Nakahara - Geometry, Topology and Physics

Chapter 8: Complex Manifolds

Exercise 1. $(T_pM^{\mathbb{C}})^*$ is the space of linear functions $T_pM^{\mathbb{C}} \to \mathbb{C}$. We have

$$\begin{aligned}\dim_{\mathbb{C}} T_pM^{\mathbb{C}} = \dim_{\mathbb{R}} T_pM &= \dim_{\mathbb{R}} T_p^*M \\ &= \dim_{\mathbb{C}}(T_p^*M)^{\mathbb{C}}\end{aligned}$$

Let $\{e_k\}$ be a basis on T_pM, and $\{\theta^k\}$ the dual basis on T_p^*M, i.e. $\theta^k(e_j) = \delta_j^k$. Now, we can also regard $\{\theta^k\}$ as a basis for $(T_p^*M)^{\mathbb{C}}$ as discussed above, and $\theta^k(e_j) = \delta_j^k$ is still true. So we must have

$$(T_pM^{\mathbb{C}})^* \cong (T_p^*M)^{\mathbb{C}}$$

with the obvious identification. We can use this fact to write $T_p^*M^{\mathbb{C}}$ without ambiguity.

Exercise 2. Let $z^\mu = x^\mu + iy^\mu$ and $w^\nu = u^\nu + iv^\nu$ be two overlapping coordinate systems on a complex manifold M. The statement, in the first coordinate system, that X is a holomorphic vector, is

$$X = X^\mu \frac{\partial}{\partial z^\mu}$$

That is, we have

$$\begin{aligned}X &= \frac{1}{2}X^\mu\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right) \\ &= \frac{1}{2}X^\mu\left(\frac{\partial u^\nu}{\partial x^\mu}\frac{\partial}{\partial u^\nu} + \frac{\partial v^\nu}{\partial x^\mu}\frac{\partial}{\partial v^\nu} - i\frac{\partial u^\nu}{\partial y^\mu}\frac{\partial}{\partial u^\nu} - i\frac{\partial v^\nu}{\partial y^\mu}\frac{\partial}{\partial v^\nu}\right) \\ &= \frac{1}{2}X^\mu\left(\frac{\partial u^\nu}{\partial x^\mu}\frac{\partial}{\partial u^\nu} - \frac{\partial u^\nu}{\partial y^\mu}\frac{\partial}{\partial v^\nu} - i\frac{\partial u^\nu}{\partial y^\mu}\frac{\partial}{\partial u^\nu} - i\frac{\partial u^\nu}{\partial x^\mu}\frac{\partial}{\partial v^\nu}\right) \\ &= \frac{1}{2}X^\mu\left(\frac{\partial u^\nu}{\partial x^\mu} - i\frac{\partial u^\nu}{\partial y^\mu}\right)\left(\frac{\partial}{\partial u^\nu} - i\frac{\partial}{\partial v^\nu}\right) \\ &= Y^\nu\frac{\partial}{\partial w^\nu}\end{aligned}$$

which defines the functions Y^ν. Thus our notion of holomorphicity is indeed coordinate-independent. Similarly then antiholomorphicity.

Exercise 3. Let X, Y be holomorphic vector fields on M, i.e. $\mathcal{P}^+(X) = X$ and $\mathcal{P}^+(Y) = Y$. Then, locally, if we write

$$\begin{aligned} X &= X^\mu \frac{\partial}{\partial z^\mu} \\ Y &= Y^\mu \frac{\partial}{\partial z^\mu} \end{aligned}$$

Then we have

$$[X, Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial z^\mu} - Y^\mu \frac{\partial X^\nu}{\partial z^\mu} \right) \frac{\partial}{\partial z^\nu}$$

So

$$\begin{aligned} \mathcal{P}^+([X, Y]) &= \left(X^\mu \frac{\partial Y^\nu}{\partial z^\mu} - Y^\mu \frac{\partial X^\nu}{\partial z^\mu} \right) \mathcal{P}^+ \left(\frac{\partial}{\partial z^\nu} \right) \\ &= \left(X^\mu \frac{\partial Y^\nu}{\partial z^\mu} - Y^\mu \frac{\partial X^\nu}{\partial z^\mu} \right) \frac{\partial}{\partial z^\nu} \\ &= [X, Y] \end{aligned}$$

Therefore the Lie bracket of two holomorphic vector fields is itself holomorphic. Similarly for antiholomorphicity.

Exercise 4.

(i) Let $\omega \in \Omega_p^q(M)^{\mathbb{C}}$, and write $\omega = \alpha + i\beta$. Let $V_1, ..., V_q \in T_pM^{\mathbb{C}}$, and write $V_i = X_i + iY_i$. Then

$$\begin{aligned} \bar{\omega}(V_1, ..., V_q) &= \alpha(V_1, ..., V_q) - i\beta(V_1, ..., V_q) \\ &= \alpha(X_1, ..., X_q) + i\alpha(Y_1, ..., Y_q) - i\beta(X_1, ..., X_q) + \beta(Y_1, ..., Y_q) \end{aligned}$$

On the other hand,

$$\begin{aligned} \omega(\bar{V}_1, ..., \bar{V}_q) &= \omega(X_1, ..., X_q) - i\omega(Y_1, ..., Y_q) \\ &= \alpha(X_1, ..., X_q) + i\beta(X_1, ..., X_q) - i\alpha(Y_1, ..., Y_q) + \beta(Y_1, ..., Y_q) \end{aligned}$$

So

$$\overline{\omega(\bar{V}_1, ..., \bar{V}_q)} = \alpha(X_1, ..., X_q) - i\beta(X_1, .., X_q) + i\alpha(Y_1, ..., Y_q) + \beta(Y_1, ..., Y_q)$$

That is,

$$\bar{\omega}(V_1, ..., V_q) = \overline{\omega(\bar{V}_1, ..., \bar{V}_q)}$$

(ii) Let $\omega, \eta \in \Omega_p^q(M)^{\mathbb{C}}$. If $\omega = \alpha + i\beta$ and $\eta = \gamma + i\delta$, we have

$$\begin{aligned} \omega + \eta &= \alpha + \gamma + i(\beta + \delta) \\ \overline{\omega + \eta} &= \alpha + \beta - i(\beta + \delta) \\ &= \bar{\omega} + \bar{\eta} \end{aligned}$$

(iii) Let also $\lambda \in \mathbb{C}$. If $\lambda = a + bi$, we have

$$\begin{aligned} \lambda\omega &= a\alpha - b\beta + i(a\beta + b\alpha) \\ \overline{\lambda\omega} &= a\alpha - b\beta - i(a\beta + b\alpha) \\ &= \bar{\lambda}\bar{\omega} \end{aligned}$$

(iv) We finally have

$$\begin{aligned} \bar{\omega} &= \alpha - i\beta \\ \bar{\bar{\omega}} &= \alpha + i\beta = \omega \end{aligned}$$

Exercise 5. Let $\omega \in \Omega^q(M)^{\mathbb{C}}$ and $\xi \in \Omega^r(M)^{\mathbb{C}}$. Write $\omega = \alpha + i\beta$ and $\xi = \gamma + i\delta$. Then

$$\begin{aligned} \omega \wedge \xi &= \alpha \wedge \gamma - \beta \wedge \delta + i(\alpha \wedge \delta + \beta \wedge \gamma) \\ &= (-1)^{qr}(\gamma \wedge \alpha - \delta \wedge \beta + i(\delta \wedge \alpha + \gamma \wedge \beta)) \\ &= (-1)^{qr}\xi \wedge \omega \end{aligned}$$

as expected. Similarly,

$$\begin{aligned} d(\omega \wedge \xi) &= d(\alpha \wedge \gamma - \beta \wedge \delta + i(\alpha \wedge \delta + \beta \wedge \gamma)) \\ &= d(\alpha \wedge \gamma) - d(\beta \wedge \delta) + i(d(\alpha \wedge \delta) + d(\beta \wedge \gamma)) \\ &= d\alpha \wedge \gamma + (-1)^q \alpha \wedge d\delta - d\beta \wedge \gamma - (-1)^q \beta \wedge d\delta \\ &\quad + i(d\alpha \wedge \delta + (-1)^q \alpha \wedge d\delta + d\beta \wedge \gamma + (-1)^q \beta \wedge d\delta) \\ &= d\omega \wedge \xi + (-1)^q \omega \wedge d\xi \end{aligned}$$

Exercise 6. Let $\dim_{\mathbb{C}} M = m$. A basis for $\Omega_p^{r,s}(M)$ is

$$\{dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}\}$$

So if $r, s \leq m$, we have

$$\dim_{\mathbb{R}} \Omega_p^{r,s}(M) = \binom{m}{r}\binom{m}{s}$$

and zero otherwise.
We also have
$$\Omega_p^q(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega_p^{r,s}(M)$$
So
$$\begin{aligned}
\dim_{\mathbb{R}} \Omega_p^q(M)^{\mathbb{C}} &= \sum_{r+s=q} \dim_{\mathbb{R}} \Omega_p^{r,s}(M) \\
&= \sum_{r+s=q} \binom{m}{r}\binom{m}{s} \\
&= \binom{2m}{q}
\end{aligned}$$

Exercise 7. We have
$$\begin{aligned}
R_{\bar{\kappa}\lambda\bar{\mu}\nu} &= g_{\bar{\kappa}}\xi R^{\xi}{}_{\lambda\bar{\mu}\nu} \\
&= g_{\bar{\kappa}\xi}\partial_{\bar{\mu}}(g^{\bar{\rho}\xi}\partial_\nu g_{\lambda\bar{\rho}}) \\
&= g_{\bar{\kappa}\xi}(\partial_{\bar{\mu}} g^{\bar{\rho}\xi}\partial_\nu g_{\lambda\bar{\rho}} + g^{\bar{\rho}\xi}\partial_{\bar{\mu}}\partial_\nu g_{\lambda\bar{\rho}}) \\
&= \partial_{\bar{\mu}}\partial_\nu g_{\lambda\bar{\kappa}}
\end{aligned}$$
Then
$$\begin{aligned}
R_{\kappa\bar{\lambda}\mu\bar{\nu}} &= g_{\kappa\bar{\xi}} R^{\bar{\xi}}{}_{\bar{\lambda}\mu\bar{\nu}} \\
&= g_{\kappa\bar{\xi}}\partial_\mu(g^{\bar{\xi}\rho}\partial_{\bar{\nu}} g_{\rho\bar{\lambda}}) \\
&= g_{\kappa\bar{\xi}}(\partial_\mu g^{\bar{\xi}\rho}\partial_{\bar{\nu}} g_{\rho\bar{\lambda}} + g^{\bar{\xi}\rho}\partial_\mu\partial_\nu g_{\rho\bar{\lambda}}) \\
&= \partial_\mu\partial_{\bar{\nu}} g_{\kappa\bar{\lambda}} - g^{\xi\rho}\partial_\mu g_{\kappa\bar{\xi}}\partial_{\bar{\nu}} g_{\rho\bar{\lambda}}
\end{aligned}$$
Next,
$$\begin{aligned}
R_{\bar{\kappa}\lambda\mu\bar{\nu}} &= g_{\bar{\kappa}\xi} R^{\xi}{}_{\lambda\mu\bar{\nu}} \\
&= -g_{\bar{\kappa}\xi} R^{\xi}{}_{\lambda\bar{\nu}\mu} \\
&= -R_{\bar{\kappa}\lambda\bar{\nu}\mu}
\end{aligned}$$
Finally
$$\begin{aligned}
R_{\kappa\bar{\lambda}\bar{\mu}\nu} &= g_{\kappa\bar{\xi}} R^{\bar{\xi}}{}_{\bar{\lambda}\bar{\mu}\nu} \\
&= -g_{\kappa\bar{\xi}} R^{\bar{\xi}}{}_{\bar{\lambda}\nu\bar{\mu}} \\
&= -R_{\kappa\bar{\lambda}\nu\bar{\mu}}
\end{aligned}$$

Next we want to check antisymmetry of the first two indices. Firstly, we have

$$\begin{aligned}
R_{\bar{\kappa}\lambda\bar{\mu}\nu} &= g_{\bar{\kappa}\rho} R^{\rho}{}_{\lambda\bar{\mu}\nu} \\
&= g_{\bar{\kappa}\rho} \partial_{\bar{\mu}} \Gamma^{\rho}{}_{\lambda\nu} \\
&= g_{\bar{\kappa}\rho} \partial_{\bar{\mu}} (g^{\bar{\xi}\rho} \partial_{\lambda} g_{\nu\bar{\xi}}) \\
&= g_{\bar{\kappa}\rho} \partial_{\bar{\mu}} g^{\bar{\xi}\rho} \partial_{\lambda} g_{\nu\bar{\xi}} + \partial_{\bar{\mu}} \partial_{\lambda} g_{\nu\bar{\kappa}}
\end{aligned}$$

While

$$\begin{aligned}
R_{\lambda\bar{\kappa}\bar{\mu}\nu} &= g_{\lambda\bar{\rho}} R^{\bar{\rho}}{}_{\bar{\kappa}\bar{\mu}\nu} \\
&= -g_{\lambda\bar{\rho}} \partial_{\nu} \Gamma^{\bar{\rho}}{}_{\bar{\kappa}\bar{\mu}} \\
&= -g_{\lambda\bar{\rho}} \partial_{\nu} (g^{\bar{\rho}\xi} \partial_{\bar{\kappa}} g_{\xi\bar{\mu}}) \\
&= -g_{\lambda\bar{\rho}} \partial_{\nu} g^{\bar{\rho}\xi} \partial_{\bar{\kappa}} g_{\xi\bar{\mu}} - \partial_{\nu} \partial_{\bar{\kappa}} g_{\lambda\bar{\mu}}
\end{aligned}$$

Can't see how to proceed??