Nakahara - Geometry, Topology and Physics

Chapter 4: Homotopy Groups

1 Supplementary Notes

Proof of Theorem 4.3.
Let X and Y be topological spaces of the same homotopy type, and $f : X \to Y$ a homotopy equivalence.
If α is a loop in X at x_0, then $f \circ \alpha$ is a loop in Y and $f(x_0)$. Define $P_f : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ by $P_f([\alpha]) = [f \circ \alpha]$. Let g be a homotopy inverse of f, and define $P_g : \pi_1(Y, y_0) \to \pi_1(X, g(y_0))$ by $P_g([\beta]) = [g \circ \beta]$. Then, if $y_0 = f(x_0)$, then $g(y_0)$ is in the same connected component of X as x_0, so $\pi_1(X, g(y_0)) \cong \pi_1(X, x_0)$, by Theorem 4.2, and

$$\begin{aligned} P_g \circ P_f([\alpha]) &= P_g([f \circ \alpha]) \\ &= [g \circ (f \circ \alpha)] \end{aligned}$$

Since $g \circ f \sim \mathrm{id}_X$, $g \circ (f \circ \alpha) \sim \alpha$, so

$$P_g \circ P_f([\alpha]) = [\alpha]$$

i.e. $P_g \circ P_f = \mathrm{id}_{\pi_1(X, x_0)}$. Similarly, using $f \circ g \sim \mathrm{id}_Y$ gives us $P_f \circ P_g = \mathrm{id}_{\pi_1(Y, f(x_0))}$. Therefore P_f has both a left- and right-inverse, and therefore is a bijection.
Next, consider

$$P_f([\alpha] \star [\beta]) = [f \circ (\alpha \star \beta)]$$

But clearly $f \circ (\alpha \star \beta) \sim (f \circ \alpha) \star (f \circ \beta)$, so

$$\begin{aligned} P_f([\alpha] \star [\beta]) &= [f \circ \alpha] \star [f \circ \beta] \\ &= P_f([\alpha]) \star P_f([\beta]) \end{aligned}$$

so P_f is a homomorphism. Therefore P_f is an isomorphism.

Elaboration on Figure 4.8.
Let X be the punctured disk of radius 2, and R the circle of unit radius. Use polar coordinates, and let $f(r, \theta) = (1, \theta)$. This is continuous, with $f|_R = \mathrm{id}_R$. Therefore R is a retract and f a retraction. Indeed, by defining

$$H(r, \theta, t) = ((1 - t)\, r + t, \theta)$$

we have $H(r,\theta,0) = (r,\theta)$, $H(r,\theta,1) = (1,\theta) \in R$, and if $r = 1$, $H(r,\theta,t) = (1,\theta)$. Therefore R is in fact a deformation retract.
Suppose we introduce a hole in the disk, such that it is encircled by the circle. f is still continuous, and hence the circle is still a retract. However, H no longer exists. To see this, realise that if we regard $H(r_0,\theta_0,t)$ as a curve $\alpha_{(r_0,\theta_0)}(t)$ for each $(r_0,\theta_0) \in X$, then it is impossible to define H such that all the $\alpha_{(r_0,\theta_0)}(t)$ encircle the hole the same number of times. Therefore introducing the hole the circle ceases to be a deformation retract.

2 Exercises

Exercise 1.

(i) Let η, ζ be homotopic paths from x_0 to x_1, and $F(s,t)$ the homotopy between them. We have maps P_η defined as in the previous theorem, and P_ζ analogously. Define $F'(s,t) = F(1-s,t)$. Then

$$\begin{aligned} F'(s,0) &= F(1-s,0) = \eta(1-s) = \eta^{-1}(s) \\ F'(s,1) &= F(1-s,1) = \zeta(1-s) = \zeta^{-1}(s) \\ F'(0,t) &= F(1,t) = x_1 \\ F'(1,t) &= F(0,t) = x_0 \end{aligned}$$

so $F'(s,t)$ is a homotopy between η^{-1} and ζ^{-1}. Then we can build a diagram like Figure **??**, but with three cells (from x_1 to x_0 to x_0 to x_1, with homotopies between η^{-1} and ζ^{-1}, α and itself, and η and ζ) to motivate the definition

$$G(s,t) = \begin{cases} F'(3s,t) & 0 \le s \le \frac{1}{3} \\ c_{x_0}(s,t) & \frac{1}{3} \le s \le \frac{2}{3} \\ F(3s-2,t) & \frac{2}{3} \le s \le 1 \end{cases}$$

Then clearly this is a homotopy between $\eta^{-1}\star\alpha\star\eta$ and $\zeta^{-1}\star\alpha\star\zeta$, so $P_\eta([\alpha]) = P_\zeta([\alpha])$ for all loops α at x_0, i.e. $P_\eta = P_\zeta$.

(ii) Let η, ζ be paths such that $\eta(1) = \zeta(0)$. We have

$$[\eta \star \zeta]^{-1} = ([\eta] \star [\zeta])^{-1} = [\zeta]^{-1} \star [\eta]^{-1}$$

Then,

$$\begin{aligned} P_\zeta \circ P_\eta([\alpha]) &= P_\zeta([\eta^{-1} \star \alpha \star \eta]) \\ &= [\zeta^{-1} \star \eta^{-1} \star \alpha \star \eta \star \zeta] \\ &= P_{\eta\star\zeta}([\alpha]) \end{aligned}$$

for any $[\alpha]$, that is, $P_{\eta\star\zeta} = P_\zeta \circ P_\eta$.

Exercise 2. Consider the punctured $n+1$-disk, $D^{n+1}\backslash\{0\}$, and the n-sphere, S^n. Define $f : D^{n+1}\backslash\{0\} \to S^n$ by

$$f(x) = \frac{x}{|x|}$$

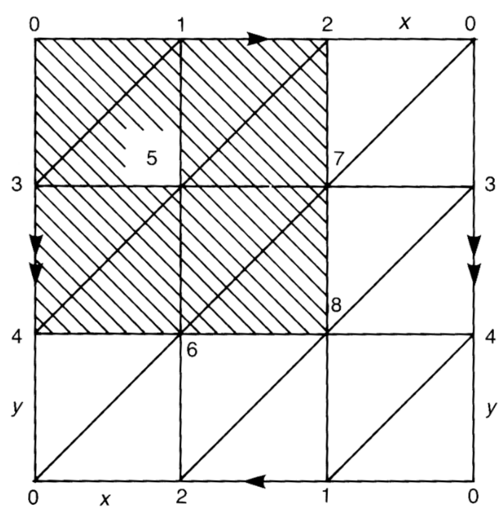
This is a retraction, so S^n is a retract of $D^{n+1}\backslash\{0\}$. Then, define $H : D^{n+1}\backslash\{0\} \times I \to D^{n+1}\backslash\{0\}$ by

$$H(x,t) = \frac{x}{|x|^t}$$

Then, $H(x,0) = x$, $H(x,1) = x/|x|$, and if $|x| = 1$, $H(x,t) = x$ for all t. Therefore S^n is in fact a deformation retract of $D^{n+1}\backslash\{0\}$.

Exercise 3. Let $X = \{a\}$. Then $F[X] = \{a^n \mid n \in \mathbb{Z}\}$, and $a^n a^m = a^{n+m}$. The identity is a^0 and the inverse of a^n is a^{-n}. Define $f : F[X] \to \mathbb{Z}$ by $f(a^n) = n$. Then $f(a^n a^m) = n+m$, so this is a homomorphism if $\mathbb{Z}$ is regarded as an additive group. It is also clearly a bijection. Therefore $F[X] \cong \mathbb{Z}$.

Figure 2.1



Exercise 4. A triangulation of the Klein bottle is shown in Figure 2.1, with L the shaded

region. Let $g_{02} = x$ and $g_{04} = y$. Then we have

$$\begin{aligned}
g_{02}g_{28} &= g_{08} && \Rightarrow \quad g_{08} = x \\
g_{03}g_{38} &= g_{08} && \Rightarrow \quad g_{38} = x \\
g_{38}g_{87} &= g_{37} && \Rightarrow \quad g_{37} = x \\
g_{34}g_{47} &= g_{37} && \Rightarrow \quad g_{47} = x \\
g_{04}g_{41} &= g_{01} && \Rightarrow \quad g_{14} = y \\
g_{47}g_{71} &= g_{41} && \Rightarrow \quad g_{17} = yx \\
g_{12}g_{27} &= g_{17} && \Rightarrow \quad g_{27} = yx \\
g_{26}g_{67} &= g_{27} && \Rightarrow \quad g_{26} = yx \\
g_{02}g_{26} &= g_{06} && \Rightarrow \quad g_{06} = xyx \\
g_{42}g_{46} &= g_{06} && \Rightarrow \quad xyxy^{-1} = 1
\end{aligned}$$

so

$$\pi_1(\text{Klein}) \cong (x, y; xyxy^{-1})$$

Exercise 5. Figure 2.2 is a triangulation of the Möbius strip. The maximal tree is $L = \{\langle v_0v_1\rangle, \langle v_0v_2\rangle, \langle v_1v_2\rangle, \langle v_1v_4\rangle, \langle v_2v_3\rangle, \langle v_2v_4\rangle, \langle v_3v_5\rangle, \langle v_4v_5\rangle\}$. Then let $g_{13} = x$. We have

$$\begin{aligned}
g_{13}g_{35} &= g_{15} && \Rightarrow g_{15} = x \\
g_{01}g_{15} &= g_{05} && \Rightarrow g_{05} = x
\end{aligned}$$

so

$$\pi_1(\text{Möbius}) \cong (x; \emptyset) \cong \mathbb{Z}$$

as expected, since the Möbius strip is of the same homotopy type as S^1 (which is a deformation retract of it).

Exercise 6. Consider $\sim$, the relation of homotopy between n-loops, and n-loops α, β, γ.

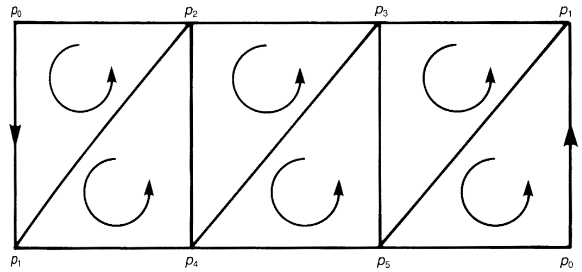
(i) $F(s_1, ..., s_n, t) = \alpha(s)$ is a homotopy $\alpha \sim \alpha$.

(ii) If $F(s_1, ..., s_n, t)$ is a homotopy $\alpha \sim \beta$, then $F(s_1, ..., s_n, 1-t)$ is a homotopy $\beta \sim \alpha$.

(iii) Let $F(s_1, ..., s_n, t)$ be a homotopy $\alpha \sim \beta$ and $G(s_1, ..., s_n, t)$ a homotopy $\beta \sim \gamma$, then define

$$H(s_1, ..., s_n, t) = \begin{cases} F(s_1, ..., s_n, 2t) & 0 \le t \le \frac{1}{2} \\ G(s_1, ..., s_n, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Figure 2.2



Then

$$\begin{aligned}
H(s_1,...,s_n,0) &= F(s_1,...,s_n,0) = \alpha(s_1,...,s_n) \\
H(s_1,...,s_n,1) &= G(s_1,...,s_n,1) = \gamma(s_1,...,s_n) \\
H(0,...,0,t) &= \begin{cases} F(0,...,0,2t) & 0 \le t \le \frac{1}{2} \\ G(0,...,0,2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \\
&= x_0 \\
H(1,...,1,t) &= \begin{cases} F(1,...,1,2t) & 0 \le t \le \frac{1}{2} \\ G(1,...,1,2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \\
&= x_0
\end{aligned}$$

so $H(s_1,...,s_n,t)$ is a homotopy $\alpha \sim \gamma$.

Thus we have an equivalence relation.

Exercise 7. Suppose $\alpha \sim \alpha'$ and $\beta \sim \beta'$ are n-loops at x_0, and $F(s_1,...,s_n,t)$ and $G(s_1,...,s_n,t)$ the corresponding homotopies. Then consider

$$H(s_1,...,s_n,t) = \begin{cases} F(2s_1,s_2,...,s_n,t) & 0 \le s_1 \le \frac{1}{2} \\ G(2s_1-1,s_2,...,s_n,t) & \frac{1}{2} \le s_1 \le 1 \end{cases}$$

Then

$$
\begin{aligned}
H(s_1, ..., s_n, 0) &= \begin{cases} F(2s_1, s_2, ..., s_n, 0) & 0 \le s_1 \le \frac{1}{2} \\ G(2s_1 - 1, s_2, ..., s_n, 0) & \frac{1}{2} \le s_1 \le 1 \end{cases} \\
&= \alpha * \beta(s_1, ..., s_n) \\
H(s_1, ..., s_n, 1) &= \begin{cases} F(2s_1, s_2, ..., s_n, 1) & 0 \le s_1 \le \frac{1}{2} \\ G(2s_1 - 1, s_2, ..., s_n, 1) & \frac{1}{2} \le s_1 \le 1 \end{cases} \\
&= \alpha' * \beta'(s_1, ..., s_n) \\
H(0, ..., 0, t) &= F(0, ..., 0, t) = x_0 \\
H(1, ..., 1, t) &= G(1, ..., 1, t) = x_0
\end{aligned}
$$

so H is a homotopy between $\alpha * \beta$ and $\alpha' * \beta'$, so

$$[\alpha] * [\beta] = [\alpha'] * [\beta']$$

and hence the product on n^{th} homotopy classes is well-defined.

Exercise 8. Proving the n^{th} homotopy group satisfies the group axioms goes exactly the same as for the fundamental group.

3 Problems

Problem 1. Consider $S^n \subset \mathbb{R}^{n+1} \backslash \{0\}$. Define $H : \mathbb{R}^{n+1} \backslash \{0\} \times I \to \mathbb{R}^{n+1} \backslash \{0\}$ by

$$H(x_1, ..., x_{n+1}, t) = (1-t)(x_1, ..., x_{n+1}) + \frac{t}{\sqrt{x_1^2 + ... + x_n^2}}(x_1, ..., x_{n+1})$$

so

$$
\begin{aligned}
H(x_1, ..., x_{n+1}, 0) &= (x_1, ..., x_{n+1}) \\
H(x_1, ..., x_{n+1}, 1) &= \frac{1}{\sqrt{x_1^2 + ... + x_{n+1}^2}}(x_1, ..., x_{n+1}) \in S^n
\end{aligned}
$$

and if $(x_1, ..., x_{n+1}) \in S^n$, then

$$H(x_1, ..., x_{n+1}, t) = (x_1, ..., x_{n+1})$$

Therefore H is a homotopy between $id_{\mathbb{R} \backslash \{0\}}$ and the map

$$f(x_1, ..., x_{n+1}) = \frac{1}{\sqrt{x_1^2 + ... + x_{n+1}^2}}(x_1, ..., x_{n+1})$$

which is continuous, and $f|_{S^n} = \mathrm{id}_{S^n}$, so it is a retraction. Therefore S^n is a deformation retract of $\mathbb{R}^{n+1}$.

Problem 2. Consider D^2 and $\partial D^2 = S^1$. Let $f : D^2 \to D^2$ be smooth, and suppose it has no fixed points. Then by hypothesis for every $p \in D^2$, a straight line can be constructed that goes through both p and $f(p)$. Let $\tilde{f} : D^2 \to S^1$ be the map that takes p to the intersection of this line with the boundary S^1 of D^2 which is closest to p, which we denote q. Then define $F : D^2 \times I \to D^2$ by

$$F(p,t) = (1-t)p + tq$$

Then

$$\begin{aligned} F(p,0) &= p \\ F(p,1) &= q \in S^1 = \partial D^2 \end{aligned}$$

and if $p \in S_1$, then $q = p$, so

$$F(p,t) = p$$

This is a homotopy between id_{D^2} and $\tilde{f}$, and makes S^1 into a deformation retract of D^2. Then $D^2 \simeq S^1$, so $\pi_1(D^2) \cong \pi_1(S^1)$. But we know that $\pi_1(D^2) = \{e\}$ by contractibility, and $\pi_1(S^1) \cong \mathbb{Z}$. Therefore this construction must fail, and f must have some fixed points.

Problem 3.

(i) The obvious map $S^3 \to S^2$ in $0 \in \mathbb{Z} \cong \pi_3(S^2)$ is the constant map $c_{x_0} : x \mapsto x_0$ for all $x \in .S^3$.

(ii) For a map in $1 \in \mathbb{Z} \cong \pi_3(S^2)$, we first need to introduce some maps involving quaternions.
Define $f : \mathbb{R}^4 \to \mathbb{H}$ by

$$f(a,b,c,d) = a + bi + cj + dk$$

This is an isomorphism. Also consider $\mathbb{H}_{pure} = \{bi + cj + dk \mid b,c,d \in \mathbb{R}\}$, i.e. the set of quaternions with no real part. If $\iota : \mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ is the inclusion $\iota(b,c,d) = (0,b,c,d)$, then $g = f \circ \iota$ is a map $\mathbb{R}^3 \to \mathbb{H}_{pure}$, which is also an isomorphism. It can be verified that, for any $r \in \mathbb{H}$, $r(xi + yj + zk)\bar{r} \in \mathbb{H}_{pure}$. Then, for each $r \in \mathbb{H}$, we can define a linear map $R_r : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$R_r(v) = g^{-1}(rg(v)\bar{r})$$

If $|r| = 1$, then $|R_r(v)| = |v|$, so in this case, R_r is a rotation. This is the standard way of identifying quaternions as rotations in $\mathbb{R}^3$. Note that if $|r| = 1$, then $f^{-1}(r) \in S^3 \subset \mathbb{R}^4$. Therefore we can define the map $h : S^3 \to S^2$ by

$$h(x) = R_{f(x)}(v_0)$$

where $v_0 = (1, 0, 0)$. This map takes a unit vector in $\mathbb{R}^4$, identifies it as a quaternion, associates to this quaternion a rotation in $\mathbb{R}^3$, and returns the result of acting with this rotation on $(1, 0, 0) \in \mathbb{R}^3$. Explicitly, if $x = (a, b, c, d)$, then

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ad - bc), 2(bd - ac))$$

This is called the Hopf fibration. By considering the dot product of $h(a, b, c, d)$ with $(1, 0, 0)$, we see that the latter is rotated by an angle θ, where

$$\cos\theta = a^2 + b^2 - c^2 - d^2$$

The RHS takes values continuously between -1 and 1, so θ takes values continuously between 0 and π. Then, by considering the cross product, we see that this rotation is around the vector $(0, ac - bd, ad - bc)$. Thus we see that as h scans S^3, it sweeps S^2 once. So h is in $1 \in \mathbb{Z} \cong \pi_3(S^2)$. Since 1 is the generator of $\mathbb{Z}$, we can then see how to find a map in any homotopy class. If a map $h^{(1)} \in 1 \in \mathbb{Z} \cong \pi_3(S^2)$, then it sweeps S^2 n times as it scans S^3 once. To obtain this, we can define

$$h^{(n)}(x) = R^n_{f(x)}(v_0)$$

Then $h^{(n)} \in n \in \mathbb{Z} \cong \pi_3(S^2)$.