Nakahara - Geometry, Topology and Physics

Chapter 10: Connections on Fibre Bundles

Exercise 1.

(i) Work in some local trivialisation, in which we can put coordinates (x^μ, y^i) on P which split into the factors $M \times G$, i.e. x^μ coordinates on M and y^i coordinates on G. Then if $X \in V_u P$ we can write

$$X = X^i \frac{\partial}{\partial y^i}$$

only. But

$$\pi_* X = X^i \frac{\partial x^\mu}{\partial y^i} \frac{\partial}{\partial x^\mu} = 0$$

(ii) Let $A, B \in \mathfrak{g}$ and define $\phi_t : P \to P$ by $\phi_t(p) = p \exp(tA)$. Then ϕ_t generates $A^\#$, and we have

$$\begin{aligned}
[A^\#, B^\#]_u &= \frac{d}{dt} \phi_{t*}^{-1}(B_{\phi_t(u)}) \Big|_{t=0} \\
&= \frac{d}{dt} \frac{d}{ds} \phi_t(p) \exp(sB) \exp(tA)^{-1} \Big|_{s=t=0} \\
&= \frac{d}{dt} \frac{d}{ds} u \exp(tA) \exp(sB) \exp(tA)^{-1} \Big|_{s=t=0} \\
&= \frac{d}{dt} \frac{d}{ds} u \exp\Big(\exp(tA) sB \exp(tA)^{-1} \Big) \Big|_{s=t=0} \\
&= \frac{d}{dt} \frac{d}{ds} u \exp\big\{ s \operatorname{Ad}_{\exp(tA)} B \big\} \Big|_{s=t=0} \\
&= \frac{d}{ds} u \exp\left\{ s \frac{d}{dt} \left(\operatorname{Ad}_{\exp(tA)} B \right) \right\} \Big|_{s=t=0} \\
&= \frac{d}{ds} u \exp\{ s[A, B] \} \Big|_{s=0} \\
&= [A, B]^*_u
\end{aligned}$$

Exercise 2. Haven't we just done this with $g = t_{ij}$?

Exercise 3.

(i) Let $\tilde{\gamma}$ be the horizontal lift of γ, and $u_0 \in \pi^{-1}(\gamma(0))$. Then we have some $u_1 = \Gamma(\tilde{\gamma})(u_0)$. We then want to construct $u_2 = \Gamma(\tilde{\gamma}^{-1})(u_1)$, where $\gamma^{-1}(t) = \gamma(1-t)$. But by the uniqueness of horizontal lifts of γ through a given point, it must be the case that $\tilde{\gamma}^{-1}(t) = \tilde{\gamma}(1-t)$. Then clearly $u_2 = u_0$, i.e. $\Gamma(\tilde{\gamma}^{-1}) = \Gamma(\tilde{\gamma})^{-1}$.

(ii) Let $\alpha, \beta : [0,1] \to M$ be two curves in M, such that $\alpha(1) = \beta(0)$, so we can define a product by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then the map $\Gamma(\widetilde{\alpha * \beta})$ is defined by the solutions to two separate differential equations, with boundary conditions appropriately matched. This is equivalent to solving the differential equations for $\Gamma(\tilde{\alpha})$ and $\Gamma(\tilde{\beta})$ separately, and matching up their boundary conditions appropriately. Then by the uniqueness theorem of ODEs, we have

$$\Gamma(\widetilde{\alpha * \beta}) = \Gamma(\tilde{\beta}) \circ \Gamma(\tilde{\alpha})$$

Exercise 4. Let $u, v \in P$ satisfy the equivalence relation $u \sim v$ iff they are on the same horizontal lift of a curve γ passing through $\pi(u)$ and $\pi(v)$. Then obviously $u \sim u$, and $u \sim v$ implies $v \sim u$. Lastly, by the uniqueness theorem of ODEs, $u \sim v$ and $v \sim w$ must imply $u \sim w$, that is, if a horizontal lift passes through u and v and another through v and w, they must be identical.

Exercise 5.

(i) Write $t_\alpha(u) = ug_\alpha$, i.e. $\Phi_\alpha\{g_\alpha\}$. Then

$$\begin{aligned} t_\alpha(ug) &= t_\alpha(u)g \\ &= ug_\alpha g \\ &= ugg^{-1}g_\alpha g \\ &= ug \operatorname{ad}_g g_\alpha \end{aligned}$$

So

$$\begin{aligned} \Phi_{ua} &= \{g \in G \mid \tau_\gamma(ua) = uag\} \\ &= \{g \in G \mid ua \operatorname{ad}_a g_\gamma = uag\} \\ &= \{\operatorname{ad}_a g_\gamma\} \\ &= a^{-1}\Phi_u a \end{aligned}$$

Further, as we will use in (iii), since a is invertible,

$$\Phi_u \cong \Phi_{ua}$$

Holonomy groups on the same fibre are isomorphic.

(ii) Let u and u' be on the same horizontal lift $\tilde{\gamma}$. Then $\pi(u)$ and $\pi(u')$ are connected by γ (for simplicity restrict γ to not extend beyond these points). Then for every loop α at $\pi(u)$, we can construct some loop $\beta = \gamma\alpha\gamma^{-1}$ at $\pi(u')$, and vice versa. Then we have

$$\begin{aligned}
\tau_\beta = \Gamma(\beta) &= \Gamma(\gamma\alpha\gamma^{-1}) \\
&= \Gamma(\gamma)\Gamma(\alpha)\Gamma(\gamma^{-1}) \\
&= \Gamma(\gamma)\tau_\alpha\Gamma(\gamma)^{-1} \\
\{\tau_\beta\} &= \Gamma(\gamma)\{\tau_\alpha\}\Gamma(\gamma)^{-1} \\
&\cong \{\tau_\alpha\} \\
\Phi_{u'} &\cong \Phi_u
\end{aligned}$$

Holonomy groups on the same horizontal lift are isomorphic.

(iii) We have already found some cases in which holonomy groups are isomorphic. Suppose neither apply, i.e. that u and u' are not on the same fibre or horizontal lift. Let γ go through $\pi(u)$ and $\pi(u')$, which is always possible if M is connected (or more generally if $\pi(u)$ and $\pi(u')$ are in the same connected component of M) and $\tilde{\gamma}$ the lift through u. Then it also goes through some u'' such that $\pi(u'') = \pi(u')$. From (ii) we have

$$\Phi_u \cong \Phi_{u''}$$

Furthermore, from (i) we have

$$\Phi_{u''} \cong \Phi_{u'}$$

Therefore,

$$\Phi_u \cong \Phi_{u'}$$

Exercise 6. Let $\mathcal{A}_i = \mathcal{A}_{i\mu} dx^\mu$ be a local gauge connection over U_i, γ a curve completely in U_i, and σ_i a section over U_i. Choose some u above $\gamma(0)$, and σ_i such that $\sigma_i(\gamma(0)) = u$. Then the horizontal lift of γ through u can be written

$$\tilde{\gamma}(t) = \sigma_i(\gamma(t)) g_i(\gamma(t))$$

where $g_i(\gamma(0)) = e$. Further suppose γ is a loop. Then we have $\tau_\gamma(u) = \tilde{\gamma}(1)$. Using the earlier result,

$$\begin{aligned}
\tau_\gamma(u) = \tilde{\gamma}(1) &= \sigma_i(\gamma(1))\mathcal{P}\exp\left\{-\oint_\gamma \mathcal{A}_i(\gamma(t))\right\} \\
&= \sigma_i(\gamma(0))\mathcal{P}\exp\left\{-\oint_\gamma \mathcal{A}_i(\gamma(t))\right\} \\
&= u\mathcal{P}\exp\left\{-\oint_\gamma \mathcal{A}_i(\gamma(t))\right\}
\end{aligned}$$

So if we write

$$\tau_\gamma(u) = ug_\gamma$$

we have found

$$g_\gamma = \mathcal{P}\exp\left\{-\oint_\gamma \mathcal{A}_i(\gamma(t))\right\}$$

Exercise 7. Suppose $\mathcal{A}$ is pure gauge, i.e. $\mathcal{A} = g^{-1}dg$ for some g. Then the associated field strength is

$$\begin{aligned}
\mathcal{F} &= d(g^{-1}dg) + g^{-1}dg \wedge g^{-1}dg \\
&= dg^{-1} \wedge dg + g^{-1}dg \wedge g^{-1}dg \\
&= -g^{-1}dg\ g^{-1} \wedge dg + g^{-1}dg \wedge g^{-1}dg \\
&= 0
\end{aligned}$$

Exercise 8. Let $a_i \in \mathbb{R}$, $s, s' \in \Gamma(M, E)$ and $f \in C^\infty(M)$. Let γ be a curve with $\gamma(0) = p$ and $\gamma'(0) = X_p$. Parameterise s_1 and s_2 along γ by

$$\begin{aligned}
s_1(\gamma(t)) &= [(\tilde{\gamma}(t), \eta(\gamma(t)))] \\
s_2(\gamma(t)) &= [(\tilde{\gamma}(t), \zeta(\gamma(t)))]
\end{aligned}$$

(i) We have

$$\begin{aligned}
\nabla_{X_p}(a_1 s_1 + a_2 s_2) &= \nabla_{X_p}[(\tilde{\gamma}(t), a_1\eta(\gamma(t)) + a_2\zeta(\gamma(t)))] \\
&= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(a_1\eta(\gamma(t)) + a_2\zeta\gamma(t))\Big|_{t=0}\right)\right] \\
&= \left[\left(\tilde{\gamma}(0), a_1\frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right)\right] + \left[\left(\tilde{\gamma}(0), a_2\frac{d}{dt}\zeta(t)\Big|_{t=0}\right)\right] \\
&= a_1\nabla_{X_p}s_1 + a_2\nabla_{X_p}s_2
\end{aligned}$$

This holds for all p, so

$$\nabla_X(a_1s_1 + a_2s_2) = a_1\nabla_X s_1 + a_2\nabla_X s_2$$

(ii) Then,

$$\begin{aligned}\nabla(a_1s_2 + a_2s_2)(X) &= \nabla_X(a_1s_1 + a_2s_2)\\ &= a_1\nabla_X s_1 + a_2\nabla_X s_2\\ &= a_1\nabla s_1(X) + a_2\nabla s_2(X)\end{aligned}$$

So

$$\nabla(a_1s_1 + a_2s_2) = a_1\nabla s_1 + a_2\nabla s_2$$

(iii) Now define X_1, X_2, γ_1, γ_2 and γ such that $\gamma_1(0) = \gamma_2(0) = \gamma(0) = p$ and $\gamma_1'(0) = X_{1p}$, $\gamma_2'(0) = X_{2p}$ and $\gamma'(0) = a_1X_{1p} + a_2X_{2p}$. Then we will have

$$\frac{d}{dt}\eta(\gamma(t)) = \frac{d}{dt}(a_1\eta(\gamma_1(t)) + a_2\eta(\gamma_2(t)))$$

Thus we have

$$\begin{aligned}\nabla_{a_1X_{1p}+a_2X_{2p}}s &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right)\right]\\ &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(a_1\eta(\gamma_1(t)) + a_2\eta(\gamma_2(t)))\Big|_{t=0}\right)\right]\\ &= \left[\left(\tilde{\gamma}(0), a_1\frac{d}{dt}\eta(\gamma_1(t))\Big|_{t=0}\right)\right] + \left[\left(\tilde{\gamma}(0), a_2\frac{d}{dt}\eta(\gamma_2(t))\Big|_{t=0}\right)\right]\\ &= a_1\nabla_{X_{1p}}s + a_2\nabla_{X_{2p}}s\end{aligned}$$

We have this for all p, so

$$\nabla_{a_1X_1+a_2X_2}s = a_1\nabla_{X_1}s + a_2cd_{X_2}s$$

(iv) Return to the original definitions. We have

$$\begin{aligned}\nabla_{X_p}(fs) &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(f(\gamma(t))\eta(\gamma(t)))\Big|_{t=0}\right)\right]\\ &= \left[\left(\tilde{\gamma}(0), \left(\frac{d}{dt}f(\gamma(t))\right)\Big|_{t=0}\eta(p)\right)\right] + \left[\left(\tilde{\gamma}(0), f(p)\frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right)\right]\\ &= X(f)(p)s(p) + f(p)\nabla_{X_p}s\end{aligned}$$

This holds for all p, so

$$\nabla_X(fs) = X(f)s + f\nabla_X s$$

(v) Then,

$$\begin{aligned}\nabla(fs)(X) &= \nabla_X(fs)\\ &= X(f)s + f\nabla_X s\\ &= (df)(X)s + f\nabla s(X)\end{aligned}$$

for all X, so

$$\nabla(fs) = (df)s + f\nabla s$$

(vi) Here choose γ and γ_X such that $\gamma(0) = \gamma_X(0) = p$, $\gamma'(0) = f(p)X_p$ and $\gamma_X'(0) = X_p$. Then

$$\begin{aligned}\nabla_{f(p)X_p} s &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right)\right]\\ &= \left[\left(\tilde{\gamma}(0), f(p)\frac{d}{dt}\eta(\gamma_X(t))\Big|_{t=0}\right)\right]\\ &= f(p)\nabla_X s\end{aligned}$$

for all p, so

$$\nabla_{fX} s = f\nabla_X s$$

Exercise 9. Let $s \in \Gamma(M, E)$ be a generic section, written as

$$s(p) = [(\sigma_i(p), \xi_i(p))] = \xi_i^\alpha(p) e_\alpha$$

where $\xi_i = \xi_i^\alpha e_\alpha^0$. Then we have

$$\begin{aligned}\nabla_X s &= \nabla_X(\xi_i^\alpha e_\alpha)\\ &= X(\xi_i^\alpha)e_\alpha + \xi_i^\alpha \nabla_X e_\alpha\\ &= \frac{d\xi_i^\alpha}{dt} e_\alpha + \xi_i^\alpha A_{i\mu\alpha}^\beta \frac{dx^\mu}{dt} e_\beta\\ &= \left(\frac{d\xi_i^\alpha}{dt} + \xi_i^\beta A_{i\mu\beta}^\alpha \frac{dx^\mu}{dt}\right) e_\alpha\end{aligned}$$

We may write this as

$$\nabla_X s = \frac{dx^\mu}{dt}\left(\frac{\partial \xi_i^\alpha}{\partial x^\mu} + \xi_i^\beta A_{i\mu\beta}^\alpha\right) e_\alpha$$

or

$$\nabla_X s = \left[\left(\sigma_i(t), \frac{d\xi_i}{dt} + \mathcal{A}_i(X)\xi_i\right)\right]$$

If we mean $X = X_p \in T_pM$, we have

$$\nabla_X s = \left[\left(\sigma_i(0), \left(\frac{d\xi_i}{dt} + \mathcal{A}_i(X)\xi_i\right)\Big|_{t=0}\right)\right]$$

Exercise 10. Consider the associated bundle $E_{\mathfrak{g}} = P \times_{\text{Ad}} \mathfrak{g}$, where $\text{Ad}_g : V \mapsto gVg^{-1}$ (text has inverses the wrong way round). Choose a local section σ_i and parameterise a horizontal lift as

$$\tilde{\gamma}(t) = \sigma_i(t)g(t)$$

Let a generic section $s \in \Gamma(M, E)$ be defined by

$$s(p) = [(\sigma_i(p), V(p))]$$

Now define $\mathcal{D}_X s$ by

$$\mathcal{D}_X s = \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(\text{Ad}_{g(t)^{-1}} V(t))\Big|_{t=0}\right)\right]$$

We have

$$\begin{aligned}\frac{d}{dt}(\text{Ad}_{g(t)^{-1}} V(t)) &= \frac{d}{dt}(g(t)^{-1}V(t)g(t)) \\ &= -g(t)^{-1}\frac{dg}{dt}g(t)^{-1}V(t)g(t) + g(t)^{-1}\frac{dV}{dt}g(t) + g(t)^{-1}V(t)\frac{dg}{dt}\end{aligned}$$

Now,

$$\mathcal{D}_X s = \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(\text{Ad}_{g(t)^{-1}} V(t))\Big|_{t=0}\right)\right] = \left[\left(\sigma_i(0), \text{Ad}_{g(0)}\frac{d}{dt}(\text{Ad}_{g(t)^{-1}} V(t))\Big|_{t=0}\right)\right]$$

We have

$$\begin{aligned}\text{Ad}_{g(0)}\frac{d}{dt}(\text{Ad}_{g(t)^{-1}} V(t))\Big|_{t=0} &= \left[-\frac{dg}{dt}g(t)^{-1}V(t)g(t) + \frac{dV}{dt}g(t) + V(t)\frac{dg}{dt}\right]\Big|_{t=0}g(0)^{-1} \\ &= \frac{dV}{dt}\Big|_{t=0} + \left[-\frac{dg}{dt}g(t)^{-1}V(t) + V(t)\frac{dg}{dt}g(t)^{-1}\right]\Big|_{t=0} \\ &= \frac{dV}{dt}\Big|_{t=0} - \left[\frac{dg}{dt}g(t)^{-1}, V(t)\right]\Big|_{t=0} \\ &= \frac{dV}{dt}\Big|_{t=0} + [\mathcal{A}_i(X), V(t)]|_{t=0}\end{aligned}$$

where in the last line we have recalled that $\mathcal{A}_i$ is obtained by solving

$$\frac{dg_i(\gamma(t))}{dt} = -\mathcal{A}_i(X)g_i(\gamma(t))$$

Thus we have

$$\begin{aligned}\mathcal{D}_X s &= \left[\left(\sigma_i(0), \left(\frac{dV}{dt} + [\mathcal{A}(X), V(t)]\right)\Big|_{t=0}\right)\right] \\ &= \left[\left(\sigma_i(0), X^\mu(\partial_\mu V^\alpha T_\alpha + A^\alpha_{i\mu}V^\beta f^\gamma_{\alpha\beta}T_\gamma)\right)\right] \\ &= X^\mu(\partial_\mu V^\alpha + f^\alpha_{\beta\gamma}A^\beta_{i\mu}V^\gamma)e_\alpha\end{aligned}$$

where

$$e_\alpha = [(\sigma_i(0), T_\alpha)]$$

Exercise 11. Let $s = \xi^\alpha e_\alpha$. Then

$$\begin{aligned}
\nabla\nabla s &= \nabla\nabla(\xi^\alpha e_\alpha) \\
&= \nabla(e_\alpha \otimes d\xi^\alpha + \xi^\alpha \nabla e_\alpha) \\
&= (\nabla e_\alpha) \wedge d\xi^\alpha + d\xi^\alpha \wedge \nabla e_\alpha + \xi^\alpha \nabla\nabla e_\alpha \\
&= \xi^\alpha \nabla\nabla e_\alpha \\
&= e_\beta \otimes F^\beta_{i\alpha} \xi^\alpha
\end{aligned}$$