Nakahara - Geometry, Topology and Physics

Chapter 3: Homology Groups

1 Supplementary Notes

[Section 3.4.2] It is instructive to compare homology groups of the cylinder, Möbius strip, torus, and Klein bottle. We have

	H_0	H_1	H_2
Cylinder	$\mathbb{Z}$	$\mathbb{Z}$	$\{0\}$
Torus	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$
Möbius strip	$\mathbb{Z}$	$\mathbb{Z}$	$\{0\}$
Klein bottle	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\{0\}$

We know that homology basically counts holes, and that finite factors (as in the first homology group of the Klein bottle) indicate some sort of twisting, and are together known as the torsion subgroup. But there is no such indication of twisting in the homology of the Möbius strip, which in fact is identical to that of the cylinder. Indeed it is easy to see by thinking about possible 1-cycles that this should be the case. Where we see torsion proper is when we glue the edges to each other: in the case of the cylinder, we produce the torus, which exhibits no torsion; however, in the case of the Möbius strip, we get the Klein bottle, which does. To see this, consider a 1-cycle around the handle of the Klein bottle. We can smoothly translate this along the handle, and all the way around - but then we have reversed its direction. In other words, if this 1-cycle is z, then $z \sim (-z)$, and hence $[2nz] \sim 0$, for $n \in \mathbb{Z}$. Thus we have the $\mathbb{Z}_2$ factor. The torsion shows up when we glue the ends of the Möbius strip to close the new hole.

2 Exercises

Exercise 1. Let $K = \{p_0, p_1\}$. Then $Z_0(K) = C_0(K) = \{ip_0 + jp_1 \mid i, j \in \mathbb{Z}\} = \mathbb{Z} \oplus \mathbb{Z}$. Similarly neither p_0 nor p_1 is a boundary of anything, so $B_0(K) = \{0\}$. Thus

$$H_0(K) = C_0(K)/\{0\} = C_0(K) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Exercise 2. Let $K = \{p_0, p_1, p_2, p_3, (p_0p_1), (p_1p_2), (p_2p_3), (p_3p_0)\}$. This is a square, so is another triangulation of S^1. Therefore we expect the same homology groups as in the previous example. $H_1(K) = Z_1(K)$. Let $z = i(p_0p_1) + j(p_1p_2) + k(p_2p_3) + l(p_3p_0)$. Then

$$\partial_1 z = (l - i)p_0 + (i - j)p_1 + (j - k)p_2 + (k - i)p_3 = 0$$

so $i = j = k = l$, and hence $Z_1(K) \cong \mathbb{Z}$, i.e.

$$H_1(K) \cong \mathbb{Z}$$

as expected. Then, $Z_0(K) = C_0(K)$, and

$$B_0(K) = \{(l - i)p_0 + (i - j)p_1 + (j - k)p_2 + (k - l)p_3 \mid i, j, k, l \in \mathbb{Z}\}$$

Define $f : Z_0 \to \mathbb{Z}$ by

$$f(ip_0 + jp_1 + kp_2 + lp_3) = i + j + k + l$$

This is surjective and a homomorphism, and $\ker f = B_0(K)$. So

$$\begin{aligned} H_0(K) = Z_0(K)/\ker f &\cong \operatorname{Im} f \\ &\cong \mathbb{Z} \end{aligned}$$

again as expected.

Exercise 3. Let

$$\begin{aligned} K = \{&p_0, p_1, p_2, p_3, (p_0p_1), (p_0p_2), (p_0p_3), (p_1p_2), (p_1p_3), (p_2, p_3), \\ &(p_0p_1p_2), (p_0p_1p_3), (p_0p_2p_3), (p_1p_2p_3)\} \end{aligned}$$

This describes a tetrahedron, which is a triangulation of S^2. Calculating $H_0(K)$ explicitly, we have $Z_0(K) = C_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and

$$\begin{aligned} B_0(K) &= \{i\partial_1(p_0p_1) + j\partial_1(p_0p_2) + k\partial_1(p_0p_3) + l\partial_1(p_1p_2) + m\partial_1(p_1p_3) + n\partial_1(p_2p_3)\} \\ &= \{i(p_1 - p_0) + j(p_2 - p_0) + k(p_3 - p_0) + l(p_2 - p_1) + m(p_3 - p_1) + n(p_3 - p_2)\} \\ &= \{-(i + j + k)p_0 + (i - l - m)p_1 + (j + l - n)p_2 + (k + m + n)p_3\} \\ &= \{ap_0 + bp_1 + cp_2 - (a + b + c)p_3\} \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

so indeed

$$\begin{aligned} H_0(K) &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/Z \oplus \mathbb{Z} \oplus \mathbb{Z} \\ &\cong \mathbb{Z} \end{aligned}$$

Let

$$z_1 = i(p_0p_1) + j(p_0p_2) + k(p_0p_3) + l(p_1p_2) + m(p_1p_3) + n(p_2p_3) \in Z_1(K)$$

Then $\partial_1 z_1 = 0$, so $i + j + k = 0$, $i - l - m = 0$, $j + l - n = 0$, and $k + m + n = 0$. Then we can rewrite any two of the coefficients in terms of the others. Thus $Z_1(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. We also have

$$B_1(K) = \{i\partial_2(p_0p_1p_2) + j\partial_2(p_0p_1p_3) + k\partial_2(p_0p_2p_3) + l\partial_2(p_1p_2p_3)\}$$

and it is easy but slightly longwinded to check that indeed $B_1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Thus

$$H_1(K) \cong \{0\}$$

That is, S^2 has no two-dimensional holes. Then, $B_2(K) = \{0\}$ so $H_2(K) = Z_2(K)$. Let $z_2 = i(p_0p_1p_2) + j(p_0p_1p_3) + k(p_0p_2p_3) + l(p_1p_2p_3) \in Z_2(K)$. Then $\partial_2 z_2 = 0$ implies $i + j = 0$, $i + l = 0$, $i - k = 0$, $j - l = 0$, i.e. $z_2 = i[(p_0p_1p_2) - (p_0p_1p_3) + (p_0p_2p_3) - (p_1p_2p_3)]$, so $Z_2(K) \cong \mathbb{Z}$. Thus

$$H_2(K) \cong \mathbb{Z}$$

There is one three-dimensional hole in S^2.

3 Problems

Problem 1. The most general orientable two-dimensional surface is a 2-sphere with h handles and q holes. We have earlier found that $H_0(S^2) \cong \mathbb{Z}$, $H_1(S^2) \cong \{0\}$, $H_2(S^2) \cong \mathbb{Z}$, in agreement with the observation that S^2 is connected, has no one-dimensional holes, and one two dimensional hole. If we add h handles, we turn S^2 into Σ_h, so the 2-sphere with h handles has $H_1(K) \cong \mathbb{Z} \oplus ... \oplus \mathbb{Z}$. Then if we add q holes, we mean that we add q one-dimensional holes. Neither handles nor (one-dimensional) holes add any three-dimensional holes or change the fact of connectedness, so

$$H_0(K) \cong \mathbb{Z}, \quad H_1(K) \cong \mathbb{Z} \oplus \overset{2h+q}{\cdots} \mathbb{Z}, \quad H_2(K) \cong \mathbb{Z}$$

Therefore,

$$b_0(K) = 1, \quad b_1(K) = 2h + q, \quad b_2(K) = 1$$

and then

$$\begin{aligned} \chi(K) &= 1 - (2h + q) + 1 \\ &= 2 - 2h - q \end{aligned}$$

Problem 2. Consider the 2-sphere with a circular hole, on which opposite points are identified. This hole is called a crosscap. By expanding the hole to the equator, we make the sphere into the disk with opposite points identified, i.e. the real projective plane. Then by comparing the homology groups of S^2 and $\mathbb{R}P^2$, we deduce that if we add q crosscaps to the 2-sphere, the resulting homology groups are

$$H_0(K) \cong \mathbb{Z}, \quad H_1(K) \cong \mathbb{Z}_2 \oplus \overset{q}{\ldots} \oplus \mathbb{Z}_2, \quad H_2(K) \cong \{0\}$$

Therefore,

$$b_0(K) = 1, \quad b_1(K) = 0, \quad b_2(K) = 0$$

and so

$$\chi(K) = 1$$