Nakahara - Geometry, Topology and Physics

Chapter 9: Fibre Bundles

Exercise 1. Let $\{E_i\}$ be vector bundles over M with fibres $\mathbb{R}^{k_i}$ and structure groups G_i, and $\{U_\alpha\}$ be an open cover of M. Then we can write

$$E_i = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^{k_i}/G_i$$

We have

$$\begin{aligned} E_i \oplus E_j &= \bigsqcup_\alpha U_\alpha \times (\mathbb{R}^{k_i} \oplus \mathbb{R}^{k_j})/(G_i \oplus G_j) \\ E_i \otimes E_j &= \bigsqcup_\alpha U_\alpha \times (\mathbb{R}^{k_i} \otimes \mathbb{R}^{k_j})/(G_i \otimes G_j) \end{aligned}$$

Thus, since $\otimes$ is distributive over $\oplus$ for sets and groups,

$$\begin{aligned} V_1 \otimes (V_2 \oplus V_3) &= (V_1 \otimes V_2) \oplus (V_1 \otimes V_3) \\ G_1 \otimes (G_2 \oplus G_3) &= (G_1 \otimes G_2) \oplus (G_1 \otimes G_3) \end{aligned}$$

we have the same result for vector bundles:

$$E_1 \otimes (E_2 \oplus E_3) = (E_1 \otimes E_2) \oplus (E_1 \otimes E_3)$$

The bundle $E_1 \otimes (E_2 \oplus E_3)$ has transition functions of the form

$$T_{ij}(p) = \begin{pmatrix} t_{ij}^{E_1}(p) \otimes t_{ij}^{E_2}(p) & \\ & t_{ij}^{E_1}(p) \otimes t_{ij}^{E_3}(p) \end{pmatrix}$$

Problem 1. Let $L = S^1 \times \mathbb{R}$ be the trivial line bundle over S^1. Consider the Whitney sum $L \oplus L$. We have

$$\begin{aligned} L \oplus L &= \{(u, u') \in L \times L \mid \pi(u) = \pi(u')\} \\ &= \{(\theta, t, \phi, s) \in S^1 \times \mathbb{R} \times S^1 \times \mathbb{R} \mid t = s\} \\ &\cong \{(\theta, t, \psi) \in S^1 \times \mathbb{R} \times S^1\} \end{aligned}$$

That is, $L \oplus L \cong S^1 \times T^2$, so the sum bundle is trivial.
Suppose E is the trivial $\mathbb{R}^2$ bundle over $[0, 2\pi]$, and consider the sections

$$s_1(t) = (\cos t/2, \sin t/2), \quad s_2(t) = (-\sin t/2, \cos t/2)$$

These two span linear subspaces of E which are orthogonal at every t and rotate by π between $t = 0$ and $t = 2\pi$. Then identifying $0 \sim 2\pi$, these two subspaces of E become Möbius bundles over S^1. That is, if L is the Möbius bundle, $L \oplus L$ is the trivial $\mathbb{R}^2$-bundle: $L \oplus L \cong S^1 \times \mathbb{R}^2$.

Problem 2. Let Ω_n be the volume element on S^n normalised by $\int_{S^n} \Omega_n = 1$, and $f : S^{2n-1} \to S^n$ a smooth map.

(a) We have $d\Omega_n = 0$, so $df^*\Omega_n = f^* d\Omega_n = 0$. But $H^n(S^{2n-1}) = 0$, so every closed form is exact and we can write $f^*\Omega_n = d\omega_{n-1}$ for some $\omega_{n-1} \in \Omega^{n-1}(S^{2n-1})$.

(b) Define the Hopf invariant

$$H(f) = \int_{S^{2n-1}} \omega_{n-1} \wedge d\omega_{n-1}$$

We have the freedom from the definition to shift

$$\omega_{n-1} \to \omega_{n-1} + d\gamma_{n-2}$$

Under this,

$$\begin{aligned} H(f) \to & \int_{S^{2n-1}} (\omega_{n-1} + d\gamma_{n-2}) \wedge d\omega_{n-1} \\ &= \int_{S^{2n-1}} (\omega_{n-1} \wedge d\omega_{n-1} + d(\gamma_{n-2} \wedge d\omega_{n-1})) \\ &= \int_{S^{2n-1}} \omega_{n-1} \wedge d\omega_{n-1} \\ &= H(f) \end{aligned}$$

Thus $H(f)$ is well-defined.

(c) We know that the pullback of a form by two homotopic maps can only differ by an exact form. Thus if g is homotopic to f, $g^*\Omega_n = f^*\Omega_n + d\alpha_{n-1}$. Thus

$$g^*\Omega_n = d\omega_{n-1} + d\alpha_{n-1}$$

This amounts to shifting $\omega_{n-1} \to \omega_{n-1} + \alpha_{n-1}$? α_{n-1} isn't exact in general??

(d) If n is odd,

$$
\begin{aligned}
d(\omega_{n-1} \wedge \omega_{n-1}) &= d\omega_{n-1} \wedge \omega_{n-1} - \omega_{n-1} \wedge d\omega_{n-1} \\
&= 2d\omega_{n-1} \wedge \omega_{n-1} \\
&= 2(-1)^{n(n-1)}\omega_{n-1} \wedge d\omega_{n-1} \\
&= 2\omega_{n-1} \wedge d\omega_{n-1}
\end{aligned}
$$

So we have

$$
\begin{aligned}
H(f) &= \int_{S^{2n-1}} \frac{1}{2} d(\omega_{n-1} \wedge \omega_{n-1}) \\
&= 0
\end{aligned}
$$

(e) Consider the case $f = \pi : S^3 \to S^2$. The canonical volume form on S^2 induced from the standard embedding in $\mathbb{R}^3$, normalised to give 1 upon integration, is

$$
\Omega = \frac{1}{4\pi}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)
$$

Put coordinates (X, Y, Z, T) on $S^3 \subset \mathbb{R}^4$. Then our map $\pi : S^3 \to S^2$ is defined by

$$
\begin{aligned}
\pi^*(x) &= 2(XZ + YT) \\
\pi^*(y) &= 2(YZ - XT) \\
\pi^*(z) &= X^2 + Y^2 - Z^2 - T^2
\end{aligned}
$$

Then

$$
\begin{aligned}
\pi^*(dx) &= 2(ZdX + XdZ + TdY + YdT) \\
\pi^*(dy) &= 2(ZdY + YdZ - TdX - XdT) \\
\pi^*(dz) &= 2(XdX + YdY - ZdZ - TdT)
\end{aligned}
$$

Thus we can calculate

$$
\begin{aligned}
\pi^*(dy \wedge dz) = 4\Big[&(-XZ - YT)dX \wedge dY + (ZT - XY)dX \wedge dZ + (X^2 + T^2)dX \wedge dT \\
&+ (-Y^2 - Z^2)dY \wedge dZ + (XY - ZT)dY \wedge dT + (-YT - XZ)dZ \wedge dT\Big] \\
\pi^*(dz \wedge dx) = 4\Big[&(XT - YZ)dX \wedge dY + (X^2 + Z^2)dX \wedge dZ + (XY + ZT)dX \wedge dT \\
&+ (XY + ZT)dY \wedge dZ + (Y^2 + T^2)dY \wedge dT + (-YZ - XT)dZ \wedge dT\Big] \\
\pi^*(dx \wedge dy) = 4\Big[&(Z^2 + T^2)dX \wedge dY + (YZ + XT)dX \wedge dZ + (YT - XZ)dX \wedge dT \\
&+ (YT - XZ)dY \wedge dZ + (-XT - YZ)dY \wedge dT + (-X^2 - Y^2)dZ \wedge dT\Big]
\end{aligned}
$$

Writing $\pi^*\Omega = \tilde{\Omega}$, we then have

$$\begin{aligned}
\tilde{\Omega}_{XY} &= \frac{1}{\pi}\left[2(XZ+YT)(-XZ-YT)+2(YZ-XT)(XT-YZ)+(X^2+Y^2-Z^2-T^2)(Z^2+T^2)\right] \\
&= \frac{1}{\pi}\left[(-2(X^2+Y^2)+X^2+Y^2-Z^2-T^2)(Z^2+T^2)\right] \\
&= -\frac{1}{\pi}(X^2+T^2) \\
&= \frac{1}{\pi}(X^2+Y^2)
\end{aligned}$$

Similarly, with a bit of work we can calculate

$$\begin{aligned}
\tilde{\Omega}_{XZ} &= \frac{1}{\pi}(YZ-XT) \\
\tilde{\Omega}_{XT} &= \frac{1}{\pi}(XZ+YT) \\
\tilde{\Omega}_{YZ} &= -\frac{1}{\pi}(XZ+YT) \\
\tilde{\Omega}_{YT} &= \frac{1}{\pi}(YZ-XT) \\
\tilde{\Omega}_{ZT} &= -\frac{1}{\pi}(1-Z^2-T^2)
\end{aligned}$$

Actually, by checking $d\tilde{\Omega} = 0$, there must be some sign errors here - I will assume they are corrected in:

$$\begin{aligned}
\pi^*\tilde{\Omega} = \frac{1}{\pi}\big[&(X^2+Y^2)dX\wedge dY + (YZ-XT)dX\wedge dZ - (XZ+YT)dX\wedge dT \\
&+ (XZ+YT)dY\wedge dZ + (YZ-XT)dY\wedge dT + (1-Z^2-T^2)dZ\wedge dT\big]
\end{aligned}$$

Now this is indeed closed, and we can therefore find a 1-form ω such that $\tilde{\Omega} = d\omega$. The XY and ZT parts are taken care of by

$$-X^2YdX + XY^2dY - T(1-Z^2)dZ - ZT^2dT$$

Then we have

$$\begin{aligned}
YZdX\wedge dZ + XZdY\wedge dZ &= d(XYZdZ) \\
-XTdX\wedge dZ - XZdX\wedge dT &= d(XZTdX) \\
-YTdX\wedge dT - XTdY\wedge dT &= d(-XYTdT) \\
YTdY\wedge dZ + YZdY\wedge dT &= d(-YZTdY)
\end{aligned}$$

Thus, we may define

$$\omega = \frac{1}{\pi}\big[(XZT-X^2Y)dX+(XY^2-XYT)dY+(XYZ-T(1-Z^2))dZ-(ZT^2+XYT)dT\big]$$

Now, we want to integrate $\omega\wedge\tilde{\Omega}$. However, I have run out of steam.