# Blumenhagen, Lüst and Theisen - Basic Concepts of String Theory

Supplementary Notes

January 28, 2019

**Abstract**

These are my personal notes to supplement my reading of the book, made as I go. Notation as used there and page numbers follow first edition paperback.

# Contents

# 2 The Classical Bosonic String

## 2.3 The Polyakov Action and its Symmetries

**Page 15**
We have the Gauss-Bonnet term

$$S_2 = \frac{\lambda_2}{4\pi} \int_\Sigma d^2\sigma \, \sqrt{-h} R$$

which by the Gauss-Bonnet theorem is just $\lambda_2 \chi(\Sigma)$. (For instance, this vanishes for the classical closed string, or equivalently at tree-level for the quantised closed string, for which $\Sigma$ is just a cylinder.) It can be shown that under a Weyl rescaling $h_{\alpha\beta} \to e^{2\Lambda} h_{\alpha\beta}$ we have

$$\sqrt{-h} R \to \sqrt{-h}(R - 2\nabla_\alpha \nabla^\alpha \Lambda)$$

That is,

$$\delta(\sqrt{-h} R) = -2\sqrt{-h} \nabla_\alpha \nabla^\alpha \Lambda$$

Now we note that, for any $v^\alpha$,

$$\nabla_\alpha v^\alpha = \partial_\alpha v^\alpha + \Gamma^\alpha_{\alpha\beta} v^\beta$$

while

$$\partial_\alpha(\sqrt{-h} v^\alpha) = (\partial_\alpha \sqrt{-h}) + \sqrt{-h} \partial_\alpha v^\alpha$$

where

$$\begin{aligned}
\partial_\alpha \sqrt{-h} &= \frac{1}{2} \frac{1}{\sqrt{-h}} h h^{\beta\gamma} \partial_\alpha h_{\beta\gamma} \\
&= \frac{1}{2} \sqrt{-h} h^{\beta\gamma} \partial_\alpha h_{\beta\gamma} \\
&= \frac{1}{2} \sqrt{-h} h^{\beta\gamma} (\Gamma^\delta_{\alpha\beta} h_{\delta\gamma} + \Gamma^\delta_{\alpha\gamma} h_{\beta\delta}) \\
&= \sqrt{-h} h^{\beta\gamma} \Gamma^\delta_{\alpha\beta} h_{\delta\gamma} \\
&= \sqrt{-h} \Gamma^\beta_{\alpha\beta}
\end{aligned}$$

where in the third line we have assumed metric compatibility of $h$ (see below). Then, assuming $\Gamma$ is torsion free (hence Levi-Civita, which it is since it is proportional to the induced metric from the target space, which we certainly will want to be Levi-Civita), we finally have that

$$\sqrt{-h} \nabla_\alpha v^\alpha = \partial_\alpha(\sqrt{-h} v^\alpha)$$

So the variation of the integrand of $S_2$ under Weyl transformations is a total derivative, and therefore $S_2$ is Weyl-invariant if $\partial\Sigma = 0$. Otherwise (Polchinski Exercise 1.3) we must include the additional term

$$S_3 = \frac{1}{2\pi}\int_{\partial\Sigma} ds\ k$$

where

$$k = t^\alpha n_\beta \nabla_\alpha t^\beta$$

in which $t^\alpha$ and $n^\alpha$ are unit vectors tangent and normal, respectively, to $\partial\Sigma$. Now, using Stokes' the variation of $S_2$ with boundary is

$$\delta S_2 = -\frac{1}{2\pi}\int_{\partial\Sigma} ds\ n_\alpha \partial^\alpha \Lambda$$

where $ds = \sqrt{-h_{\tau\tau}}d\tau$ is the induced integration element on the boundary. We want this to cancel with $\delta S_3$. First we note that

$$ds \to \sqrt{-e^{2\Lambda}h_{\tau\tau}}d\tau = e^\Lambda ds$$

Next, if $v^\alpha$ is a unit vector, under Weyl,

$$\gamma_{\alpha\beta}v^\alpha v^\beta \to e^{2\Lambda}v'^\alpha v'^\beta = \pm 1$$

i.e. $v'^\alpha = e^{-\Lambda}v^\alpha$. This is true for both $t^\alpha$ and $n^\alpha$, and lowering with the new metric, $n'_\alpha = e^\Lambda n_\alpha$. So $t^\alpha n_\beta$ is Weyl-invariant. Now we just need to consider $\nabla_\alpha t^\beta$. We have

$$\begin{aligned}
\nabla_\alpha t^\beta &\to \nabla'_\alpha t'^\beta \\
&= \partial_\alpha(e^{-\Lambda}t^\beta) + \Gamma'^\beta_{\alpha\gamma}e^{-\Lambda}t^\gamma \\
&= e^{-\Lambda}\left(-(\partial_\alpha\Lambda)t^\beta + (\partial_\alpha t^\beta + \Gamma'^\beta_{\alpha\gamma}t^\gamma)\right)
\end{aligned}$$

where $\Gamma'$ are the new connection coefficients,

$$\begin{aligned}
\Gamma'^\beta_{\alpha\gamma} &= \frac{1}{2}h'^{\beta\delta}(\partial_\alpha h\gamma\delta' + \partial_\gamma h\alpha\delta' - \partial_\delta h\alpha\gamma') \\
&= \frac{1}{2}e^{-2\Lambda}h^{\beta\delta}(\partial_\alpha(e^{2\Lambda}h\gamma\delta) + \partial_\gamma(e^{2\Lambda}h\alpha\delta) - \partial_\delta(e^{2\Lambda}h\alpha\gamma)) \\
&= \frac{1}{2}h^{\beta\delta}(2(\partial_\alpha\Lambda)h\gamma\delta + \partial_\alpha h\gamma\delta + 2(\partial_\gamma\Lambda)h\alpha\delta + \partial_\gamma h\alpha\delta - 2(\partial_\delta)h\alpha\gamma - \partial_\delta h\alpha\gamma) \\
&= \Gamma^\beta_{\alpha\gamma} + \partial_\alpha\Lambda\delta^\beta_\gamma + \partial_\gamma\Lambda\delta^\beta_\alpha - (\partial_\delta\Lambda)h^{\beta\delta}h_{\alpha\gamma}
\end{aligned}$$

That is,

$$\begin{aligned}
\nabla_\alpha t^\beta &\to e^{-\Lambda}\left(-(\partial_\alpha\Lambda)t^\beta + \nabla_\alpha t^\beta + (\partial_\alpha\Lambda\delta^\beta_\gamma + \partial_\gamma\Lambda\delta^\beta_\alpha - (\partial_\delta\Lambda)h^{\beta\delta}h_{\alpha\gamma})t^\gamma\right) \\
&= e^{-\Lambda}\left(\nabla_\alpha t^\beta + (\partial_\gamma\Lambda)t^\gamma\delta^\beta_\alpha - (\partial_\gamma\Lambda)h^{\beta\gamma}t_\alpha\right)
\end{aligned}$$

So, we finally have

$$k\ ds \to \pm t^\alpha n_\beta \left(\nabla_\alpha t^\beta + (\partial_\gamma \Lambda) t^\gamma \delta^\beta_\alpha - (\partial_\gamma \Lambda) h^{\beta\gamma} t_\alpha\right) ds$$

That is,

$$\begin{aligned}
\delta S_3 &= \frac{1}{2\pi}\int_{\partial M} t^\alpha n_\beta (\partial_\gamma \Lambda)(t^\gamma \delta^\beta_\alpha - \gamma^{\beta\gamma} t_\alpha) ds \\
&= \frac{1}{2\pi}\int_{\partial M} (\partial_\gamma \Lambda)(t^\beta n_\beta t^\gamma - t^\alpha t_\alpha n^\gamma) ds \\
&= \frac{1}{2\pi}\int_{\partial M} n^\alpha \partial_\alpha \Lambda
\end{aligned}$$

where in the last line we have used that $t^a$ and $n^a$ are orthogonal, and that $t^a$ is a unit vector, which on a timelike curve $\partial M$ means $t^a t_a = -1$. This is indeed $-\partial S_2$, so the full Gauss-Bonnet term

$$\chi = \frac{1}{4\pi}\int_M d\tau\ d\sigma\ \sqrt{-\gamma} R + \frac{1}{2\pi}\int_{\partial M} ds\ k$$

is Weyl-invariant.

**Page 15** Note that the reparameterisation of $X^\mu$ can be written as

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma)$$

which makes it clear that $X^\mu$ is a worldsheet scalar.
In calculating $\delta h_{\alpha\beta}$ is has been assumed (naturally) that $\nabla_\alpha$ is metric-compatible, from which we have

$$\partial_\alpha h_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} h_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} h_{\beta\delta} = 0$$

**Page 25** On deducing the Poisson brackets of Equation 2.81 see solutions to Exercises 1.7-9 of Polchinski.

# 3 The Quantised Bosonic String

## 3.1 Canonical Quantisation of the Bosonic String

**Page 39**

**Theorem 3.1.** *The $L_n$ satisfy the **Virasoro algebra**, $\hat{v}$:*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$$

*where $c$ is called the **central charge**.*

*Proof.* Will we proceed differently to BLT in the appendix to this chapter, and in a slightly indirect way: first we will find the Witt algebra, the classical limit of the Virasoro algebra. We start with the Poisson brackets for the oscillators:

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{\text{PB}} = im\eta^{\mu\nu}\delta_{m+n}$$

The Virasoro generators are

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}^\mu \alpha_{m\mu}$$

Now we compute:

$$\begin{aligned}
\{L_m, L_n\}_{\text{PB}} &= \frac{1}{4}\sum_{p,q\in\mathbb{Z}} \left\{\alpha_{m-p}^\mu \alpha_{p\mu}, \alpha_{n-q}^\nu \alpha_{q\nu}\right\}_{\text{PB}} \\
&= \frac{1}{4}\eta_{\mu\rho}\eta_{\nu\sigma}\sum_{p,q\in\mathbb{Z}} \left(\alpha_{m-p}^\mu \left\{\alpha_p^\rho, \alpha_{n-q}^\nu \alpha_q^\sigma\right\}_{\text{PB}} + \left\{\alpha_{m-p}^\mu, \alpha_{n-q}^\nu \alpha_q^\sigma\right\}_{\text{PB}} \alpha_p^\rho\right) \\
&= \frac{1}{4}\eta_{\mu\rho}\eta_{\nu\sigma}\sum_{p,q\in\mathbb{Z}} \Big[\alpha_{m-p}^\mu \left(\alpha_{n-q}^\nu \left\{\alpha_p^\rho, \alpha_q^\sigma\right\}_{\text{PB}} + \left\{\alpha_p^\rho, \alpha_{n-q}^\nu\right\}_{\text{PB}} \alpha_q^\sigma\right) \\
&\qquad + \left(\alpha_{n-q}^\nu \left\{\alpha_{m-p}^\mu, \alpha_q^\sigma\right\}_{\text{PB}} + \left\{\alpha_{m-p}^\mu, \alpha_{n-q}^\nu\right\}_{\text{PB}} \alpha_q^\sigma\right)\alpha_p^\rho\Big] \\
&= \frac{1}{4}\eta_{\mu\rho}\eta_{\nu\sigma}\sum_{p,q\in\mathbb{Z}} \Big[\alpha_{m-p}^\mu \left(\alpha_{n-q}^\nu ip\eta^{\rho\sigma}\delta_{p+q} + ip\eta^{\rho\nu}\delta_{p+n-q}\alpha_q^\sigma\right) \\
&\qquad + \left(\alpha_{n-q}^\nu i(m-p)\eta^{\mu\sigma}\delta_{m-p+q} + i(m-p)\eta^{\mu\nu}\delta_{m-p+n-q}\alpha_q^\sigma\right)\alpha_p^\rho\Big] \\
&= \frac{i}{4}\eta_{\mu\rho}\eta_{\nu\sigma}\sum_{p\in\mathbb{Z}} \Big[\alpha_{m-p}\left(\alpha_{n+p}^\nu p\eta^{\rho\sigma} + p\eta^{\rho\nu}\alpha_{n+p}^\sigma\right) \\
&\qquad + \left(\alpha_{n-p+m}^\nu (m-p)\eta^{\mu\sigma} + (m-p)\eta^{\mu\nu}\alpha_{m-p+n}^\sigma\right)\alpha_p^\rho\Big] \\
&= \frac{i}{4}\eta_{\mu\rho}\sum_{p\in\mathbb{Z}} \Big[p\alpha_{m-p}^\mu \left(\alpha_{n+p}^\nu \delta_\nu^\rho + \delta_\sigma^\rho \alpha_{n+p}^\sigma\right) + (m-p)\left(\alpha_{n-p+m}^\nu \delta_\nu^\mu + \delta_\sigma^\mu \alpha_{m-p+n}^\sigma\right)\alpha_p^\rho\Big]
\end{aligned}$$

$$= \frac{i}{4} \sum_{p \in \mathbb{Z}} \left[ p \left( \alpha^{\mu}_{m-p} \alpha_{n+p,\mu} + \alpha^{\mu}_{m-p} \alpha_{n+p,\mu} \right) + (m-p) \left( \alpha^{\mu}_{n-p+m} \alpha_{p\mu} + \alpha^{\mu}_{n-p+m} \alpha_{p\mu} \right) \right]$$

$$= \frac{i}{2} \left[ \sum_{p \in \mathbb{Z}} p \alpha^{\mu}_{m-p} \alpha_{n+p,\mu} + \sum_{p \in \mathbb{Z}} (m-p) \alpha^{\mu}_{n-p+m} \alpha_{p\mu} \right]$$

Now, in the first sum shift $p \to p - n$. Then we have

$$\begin{aligned} \{L_m, L_n\}_{\text{PB}} &= \frac{i}{2} \sum_{p \in \mathbb{Z}} (p - n + m - p) \alpha^{\mu}_{n+m-p} \alpha_{p\mu} \\ &= i(m-n) \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha^{\mu}_{n+m-p} \alpha_{p\mu} \\ &= i(m-n) L_{m+n} \end{aligned}$$

Now we want to see how to upgrade this to the quantum case, thereby obtaining the full Virasoro algebra. In canonical quantisation, $\{,\}_{\text{PB}} \to (-i)[,]$, so we expect to get

$$[L_m, L_n] = (m-n) L_{m+n} + \text{quantum corrections}$$

We must now motivate the form of the quantum corrections by making several observations.

(i) All terms must be antisymmetric in $m, n$.

(ii) Since in the classical theory

$$L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha^{\mu}_{m-k} \alpha_{k\mu}$$

quantisation is straightforward except when $m = 0$, when we need to be careful about normal ordering. Therefore we expect corrections to the commutator giving $L_{m+n}$ to vanish unless $m + n = 0$. We can therefore write it as $A(m,n)\delta_{m+n}$ where $f$ is antisymmetric in $m, n$.

(iii) The correction, since it is just due to normal ordering, must simply be a $c$-number. We can then use the $\delta$ to just regard it as $A(m)\delta_{m+n}$.

(iv) Under $m \leftrightarrow n$, we have

$$A(m)\delta_{m+n} \to A(n)\delta_{n+m} = A(-m)\delta_{n+m}$$

where we have used $\delta_{n+m}$ to get the second equality. But by antisymmetry, this must be $-A(m)\delta_{m+n}$. That is,

$$A(-m) = -A(m)$$

(v) By the Jacobi identity,

$$[L_k, [L_m, L_m]] + [L_m, [L_n, L_k]] + [L_n, [L_k, L_m]] = 0$$

Using the Virasoro algebra as we so far have it, this is

$$\begin{aligned} 0 = {} & (m-n)\left[(k-m-n)L_{k+m+n} + A(k)\delta_{k+m+n}\right] + (n-k)\left[(m-n-k)L_{m+n+k} + A(m)\delta_{m+n+k}\right] \\ & + (k-m)\left[(n-k-m)L_{n+k+m} + A(n)\delta_{n+k+m}\right] \end{aligned}$$

In particular, for the case $k+m+n=0$, we have

$$\begin{aligned} 0 = {} & \left[(m-n)(k-m-n) + (n-k)(m-n-k) + (k-m)(n-k-m)\right] L_0 \\ & + (m-n)A(k) + (n-k)A(m) + (k-m)A(n) \end{aligned}$$

The $L_0$ part must vanish, since this is all there would be in the Witt algebra Jacobi identity. This leaves us with

$$(m-n)A(k) + (n-k)A(m) + (k-m)A(n) = 0$$

Consider the case $k=1$, which makes $m+n=-1$. We can then rearrange to get

$$A(m+1) = \frac{1}{m-1}\left((m+2)A(m) - (2m+1)A(1)\right)$$

This is a recursion relation giving $A(m)$ in terms of $A(1)$ and $A(2)$ for all $m \geq 3$. Therefore we must be able to write $A(m)$ as a power series with only two non-zero coefficients. Remembering the requirement of antisymmetry, the obvious ansatz is

$$A(m) = c_1 m + c_3 m^3$$

Indeed this can be plugged into the recursion relation and verified.

(vi) It just remains to find the relationship between $c_1$ and $c_3$. We will however go slightly further and find what these actually are. Consider

$$\langle 0;0|\, [L_m, L_{-m}]\, |0;0\rangle$$

for $m>0$. We have

$$\langle 0;0|\, [L_m, L_{-m}]\, |0;0\rangle = \frac{1}{4}\sum_{p,q\in\mathbb{Z}} \langle 0;0|\, \alpha^{\mu}_{m-p}\alpha_{p\mu}\alpha^{\nu}_{-m-q}\alpha_{q\nu}\, |0;0\rangle$$

If this is non-zero, we must have $q<0$ and $-m-q<0$. So $m>|q|$, and therefore we can say $q \leq -1$ and $m \geq 2$. So if $m=1$ we get zero:

$$\langle 0;0|\, [L_1, L_{-1}]\, |0;0\rangle = 0$$

On the other hand, if $m = 2$ we expect a non-zero result. We calculate

$$
\begin{aligned}
\langle 0;0|\, [L_2, L_{-2}]\, |0;0\rangle &= \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\, \langle 0;0|\, \alpha_1^\mu \alpha_1^\nu \alpha_{-1}^\rho \alpha_{-1}^\sigma\, |0;0\rangle \\
&= \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\, \langle 0;0|\, \alpha_1^\mu [\alpha_1^\nu, \alpha_{-1}^\rho] \alpha_{-1}^\sigma\, |0;0\rangle \\
&\quad + \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\, \langle 0;0|\, \alpha_1^\mu \alpha_{-1}^\rho \alpha_1^\nu \alpha_{-1}^\sigma\, |0;0\rangle \\
&= \frac{1}{4}\, \langle 0;0|\, \alpha_1^\mu \alpha_{-1\mu}\, |0;0\rangle + \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\, \langle 0;0|\, [\alpha_1^\mu, \alpha_{-1}^\rho][\alpha_1^\nu, \alpha_{-1}^\sigma]\, |0;0\rangle \\
&= \frac{1}{4}\, \langle 0;0|\, [\alpha_1^\mu, \alpha_{-1\mu}\, |0;0\rangle + \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma} \\
&= \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu} \\
&= \frac{d}{2}
\end{aligned}
$$

where $d$ is the dimensionality of the target space. Now, the point of this is that we have

$$\langle 0;0|\, [L_m, L_n]\, |0;0\rangle = \langle 0;0|\, (2mL_0 + A(m))\, |0;0\rangle$$

Now, $L_0$ has a $p^\mu p_\mu$ piece and an $N$ piece (slightly different for open and closed strings of course). But both annihilate the vacuum, so

$$\langle 0;0|\, [L_m, L_n]\, |0;0\rangle = A(m)$$

Thus we have found two things:

$$A(1) = c_1 + c_3 = 0$$

$$A(2) = 2c_1 + 8c_3 = \frac{d}{2}$$

Thus finally $c_1 = -d/12$ and $c_3 = d/12$, so

$$A(m) = \frac{d}{12} m(m^2 - 1)$$

We have finally computed, therefore, the Virasoro algebra:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{d}{12} m(m^2 - 1)\delta_{m+n}$$

which has central charge $c = d$, the target space dimensionality (which is indeed the number of free scalar fields $X^\mu$ in the theory). □

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Note that the Virasoro constraints are $ISO(1, d-1)$ invariant. That is,

$$[J^{\mu\nu}, L_m] = [P^\mu, L_m] = 0$$

## 3.2 Light-Cone Quantisation of the Bosonic String

**Page 42**
For some nuts and bolts of this procedure, see solutions to Polchinski, Exercise 1.7 for the open string and 1.8-9 for the closed string.

## 3.3 Spectrum of the Bosonic String

**Page 46**
Again, see the same Polchinski exercises for some detail.

**Page 47, Table 3.1**

(i) What's going on in the $N = 0$ and $N = 1$ levels is obvious.

(ii) For $N = 2$, we have the $SO(d-2)$ vector $\alpha^i_{-2}\left|0\right\rangle$ - i.e. the $(d-2)$ representation - and the symmetric rank-2 tensor $\alpha^i_{-1}\alpha^j_{-1}\left|0\right\rangle$. For the latter we decompose into symmetric traceless - the $\frac{1}{2}(d-2)(d-1)$ - and trace - the singlet. Then we observe that this together gives the symmetric traceless rank-2 representation of $SO(d-1)$ - i.e. the $\frac{1}{2}(d-1)d$: for some generic matrix $M$ in this latter, we can write

$$M = \begin{pmatrix} N & v \\ v^T & c \end{pmatrix}$$

$N$ is in the symmetric rank-2 of $SO(d-2)$, which we can decompose as above, $v$ is in the vector of $SO(d-2)$, and $c = -\operatorname{tr} N$ is not independent, being fully determined by $N$.

(iii) For $N = 3$, we have the $SO(d-2)$ vector $\alpha^i_{-3}\left|0\right\rangle$, the rank-2 tensor $\alpha^i_{-2}\alpha^j_{-1}\left|0\right\rangle$ and the rank-3 tensor $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\left|0\right\rangle$. The rank-2 tensor - $(d-2)(d-2)$ has no exact symmetry, and therefore can be decomposed in the generic way into symmetric traceless - $\frac{1}{2}(d-2)(d-1)$ - antisymmetric - $\frac{1}{2}(d-2)(d-3)$ - and trace. The rank-3 tensor - $\frac{1}{3!}(d-2)(d-1)d$ - can be decomposed by writing

$$\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1} = \begin{cases} \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1} & i \neq j \neq k \\ \left(\sum_i \alpha^i_{-1}\alpha^i_{-1}\right)\alpha^j_{-1} & \end{cases}$$

That is, we get a fully symmetric rank-3 tensor - $\frac{1}{3!}(d-3)(d-2)(d+2)$ - and another vector. Now, consider $SO(d-1)$ representations. Firstly, the antisymmetric rank-2 - $\frac{1}{2}(d-1)(d-s)$ - decomposes into an $SO(d-2)$ antisymmetric rank-2 and vector. This takes care of two parts of $\alpha^i_{-2}\alpha^j_{-1}\left|0\right\rangle$. Also consider a fully symmetric rank-3 of

$SO(d-1)$. If $a,b,c$ are $SO(25)$ indices and $i,j,k$ $SO(24)$ indices, this tensor can be written $\psi_{abc}$, which decomposes into

$$\psi_{(ijk)}$$
$$\psi_{(ij)a} = \psi_{(i)a(j)} = \psi_{a(ij)}$$
$$\psi_{iaa} = \psi_{aia} = \psi_{aai}$$

where now $a$ is a fixed index. That is, we have an $SO(d-2)$ symmetric rank-3, symmetric rank-2, and vector. Now we have recovered the full $SO(d-2)$ content. The symmetric rank-2 can of course be decomposed into symmetric traceless and trace.

(iv) For $N=4$, we have the $SO(d-2)$ vector $\alpha^i_{-4}|0\rangle$, mixed symmetry rank-2 tensor $\alpha^i_{-3}\alpha^j_{-1}|0\rangle$, symmetric rank-2 tensor $\alpha^i_{-2}\alpha^j_{-2}|0\rangle$, mixed symmetry rank-3 $\alpha^i_{-2}\alpha^j_{-1}\alpha^k_{-1}|0\rangle$ and symmetric rank-4 $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\alpha^l_{-1}|0\rangle$. The rank-2 tensors can be decomposed as before. For the rank-3, similarly to the $N=3$ case we get vectors. There the full symmetry meant they were all identical; here this is not the case, and so we get two vectors (corresponding to $i=j$ and $i=k$). The remaining part of the rank-3 tensor can be decomposed in the normal way, but note that it has no fully antisymmetric part. Finally, the symmetric rank-4 is decomposed into a 'traceless' symmetric rank-4 (of the form $\eta_{ij}\psi^{ijkl}$) and a symmetric rank-2, which in turn becomes a traceless symmetric rank-2 and a trace. So our total $SO(d-2)$ content is: 3 singlets; 3 vectors; 3 traceless symmetric rank-2; 1 antisymmetric rank-2; 1 symmetric rank-3; 1 mixed symmetry rank-3; 1 'traceless' symmetric rank-4. Now we want to organise this into complete $SO(d-1)$ representations. First consider a fully symmetric rank-4 $SO(d-1)$ tensor, $\psi_{abcd}$. Ignoring permutations, this decomposes into

$$\psi_{(ijkl)}, \quad \psi_{(ijk)a}, \quad \psi_{(ij)aa}, \quad \psi_{iaaa}, \quad \psi_{aaaa}$$

where $a$ is now fixed. These are $SO(d-2)$: symmetric rank-4; symmetric rank-3; symmetric rank-2; vector; scalar. The remaining $SO(d-2)$ content we need to find is now: 1 singlet; 2 vectors; 1 symmetric rank-2; 1 mixed symmetry rank-3. To account for the mixed symmetry tensor we will need a mixed symmetry rank-3 $SO(d-1)$ tensor, with the same symmetry properties. Call this $\phi_{abc} = \phi_{bac} = -\phi_{cba}$. This decomposes into

$$\phi_{ijk}$$
$$\phi_{ija} = \phi_{jia} = -\phi_{aji} = -\phi_{jai} = \phi_{iaj} = \phi_{aij}$$
$$\phi_{iaa} = \phi_{aia} = -\phi_{aai}$$
$$\phi_{aaa}$$

where $a$ is fixed. That is, an $SO(d-2)$: mixed symmetry rank-3; a vanishing symmetric rank-2; vector; scalar. Finally we are left to account for an $SO(d-2)$ vector and a symmetric rank-2. We already know this is what comes from an $SO(d-1)$ symmetric rank-2 (which in turn can be decomposed into a traceless symmetric rank-2 and a trace singlet). Thus in summary, our $SO(d-1)$ content is a symmetric rank-4 tensor (from which we can take out a symmetric rank-2, from which we can take out a trace) and a mixed symmetry rank-3 tensor.