Nakahara - Geometry, Topology and Physics

Chapter 5: De Rham Cohomology Groups

1 Exercises

Exercise 1. Consider $M = \mathbb{R}^3$.

(i) Let $\omega = adx + bdy + cdz$ be a 1-form, and S a surface in $\mathbb{R}^3$, i.e. $\dim S = 2$. Then Stokes' theorem states that

$$\int_S d\omega = \int_C \omega$$

where $C = \partial S$. Now, on the LHS,

$$\begin{aligned} d\omega &= \left(\frac{\partial a}{\partial y}dy + \frac{\partial a}{\partial z}dz\right) \wedge dx + \left(\frac{\partial b}{\partial x}dx + \frac{\partial b}{\partial z}dz\right) \wedge dy + \left(\frac{\partial c}{\partial x}dx + \frac{\partial c}{\partial y}dy\right) \wedge dz \\ &= \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy + \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\right) dy \wedge dz + \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\right) dz \wedge dx \end{aligned}$$

Define the vector $\boldsymbol{\omega} = (a, b, c)$, and the surface element

$$d\boldsymbol{S} = (dy \wedge dz, dz \wedge dx, dx \wedge dy)$$

Then we see that this is simply

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \cdot d\boldsymbol{S}$$

For the RHS, we note that, with $d\boldsymbol{x} = (dx, dy, dz)$,

$$\boldsymbol{\omega} \cdot d\boldsymbol{x} = \omega$$

Therefore, Stokes' theorem in this case reads

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{\omega} \cdot d\boldsymbol{x}$$

(ii) Now consider the 2-form $\psi = \frac{1}{2}\psi_{\mu\nu}dx^\mu dx^\nu$. Stokes' theorem tells us

$$\int_V d\psi = \int_S \psi$$

where $S = \partial V$. On the LHS,

$$\begin{aligned} d\psi &= \frac{1}{2}\frac{\partial}{\partial x^\lambda}\psi_{\mu\nu}dx^\lambda \wedge dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\frac{\partial \psi_{\mu\nu}}{\partial x^\lambda}\varepsilon^{\mu\nu\lambda}dx^1 \wedge dx^2 \wedge dx^3 \\ &= \frac{1}{2}\frac{\partial \psi^\lambda}{\partial x^\lambda}dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

where $\psi^\lambda = \varepsilon^{\lambda\mu\nu}\psi_{\mu\nu}$. Then regarding this as the index notation form of a vector $\boldsymbol{\psi}$, this is

$$d\psi = \frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{\psi}dx^1 \wedge dx^2 \wedge dx^3$$

So the LHS is

$$\frac{1}{2}\int_V \boldsymbol{\nabla}\cdot\boldsymbol{\psi}\ dV$$

On the RHS, we see that

$$\begin{aligned} \psi &= \frac{1}{2}(\psi_{12} - \psi_{21})dx \wedge dy + \frac{1}{2}(\psi_{23} - \psi_{32})dy \wedge dz + \frac{1}{2}(\psi_{31} - \psi_{13})dz \wedge dx \\ &= \frac{1}{2}\psi^3 dx \wedge dy + \frac{1}{2}\psi^1 dy \wedge dz + \frac{1}{2}\psi^2 dz \wedge dx \\ &= \frac{1}{2}\boldsymbol{\psi}\cdot d\boldsymbol{S} \end{aligned}$$

Thus, Stokes' theorem here states

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{\psi}\ dV = \oint_S \boldsymbol{\psi}\cdot d\boldsymbol{S}$$

Exercise 2.

(a) Let $\omega \in Z^r(M)$ and $\psi \in Z^s(M)$. Then $d\omega = 0$ and $d\psi = 0$. Consider $\omega\wedge\psi \in \Omega^{r+s}(M)$. We have

$$\begin{aligned} d(\omega \wedge \psi) &= d\omega \wedge \psi + (-1)^r \omega \wedge d\psi \\ &= 0 \end{aligned}$$

so $\omega \wedge \psi \in Z^{r+s}(M)$.

(b) Let $\omega \in Z^r(M)$ and $\psi =\in B^s(M)$. Then $d\omega = 0$ and there exists some $\alpha \in \Omega^{s-1}(M)$ such that $\psi = d\alpha$. Consider $\omega \wedge \psi \in \Omega^{r+s}(M)$. We have

$$\begin{aligned}\omega \wedge \psi &= \omega \wedge d\alpha \\ &= (-1)^r(d\omega \wedge \alpha + (-1)^r \omega \wedge d\alpha) \\ &= (-1)^r d(\omega \wedge \alpha) \\ &= d((-1)^r \omega \wedge \alpha)\end{aligned}$$

so $\omega \wedge \psi \in B^{r+s}(M)$.

(c) Let $\omega \in B^r(M)$ and $\psi \in B^s(M)$. Then there exist some $\alpha \in \Omega^{r-1}(M)$ and $\beta \in \Omega^{s-1}(M)$ such that $\omega = d\alpha$ and $\psi = d\beta$. Consider $\omega \wedge \psi \in \Omega^{r+s}(M)$. We have

$$\begin{aligned}\omega \wedge \psi &= d\alpha \wedge d\beta \\ &= (-1)^r(d^2\alpha \wedge \beta + (-1)^r d\alpha \wedge d\beta) \\ &= (-1)^r d(d\alpha \wedge \beta) \\ &= d((-1)^r d\alpha \wedge \beta)\end{aligned}$$

so $\omega \wedge \psi \in B^{r+s}(M)$.

Exercise 3. Let $M = \mathbb{R}^2 \backslash \{0\}$, and define $\omega \in \Omega^1(M)$ by

$$\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

(a) We have

$$\begin{aligned}d\omega &= d\left(\frac{-y}{x^2+y^2}\right) \wedge dx + d\left(\frac{x}{x^2+y^2}\right) \wedge dy \\ &= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) dy \wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy \\ &= \left(\frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right) dy \wedge dx + \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right) dx \wedge dy \\ &= \left(\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right) dx \wedge dy \\ &= 0\end{aligned}$$

so ω is closed.

(b) Define $F(x, y)$ by

$$F(x, y) = \arctan \frac{y}{x}$$

Then

$$
\begin{aligned}
dF &= \frac{\partial}{\partial x}\arctan\frac{y}{x}dx + \frac{\partial}{\partial y}\arctan\frac{y}{x}dy \\
&= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \\
&= \omega
\end{aligned}
$$

However, F is not defined on the submanifold defined by $x = 0$, so it is not a smooth function on all of M, and therefore ω is not exact.

Exercise 4.

(i) Let $c \in B_r(M)$ and $\omega \in Z^r(M)$. Then $c = \partial b$ for some $b \in C_{r+1}(M)$, so

$$
\begin{aligned}
(c, \omega) &= (\partial b, \omega) \\
&= (c, d\omega) \\
&= 0
\end{aligned}
$$

(ii) Let $c \in Z_r(M)$ and $\omega \in B^r(M)$. Then $\omega = d\alpha$ for some $\alpha \in \Omega^{r-1}(M)$, so

$$
\begin{aligned}
(c, \omega) &= (c, d\alpha) \\
&= (\partial c, \alpha) \\
&= 0
\end{aligned}
$$

Exercise 5. Let $M = M_1 \times M_2$. We have

$$
\begin{aligned}
\chi(M) &= \sum_{r=0}^{m}(-1)^r b^r(M) \\
&= \sum_{r=0}^{m}\sum_{p+q=r}(-1)^p b^p(M_1)(-1)^q b^q(M_2) \\
&= \sum_{p=0}^{m_1}(-1)^p b^p(M_1)\sum_{q=0}^{m_2}(-1)^q b^q(M_2) \\
&= \chi(M_1)\chi(M_2)
\end{aligned}
$$