Nakahara - Geometry, Topology and Physics

Chapter 7: Riemannian Geometry

Exercise 1. Let

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

To diagonalise this we just need to diagonalise the upper left submatrix. Its eigenvalues are ± 1, so the full diagonalised matrix is $\mathrm{diag}(-1, 1, 1, 1)$. To perform this diagonalisation, we have noted that the eigenvectors are $(\pm 1, 1)^T/\sqrt{2}$. In terms of the usual basis vectors, this is

$$e_\pm = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}(e_1 \pm e_0)$$

If we have a vector V with components (V^+, V^-, V^2, V^3), the corresponding 1-form has components

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} V^+ \\ V^- \\ V^2 \\ V^3 \end{pmatrix} = \begin{pmatrix} V^- \\ V^+ \\ V^2 \\ V^3 \end{pmatrix}$$

Exercise 2. Consider $T^2 \subset \mathbb{R}^3$, with the Euclidean metric on $\mathbb{R}^3$ and the embedding

$$f : (\theta, \phi) \mapsto ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$$

for $R > r$. The induced metric is

$$
\begin{aligned}
g &= \delta_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} dx^\mu \otimes dx^\nu \\
&= (-r \sin\theta \cos\phi)^2 d\theta \otimes d\theta + 2(-r\cos\theta\cos\phi)(R + r\cos\theta)(-\sin\phi) d\theta \otimes d\phi + (R + r\cos\theta)^2 (-\sin\phi)^2 d\phi \otimes d\phi \\
&\quad + (-r\sin\theta\sin\phi)^2 d\theta \otimes d\theta + 2(-r\sin\theta\sin\phi)(R + r\cos\theta)\cos\phi d\theta \otimes d\phi + (R + r\cos\theta)^2 \cos\phi^2 d\phi \otimes d\phi \\
&\quad + (r\cos\theta)^2 d\theta \otimes d\theta \\
&= (r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2\cos^2\theta) d\theta \otimes d\theta \\
&\quad + 2(r\sin\theta\sin\phi\cos\phi(R + r\cos\theta) - r\sin\theta\sin\phi\cos\phi(R + r\cos\theta)) d\theta \otimes d\phi \\
&\quad + ((R + r\cos\theta)^2 \sin^2\phi + (R + r\cos\theta)^2\cos^2\phi) d\phi \otimes d\phi \\
&= r^2 d\theta \otimes d\theta + (R + r\cos\theta)^2 d\phi \otimes d\phi
\end{aligned}
$$

Exercise 3. Under the affine transformation $t \to at + b = t'$, we have

$$
\begin{aligned}
\frac{dx^\mu}{dt} &\to \frac{dx^\mu}{dt'} \frac{dt'}{dt} \\
&= \frac{1}{a} \frac{dx^\mu}{dt}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{d^2x^\mu}{dt^2} &\to \frac{1}{a} \frac{d}{dt'} \left(\frac{1}{a} \frac{dx^\mu}{dt} \right) \\
&= \frac{1}{a^2} \frac{d^2 x^\mu}{dt^2}
\end{aligned}
$$

Therefore the left-action of this transformation on the LHS of the geodesic equation is

$$
\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda} \frac{dx^\nu}{dt} \frac{dx^\lambda}{dt} \to \frac{1}{a^2} \frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda} \frac{1}{a} \frac{dx^\nu}{dt} \frac{1}{a} \frac{dx^\lambda}{dt}
$$

That is,

$$
(L_{a,b})_* (\nabla_V V) \mid_{c(t)} = \frac{1}{a}^2 \nabla_V V \mid_{c(at+b)}
$$

so the geodesic equation, $\nabla_V V = 0$, is preserved by affine transformations.

Exercise 4. A metric tensor g is type-(0,2), so we have

$$
\begin{aligned}
\nabla_\nu g &= \nabla_\nu (g_{\mu\lambda} dx^\mu \otimes dx^\lambda) \\
&= \nabla_\nu (g_{\mu\nu}) dx^\mu \otimes dx^\lambda + g_{\mu\lambda} (\nabla_\nu dx^\mu \otimes dx^\lambda + dx^\mu \otimes \nabla_\nu dx^\lambda) \\
&= \partial_\nu g_{\mu\lambda} dx^\mu \otimes dx^\lambda + g_{\mu\lambda} (-\Gamma^\mu{}_{\nu\kappa} dx^\kappa \otimes dx^\lambda - dx^\mu \otimes \Gamma^\lambda{}_{\nu\kappa} dx^\kappa) \\
&= \partial_\nu g_{\mu\lambda} dx^\mu \otimes dx^\lambda - \Gamma^\mu{}_{\nu\kappa} g_{\mu\nu} dx^\kappa \otimes dx^\lambda - \Gamma^\lambda{}_{\nu\kappa} g_{\mu\lambda} dx^\mu \otimes dx^\kappa
\end{aligned}
$$

So

$$(\nabla_\nu g)_{\mu\lambda} = \partial_\nu g_{\mu\lambda} - \Gamma^\kappa{}_{\nu\mu} g_{\kappa\lambda} - \Gamma^\kappa{}_{\nu\lambda} g_{\kappa\mu}$$

Exercise 5. Let $\Gamma^\lambda{}_{\mu\nu}$ be a connection and t a type-(1,2) tensor. We have

$$\tilde{\Gamma}^\gamma{}_{\alpha\beta} = \left(\frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \Gamma^\nu{}_{\lambda\mu} \right) \frac{\partial y^\gamma}{\partial x^\nu}$$

and

$$\begin{aligned} t &= t^\lambda_{\mu\nu} \frac{\partial}{\partial x^\lambda} \otimes dx^\mu \otimes dx^\nu \\ &= t^\lambda_{\mu\nu} \frac{\partial y^\alpha}{\partial x^\lambda} \frac{\partial}{\partial y^\alpha} \otimes \frac{\partial x^\mu}{\partial y^\beta} dy^\beta \otimes \frac{\partial x^\nu}{\partial y^\gamma} dy^\gamma \\ &= t^\lambda_{\mu\nu} \frac{\partial y^\alpha}{\partial x^\lambda} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\gamma} \frac{\partial}{\partial y^\alpha} \otimes dy^\beta \otimes dy^\gamma \end{aligned}$$

i.e.

$$\tilde{t}^\gamma_{\alpha\beta} = t^\lambda_{\mu\nu} \frac{\partial y^\alpha}{\partial x^\lambda} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\gamma}$$

Then

$$\begin{aligned} (\widetilde{\Gamma^\gamma{}_{\alpha\beta} + t^\gamma_{\alpha\beta}}) &= \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} \Gamma^\nu{}_{\lambda\mu} + \frac{\partial y^\gamma}{\partial x^\lambda} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} t^\lambda_{\mu\nu} \\ &= \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} (\Gamma^\nu{}_{\lambda\mu} + t^\nu_{\lambda\mu}) \end{aligned}$$

Therefore $\Gamma^\gamma{}_{\alpha\beta} + t^\gamma_{\alpha\beta}$ is a connection.
Now, suppose Γ and $\overline{\Gamma}$ are connections. Then

$$\begin{aligned} (\widetilde{\Gamma^\gamma{}_{\alpha\beta} - \overline{\Gamma}^\gamma{}_{\alpha\beta}}) &= \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} \Gamma^\nu{}_{\lambda\mu} - \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} \overline{\Gamma}^\nu{}_{\lambda\mu} \\ &= \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} (\Gamma^\nu{}_{\lambda\mu} - \overline{\Gamma}^\nu{}_{\lambda\mu}) \end{aligned}$$

So the difference between connections transforms like a type-(1,2) tensor.

Exercise 6.

(i) By the previous exercise, $T^\kappa{}_{\lambda\mu} = \Gamma^\kappa{}_{\lambda\mu} - \Gamma^\kappa{}_{\mu\lambda}$ obeys the transformation rule for a type-(1,2) tensor.

(ii) We have

$$
\begin{aligned}
K^{\kappa}{}_{[\mu\nu]} &= \frac{1}{2}(K^{\kappa}{}_{\mu\nu} - K^{\kappa}{}_{\nu\mu}) \\
&= \frac{1}{4}(T^{\kappa}{}_{\mu\nu} + T_{\mu}{}^{\kappa}{}_{\nu} + T_{\nu}{}^{\kappa}{}_{\mu} - T^{\kappa}{}_{\nu\mu} - T_{\nu}{}^{\kappa}{}_{\mu} - T_{\mu}{}^{\kappa}{}_{\nu}) \\
&= \frac{1}{4}(2T^{\kappa}{}_{\mu\nu}) \\
&= \frac{1}{2}T^{\kappa}{}_{\mu\nu}
\end{aligned}
$$

(iii) We have

$$
\begin{aligned}
K_{\kappa\mu\nu} &= g_{\kappa\lambda}K^{\lambda}{}_{\mu\nu} \\
&= \frac{1}{2}(T_{\kappa\mu\nu} + T_{\mu\kappa\nu} + T_{\nu\kappa\mu}) \\
&= -\frac{1}{2}(T_{\kappa\nu\mu} + T_{\mu\nu\kappa} + T_{\nu\mu\kappa}) \\
&= -K_{\nu\mu\kappa}
\end{aligned}
$$

Exercise 7. Consider

$$
\begin{aligned}
T(fX, gY) &= f\nabla_X(gY) - g\nabla_Y(fX) - fg[X,Y] - fX[g]Y + gY[f]X \\
&= fX[g]Y + fg\nabla_X Y - gY[f]X - gf\nabla_Y X - fg[X,Y] - fX[g]Y + gY[f]X \\
&= fg(\nabla_X Y - \nabla_Y X - [X,Y]) \\
&= fgT(X,Y)
\end{aligned}
$$

where we have used

$$[fX, gY] = fg[X,Y] + fX[g]Y - gY[f]X$$

Thus

$$T(X,Y) = X^{\mu}Y^{\nu}T(e_{\mu}, e_{\nu})$$

and T is a type-(1,2) tensor.

Exercise 8. Let ∇ be a Levi-Civita connection on M.

(a) Let $f \in C^{\infty}(M)$. We have

$$
\begin{aligned}
\nabla_{\nu}\nabla_{\nu}f &= \nabla_{\mu}\partial_{\nu}f \\
&= \partial_{\mu}\partial_{\nu}f + \Gamma^{\lambda}{}_{\mu\nu}\partial_{\lambda}f \\
&= \partial_{\nu}\partial_{\mu}f + \Gamma^{\lambda}{}_{\nu\mu}\partial_{\lambda}f \\
&= \nabla_{\nu}\nabla_{\mu}f
\end{aligned}
$$

(b) Let $\omega \in \Omega^1(M)$, in local coordinates $\omega = \omega_\mu dx^\mu$. Then

$$d\omega = \partial_\mu \omega_\nu dx^\mu \wedge dx^\nu$$

We also have

$$(\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu}\omega_\lambda$$

Now, consider

$$(\nabla_\mu \omega)_\nu dx^\mu \wedge dx^\nu = (\partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu}\omega_\lambda) dx^\mu \wedge dx^\nu$$

The second term in the brackets vanishes since Γ is symmetric in μ and ν, but $dx^\mu \wedge dx^\nu$ is antisymmetric in these indices. Thus

$$d\omega = (\nabla_\mu \omega)_\nu dx^\mu \wedge dx^\nu$$

(c) Let $\omega \in \Omega^1(M)$ and $U \in \mathrm{Vect}(M)$ be the corresponding vector field, i.e. $U^\mu = g^{\mu\nu}\omega_\nu$. Then for any $X \in \mathrm{Vect}(M)$,

$$\begin{aligned}
(\nabla_X U)^\mu &= X^\lambda(\partial_\lambda U^\mu + U^\rho \Gamma^\mu{}_{\lambda\rho}) \\
g_{\mu\nu}(\nabla_X U)^\mu &= X^\lambda g_{\mu\nu}\partial_\lambda U^\mu + X^\lambda U^\rho \Gamma^\mu{}_{\lambda\rho} \\
&= X^\lambda(\partial_\lambda(g_{\mu\nu}U^\mu) - U^\mu \partial_\lambda g_{\mu\nu}) + X^\lambda U^\rho \Gamma^\mu{}_{\lambda\rho} g_{\mu\nu} \\
&= X^\lambda \partial_\lambda \omega_\nu - X^\lambda U^\mu \partial_\lambda g_{\mu\nu} + X^\lambda g_{\mu\nu} U^\rho \Gamma^\mu{}_{\lambda\rho} \\
&= X^\lambda \partial_\lambda \omega_\nu + X^\lambda U^\mu(-\partial_\lambda g_{\mu\nu} + g_{\mu\nu}\Gamma^\mu{}_{\lambda\rho})
\end{aligned}$$

We also have

$$\begin{aligned}
(\nabla_X \omega)_\nu &= X^\lambda \partial_\lambda \omega_\nu - X^\lambda \Gamma^\rho{}_{\lambda\nu}\omega_\rho \\
&= X^\lambda \partial_\lambda \omega_\nu - X^\lambda U^\mu g_{\mu\rho}\Gamma^\rho{}_{\lambda\nu}
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma^\mu{}_{\lambda\rho} &= \frac{1}{2} g^{\mu\kappa}(\partial_\lambda g_{\kappa\rho} + \partial_\rho g_{\kappa\mu} - \partial_\kappa g_{\lambda\rho}) \\
g_{\mu\nu}\Gamma^\mu{}_{\lambda\rho} &= \frac{1}{2}(\partial_\lambda g_{\nu\rho} + \partial_\rho g_{\nu\mu} - \partial_\nu g_{\lambda\rho}) \\
g_{\mu\nu}\Gamma^\mu{}_{\lambda\rho} - \partial_\lambda g_{\mu\nu} &= \frac{1}{2}(\partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\mu} - \partial_\nu g_{\lambda\rho}) \\
&= -g_{\mu\rho}\Gamma^\rho{}_{\lambda\nu}
\end{aligned}$$

Thus

$$g_{\mu\nu}(\nabla_X U)^\mu = (\nabla_X \omega)_\nu$$

That is,

$$g(\nabla_X U, V) = \langle \nabla_X \omega, V \rangle$$

for all vector fields V.

Exercise 9. Consider the cylinder, $M = S^1 \times \mathbb{R}$, with the usual metric

$$g = d\phi \otimes d\phi + dz \otimes dz$$

Clearly the Christoffel symbols vanish, giving the geodesic equations

$$\frac{d^2\phi}{dt^2} = 0$$
$$\frac{d^2 z}{dt^2} = 0$$

which are solved by

$$(\phi, z) = (at + b, ct + d)$$

This describes a helix.

Exercise 10. I am only going to calculate Christoffels here and save myself the pain of calculating the full Riemann tensor.

(a) Consider the metric

$$g = dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)$$

on $\mathbb{R}^3$. We have

$$F = \frac{1}{2}(r'^2 + r^2\theta'^2 + r^2\sin^2\theta\phi'^2)$$

The EL equations are:

$$\frac{d}{ds}(r') - r\theta'^2 - r\sin^2\theta\phi'^2 = 0$$
$$\frac{d^2r}{ds^2} - r\frac{d\theta}{ds}\frac{d\theta}{ds} - r\sin^2\theta\frac{d\phi}{ds}\frac{d\phi}{ds} = 0$$

giving

$$\Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

Then

$$\frac{d}{ds}(r^2\theta') - r^2\sin\theta\cos\theta\phi'^2 = 0$$
$$2r\frac{dr}{ds}\frac{d\theta}{ds} + r^2\frac{d^2\theta}{ds^2} - r^2\sin\theta\cos\theta\frac{d\phi}{ds}\frac{d\phi}{ds} = 0$$
$$\frac{d^2\theta}{ds^2} + \frac{1}{2r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\frac{d\phi}{ds}\frac{d\phi}{ds} = 0$$

giving

$$\Gamma^{\theta}{}_{r\theta} = \frac{1}{r}, \quad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

Lastly,

$$\begin{aligned}
\frac{d}{ds}(r^2\sin^2\theta\phi') &= 0 \\
2r\frac{dr}{ds}\sin^2\theta\frac{d\phi}{ds} + r^2 2\sin\theta\cos\theta\frac{d\theta}{ds}\frac{d\phi}{ds} + r^2\sin^2\theta\frac{d^2\phi}{ds^2} &= 0 \\
\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} &= 0
\end{aligned}$$

giving

$$\Gamma^{\phi}{}_{r\phi} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$$

and all other Christoffels are zero.

(b) Consider $\mathbb{R}^4$ with the Robertson-Walker metric

$$g = -dt\otimes dt + a^2(t)\left(\frac{dr\otimes dr}{1-kr^2} + r^2(d\theta\otimes d\theta + \sin^2\theta d\phi\otimes d\phi)\right)$$

where $k = \pm 1$ or 0. We have

$$F = \frac{1}{2}\left(t'^2 + a^2\left(\frac{r'^2}{1-kr^2} + r^2(\theta'^2 + \sin^2\theta\phi'^2)\right)\right)$$

So

$$\frac{d^2t}{ds^2} - 2a\frac{da}{dt}\left(\frac{1}{1-kr^2}\frac{dr}{ds}\frac{dr}{ds} + r^2\left(\frac{d\theta}{ds}\frac{d\theta}{ds} + \sin^2\theta\frac{d\phi}{ds}\frac{d\phi}{ds}\right)\right) = 0$$

gives

$$\begin{aligned}
\Gamma^{t}{}_{rr} &= -2a\frac{da}{dt}\frac{1}{1-kr^2} \\
\Gamma^{t}{}_{\theta\theta} &= -2a\frac{da}{dt}r^2 \\
\Gamma^{t}{}_{\phi\phi} &= -2\frac{da}{dt}r^2\sin^2\theta
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{ds}\left(a^2\frac{1}{1-kr^2}\frac{dr}{ds}\right) - r\left(\frac{d\theta}{ds}\frac{d\theta}{ds} + \sin^2\theta\frac{d\phi}{ds}\frac{d\phi}{ds}\right) &= 0 \\
2a\frac{da}{dt}\frac{dt}{ds}\frac{1}{1-kr^2}\frac{dr}{ds} + a^2\frac{2kr}{(1-kr^2)^2}\frac{dr}{ds}\frac{dr}{ds} + a^2\frac{1}{1-kr^2}\frac{d^2r}{ds^2} - r\frac{d\theta}{ds}\frac{d\theta}{ds} - r\sin^2\theta\frac{d\phi}{ds}\frac{d\phi}{ds} &= 0 \\
\frac{d^2r}{ds^2} + \frac{2}{a}\frac{da}{dt}\frac{dt}{ds}\frac{dr}{ds} + \frac{2kr}{1-kr^2}\frac{dr}{ds}\frac{dr}{ds} - r\frac{d\theta}{ds}\frac{d\theta}{ds} - r\sin^2\theta\frac{d\phi}{ds}\frac{d\phi}{ds} &= 0
\end{aligned}$$

gives

$$\begin{aligned}
{\Gamma^r}_{tr} &= \frac{1}{a}\frac{da}{dt} \\
{\Gamma^r}_{rr} &= \frac{2kr}{1-kr^2} \\
{\Gamma^r}_{\theta\theta} &= -r \\
{\Gamma^r}_{\phi\phi} &= -r\sin^2\theta
\end{aligned}$$

Next

$$\begin{aligned}
\frac{d}{ds}\left(a^2r^2\frac{d\theta}{ds}\right) - a^2r^2\sin\theta\cos\theta\frac{d\phi}{dt}\frac{d\phi}{dt} &= 0 \\
2a\frac{da}{dt}\frac{dt}{ds}r^2\frac{d\theta}{ds} + a^2 2r\frac{dr}{ds}\frac{d\theta}{ds} + a^2r^2\frac{d^2\theta}{ds^2} - a^2r^2\sin\theta\cos\theta\frac{d\phi}{dt}\frac{d\phi}{dt} &= 0 \\
\frac{d^2\theta}{ds^2} + \frac{2}{a}\frac{da}{dt}\frac{dt}{ds}\frac{d\theta}{ds} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\frac{d\phi}{ds}\frac{d\phi}{ds} &= 0
\end{aligned}$$

gives

$$\begin{aligned}
{\Gamma^\theta}_{t\theta} &= \frac{1}{a}\frac{da}{dt} \\
{\Gamma^\theta}_{r\theta} &= \frac{1}{r} \\
{\Gamma^\theta}_{\phi\phi} &= -\sin\theta\cos\theta
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{d}{ds}\left(a^2r^2\sin^2\theta\frac{d\phi}{ds}\right) &= 0 \\
2a\frac{da}{dt}\frac{dt}{ds}r^2\sin^2\theta\frac{d\phi}{ds} + a^2 2r\frac{dr}{ds}\sin^2\theta\frac{d\phi}{ds} + a^2r^2 2\sin\theta\cos\theta\frac{d\theta}{ds}\frac{d\phi}{ds} + a^2r^2\sin^2\theta\frac{d^2\phi}{ds^2} &= 0 \\
\frac{d^2\phi}{ds^2} + \frac{2}{a}\frac{da}{dt}\frac{dt}{ds}\frac{d\phi}{ds} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} &= 0
\end{aligned}$$

gives

$$\begin{aligned}
{\Gamma^\phi}_{t\phi} &= \frac{1}{a}\frac{da}{dt} \\
{\Gamma^\phi}_{r\phi} &= \frac{1}{r} \\
{\Gamma^\phi}_{\theta\phi} &= \cot\theta
\end{aligned}$$

(c) Consider $\mathbb{R}^4$ with the Schwarzschild metric

$$g = Bdt \otimes dt + \frac{1}{B}dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)$$

where

$$B = 1 - \frac{2M}{r}$$

We have

$$F = \frac{1}{2}\left(Bt'^2 + \frac{1}{B}r'^2 + r^2(\theta'^2 + \sin^2\theta\phi'^2)\right)$$

Firstly,

$$\frac{d}{ds}\left(B\frac{dt}{ds}\right) = 0$$

$$\frac{dB}{dr}\frac{dr}{ds}\frac{dt}{ds} + B\frac{d^2t}{ds^2} = 0$$

so

$$\Gamma^t{}_{rt} = \frac{1}{B}\frac{dB}{dr} = \frac{r}{r-2M}\frac{2M}{r^2} = \frac{2M}{r(r-2M)} = \frac{1}{r(r/2M-1)}$$

Then,

$$\frac{d}{ds}\left(\frac{1}{B}\frac{dr}{ds}\right) - \frac{1}{2}\frac{dB}{dr}\left(\frac{dt}{ds}\right)^2 - r\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$-\frac{1}{B^2}\frac{dB}{dr}\left(\frac{dr}{ds}\right)^2 + \frac{1}{B}\frac{d^2r}{ds^2} - \frac{1}{2}\frac{dB}{dr}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}\frac{1}{B^2}\frac{dB}{dr}\left(\frac{dr}{ds}\right)^2 - r\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2r}{ds^2} - \frac{1}{B}\frac{dB}{dr}\left(\frac{dr}{ds}\right)^2 - \frac{1}{2B}\frac{dB}{dr}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2B}\frac{dB}{dr}\left(\frac{dr}{ds}\right)^2 - Br\left(\frac{d\theta}{ds}\right)^2 - Br\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

so

$$\Gamma^r{}_{rr} = \frac{1}{B}\frac{dB}{dr} = \frac{1}{r(r/2M-1)}$$

$$\Gamma^r{}_{tt} = -\frac{1}{2B}\frac{dB}{dr} = -\frac{2}{r(r/2M-1)}$$

$$\Gamma^r{}_{\theta\theta} = -Br = 2M - r$$

$$\Gamma^r{}_{\phi\phi} = -Br\sin^2\theta = (2M-r)\sin^2\theta$$

Next,

$$
\begin{aligned}
\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right)-r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2&=0\\
2r\frac{dr}{ds}\frac{d\theta}{ds}+r^2\frac{d^2\theta}{ds^2}-r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2&=0\\
\frac{d^2\theta}{ds^2}+\frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds}-\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2&=0
\end{aligned}
$$

so

$$
\begin{aligned}
\Gamma^{\theta}{}_{r\theta}&=\frac{1}{r}\\
\Gamma^{\theta}{}_{\phi\phi}&=-\sin\theta\cos\theta
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
\frac{d}{ds}\left(r^2\sin^2\theta\frac{d\phi}{ds}\right)&=0\\
2r\frac{dr}{ds}\sin^2\theta\frac{d\phi}{ds}+r^2 2\sin\theta\cos\theta\frac{d\theta}{ds}\frac{d\phi}{ds}+r^2\sin^2\theta\frac{d^2\phi}{ds^2}&=0\\
\frac{d^2\phi}{ds^2}+\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds}+2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds}&=0
\end{aligned}
$$

so

$$
\begin{aligned}
\Gamma^{\phi}{}_{r\phi}&=\frac{1}{r}\\
\Gamma^{\phi}{}_{\theta\phi}&=\cot\theta
\end{aligned}
$$

Exercise 11. We have

$$
\begin{aligned}
R^{\rho}{}_{\lambda\mu\nu}&=\partial_\mu\Gamma^{\rho}{}_{\nu\lambda}+\Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\rho}{}_{\mu\eta}-(\mu\leftrightarrow\nu)\\
R_{\kappa\lambda\mu\nu}&=g_{\kappa\rho}(\partial_\mu\Gamma^{\rho}{}_{\nu\lambda}+\Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\rho}{}_{\mu\eta})-(\mu\leftrightarrow\nu)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Gamma^{\rho}{}_{\nu\lambda}&=\frac{1}{2}g^{\rho\sigma}(\partial_\nu g_{\lambda\sigma}+\partial_\lambda g_{\nu\sigma}-\partial_\sigma g_{\nu\lambda})\\
\partial_\mu\Gamma^{\rho}{}_{\nu\lambda}&=\frac{1}{2}\partial_\mu g^{\rho\sigma}(\partial_\nu g_{\lambda\sigma}+\partial_\lambda g_{\nu\sigma}-\partial_\sigma g_{\nu\lambda})\\
&\quad+\frac{1}{2}g^{\rho\sigma}(\partial_\mu\partial_\nu g_{\lambda\sigma}+\partial_\mu\partial_\lambda g_{\nu\sigma}-\partial_\mu\partial_\sigma g_{\nu\lambda})
\end{aligned}
$$

and

$$\Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\rho}{}_{\mu\eta} = \frac{1}{4}g^{\eta\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda})$$
$$g^{\rho\tau}(\partial_\mu g_{\tau\eta} + \partial_\eta g_{\tau\mu} - \partial_\tau g_{\eta\mu})$$

Thus

$$\begin{aligned}
R_{\kappa\lambda\mu\nu} =& \frac{1}{2}\partial_\mu g^{\rho\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) + \frac{1}{2}g^{\rho\sigma}(\partial_\mu\partial_\nu g_{\lambda\sigma} + \partial_\mu\partial_\lambda g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\nu\lambda}) \\
&+ \frac{1}{4}g^{\eta\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda})g^{\rho\tau}(\partial_\mu g_{\tau\eta} + \partial_\eta g_{\tau\mu} - \partial_\tau g_{\eta\mu}) - (\mu \leftrightarrow \nu) \\
=& -\frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\kappa\rho}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) + \frac{1}{2}(\partial_\mu\partial_\lambda g_{\nu\kappa} - \partial_\mu\partial_\kappa g_{\nu\lambda}) \\
&+ \frac{1}{4}g^{\rho\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda})(\partial_\mu g_{\kappa\rho} + \partial_\rho g_{\kappa\mu} - \partial_\kappa g_{\rho\mu}) - (\mu \leftrightarrow \nu) \\
=& -\frac{1}{4}g^{\rho\sigma}\partial_\mu g_{\kappa\rho}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) + \frac{1}{2}(\partial_\mu\partial_\lambda g_{\nu\kappa} - \partial_\mu\partial_\kappa g_{\nu\lambda}) \\
&+ \frac{1}{4}g^{\rho\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda})(\partial_\rho g_{\kappa\mu} - \partial_\kappa g_{\rho\mu}) - (\mu \leftrightarrow \nu) \\
=& \frac{1}{2}(\partial_\mu\partial_\lambda g_{\nu\kappa} - \partial_\mu\partial_\kappa g_{\nu\lambda}) + \frac{1}{4}g^{\rho\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda})(\partial_\rho g_{\kappa\mu} - \partial_\kappa g_{\rho\mu} - \partial_\mu g_{\kappa\rho}) - (\mu \leftrightarrow \nu) \\
=& \frac{1}{2}(\partial_\mu\partial_\lambda g_{\nu\kappa} - \partial_\mu\partial_\kappa g_{\nu\lambda}) - g_{\tau\eta}\Gamma^{\tau}{}_{\kappa\mu}\Gamma^{\eta}{}_{\lambda\nu} - (\mu \leftrightarrow \nu)
\end{aligned}$$

(Correct up to an overall sign.)

Exercise 12. Consider $\mathbb{R}^2$ with the metric

$$g = -dt \otimes dt + R^2(t)dx \otimes dx$$

We have

$$F = \frac{1}{2}(-t'^2 + R^2x'^2)$$

The first EL equation is

$$-\frac{d^2t}{ds^2} - R\frac{dR}{dt}\left(\frac{dx}{dt}\right)^2 = 0$$

giving

$$\Gamma^{t}{}_{xx} = R\frac{dR}{dt}$$

and the second is

$$\frac{d}{ds}\left(R^2\frac{dx}{ds}\right) = 0$$

$$2R\frac{dR}{dt}\frac{dt}{ds}\frac{dx}{ds} + R^2\frac{d^2x}{ds^2} = 0$$

giving

$$\Gamma^x{}_{xt} = \frac{1}{R}\frac{dR}{dt}$$

Now, we have

$$\begin{aligned}
R^x{}_{txt} &= \partial_x \Gamma^x{}_{tt} - \partial_t \Gamma^x{}_{xt} + \Gamma^\eta{}_{tt}\Gamma^x{}_{x\eta} - \Gamma^\eta{}_{xt}\Gamma^x{}_{t\eta} \\
&= 0 - \frac{d}{dt}\left(\frac{1}{R}\frac{dR}{dt}\right) + 0 - \frac{1}{R}\frac{dR}{dt}\frac{1}{R}\frac{dR}{dt} \\
&= \frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 - \frac{1}{R}\frac{d^2R}{dt^2} - \frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 \\
&= -\frac{1}{R}\frac{d^2R}{dt^2} \\
&\equiv A
\end{aligned}$$

where the last line defines A. Now, the Ricci tensor, regarded as a matrix, is

$$\mathrm{Ric}_{\mu\nu} = \begin{pmatrix} R^t{}_{ttt} + R^x{}_{txt} & R^t{}_{xtt} + R^x{}_{xxt} \\ R^t{}_{ttx} + R^x{}_{txx} & R^t{}_{xtx} + R^x{}_{xxx} \end{pmatrix}$$

We have

$$R^t{}_{ttt} = R^t{}_{xtt} = R^x{}_{xxt} = R^t{}_{ttx} = R^x{}_{txx} = R^x{}_{xxx} = 0$$

by antisymmetry properties of the Riemann tensor, leaving just

$$\mathrm{Ric}_{\mu\nu} = \begin{pmatrix} R^x{}_{txt} & 0 \\ 0 & R^t{}_{xtx} \end{pmatrix}$$

Now, we have

$$\begin{aligned}
R_{xtxt} &= g_{x\mu} R^\mu{}_{txt} \\
&= R^2 R^x{}_{txt} \\
&= R^2 A
\end{aligned}$$

and $R_{xtxt} = R_{txtx}$, so

$$\begin{aligned}
R^2 A &= R_{txtx} \\
&= g_{t\mu} R^\mu{}_{xtx} \\
&= -R^t{}_{xtx}
\end{aligned}$$

Therefore

$$\mathrm{Ric}_{\mu\nu} = \begin{pmatrix} A & \\ & -R^2 A \end{pmatrix}$$

Then

$$
\begin{aligned}
\mathcal{R} &= g^{\mu\nu}\,\mathrm{Ric}_{\mu\nu} \\
&= -A + \frac{1}{R^2}(-R^2 A) \\
&= -2A
\end{aligned}
$$

Therefore the Einstein tensor, regarded as a matrix, is

$$
\begin{aligned}
G_{\mu\nu} &= \begin{pmatrix} A & \\ & -R^2 A \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & \\ & R^2 \end{pmatrix}(-2A) \\
&= 0
\end{aligned}
$$

Exercise 13. Let (M, g) be two-dimensional. Then $F(2) = 1$, so the Riemann tensor only has a single independent component. We claim that

$$
R_{\kappa\lambda\mu\nu} = K(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})
$$

for some function K. That is, the factor in the brackets spans the space of possible Riemann tensors allowed by symmetry constraints. Since this space is one-dimensional, we just need to show that this is in it, i.e. that it exhibits the appropriate symmetries. Clearly $R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\mu\nu}$ and $R_{\kappa\lambda\nu\mu} = -R_{\kappa\lambda\mu\nu}$. Then

$$
\begin{aligned}
R^{\kappa}{}_{\lambda\mu\nu} &= g^{\kappa\sigma}R_{\sigma\lambda\mu\nu} \\
&= Kg^{\kappa\sigma}(g_{\sigma\mu}g_{\lambda\nu} - g_{\sigma\nu}g_{\lambda\mu}) \\
&= K(\delta^{\kappa}_{\mu}g_{\lambda\nu} - \delta^{\kappa}_{\nu}g_{\lambda\mu})
\end{aligned}
$$

So

$$
\begin{aligned}
R^{\kappa}{}_{[\lambda\mu\nu]} &= \frac{K}{3!}\big(\delta^{\kappa}_{\mu}g_{\lambda\nu} - \delta^{\kappa}_{\nu}g_{\lambda\mu} + \delta^{\kappa}_{\nu}g_{\mu\lambda} - \delta^{\kappa}_{\lambda}g_{\mu\nu} \\
&\qquad + \delta^{\kappa}_{\lambda}g_{\mu\nu} - \delta^{\kappa}_{\mu}g_{\nu\lambda} - \delta^{\kappa}_{\nu}g_{\mu\lambda} + \delta^{\kappa}_{\mu}g_{\lambda\nu} \\
&\qquad - \delta^{\kappa}_{\lambda}g_{\nu\mu} + \delta^{\kappa}_{\nu}g_{\mu\lambda} - \delta^{\kappa}_{\mu}g_{\lambda\nu} + \delta^{\kappa}_{\lambda}g_{\nu\mu}\big) \\
&= 0
\end{aligned}
$$

Therefore this is indeed the correct form for the Riemann tensor. To find K, first contract to obtain the Ricci tensor

$$
\begin{aligned}
\mathrm{Ric}_{\kappa\mu} &= g^{\lambda\nu}R_{\kappa\lambda\mu\nu} \\
&= Kg^{\lambda\nu}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}) \\
&= K(2g_{\kappa\mu} - g_{\kappa\mu}) \\
&= Kg_{\kappa\mu}
\end{aligned}
$$

and then again to obtain the scalar curvature

$$\begin{aligned}\mathcal{R} &= g^{\mu\nu} K g_{\mu\nu} \\ &= 2K\end{aligned}$$

so

$$K = \frac{1}{2}\mathcal{R}$$

Thus in two dimensions, the Riemann tensor is

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2}\mathcal{R}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})$$

Exercise 14. Consider the upper half-plane with the Poincaré metric and Levi-Civita connection.

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

We have

$$F = \frac{1}{2y^2}(x'^2 + y'^2)$$

The first EL equation is

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{y^2}\frac{dx}{ds}\right) &= 0 \\ -\frac{2}{y^3}\frac{dy}{ds}\frac{dx}{ds} + \frac{1}{y^2}\frac{d^2x}{ds^2} &= 0\end{aligned}$$

giving

$$\Gamma^x{}_{xy} = -\frac{1}{y}$$

The second EL equation is

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{y^2}\frac{dy}{ds}\right) + \frac{1}{y^3}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) &= 0 \\ -\frac{2}{y^3}\left(\frac{dy}{ds}\right)^2 + \frac{1}{y^2}\frac{d^2y}{ds^2} + \frac{1}{y^3}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) &= - \\ \frac{d^2y}{ds^2} + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}\left(\frac{dy}{ds}\right)^2 &= 0\end{aligned}$$

giving

$$\Gamma^y{}_{xx} = \frac{1}{y}, \quad \Gamma^y{}_{yy} = -\frac{1}{y}$$

To calculate the holonomy of this connection, we will consider the parallel transport of a vector $X = X^x e_x + X^y e_y$ around a rectangle $pqrs$, where, WLOG,

$$\begin{aligned}
p &= (0, 1) \\
q &= (a, 1) \\
r &= (a, 1 + b) \\
s &= (0, 1 + b)
\end{aligned}$$

(i) On the path pq, we have $\nabla_x X_{pq} = 0$, so

$$(\partial_x X^\mu_{pq} + X^\nu_{pq} \Gamma^\mu{}_{x\nu}) e_\mu = 0$$

This is the pair of equations

$$\begin{aligned}
\partial_x X^x_{pq} - \frac{1}{y} X^y_{pq} &= 0 \\
\partial_x X^y_{pq} + \frac{1}{y} X^x_{pq} &= 0
\end{aligned}$$

Then we have

$$\partial_x^2 X^x_{pq} + \frac{1}{y^2} X^x_{pq} = 0$$

which has a general solution

$$X^x_{pq}(x) = A \cos \frac{x}{y} + B \sin \frac{x}{y}$$

Setting $X^x_{pq}(0) = X^x_p$ and $X^y_{pq}(0) = X^y_p$, we have $A = X^x_p$ and $B = X^y_p$, and parameterising $x = t$, $y = 1$, this is

$$\begin{aligned}
X^x_{pq}(t) &= X^x_p \cos t + X^y_p \sin t \\
X^y_{pq}(t) &= -X^x_p \sin t + X^y_p \cos t
\end{aligned}$$

(ii) On the path qr, we have $\nabla_y X_{qr} = 0$, so

$$(\partial_y X^\mu_{qr} + X^\nu_{qr} \Gamma^\mu{}_{y\nu}) e_\mu = 0$$

This is the pair of equations

$$\begin{aligned}
\partial_y X^x_{qr} - \frac{1}{y} X^x_{qr} &= 0 \\
\partial_y X^y_{qr} - \frac{1}{y} X^y_{qr} &= 0
\end{aligned}$$

Then we have

$$X^x_{qr}(y) = Ay$$
$$X^y_{qr}(y) = By$$

Setting $X^x_{qr}(1) = X^x_q$ and $X^y_{qr}(1) = X^y_q$, and parameterising $x = 1$, $y = 1 + tb$, this is

$$X^x_{qr}(t) = X^x_q(1 + tb)$$
$$X^y_{qr}(t) = X^y_q(1 + tb)$$

(iii) On the path rs, we have $-\nabla_x X_{rs} = 0$, so as on pq, we have the general solution

$$X^x_{rs}(x) = A\cos\frac{x}{y} + B\sin\frac{x}{y}$$
$$X^y_{rs}(x) = -A\sin\frac{x}{y} + B\cos\frac{x}{y}$$

Setting $X^x_{rs}(a) = X^x_r$ and $X^y_{rs}(a) = X^y_r$, and fixing $y = 1 + b$, we have

$$A = X^x_r \cos\frac{a}{1+b} - X^y_r \sin\frac{a}{1+b}$$
$$B = X^x_r \sin\frac{a}{1+b} + X^y_r \cos\frac{a}{1+b}$$

Parameterising $t = 1 - x/a$, we have

$$X^x_{rs}(t) = A\cos\frac{a(1-t)}{1+b} + B\sin\frac{a(1-t)}{1+b}$$
$$X^y_{rs}(t) = -A\sin\frac{a(1-t)}{1+b} + B\cos\frac{a(1-t)}{1+b}$$

(iv) On the path sp, we have $-\nabla_y X_{sp} = 0$, so as on qr, we have the general solution

$$X^x_{sp}(y) = Cy$$
$$X^y_{sp}(y) = Dy$$

Setting $X^x_{sp}(1+b) = X^x_s$ and $X^y_{sp}(1+b) = X^y_s$, and parameterising $t = 1 - (y-1)/b$, this is

$$X^x_{sp}(t) = X^x_s\left(1 - \frac{bt}{1+b}\right)$$
$$X^y_{sp}(t) = X^y_s\left(1 - \frac{bt}{1+b}\right)$$

Now we want to patch this all together. First, set $X_q = X_{pq}(t=1)$, i.e.

$$X_q^x = X_p^x \cos 1 + X_p^y \sin 1$$
$$X_q^y = -X_p^x \sin 1 + X_p^y \cos 1$$

Then, set $X_r = X_{qr}(t=1)$, i.e.

$$X_r^x = (1+b)(X_p^x \cos 1 + X_p^y \sin 1)$$
$$X_r^y = (1+b)(-X_p^x \sin 1 + X_p^y \cos 1)$$

so

$$A = (1+b)(X_p^x \cos 1 + X_p^y \sin 1) \cos \frac{a}{1+b} - (1+b)(-X_p^x \sin 1 + X_p^y \cos 1) \sin \frac{a}{1+b}$$
$$B = (1+b)(X_p^x \cos 1 + X_p^y \sin 1) \sin \frac{a}{1+b} + (1+b)(-X_p^x \sin 1 + X_p^y \cos 1) \cos \frac{a}{1+b}$$

Lastly, set $X_s = X_{rs}(t=1)$, i.e.

$$X_s^x = A$$
$$X_s^y = B$$

Then finally we have the result of parallel transport around the full rectangle:

$$\tilde{X}^x = X_{sp}^x(t=1) = \frac{A}{1+b}$$
$$\tilde{X}^y = X_{sp}^y(t=1) = \frac{B}{1+b}$$

That is,

$$\begin{pmatrix} \tilde{X}^x \\ \tilde{X}^y \end{pmatrix} = \begin{pmatrix} \cos 1 \cos \frac{a}{1+b} + \sin 1 \sin \frac{a}{1+b} & \sin 1 \cos \frac{a}{1+b} - \cos 1 \sin \frac{a}{1+b} \\ \cos 1 \sin \frac{a}{1+b} - \sin 1 \cos \frac{a}{1+b} & \sin 1 \sin \frac{a}{1+b} + \cos 1 \cos \frac{a}{1+b} \end{pmatrix} \begin{pmatrix} X_p^x \\ X_p^y \end{pmatrix}$$

This matrix always has determinant 1, and is parameterised by the single number $a/(1+b)$. (It is easy to see that more generally this will be $(x_0+a)/(y_0+b)$ if we are finding the holonomy at (x_0, y_0).) Therefore $H(0,1) \cong SO(2)$, and since the upper half plane is connected,

$$H(p) \cong SO(2)$$

for all p.

Exercise 15. Whether a vector is timelike, null, or spacelike depends on whether its norm is negative, zero, or positive. But $e^{2\sigma}$ is positive-definite, so this is invariant under conformal transformations.

Exercise 16. The Milne universe has the metric

$$g = -dt \otimes dt + t^2 dx \otimes dx$$

If we change coordinates such that $|t| \to e^\eta$, $|dt| \to d(e^\eta) = e^\eta d\eta$, so

$$\begin{aligned} g &= -e^{2\eta} d\eta \otimes d\eta + e^{2\eta} dx \otimes dx \\ &= e^{2\eta}(-d\eta \otimes d\eta + dx \otimes dx) \end{aligned}$$

so g is conformally Lorentz-flat. Next, consider the transformation

$$(\eta, x) \mapsto (u = e^\eta \sinh x, v = e^\eta \cosh x)$$

We have

$$\begin{aligned} du &= e^\eta \sinh x d\eta + e^\eta \cosh x dx = ud\eta + vdx \\ dv &= e^\eta \cosh x d\eta + e^\eta \sinh x dx = vd\eta + udx \end{aligned}$$

Rearranging,

$$\begin{aligned} dx &= \frac{1}{u^2 - v^2}(udv - vdu) \\ d\eta &= \frac{1}{u^2 - v^2}(udu - vdv) \end{aligned}$$

Then

$$\begin{aligned} d\eta \otimes d\eta &= \frac{1}{(u^2 - v^2)^2}(u^2 du \otimes du - 2uvdu \otimes dv + v^2 dv \otimes dv) \\ dx \otimes dx &= \frac{1}{(u^2 - v^2)^2}(u^2 dv \otimes dv - 2uvdu \otimes dv + v^2 dv \otimes dv) \end{aligned}$$

so

$$\begin{aligned} -d\eta \otimes d\eta + dx \otimes dx &= \frac{1}{(u^2 - v^2)^2}(u^2 - v^2)(du \otimes du + dv \otimes dv) \\ &= \frac{1}{u^2 - v^2}(du \otimes du + dv \otimes dv) \end{aligned}$$

Thus

$$g = e^{2u} \frac{1}{u^2 - v^2}(du \otimes du + dv \otimes dv)$$

Exercise 17. Let ∇ be the Levi-Civita connection. Consider

$$\begin{aligned}(\nabla_\mu X)_\nu + (\nabla_\nu X)_\mu &= \partial_\mu X_\nu + \partial_\nu X_\mu - 2\Gamma^\lambda{}_{\mu\nu} X_\lambda \\ &= \partial_\mu X_\nu + \partial_\nu X_\mu - g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) X_\lambda\end{aligned}$$

We have

$$\begin{aligned}\partial_\mu X_\nu &= \partial_\mu (g_{\nu\lambda} X^\lambda) \\ &= \partial_\mu g_{\nu\lambda} X^\lambda + g_{\nu\lambda} \partial_\mu X^\lambda\end{aligned}$$

so this is

$$\begin{aligned}&\partial_\mu g_{\nu\lambda} X^\lambda + \partial_\nu g_{\mu\lambda} X^\lambda + g_{\nu\lambda}\partial_\mu X^\lambda + g_{\mu\lambda}\partial_\nu X^\lambda - (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) X^\kappa \\ &= g_{\nu\lambda}\partial_\mu X^\lambda + g_{\mu\lambda}\partial_\nu X^\lambda - \partial_\kappa g_{\mu\nu} X^\kappa\end{aligned}$$

But this is just the LHS of the Killing equation. Thus the Killing equation is equivalent to

$$(\nabla_{(\mu} X)_{\nu)} = 0$$

Exercise 18. Knowing the symmetries of $(\mathbb{R}^2, \delta)$, it is easy to guess what the Killing vector fields are.

(i) Let $X = e_x$. Then

$$\begin{aligned}(\mathcal{L}_X g)_{\mu\nu} &= X^\lambda \partial_\lambda \delta_{\mu\nu} + \partial_\mu X^\lambda \delta_{\lambda\nu} + \partial_\nu X^\lambda \delta_{\mu\lambda} \\ &= 0\end{aligned}$$

corresponding to x-translations.

(ii) Similarly, $Y = e_y$ is the Killing vector field corresponding to y-translations.

(iii) Let $Z = -ye_x + xe_y$. Then

$$\begin{aligned}(\mathcal{L}_Z g)_{\mu\nu} &= \partial_\mu Z^\lambda \delta_{\lambda\nu} + \partial_\nu X^\lambda \delta_{\mu' lambda} \\ &= \partial_\mu Z_\nu + \partial_\nu Z_\mu\end{aligned}$$

So

$$\begin{aligned}(\mathcal{L}_Z g)_{11} &= 0 \\ (\mathcal{L}_Z g)_{12} &= 1 - 1 = 0 \\ (\mathcal{L}_Z g)_{21} &= -1 + 1 = 0 \\ (\mathcal{L}_Z g)_{22} &= 0\end{aligned}$$

So this is a Killing vector, corresponding to rotations.

Exercise 19.

(a) We have

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{\alpha\beta} \hat{\theta}^\alpha \otimes \hat{\theta}^\beta$$

Then

$$\begin{aligned} g(\hat{e}_\alpha, \hat{e}_\beta) = g_{\mu\nu} dx^\mu(\hat{e}_\alpha) \otimes dx^\nu(\hat{e}_\beta) = \delta_{\alpha\beta} \\ g_{\mu\nu} dx^\mu(e_\alpha{}^\rho e_\rho) \otimes dx^\nu(e_\beta{}^\sigma e_\sigma) = \delta_{\alpha\beta} \\ g_{\mu\nu} e_\alpha{}^\mu e_\beta{}^\nu = \delta_{\alpha\beta} \end{aligned}$$

(b) In flat spacetime, the γ^α satisfy

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$$

Then define their curved spacetime analogues $\gamma^\mu = e_\alpha{}^\mu \gamma^\alpha$. Then we have

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= \{e_\alpha{}^\mu, \gamma^\alpha, e_\beta{}^\nu \gamma^\beta\} \\ &= e_\alpha{}^\mu e_\beta{}^\nu \{\gamma^\alpha, \gamma^\beta\} \\ &= e_\alpha{}^\mu e_\beta{}^\nu 2\eta^{\alpha\beta} \\ &= 2g^{\mu\nu} \end{aligned}$$

Exercise 20. In the non-coordinate basis, $\{\hat{\theta}^\alpha\} = \{e^\alpha{}_\mu dx^\mu\}$, we have

$$\begin{aligned} g &= \delta_{\alpha\beta} \hat{\theta}^\alpha \otimes \hat{\theta}^\beta \\ |g| &= |e|^2 \end{aligned}$$

where $e = \det e^\alpha{}_\mu$. Then

$$\begin{aligned} \Omega_M &= |e| dx^1 \wedge ... \wedge dx^m \\ &= \hat{\theta}^1 \wedge ... \wedge \hat{\theta}^m \end{aligned}$$

Exercise 21. Let ω be a 1-form. We have

$$\begin{aligned}
*\omega &= \frac{\sqrt{|g|}}{(m-1)!}\omega_\mu g^{\mu\nu_1}\varepsilon_{\nu_1\nu_2\ldots\nu_m}dx^{\nu_2}\wedge\ldots\wedge dx^{\nu_m}\\
d*\omega &= \frac{1}{(m-1)!}\partial_\nu\left(\sqrt{|g|}\omega_\mu g^{\mu\nu_1}\right)\varepsilon_{\nu_1\nu_2\ldots\nu_m}dx^{\nu}\wedge dx^{\nu_2}\wedge\ldots\wedge dx^{\nu_m}\\
&= \partial_\nu\left(\sqrt{|g|}\omega_\mu g^{\mu\nu}\right)dx^1\wedge\ldots\wedge dx^m\\
d\omega &= \partial_\nu\left(\sqrt{|g|}\omega_\mu g^{\mu\nu}\right)\\
d^\dagger\omega &= (-1)^{3m+1}\partial_\nu\left(\sqrt{|g|}\omega_\mu g^{\mu\nu}\right)\\
&= (-1)^{m+1}\partial_\nu\left(\sqrt{|g|}\omega_\mu g^{\mu\nu}\right)\\
dd^\dagger\omega &= (-1)^{m+1}\partial_\lambda\partial_\nu\left(\sqrt{|g|}\omega_\mu g^{\mu\nu}\right)dx^\lambda
\end{aligned}$$

So, for $(\mathbb{R}^m,\delta)$,

$$dd^\dagger\omega = (-1)^{m+1}\frac{\partial^2\omega_\nu}{\partial x^\nu\partial x^\mu}dx^\mu$$

Then,

$$\begin{aligned}
d\omega &= \partial_\nu\omega_\mu dx^\nu\wedge dx^\mu\\
*d\omega &= \partial_{\mu_1}\omega_{\mu_2}\frac{\sqrt{|g|}}{(m-2)!}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\varepsilon_{\nu_1\nu_2\nu_3\ldots\nu_m}dx^{\nu_3}\wedge\ldots\wedge dx^{\nu_m}\\
d*d\omega &= \frac{1}{(m-2)!}\partial_\lambda\left(\partial_{\mu_1}\omega_{\mu_2}\sqrt{|g|}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\right)\varepsilon_{\nu_1\nu_2\nu_3\ldots\nu_m}dx^\lambda\wedge dx^{\nu_3}\wedge\ldots\wedge dx^{\nu_m}\\
*d*d\omega &= \frac{1}{(m-2)!}\partial_\lambda\left(\partial_{\mu_1}\omega_{\mu_2}\sqrt{|g|}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\right)\varepsilon_{\nu_1\nu_2\nu_3\ldots\nu_m}\varepsilon^{\lambda\nu_3\ldots\nu_m\sigma}g_{\sigma\rho}dx^\rho\\
&= -\frac{1}{(m-2)!}\partial_\lambda\left(\partial_{\mu_1}\omega_{\mu_2}\sqrt{|g|}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\right)\varepsilon_{\nu_3\ldots\nu_m\nu_1\nu_2}\varepsilon^{\nu_3\ldots\nu_m\lambda\sigma}g_{\rho\sigma}dx^\rho\\
&= -\partial_\lambda\left(\partial_{\mu_1}\omega_{\mu_2}\sqrt{|g|}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\right)\delta^{\lambda\sigma}_{\nu_1\nu_2}g_{\sigma\rho}dx^\rho\\
&= -\partial_\lambda\left(\partial_\mu\omega_\nu\sqrt{|g|}g^{\mu\lambda}g^{\nu\sigma}\right)g_{\sigma\rho}dx^\rho\\
d^\dagger d\omega &= (-1)^{3m+1}\partial_\lambda\left(\partial_\mu\omega_\nu\sqrt{|g|}g^{\mu\lambda}g^{\nu\sigma}\right)g_{\sigma\rho}dx^\rho\\
&= (-1)^{m+1}\partial_\lambda\left(\partial_\mu\omega_\nu\sqrt{|g|}g^{\mu\lambda}g^{\nu\sigma}\right)g_{\sigma\rho}dx^\rho
\end{aligned}$$

So, for $(\mathbb{R}^m,\delta)$,

$$d^\dagger d\omega = (-1)^{m+1}\frac{\partial^2\omega_\nu}{\partial x^\mu\partial x^\mu}dx^\nu$$

This gives the correct answer (up to $(-1)^m$), but the contribution due to $dd^\dagger\omega$ is wrong/superfluous?

Exercise 22. We have

$$(d\alpha_{r-1}, d^\dagger\beta_{r+1}) = (\alpha_{r-1}, d^{\dagger 2}\beta_{r+1}) = 0$$

Next

$$(d\alpha_{r-1}, \gamma_r) = (\alpha_{r-1}, d^\dagger\gamma_r)$$

but if γ_r is harmonic, it is coclosed, so this is also zero. Then,

$$(d^\dagger\beta_{r+1}, \gamma_r) = (\beta_{r+1}, d\gamma_r)$$

but if γ_r is harmonic, it is closed, so this is also zero.
Suppose that $\omega_r \in \Omega^r(M)$ satisfies

$$(d\alpha_{r-1}, \omega_r) = 0$$

for any $\alpha_{r-1} \in d\Omega^{r-1}(M)$. Then $d^\dagger\omega_r = 0$, so ω_r is coclosed. If it satisfies

$$(d^\dagger\beta_{r+1}, \omega_r) = 0$$

for any $\beta_{r+1} \in d^\dagger\Omega^{r+1}(M)$, then $d\omega_r = 0$, so ω_r is closed. So if it satisfies both of these, it is harmonic. Then if also

$$(\gamma_r, \omega_r) = 0$$

for any $\gamma_r \in \mathrm{Harm}^r(M)$, then in particular for $\gamma_r = \omega_r$, so ω_r must be zero.

Exercise 23. Suppose $\omega_r \in \Omega^r(M)$ can be written as $\omega_r = \Delta\psi_r$ for some $\psi_r \in \Omega^r(M)$. Then, for any harmonic γ_r,

$$\begin{aligned}
(\omega_r, \gamma_r) &= (\Delta\psi_r, \gamma_r) \\
&= (dd^\dagger\psi_r, \gamma_r) + (d^\dagger d\psi_r, \gamma_r) \\
&= (d^\dagger\psi_r, d^\dagger\gamma_r) + (d\psi_r, d\gamma_r) \\
&= (\psi_r, (dd^\dagger + d^\dagger d)\gamma_r) \\
&= 0
\end{aligned}$$

That is, if ω_r can be written as the Laplacian of another r-form, it is orthogonal to all harmonic forms.