

Candelas - Lectures on Complex Manifolds

Exercises - Sections I to VII

Exercise 1. The Hodge star $*$ is defined by

$$*dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = \frac{1}{(n-p)!} \sqrt{g} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} \varepsilon_{\nu_1 \dots \nu_p \nu_{p+1} \dots \nu_n} dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_n}$$

in n dimensions. We will find $*^2$ on a basis element for simplicity. Rewrite the definition as

$$*dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = \frac{1}{(n-p)!} \sqrt{g} \varepsilon^{\mu_1 \dots \mu_p}_{\nu_{p+1} \dots \nu_n} dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_n}$$

Then applying $*$ again,

$$\begin{aligned} *^2 dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} &= \frac{1}{(n-p)!} \sqrt{g} \varepsilon^{\mu_1 \dots \mu_p}_{\nu_{p+1} \dots \nu_n} * dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_n} \\ &= \frac{1}{(n-p)! p!} \sqrt{g} \varepsilon^{\mu_1 \dots \mu_p}_{\nu_{p+1} \dots \nu_n} \sqrt{g} \varepsilon^{\nu_{p+1} \dots \nu_n}_{\lambda_1 \dots \lambda_p} dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_p} \\ &= \frac{1}{(n-p)! p!} (-1)^{p(n-p)} \sqrt{g} \varepsilon_{\nu_{p+1} \dots \nu_n}^{\mu_1 \dots \mu_p} \sqrt{g} \varepsilon^{\nu_{p+1} \dots \nu_n}_{\lambda_1 \dots \lambda_p} dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_p} \\ &= (-1)^{p(n-p)} \frac{1}{p!} \delta_{\lambda_1 \dots \lambda_p}^{\mu_1 \dots \mu_p} dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_p} \\ &= (-1)^{p(n-p)} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \end{aligned}$$

Thus

$$*^2 = (-1)^{p(n-p)}$$

(Note: on a pseudo-Riemannian manifold with signature $(n-s, s)$, we would also have an additional factor $(-1)^s$, introduced in the second to last line.)

Exercise 2. The adjoint exterior derivative acting on p -forms on an n -manifold is

$$d^\dagger = (-1)^{p(n-p+1)} * d *$$

Consider a p -form ω .

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

Assuming a torsion-free connection, we have

$$\begin{aligned}
*\omega &= \frac{1}{p!(n-p)!} \omega_{\mu_1 \dots \mu_p} \sqrt{g} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} \varepsilon_{\nu_1 \dots \nu_p \nu_{p+1} \dots \nu_n} dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_n} \\
d * \omega &= \frac{1}{p!(n-p)!} \nabla_\lambda (\omega_{\mu_1 \dots \mu_p} \sqrt{g} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p}) \varepsilon_{\nu_1 \dots \nu_p \nu_{p+1} \dots \nu_n} dx^\lambda \wedge dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_n} \\
*d * \omega &= \frac{1}{p!(n-p)!(p-1)!} \nabla_\lambda (\omega_{\mu_1 \dots \mu_p} \sqrt{g} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p}) \varepsilon_{\nu_1 \dots \nu_p \nu_{p+1} \dots \nu_n} \\
&\quad \sqrt{g} g^{\lambda \kappa_p} g^{\nu_{p+1} \kappa_{p+1}} \dots g^{\nu_n \kappa_n} \varepsilon_{\kappa_p \dots \kappa_n \kappa_1 \dots \kappa_{p-1}} dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_{p-1}} \\
d^\dagger \omega &= (-1)^{n(p-1)-1} \frac{1}{p!(n-p)!(p-1)!} \nabla_\lambda (\omega_{\mu_1 \dots \mu_p} \sqrt{g} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p}) \varepsilon_{\nu_1 \dots \nu_p \nu_{p+1} \dots \nu_n} \\
&\quad \sqrt{g} g^{\lambda \kappa_p} g^{\nu_{p+1} \kappa_{p+1}} \dots g^{\nu_n \kappa_n} \varepsilon_{\kappa_p \dots \kappa_n \kappa_1 \dots \kappa_{p-1}} dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_{p-1}}
\end{aligned}$$

Now, we assume our connection is also metric-compatible, and hence Levi-Civita.

$$\begin{aligned}
d^\dagger \omega &= (-1)^{n(p-1)-1} \frac{1}{p!(n-p)!(p-1)!} \nabla^{\kappa_p} \omega_{\mu_1 \dots \mu_p} \\
&\quad \sqrt{g} \varepsilon^{\mu_{p_1} \dots \mu_p \kappa_{p+1} \dots \kappa_n} \sqrt{g} \varepsilon_{\kappa_p \dots \kappa_n \kappa_1 \dots \kappa_{p-1}} dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_{p-1}} \\
&= (-1)^{n(p-1)-1} \frac{1}{p!(n-p)!(p-1)!} \nabla^{\kappa_p} \omega_{\mu_1 \dots \mu_p} \\
&\quad \sqrt{g} (-1)^{p(n-p)} \varepsilon^{\kappa_{p+1} \dots \kappa_n \mu_1 \dots \mu_p} \sqrt{g} (-1)^{n-p} \varepsilon_{\kappa_{p+1} \dots \kappa_n \kappa_p \kappa_1 \dots \kappa_{p-1}} dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_{p-1}} \\
&= (-1)^{n(p-1)-1+p(n-p)+n-p} \frac{1}{p!(p-1)!} \nabla^{\kappa_p} \omega_{\mu_1 \dots \mu_p} \delta^{\mu_1 \mu_2 \dots \mu_p}_{\kappa_p \kappa_1 \dots \kappa_{p-1}} dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_{p-1}} \\
&= -\frac{1}{(p-1)!} \nabla^{\mu_1} \omega_{\mu_1 \dots \mu_p} dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}
\end{aligned}$$

Exercise 3. Let

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

We want to find the components of $\Delta\omega$, where $\Delta = d^\dagger d + dd^\dagger$. Firstly, we have

$$\begin{aligned}
d\omega &= \frac{1}{p!} \nabla_\nu \omega_{\mu_1 \dots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \\
d^\dagger d\omega &= -\frac{1}{p!} \nabla^\lambda \nabla_\lambda \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}
\end{aligned}$$

(We have assumed that ∇ is torsion-free in the first line.) So

$$(d^\dagger d\omega)_{\mu_1 \dots \mu_p} = -\nabla^\lambda \nabla_\lambda \omega_{\mu_1 \dots \mu_p}$$

Now turn to the other half of Δ . We have

$$\begin{aligned} d^\dagger \omega &= -\frac{1}{(p-1)!} \nabla^\nu \omega_{\nu\mu_1\dots\mu_p} dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p} \\ dd^\dagger \omega &= -\frac{1}{(p-1)!} \nabla_{\mu_1} \nabla^\nu \omega_{\nu\mu_2\dots\mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \\ &= -\frac{1}{(p-1)!} \nabla_{\mu_1} \nabla_\nu \omega_{\mu_2\dots\mu_p}^\nu dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \end{aligned}$$

(We have assumed that ∇ is metric-compatible in the third line.) Rewriting $\nabla_{\mu_1} \nabla_\nu = [\nabla_{\mu_1}, \nabla_\nu] + \nabla_\nu \nabla_{\mu_1}$, we have

$$dd^\dagger \omega = -\frac{1}{(p-1)!} \left([\nabla_{\mu_1}, \nabla_\nu] \omega_{\mu_2\dots\mu_p}^\nu + \nabla_\nu \nabla_{\mu_1} \omega_{\mu_2\dots\mu_p}^\nu \right) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

Now, we have

$$[\nabla_{\mu_1}, \nabla_\nu] \omega_{\mu_2\dots\mu_p}^\nu = R_{\lambda\mu_1\nu}^\nu \omega_{\mu_2\dots\mu_p}^\lambda - R_{\mu_2\mu_1\nu}^\lambda \omega_{\lambda\mu_3\dots\mu_p}^\nu - R_{\mu_3\mu_1\nu}^\lambda \omega_{\lambda\mu_2\mu_4\dots\mu_p}^\nu - \dots$$

The negative terms are symmetric in the μ_i for $i \geq 2$, so upon contracting with the forms all vanish. The first term simplifies to

$$-R_{\lambda\mu_1} \omega_{\mu_2\dots\mu_p}^\lambda$$

Thus we have

$$dd^\dagger \omega = -\frac{1}{(p-1)!} \left(-R_{\lambda\mu_1} \omega_{\mu_2\dots\mu_p}^\lambda + \nabla_\nu \nabla_{\mu_1} \omega_{\mu_2\dots\mu_p}^\nu \right) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

The first term leads to a contribution to $(\Delta\omega)_{\mu_1\dots\mu_p}$ of

$$p R_{\lambda[\mu_1} \omega_{\mu_2\dots\mu_p]}^\lambda$$

This is off by a sign. The remaining part should be

$$-\frac{1}{2} \frac{1}{(p-2)!} R_{\lambda\kappa\mu_1\mu_2} \omega_{\mu_3\dots\mu_p}^{\lambda\kappa} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

leading to a contribution to $(\Delta\omega)_{\mu_1\dots\mu_p}$ of

$$-\frac{1}{2} p(p-1) R_{\lambda\kappa[\mu_1\mu_2} \omega_{\mu_3\dots\mu_p]}^{\lambda\kappa}$$

but I can't see how to get this. The final result should then be

$$(\Delta\omega)_{\mu_1\dots\mu_p} = -\nabla^\lambda \nabla_\lambda \omega_{\mu_1\dots\mu_p} - p R_{\lambda[\mu_1} \omega_{\mu_2\dots\mu_p]}^\lambda - \frac{1}{2} p(p-1) R_{\lambda\kappa[\mu_1\mu_2} \omega_{\mu_3\dots\mu_p]}^{\lambda\kappa}$$

Exercise 4. Let ω be a 1-form on T^2 .

$$\omega = u(\theta, \phi)d\theta + v(\theta, \phi)d\phi$$

The functions u and v must be periodic in θ and ϕ . Therefore we can Fourier decompose and write

$$\omega = \sum_{m,n=0}^{\infty} (u_{mn}d\theta + v_{mn}d\phi)e^{i(m\theta+n\phi)}$$

Now we want to confirm that the Hodge decomposition of this looks like

$$\begin{aligned} \omega &= u_{00}d\theta + v_{00}d\phi \\ &+ d \left[-i \sum' \frac{mu_{mn} + nv_{mn}}{m^2 + n^2} e^{i(m\theta+n\phi)} \right] \\ &+ d^\dagger \left[-i \sum' \frac{nu_{mn} - mv_{mn}}{m^2 + n^2} e^{i(m\theta+n\phi)} d\theta \wedge d\phi \right] \end{aligned}$$

Clearly this is in the right form - we just have to check that this is indeed true. First we have

$$\begin{aligned} &d \left[-i \sum' \frac{mu_{mn} + nv_{mn}}{m^2 + n^2} e^{i(m\theta+n\phi)} \right] \\ &= \frac{m(mu_{mn} + nv_{mn})}{m^2 + n^2} e^{i(m\theta+n\phi)} d\theta + \frac{n(mu_{mn} + nv_{mn})}{m^2 + n^2} e^{i(m\theta+n\phi)} d\phi \end{aligned}$$

where \sum' means excluding the $(0,0)$ term. Next,

$$\begin{aligned} &d^\dagger \left[-i \sum' \frac{nu_{mn} - mv_{mn}}{m^2 + n^2} e^{i(m\theta+n\phi)} d\theta \wedge d\phi \right] \\ &= - * d \left[-i \sum' \frac{nu_{mn} - mv_{mn}}{m^2 + n^2} e^{i(m\theta+n\phi)} \right] \\ &= - * \left[\frac{m(nu_{mn} - mv_{mn})}{m^2 + n^2} e^{i(m\theta+n\phi)} d\theta + \frac{n(nu_{mn} - mv_{mn})}{m^2 + n^2} e^{i(m\theta+n\phi)} d\phi \right] \\ &= - \frac{m(nu_{mn} - mv_{mn})}{m^2 + n^2} e^{i(m\theta+n\phi)} d\phi + \frac{n(nu_{mn} - mv_{mn})}{m^2 + n^2} e^{i(m\theta+n\phi)} d\theta \end{aligned}$$

We can then confirm that everything does indeed add to ω .

Exercise 5. Don't understand the question.

Exercise 6. The connection 1-forms are

$$\omega_c^a = \Gamma_{bc}^a e^b$$

Now, we have

$$\nabla_a e^b = -\Gamma_{ac}^b e^c$$

So

$$\begin{aligned} e_a^m \nabla_m e_n^b dx^n &= -\Gamma_{ac}^b e_n^c dx^n \\ e_a^m \nabla_m e_n^b &= -\Gamma_{ac}^b e_n^c \\ e_a^m e_d^n \nabla_m e_n^b &= -\Gamma_{ad}^b \\ e_a^m e_d^n (\nabla_m e_n^b) e_q^a dx^q &= -\Gamma_{ad}^b e^a \\ e_d^n (\nabla_m e_n^b) dx^m &= -\Gamma_{ad}^b e^a \end{aligned}$$

That is,

$$\begin{aligned} \Gamma_{bc}^a e^b &= -(e_n^c \nabla_m e_n^a) dx^m \\ &= (e_n^a \nabla_m e_n^c) dx^m \\ \omega_c^a &= (e_n^a \nabla_m e_n^c) dx^m \end{aligned}$$

where in the second line we have used $e_n^a e_n^c = \delta_a^c$ to integrate by parts.

Exercise 7. We have

$$dR_b^a + \omega_c^a \wedge R_b^c - R_c^a \wedge \omega_b^c = 0$$

These terms can be rewritten.

$$\begin{aligned} R_b^a &= \frac{1}{2} R_{mn}^{k} e_k^a e_b^l dx^m \wedge dx^n \\ &= \frac{1}{2} g^{kp} R_{mnp l} e_k^a e_b^l dx^m \wedge dx^n \\ dR_b^a &= \frac{1}{2} g^{kp} \nabla_q (R_{mnp l} e_k^a e_b^l) dx^q \wedge dx^m \wedge dx^n \\ \omega_c^a \wedge R_b^c &= \frac{1}{2} \Gamma_{dc}^a e_d^c g^{kp} R_{mnp l} e_k^c e_b^l dx^q \wedge dx^m \wedge dx^n \\ R_c^a \wedge \omega_b^c &= \frac{1}{2} g^{kp} R_{mnp l} e_k^a e_c^l \Gamma_{db}^c e_d^c dx^q \wedge dx^m \wedge dx^n \end{aligned}$$

Our identity now reads

$$\left(\nabla_q (R_{mnp l} e_k^a e_b^l) + R_{mnp l} (\Gamma_{dc}^a e_d^c e_k^c e_b^l - \Gamma_{db}^c e_l^c e_c^l e_d^c) \right) dx^q \wedge dx^m \wedge dx^n$$

The first term is

$$\nabla_q(R_{mnpl}e_k^a e_b^l) = (\nabla_q R_{mnpl})e_k^a e_b^l + R_{mnpl} \left((\nabla_q e_k^a) e_b^l + e_k^a \nabla_q e_b^l \right)$$

Now, from the definition, we have

$$\begin{aligned} \nabla_b e_c &= \Gamma_{bc}^a e_a \\ e_b^m \nabla_m (e_c^n \partial_n) &= \Gamma_{bc}^a e_a^n e_m^b \\ \nabla_m e_c^n &= \Gamma_{bc}^a e_a^n e_m^b \end{aligned}$$

and similarly

$$\begin{aligned} \nabla_b e^a &= -\Gamma_{bc}^a e^c \\ e_b^m \nabla_m e_n^a dx^n &= -\Gamma_{bc}^a e_n^c e_m^b dx^n \\ \nabla_m e_n^a &= -\Gamma_{bc}^a e_n^c e_m^b \end{aligned}$$

Thus this first term becomes

$$(\nabla_q R_{mnpl})e_k^a e_b^l + R_{mnpl} \left(-\Gamma_{cd}^a e_k^d e_q^c e_b^l + e_k^a \Gamma_{cb}^d e_d^l e_q^c \right)$$

The second two terms cancel with the second two terms of our identity. Thus we just have

$$(\nabla_q R_{mnpl})e_k^a e_b^l dx^q \wedge dx^m \wedge dx^n = 0$$

The vielbeins are invertible, so we can drop them. Furthermore, we have the symmetry $R_{mnpl} = R_{plmn}$ of the Riemann tensor, so

$$\nabla_q R_{plmn} dx^q \wedge dx^m \wedge dx^n = 0$$

In other words,

$$R_{pl[mn;q]} = 0$$

where ; denotes covariant differentiation. This is just the (differential) Bianchi identity.

Exercise 8. Under a change of frame,

$$e^a \rightarrow \Phi_b^a e^b$$

i.e.

$$e_m^a \rightarrow \Phi_b^a e_m^b$$

so also

$$e_a^m \rightarrow (\Phi^{-1})_a^b e_b^m$$

Now, we have

$$\omega_b^a = (e_k^a \nabla_m e_b^k) dx^m$$

so

$$\begin{aligned} \omega_b^a &\rightarrow (\Phi_c^a e_k^c \nabla_m ((\Phi^{-1})_b^d e_d^k)) dx^m \\ &= (\Phi_c^a e_k^c (\nabla_m e_d^k) (\Phi^{-1})_b^d + \Phi_c^a e_k^c e_d^k \nabla_m (\Phi^{-1})_b^d) dx^m \\ &= \Phi_c^a \omega_d^c (\Phi^{-1})_b^d + \Phi_c^a (d\Phi^{-1})_b^c \end{aligned}$$

Also, since

$$R_b^a = \frac{1}{2} R_{mn}{}^k e_k^a e_b^l dx^m \wedge dx^n$$

we have

$$\begin{aligned} R_b^a &\rightarrow \frac{1}{2} R_{mn}{}^k \Phi_c^a e_k^c (\Phi^{-1})_b^d e_d^l dx^m \wedge dx^n \\ &= \Phi_c^a (\Phi^{-1})_b^d R_d^c \end{aligned}$$

Lastly, we have

$$D\nu_b^a = d\nu_b^a + \omega_c^a \wedge \nu_b^c - \omega_b^c \wedge \nu_c^a$$

Now,

$$\nu_b^a \rightarrow \Phi_c^a (\Phi^{-1})_b^d \nu_d^c$$

so

$$d\nu_b^a \rightarrow d(\Phi_c^a (\Phi^{-1})_b^d) \nu_d^c + \Phi_c^a (\Phi^{-1})_b^d d\nu_d^c$$

Then,

$$\begin{aligned} \omega_c^a \wedge \nu_b^c &\rightarrow (\Phi_d^a \omega_e^d (\Phi^{-1})_c^e + \Phi_d^a (d\Phi^{-1})_c^d) \wedge \Phi_f^c (\Phi^{-1})_b^g d\nu_g^f \\ &= \Phi_d^a (\Phi^{-1})_b^g \omega_e^d \wedge d\nu_g^e + \Phi_d^a \Phi_f^c (\Phi^{-1})_b^g (d\Phi^{-1})_c^d \wedge d\nu_g^f \\ &= \Phi_d^a (\Phi^{-1})_b^g \omega_e^d \wedge d\nu_g^e - \Phi_d^a (d\Phi_f^c) (\Phi^{-1})_b^g (\Phi^{-1})_c^d \wedge d\nu_g^f \\ &= \Phi_c^a (\Phi^{-1})_b^d \omega_e^c \wedge d\nu_d^e - d\Phi_f^a (\Phi^{-1})_b^d \wedge d\nu_d^f \end{aligned}$$

And

$$\begin{aligned} \omega_b^c \wedge \nu_c^a &\rightarrow (\Phi_d^c \omega_e^d (\Phi^{-1})_b^e + \Phi_d^c (d\Phi^{-1})_b^d) \wedge \Phi_f^a (\Phi^{-1})_c^g d\nu_g^f \\ &= \Phi_c^a (\Phi^{-1})_b^d \omega_e^c \wedge d\nu_d^e + \Phi_f^a (d\Phi^{-1})_b^d \wedge d\nu_d^f \end{aligned}$$

Thus

$$\begin{aligned} D\nu_b^a &\rightarrow \Phi_c^a (\Phi^{-1})_b^d (d\nu_d^c + \omega_e^c \wedge d\nu_d^e - \omega_d^e \wedge d\nu_e^c) + d(\Phi_c^a (\Phi^{-1})_b^d) \nu_d^c \\ &\quad - (d\Phi_f^a (\Phi^{-1})_b^d + \Phi_f^a (d\Phi^{-1})_b^d) \wedge d\nu_d^f \\ &= \Phi_c^a (\Phi^{-1})_b^d D\nu_d^c + d(\Phi_c^a (\Phi^{-1})_b^d) \nu_d^c - d(\Phi_f^a (\Phi^{-1})_b^d) \wedge d\nu_d^f \\ &= \Phi_c^a (\Phi^{-1})_b^d D\nu_d^c \end{aligned}$$

Exercise 9. Regard the exterior derivative d as the operator

$$d = dx^i \wedge \frac{\partial}{\partial x^i}$$

in real coordinates x^i . We can insert $P + Q = 1$ at will:

$$d = (P_j^i + Q_j^i) dx^j \wedge \frac{\partial}{\partial x^i}$$

Now, precisely if the Nijenhuis tensor vanishes, we can write this as

$$d = (\theta_\mu^i dz^\mu + \phi_\nu^i d\bar{z}^\nu) \wedge \frac{\partial}{\partial x^i}$$

Further, we can rewrite this

$$d = (\theta_\mu^i dz^\mu + \phi_\nu^i d\bar{z}^\nu) \wedge \left(\frac{\partial z^\lambda}{\partial x^i} \frac{\partial}{\partial z^\lambda} + \frac{\partial \bar{z}^\lambda}{\partial x^i} \frac{\partial}{\partial \bar{z}^\lambda} \right)$$

Then clearly the action of d on a (p, q) -form ω gives us a sum of a $(p+1, q)$ -form and a $(p, q+1)$ -form. That is, we can write

$$d = \partial + \bar{\partial}$$

and there are no further contributions to d . Then in the usual way $\partial^2 = 0$ follows from $d^2 = 0$. That is, $\partial^2 = 0$ iff $N = 0$.

Exercise 10. Let z^m be complex coordinates, in which an arbitrary connection has components Γ_{nk}^m . Then perform a holomorphic change of coordinates to w^i . Then we have

$$\nabla_i \frac{\partial}{\partial w^k} = \Gamma_{ij}^k \frac{\partial}{\partial w^j}$$

The LHS is

$$\begin{aligned} \nabla_i \frac{\partial}{\partial w^j} &= \nabla_i \left(\frac{\partial z^m}{\partial w^j} \frac{\partial}{\partial z^m} \right) \\ &= \frac{\partial^2 z^m}{\partial w^i \partial w^j} \frac{\partial}{\partial z^m} + \frac{\partial z^m}{\partial w^j} \frac{\partial z^n}{\partial w^i} \nabla_n \frac{\partial}{\partial z^m} \\ &= \frac{\partial^2 z^m}{\partial w^i \partial w^j} \frac{\partial}{\partial z^m} + \frac{\partial z^m}{\partial w^j} \frac{\partial z^n}{\partial w^i} \Gamma_{nm}^p \frac{\partial}{\partial z^p} \\ &= \left(\frac{\partial^2 z^m}{\partial w^i \partial w^j} \frac{\partial}{\partial z^m} + \frac{\partial z^p}{\partial w^j} \frac{\partial z^n}{\partial w^i} \Gamma_{np}^m \right) \frac{\partial}{\partial z^m} \end{aligned}$$

The RHS is

$$\Gamma^k_{ij} \frac{\partial}{\partial w^k} = \Gamma^k_{ij} \frac{\partial z^m}{\partial w^k} \frac{\partial}{\partial z^m}$$

Thus we have

$$\Gamma^k_{ij} = \left(\frac{\partial^2 z^m}{\partial w^i \partial w^j} \frac{\partial}{\partial z^m} + \frac{\partial z^p}{\partial w^j} \frac{\partial z^n}{\partial w^i} \Gamma^m_{np} \right) \frac{\partial w^k}{\partial z^m}$$

If the first term vanishes, this is just standard tensor transformation. Now, if $k = \bar{\kappa}$, then since $w(z)$ is holomorphic, we must have $m = \bar{\mu}$. But then the first term vanishes unless both i and j are complex components. Therefore we have that $\Gamma^{\bar{\kappa}}_{\mu\nu}, \Gamma^{\bar{\kappa}}_{\bar{\mu}\nu}, \Gamma^{\bar{\kappa}}_{\mu\bar{\nu}}$ transform as tensors. Now suppose that $i = \bar{\nu}$. Then we must have $m = \bar{\mu}$ and $j = \bar{\lambda}$ for the first term not to vanish. But then we must have $k = \bar{\kappa}$, in which case we are no longer looking at a mixed component. So we also have that $\Gamma^{\kappa}_{\bar{\nu}\lambda}, \Gamma^{\kappa}_{\bar{\nu}\bar{\lambda}}, \Gamma^{\kappa}_{\bar{\nu}\lambda}$ transform as tensors. Similarly for the mixed components with $j = \bar{\lambda}$, and we are done. All mixed components transform as tensors.

Exercise 11. The Ricci form on a Hermitean manifold is

$$\mathcal{R} = i\partial\bar{\partial} \ln \sqrt{g}$$

If we are in the intersection of two coordinate patches and we change coordinates, g picks up Jacobian factors:

$$g \rightarrow \left| \frac{\partial z^\mu}{\partial w^\alpha} \right| \left| \frac{\partial \bar{z}^\nu}{\partial \bar{w}^\beta} \right| g$$

Thus

$$\ln \sqrt{g} \rightarrow \ln \sqrt{g} + \frac{1}{2} \left(\ln \left| \frac{\partial z^\mu}{\partial w^\alpha} \right| + \ln \left| \frac{\partial \bar{z}^\nu}{\partial \bar{w}^\beta} \right| \right)$$

The new terms are destroyed by $\bar{\partial}$ and ∂ respectively, so \mathcal{R} is invariant, i.e. globally defined.

Exercise 12. Consider $\mathbb{C}P^n$. Let U_j be the patch on with $z^j \neq 0$, and use coordinates

$$\zeta_j^m = \frac{z^m}{z^j}$$

We then introduce the local Kähler potential

$$\phi_j = \ln \left(\sum_{m=1}^{n+1} |\zeta_j^m|^2 \right)$$

The Fubini-Study metric is then given by

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \phi_j$$

Let

$$\sigma = \sum_{m=1}^{n+1} |\zeta_j^m|^2$$

Now, introduce some slightly different notation and write this as

$$\sigma = \zeta^\mu \zeta_\mu + 1$$

Here, lower μ means upper $\bar{\mu}$, and the 1 has come from the fact that μ runs over the n non- j indices only. Now, we have

$$\begin{aligned} g_{\mu\bar{\nu}} &= \frac{\partial^2 \ln \sigma}{\partial \zeta^\mu \partial \zeta_\nu} \\ &= \frac{\partial}{\partial \zeta^\mu} \left(\frac{1}{\sigma} \frac{\partial \sigma}{\partial \zeta_\nu} \right) \\ &= -\frac{1}{\sigma^2} \frac{\partial \sigma}{\partial \zeta^\mu} \frac{\partial \sigma}{\partial \zeta_\nu} + \frac{1}{\sigma} \frac{\partial^2 \sigma}{\partial \zeta^\mu \partial \zeta_\nu} \\ &= \frac{1}{\sigma} \left(\frac{\partial^2 \sigma}{\partial \zeta^\mu \partial \zeta_\nu} - \frac{1}{\sigma} \frac{\partial \sigma}{\partial \zeta^\mu} \frac{\partial \sigma}{\partial \zeta_\nu} \right) \end{aligned}$$

Then we have

$$\begin{aligned} \frac{\partial \sigma}{\partial \zeta^\mu} &= \zeta_\mu \\ \frac{\partial^2 \sigma}{\partial \zeta^\mu \partial \zeta_\nu} &= \delta_{\mu\bar{\nu}} \end{aligned}$$

Thus

$$g_{\mu\bar{\nu}} = \frac{1}{\sigma} \left(\delta_{\mu\bar{\nu}} - \frac{\zeta_\mu \zeta_{\bar{\nu}}}{\sigma} \right)$$

We want to make sure this is indeed a metric - i.e. positive-definite. Let X be some vector

$$X = X^\mu \frac{\partial}{\partial \zeta^\mu} + X^{\bar{\mu}} \frac{\partial}{\partial \zeta_{\bar{\mu}}}$$

Then

$$\begin{aligned} g(X, X) &= \frac{2}{\sigma} \left(\delta_{\mu\bar{\nu}} - \frac{\zeta_\mu \zeta_{\bar{\nu}}}{\sigma} \right) X^\mu X^{\bar{\nu}} \\ &= \frac{2}{\sigma^2} (X^\mu X_\mu \sigma - X^\mu \zeta_\mu X^{\bar{\nu}} \zeta_{\bar{\nu}}) \\ &= \frac{2}{\sigma^2} (X^\mu X_\mu (\zeta^\nu \zeta_\nu + 1) - X^\mu \zeta_\mu X^{\bar{\nu}} \zeta_{\bar{\nu}}) \end{aligned}$$

But by the Schwarz inequality,

$$(X^\mu X_\mu)(\zeta^\nu \zeta_\nu) \geq (X^\mu \zeta_\mu)(X^\nu \zeta_\nu)$$

Thus $g(X, X) \geq 0$, and zero only if $X = 0$.

Exercise 13. The Fubini-Study metric is

$$g_{\mu\bar{\nu}} = \frac{1}{\sigma^2}(\sigma\delta_{\mu\bar{\nu}} - \zeta_\mu\zeta_{\bar{\nu}})$$

We want to find

$$\mathcal{R} = i\partial\bar{\partial} \ln \sqrt{g}$$

If we write

$$M_{ij} = \sigma\delta_{ij} - \zeta_i\zeta_j$$

we have

$$|g| = \frac{1}{\sigma^{2n}}|M|$$

To calculate $|M|$, note that it is the sum of a scalar multiple of the identity and a symmetric matrix of rank 1. Therefore it is possible to diagonalise M as

$$\begin{aligned} SMS^{-1} &= \begin{pmatrix} \sigma & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma \end{pmatrix} - \begin{pmatrix} \zeta^i \zeta_i & & & \\ & 0 & & \\ & & \cdot & \\ & & & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma - \zeta^i \zeta_i & & & \\ & \sigma & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \sigma \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ & \sigma & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma \end{pmatrix} \end{aligned}$$

Thus

$$|M| = \sigma^{n-1}$$

so

$$|g| = \frac{1}{\sigma^{n+1}}$$

Thus

$$\begin{aligned}\mathcal{R} &= i\partial\bar{\partial}\ln\sqrt{\frac{1}{\sigma^{n+1}}} \\ &= -\frac{i}{2}(n+1)\partial\bar{\partial}\ln\sigma\end{aligned}$$

But

$$g_{\mu\bar{\nu}} = \frac{\partial^2 \ln \sigma}{\partial \zeta^\mu \partial \bar{\zeta}^\nu}$$

So

$$\begin{aligned}\mathcal{R} &= -\frac{i}{2}(n+1)g_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu \\ &= -(n+1)J\end{aligned}$$

The Ricci form is therefore harmonic, since J is, and so not exact. Consequently the first Chern class does not vanish.

Exercise 14. We want to show that Dolbeault cohomology is equivalent to de Rham cohomology on a Kähler manifold. In order to do this it is sufficient to show that a form is $\bar{\partial}$ -harmonic iff it is d -harmonic, so we want to show that $\Delta \propto \Delta_{\bar{\partial}}$. Let ω be an (r, s) -form, and $p = r + s$.

$$\omega = \frac{1}{r!s!}\omega_{\mu_1\dots\mu_r\bar{\nu}_1\dots\bar{\nu}_s}dz^{\mu_1}\wedge\dots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\dots\wedge d\bar{z}^{\nu_s}$$

Recall our earlier result for the usual Hodge-de Rham Laplacian on a real manifold:

$$\Delta\omega = -\frac{1}{p!}\left(\nabla^k\nabla_k\omega_{m_1\dots m_p} + pR_{\lambda m_1}\omega^{\lambda}_{m_2\dots m_p} + \frac{1}{2}p(p-1)R_{klm_1m_2}\omega^{kl}_{m_3\dots m_p}\right)dx^{m_1}\wedge\dots\wedge dx^{m_p}$$

Adapting this for an (r, s) form on a complex manifold, we get

$$\begin{aligned}\Delta\omega &= -\frac{1}{r!s!}\left[\nabla^k\nabla_k\omega_{\mu_1\dots\mu_r\bar{\nu}_1\dots\bar{\nu}_s} + rR_{\bar{\kappa}\mu_1}\omega^{\bar{\kappa}}_{\mu_2\dots\mu_r\bar{\nu}_1\dots\bar{\nu}_s} - sR_{\kappa\bar{\nu}_1}\omega^{\kappa}_{\mu_1\dots\mu_r\bar{\nu}_2\dots\bar{\nu}_s} \right. \\ &\quad + \frac{1}{2}r(r-1)R_{\bar{\kappa}\bar{\lambda}\mu_1\mu_2}\omega^{\bar{\kappa}\bar{\lambda}}_{\mu_3\dots\mu_r\bar{\nu}_1\dots\bar{\nu}_s} + \frac{1}{2}s(s-1)R_{\kappa\lambda\bar{\nu}_1\bar{\nu}_2}\omega^{\kappa\lambda}_{\mu_1\dots\mu_r\bar{\nu}_3\dots\bar{\nu}_s} \\ &\quad \left. + \frac{1}{2}rs\left(R_{\bar{\kappa}\lambda\mu_1\bar{\nu}_1}\omega^{\bar{\kappa}}_{\mu_2\dots\mu_r\bar{\nu}_2\dots\bar{\nu}_s} + R_{\kappa\bar{\lambda}\bar{\nu}_1\mu_1}\omega^{\bar{\lambda}}_{\mu_2\dots\mu_r\bar{\nu}_2\dots\bar{\nu}_s}\right)\right] \\ &\quad dz^{\mu_1}\wedge\dots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\dots\wedge d\bar{z}^{\nu_s}\end{aligned}$$

where Latin k may be holomorphic or antiholomorphic. All the Riemann tensors vanish, since $R_{klmn} = 0$ unless exactly one of k, l, m, n is holomorphic or antiholomorphic. This just leaves

$$\Delta\omega = -\frac{1}{r!s!} \left[\nabla^k \nabla_k \omega_{\mu_1 \dots \mu_r \bar{\nu}_1 \dots \bar{\nu}_s} + r R_{\bar{\kappa}\mu_1} \omega_{\mu_2 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_s} - s R_{\kappa \bar{\nu}_1} \omega_{\mu_1 \dots \mu_p}{}^{\kappa}{}_{\bar{\nu}_2 \dots \bar{\nu}_s} \right] dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_p}$$

Furthermore, on a Kähler manifold the Ricci tensor is precisely the Ricci form, so

$$R_{\bar{\kappa}\mu} = \frac{i}{2} \partial_{\mu} \partial_{\bar{\kappa}} \ln g$$

Thus we finally have

$$\begin{aligned} \Delta\omega = -\frac{1}{r!s!} & \left[\nabla^{\kappa} \nabla_{\kappa} \omega_{\mu_1 \dots \mu_r \bar{\nu}_1 \dots \bar{\nu}_s} + \nabla^{\bar{\kappa}} \nabla_{\bar{\kappa}} \omega_{\mu_1 \dots \mu_r \bar{\nu}_1 \dots \bar{\nu}_s} \right. \\ & \left. + \frac{i}{2} (r(\partial_{\mu_1} \partial_{\bar{\kappa}} g) \omega_{\mu_2 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_s} - s(\partial_{\bar{\nu}_1} \partial_{\kappa} g) \omega_{\mu_1 \dots \mu_p}{}^{\kappa}{}_{\bar{\nu}_2 \dots \bar{\nu}_s}) \right] \\ & dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_p} \end{aligned}$$

Now we turn to $\Delta_{\bar{\partial}}$. First we need to say what ∂^{\dagger} and $\bar{\partial}^{\dagger}$ are. On the one hand, we have

$$d^{\dagger} = \partial^{\dagger} + \bar{\partial}^{\dagger}$$

On the other,

$$d^{\dagger} = - * (\partial + \bar{\partial}) *$$

(any complex manifold has even real dimension). Let $\omega \in \Omega^{r,s}(M)$. Then

$$\partial^{\dagger} \omega + \bar{\partial}^{\dagger} \omega = - * \partial * \omega - * \bar{\partial} * \omega$$

On the LHS, the forms are $(r-1, s)$ and $(r, s-1)$ respectively. On the RHS,

$$\begin{aligned} * \omega & \in \Omega^{m-s, m-r}(M) \\ \partial * \omega & \in \Omega^{m-s+1, m-r}(M) \\ * \partial * \omega & \in \Omega^{r, s-1}(M) \end{aligned}$$

and similarly

$$* \bar{\partial} * \omega \in \Omega^{r-1, s}(M)$$

Thus we have

$$\begin{aligned} \partial^{\dagger} & = - * \bar{\partial} * \\ \bar{\partial}^{\dagger} & = - * \partial * \end{aligned}$$

Firstly, we have

$$\bar{\partial}\omega = (-1)^r \frac{1}{r!s!} \nabla_{\bar{\lambda}} \omega_{\mu_1 \dots \mu_r \bar{\nu}_1 \dots \bar{\nu}_s} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_r} \wedge d\bar{z}^{\lambda} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_s}$$

Just as we calculate d^\dagger , we can find that

$$\bar{\partial}^\dagger \omega = -(-1)^s \frac{1}{r!(s-1)!} \nabla^{\bar{\nu}_1} \omega_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_s} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_2} \wedge \dots \wedge d\bar{z}^{\nu_s}$$

Thus

$$\bar{\partial}^\dagger \bar{\partial} \omega = (-1)^{r+s+1} \frac{1}{r!s!} \nabla^{\bar{\lambda}} \nabla_{\bar{\lambda}} \omega_{\mu_1 \dots \mu_r \bar{\nu}_1 \dots \bar{\nu}_s} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_s}$$

Similarly,

$$\bar{\partial} \bar{\partial}^\dagger \omega = (-1)^{r+s+1} \frac{1}{r!(s-1)!} \nabla_{\bar{\nu}_1} \nabla^{\bar{\kappa}} \omega_{\mu_1 \dots \mu_p \bar{\kappa} \bar{\nu}_2 \dots \bar{\nu}_s} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_s}$$

This term will lead to contributions from the Ricci tensor. Similarly we can calculate Δ_∂ . We will see that in fact

$$dd^\dagger + d^\dagger d = \partial\bar{\partial}^\dagger + \bar{\partial}^\dagger\partial + \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$$

Furthermore, the Dolbeault Laplacians are real(?) so in fact

$$dd^\dagger + d^\dagger d = 2(\partial\bar{\partial}^\dagger + \bar{\partial}^\dagger\partial) = 2(\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial})$$