\sum_{i=1}^{n} y_i^2}$$

Then

$$\phi_S \circ \phi_N^{-1}(\boldsymbol{y}) = \frac{1+y_{n+1}}{1-y_{n+1}}\boldsymbol{y}$$

This is indeed smooth with smooth inverse on $\phi_N(U_N) \cap \phi_S(U_S)$. Since the other transition function is just its inverse, we are done. Therefore we have constructed the atlas $\{(U_N,\phi_N),(U_S,\phi_S)\}$ on S^n, making S^n into an n-dimensional manifold.

Exercise 4. Let M be an n-manifold, and U an open subset of M with the induced topology. We have some atlas $\{(U_i,\phi_i) \mid 1 \le i \le I\}$ on M. Define $J \le I$ as the largest integer such that

$$\bigcup_{i=1}^{J} U_i \supseteq U$$

(Keeping I is fine, but in general superfluous if we ignore the extension to maximal atlases.) Define $V_i = U_i \cap U$ and $\psi_i = \phi_i \mid_{V_i}$ for each $i = 1, ..., J$. Then $\{(V_i, \psi_i) \mid 1 \leq i \leq J\}$ is an atlas on U, since $\{V_i \mid 1 \leq i \leq J\}$ is an open cover of U, each ψ_i is a homeomorphism of the open set U_i to $\mathbb{R}^n$, and the smoothness of transition functions is inherited from the atlas on M. Thus U is a manifold.

Exercise 5. Let M and N be manifolds, of dimensions m and n, respectively. We have some atlas $\{(U_i, \phi_i) \mid 1 \leq i \leq I\}$ on M and $\{(V_j, \psi_j) \mid 1 \leq j \leq J\}$ on N. Consider $M \times N$, with the product topology. Then each $U_i \times V_j$ is open in $M \times N$, and $\{U_i \times V_j \mid 1 \leq i \leq I, 1 \leq j \leq J\}$ is an open cover of $M \times N$. Then $\phi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ is a homeomorphism. Compatibility of charts follows from the original atlases, and therefore we have an atlas on $M \times N$. So $M \times N$ is a manifold, of dimension $m + n$.

Exercise 6. Let M and N be n-manifolds. We have atlases as in the previous exercise. Consider $M \cup N$ with the disjoint topology. Then any $U_i \cup V_j$ is open in $M \cup N$, and $\{U_i, V_j \mid 1 \leq i \leq I, 1 \leq j \leq J\}$ is an open cover of $M \cup N$. Then, define maps $\chi_{ij} : U_i \cup V_j \to \mathbb{R}^n$ by

$$\chi_{ij}(p) = \begin{cases} \phi_i(p) & p \in U_i \\ \psi_j(p) & p \in V_j \end{cases}$$

which is well-defined precisely because the union is disjoint. Compatibility is inherited (and there is no requirement that ϕ maps behave nicely with ψ maps, again due to the union being disjoint). Then we have the atlas $\{(U_i \cup V_j, \chi_{ij}) \mid 1 \leq i \leq I, 1 \leq j \leq J\}$, making $M \cup N$ into an n-manifold.

3 Vector Fields

Exercise 7. Let $v, w \in \mathrm{Vect}(M)$, $f, g \in C^\infty(M)$ and $\alpha \in \mathbb{R}$. Then

(i) $v + w$ is defined by

$$(v + w)(f) = v(f) + w(f)$$

for all f, so

$$\begin{aligned}
(v+w)(f+g) &= v(f+g)+w(f+g) \\
&= v(f)+v(g)+w(f)+w(g) \\
&= (v+w)(f)+(v+w)(g) \\
(v+w)(\alpha f) &= v(\alpha f)+w(\alpha f) \\
&= \alpha v(f)+\alpha w(f) \\
&= \alpha(v+w)(f) \\
(v+w)(fg) &= v(fg)+w(fg) \\
&= v(f)g+fv(g)+w(f)g+fw(g) \\
&= (v+w)(f)g+f(v+w)(g)
\end{aligned}$$

so $(v+w) \in \mathrm{Vect}(M)$.

(ii) gw is defined by

$$(gw)(f) = gw(f)$$

Then

$$\begin{aligned}
(gw)(f+f') &= gw(f+f') \\
&= gw(f)+gw(f') \\
&= (gw)(f)+(gw)(f') \\
(gw)(\alpha f) &= gw(\alpha f) \\
&= g\alpha w(f) \\
&= \alpha(gw)(f) \\
(gw)(ff') &= gw(ff') \\
&= gw(f)f'+gfw(f') \\
&= (gw)(f)f'+f(gw)(f')
\end{aligned}$$

so $gw \in \mathrm{Vect}(M)$.

Exercise 8. Let $v, w \in \mathrm{Vect}(M)$ and $f, g, h \in C^\infty(M)$. Then

(i)

$$\begin{aligned}
(f(v+w))(h) &= f(v+w)(h) \\
&= f(v(h)+w(h)) \\
&= fv(h)+fw(h) \\
&= (fv)(h)+(fw)(h) \\
&= (fv+fw)(h)
\end{aligned}$$

so $f(v+w) = fv + fw$.

(ii)

$$\begin{aligned} ((f+g)v)(g) &= (fv+gv)(h) \\ &= fv(h) + gv(h) \\ &= (fv)(h) + (gv)(h) \end{aligned}$$

so $(f+g)v = fv + gv$.

(iii)

$$\begin{aligned} ((fg)v)(h) &= (fg)v(h) \\ &= f(gv(h)) \\ &= f(gv)(h) \\ &= (f(gv))(h) \end{aligned}$$

so $(fg)v = f(gv)$.

(iv)

$$\begin{aligned} (1v)(h) &= 1v(h) \\ &= v(h) \end{aligned}$$

so $1v = v$.

Exercise 9. Suppose $v^\mu \partial_\mu f = 0$ for all $f \in C^\infty(M)$. Then in particular, for any coordinate function x^ν,

$$\begin{aligned} v^\mu \partial_\mu x^\nu &= 0 \\ v^\nu &= 0 \end{aligned}$$

so all the $v^\mu = 0$. Therefore $\{\partial_\mu\}$ is a linearly independent set.

Exercise 10. Let $v, w \in \mathrm{Vect}(M)$.
Suppose $v = w$. Then $v_p(f) = v(f)(p) = w(f)(p) = w_p(f)$, for all $p \in M$ and $f \in C^\infty(M)$, so $v_p = w_p$ for all $p \in M$.
Suppose $v_p = w_p$ for all $p \in M$. Then $v(f)(p) = w(f)(p)$ for all $p \in M$ and $f \in C^\infty(M)$, so $v = w$.
Therefore $v = w$ iff $v_p = w_p$ for all $p \in M$.

Exercise 11. Let $v, w \in T_pM$ and $\alpha, \beta \in \mathbb{R}$. Then

- $(v+w)$ is defined by
$$(v+w)(f) = v(f) + w(f)$$
Then
$$\begin{aligned}
(v+w)(f+g) &= v(f+g) + w(f+g) \\
&= v(f) + v(g) + w(f) + w(g) \\
&= (v+w)(f) + (v+w)(g) \\
(v+w)(\alpha f) &= v(\alpha f) + w(\alpha f) \\
&= \alpha v(f) + \alpha w(f) \\
&= \alpha(v+w)(f) \\
(v+w)(fg) &= v(fg) + w(fg) \\
&= v(f)g + fv(g) + w(f)g + fw(g) \\
&= (v+w)(F)g + f(v+w)(g)
\end{aligned}$$
So $(v+w) \in T_pM$, and hence we have closure under addition. Moreover, this operation is clearly associative and commutative. The identity element is 0, the tangent vector sending every function to zero. The additive inverse of v is $-v$, defined by $(-v)(f) = -v(f)$ for all $f \in C^\infty$.

- αv is defined by
$$(\alpha v)(f) = \alpha v(f)$$
then
$$\begin{aligned}
(\alpha v)(f+g) &= \alpha v(f+g) \\
&= \alpha v(f) + \alpha v(g) \\
&= (\alpha v)(f) + (\alpha v)(g) \\
(\alpha v)(\beta f) &= \alpha v(\beta f) \\
&= \alpha\beta v(f) \\
&= \beta(\alpha v)(f) \\
(\alpha v)(fg) &= \alpha v(fg) \\
&= \alpha v(f)g + \alpha f v(g) \\
&= (\alpha v)(f)g + f(\alpha v)(g)
\end{aligned}$$
So $\alpha v \in T_pM$, and hence we have closure under scalar multiplication. The identity is clearly 1.

- Lastly,

$$\begin{aligned}\alpha(v+w)(f) &= \alpha v(f) + \alpha w(f)\\ &= (\alpha v)(f) + (\alpha w)(f)\end{aligned}$$

and

$$\begin{aligned}(\alpha+\beta)v(f) &= \alpha v(f) + \beta v(f)\\ &= (\alpha v)(f) + (\beta v)(f)\end{aligned}$$

So we have distributivity.

Thus T_pM is a vector space.

Exercise 12.

$$\gamma'(t) : f \mapsto \frac{d}{dt} f(\gamma(t))$$

for all $f \in C^\infty$. Then

$$\begin{aligned}\gamma'(t)(f+g) &= \frac{d}{dt}((f+g)(\gamma(t)))\\ &= \frac{d}{dt}(f(\gamma(t)) + g(\gamma(t)))\\ &= \gamma'(t)(f) + \gamma'(t)(g)\\ \gamma'(t)(\alpha f) &= \frac{d}{dt}(\alpha f(\gamma(t)))\\ &= \alpha\gamma'(t)(f)\\ \gamma'(t)(fg) &= \frac{d}{dt}(f(\gamma(t))g(\gamma(t)))\\ &= \frac{d}{dt} f(\gamma(t)) g(\gamma(t)) + f(\gamma(t)) \frac{d}{dt} g(\gamma(t))\\ &= \gamma'(t)(f)g + f\gamma'(t)(g)\end{aligned}$$

So $\gamma'(t) \in T_{\gamma(t)}M$.

Exercise 13. Let $\phi : \mathbb{R} \to \mathbb{R}$ be $\phi(t) = e^t$, and x the usual coordinate function on $\mathbb{R}$, i.e. explicitly, $x(t) = t$. Then $\phi^* x = x \circ \phi$, so $\phi^* x(t) = x(e^t) = e^t = e^x(t)$. So $\phi^* x = e^x$.

Exercise 14. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a CCW rotation by θ, and x, y the usual coordinates on $\mathbb{R}^2$, i.e. $x(s,t) = s$ and $y(s,t) = t$. Then

$$\begin{aligned}\phi^* x(s,t) &= x \circ \phi(s,t) \\ &= x(s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta) \\ &= s\cos\theta - t\sin\theta \\ &= (x\cos\theta - y\sin\theta)(s,t)\end{aligned}$$

so

$$\phi^* x = x\cos\theta - y\sin\theta$$

Then,

$$\begin{aligned}\phi^* y(s,t) &= y(s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta) \\ &= s\sin\theta + t\cos\theta \\ &= (x\sin\theta + y\cos\theta)(s,t)\end{aligned}$$

so

$$\phi^* y = x\sin\theta + y\cos\theta$$

Exercise 15. $\phi : M \to N$ is smooth if $f \in C^\infty(N)$ implies $\phi^* f \in C^\infty(M)$.

(i) Earlier we said that $f : M \to \mathbb{R}$ is smooth if $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth for all charts $\phi : M \to \mathbb{R}^n$. That is, $(f \circ \phi^{-1}) \in C^\infty(\mathbb{R}^n)$ implies $f = (f \circ \phi^{-1}) \circ \phi = \phi^*(f \circ \phi^{-1}) \in C^\infty(M)$. Thus this is in agreement with our new general definition.

(ii) We also said that a curve $\gamma : \mathbb{R} \to M$ is smooth if $f(\gamma(t))$ is smooth with t for any $f \in C^\infty(M)$. That is, $f \in C^\infty(M)$ implies $f \circ \gamma = \gamma^* f \in C^\infty(\mathbb{R})$. Thus this is also in agreement with the new general definition.

Exercise 16. $\gamma'(t) \in T_{\gamma(t)}M$, so $\phi_*(\gamma'(t))$ is defined by

$$\phi_*(\gamma'(t))(f) = \gamma'(t)(\phi^* f)$$

for all $f \in C^\infty(N)$. Then

$$\phi_*(\gamma'(t))(f) = \gamma'(t)(f \circ \phi)$$

and

$$\begin{aligned}(\phi \circ \gamma)'(t) &= \frac{d}{dt}(f(\phi \circ \gamma(t))) \\ &= \frac{d}{dt}(f \circ \phi(\gamma(t))) \\ &= \gamma'(t)(f \circ \phi)\end{aligned}$$

So

$$\phi_*(\gamma'(t))(f) = (\phi \circ \gamma)'(t)(f)$$

for all $f \in C^\infty(N)$, so

$$\phi_*(\gamma'(t)) = (\phi \circ \gamma)'(t)$$

Exercise 17. Let $v, w \in T_pM$. Then $\phi_*(v+w)$ is defined by

$$\begin{aligned}(\phi_*(v+w))(f) &= (v+w)(\phi^* f)\\ &= v(\phi^* f) + w(\phi^* f)\\ &= (\phi_* v)(f) + (\phi_* w)(f)\end{aligned}$$

for all $f \in C^\infty(M)$. Also,

$$\begin{aligned}\phi_*(\alpha v)(f) &= (\alpha v)(\phi^* f)\\ &= \alpha v(\phi^* f)\\ &= \alpha(f_* v)(f)\end{aligned}$$

Therefore $\phi_* : T_pM \to T_{\phi(p)}N$ is linear.

Exercise 18. Given $\phi : M \to N$, $v_p \in T_pM$, we have $\phi_* v_p \in T_{\phi(p)}N$. Define $\phi_* v$ by

$$(\phi_* v)(f)(q) = (\phi_* v_p)(f)$$

where $q = \phi(p)$, for all $f \in C^\infty(N)$. Then $\phi_* v$ is a function $C^\infty(N) \to \mathbb{R}$, satisfying

$$\begin{aligned}(\phi_* v)(f+g)(q) &= (\phi_* v_p)(f+g)\\ &= (\phi_* v_p)(f) + (\phi_* v_p)(g)\\ &= (\phi_* v)(f)(q) + (\phi_* v)(g)(q)\end{aligned}$$

so

$$(\phi_* v)(f+g) = (\phi_* v)(f) + (\phi_* v)(g)$$

Then,

$$\begin{aligned}(\phi_* v)(\alpha f)(q) &= (\phi_* v_p)(\alpha f)\\ &= \alpha(\phi_* v_p)(f)\\ &= \alpha(\phi_* v)(f)(q)\end{aligned}$$

so

$$(\phi_* v)(\alpha f) = \alpha(\phi_* v)(f)$$

And finally,

$$\begin{aligned}(\phi_* v)(fg)(q) &= (\phi_* v_p)(fg)\\ &= (\phi_* v_p)(f)g(p) + f(p)(\phi_* v_p)(g)\\ &= (\phi_* v)(f)(q)g(p) + f(p)(\phi_* v)(g)(q)\end{aligned}$$

so

$$(\phi_* v)(fg) = (\phi_* v)(f)g + f(\phi_* v)(g)$$

So indeed $\phi_* v \in \mathrm{Vect}(N)$.

Exercise 19. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\phi(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) = (x', y')$$

$\partial_x, \partial_y \in \mathrm{Vect}(\mathbb{R}^2)$. Then we have $\phi_*\partial_x \in \mathrm{Vect}(N)$ defined by

$$\begin{aligned}(\phi_*\partial_x)(f)(x',y') &= (\phi_*\partial_x)_{(x,y)}(f)\\ &= (\partial_x)_{(x,y)}(\phi^* f)\\ &= (\partial_x)_{(x,y)}(f\circ\phi)\\ &= \left(\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}\right)\Big|_{(x,y)}\\ &= (\cos\theta\partial_{x'} f + \sin\theta\partial_{y'} f)\,|_{(x,y)}\\ &= (\cos\theta\partial_x f + \sin\theta\partial_y f)\,|_{(x,y)}\end{aligned}$$

Thus

$$\phi_*\partial_x = \cos\theta\partial_x + \sin\theta\partial_y$$

Similarly, $\phi_*\partial_y \in \mathrm{Vect}(\mathbb{R}^2)$ is defined by

$$\begin{aligned}(\phi_*\partial_y)(f)(x',y') &= \left(\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y}\right)\Big|_{(x,y)}\\ &= (-\sin\theta\partial_{x'} f + \cos\theta\partial_{y'} f)\,|_{(x,y)}\end{aligned}$$

so

$$\phi_*\partial_y = -\sin\theta\partial_x + \cos\theta\partial_y$$

(There is an error in the text.)

Exercise 20. Let $v \in \mathrm{Vect}(\mathbb{R}^2)$ be the vector field

$$v = x^2\partial_x + y\partial_y$$

If $\gamma(t) = (x(t), y(t))$ is an integral curve of v,

$$\frac{dx}{dt} = x^2 \qquad \frac{dy}{dt} = y$$

The solution to these, with initial condition $\gamma(0) = (x_0, y_0)$, is

$$\gamma(t) = \left(\frac{x_0}{1 - x_0 t}, y_0 e^t\right)$$

This is well-defined unless $t = 1/x_0$.

Exercise 21. $\phi_0(p) = p$, since p is simply the point at which we define the integral curve ϕ_t to be at at $t = 0$. So $\phi_0 = \mathrm{id}$.
$\phi_{s+t}(p)$ is the $q \in M$ reached by flowing along v from p for 'time' $s + t$. $\phi_t \circ \phi_s(p)$ is the $q' \in M$ reached by flowing along v from p for 'time' s, and then for a further t. By the uniqueness of solutions to differential equations given identical initial conditions, these must be the same.

$$\phi_{s+t} = \phi_t \circ \phi_s$$

Note that this makes $\{\phi_t\}$ into an Abelian group, with $\phi_t^{-1} = \phi_{-t}$.

Exercise 22. Write

$$v = v^i \frac{\partial}{\partial x^i}, \qquad w = w^i \frac{\partial}{\partial x^i}$$

Then

$$w(f) = w^i \frac{\partial f}{\partial x^i}$$
$$v(w(f)) = v^j \left(\frac{\partial w^i}{\partial x^j}\frac{\partial f}{\partial x^i} + w^i \frac{\partial^2 f}{\partial x^i \partial x^j}\right)$$

and

$$v(f) = v^j \frac{\partial f}{\partial x^j}$$
$$w(v(f)) = w^i \left(\frac{\partial v^j}{\partial x^i}\frac{\partial f}{\partial x^j} + v^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right)$$

Thus,

$$[v, w](f) = \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\right)\frac{\partial f}{\partial x^i}$$

Here, on $\mathbb{R}^2$,

$$v = \frac{1}{\sqrt{x^2+y^2}}(x\partial_x + y\partial_y)$$
$$w = \frac{1}{\sqrt{x^2+y^2}}(x\partial_y - y\partial_x)$$

By fairly tortuous application of the above result, we find

$$[v,w](f) = \frac{1}{x^2+y^2}\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)$$

That is,

$$[v,w] = \frac{1}{x^2+y^2}(y\partial_x - x\partial_y)$$

Exercise 23. Let ϕ_t be the flow corresponding to v, and ψ_t that corresponding to w. Then

$$(v(f))(p) = \frac{d}{dt}f(\phi_t(p)) \mid_{t=0}$$
$$(w(f))(p) = \frac{d}{ds}f(\psi_s(p)) \mid_{s=0}$$

We have

$$\begin{aligned}(v(w(f)))(p) &= \frac{d}{dt}w(f(\phi_t(p))) \mid_{t=0} \\ &= \frac{d}{dt}\frac{d}{ds}f(\psi_s(\phi_t(p))) \mid_{s=t=0} \\ (v(w(f)))(p) &= \frac{d}{ds}\frac{d}{dt}f(\phi_t(\psi_s(p))) \mid_{s=t=0}\end{aligned}$$

So indeed

$$[v,w](f)(p) = \frac{d^2}{dtds}\left[f(\psi_s(\phi_t(p))) - f(\phi_t(\psi_s(p)))\right] \mid_{s=t=0}$$

(Sign error in text.)

Exercise 24. Recall from Exercise 22 that

$$[v,w] = \left(v^j\frac{\partial w^i}{\partial x^j} - w^j\frac{\partial v^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}$$

(i) It is clear from this that $[v,w]$ is antisymmetric under $v \leftrightarrow w$, that is

$$[v,w] = -[w,v]$$

(ii) For any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
[u, \alpha v + \beta w] &= \left(u^j \frac{\partial(\alpha v + \beta w)^i}{\partial x^j} - (\alpha v + \beta w)^j \frac{\partial u^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \\
&= \left(\alpha u^j \frac{\partial v^i}{\partial x^j} + \beta u^j \frac{\partial w^i}{\partial x^j} - \alpha v^j \frac{\partial u^i}{\partial x^j} - \beta w^j \frac{\partial u^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \\
&= \alpha[u, v] + \beta[v, w]
\end{aligned}$$

So we have linearity in the second argument. Note that together with antisymmetry this implies linearity in the first argument.

(iii) Consider

$$[u, [v, w]](f) + [v, [w, u]](f) + [w, [u, v]](f)$$

for any $f \in C^\infty(M)$. Fully expanding this out, every term cancels. Thus we have the Jacobi identity:

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]]$$

4 Differential Forms

Exercise 25. Let $\omega, \mu \in \Omega^1(M)$, $f, g \in C^\infty(M)$.

(i) $\omega + \mu$ is defined by

$$(\omega + \mu)(v) = \omega(v) + \omega(\mu)$$

for all $v \in \mathrm{Vect}(M)$. So

$$\begin{aligned}
(\omega + \mu)(v + w) &= \omega(v + w) + \mu(v + w) \\
&= \omega(v) + \omega(u) + \mu(v) + \mu(u) \\
&= (\omega + \mu)(v) + (\omega + \mu)(u)
\end{aligned}$$

and

$$\begin{aligned}
(\omega + \mu)(gv) &= \omega(gv) + \mu(gv) \\
&= g\omega(v) + g\mu(v) \\
&= g(\omega + \mu)(v)
\end{aligned}$$

So $(\omega + \mu) \in \Omega^1(M)$.

(ii) $f\omega$ is defined by

$$(f\omega)(v) = f\omega(v)$$

for all $v \in \mathrm{Vect}(M)$. So

$$\begin{aligned}(f\omega)(v+u) &= f\omega(v+u) \\ &= f\omega(v) + f\omega(u) \\ &= (f\omega)(v) + (f\omega)(u)\end{aligned}$$

and

$$\begin{aligned}(f\omega)(gv) &= f\omega(gv) \\ &= fg\omega(v) \\ &= g(f\omega)(v)\end{aligned}$$

So $f\omega \in \Omega^1(M)$.

Exercise 26. For all $v \in \mathrm{Vect}(M)$, we have

$$\begin{aligned}(f(\omega+\mu))(v) &= f(\omega+\mu)(v) \\ &= f\omega(v) + f\mu(v) \\ &= (f\omega)(v) + (f\mu)(v)\end{aligned}$$

so

$$f(\omega+\mu) = f\omega + f\mu$$

Then,

$$\begin{aligned}((f+g)\omega)(v) &= (f\omega + g\omega)(v) \\ &= f\omega(v) + g\omega(v) \\ &= (f\omega)(v) + (g\omega)(v)\end{aligned}$$

so

$$(f+g)\omega = f\omega + g\omega$$

Next,

$$\begin{aligned}((fg)\omega)(v) &= fg\omega(v) \\ &= f(g\omega)(v)\end{aligned}$$

so

$$(fg)\omega = f(g\omega)$$

Lastly,

$$\begin{aligned}(1\omega)(v) &= 1(\omega(v)) \\ &= \omega(v)\end{aligned}$$

so

$$1\omega = \omega$$

Thus $\Omega^1(M)$ is a module over $C^\infty(M)$.

Exercise 27. Let $f, g, h \in C^\infty(M)$, $\alpha \in \mathbb{R}$ and $v \in \mathrm{Vect}(M)$. We have the linearity rules

$$\begin{aligned}
d(f+g)(v) &= v(f+g) \\
&= v(f) + v(g) \\
&= df(v) + dg(v) \Rightarrow d(f+g) &= df + dg \\
d(\alpha f)(v) &= v(\alpha f) \\
&= \alpha v(f) \\
&= \alpha df(v) \\
\Rightarrow d(\alpha f) &= \alpha df \\
(f+g)dh(v) &= (f+g)v(h) \\
&= fv(h) + gv(h) \\
&= fdh(v) + gdh(v) \\
\Rightarrow (f+g)dh &= fdh + gdh
\end{aligned}$$

and the Leibniz rule

$$\begin{aligned}
d(fg)(v) &= v(fg) \\
&= v(f)g + fv(g) \\
&= df(v)g + fdg(v) \\
\Rightarrow d(fg) &= gdf + fdg
\end{aligned}$$

Exercise 28. Let $v \in \mathrm{Vect}(\mathbb{R}^n)$ be $v = v^\mu \partial_\mu$, and $f \in C^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
df(v) &= v(f) \\
&= v^\mu \partial_\mu f
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\partial_\mu f dx^\mu)(v) &= (\partial_\mu f) dx^\mu(v) \\
&= \partial_\mu f v(x^\mu) \\
&= \partial_\mu f v^\nu \partial_\nu x^\mu \\
&= \partial_\mu f v^\nu \delta^\mu_\nu \\
&= v^\mu \partial_\mu f
\end{aligned}$$

So indeed

$$df(v) = (\partial_\mu f dx^\mu)(v)$$

for all v, that is,

$$df = \partial_\mu f dx^\mu$$

Exercise 29. Suppose $\omega = \omega_\mu dx^\mu = 0$. Then $\omega(v) = 0$ for all $v \in \mathrm{Vect}(M)$. This is

$$\begin{aligned} \omega(v) &= \omega_\mu dx^\mu(v) \\ &= \omega_\mu v^\nu \partial_\nu x^\mu \\ &= \omega_\mu v^\mu = 0 \end{aligned}$$

for all functions v^μ. Therefore all the ω_μ must be zero. Thus the $\{dx^\mu\}$ are linearly independent.

Exercise 30. Given $v \in \mathrm{Vect}(M)$ w have $v_p \in T_pM$ for every $p \in M$, defined by

$$v_p(f) = v(f)(p)$$

for all $f \in C^\infty(M)$. Then, given $\omega \in \Omega^1(M)$, we define $\omega_p \in T_p^*M$ for each $p \in M$ by

$$\omega_p(v_p) = \omega(v)(p)$$

(i) Consider $u \in \mathrm{Vect}(M)$, defined such that $u_q = 0$ only if $q = p$ for some p. Then consider

$$\begin{aligned} \omega_q(v_q + u_q) &= \omega(v + u)(q) \\ &= \omega(v)(q) + \omega(u)(q) \\ &= \omega_q(v_q) + \omega_q(u_q) \end{aligned}$$

at $q = p$, this is

$$\omega_p(v_p + u_p) = \omega_p(v_p)$$

In other words, we could shift $v \to v + u$, and as long as $u_p = 0$, $\omega_p(v_p)$ is unchanged. Therefore ω_p is well-defined in the sense that its action depends only on objects in T_pM, and not in other tangent spaces.

(ii) Suppose $\omega_p = \nu_p$ for all $p \in M$. Then for all $v \in \mathrm{Vect}(M)$ and $p \in M$ we have

$$\omega(v)(p) = \omega_p(v_p) = \nu_p(v_p) = \nu(v)(p)$$

and hence $\omega = \nu$. That is, 1-forms are uniquely defined by giving the corresponding cotangent vectors everywhere.

Exercise 31. Given a map of vector spaces $f : V \to W$ we have the dual map $f^* : W^* \to V^*$, defined by

$$(f^*w)(v) = w(f(v))$$

for all $v \in V, w \in W$.

(i) Let $W = V$ and f be the identity. Then $w(f(v)) = w(v)$, so $(f^*w)(v) = w(v)$, and hence f^* is also the identity. The dual of the identity is the identity.

(ii) Given $f : V \to W$ and $g : W \to X$, we have the composition $g \circ f : V \to X$, and hence also $(g \circ f)^* : X^* \to V^*$, defined by

$$((g \circ f)^*x)(v) = x((g \circ f)(v))$$

for all $x \in X^*$ and $v \in V$. On the other hand,

$$(g^*x)(w) = x(g(w))$$

for all $w \in W$, and

$$\begin{aligned}(f^*(g^*x))(v) &= (g^*x)(f(v)) \\ &= x(g(f(v))) \\ &= ((g \circ f)^*x)(v)\end{aligned}$$

So we have

$$(g \circ f)^* = f^* \circ g^*$$

The dual map is contravariant.

Exercise 32. Given $\phi : M \to N$ we have $\phi^* : T^*_{\phi(p)}N \to T^*_pM$, defined by

$$(\phi^*\omega)_p = \phi^*(\omega_{\phi(p)})$$

Then given $\omega \in \Omega^1(N)$, we want to check that $\phi^*\omega \in \Omega^1(M)$, i.e. that $(\phi^*\omega)(v) \in C^\infty(M)$ for all $v \in \mathrm{Vect}(M)$. Thus we have to show that the map $p \mapsto \omega(\phi_*(v_p))$ is smooth for all $p \in M$ and $v_p \in T_pM$. It suffices to show this locally, so let x^μ be coordinates near $p \in M$, and y^α coordinates near $\phi(p) \in N$. Then this map is

$$\begin{aligned}x^\mu &\mapsto \omega(\phi_*(v(x))) \\ &= \omega_\alpha dy^\alpha(\phi_*(v^\mu(x)\partial_\mu)) \\ &= \omega_\alpha dy^\alpha\left(v^\mu(x)\frac{\partial y^\beta}{\partial x^\mu}\partial_\beta\right) \\ &= \omega_\beta(\phi(x))v^\mu(x)\frac{\partial y^\beta}{\partial x^\mu}\end{aligned}$$

ω_β and v^μ are smooth, and, because ϕ is, so is the Jacobian.
We should also have

$$(\phi^*\omega)(v+u) = (\phi^*\omega)(v) + (\phi^*\omega)(u)$$
$$(\phi^*\omega)(gv) = g(\phi^*\omega)(v)$$

for all $v, u \in \mathrm{Vect}(M)$ and $g \in C^\infty(M)$. But these linearity conditions are clearly inherited from $\omega^1(N)$.
This establishes the existence of the induced map.
To see its uniqueness, we just proceed as in the second part of Exercise 30.

Exercise 33. Consider the map $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = \sin t$. Then let x be the coordinate used on the codomain. We have

$$\begin{aligned}\phi^*(dx) &= d(\phi^* x)\\ &= d(\sin t)\\ &= \cos t dt\end{aligned}$$

Exercise 34. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\phi(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

Then

$$\begin{aligned}\phi^*(dx) &= d(\phi^* x)\\ &= d(x\cos\theta - y\sin\theta)\\ &= \cos\theta dx - \sin\theta dy\end{aligned}$$

Similarly,

$$\begin{aligned}\phi^*(dy) &= d(\phi^* y)\\ &= d(x\sin\theta + y\cos\theta)\\ &= \sin\theta dx + \cos\theta dy\end{aligned}$$

Exercise 35. We have

$$(dx^\mu)(\partial_\nu) = \phi^*(dx^\mu)(\phi_*^{-1}\partial_\nu)$$

If dx^μ really is the derivative of x^μ, $dx^\mu(\partial_\nu) = \partial_\nu x^\mu = \delta^\mu_\nu$. The RHS at a point p is

$$\begin{aligned}\phi^*(dx^\mu)(\phi_*\partial_\nu)(p) &= dx^\mu(\phi_*\phi_*^{-1}(\partial_\nu))(p)\\ &= dx^\mu(\partial_\nu)(p)\\ &= \delta^\mu_\nu\end{aligned}$$

which confirms this.

Exercise 36. Let $\{dx^\mu\}$ and $\{dx'^\nu\}$ be bases of $\Omega^1(M)$.

(i) Then for some T^ν_μ we can write

$$dx'^\nu = T^\nu_\mu dx^\mu$$

Act with this on ∂_λ:

$$\begin{aligned}dx'^\nu(\partial_\lambda) &= T^\nu_\mu \delta^\mu_\lambda\\ \partial_\lambda(x'^\nu) &= T^\nu_\lambda\end{aligned}$$

So

$$dx'^\nu = \frac{\partial x'^\nu}{\partial x^\lambda} dx^\lambda$$

(ii) Then, if we have the 1-form

$$\omega = \omega_\mu dx^\mu = \omega'_\nu dx'^\nu$$

this is

$$\omega_\mu dx^\mu = \omega'_\nu \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu$$

So

$$\begin{aligned}\omega_\mu &= \omega'_\nu \frac{\partial x'^\nu}{\partial x^\mu}\\ \omega'_\nu &= \frac{\partial x^\mu}{\partial x'^\nu}\omega_\mu\end{aligned}$$

Exercise 37. We claim

$$\phi^*(dx'^\nu) = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu$$

Act with the LHS on ∂_λ:

$$\begin{aligned}
\phi^*(dx'^\nu)(\partial_\lambda) &= dx'^\nu(\phi_*\partial_\lambda) \\
&= dx'^\nu\left(\frac{\partial x'^\mu}{\partial x^\lambda}\partial'_\mu\right) \\
&= \frac{\partial x'^\mu}{\partial x^\lambda}\delta^\nu_\mu \\
&= \frac{\partial x'^\nu}{\partial x^\lambda}
\end{aligned}$$

Now act with the RHS on ∂_λ:

$$\begin{aligned}
\frac{\partial x'^\nu}{\partial x^\mu}dx^\mu(\partial_\lambda) &= \frac{\partial x'^\nu}{\partial x^\mu}\delta^\mu_\lambda \\
&= \frac{\partial x'^\nu}{\partial x^\lambda}
\end{aligned}$$

So the claim is true on any ∂_λ, and hence true in general.

Exercise 38. If $\{\partial_\nu\}$ is a basis for $\mathrm{Vect}(U)$, it provides a basis for each T_pM for $p \in U$, and similarly $\{e_\mu\}$ (specifically, the basis given to T_pM is $\{e_\mu(p)\}$). Then we have two bases for each vector space T_pM, and hence they must be related by an invertible transformation, i.e.

$$e_\mu = T^\nu_\mu \partial_\nu$$

for some invertible T^ν_μ.

Exercise 39. Given a basis $\{e_\mu\}$ on $\mathrm{Vect}(U)$, we have a dual basis $\{f^\mu\}$ on $\Omega^1(U)$ defined by

$$f^\mu(e_\nu) = \delta^\mu_\nu$$

To see that this is indeed a basis, suppose $\omega_\mu f^\mu = 0$. Then

$$\begin{aligned}
(\omega_\mu f^\mu)(e_\nu) &= \omega_\mu f^\mu(e_\nu) \\
&= \omega_\mu \delta^\mu_\nu \\
&= \omega_\nu = 0
\end{aligned}$$

So the f^μ are linearly independent. There are $\dim U$ of them, so indeed they give a basis. Now, using the previous exercise, we claim that if

$$e_\mu = T^\nu_\mu \partial_\nu$$

then

$$f^\mu = (T^{-1})^\mu_\nu dx^\nu$$

To see this, act on e_ν:

$$\begin{aligned} f^\mu(e_\nu) &= (T^{-1})^\mu_\lambda dx^\lambda (T^\rho_\nu \partial_\rho) \\ &= (T^{-1})^\mu_\lambda T^\rho_\nu \delta^\lambda_\rho \\ &= (T^{-1})^\mu_\lambda T^\lambda_\nu \\ &= \delta^\mu_\nu \end{aligned}$$

So this is correct, and the dual basis exists. For its uniqueness, we simply note that dual bases are defined exactly and completely by their action on the basis to which they are dual. That is, if $f^\mu(e_\nu) = \delta^\mu_\nu$ and $g^\mu(e_\nu) = \delta^\mu_\nu$, then in fact $g^\mu = f^\mu$.

Exercise 40. Let $\{e_\mu\}$ be a basis for $\mathrm{Vect}(U)$, and $\{f^\mu\}$ the dual basis on $\Omega^1(U)$. Let

$$e'_\mu = T^\nu_\mu e_\nu$$

also be a basis for $\mathrm{Vect}(U)$, and its dual be $\{f'^\mu\}$. Then

$$f^\mu = (T^{-1})^\mu_\nu dx^\nu$$

Consider

$$\begin{aligned} (T^{-1})^\mu_\nu f^\nu(e'_\lambda) &= (T^{-1})^\mu_\nu T^\rho_\lambda f^\nu(e_\rho) \\ &= (T^{-1})^\mu_\nu T^\rho_\lambda \delta^\nu_\rho \\ &= (T^{-1})^\mu_\nu T^\nu_\lambda \\ &= \delta^\mu_\lambda \end{aligned}$$

So indeed

$$f'^\mu = (T^{-1})^\mu_\nu f^\nu$$

Now, if $v = v^\mu e_\mu = v'^\mu e'_\mu$,

$$v^\mu e_\mu = v'^\mu T^\nu_\mu e_\nu$$

Since T^ν_μ is invertible, this is

$$v'^\mu = (T^{-1})^\mu_\nu v^\nu$$

Similarly, if $\omega = \omega_\mu f^\mu = \omega'_\mu f'^\mu$,

$$\omega_\mu f^\mu = \omega'_\mu (T^{-1})^\mu_\nu f^\nu$$

so by invertibility

$$\omega'_\mu = T^\nu_\mu \omega_\nu$$

Exercise 41. Let

$$\begin{aligned} u &= u_x dx + u_y dy + u_z dz \\ v &= v_x dx + v_y dy + v_z dz \\ w &= w_x dx + w_y dy + w_z dz \end{aligned}$$

Then

$$v \wedge w = (v_x w_y - v_y w_x) dx \wedge dy - (v_x w_z - v_z w_x) dz \wedge dx + (v_y w_z - v_z w_y) dy \wedge dz$$

and

$$\begin{aligned} u \wedge v \wedge w &= u_z(v_x w_y - v_y w_x) dx \wedge dy \wedge dz - u_y(v_x w_z - v_z w_x) dx \wedge dy \wedge dz + u_x(v_y w_z - v_z w_y) dx \wedge dy \wedge dz \\ &= \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} dx \wedge dy \wedge dz \end{aligned}$$

Notice that

$$\det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$$

where $\boldsymbol{u} = (u_x, u_y, u_z)$ and so on.

Exercise 42. Let a, b, c, d be vectors in a 3-dimensional vector space. Then they cannot be linearly independent, and at the least we can write

$$d = \alpha a + \beta b + \gamma c$$

where $\alpha, \beta, \gamma \in \mathcal{F}$, the field over which V is a vector space. Then

$$\begin{aligned} a \wedge b \wedge c \wedge d &= a \wedge b \wedge c \wedge (\alpha a + \beta b + \gamma c) \\ &= 0 \end{aligned}$$

Exercise 43. Let $\dim V = 1$, so $\{v\}$ is a basis for V for any $v \in V$. Then $\bigwedge V$ has a basis $\{1, v\}$, and this is all, since $v \wedge v = 0$.
Now let $\dim V = 2$, with basis $\{v, w\}$. The $\bigwedge V$ has basis $\{1, v, w, v \wedge w\}$ since $w \wedge v = -v \wedge w$, and any wedge of three factors is zero as in the previous exercise.
Now let $\dim V = 4$, with basis $\{v_\mu\}$. $\bigwedge V$ has basis

$$\{1, v_\mu, v_\mu \wedge v_\nu, v_\mu \wedge v_\nu \wedge v_\lambda, v_1 \wedge v_2 \wedge v_3 \wedge v_4 \mid \mu \neq \nu \neq \lambda\}$$

Exercise 44. Let $\dim V = n$, and $\{v_\mu\}$ be a basis. An element of $\bigwedge^p V$ is a product $\alpha v_{\mu_1} \wedge \dots \wedge v_{\mu_p}$, where $\alpha \in \mathcal{F}$, the relevant field. If $p > n$, this is clearly zero, as in Exercise 42. Otherwise, if $p \leq n$, all the μ_i must be distinct, so we have

$$\dim \bigwedge^p V = \binom{n}{p} = \frac{n!}{p!(n-p)!}$$

Exercise 45. It is clear from the last few exercises that

$$\bigwedge V = \bigoplus_{p=1}^{n} \bigwedge^p V$$

Then

$$\begin{aligned} \dim \bigwedge V &= \sum_{p=1}^{n} \dim \bigwedge^p V \\ &= \sum_{p=1}^{n} \binom{n}{p} \\ &= 2^n \end{aligned}$$

Exercise 46. Let $\omega \in \bigwedge^p V$, $\mu \in \bigwedge^q V$. Write

$$\begin{aligned} \omega &= \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} v^{\mu_1} \wedge \dots \wedge v^{\mu_p} \\ \mu &= \frac{1}{q!} \mu_{\nu_1 \dots \nu_q} v^{\nu_1} \wedge \dots \wedge v^{\nu_q} \end{aligned}$$

Then

$$\begin{aligned} \omega \wedge \mu &= \frac{1}{p!q!} \omega_{\mu_1 \dots \mu_p} \nu_{\nu_1 \dots \nu_q} v^{\mu_1} \wedge \dots \wedge v^{\mu_p} \wedge v^{\nu_1} \wedge \dots \wedge v^{\nu_q} \\ &= \frac{1}{p!q!} \omega_{\mu_1 \dots \mu_p} \nu_{\nu_1 \dots \nu_q} (-1)^{pq} v^{\nu_1} \wedge \dots \wedge v^{\nu_q} \wedge v^{\mu_1} \wedge \dots \wedge v^{\mu_p} \\ &= (-1)^{pq} \mu \wedge \omega \end{aligned}$$

So $\bigwedge V$ is a graded commutative, or supercommutative, algebra.
Then if $\omega, \mu \in \Omega(M)$, ω_x, μ_x are elements of some $\bigwedge^p T_x^* M$ and $\bigwedge^q T_x^* M$, respectively. Then, working in $\bigwedge T_x^* M$, we have

$$\omega_p \wedge \mu_p = (-1)^{pq} \mu_p \wedge \omega_p$$

This is true for all $x \in M$, and we know that a form is fully defined by its value at each point of the manifold, so in fact

$$\omega \wedge \mu = (-1)^{pq} \mu \wedge \omega$$

Therefore $\Omega(M)$ is graded commutative.

Exercise 47. Given $\phi : M \to N$, we want to show the existence and uniqueness of a pullback

$$\phi^* : \Omega(N) \to \Omega(M)$$

agreeing with the pullbacks on 0- and 1-forms already established, and satisfying

$$\begin{aligned} \phi^*(\alpha\omega) &= \alpha\phi^*\omega \\ \phi^*(\omega + \mu) &= \phi^*\omega + \phi^*\mu \\ \phi^*(\omega \wedge \mu) &= \phi^*\omega \wedge \phi^*\mu \end{aligned}$$

for all $\omega, \mu \in \Omega(M)$ and $\alpha \in \mathbb{R}$.
Let $\omega \in \Omega^p(N)$ have local coordinate expansion

$$\omega_\mu = \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$$

Then define $\phi^*\omega$ by

$$\begin{aligned} \phi^*\omega &= (\phi^*\omega_{\mu_1 \ldots \mu_p})(\phi^* dx^{\mu_1}) \wedge \ldots \wedge (\phi^d x^{\mu_p}) \\ &= (\omega_{\mu_1 \ldots \mu_p} \circ \phi) d\phi^* x^{\mu_1} \wedge \ldots \wedge d\phi^* x^{\mu_p} \end{aligned}$$

Then certainly $\phi^*\omega \in \Omega(M)$. If ω is a 0-form, this is just

$$\phi^*\omega = \omega \circ \phi$$

as desired, and if ω is a 1-form, we have

$$\phi^*\omega = (\phi^*\omega_\mu) d\phi^* x^\mu$$

also as we would like. Then, we have

$$\begin{aligned} \phi^*(\alpha\omega) &= (\phi^*(\alpha\omega_{\mu_1 \ldots \mu_p}))(\phi^* dx^{\mu_1}) \wedge \ldots \wedge (\phi^d x^{\mu_p}) \\ &= (\alpha\omega_{\mu_1 \ldots \mu_p} \circ \phi)(\phi^* dx^{\mu_1}) \wedge \ldots \wedge (\phi^d x^{\mu_p}) \\ &= \alpha\phi^*\omega \end{aligned}$$

Clearly $\phi^*(\omega + \mu) = \phi^*\omega + \phi^*\mu$. Lastly,

$$\begin{aligned} \phi^*(\omega \wedge \mu) &= \phi^* \left(\frac{1}{p!q!} \omega_{\mu_1 \ldots \mu_p} \mu_{\nu_1 \ldots \nu_q} \right) (\phi^* dx^{\mu_1}) \wedge \ldots \wedge (\phi^* dx^{\mu_p}) \wedge (\phi^* dx^{\nu_1}) \wedge \ldots \wedge (\phi^* dx^{\nu_q}) \\ &= \phi^*\omega \wedge \phi^*\mu \end{aligned}$$

So certainly the pullback on $\Omega(N)$ exists, and its uniqueness is clear from that of its constituent parts.

Exercise 48. Let $P : \mathbb{R}^3 \to \mathbb{R}^3; \boldsymbol{x} \mapsto -\boldsymbol{x}$ be the parity transformation on $\mathbb{R}^3$. Then

$$\begin{aligned} P^*(dx^\mu) &= d(P^* x^\mu) \\ &= d(-x^\mu) \\ &= -dx^\mu \end{aligned}$$

So

$$P^*(\omega_\mu dx^\mu = -\omega_\mu dx^\mu$$

but

$$P^* \left(\frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu \right) = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu$$

Exercise 49. Let $\omega = \omega_\mu dx^\mu$ be a 1-form. Then

$$\begin{aligned} d\omega &= d\omega_\mu \wedge dx^\mu + \omega_\mu d^2 x^\mu \\ &= d\omega_\mu \wedge dx^\mu \\ &= \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu \end{aligned}$$