Nakahara - Geometry, Topology and Physics

Chapter 5: Manifolds

1 Exercises

Exercise 1. Consider S^2. Define charts (U_N, ϕ_N) and (U_S, ϕ_S) by

$$U_N = S^2 \backslash \{(0,0,-1)\}, \quad \phi_N(x_1, x_2, x_3) = \frac{1}{1+x_3}(x_1, x_2)$$

and

$$U_S = S^2 \backslash \{(0,0,1)\}, \quad \phi_S(x_1, x_2, x_3) = \frac{1}{1-x_3}(x_1, x_2)$$

Then use coordinates (y_1, y_2) on $\phi_N(U_N) \subset \mathbb{R}^2$

$$\phi_N^{-1}(y_1, y_2) = \left(1 + \sqrt{1 - y_1^2 - y_2^2}\right)(y_1, y_2)$$

Define $y_3 = \sqrt{1 - y_1^2 - y_2^2}$. Then

$$\phi_S \cdot \phi_N^{-1}(y_1, y_2) = \frac{1+y_3}{1-y_3}(y_1, y_2)$$

This is smooth with smooth inverse on $U_N \cap U_S$. Similarly,

$$\phi_N \cdot \phi_S^{-1}(y_1, y_2) = \frac{1-y_3}{1+y_3}(y_1, y_2)$$

is smooth with smooth inverse on $U_N \cap U_S$. Therefore these two charts are compatible. Furthermore, $U_N \cup U_S = S^2$, so they define an atlas, and hence a maximal atlas. Therefore we have a differential structure on S^2.

Exercise 2. Let $f : M \to N$ be a map of manifolds, and (U, ϕ) a chart on M, and (V_1, ψ_1) and (V_2, ψ_2) charts on N such that $V_1 \cap V_2 \cap f(U) \neq \emptyset$. f has the representative maps $\psi_1 \circ f \circ \phi$ and $\psi_2 \circ f \circ \phi$. But

$$\psi_1 \circ f \circ \phi = (\psi_1 \circ \psi_2^{-1}) \circ (\psi_2 \circ f \circ \phi)$$

If the charts on N are in the same (maximal) atlas, then by definition the transition function $\psi_1 \circ \psi_2^{-1}$ is smooth. Then clearly $\psi_1 \circ f \circ \phi$ is smooth iff $\psi_2 \circ f\phi$ is. Therefore (in combination with the same result for charts on M obtained in the text) the notion of the smoothness of f is well-defined.

Exercise 3. Let $f : M \to N$ and $g : N \to P$ be smooth maps, and $u \in T_pM$. Then $f_* u = v \in T_{f(p)}N$ satisfies

$$v(\alpha) = u(\alpha \circ f)$$

for all smooth functions α on N. Next, $g_* v = w \in T_{g(f(p))}P$ satisfies

$$\begin{aligned} w(\beta) &= v(\beta \circ g) \\ &= u(\beta \circ g \circ f) \end{aligned}$$

for all smooth functions β on P. So clearly $w = (g \circ f)_* u$. That is, induced maps are covariantly functorial:

$$(g \circ f)_* = g_* \circ f_*$$

Exercise 4. Let $f : M \to N$ be smooth, and $\omega = \omega_\alpha dy^\alpha \in T^*_{f(p)}N$. Let x^μ be local coordinates on M near p. Then write $V \in T_pM$ as

$$V = V^\mu \frac{\partial}{\partial x^\mu}$$

Then

$$f_* V = V^\nu \frac{\partial y^\alpha}{\partial x^\nu} \frac{\partial}{\partial y^\alpha}$$

so

$$\begin{aligned} \langle \omega, f_* V \rangle &= \omega_\beta V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \left\langle dy^\beta, \frac{\partial}{\partial y^\alpha} \right\rangle \\ &= \omega_\beta V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \delta^\beta_\alpha \\ &= \omega_\alpha V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \end{aligned}$$

Meanwhile, writing $f^* \omega = \xi_\mu dx^\mu$, we have

$$\begin{aligned} \langle f^* \omega, V \rangle &= \xi_\mu V^\nu \left\langle dx^\mu, \frac{\partial}{\partial x_\nu} \right\rangle \\ &= \xi_\mu V^\mu \end{aligned}$$

Therefore,

$$\xi_\mu V^\mu = \omega_\alpha V^\mu \frac{\partial y^\alpha}{\partial x^\mu}$$

for all $v \in T_pM$, so

$$\xi_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}$$

Exercise 5. Let $f : M \to N$ and $g : N \to P$ be smooth, and use coordinates x^μ, y^α, x^i on M, N, P, respectively. Let $\omega \in T^*_{g(f(p))}P$, and write $\omega = \omega_i dz^i$. Then

$$g^*\omega = \omega_i \frac{\partial z^i}{\partial y^\alpha} dy^\alpha \in T^*_{f(p)}N$$
$$f^* \circ g^*\omega = \omega_i \frac{\partial z^i}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu} dx^\mu \in T^*_pM$$

But

$$\begin{aligned}(g \circ f)^*\omega &= \omega_i \frac{\partial z^i}{\partial x^\mu} dx^\mu \\ &= \omega_i \frac{\partial z^i}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu} dx^\mu\end{aligned}$$

Therefore pullbacks are contravariantly functorial:

$$(g \circ f)^* = f^* \circ g^*$$

Exercise 6. Let T be a type $(1,1)$ tensor on M, i.e. $T \in T_pM \otimes T^*_pM$. Write

$$T = T^\mu_\nu \frac{\partial}{\partial x^\mu} \otimes dx^\nu$$

and $f : M \to N$ a diffeomorphism. This induces a map $f_* : T_pM \otimes T^*_pM \to T_{f(p)}N \otimes T^*_{f(p)}N$. We can decompose into each part, so

$$\begin{aligned} f_*\left(T^\mu_\nu \frac{\partial}{\partial x^\mu} \otimes dx^\nu\right) &= f_*\left(T^\mu_\nu \frac{\partial}{\partial x^\mu}\right) \otimes f_*(dx^\nu) \\ &= \left(T^\mu_\nu \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial}{\partial y^\alpha}\right) \otimes \left(\frac{\partial x^\nu}{\partial y^\beta} dy^\beta\right) \\ &= T^\mu_\nu \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial}{\partial y^\alpha} \otimes dy^\beta \end{aligned}$$

Exercise 7. Consider $M = \mathbb{R}^2$, and the vector field

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Denote the corresponding flow $\sigma(t) = (x(t), y(t))$. Then

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x$$

This is solved by

$$\begin{aligned} x(t) &= A\cos t + B\sin t \\ y(t) &= B\cos t - A\sin t \end{aligned}$$

With initial conditions $\sigma(0) = (x_0, y_0)$,

$$\begin{aligned} x(t) &= x_0\cos t + y_0\sin t \\ y(t) &= y_0\cos t - x_0\sin t \end{aligned}$$

Exercise 8. The Lie derivative of a vector field Y with respect to a vector field X, with flow σ, is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_* Y\mid_{\sigma_\epsilon(x)} - Y\mid_x\right]$$

The limit outside the brackets means we can precompose with $(\sigma_\epsilon)_*$ inside the brackets (since σ_0 is the identity):

$$\begin{aligned} \mathcal{L}_X Y &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\sigma_\epsilon\circ\sigma_{-\epsilon})_* Y\mid_{\sigma_{-\epsilon}\circ\sigma_\epsilon(x)} - (\sigma_\epsilon)_* Y\mid_{\sigma_{-\epsilon}(x)}\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[Y\mid_x - (\sigma_\epsilon)_* Y\mid_{\sigma_{-\epsilon}(x)}\right] \end{aligned}$$

Then we can also flow $x \to \sigma_\epsilon(x)$ for the same reason:

$$\begin{aligned} \mathcal{L}_X Y &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[Y\mid_{\sigma_\epsilon(x)} - (\sigma_\epsilon)_* Y\mid_{\sigma_{-\epsilon}\circ\sigma_\epsilon(x)}\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[Y\mid_{\sigma_\epsilon(x)} - (\sigma_\epsilon)_* Y\mid_x\right] \end{aligned}$$

Exercise 9. Let X and Y be vector fields, with local coordinate expansions

$$X = X^\mu\frac{\partial}{\partial x^\mu}, \quad Y = Y^\mu\frac{\partial}{\partial x^\mu}$$

Their Lie bracket is $[X, Y]$, where

$$[X, Y]f = X[Y[f]] - Y[X[f]]$$

for all smooth functions f. This is

$$\begin{aligned}
X[Y[f]] - Y[X[f]] &= X\left[Y^\mu \frac{\partial f}{\partial x^\mu}\right] - Y\left[X^\mu \frac{\partial f}{\partial x^\mu}\right] \\
&= X^\nu \frac{\partial}{\partial x^\nu}\left(Y^\mu \frac{\partial f}{\partial x^\mu}\right) - Y^\nu \frac{\partial}{\partial x^\nu}\left(X^\mu \frac{\partial f}{\partial x^\mu}\right) \\
&= X^\nu \frac{\partial Y^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} \\
&= \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial f}{\partial x^\mu}
\end{aligned}$$

Therefore

$$[X, Y] = \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu}$$

Exercise 10. From the result of the previous exercise, it is clear that the Lie bracket is antisymmetric:

$$[X, Y] = -[Y, X]$$

Then, consider

$$\begin{aligned}
[X, c_1Y_1 + c_2Y_2] &= \left[X^\nu \frac{\partial}{\partial x^\nu}(c_1Y_1^\mu + c_2Y_2^\mu) - (c_1Y_1^\nu + c_2Y_2^\nu)\frac{\partial X^\mu}{\partial x^\nu}\right]\frac{\partial}{\partial x^\mu} \\
&= \left[X^\nu \frac{\partial(c_1Y_1^\mu)}{\partial x^\nu} - c_1Y_1^\nu \frac{\partial X^\mu}{\partial x^\nu}\right]\frac{\partial}{\partial x^\mu} + \left[X^\nu \frac{\partial(c_2Y_2^\mu)}{\partial x^\nu} - c_2Y_2^\nu \frac{\partial X^\mu}{\partial x^\nu}\right]\frac{\partial}{\partial x^\mu} \\
&= [X, c_1Y_1] + [X, c_2Y_2]
\end{aligned}$$

So the Lie bracket is linear in its second argument. But by antisymmetry, it must also be linear in its first argument, so it is bilinear. Lastly,

$$\begin{aligned}
[[X, Y], Z]f &= [X, Y]X[f] - Z[X, Y][f] \\
&= X[Y[Z[f]]] - Y[X[Z[f]]] - Z[X[Y[f]]] + Z[Y[X[f]]]
\end{aligned}$$

so we see that in

$$[[X, Y], Z]f + [[Z, X], Y]f + [[Y, Z], X]f$$

everything cancels. That is,

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

so the Lie bracket satisfies the Jacobi identity.

Exercise 11.

(a) Let X and Y be vector fields on M, and f a smooth function on M. We have

$$\begin{aligned}
\mathcal{L}_{fX}Y &= [fX, Y] \\
&= \left(fX^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial (fX^\mu)}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu} \\
&= \left(fX^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial x^\mu}{\partial x^\nu} - Y^\nu \frac{\partial f}{\partial x^\nu} X^\mu \right) \frac{\partial}{\partial x^\mu} \\
&= f[X, Y] - Y[f]X
\end{aligned}$$

Using this, and the antisymmetry of the Lie bracket,

$$\begin{aligned}
\mathcal{L}_X(fY) &= [X, fY] \\
&= -[fY, X] \\
&= -f[Y, X] + X[f]Y \\
&= f[X, Y] + X[f]Y
\end{aligned}$$

(b) Let $f : M \to N$ be a smooth map, and introduce coordinates y^μ on N. We have

$$\begin{aligned}
f_*X &= X^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \\
f_*Y &= Y^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}
\end{aligned}$$

so

$$\begin{aligned}
[f_*, f_*Y] &= \left[X^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \left(Y^\rho \frac{\partial y^\sigma}{\partial x^\rho} \right) - Y^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \left(X^\rho \frac{\partial y^\sigma}{\partial x^\rho} \right) \right] \frac{\partial}{\partial y^\sigma} \\
&= \left[X^\mu \frac{\partial y^\nu}{\partial x^\mu} \left(\frac{\partial Y^\rho}{\partial y^\nu} \frac{\partial y^\sigma}{\partial x^\rho} + Y^\rho \frac{\partial^2 y^\sigma}{\partial x^\nu \partial x^\rho} \right) - Y^\mu \frac{\partial y^\nu}{\partial x^\mu} \left(\frac{\partial X^\rho}{\partial y^\nu} \frac{\partial y^\sigma}{\partial x^\rho} + X^\rho \frac{\partial^2 y^\sigma}{\partial y^\nu \partial x^\rho} \right) \right] \frac{\partial}{\partial y^\sigma} \\
&= \left(X^\mu \frac{\partial Y^\rho}{\partial y^\nu} - Y^\mu \frac{\partial X^\rho}{\partial y^\nu} \right) \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial y^\sigma}{\partial x^\rho} \frac{\partial}{\partial y^\sigma}
\end{aligned}$$

On the other hand,

$$[X, Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu}$$

so

$$\begin{aligned}
f_*[X, Y] &= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial y^\sigma}{\partial x^\nu} \frac{\partial}{\partial y^\sigma} \\
&= \left(X^\mu \frac{\partial Y^\nu}{\partial y^\rho} - Y^\mu \frac{\partial X^\nu}{\partial y^\rho} \right) \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial y^\sigma}{\partial x^\nu} \frac{\partial}{\partial y^\sigma} \\
&= [f_*X, f_*Y]
\end{aligned}$$

Exercise 12. A type (p,q) tensor t can be written as a tensor product

$$t = t^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_p}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_q}$$

Then

$$\begin{aligned}\mathcal{L}_X t =& \mathcal{L}_X (t^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}) \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_p}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_q} \\ &+ t^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_p}} \right) \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_q} \\ &+ t^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_p}} \otimes \mathcal{L}_X \left(dx^{\nu_1} \otimes \dots \otimes dx^{\nu_q} \right)\end{aligned}$$

First consider functions.

$$\begin{aligned}\mathcal{L}_Y f &= Y[f] = Y^\mu \frac{\partial f}{\partial x^\mu} \\ \mathcal{L}_X \mathcal{L}_Y f &= X^\nu \frac{\partial}{\partial x^\nu} \left(Y^\mu \frac{\partial f}{\partial x^\mu} \right) \\ &= X^\nu \left(\frac{\partial Y^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + Y^\mu \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \right)\end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f &= \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right) \frac{\partial f}{\partial x^\mu} \\ &= \mathcal{L}_{[X,Y]} f\end{aligned}$$

Next consider vector fields. We have

$$\begin{aligned}\mathcal{L}_Y \frac{\partial}{\partial x^\mu} &= \left[X, \frac{\partial}{\partial x^\mu} \right] \\ &= -\frac{\partial Y^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu}\end{aligned}$$

so

$$\begin{aligned}\mathcal{L}_X \mathcal{L}_Y \frac{\partial}{\partial x^\mu} &= \left[X, -\frac{\partial Y^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right] \\ &= \left(-X^\nu \frac{\partial^2 Y^\rho}{\partial x^\mu \partial x^\nu} + \frac{\partial Y^\nu}{\partial x^\mu} \frac{\partial X^\rho}{\partial x^\nu} \right) \frac{\partial}{\partial x^\rho}\end{aligned}$$

Then

$$\mathcal{L}_X\mathcal{L}_Y\frac{\partial}{\partial x^\mu} - \mathcal{L}_Y\mathcal{L}_X\frac{\partial}{\partial x^\mu} = \left(\frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial X^\rho}{\partial x^\nu} - X^\nu\frac{\partial^2 Y^\rho}{\partial x^\mu \partial x^\nu} + Y^\nu\frac{\partial^2 X^\rho}{\partial x^\mu \partial x^\nu} - \frac{\partial X^\nu}{\partial x^\mu}\frac{\partial Y^\rho}{\partial x^\nu}\right)\frac{\partial}{\partial x^\rho}$$
$$= \mathcal{L}_{[X,Y]}\frac{\partial}{\partial x^\mu}$$

Lastly, consider 1-forms. We have

$$\mathcal{L}_Y dx^\mu = \frac{\partial Y^\mu}{\partial x^\nu}dx^\nu$$

so

$$\mathcal{L}_X\mathcal{L}_Y dx^\mu = \left(X^\rho\frac{\partial^2 Y^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial X^\rho}{\partial x^\nu}\frac{\partial Y^\mu}{\partial x^\rho}\right)dx^\nu$$

Then

$$\mathcal{L}_X\mathcal{L}_Y dx^\mu - \mathcal{L}_Y\mathcal{L}_X dx^\mu = \left(X^\rho\frac{\partial^2 Y^\mu}{\partial x^\nu \partial x^\rho} + \frac{\partial X^\rho}{\partial x^\nu}\frac{\partial Y^\mu}{\partial x^\rho} - Y^\rho\frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\rho} - \frac{\partial X^\rho}{\partial x^\nu}\frac{\partial X^\mu}{\partial x^\rho}\right)dx^\nu$$
$$= \mathcal{L}_{[X,Y]}dx^\mu$$

Now, suppose a and b are two objects satisfying this rule confirmed in the cases above. Then

$$\begin{aligned}
\mathcal{L}_{[X,Y]}(a\otimes b) &= (\mathcal{L}_{[X,Y]}a)\otimes b + a\otimes\mathcal{L}_{[X,Y]}b \\
&= (\mathcal{L}_X\mathcal{L}_Y a - \mathcal{L}_Y\mathcal{L}_X a)\otimes b + a\otimes(\mathcal{L}_X\mathcal{L}_Y b - \mathcal{L}_Y\mathcal{L}_X b) \\
&= (\mathcal{L}_X\mathcal{L}_Y a)\otimes b + a\otimes\mathcal{L}_X\mathcal{L}_y b - (\mathcal{L}_Y\mathcal{L}_X a)\otimes b - a\otimes\mathcal{L}_Y L_X b \\
&= \mathcal{L}_X\mathcal{L}_Y(a\otimes b) - \mathcal{L}_Y\mathcal{L}_X(a\otimes b)
\end{aligned}$$

Thus the same rule holds for $a\otimes b$. But then we have proved that

$$\mathcal{L}_{[X,Y]}t = \mathcal{L}_X\mathcal{L}_Y t - \mathcal{L}_Y\mathcal{L}_X t$$

Exercise 13. Consider $M = \mathbb{R}^2$, and the 2-form which is $dx\wedge dy$ in Cartesian coordinates. Introduce polar coordinates by

$$(x, y) = (r\cos\theta, r\sin\theta)$$

Then

$$\begin{aligned}
dx &= \cos\theta dr - r\sin\theta d\theta \\
dy &= \sin\theta dr + r\cos\theta d\theta
\end{aligned}$$

so

$$
\begin{aligned}
dx \wedge dy &= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= r\cos^2\theta dr \wedge d\theta - r\sin^2\theta d\theta \wedge dr \\
&= r dr \wedge d\theta
\end{aligned}
$$

(Strictly, we are introducing a mapping $f : M \to M$, using polars on the domain and Cartesian coordinates on the codomain, and considering the pullback of the 2-form.)

Exercise 14. Let ξ be a q-form, and η an r-form. Write

$$
\begin{aligned}
\xi &= \xi_{\mu_1 \dots \mu_q} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \\
\eta &= \eta_{\nu_1 \dots \nu_r} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_r}
\end{aligned}
$$

Then

$$
\begin{aligned}
\xi \wedge \eta &= \xi_{\mu_1 \dots \mu_q} \eta_{\nu_1 \dots \nu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_r} \\
&= \eta_{\nu_1 \dots \nu_r} \xi_{\mu_1 \dots \mu_q} (-1)^{qr} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_r} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \\
&= (-1)^{qr} \eta \wedge \xi
\end{aligned}
$$

In particular, if $\eta = \xi$, this is

$$\xi \wedge \xi = (-1)^{q^2} \xi \wedge \xi$$

so $\xi \wedge \xi = 0$ if q^2 is odd. Therefore $\xi \wedge \xi = 0$ if q is odd.
To prove associativity we will first prove a lemma. Introduce an s-form ω, and tensors a, b, c such that $\xi = \pi(a)$, $\eta = \pi(b)$ and $\omega = \pi(c)$, where π is the projection from the space of tensors to the space of forms, i.e.

$$\pi(a) = \frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{sgn}\sigma(a)$$

where $\sigma(a)$ is a with indices permuted. Now we claim that

$$\pi(\pi(a) \otimes b) = \pi(a \otimes b)$$

To see this, note first that

$$
\begin{aligned}
\pi(a) \otimes b &= \left(\frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{sgn}(\sigma)\sigma(a) \right) \otimes b \\
&= \frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{sgn}(\sigma)\sigma(a) \otimes b
\end{aligned}
$$

Regard S_q as the subset of S_{q+r} leaving the last r indices fixed. Then this is

$$\pi(a) \otimes b = \frac{1}{q!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma)\sigma(a \otimes b)$$

Then,

$$\begin{aligned}
\pi(\pi(a) \otimes b) = \pi \left(\frac{1}{q!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma)\sigma(a \otimes b) \right) \\
= \frac{1}{q!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma)\pi(\sigma(a \otimes b))
\end{aligned}$$

Now,

$$\pi(\sigma(a \otimes b)) = \frac{1}{(q+r)!} \sum_{\sigma' \in S_{q+r}} \operatorname{sgn}(\sigma')\sigma'(\sigma(a \otimes b))$$

Define $\sigma'' = \sigma'\sigma$. Then $\operatorname{sgn}(\sigma'') = \operatorname{sgn}(\sigma')\operatorname{sgn}(\sigma)$, since sgn is a homomorphism. Then

$$\pi(\sigma(a \otimes b)) = \frac{1}{(q+r)!} \sum_{\sigma'' \in S_{q+r}} \frac{\operatorname{sgn}(\sigma'')}{\operatorname{sgn}(\sigma)} \sigma''(a \otimes b)$$

So

$$\begin{aligned}
\pi(\pi(a) \otimes b) &= \frac{1}{q!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) \frac{1}{(q+r)!} \sum_{\sigma'' \in S_{q+r}} \frac{\operatorname{sgn}(\sigma'')}{\operatorname{sgn}(\sigma)} \sigma''(a \otimes b) \\
&= \frac{1}{q!(q+r)!} \sum_{\sigma \in S_q} \sum_{\sigma' \in S_{q+r}} \operatorname{sgn}(\sigma')\sigma'(a \otimes b) \\
&= \frac{1}{(q+r)!} \sum_{\sigma' \in S_{q+r}} \operatorname{sgn}(\sigma')\sigma'(a \otimes b) \\
&= \pi(a \otimes b)
\end{aligned}$$

So we have the lemma. Then,

$$\begin{aligned}
(\xi \wedge \eta) \wedge \omega &= \pi(\pi(a \otimes b) \otimes c) \\
&= \pi(a \otimes b \otimes c) \\
&= \pi(a \otimes \pi(b \otimes c)) \\
&= \xi \wedge (\eta \wedge \omega)
\end{aligned}$$

Exercise 15. Let ξ and ω be q- and r-forms, respectively. Write

$$\begin{aligned}\xi &= \xi_{\mu_1 \ldots \mu_q} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \\ \omega &= \omega_{\nu_1 \ldots \nu_r} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_q}\end{aligned}$$

Then

$$\begin{aligned}d(\xi \wedge \omega) &= d\left(\xi_{\mu_1 \ldots \mu_q} \omega_{\nu_1 \ldots \nu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_q}\right) \\ &= \frac{\partial}{\partial x^\rho}\left(\xi_{\mu_1 \ldots \mu_q} \omega_{\nu_1 \ldots \nu_r}\right) dx^\rho \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_q} \\ &= \frac{\partial}{\partial x^\rho}(\xi_{\mu_1 \ldots \mu_q}) \omega_{\nu_1 \ldots \nu_r} dx^\rho \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_q} \\ &\quad + \xi_{\mu_1 \ldots \mu_q} \frac{\partial}{\partial x^\rho}(\omega_{\nu_1 \ldots \nu_r})(-1)^q dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \wedge dx^\rho \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_r} \\ &= d\xi \wedge \omega + (-1)^q \xi \wedge d\omega\end{aligned}$$

Exercise 16. Let ξ, ω be r-forms on N, and $f : M \to N$ a smooth map. Then we have r-forms $f^*\xi$ and $f^*\omega$ on M. Consider the case $r = 0$. Then $d\omega$ is a 1-form, and we have

$$f^*(d\omega) = d(\omega \circ f) = d(f^*\omega)$$

Now let $r > 0$, and write $\omega = \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1 \cdots \mu_r}$. Then

$$\begin{aligned}f^*\omega &= f^*(\omega_{\mu_1 \ldots \mu_r}) f^*(dx^{\mu_1}) \wedge \ldots \wedge f^*(dx^{\mu_r}) \\ &= (\omega_{\mu_1 \ldots \mu_r} \circ f) f^*(dx^{\mu_1}) \wedge \ldots \wedge f^*(dx^{\mu_r})\end{aligned}$$

Then from the result on 1-forms we know that $d(f^*(dx^\mu)) = f^*(d^2 x^\mu) = 0$, so taking the exterior derivative we just have

$$\begin{aligned}d(f^*\omega) &= d(\omega_{\mu_1 \ldots \mu_r} \circ f) f^*(dx^{\mu_1}) \wedge \ldots \wedge f^*(dx^{\mu_r}) \\ &= f^*(d\omega_{\mu_1 \ldots \mu_r}) f^*(dx^{\mu_1}) \wedge \ldots \wedge f^*(dx^{\mu_r}) \\ &= f^*(d\omega)\end{aligned}$$

Introduce tensors a and b such that $\xi = \pi(a)$ and $\omega = \pi(b)$, where π is the projection from tensors to forms. Then

$$\begin{aligned}\xi \wedge \omega &= \pi(a \otimes b) \\ f^*(\xi \wedge \omega) &= \pi(f^*(a \otimes b)) \\ &= \pi(f^* a \otimes f^* b) \\ &= f^*\xi \wedge f^*\omega\end{aligned}$$

Exercise 17.

(i) There is an error in the text: the identity we have to prove is

$$i_{[X,Y]}\omega = \mathcal{L}_X i_Y \omega - i_Y \mathcal{L}_X \omega$$

Now, suppose ω is a 1-form, and write $\omega = \omega_\mu dx^\mu$. We have

$$\begin{aligned} i_Y\omega &= Y^\mu \omega_\mu \mathcal{L}_X i_Y \omega \qquad\qquad = X^\nu \frac{\partial}{\partial x^\nu}(Y^\mu \omega_\mu) \\ &= X^\nu \left(\frac{\partial Y^\mu}{\partial x^\nu}\omega_\mu + Y^\mu \frac{\partial \omega_\mu}{\partial x^\nu} \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_X\omega &= \mathcal{L}_X(\omega_\mu)dx^\mu + \omega_\mu \mathcal{L}_X dx^\mu \\ &= X^\nu \frac{\partial \omega_\mu}{\partial x^\nu} dx^\mu + \omega_\mu \frac{\partial X^\mu}{\partial x^\nu} dx^\nu \\ i_Y \mathcal{L}_X \omega &= Y^\mu X^\nu \frac{\partial \omega_\mu}{\partial x^\nu} + Y^\nu \omega_\mu \frac{\partial X^\mu}{\partial x^\nu} \end{aligned}$$

So

$$\mathcal{L}_X i_Y \omega - i_Y \mathcal{L}_X \omega = \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right) \omega_\mu$$

We also have

$$\begin{aligned} i_{[X,Y]}\omega &= [X,Y]^\mu \omega_\mu \\ &= \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right) \omega_\mu \end{aligned}$$

Therefore on 1-forms we have the identity.
Now, suppose we have the identity for $(n-1)$-forms, and let ω be such a form, and α a 1-form, which therefore also obeys the identity. Then

$$\begin{aligned} i_{[X,Y]}(\omega \wedge \alpha) &= i_{[X,Y]}\omega \wedge \alpha + (-1)^{n-1}\omega \wedge i_{[X,Y]}\alpha \\ &= (\mathcal{L}_X i_Y \omega - i_Y \mathcal{L}_X \omega) \wedge \alpha + (-1)^{n-1} \omega \wedge (\mathcal{L}_X i_I \alpha - i_Y \mathcal{L}_X \alpha) \end{aligned}$$

We also have

$$\begin{aligned} \mathcal{L}_X i_Y(\omega \wedge \alpha) &= \mathcal{L}_X(i_Y\omega \wedge \alpha + (-1)^{n-1}\omega \wedge i_Y \alpha) \\ &= \mathcal{L}_X i_Y \omega \wedge \alpha + i_Y \omega \wedge \mathcal{L}_X \alpha + (-1)^n (\mathcal{L}_X \omega \wedge i_Y \alpha + \omega \wedge \mathcal{L}_X i_Y \alpha) \end{aligned}$$

and

$$
\begin{aligned}
i_Y \mathcal{L}_X(\omega \wedge \alpha) &= i_Y(\mathcal{L}_X \omega \wedge \alpha + \omega \wedge \mathcal{L}_X \alpha) \\
&= (i_Y \mathcal{L}_X \omega) \wedge \alpha + (-1)^{n-1} \mathcal{L}_X \omega \wedge i_Y \alpha \\
&\quad + i_Y \omega \wedge \mathcal{L}_X \alpha + (-1)^{n-1} \omega \wedge i_Y \mathcal{L}_Y \alpha
\end{aligned}
$$

So

$$
\begin{aligned}
&\mathcal{L}_X i_Y(\omega \wedge \alpha) - i_Y \mathcal{L}_X(\omega \wedge \alpha) \\
&\qquad = \mathcal{L}_X i_Y \omega \wedge \alpha - i_Y \mathcal{L}_X \omega \wedge \alpha + (-1)^{n-1}(\omega \wedge \mathcal{L}_X i_Y \alpha - \omega \wedge i_Y \mathcal{L}_X \alpha) \\
&\qquad = i_{[X,Y]}(\omega \wedge \alpha)
\end{aligned}
$$

So we have the identity for $\omega \wedge \alpha$. Any n-form can be written as a sum of such products, and the interior product is linear, so we have the identity for all n-forms. We have this for $n = 2$, and by induction for all n.

(ii) Let $\omega = \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$ and $\eta = \eta_{\nu_1 \ldots \nu_p} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p}$. Then

$$
\begin{aligned}
i_X(\omega \wedge \eta) &= i_X(\omega_{\mu_1 \ldots \mu_r} \eta_{\nu_1 \ldots \nu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p}) \\
&= \sum_{s=1}^{r} X^{\mu_s} \omega_{\mu_1 \ldots \mu_s \ldots \mu_r} \eta_{\nu_1 \ldots \nu_p} (-1)^{s-1} dx^{\mu_1} \wedge \ldots \wedge \widehat{dx^{\mu_s}} \wedge \ldots \wedge dx^{\mu_r} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p} \\
&\quad + \sum_{s=1}^{p} X^{\nu_s} \omega_{\mu_1 \ldots \mu_r} \eta_{\nu_1 \ldots \nu_s \ldots \nu_p} (-1)^{r+s-1} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \wedge dx^{\nu_1} \wedge \ldots \wedge \widehat{dx^{\nu_s}} \wedge \ldots \wedge dx^{\nu_p} \\
&= i_X \omega \wedge \eta + (-1)^r \omega \wedge i_X \eta
\end{aligned}
$$

(iii) We have

$$
\begin{aligned}
i_X i_X \omega &= i_X \left(\sum_{s=1}^{r} X^{\mu_s} \omega_{\mu_1 \ldots \mu_s \ldots \mu_r} (-1)^{s-1} dx^{\mu_1} \wedge \ldots \wedge \widehat{dx^{\mu_s}} \wedge \ldots \wedge dx^{\mu_r} \right) \\
&= \sum_{s,t<s} X^{\mu_t} X^{\mu_s} \omega_{\mu_1 \ldots \mu_t \ldots \mu_s \ldots \mu_r} (-1)^{t-1} (-1)^{s-1} dx^{\mu_1} \wedge \ldots \wedge \widehat{dx^{\mu_t}} \wedge \ldots \wedge \widehat{dx^{\mu_s}} \wedge \ldots \wedge dx^{\mu_r} \\
&\quad + \sum_{s,t>s} X^{\mu_s} X^{\mu_t} \omega_{\mu_1 \ldots \mu_s \ldots \mu_t \ldots \mu_r} (-1)^{s-1} (-1)^{t} dx^{\mu_1} \wedge \ldots \wedge \widehat{dx^{\mu_s}} \wedge \ldots \wedge \widehat{dx^{\mu_t}} \wedge \ldots \wedge dx^{\mu_r}
\end{aligned}
$$

Redefining dummy indices, the two terms differ by a sign, and therefore cancel, so

$$
i_X i_X \omega = 0
$$

(iv) Using this result and Cartan's magic formula, we have

$$\begin{aligned}\mathcal{L}_X i_Y \omega &= d i_X i_X \omega + i_X d i_X \omega \\ &= i_X d i_X \omega\end{aligned}$$

and

$$\begin{aligned}i_X \mathcal{L}_X \omega &= i_X d i_X \omega + i_X i_X d\omega \\ &= i_X d i_X \omega\end{aligned}$$

so

$$\mathcal{L}_X i_X \omega = i_X \mathcal{L}_X \omega$$

Exercise 18. Let t be a type (n, m) tensor, and write

$$t = t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_n}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}$$

Then

$$\begin{aligned}\mathcal{L}_X(t) = \mathcal{L}_X(t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m}) &\frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_n}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m} \\ &+ t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m} \mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_n}} \right) dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m} \\ &+ t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_n}} \mathcal{L}_X (dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m})\end{aligned}$$

First, we have

$$\mathcal{L}_X(t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m}) = X^\lambda \frac{\partial}{\partial x^\lambda} t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m}$$

so

$$\mathcal{L}_X(t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m}) \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_n}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m} = X^\lambda \frac{\partial}{\partial x^\lambda} t$$

Second,

$$\begin{aligned}\mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_n}} \right) &= \sum_{s=1}^{n} \mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_s}} \right) \frac{\partial}{\partial x^{\mu_1}} \cdots \widehat{\frac{\partial}{\partial x^{\mu_s}}} \cdots \frac{\partial}{\partial x^{\mu_n}} \\ &= \sum_{s=1}^{n} \left(-\frac{\partial X^\lambda}{\partial x^{\mu_s}} \right) \frac{\partial}{\partial x^{\mu_1}} \cdots \widehat{\frac{\partial}{\partial x^{\mu_s}}} \cdots \frac{\partial}{\partial x^{\mu_n}}\end{aligned}$$

so

$$t^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_m} \mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_n}} \right) = -\sum_{s=1}^{n} t^{\mu_1 \cdots \mu_s \cdots \mu_n}_{\nu_1 \dots \nu_m} \frac{\partial X^\lambda}{\partial x^{\mu_s}} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^{\mu_1}} \cdots \widehat{\frac{\partial}{\partial x^{\mu_s}}} \cdots \frac{\partial}{\partial x^{\mu_n}}$$

Swapping dummy indices $\mu_s \leftrightarrow \lambda$, this is

$$t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} \right) = - \sum_{s=1}^{n} t^{\mu_1 \dots \lambda \dots \mu_n}_{\nu_1 \dots \nu_m} \frac{\partial X^{\mu}_s}{\partial x^{\lambda}} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}}$$

so

$$\begin{aligned} &t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} \right) dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m} \\ &\qquad = - \sum_{s=1}^{n} \frac{\partial X^{\mu_s}}{\partial x^{\lambda}} t^{\mu_1 \dots \lambda \dots \mu_n}_{\nu_1 \dots \nu_m} \frac{\partial X^{\mu}_s}{\partial x^{\lambda}} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m} \end{aligned}$$

and hence

$$\left(t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \mathcal{L}_X \left(\frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} \right) dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m} \right)^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \beta_m} = - \sum_{s=1}^{n} \frac{\partial X^{\alpha}_s}{\partial x^{\lambda}} X^{\alpha_s} t^{\alpha_1 \dots \lambda \dots \alpha_n}_{\beta_1 \dots \beta_m}$$

Lastly,

$$\begin{aligned} \mathcal{L}_X(dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}) &= \sum_{s=1}^{m} (-1)^{s-1} \mathcal{L}_X(dx^{\nu_s}) dx^{\nu_1} \wedge \dots \wedge \widehat{dx^{\nu_s}} \wedge \dots \wedge dx^{\nu_m} \\ &= \sum_{s=1}^{m} (-1)^{s-1} \frac{\partial X^{\nu_s}}{\partial x^{\lambda}} dx^{\lambda} \wedge dx^{\nu_1} \wedge \dots \wedge \widehat{dx^{\nu_s}} \wedge \dots \wedge dx^{\nu_m} \end{aligned}$$

so

$$t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \mathcal{L}_X(dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}) = \sum_{s=1}^{m} (-1)^{s-1} t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_s \dots \nu_m} \frac{\partial X^{\nu_s}}{\partial x^{\lambda}} dx^{\lambda} \wedge dx^{\nu_1} \wedge \dots \wedge \widehat{dx^{\nu_s}} \wedge \dots \wedge dx^{\nu_m}$$

Again, swap dummy indices $\nu_s \leftrightarrow \lambda$, so

$$t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \mathcal{L}_X(dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}) = \sum_{s=1}^{m} (-1)^{s-1} t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \lambda \dots \nu_m} \frac{\partial X^{\lambda}}{\partial x^{\nu_s}} dx^{\nu_s} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}$$

so

$$t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} \mathcal{L}_X(dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}) = \sum_{s=1}^{m} (-1)^{s-1} t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \lambda \dots \nu_m} \frac{\partial X^{\lambda}}{\partial x^{\nu_s}} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}$$

and hence

$$\left(t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} \mathcal{L}_X(dx^{\nu_1} \wedge \dots \wedge dx^{\nu_m}) \right)^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \beta_m} = \sum_{s=1}^{n} \frac{\partial X^{\lambda}}{\partial x^{\alpha_s}} X^{\lambda} t^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \lambda \dots \beta_m}$$

Putting everything together, we have that

$$(\mathcal{L}_X t)^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} = X^{\lambda} \frac{\partial}{\partial x^{\lambda}} t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} - \sum_{s=1}^{n} \frac{\partial X^{\mu_s}}{\partial x^{\lambda}} t^{\mu_1 \dots \lambda \dots \mu_s}_{\nu_1 \dots \nu_m} + \sum_{s=1}^{n} \frac{\partial X^{\lambda}}{\partial x^{\nu_s}} t^{\mu_1 \dots \mu_n}_{\nu_1 \dots \lambda \dots \nu_m}$$

Exercise 19.

(a) Consider the manifold $\mathbb{R}^+$. The set $\mathbb{R}^+$ is closed under multiplication, the identity is 1, and for each $x \in \mathbb{R}^+$ we have $x^{-1} = 1/x \in \mathbb{R}^+$. So $\mathbb{R}^+$ is also a group, and hence a Lie group.

(b) Consider the manifold $\mathbb{R}$. The set $\mathbb{R}$ is closed under multiplication, the identity is 0, and for each $x \in \mathbb{R}$ we have$x^{-1} = -x \in \mathbb{R}$. So $\mathbb{R}$ is also a group, and hence a Lie group.

(c) Consider the manifold $\mathbb{R}^2$. The set $\mathbb{R}^2$ is closed under componentwise addition, the identity is $(0, 0)$, and for each $(x, y) \in \mathbb{R}$ we have $(x, y)^{-1} = (-x, -y) \in \mathbb{R}$. So $\mathbb{R}$ is also a group, and hence a Lie group.

Exercise 20. Consider the matrix group

$$O(1,3) = \{M \in GL(4;\mathbb{R}) \mid M\eta M^T = \eta\}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. If $M \in O(1, 3)$,

$$\begin{aligned} \det\bigl(M\eta M^T\bigr) &= \det\eta \\ \Rightarrow \det M &= \pm 1 \end{aligned}$$

Furthermore, we have

$$\begin{aligned} M_{0i}\eta_{ij}(M^T)_{j0} = \eta_{00} = -1 \\ -(M_{00})^2 + \sum_{i=1}^{3}(M_{ii})^2 = -1 \\ (M_{00})^2 = 1 + \sum_{i=1}^{3}(M_{ii})^2 \geq 1 \end{aligned}$$

so either $M_{00} \geq 1$ or $M_{00} \leq -1$. Overall then, $O(1, 3)$ has four connected components.

$$\begin{aligned} O_+^{\uparrow}(1,3) &= \{M \in O(1,3) \mid \det M = 1, M_{00} \geq 1\} \\ O_+^{\downarrow}(1,3) &= \{M \in O(1,3) \mid \det M = 1, M_{00} \leq -1\} \\ O_-^{\uparrow}(1,3) &= \{M \in O(1,3) \mid \det M = -1, M_{00} \geq 1\} \\ O_-^{\downarrow}(1,3) &= \{M \in O(1,3) \mid \det M = -1, M_{00} \leq -1\} \end{aligned}$$

Consider $O_+^\uparrow$, the proper orthochronous group. A subgroup describes Lorentz transformations in a single direction, parameterised by $v/c \in (-1,+1)$. But any open interval is homeomorphic to all of $\mathbb{R}$, so so is this subgroup. Then it is not compact, and hence neither is the proper orthochronous group, and indeed the full Lorentz group.

Exercise 21. Let X be a left-invariant vector field on G. We have

$$(L_a)_* X \mid_g = X^\mu(g) \frac{\partial x^\nu(ag)}{\partial x^\mu(g)} \frac{\partial}{\partial x^\nu} \mid_{ag}$$

and

$$X \mid_{ag} = X^\nu(ag) \frac{\partial}{\partial x^\nu} \mid_{ag}$$

so

$$X^\mu(g) \frac{\partial x^\nu(ag)}{\partial x^\mu(g)} \frac{\partial}{\partial x^\nu} \mid_{ag} = X^\nu(ag) \frac{\partial}{\partial x^\nu} \mid_{ag}$$

Exercise 22. Let $c : (-\varepsilon, +\varepsilon) \to SO(3)$ be a curve defined by

$$c(s) = \begin{pmatrix} \cos s & -\sin s & \\ \sin s & \cos s & \\ & & -1 \end{pmatrix}$$

Then

$$c'(s) = \begin{pmatrix} -\sin s & -\cos s & \\ \cos s & -\sin s & \\ & & 0 \end{pmatrix}$$

so the tangent vector to c at I_3 is

$$c'(s) = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$$

As expected this is antisymmetric.

Exercise 23.

(a) The Lie bracket is antisymmetric, so $c_{\mu\nu}{}^\lambda = -c_{\nu\mu}{}^\lambda$.

(b) The Jacobi identity on the Lie bracket is

$$\begin{aligned} [[X_\mu, X_\nu], X_\lambda] + [[X_\nu, X_\lambda], X_\mu] + [[X_\lambda, X_\mu], X_\nu] &= 0 \\ [c_{\mu\nu}{}^\rho X_\rho, X_\lambda] + [c_{\nu\lambda}{}^\rho X_\rho, X_\mu] + [c_{\lambda\mu}{}^\rho X_\rho, X_\nu] &= 0 \\ c_{\mu\nu}{}^\rho c_{\rho\lambda}{}^\sigma X_\sigma + c_{\nu\lambda}{}^\rho c_{\rho\mu}{}^\sigma X_\sigma + c_{\lambda\mu}{}^\rho c_{\rho\nu}{}^\sigma X_\sigma &= 0 \end{aligned}$$

for all X_μ, so

$$c_{\mu\nu}{}^{\rho}c_{\rho\lambda}{}^{\sigma} + c_{\nu\lambda}{}^{\rho}c_{\rho\mu}{}^{\sigma} + c_{\lambda\mu}{}^{\rho}c_{\rho\nu}{}^{\sigma} = 0$$

Exercise 24.

(a) Consider

$$A = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

and define $X(x) = \sigma_\mu x^\mu$. Then we have the action

$$\begin{aligned} \sigma(A,x) &= AX(x)A^\dagger \\ &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \\ &= \begin{pmatrix} x^0+x^3 & e^{-i\theta}(x^1-ix^2) \\ e^{i\theta}(x^1+ix^2) & x^0-x^3 \end{pmatrix} \end{aligned}$$

So

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} x^0 \\ \cos\theta x^1 - \sin\theta x^2 \\ \cos\theta x^2 + \sin\theta x^1 \\ x^3 \end{pmatrix}$$

This is a rotation around the z-axis by θ.

(b) Consider

$$A = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix}$$

We have

$$\begin{aligned} \sigma(A,x) &= AX(x)A^\dagger \\ &= \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{\alpha}(x^0+x^3) & x^1-ix^2 \\ x^1+ix^2 & e^{-\alpha}x^0-x^3 \end{pmatrix} \end{aligned}$$

so

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(e^{\alpha}(x^0+x^3) + e^{-\alpha}(x^0-x^3)) \\ x^1 \\ x^2 \\ \frac{1}{2}(e^{\alpha}(x^0+x^3) - e^{-\alpha}(x^0-x^3)) \end{pmatrix} = \begin{pmatrix} \cosh\alpha x^0 + \sinh\alpha x^3 \\ x^1 \\ x^2 \\ \sinh\alpha x^0 + \cosh\alpha x^3 \end{pmatrix}$$

This is a boost along the x^3-axis with velocity $v = \tanh\alpha$.

Exercise 25. For any $p_1, p_2 \in G$, $R_g p_1 = p_2$ for $g = p_2^{-1} p_1 \in G$, and $L_g p_1 = p_2$ for $g = p_2 p_1^{-1} \in G$, so R_g and L_g are transitive. Then, $R_g p = pg = p$ iff $g = e$, and $L_g p = gp = p$ iff $g = e$, so they are also free actions.

Exercise 26. Let $\sigma : G \times M \to M$ be a free action. Then $\sigma(g, p) = p$ iff $g = e$, so $H(p) = \{e\}$.

Exercise 27. Consider the usual action of $SO(2)$ on $\mathbb{R}^2$. Let

$$V = \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix} \in \mathfrak{so}(2)$$

(a) Notice that $V^2 = -I$. Then

$$\begin{aligned}
\exp\{tV\} &= I + tV - \frac{1}{2}t^2 I - \frac{1}{3!}t^3 V + ... \\
&= \left(1 - \frac{1}{2}t^2 + ...\right) I + \left(t - \frac{1}{3!}t^3 + ...\right) V \\
&= \cos tI + \sin tV \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}
\end{aligned}$$

So the induced flow is

$$\begin{aligned}
\sigma(t, \mathbf{x}) &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x\cos t - y \sin t \\ x \sin t + y \cos t \end{pmatrix}
\end{aligned}$$

(b) Then,

$$\begin{aligned}
V^\# \mid_{\mathbf{x}} &= \frac{d}{dt} \begin{pmatrix} x\cos t - y \sin t \\ x \sin t + y \cos t \end{pmatrix} \mid_{t=0} \\
&= \begin{pmatrix} -x\sin t - y \cos t \\ x \cos t - y \sin t \end{pmatrix} \mid_{t=0} \\
&= \begin{pmatrix} -y \\ x \end{pmatrix}
\end{aligned}$$

i.e.

$$V^\# = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Exercise 28. We have

$$\mathrm{ad}_a(g_1g_2) = ag_1g_2a^{-1} = ag_1a^{-1}ag_2a^{-1} = \mathrm{ad}_a(g_1)\,\mathrm{ad}_a(g_2)$$

so this is indeed a homomorphism. Define $\sigma : G \times G \to G$ by $\sigma(a, g) = \mathrm{ad}_a\, g$. Then

(i) $\sigma(e, g) = ege^{-1} = g$ for all $g \in G$

(ii) $\sigma(g_1, \sigma(g_2, g_3)) = \sigma(g_1, g_2g_3g_2^{-1}) = g_1g_2g_3g_2^{-1}g_1^{-1} = \sigma(g_1g_2, g_3)$

so σ is an action of G on itself.

2 Problems

Problem 1. An element of the Stiefel manifold $V(m, r)$, where $r \le m$, is a set

$$\{\mathbf{e}_i \in \mathbb{R}^m \mid 1 \le i \le r, \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\}$$

We can represent these by $m \times r$ matrices $(\mathbf{e}_1, ..., \mathbf{e}_r)$. Since $SO(m)$ can be used to relate any two orthonormal bases of $\mathbb{R}^m$ (unit determinant preserves normality), it is in particular transitive on $V(m; r)$. Let

$$A_0 = \begin{pmatrix} 1 & 0 & & & 0 \\ 0 & 1 & & & \\ & & \cdot & & \\ & & & 1 & 0 \\ & & & 0 & 1 \\ & & & & 0 \\ & & & & \cdot \\ 0 & & & & 0 \end{pmatrix}$$

An element of $SO(m)$ leaving this invariant must be the identity on the upper $r \times r$ component, with no restriction on the action on the lower $(m - r) \times r$ component. This means it is the identity in its top-left $r \times r$ component, and an element of $SO(m - r)$ in its bottom-right $(m - r) \times (m - r)$ component (the top-left component contributes a factor 1 to the determinant). Therefore $H(A_0) = SO(m - r)$. Thus we have

$$V(m, r) \cong SO(m)/SO(m - r)$$

Indeed, when $r = 1$, $V(m,1) \cong S^{m-1}$ and this agrees with the result established earlier. Then the dimension of the Stiefel manifold is

$$\begin{aligned}\dim V(m,r) = \frac{1}{2}m(m-1) - \frac{1}{2}(m-r)(m-r-1) \\ &= mr - \frac{1}{2}r(r+1) \\ &= \frac{1}{2}r(r-1) + r(m-r)\end{aligned}$$

Problem 2.

(a) The electromagnetic vector potential A is the 1-form $A = A_\mu dx^\mu$, where $A_\mu = (-\phi, \mathbf{A})$. We have

$$\begin{aligned}F = dA &= d(A_\mu dx^\mu) \\ &= \frac{\partial A_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu \\ &= -\frac{\partial A_0}{\partial x^i} dx^i \wedge dx^0 + \frac{\partial A_j}{\partial x^i} dx^i \wedge dx^j - \frac{\partial A_i}{\partial x^0} dx^0 \wedge dx^i \\ &= \left(-\frac{\partial A_i}{\partial x^0} + \frac{\partial A_0}{\partial x^i}\right) dx^0 \wedge dx^i + \frac{\partial A_j}{\partial x^i} dx^i \wedge dx^j \\ &= E_i dx^0 \wedge dx^i - B_i dx^j \wedge dx^k\end{aligned}$$

where (i,j,k) is an even permutation of $(1,2,3)$ in the second term. Then,

$$\begin{aligned}*(E_i dx^0 \wedge dx^i) &= E_i dx^j \wedge dx^k \\ &= E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2\end{aligned}$$

and

$$*(-B_i dx^j \wedge dx^k) = -B_1 dx^1 \wedge dx^0 - B_2 dx^2 \wedge dx^0 - B_3 dx^3 \wedge dx^0$$

So

$$\begin{aligned}*F = &E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2 \\ &- B_1 dx^1 \wedge dx^0 - B_2 dx^2 \wedge dx^0 - B_3 dx^3 \wedge dx^0\end{aligned}$$

Then

$$
\begin{aligned}
d*F = &\left(\frac{\partial E_1}{\partial x^1}+\frac{\partial E_2}{\partial x^2}+\frac{\partial E_3}{\partial x^3}\right)dx^1\wedge dx^2\wedge dx^3\\
&+\left(-\frac{\partial E_1}{\partial x^0}+\frac{\partial B_2}{\partial x^3}-\frac{\partial B_3}{\partial x^2}\right)dx^0\wedge dx^2\wedge dx^3\\
&+\left(-\frac{\partial E_2}{\partial x^0}-\frac{\partial B_1}{\partial x^3}+\frac{\partial B_3}{\partial x_1}\right)dx^0\wedge dx^3\wedge dx^3\\
&+\left(-\frac{\partial E_3}{\partial x^0}+\frac{\partial B_1}{\partial x^2}-\frac{\partial B_2}{\partial x^1}\right)dx^0\wedge dx^1\wedge dx^2
\end{aligned}
$$

Meanwhile, we have the current 1-form $J=\rho dx^0+j_k dx^k$, so

$$*J=-\rho dx^1\wedge dx^2\wedge dx^3-j_1dx^0\wedge dx^2\wedge dx^3-j_2dx^0\wedge dx^3\wedge dx^1-j_3dx^0\wedge dx^1\wedge dx^2$$

Then if $d*F=-*J$ (metric signature sign compared to the book),

$$
\begin{aligned}
\nabla\cdot\mathbf{E}&=\rho\\
-\frac{\partial\mathbf{E}}{\partial t}+\nabla\times\mathbf{B}&=\mathbf{j}
\end{aligned}
$$

(b) Now, $d(d*F)=0$, so $d*J=0$. We have

$$d*J=\frac{\partial\rho}{\partial x^0}dx^0\wedge dx^1\wedge dx^2\wedge dx^3+\left(\frac{\partial j_1}{\partial x^1}+\frac{\partial j_2}{\partial x^2}+\frac{\partial j_3}{\partial x^3}\right)dx^0\wedge dx^1\wedge dx^2\wedge dx^3$$

so

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0$$

(c) $A=A_\mu dx^\mu=A_0dx^0+A_idx^i$, so

$$*A=-A_0dx^1\wedge dx^2\wedge dx^3-A_idx^0\wedge dx^j\wedge dx^k$$

where (i,j,k) is an even permutation of $(1,2,3)$. Then

$$d*A=\left(-\frac{\partial A_0}{\partial x^0}-\frac{\partial A_1}{\partial x^1}-\frac{\partial A_2}{\partial x^2}-\frac{\partial A_3}{\partial x^3}\right)dx^0\wedge dx^1\wedge dx^2\wedge dx^3$$

Then

$$d*A=0\Longleftrightarrow\partial_\mu A^\mu=0$$