Candelas - Lectures on Complex Manifolds

Exercises - Sections I to VII

Exercise 1. The Hodge star $*$ is defined by

$$*dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} = \frac{1}{(n-p)!}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p}\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n}dx^{\nu_{p+1}} \wedge ... \wedge dx^{\nu_n}$$

in n dimensions. We will find $*^2$ on a basis element for simplicity. Rewrite the definition as

$$*dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} = \frac{1}{(n-p)!}\sqrt{g}\varepsilon^{\mu_1...\mu_p}{}_{\nu_{p+1}...\nu_n}dx^{\nu_{p+1}} \wedge ... \wedge dx^{\nu_n}$$

Then applying $*$ again,

$$\begin{aligned}
*^2dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} &= \frac{1}{(n-p)!}\sqrt{g}\varepsilon^{\mu_1...\mu_p}{}_{\nu_{p+1}...\nu_n} * dx^{\nu_{p+1}} \wedge ... \wedge dx^{\nu_n} \\
&= \frac{1}{(n-p)!p!}\sqrt{g}\varepsilon^{\mu_1...\mu_p}{}_{\nu_{p+1}...\nu_n}\sqrt{g}\varepsilon^{\nu_{p+1}...\nu_n}{}_{\lambda_1...\lambda_p}dx^{\lambda_1} \wedge ... \wedge dx^{\lambda_p} \\
&= \frac{1}{(n-p)!p!}(-1)^{p(n-p)}\sqrt{g}\varepsilon_{\nu_{p+1}...\nu_n}{}^{\mu_1...\mu_p}\sqrt{g}\varepsilon^{\nu_{p+1}...\nu_n}{}_{\lambda_1...\lambda_p}dx^{\lambda_1} \wedge ... \wedge dx^{\lambda_p} \\
&= (-1)^{p(n-p)}\frac{1}{p!}\delta^{\mu_1...\mu_p}_{\lambda_1...\lambda_p}dx^{\lambda_1} \wedge ... \wedge dx^{\lambda_p} \\
&= (-1)^{p(n-p)}dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}
\end{aligned}$$

Thus

$$*^2 = (-1)^{p(n-p)}$$

(Note: on a pseudo-Riemannian manifold with signature $(n-s, s)$, we would also have an additional factor $(-1)^s$, introduced in the second to last line.)

Exercise 2. The adjoint exterior derivative acting on p-forms on an n-manifold is

$$d^\dagger = (-1)^{p(n-p+1)} * d*$$

Consider a p-form ω.

$$\omega = \frac{1}{p!}\omega_{\mu_1...\mu_p}dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}$$

We have

$$
\begin{aligned}
*\omega &= \frac{1}{p!(n-p)!}\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p}\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n}dx^{\nu_{p+1}}\wedge...\wedge dx^{\nu_n} \\
d*\omega &= \frac{1}{p!(n-p)!}\nabla_\lambda(\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p})\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n}dx^{\lambda}\wedge dx^{\nu_{p+1}}\wedge...\wedge dx^{\nu_n} \\
d\omega &= \frac{1}{p!(n-p)!(p-1)!}\nabla_\lambda(\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p})\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n} \\
&\qquad \sqrt{g}g^{\lambda\kappa_p}g^{\nu_{p+1}\kappa_{p+1}}...g^{\nu_n\kappa_n}\varepsilon_{\kappa_p...\kappa_n\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge...\wedge dx^{\kappa_{p-1}} \\
d^\dagger\omega &= (-1)^{p(n-p+1)}\frac{1}{p!(n-p)!(p-1)!}\nabla_\lambda(\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p})\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n} \\
&\qquad \sqrt{g}g^{\lambda\kappa_p}g^{\nu_{p+1}\kappa_{p+1}}...g^{\nu_n\kappa_n}\varepsilon_{\kappa_p...\kappa_n\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge...\wedge dx^{\kappa_{p-1}}
\end{aligned}
$$

Now, we assume our connection is metric-compatible.

$$
\begin{aligned}
d^\dagger\omega &= (-1)^{p(n-p+1)}\frac{1}{p!(n-p)!(p-1)!}\nabla^{\kappa_p}\omega_{\mu_1...\mu_p} \\
&\qquad \sqrt{g}\varepsilon^{\mu_{p_1}...\mu_p\kappa_{p+1}...\kappa_n}\sqrt{g}\varepsilon_{\kappa_p...\kappa_n\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge...\wedge dx^{\kappa_{p-1}} \\
&= (-1)^{p(n-p+1)}\frac{1}{p!(n-p)!(p-1)!}\nabla^{\kappa_p}\omega_{\mu_1...\mu_p} \\
&\qquad \sqrt{g}(-1)^{p(n-p)}\varepsilon^{\kappa_{p+1}...\kappa_n\mu_1...\mu_p}\sqrt{g}(-1)^{n-p-1}\varepsilon_{\kappa_{p+1}...\kappa_n\kappa_p\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge...\wedge dx^{\kappa_{p-1}} \\
&= (-1)^{p(n-p+1)+p(n-p)+n-p+1}\frac{1}{p!(p-1)!}\nabla^{\kappa_p}\omega_{\mu_1...\mu_p}\delta^{\mu_1\mu_2...\mu_p}_{\kappa_p\kappa_1...\kappa_{p-1}}dx^{\kappa}_1\wedge...\wedge dx^{\kappa_{p-1}} \\
&= (-1)^{2pn-2p^2+n+1}\frac{1}{(p-1)!}\nabla^{\mu_1}\omega_{\mu_1...\mu_p}dx^{\mu_2}\wedge...\wedge dx^{\mu_p} \\
&= -(-1)^n\frac{1}{(p-1)!}\nabla^{\mu_1}\omega_{\mu_1...\mu_p}dx^{\mu_2}\wedge...\wedge dx^{\mu_p}
\end{aligned}
$$

This is correct up to a factor $(-1)^n$.

Exercise 3. Let

$$\omega = \frac{1}{p!}\omega_{\mu_1...\mu_p}dx^{\mu_1}\wedge...\wedge dx^{\mu_p}$$

We want to find the components of $\Delta\omega$, where $\Delta = d^\dagger d + dd^\dagger$. Firstly, we have

$$
\begin{aligned}
d\omega &= \frac{1}{p!}\nabla_\nu\omega_{\mu_1...\mu_p}dx^\nu\wedge dx^{\mu_1}\wedge...\wedge dx^{\mu_p} \\
d^\dagger d\omega &= -\frac{1}{p!}\nabla^\lambda\nabla_\lambda\omega_{\mu_1...\mu_p}dx^{\mu_1}\wedge...\wedge dx^{\mu_p}
\end{aligned}
$$

(We have assumed that ∇ is torsion-free in the first line.) So

$$(d^\dagger d\omega)_{\mu_1\ldots\mu_p} = -\nabla^\lambda \nabla_\lambda \omega_{\mu_1\ldots\mu_p}$$

Now turn to the other half of Δ. We have

$$\begin{aligned}
d^\dagger\omega &= -\frac{1}{(p-1)!}\nabla^\nu \omega_{\nu\mu_1\ldots\mu_p} dx^{\mu_2}\wedge \ldots \wedge dx^{\mu_p} \\
dd^\dagger\omega &= -\frac{1}{(p-1)!}\nabla_{\mu_1}\nabla^\nu \omega_{\nu\mu_2\ldots\mu_p} dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_p} \\
&= -\frac{1}{(p-1)!}\nabla_{\mu_1}\nabla_\nu \omega^\nu{}_{\mu_2\ldots\mu_p} dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_p}
\end{aligned}$$

(We have assumed that ∇ is metric-compatible in the third line.) Rewriting $\nabla_{\mu_1}\nabla_\nu = [\nabla_{\mu_1},\nabla_\nu] + \nabla_\nu\nabla_{\mu_1}$, we have

$$dd^\dagger\omega = -\frac{1}{(p-1)!}\left([\nabla_{\mu_1},\nabla_\nu]\omega^\nu{}_{\mu_2\ldots\mu_p} + \nabla_\nu\nabla_{\mu_1}\omega^\nu{}_{\mu_2\ldots\mu_p}\right) dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_p}$$

Now, we have

$$[\nabla_{\mu_1},\nabla_\nu]\omega^\nu{}_{\mu_2\ldots\mu_p} = R^\nu{}_{\lambda\mu_1\nu}\omega^\lambda{}_{\mu_2\ldots\mu_p} - R^\lambda{}_{\mu_2\mu_1\nu}\omega^\nu{}_{\lambda\mu_3\ldots\mu_p} - R^\lambda{}_{\mu_3\mu_1\nu}\omega^\nu{}_{\lambda\mu_2\mu_4\ldots\mu_p} - \ldots$$

The negative terms are symmetric in the μ_i for $i \geq 2$, so upon contracting with the forms all vanish. The first term simplifies to

$$-R_{\lambda\mu_1}\omega^\lambda{}_{\mu_2\ldots\mu_p}$$

Thus we have

$$dd^\dagger\omega = -\frac{1}{(p-1)!}\left(-R_{\lambda\mu_1}\omega^\lambda{}_{\mu_2\ldots\mu_p} + \nabla_\nu\nabla_{\mu_1}\omega^\nu{}_{\mu_2\ldots\mu_p}\right) dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_p}$$

The first term leads to a contribution to $(\Delta\omega)_{\mu_1\ldots\mu_p}$ of

$$pR_{\lambda[\mu_1}\omega^\lambda{}_{\mu_2\ldots\mu_p]}$$

This is off by a sign. The remaining part should be

$$-\frac{1}{2}\frac{1}{(p-2)!}R_{\lambda\kappa\mu_1\mu_2}\omega^{\lambda\kappa}{}_{\mu_3\ldots\mu_p} dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_p}$$

leading to a contribution to $(\Delta\omega)_{\mu_1\ldots\mu_p}$ of

$$-\frac{1}{2}p(p-1)R_{\lambda\kappa[\mu_1\mu_2}\omega^{\lambda\kappa}{}_{\mu_3\ldots\mu_p]}$$

but I can't see how to get this. The final result should then be

$$(\Delta\omega)_{\mu_1\ldots\mu_p} = -\nabla^\lambda\nabla_\lambda\omega_{\mu_1\ldots\mu_p} - pR_{\lambda[\mu_1}\omega^\lambda_{\mu_2\ldots\mu_p]} - \frac{1}{2}p(p-1)R_{\lambda\kappa[\mu_1\mu_2}\omega^{\lambda\kappa}{}_{\mu_3\ldots\mu_p]}$$

Exercise 4. Let ω be a 1-form on T^2.

$$\omega = u(\theta, \phi)d\theta + v(\theta, \phi)d\phi$$

The functions u and v must be periodic in θ and ϕ. Therefore we can Fourier decompose and write

$$\omega = \sum_{m,n=0}^{\infty} (u_{mn}d\theta + v_{mn}d\phi)e^{i(m\theta+n\phi)}$$

Now we want to confirm that the Hodge decomposition of this looks like

$$\begin{aligned}\omega &= u_{00}d\theta + v_{00}d\phi \\ &\quad + d\left[-i{\sum}' \frac{mu_{mn} + nv_{mn}}{m^2+n^2}e^{i(m\theta+n\phi)}\right] \\ &\quad + d^\dagger\left[-i{\sum}' \frac{nu_{mn} - mv_{mn}}{m^2+n^2}e^{i(m\theta+n\phi)}d\theta \wedge d\phi\right]\end{aligned}$$

Clearly this is in the right form - we just have to check that this is indeed true. First we have

$$\begin{aligned}&d\left[-i{\sum}' \frac{mu_{mn} + nv_{mn}}{m^2+n^2}e^{i(m\theta+n\phi)}\right] \\ &\quad = \frac{m(mu_{mn} + nv_{mn})}{m^2+n^2}e^{i(m\theta+n\phi)}d\theta + \frac{n(mu_{mn} + nv_{mn})}{m^2+n^2}e^{i(m\theta+n\phi)}d\phi\end{aligned}$$

where $\sum'$ means excluding the $(0,0)$ term. Next,

$$\begin{aligned}&d^\dagger\left[-i{\sum}' \frac{nu_{mn} - mv_{mn}}{m^2+n^2}e^{i(m\theta+n\phi)}d\theta \wedge d\phi\right] \\ &\quad = -*d\left[-i{\sum}' \frac{nu_{mn} - mv_{mn}}{m^2+n^2}e^{i(m\theta+n\phi)}\right] \\ &\quad = -*\left[\frac{m(nu_{mn} - mv_{mn})}{m^2+n^2}e^{i(m\theta+n\phi)}d\theta + \frac{n(nu_{mn} - mv_{mn})}{m^2+n^2}e^{i(m\theta+n\phi)}d\phi\right] \\ &\quad = -\frac{m(nu_{mn} - mv_{mn})}{m^2+n^2}e^{i(m\theta+n\phi)}d\phi + \frac{n(nu_{mn} - mv_{mn})}{m^2+n^2}e^{i(m\theta+n\phi)}d\theta\end{aligned}$$

We can then confirm that everything does indeed add to ω.

Exercise 5. Don't understand the question.

Exercise 6. The connection 1-forms are

$$\omega^a{}_c = \Gamma^a{}_{bc} e^b$$

Now, we have

$$\nabla_a e^b = -\Gamma^b{}_{ac} e^c$$

So

$$\begin{aligned}
e_a{}^m \nabla_m e^b{}_n dx^n &= -\Gamma^b{}_{ac} e^c{}_n dx^n \\
e_a{}^m \nabla_m e^b{}_n &= -\Gamma^b{}_{ac} e^c{}_n \\
e_a{}^m e^n{}_d \nabla_m e^b{}_n &= -\Gamma^b{}_{ad} \\
e_a{}^m e^n{}_d (\nabla_m e^b{}_n) e^a{}_q dx^q &= -\Gamma^b{}_{ad} e^a \\
e^n{}_d (\nabla_m e^b{}_n) dx^m &= -\Gamma^b{}_{ad} e^a
\end{aligned}$$

That is,

$$\begin{aligned}
\Gamma^a{}_{bc} e^b &= -(e^n{}_c \nabla_m e^a{}_n) dx^m \\
&= (e^a{}_n \nabla_m e^n{}_c) dx^m \\
\omega^a{}_c &= (e^a{}_n \nabla_m e^n{}_c) dx^m
\end{aligned}$$

where in the second line we have used $e^a{}_n e^n{}_c = \delta^c_a$ to integrate by parts.

Exercise 7. We have

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$$

These terms can be rewritten.

$$\begin{aligned}
R^a{}_b &= \frac{1}{2} R_{mn}{}^k{}_l e^a{}_k e_b{}^l dx^m \wedge dx^n \\
&= \frac{1}{2} g^{kp} R_{mnpl} e^a{}_k e_b{}^l dx^m \wedge dx^n \\
dR^a{}_b &= \frac{1}{2} g^{kp} \nabla_q (R_{mnpl} e^a{}_k e_b{}^l) dx^q \wedge dx^m \wedge dx^n \\
\omega^a{}_c \wedge R^c{}_b &= \frac{1}{2} \Gamma^a{}_{dc} e^d{}_q g^{kp} R_{mnpl} e^c{}_k e_b{}^l dx^q \wedge dx^m \wedge dx^n \\
R^a{}_c \wedge \omega^c{}_b &= \frac{1}{2} g^{kp} R_{mnpl} e^a{}_k e_c{}^l \Gamma^c{}_{db} e^d{}_q dx^q \wedge dx^m \wedge dx^n
\end{aligned}$$

Our identity now reads

$$\left(\nabla_q (R_{mnpl} e^a{}_k e_b{}^l) + R_{mnpl} (\Gamma^a{}_{dc} e^d{}_q e^c{}_k e_b{}^l - \Gamma^c{}_{db} e^a{}_l e_c{}^l e^d{}_q) \right) dx^q \wedge dx^m \wedge dx^n$$

The first term is

$$\nabla_q(R_{mnpl}e^a{}_k e_b{}^l) = (\nabla_q R_{mnpl})e^a{}_k e_b{}^l + R_{mnpl}\left((\nabla_q e^a{}_k)e_b{}^l + e^a{}_k \nabla_q e_b{}^l\right)$$

Now, from the definition, we have

$$\begin{aligned}
\nabla_b e_c &= \Gamma^a{}_{bc} e_a \\
e_b{}^m \nabla_m (e_c{}^n \partial_n) &= \Gamma^a{}_{bc} e_a{}^n e^b{}_m \\
\nabla_m e_c{}^n &= \Gamma^a{}_{bc} e_a{}^n e^b{}_m
\end{aligned}$$

and similarly

$$\begin{aligned}
\nabla_b e^a &= -\Gamma^a{}_{bc} e^c \\
e_b{}^m \nabla_m e^a{}_n dx^n &= -\Gamma^a{}_{bc} e^c{}_n dx^n \\
\nabla_m e^a{}_n &= -\Gamma^a{}_{bc} e^c{}_n e^b{}_m
\end{aligned}$$

Thus this first term becomes

$$(\nabla_q R_{mnpl})e^a{}_k e_b{}^l + R_{mnpl}\left(-\Gamma^a{}_{cd} e^d{}_k e^c{}_q e_b{}^l + e^a{}_k \Gamma^d{}_{cb} e_d{}^l e^c{}_q\right)$$

The second two terms cancel with the second two terms of our identity. Thus we just have

$$(\nabla_q R_{mnpl})e^a{}_k e_b{}^l dx^q \wedge dx^m \wedge dx^n = 0$$

The vielbeins are invertible, so we can drop them. Furthermore, we have the symmetry $R_{mnpl} = R_{plmn}$ of the Riemann tensor, so

$$\nabla_q R_{plmn} dx^q \wedge dx^m \wedge dx^n = 0$$

In other words,

$$R_{pl[mn;q]} = 0$$

where ; denotes covariant differentiation. This is just the (differential) Bianchi identity.

Exercise 8. Under a change of frame,

$$e^a \to \Phi^a{}_b e^b$$

i.e.

$$e^a{}_m \to \Phi^a{}_b e^b{}_m$$

so also

$$e_a{}^m \to (\Phi^{-1})_a{}^b e_b{}^m$$

Now, we have

$$\omega^a{}_b = (e^a{}_k \nabla_m e_b{}^k) dx^m$$

so

$$\begin{aligned}
\omega^a{}_b &\to (\Phi^a{}_c e^c{}_k \nabla_m ((\Phi^{-1})_b{}^d e_d{}^k)) dx^m \\
&= (\Phi^a{}_c e^c{}_k (\nabla_m e_d{}^k)(\Phi^{-1})_b{}^d + \Phi^a{}_c e^c{}_k e_d{}^k \nabla_m (\Phi^{-1})_b{}^d) dx^m \\
&= \Phi^a{}_c \omega^c{}_d (\Phi^{-1})_b{}^d + \Phi^a{}_c (d\Phi^{-1})_b{}^c
\end{aligned}$$

Also, since

$$R^a{}_b = \frac{1}{2} R_{mn}{}^k{}_l e^a{}_k e_b{}^l dx^m \wedge dx^n$$

we have

$$\begin{aligned}
R^a{}_b &\to \frac{1}{2} R_{mn}{}^k{}_l \Phi^a{}_c e^c{}_k (\Phi^{-1})^b{}_d e^d{}_d dx^m \wedge dx^n \\
&= \Phi^a{}_c (\Phi^{-1})_b{}^d R^c{}_d
\end{aligned}$$

Lastly, we have

$$D\nu^a{}_b = d\nu^a{}_b + \omega^a{}_c \wedge \nu^c{}_b - \omega^c{}_b \wedge \nu^a{}_c$$

Now,

$$\nu^a{}_b \to \Phi^a{}_c (\Phi^{-1})_b{}^d \nu^c{}_d$$

so

$$d\nu^a{}_b \to d(\Phi^a{}_c (\Phi^{-1})_b{}^d) \nu^c{}_d + \Phi^a{}_c (\Phi^{-1})_b{}^d d\nu^c{}_d$$

Then,

$$\begin{aligned}
\omega^a{}_c \wedge \nu^c{}_b &\to (\Phi^a{}_d \omega^d{}_e (\Phi^{-1})_c{}^e + \Phi^a{}_d (d\Phi^{-1})_c{}^d) \wedge \Phi^c{}_f (\Phi^{-1})_b{}^g d\nu^f{}_g \\
&= \Phi^a{}_d (\Phi^{-1})_b{}^g \omega^d{}_e \wedge d\nu^e{}_g + \Phi^a{}_d \Phi^c{}_f (\Phi^{-1})_b{}^g (d\Phi^{-1})_c{}^d \wedge d\nu^f{}_g \\
&= \Phi^a{}_d (\Phi^{-1})_b{}^g \omega^d{}_e \wedge d\nu^e{}_g - \Phi^a{}_d (d\Phi^c{}_f)(\Phi^{-1})_b{}^g (\Phi^{-1})_c{}^d \wedge d\nu^f{}_g \\
&= \Phi^a{}_c (\Phi^{-1})_b{}^d \omega^c{}_e \wedge d\nu^e{}_d - d\Phi^a{}_f (\Phi^{-1})_b{}^d \wedge d\nu^f{}_d
\end{aligned}$$

And

$$\begin{aligned}
\omega^c{}_b \wedge \nu^a{}_c &\to (\Phi^c{}_d \omega^d{}_e (\Phi^{-1})_b{}^e + \Phi^c{}_d (d\Phi^{-1})_b{}^d) \wedge \Phi^a{}_f (\Phi^{-1})_c{}^g d\nu^f{}_g \\
&= \Phi^a{}_c (\Phi^{-1})_b{}^d \omega^e{}_d \wedge d\nu^c{}_e + \Phi^a{}_f (d\Phi^{-1})_b{}^d \wedge d\nu^f{}_d
\end{aligned}$$

Thus

$$\begin{aligned}
D\nu^a{}_b &\to \Phi^a{}_c (\Phi^{-1})_b{}^d (d\nu^c{}_d + \omega^c{}_e \wedge d\nu^e{}_d - \omega^e{}_d \wedge d\nu^c{}_e) + d(\Phi^a{}_c (\Phi^{-1})_b{}^d) \nu^c{}_d \\
&\qquad - (d\Phi^a{}_f (\Phi^{-1})_b{}^d + \Phi^a{}_f (d\Phi^{-1})_b{}^d) \wedge d\nu^f{}_d \\
&= \Phi^a{}_c (\Phi^{-1})_b{}^d D\nu^c{}_d + d(\Phi^a{}_c (\Phi^{-1})_b{}^d) \nu^c{}_d - d(\Phi^a{}_f (\Phi^{-1})_b{}^d) \wedge d\nu^f{}_d \\
&= \Phi^a{}_c (\Phi^{-1})_b{}^d D\nu^c{}_d
\end{aligned}$$

Exercise 9. Regard the exterior derivative d as the operator

$$d = dx^i \wedge \frac{\partial}{\partial x^i}$$

in real coordinates x^i. We can insert $P + Q = 1$ at will:

$$d = (P_j{}^i + Q_j{}^i) dx^j \wedge \frac{\partial}{\partial x^i}$$

Now, precisely if the Nijenhuis tensor vanishes, we can write this as

$$d = (\theta_\mu{}^i dz^\mu + \phi_{\bar{\nu}}{}^i d\bar{z}^\nu) \wedge \frac{\partial}{\partial x^i}$$

Further, we can rewrite this

$$d = (\theta_\mu{}^i dz^\mu + \phi_{\bar{\nu}}{}^i d\bar{z}^\nu) \wedge \left(\frac{\partial z^\lambda}{\partial x^i} \frac{\partial}{\partial z^\lambda} + \frac{\partial z^{\bar{\lambda}}}{\partial x^i} \frac{\partial}{\partial z^{\bar{\lambda}}} \right)$$

Then clearly the action of d on a (p,q)-form ω gives us a sum of a $(p+1,q)$-form and a $(p,q+1)$-form. That is, we can write

$$d = \partial + \bar{\partial}$$

and there are no further contributions to d. Then in the usual way $\partial^2 = 0$ follows from $d^2 = 0$. That is, $\partial^2 = 0$ iff $N = 0$.

Exercise 10. Let z^m be complex coordinates, in which an arbitrary connection has components $\Gamma^m{}_{nk}$. Then perform a holomorphic change of coordinates to w^i. Then we have

$$\nabla_i \frac{\partial}{\partial w^k} = \Gamma^k{}_{ij} \frac{\partial}{\partial w^k}$$

The LHS is

$$\begin{aligned}
\nabla_i \frac{\partial}{\partial w^j} &= \nabla_i \left(\frac{\partial z^m}{\partial w^j} \frac{\partial}{\partial z^m} \right) \\
&= \frac{\partial^2 z^m}{\partial w^i \partial w^j} \frac{\partial}{\partial z^m} + \frac{\partial z^m}{\partial w^j} \frac{\partial z^n}{\partial w^i} \nabla_n \frac{\partial}{\partial z^m} \\
&= \frac{\partial^2 z^m}{\partial w^i \partial w^j} \frac{\partial}{\partial z^m} + \frac{\partial z^m}{\partial w^j} \frac{\partial z^n}{\partial w^i} \Gamma^p{}_{nm} \frac{\partial}{\partial z^p} \\
&= \left(\frac{\partial^2 z^m}{\partial w^i \partial w^j} \frac{\partial}{\partial z^m} + \frac{\partial z^p}{\partial w^j} \frac{\partial z^n}{\partial w^i} \Gamma^m{}_{np} \right) \frac{\partial}{\partial z^m}
\end{aligned}$$

The RHS is

$$\Gamma^k{}_{ij}\frac{\partial}{\partial w^k} = \Gamma^k{}_{ij}\frac{\partial z^m}{\partial w^k}\frac{\partial}{\partial z^m}$$

Thus we have

$$\Gamma^k{}_{ij} = \left(\frac{\partial^2 z^m}{\partial w^i \partial w^j}\frac{\partial}{\partial z^m} + \frac{\partial z^p}{\partial w^j}\frac{\partial z^n}{\partial w^i}\Gamma^m{}_{np}\right)\frac{\partial w^k}{\partial z^m}$$

If the first term vanishes, this is just standard tensor transformation. Now, if $k = \bar{\kappa}$, then since $w(z)$ is holomorphic, we must have $m = \bar{\mu}$. But then the first term vanishes unless both i and j are complex components. Therefore we have that $\Gamma^{\bar{\kappa}}{}_{\mu\nu}, \Gamma^{\bar{\kappa}}{}_{\bar{\mu}\nu}, \Gamma^{\bar{\kappa}}{}_{\mu\bar{\nu}}$ transform as tensors. Now suppose that $i = \bar{\nu}$. Then we must have $m = \bar{\mu}$ and $j = \bar{\lambda}$ for the first term not to vanish. But then we must have $k = \bar{\kappa}$, in which case we are no longer looking at a mixed component. So we also have that $\Gamma^{\kappa}{}_{\bar{\nu}\lambda}, \Gamma^{\bar{\kappa}}{}_{\bar{\nu}\lambda}, \Gamma^{\kappa}{}_{\bar{\nu}\bar{\lambda}}$ transform as tensors. Similarly for the mixed components with $j = \bar{\lambda}$, and we are done. All mixed components transform as tensors.

Exercise 11. The Ricci form on a Hermitean manifold is

$$\mathcal{R} = i\partial\bar{\partial}\ln\sqrt{g}$$

If we are in the intersection of two coordinate patches and we change coordinates, g picks up Jacobian factors:

$$g \to \left|\frac{\partial z^\mu}{\partial w^\alpha}\right|\left|\frac{\partial \bar{z}^\nu}{\partial \bar{w}^\beta}\right| g$$

Thus

$$\ln\sqrt{g} \to \ln\sqrt{g} + \frac{1}{2}\left(\ln\left|\frac{\partial z^\mu}{\partial w^\alpha}\right| + \ln\left|\frac{\partial \bar{z}^\nu}{\partial \bar{w}^\beta}\right|\right)$$

The new terms are destroyed by $\bar{\partial}$ and ∂ respectively, so $\mathcal{R}$ is invariant, i.e. globally defined.

Exercise 12. Consider $\mathbb{C}P^n$. Let U_j be the patch on with $z^j \neq 0$, and use coordinates

$$\zeta_j^m = \frac{z^m}{z^j}$$

We then introduce the local Kähler potential

$$\phi_j = \ln\left(\sum_{m=1}^{n+1}|\zeta_j^m|^2\right)$$

The Fubini-Study metric is then given by

$$g_{\mu\bar{\nu}} = \partial_\mu\partial_{\bar{\nu}}\phi_j$$

Let

$$\sigma = \sum_{m=1}^{n+1} |\zeta_j^m|^2$$

Now, introduce some slightly different notation and write this as

$$\sigma = \zeta^\mu \zeta_\mu + 1$$

Here, lower μ means upper $\bar{\mu}$, and the 1 has come from the fact that μ runs over the n non-j indices only. Now, we have

$$\begin{aligned}
g_{\mu\bar{\nu}} &= \frac{\partial^2 \ln \sigma}{\partial \zeta^\mu \partial \zeta_\mu} \\
&= \frac{\partial}{\partial \zeta^\mu} \left(\frac{1}{\sigma} \frac{\partial \sigma}{\partial \zeta_\nu} \right) \\
&= -\frac{1}{\sigma^2} \frac{\partial \sigma}{\partial \zeta^\mu} \frac{\partial \sigma}{\partial \zeta_\nu} + \frac{1}{\sigma} \frac{\partial^2 \sigma}{\partial \zeta^\mu \partial \zeta_\nu} \\
&= \frac{1}{\sigma} \left(\frac{\partial^2 \sigma}{\partial \zeta^\mu \partial \zeta_\nu} - \frac{1}{\sigma} \frac{\partial \sigma}{\partial \zeta^\mu} \frac{\partial \sigma}{\partial \zeta_\nu} \right)
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{\partial \sigma}{\partial \zeta^\mu} &= \zeta_\mu \\
\frac{\partial^2 \sigma}{\partial \zeta^\mu \partial \zeta_\nu} &= \delta_{\mu\bar{\nu}}
\end{aligned}$$

Thus

$$g_{\mu\bar{\nu}} = \frac{1}{\sigma} \left(\delta_{\mu\bar{\nu}} - \frac{\zeta_\mu \zeta_{\bar{\nu}}}{\sigma} \right)$$

We want to make sure this is indeed a metric - i.e. positive-definite. Let X be some vector

$$X = X^\mu \frac{\partial}{\partial \zeta^\mu} + X^{\bar{\mu}} \frac{\partial}{\partial \zeta^{\bar{\mu}}}$$

Then

$$\begin{aligned}
g(X, X) &= \frac{2}{\sigma} \left(\delta_{\mu\bar{\nu}} - \frac{\zeta_\mu \zeta_{\bar{\nu}}}{\sigma} \right) X^\mu X^{\bar{\nu}} \\
&= \frac{2}{\sigma^2} \left(X^\mu X_\mu \sigma - X^\mu \zeta_\mu X^{\bar{\nu}} \zeta_{\bar{\nu}} \right) \\
&= \frac{2}{\sigma^2} \left(X^\mu X_\mu (\zeta^\nu \zeta_\nu + 1) - X^\mu \zeta_\mu X^{\bar{\nu}} \zeta_{\bar{\nu}} \right)
\end{aligned}$$

But by the Schwarz inequality,

$$(X^\mu X_\mu)(\zeta^\nu \zeta_\nu) \geq (X^\mu \zeta_\mu)(X^{\bar\nu} \zeta_{\bar\nu})$$

Thus $g(X,X) \geq 0$, and zero only if $X = 0$.

Exercise 13. The Fubini-Study metric is

$$g_{\mu\bar\nu} = \frac{1}{\sigma^2}(\sigma\delta_{\mu\bar\nu} - \zeta_\mu \zeta_{\bar\nu})$$

We want to find

$$\mathcal{R} = i\partial\bar\partial \ln \sqrt{g}$$

If we write

$$M_{ij} = \sigma\delta_{ij} - \zeta_i\zeta_j$$

we have

$$|g| = \frac{1}{\sigma^{2n}}|M|$$

To calculate $|M|$, note that it is the sum of a scalar multiple of the identity and a symmetric matrix of rank 1. Therefore it is possible to diagonalise M as

$$\begin{aligned}
SMS^{-1} &= \begin{pmatrix} \sigma & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma \end{pmatrix} - \begin{pmatrix} \zeta^i\zeta_i & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma - \zeta^i\zeta_i & & & & & \\ & \sigma & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \sigma \end{pmatrix} \\
&= \begin{pmatrix} 1 & & & & & \\ & \sigma & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \sigma \end{pmatrix}
\end{aligned}$$

Thus

$$|M| = \sigma^{n-1}$$

so

$$|g| = \frac{1}{\sigma^{n+1}}$$

Thus

$$\begin{aligned}\mathcal{R} &= i\partial\bar{\partial}\ln\sqrt{\frac{1}{\sigma^{n+1}}}\\ &= -\frac{i}{2}(n+1)\partial\bar{\partial}\ln\sigma\end{aligned}$$

But

$$g_{\mu\bar{\nu}} = \frac{\partial^2\ln\sigma}{\partial\zeta^\mu\partial\zeta_\nu}$$

So

$$\begin{aligned}\mathcal{R} &= -\frac{i}{2}(n+1)g_{\mu\bar{\nu}}dz^\mu\wedge d\bar{z}^\nu\\ &= -(n+1)J\end{aligned}$$

The Ricci form is therefore harmonic, since J is, and so not exact. Consequently the first Chern class does not vanish.

Exercise 14. We want to show that Dolbeault cohomology is equivalent to de Rham cohomology on a Kähler manifold. In order to do this it is sufficient to show that a form is $\bar{\partial}$-harmonic iff it is d-harmonic, so we want to show that $\Delta \propto \Delta_{\bar{\partial}}$. Let ω be an (r,s)-form, and $p = r+s$.

$$\omega = \frac{1}{r!s!}\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}dz^{\mu_1}\wedge\ldots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\ldots\wedge d\bar{z}^{\nu_s}$$

Recall our earlier result for the usual Hodge-de Rham Laplacian on a real manifold:

$$\Delta\omega = -\frac{1}{p!}\left(\nabla^k\nabla_k\omega_{m_1\ldots m_p} + pR_{\lambda m_1}\omega^k{}_{m_2\ldots m_p} + \frac{1}{2}p(p-1)R_{klm_1m_2}\omega^{kl}{}_{m_3\ldots m_p}\right)dx^{m_1}\wedge\ldots\wedge dx^{m_p}$$

Adapting this for an (r,s) form on a complex manifold, we get

$$\begin{aligned}\Delta\omega = -\frac{1}{r!s!}\Big[&\nabla^k\nabla_k\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s} + rR_{\bar{\kappa}\mu_1}\omega^{\bar{\kappa}}{}_{\mu_2\ldots\mu_p\bar{\nu}_1\ldots\bar{\nu}_s} - sR_{\kappa\bar{\nu}_1}\omega_{\mu_1\ldots\mu_p}{}^{\kappa}{}_{\bar{\nu}_2\ldots\bar{\nu}_s}\\ &+\frac{1}{2}r(r-1)R_{\bar{\kappa}\bar{\lambda}\mu_1\mu_2}\omega^{\bar{\kappa}\bar{\lambda}}{}_{\mu_3\ldots\mu_p\bar{\nu}_1\ldots\bar{\nu}_s} + \frac{1}{2}s(s-1)R_{\kappa\lambda\bar{\nu}_1\bar{\nu}_2}\omega_{\mu_1\ldots\mu_p}{}^{\kappa\lambda}{}_{\bar{\nu}_3\ldots\bar{\nu}_s}\\ &+\frac{1}{2}rs\left(R_{\bar{\kappa}\lambda\mu_1\bar{\nu}_1}\omega^{\bar{\kappa}}{}_{\mu_2\ldots\mu_p}{}^{\lambda}{}_{\bar{\nu}_2\ldots\bar{\nu}_s} + R_{\kappa\bar{\lambda}\bar{\nu}_1\mu_1}\omega^{\bar{\lambda}}{}_{\mu_2\ldots\mu_p}{}^{\kappa}{}_{\bar{\nu}_2\ldots\bar{\nu}_s}\right)\Big]\\ &dz^{\mu_1}\wedge\ldots\wedge dz^{\mu_p}\wedge d\bar{z}^{\nu_1}\wedge\ldots\wedge d\bar{z}^{\nu_s}\end{aligned}$$

where Latin k may be holomorphic or antiholomorphic. All the Riemann tensors vanish, since $R_{klmn} = 0$ unless exactly one of k, l, m, n is holomorphic or antiholomorphic. This just leaves

$$\Delta\omega = -\frac{1}{r!s!}\left[\nabla^k \nabla_k \omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s} + rR_{\bar{\kappa}\mu_1}\omega^{\bar{\kappa}}{}_{\mu_2\ldots\mu_p\bar{\nu}_1\ldots\bar{\nu}_s} - sR_{\kappa\bar{\nu}_1}\omega_{\mu_1\ldots\mu_p}{}^{\kappa}{}_{\bar{\nu}_2\ldots\bar{\nu}_s}\right]$$
$$dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_p}$$

Furthermore, on a Käher manifold the Ricci tensor is precisely the Ricci form, so

$$R_{\bar{\kappa}\mu} = \frac{i}{2}\partial_\mu \partial_{\bar{\kappa}} \ln g$$

Thus we finally have

$$\Delta\omega = -\frac{1}{r!s!}\Big[\nabla^\kappa \nabla_\kappa \omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s} + \nabla^{\bar{\kappa}} \nabla_{\bar{\kappa}} \omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}$$
$$+ \frac{i}{2}(r(\partial_{\mu_1}\partial_{\bar{\kappa}} g)\omega^{\bar{\kappa}}{}_{\mu_2\ldots\mu_p\bar{\nu}_1\ldots\bar{\nu}_s} - s(\partial_\kappa \partial_{\bar{\nu}_1} g)\omega_{\mu_1\ldots\mu_p}{}^{\kappa}{}_{\bar{\nu}_2\ldots\bar{\nu}_s})\Big]$$
$$dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_p}$$

Now we turn to $\Delta_{\bar{\partial}}$. First we need to say what $\partial^\dagger$ and $\bar{\partial}^\dagger$ are. On the one hand, we have

$$d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$$

On the other,

$$d^\dagger = - * (\partial + \bar{\partial}) *$$

(any complex manifold has even real dimension). Let $\omega \in \Omega^{r,s}(M)$. Then

$$\partial^\dagger \omega + \bar{\partial}^\dagger \omega = - * \partial * \omega - * \bar{\partial} * \omega$$

On the LHS, the forms are $(r-1, s)$ and $(r, s-1)$ respectively. On the RHS,

$$*\omega \in \Omega^{m-s,m-r}(M)$$
$$\partial * \omega \in \Omega^{m-s+1,m-r}(M)$$
$$*\partial * \omega \in \Omega^{r,s-1}(M)$$

and similarly

$$*\bar{\partial} * \omega \in \Omega^{r-1,s}(M)$$

Thus we have

$$\partial^\dagger = - * \bar{\partial} *$$
$$\bar{\partial}^\dagger = - * \partial *$$

Firstly, we have

$$\bar{\partial}\omega = (-1)^r \frac{1}{r!s!} \nabla_{\bar{\lambda}} \omega_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_s} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\lambda} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_s}$$

Just as we calculate $d^\dagger$, we can find that

$$\bar{\partial}^\dagger \omega = -(-1)^s \frac{1}{r!(s-1)!} \nabla^{\bar{\nu}_1} \omega_{\mu_1 \ldots \mu_p \bar{\nu}_1 \ldots \bar{\nu}_s} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_2} \wedge \ldots \wedge d\bar{z}^{\nu_s}$$

Thus

$$\bar{\partial}^\dagger \bar{\partial}\omega = (-1)^{r+s+1} \frac{1}{r!s!} \nabla^{\bar{\lambda}} \nabla_{\bar{\lambda}} \omega_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_s} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_s}$$

Similarly,

$$\bar{\partial}\bar{\partial}^\dagger \omega = (-1)^{r+s+1} \frac{1}{r!(s-1)!} \nabla_{\bar{\nu}_1} \nabla^{\bar{\kappa}} \omega_{\mu_1 \ldots \mu_p \bar{\kappa} \bar{\nu}_2 \ldots \bar{\nu}_s} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_s}$$

This term will lead to contributions from the Ricci tensor. Similarly we can calculate Δ_∂. We will see that in fact

$$dd^\dagger + d^\dagger d = \partial\partial^\dagger + \partial^\dagger\partial + \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$$

Furthermore, the Dolbeault Laplacians are real(?) so in fact

$$dd^\dagger + d^\dagger d = 2(\partial\partial^\dagger + \partial^\dagger\partial) = 2(\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial})$$