Cohomology in Physics

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Abstract

These are my personal notes on cohomology and related topics focussing on applications to physics. They are a work in progress, and certainly imperfect and incomplete, with regard to both what has been included and what hasn't. Some sections are empty but hopefully will not remain so indefinitely. Naturally these notes have not been composed out of thin air; references I have used in compiling them are given at the end of each section, and in full at the end of the document.

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1 Mathematical Basics

1.1 What is Cohomology?

This is a basic introduction to cohomology and the idea of what it tells us. Suppose we have three vector spaces V^0, V^1, V^2, and two linear maps $d_0 : V^0 \to V^1$ and $d_1 : V^1 \to V^2$. Then we have the sequence V^* given by

$$V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2$$

Suppose that the composition $d_1 d_0 = 0$. That is, d_1 is the zero map on $\operatorname{im} d_0$, so $\operatorname{im} d_0 \subseteq \ker d_1$. Then the quotient

$$\frac{\ker d_1}{\operatorname{im} d_0} =: H^1(V^*)$$

is well-defined. This is called the **first cohomology of** V^*.

We can already gain some insight into the meaning of $H^1(V^*)$. $\ker d_1$ is the space of solutions to the equation $d_1 v = 0$. Then $H^1(V^*)$ is the space of such solutions, modulo those which can be written in the form $v = d_0 v'$.

To make this more concrete, consider an open set $U \subseteq \mathbb{R}^3$ and let $d_0 = \text{grad} : C^\infty(U) \to C^\infty(U)^3$, and $d_1 = \text{curl} : C^\infty(U)^2 \to C^\infty(U)^2$; it is well known that curl grad $= 0$. $\ker d_1$ is all those vector fields on U that are curl-free. The statement that curl grad $= 0$ means that there is an obvious way to find such vector fields: define a function on U, and take its gradient. So such vector fields (conservative vector fields), are rather trivial, and therefore it makes sense, if we want to understand ker curl, to quotient them out. Thus we are interested in $H^1(V^*)$, the space of non-trivial solutions to curl $v = 0$.

The typical non-trivial curl-free vector field to mention is

$$v(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

Note that this is defined everywhere except $(0, 0)$, so we are taking $U = \mathbb{R}^2 \backslash \{0\}$. To see that this is indeed non-trivial, i.e. that $v \notin$ im grad, we first note that any closed contour integral of a gradient is zero. On the other hand, we can confirm (using the fact that $r \cdot v = 0$) that, if C is the unit circle,

$$\int_C v \cdot ds \neq 0$$

where s parameterises C. Therefore v is a non-trivial curl-free vector field. Now, if we capitalise on this discovery by defining the map f : ker curl$\to \mathbb{R}$ by

$$f(v) = \int_C v \cdot ds$$

we quickly see that this is surjective, by considering the vector fields αv, where v is the vector field above and $\alpha \in \mathbb{R}$. Furthermore, $\ker f$ is precisely the set of conservative vector fields, i.e. $\ker f = \text{im grad}$. Then by the first isomorphism theorem,

$$\mathbb{R} \cong \frac{\ker \text{curl}}{\text{im grad}} \cong H^1(V^*)$$

Thus we have computed the first cohomology itself. $\mathbb{R}$ 'counts' the non-trivial solutions of curl $v = 0$.

So a solution $v \in C^\infty(U)^3$ of curl $v = 0$ may be written as $v = \text{grad } f$ iff v is in the trivial element of $H^1(V^*)$. That is, non-trivial cohomology represents 'obstruction to lifting'. If v is instead in, say, the 3 of $H^1(V^*)$, I cannot lift it back through the sequence to $C^\infty(U)$.

1.2 De Rham Cohomology

This basic first taste of cohomology is actually a particular case of de Rham cohomology, a tool used to understand the global topological structure of manifolds (the manifold we investigated was $U = \mathbb{R}^2 \backslash \{0, 0\}$). We assume the basics of manifold theory and differential forms.

Denote the space of k-forms on an n-dimensional manifold M by $\Omega^k(M)$. We have the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$, and hence the sequence

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M)$$

It is an important property of the exterior derivative that $d^2 = 0$, and thus we are in an appropriate context to introduce cohomology.

Definition 1.1. We define the k^{th} **de Rham cohomology group** $H^k(M)$ of M as the quotient

$$H^k(M) = \frac{\ker d : \Omega^k(M) \to \Omega^{k+1}(M)}{\text{im } d : \Omega^{k-1}(M) \to \Omega^k(M)}$$

We say that a k-form ω is **closed** if $d\omega = 0$, and **exact** if $\omega = d\alpha$ for some $(k-1)$-form α. Then the k^{th} de Rham cohomology group is the space of closed k-forms modulo the space of exact k-forms. That is, it consists of equivalence classes $[\omega] = [\omega + d\alpha]$.

1.2.1 A Few Preliminary Results

Proposition 1.2. The induced wedge product and pullback on cohomology are well-defined.

Proof. The induced wedge product is

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$$

To see that this is well-defined, we just need to add exact forms to α and β.

$$\begin{aligned}
[\alpha + d\omega] \wedge [\beta + d\mu] &= [(\alpha + d\omega) \wedge (\beta + d\mu)] \\
&= [\alpha \wedge \beta + \alpha \wedge d\mu + d\omega \wedge \beta + d\omega \wedge d\mu] \\
&= [\alpha \wedge \beta + d(\alpha \wedge \mu + \omega \wedge \beta + \omega \wedge d\mu)] \\
&= [\alpha \wedge \beta]
\end{aligned}$$

The induced pullback is

$$f^*[\alpha] = [f^*\alpha]$$

To see that this is well-defined, consider

$$\begin{aligned}
f^*[\alpha + d\beta] &= [f^*(\alpha + d\beta)] \\
&= [f^*\alpha + df^*\beta] \\
&= [f^*\alpha]
\end{aligned}$$

□

Proposition 1.3. A manifold M with k connected components has

$$H^0(M) \cong \mathbb{R}^k$$

Proof. A 0-form is a function, and hence closed if it is locally constant. There are no (-1)-forms, so quotienting out exact 0-forms is trivial. Therefore $H^0(M)$ is the set of locally constant functions, which if M has k connected components, is $\mathbb{R}^k$. □

Proposition 1.4. $H^1(M)$ is trivial if M is simply connected.

Proof. Let ω be a closed 1-form. We want to know if there exists a function ϕ such that $\omega = d\phi$. If such a function exists, then, picking a basepoint $p \in M$, it is defined by

$$\phi(q) = \int_\gamma \omega$$

where $\gamma : [0, 1] \to M$ is a smooth curve in M from p to q. For this to be well-defined, the integral must be independent of the choice of γ. Now, assume M is simply connected.

Then the set of smooth curves from p to q $\gamma_s(t)$ can be regarded as a smooth family $\gamma(s,t)$. Write

$$I(s) = \int_{\gamma_s} \omega$$

That is, if $dI(s)/ds = 0$, then $I = \phi(q)$. In full,

$$I(s) = \int_0^1 \omega_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t)dt$$

Now consider

$$\begin{aligned}
\frac{dI(s)}{ds} &= \int_0^1 \left(\partial_s\omega_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t) + \omega_\mu(\gamma(s,t))\partial_s\partial_t\gamma^\mu(s,t)\right) dt \\
&= \int_0^1 \left(\partial_s\omega_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t) - \partial_t\omega_\mu(\gamma(s,t))\partial_s\gamma^\mu(s,t)\right) dt \\
&= \int_0^1 (\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)\partial_s\gamma^\mu\partial_t\gamma^\nu dt
\end{aligned}$$

For this to be zero for arbitrary paths, we must have

$$\partial_\mu\omega_\nu - \partial_\nu\omega_\mu = 0$$

But this is just $d\omega_{\mu\nu}$. Therefore it is zero automatically because ω is closed. Thus closed forms are automatically exact if this procedure does not fail - i.e. if M is simply connected. $\square$

Proposition 1.5. An orientable manifold (without boundary) has non-trivial top cohomology.

Proof. Let M be an orientable n-manifold. Then there exists a volume form ω. As a top-form it is automatically closed. Suppose it is also exact, and write $\omega = d\alpha$. Then

$$\int_M \omega = \int_{\partial M} \alpha = 0$$

since $\partial M = \emptyset$. But this contradicts that ω is a volume form. Hence it is closed but not exact, and therefore in a non-trivial cohomology class. $\square$

Proposition 1.6. Given a metric and any torsion-free connection ∇, a p-form ω, given by

$$\omega = \frac{1}{p!}\omega_{\mu_1...\mu_p} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}$$

has exterior derivative

$$d\omega = \nabla\omega := \frac{1}{p!}\nabla_\nu \omega_{\mu_1...\mu_p} dx^\nu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}$$

Proof. We have

$$\nabla\omega = \frac{1}{p!}\left(\partial_\nu \omega_{\mu_1...\mu_p} - \Gamma^\lambda{}_{\nu\mu_1}\omega_{\lambda\mu_2...\mu_p} - \Gamma^\lambda{}_{\nu\mu_2}\omega_{\mu_1\lambda\mu_3...\mu_p} - ...\right) dx^\nu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}$$

For a torsion free connection, the connection-dependent terms are symmetric in ν, μ_i, but the forms are antisymmetric in these, so only the partial derivative survives. $\square$

Corollary 1.7. *If $\omega_{\mu_1...\mu_p}$ is covariantly conserved by a torsion-free connection, $d\omega = 0$. If further $p < n$, the dimension of the manifold, this is an iff statement.*

Theorem 1.8 (The Künneth formula)**.** *If $M = M_1 \times M_2$,*

$$H^k(M) = \bigoplus_{p+q=k} H^p(M_1) \otimes H^q(M_2)$$

Proof. Let $\{\omega_i^p\}$ be a basis form $H^p(M_1)$, and $\{\eta_j^{k-p}\}$ be a basis for $H^{k-p}(M_2)$. Then each $\omega_i^p \wedge \eta_j^{k-p}$ is a closed k-form on M. Suppose it is exact. Then we can write

$$\begin{aligned}\omega_i^p \wedge \eta_j^{k-p} &= d(\alpha^{p-1} \wedge \beta^{k-p} + \gamma^p \wedge \delta^{k-p-1}) \\ &= d\alpha^{p-1} \wedge \beta^{k-p} + (-1)^{p-1}\alpha^{p-1} \wedge d\beta^{k-p} + d\gamma^p \wedge \delta^{k-p-1} + (-1)^p \gamma^p \wedge d\delta^{k-p-1}\end{aligned}$$

But acting with d and remembering that this is by hypothesis closed, we must have either $\alpha = \delta = 0$ or $\beta = \gamma = 0$. But then either way $\omega_i^p \wedge \eta_j^{k=p} = 0$. This contradicts that $\{\omega_i^p\}$ and $\{\eta_j^{k-p}\}$ are bases. Furthermore, every closed but not exact form on M must be a sum of wedge products of closed but not exact forms on M_1 and M_2. Therefore $\{\omega_i^p \wedge \eta_j^{k-p}\}$ is a basis form $H^k(M)$, i.e.

$$H^k(M) = \bigoplus_{p+q=k} H^p(M_1) \otimes H^q(M_2)$$

$\square$

Corollary 1.9. *Let M be an n-manifold and consider*

$$H^k(M \times \mathbb{R}^m) = \bigoplus_{p+q=k} H^p(M) \otimes H^q(\mathbb{R}^m)$$

Only one contribution to the direct sum survives, $q = 0$:

$$H^k(M \times \mathbb{R}^m) = H^k(M)$$

1.3 Basics of Hodge Theory

1.3.1 The Hodge Star and Adjoint Exterior Derivative

Definition 1.10. The **Hodge star**, $*$, on an n-manifold M, is a map $\Omega^p(M) \to \Omega^{n-p}(M)$ defined by

$$*dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} = \frac{1}{(n-p)!}\sqrt{g} g^{\mu_1\nu_1} ... g^{\mu_p\nu_p} \varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n} dx^{\nu_{p+1}} \wedge ... \wedge dx^{\nu_n}$$

Proposition 1.11. $*^2 : \Omega^p(M) \to \Omega^p(M)$ is just multiplication by $(-1)^{p(n-p)}$.

Proof. We have

$$*dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} = \frac{1}{(n-p)!}\sqrt{g}\varepsilon^{\mu_1...\mu_p}{}_{\nu_{p+1}...\nu_n} dx^{\nu_{p+1}} \wedge ... \wedge dx^{\nu_n}$$

Then applying $*$ again,

$$\begin{aligned} *^2 dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} &= \frac{1}{(n-p)!}\sqrt{g}\varepsilon^{\mu_1...\mu_p}{}_{\nu_{p+1}...\nu_n} * dx^{\nu_{p+1}} \wedge ... \wedge dx^{\nu_n} \\ &= \frac{1}{(n-p)!p!}\sqrt{g}\varepsilon^{\mu_1...\mu_p}{}_{\nu_{p+1}...\nu_n}\sqrt{g}\varepsilon^{\nu_{p+1}...\nu_n}{}_{\lambda_1...\lambda_p} dx^{\lambda_1} \wedge ... \wedge dx^{\lambda_p} \\ &= \frac{1}{(n-p)!p!}(-1)^{p(n-p)}\sqrt{g}\varepsilon_{\nu_{p+1}...\nu_n}{}^{\mu_1...\mu_p}\sqrt{g}\varepsilon^{\nu_{p+1}...\nu_n}{}_{\lambda_1...\lambda_p} dx^{\lambda_1} \wedge ... \wedge dx^{\lambda_p} \\ &= (-1)^{p(n-p)}\frac{1}{p!}\delta^{\mu_1...\mu_p}_{\lambda_1...\lambda_p} dx^{\lambda_1} \wedge ... \wedge dx^{\lambda_p} \\ &= (-1)^{p(n-p)} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \end{aligned}$$

Thus

$$*^2 = (-1)^{p(n-p)}$$

(Note: on a pseudo-Riemannian manifold with signature $(n-s, s)$, we would also have an additional factor $(-1)^s$, introduced in the second to last line.) $\square$

Corollary 1.12. $*$ *is invertible. Specifically,*

$$*^{-1} = (-1)^{p(n-p)} *$$

Proposition 1.13. If $\alpha, \beta \in \Omega^p(M)$, then

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} \beta^{\mu_1 \ldots \mu_p} \Omega$$

where Ω is the natural volume form on a Riemannian manifold,

$$\Omega = \sqrt{g} dx^1 \wedge \ldots \wedge dx^n$$

Proof. We have

$$\begin{aligned}
\alpha \wedge *\beta &= \frac{1}{p!p!} \alpha \mu_1 \ldots \mu_p \beta_{\nu_1 \ldots \nu_p} \frac{\sqrt{g}}{(n-p)!} g^{\nu_1 \lambda_1} \ldots g^{\nu_p \lambda_p} \\
&\quad dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \wedge dx^{\lambda_{p+1}} \wedge \ldots \wedge dx^{\lambda_n} \\
&= \frac{1}{p!p!(n-p)!} \alpha_{\mu_1 \ldots \mu_p} \beta^{\lambda_1 \ldots \lambda_p} \sqrt{\varepsilon}_{\lambda_1 \ldots \lambda_p \lambda_{p+1} \ldots \lambda_n} \varepsilon^{\mu_1 \ldots \mu_p \lambda_{p+1} \ldots \lambda_n} dx^1 \wedge \ldots \wedge dx^n \\
&= \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} \beta^{\mu_1 \ldots \mu_p} \Omega
\end{aligned}$$

This is manifestly the same as $\beta \wedge *\alpha$. $\square$

Definition 1.14. We define an inner product on the space of p-forms on a compact orientable Riemannian manifold without boundary by

$$(\alpha, \beta) = \int \alpha \wedge *\beta$$

By the previous proposition this is symmetric, non-negative, and zero iff at least one of α and β are (since $\alpha \wedge *\beta$ is proportional to a volume form).

Definition 1.15. We define the **adjoint exterior derivative** $d^\dagger$ as the adjoint operator of d with respect to this inner product, i.e. such that

$$(\alpha, d\beta) = (d^\dagger \alpha, \beta)$$

If $d^\dagger \omega = 0$, we say ω is **coclosed**. If $\omega = d^\dagger \alpha$, we say it is **coexact**.

Proposition 1.16. $d^\dagger$ is given by

$$d^\dagger = (-1)^{n(p-1)-1} * d*$$

Proof. We have

$$\begin{aligned}
(\alpha, d\beta) &= \int d\beta \wedge *\alpha \\
&= \int d(\beta \wedge *\alpha) - (-1)^{p-1} \int \beta \wedge d * \alpha \\
&= (-1)^p \int \beta \wedge d * \alpha
\end{aligned}$$

Now, α is a p-form, so $d*\alpha$ is an $(n-p+1)$-form. Thus we can insert $*^2 = (-1)^{(n-p+1)(p-1)}$ to get

$$\begin{aligned}
(\alpha, d\beta) &= (-1)^{p+(n-p+1)(p-1)} \int \beta \wedge *(*d * \alpha) \\
&= (-1)^{n(p-1)-1} \int \beta \wedge *(*d * \alpha)
\end{aligned}$$

Thus

$$d^\dagger = (-1)^{n(p-1)-1} * d*$$

(Note that on a pseudo-Riemannian manifold the additional factor $(-1)^s$ in the square of the Hodge star carries through to here.) □

Corollary 1.17. $d^{\dagger 2} = 0$.

Remark. Since $*$ is invertible, we can express that ω is coclosed by writing $d * \omega = 0$.

Proposition 1.18. If ω is a p-form given by

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

then

$$d^\dagger \omega = -\frac{1}{(p-1)!} \nabla^\nu \omega_{\nu \mu_2 \dots \mu_p} dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}$$

where ∇ is the Levi-Civita connection.

Proof. Assuming our connection is torsion-free, we can use Proposition 1.6 to get

$$*\omega = \frac{1}{p!(n-p)!}\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p}\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n}dx^{\nu_{p+1}}\wedge ... \wedge dx^{\nu_n}$$

$$d*\omega = \frac{1}{p!(n-p)!}\nabla_\lambda(\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p})\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n}dx^\lambda \wedge dx^{\nu_{p+1}}\wedge ... \wedge dx^{\nu_n}$$

$$*d*\omega = \frac{1}{p!(n-p)!(p-1)!}\nabla_\lambda(\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p})\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n}$$
$$\sqrt{g}g^{\lambda\kappa_p}g^{\nu_{p+1}\kappa_{p+1}}...g^{\nu_n\kappa_n}\varepsilon_{\kappa_p...\kappa_n\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge ... \wedge dx^{\kappa_{p-1}}$$

$$d^\dagger\omega = (-1)^{n(p-1)-1}\frac{1}{p!(n-p)!(p-1)!}\nabla_\lambda(\omega_{\mu_1...\mu_p}\sqrt{g}g^{\mu_1\nu_1}...g^{\mu_p\nu_p})\varepsilon_{\nu_1...\nu_p\nu_{p+1}...\nu_n}$$
$$\sqrt{g}g^{\lambda\kappa_p}g^{\nu_{p+1}\kappa_{p+1}}...g^{\nu_n\kappa_n}\varepsilon_{\kappa_p...\kappa_n\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge ... \wedge dx^{\kappa_{p-1}}$$

Now, we assume our connection is also metric-compatible, and hence Levi-Civita.

$$\begin{aligned}
d^\dagger\omega &= (-1)^{n(p-1)-1}\frac{1}{p!(n-p)!(p-1)!}\nabla^{\kappa_p}\omega_{\mu_1...\mu_p} \\
&\quad \sqrt{g}\varepsilon^{\mu_{p_1}...\mu_p\kappa_{p+1}...\kappa_n}\sqrt{g}\varepsilon_{\kappa_p...\kappa_n\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge ... \wedge dx^{\kappa_{p-1}} \\
&= (-1)^{n(p-1)-1}\frac{1}{p!(n-p)!(p-1)!}\nabla^{\kappa_p}\omega_{\mu_1...\mu_p} \\
&\quad \sqrt{g}(-1)^{p(n-p)}\varepsilon^{\kappa_{p+1}...\kappa_n\mu_1...\mu_p}\sqrt{g}(-1)^{n-p}\varepsilon_{\kappa_{p+1}...\kappa_n\kappa_p\kappa_1...\kappa_{p-1}}dx^{\kappa_1}\wedge ... \wedge dx^{\kappa_{p-1}} \\
&= (-1)^{n(p-1)-1+p(n-p)+n-p}\frac{1}{p!(p-1)!}\nabla^{\kappa_p}\omega_{\mu_1...\mu_p}\delta^{\mu_1\mu_2...\mu_p}_{\kappa_p\kappa_1...\kappa_{p-1}}dx^\kappa_1\wedge ... \wedge dx^{\kappa_{p-1}} \\
&= -\frac{1}{(p-1)!}\nabla^{\mu_1}\omega_{\mu_1...\mu_p}dx^{\mu_2}\wedge ... \wedge dx^{\mu_p}
\end{aligned}$$

□

Corollary 1.19. *If $\omega_{\mu_1...\mu_p}$ is covariantly conserved by the Levi-Civita connection, $d^\dagger\omega = 0$. If further $p < n$, the dimension of the manifold, this is an iff statement.*

1.3.2 The Laplacian

Definition 1.20. We define the **Laplacian** or **Hodge-de Rham operator** Δ, a map $\Delta : \Omega^p(M) \to \Omega^p(M)$, by

$$\Delta = dd^\dagger + d^\dagger d$$

If $\Delta\omega = 0$, we say ω is **harmonic**. Notice that this property is metric-dependent. The set of harmonic p-forms on M is denoted $\mathrm{Harm}^p(M)$.

Proposition 1.21. If

$$\omega = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

Then

$$\Delta\omega_{\mu_1\dots\mu_p} = -\nabla^\kappa \nabla_\kappa \omega_{\mu_1\dots\mu_p} - pR_{\kappa[\mu_1}\omega^\kappa{}_{\mu_2\dots\mu_p]} - \frac{1}{2}p(p-1)R_{\lambda\kappa[\mu_1\mu_2}\omega^{\lambda\kappa}{}_{\mu_3\dots\mu_p]}$$

Proof. Quite lengthy and can't get the last term exactly - see my solutions to Candelas' Lectures on Complex Manifolds for a partial proof. □

Proposition 1.22. A form on a compact orientable Riemannian manifold is harmonic iff it is both closed and coclosed.

Proof. Let ω be harmonic. Then $(dd^\dagger + d^\dagger d)\omega = 0$. This is true iff $(\omega, (dd^\dagger + d^\dagger d)\omega) = 0$. But this is

$$(d^\dagger\omega, d^\dagger\omega) + (d\omega, d\omega) = 0$$

This can only be true if both terms disappear independently (since the inner product is non-negative), which in turn is only true if ω is both closed and coclosed. □

Corollary 1.23. *If $\omega_{\mu_1\dots\mu_p}$ is covariantly conserved by the Levi-Civita connection on a compact orientable manifold, ω is harmonic. If further $p < n$, this is an iff statement.*

1.3.3 Hodge's Theorems

Theorem 1.24 (Hodge's decomposition theorem)**.** *The space of p-forms on a compact orientable Riemannian manifold without boundary uniquely decomposes as*

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus d^\dagger\Omega^{p+1}(M) \oplus \mathrm{Harm}^p(M)$$

That is, any p-form ω can be uniquely written in the form

$$\omega = d\alpha + d^\dagger\beta + \gamma$$

where γ is harmonic.

Proof. First we must show that the subspaces of exact, coexact, and harmonic p-forms are orthogonal to each other. Let $d\alpha$, $d^\dagger\beta$ and γ be generic exact, coexact, and harmonic p-forms. Then we have

$$\begin{aligned}(d\alpha, d^\dagger\beta) &= (d^2\alpha, \beta) = 0\\ (d\alpha, \gamma) &= (\alpha, d^\dagger\gamma) = 0\\ (d^\dagger\beta, \gamma) &= (\beta, d\gamma) = 0\end{aligned}$$

Orthogonality ensures uniqueness of the decomposition. Now we must further show that they together span all of $\Omega^p(M)$. To see this, suppose ω is a p-form which is orthogonal to each subspace. Firstly

$$(d\alpha, \omega) = (\alpha, d^\dagger\omega) = 0$$

for any $(p-1)$-form α, so ω is coclosed. Similarly,

$$(d^\dagger\beta, \omega) = (\beta, d\omega) = 0$$

for any $(p+1)$-form β, so ω is closed. Therefore it is harmonic. But by hypothesis it is orthogonal to the space of harmonic forms, so it must be zero. This ensures existence of the decomposition. □

Proposition 1.25. A de Rham cohomology class has a unique harmonic representative.

Proof. Let ω be a closed p-form, and $[\omega]$ its associated cohomology class. Generically,

$$\omega = d\alpha + d^\dagger\beta + \gamma$$

where γ is harmonic, and therefore the expected harmonic representative of $[\omega]$. Since ω is closed,

$$d\omega = dd^\dagger\beta = 0$$

Thus we have

$$\begin{aligned}(d\omega, \beta) = (dd^\dagger\beta, \beta) &= 0\\ (d^\dagger\beta, d^\dagger\beta) &= 0\end{aligned}$$

Therefore $d^\dagger\beta = 0$, and hence any closed form can be written

$$\omega = \gamma + d\alpha$$

Thus $\gamma \in [\omega]$ as expected. Indeed,

$$[\omega] = \{\gamma + d\alpha \mid \alpha \in \Omega^{p-1}(M)\}$$

Since the space of exact forms is orthogonal to the space of harmonic forms, there is precisely one harmonic representative, γ itself. □

Theorem 1.26 (Hodge's theorem)**.** *If M is a compact orientable Riemannian manifold without boundary,*

$$H^p(M) \cong \mathrm{Harm}^p(M)$$

Proof. We have just seen that a general closed form ω can be written

$$\omega = \gamma + d\alpha$$

where γ is harmonic, and further that γ is the unique harmonic representative of $[\omega]$. Define a map

$$\begin{aligned} f : H^p(M) &\to \mathrm{Harm}^p(M) \\ [\omega] &\mapsto \gamma \end{aligned}$$

in this way. Then we have found in the previous proposition that this is well-defined and injective. Furthermore, $Z^p(M) \subset \mathrm{Harm}^p(M)$, while $B^p(M) = d\Omega^{p-1}(M)$ is orthogonal to $\mathrm{Harm}^p(M)$. So we also have $f(H^p(M)) \supset \mathrm{Harm}^p(M)$, so in fact $f(H^p(M)) = \mathrm{Harm}^p(M)$. Therefore f is a bijection, and furthermore clearly an isomorphism of vector spaces. Thus

$$H^p(M) \cong \mathrm{Harm}^p(M)$$

□

Corollary 1.27. *The Künneth formula can be restated for harmonic forms:*

$$\mathrm{Harm}^k(M_1 \times M_2) = \bigoplus_{p+q=r} \mathrm{Harm}^p(M_1) \otimes \mathrm{Harm}^q(M_2)$$

Remark. Note that while it is true that each cohomology class has a unique harmonic representative, allowing the construction of this isormophism, which representative this will be is in fact metric-dependent, since $*$ is. Put another way, Hodge's theorem establishes a relationship between the topological properties of a compact orientable manifold (cohomology) and geometrical properties of the metrics it admits (harmonic forms).

Theorem 1.28 (Poincaré duality)**.** *For a compact orientable Riemannian n-manifold M, $H^p(M) \cong H^{n-p}(M)$.*

Proof. Recall that a p-form ω is harmonic iff it is both closed and coclosed, $d\omega = 0$ and $d * \omega = 0$. Then immediately $*\omega$ is closed, and indeed $*(*\omega) = \pm\omega$, so it is also coclosed. Thus $\Delta\omega = 0$ iff $\Delta * \omega = 0$. Therefore $*$ provides an isomorphism

$$\mathrm{Harm}^p(M) \cong \mathrm{Harm}^{n-p}(M)$$

and hence

$$H^p(M) \cong H^{n-p}(M)$$

$\square$

Corollary 1.29. *A connected compact orientable Riemannian manifold has a unique volume form, up to a constant factor, since $H^n(M) \cong H^0(M) \cong \mathbb{R}$. Indeed, more generally there is a unique volume form up to a constant for each connected component of a compact manifold.*

1.4 Homology and Cohomology

1.4.1 Basics of Simplicial Homology

Definition 1.30. Let M be a smooth manifold, and $\{N_i\}$ a set of smooth p-dimensional oriented submanifolds of M. We define a p**-chain** a_p as the formal sum

$$a_p = \sum_i c_i N_i$$

Here, $c_i \in \mathcal{F}$, a field which can be chosen. The set of p-chains on M is denoted $C_p(M)$.

Definition 1.31. Let ω be a p-form on M. We define its integral over a p-chain $a_p = \sum c_i N_i$ by

$$\int_{a_p} \omega = \sum_i c_i \int_{N_i} \omega$$

Definition 1.32. A **cycle** is a chain with no boundary. That is, boundaries are in the image of ∂, and cycles in its kernel. Every boundary is itself a cycle. Therefore, if $Z_p(M)$ is the set of p-cycles, and $B_p(M)$ the set of p-boundaries, $B_p(M) \subset Z_p(M)$

Definition 1.33. The $p^{\mathbf{th}}$ **simplicial homology group** of M is the quotient

$$H_p(M) = \frac{Z_p(M)}{B_p(M)} = \frac{\ker \partial : C_p(M) \to C_{p+1}(M)}{\operatorname{im} \partial : C_{p-1}(M) \to C_p(M)}$$

The p^{th} **Betti number**, b_p, is the dimension of the p^{th} homology:

$$b_p(M) = \dim H_p(M)$$

It counts the number of p-holes in M. More precisely, b_0 is the number of connected components, b_1 the number of 'circular' holes, b_2 the number of cavities, and so on.

Definition 1.34. The **Euler characteristic** of an n-manifold homeomorphic to a simplicial complex with I_p p-simplices, $r = 0, ..., n$, is given by

$$\chi = \sum_{p=0}^{n} (-1)^p I_p$$

Proposition 1.35. The Euler characteristic is given in terms of Betti numbers by

$$\chi = \sum_{p=0}^{n} (-1)^p b_p$$

Proof. We have $I_p = \dim C_p(M)$. By considering $\partial : C_p(M) \to C_{p-1}(M)$, we have

$$\begin{aligned} I_p &= \dim \ker \partial + \dim \operatorname{im} \partial \\ &= \dim Z_p(M) + \dim B_{p-1}(M) \end{aligned}$$

On the other hand,

$$b_p = \dim Z_p(M) - \dim B_p(M)$$

Thus

$$\begin{aligned} \chi &= \sum_{p=0}^{n} (-1)^p (\dim Z_p(M) + \dim B_{p-1}(M)) \\ &= \sum_{p=0}^{n} ((-1)^p \dim Z_p(M) - (-1)^p \dim B_p(M)) \\ &= \sum_{p=0}^{n} (-1)^p b_p \end{aligned}$$

where in the second line we have used the fact that $B_{-1}(M)$ is empty to shift the summation index. $\square$

1.4.2 De Rham's Theorem

Definition 1.36. We define an **inner product of p-chains and p-forms** on compact orientable manifolds by

$$(c, \omega) = \int_c \omega$$

In particular, when ω is closed, this is called a **period** of ω over c.

Remark. Stokes' theorem can now be formulated as

$$(c, d\omega) = (\partial c, \omega)$$

That is, it is the statement that d and ∂ are adjoint operators with respect to this inner product.

Proposition 1.37. This inner product induces a natural inner product on homology and cohomology. That is, the map

$$([c], [\omega]) = \int_c \omega$$

is well-defined.

Proof. We have

$$\begin{aligned}
\int_{c+\partial z} (\omega + d\alpha) &= \int_c \omega + \int_c d\alpha + \int_{\partial z} \omega + \int_{\partial z} d\alpha \\
&= \int_c \omega + \int_{\partial c} \alpha + \int_z d\omega + 0 \\
&= \int_c \omega
\end{aligned}$$

since $\partial c = d\omega = 0$. □

Theorem 1.38 (de Rham's theorem)**.** *This inner product is bilinear and non-degenerate, making $H_p(M)$ and $H^p(M)$ dual vector spaces.*

Remark. The implication of de Rham's theorem for physics is that the topological structure of spacetime (described by homology) is intimately connected to the kind of fields that can live on it (described by cohomology).

Corollary 1.39. *Betti numbers can be regarded as dimensions of cohomology, and hence the Euler characteristic can be derived from cohomology. Furthermore, if M is a compact orientable Riemannian manifold without boundary, $b_p(M)$ can be regarded as the number of independent harmonic p-forms on M.*

Corollary 1.40. *The Künneth formula may be written*

$$b_k(M_1 \times M_2) = \sum_{p+q=k} b_p(M_1) b_q(M_2)$$

In particular,

$$b_k(M \times \mathbb{R}^m) = b_k(M)$$

This expression for the Betti numbers implies

$$\begin{aligned}\chi(M_1 \times M_2) &= \sum_{k=0}^{n_1+n_2} \sum_{p+q=k} (-1)^{p+q} b_p(M_1) b_q(M_2) \\ &= \sum_{p=0}^{n_1} (-1)^p b_p(M_1) \sum_{q=0}^{n_2} (-1)^q b_q(M_2) \\ &= \chi(M_1)\chi(M_2)\end{aligned}$$

The Euler characteristic is multiplicative.

Corollary 1.41. *Poincaré duality may be expressed*

$$b_p = b_{n-p}$$

Therefore $\chi(M)$ is zero if n is odd.

Corollary 1.42.

(i) If M has k connected components, $H_k(M) \cong \mathbb{R}^k$.

(ii) If M is simply connected, $H_1(M)$ is trivial.

(iii) If M is orientable, $H_n(M)$ is non-trivial.

Remark. It is worth summarising the results for compact orientable Riemannian manifolds: $b_p = b_{n-p}$, and $b_0 = b_n =$ the number of connected components. Furthermore if we have simply connectedness, $b_1 = b_{n-1} = 0$.

This section was written with the help of [3] and [4].

2 Electric Circuits

Surprisingly, the usually almost painfully mundane topic of electric circuits is a natural setting for some discussion of homology and cohomology. This follows from the realisation that a circuit can be regarded as a 1-dimensional simplicial complex, where a node P_i represents a circuit component, and a connection P_iP_j a wire.

Proposition 2.1. The topological information about a circuit is encoded by

$$b_1 = 1 - s_0 + s_1$$

where s_0 is the number of nodes (components) and s_1 the number of connections (wires).

Proof. It is well-known that for a 1-dimensional simplicial complex the Euler characteristic is given by

$$\chi = s_0 - s_1$$

On the other hand,

$$\chi = \sum_{k=0}^{n} (-1)^k b_k$$

so here

$$\chi = b_0 - b_1$$

Assuming that the circuit is connected, we have $b_0 = 1$, and therefore

$$b_1 = 1 - s_0 + s_1$$

□

Proposition 2.2. There are b_1 basic cycles in the circuit.

Proof. Recall that $b_1 = \dim H_1 = \dim Z_1 - \dim B_1$, but there are no 1-boundaries since there are no 2-chains, and hence $b_1 = \dim Z$ is the number of independent (i.e. basic) cycles. □

Definition 2.3. The **current** I is a property of a wire, so we can regard it as a map assigning a real number to each P_iP_j. That is, it is a 1-chain, and we can therefore write

$$I = \sum_{i,j} I_{ij} P_i P_j$$

Proposition 2.4. Kirchoff's current law is equivalent to

$$\partial I = 0$$

That is, I must be a 1-cycle.

Proof. We have

$$\begin{aligned}
\partial I &= \sum_{i,j} I_{ij}(P_j - P_i) \\
&= \sum_i P_i \sum_{\text{neighbours } j \text{ of } i} (I_{ji} - I_{ij})
\end{aligned}$$

Therefore the total current entering each node is equal to that leaving iff this is zero. $\square$

Corollary 2.5. *Therefore we can write*

$$I = \sum_{k=1}^{b_1} I_k c_k$$

where $\{c_k\}$ is a basis of 1-cycles. Indeed, as in Proposition 2.2, the space of 1-cycles is just the first homology, so I can be considered to take values in $H_1 \cong \mathbb{R}^{b_1}$. There are b_1 independent currents. (Note that $b_1 < s_1$, the number of wires, for any non-trivial circuit.)

Definition 2.6. The **potential** U is a 0-cocyle (i.e. a 0-form, i.e. a function) assigning a real number to each node. That is,

$$U\left(\sum_i a_i P_i\right) = \sum_i a_i U(P_i)$$

Proposition 2.7. Kirchoff's voltage law is equivalent to

$$V = -dU$$

Proof. dU is defined by

$$dU(z) = U(\partial z)$$

for any 1-chain z. In particular,

$$dU(P_iP_j) = U(P_j - P_i) = U(P_j) - U(P_i) = -V_{ji}$$

From this it follows that

$$V(c) = -dU(c) = -U(\partial c) = 0$$

for any 1-cycle c. That is, the total voltage along a loop is zero. $\square$

Corollary 2.8. *The voltage V can be identified with an element in H^0, since shifting $U \to U + k$, where k is a constant function, leaves V invariant. But assuming connectedness, $H^0 \cong \mathbb{R}$.*

Definition 2.9. Resistance is the map of dual vector spaces $R : C_1 \to \Omega^1$, i.e. an isomorphism. We say **Ohm's law** holds whenever V and I are related in this way, i.e. when

$$V = RI$$

Definition 2.10. Given the dual vector space isomorphisms $r : C_0 \to \Omega^0$ and $R : C_1 \to \Omega^1$, and the boundary map $\partial : C_1 \to C_0$, we define $d^\dagger : \Omega^1 \to \Omega^0$ such that the diagram

$$\begin{tikzcd} C_1 \arrow[r, "\partial"] \arrow[d, "R"] & C_0 \arrow[d, "r"] \\ \Omega^1 \arrow[r, "d^\dagger"] & \Omega^0 \end{tikzcd}$$

commutes. That is, $d^\dagger R = r\partial$. This is analogous to the Hodge codifferential. We can also define a **Laplacian** on Ω^0 by $d^\dagger d$.

Proposition 2.11. Kirchoff's current law holds iff the potential U is harmonic with respect to the Laplacian just defined.

Proof. Kirchoff's current law states $\partial I = 0$. By Ohm's law, $V = RI$, and furthermore R is invertible so $I = R^{-1}V$. Therefore we have

$$\partial(-R^{-1}dU) = 0$$

But by the definition of $d^\dagger$, this is

$$-r^{-1}d^\dagger dU = 0$$

And since r is an isomorphism, this amounts to

$$d^\dagger dU = 0$$

$\square$

This section was written with the help of [1] and [7].

3 Electromagnetism

3.1 Electromagnetism in the Language of Differential Forms

Definition 3.1. **Maxwell's equations** in 3-vector notation are

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \rho \\ \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{j} \end{aligned}$$

Definition 3.2. Notice that in vacuum these equations are invariant under

$$\boldsymbol{B} \to \boldsymbol{E}, \quad \boldsymbol{E} \to -\boldsymbol{B}$$

This property of the vacuum equations is called **electromagnetic duality**.

Remark. Note that the obstruction to the duality symmetry holding in the non-vacuum case is the absence of magnetic charge.

Definition 3.3. We associate to our 3-vectors form fields: an **electric 1-form**

$$E = E_x dx + E_y dy + E_z dz$$

and a **magnetic 2-form**

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

Definition 3.4. We define the **electromagnetic 2-form**

$$F = B + E \wedge dt$$

Proposition 3.5. Writing

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

we have components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_z & 0 \end{pmatrix}$$

Theorem 3.6. *The first pair of Maxwell equations are summarised by the form equation*

$$dF = 0$$

The second pair are given by

$$*d*F = J$$

where

$$J = j_x dx + j_y dy + j_z dz - \rho dt$$

Remark. Notice that the first equation, that F is closed, is independent of the metric, while the second is dependent on the metric by virtue of the Hodge star.

Proposition 3.7. If F solves the vacuum Maxwell equations on a compact manifold, it is harmonic.

Proof. We have $dF = d*F = 0$, so F is both closed and coclosed. But on a compact manifold this is equivalent to being harmonic. □

Corollary 3.8. *On a compact manifold M, using Hodge's theorem (1.26) there are $b_2(M)$ independent solutions to the vacuum Maxwell equations.*

Corollary 3.9. *The only solution of the vacuum Maxwell equations on $\mathbb{R}^4$ which vanishes at infinity is $F = 0$.*

Proof. If we want 2-forms F on $\mathbb{R}^4$ which vanish at infinity, we can regard them as forms on S^4. This is a compact manifold, and therefore we can use the previous corollary. But $b_2(S^4) = 0$, so the only harmonic 2-form on S^4 is $F = 0$. □

Proposition 3.10. The duality transformation of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields mentioned above is precisely the transformation $F \mapsto *F$.

Corollary 3.11. *The vacuum Maxwell equations are invariant under the duality transformation $F \mapsto *F$.*

Definition 3.12. On a Riemannian manifold, $*^2 = 1$ and further in 4d $* : \Omega^2(M) \to \Omega^2(M)$, so we say that F is **self-dual** if $*F = F$, or **anti-self-dual** if $*F = -F$. On a Lorentzian manifold, $*^2 = -1$, so we have $*F = iF$ and $*F = -iF$ instead.

Proposition 3.13. In vacuum, any self-dual or anti-self-dual solution F of the first Maxwell equation, $dF = 0$, also solves the second, and vice versa.

Proposition 3.14. The continuity equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{j} + \partial_t \rho = 0$$

is equivalent to

$$d * J = 0$$

That is, the current 1-form must be coclosed.

3.2 Gauge Transformations

Proposition 3.15. The first Maxwell equation, $dF = 0$, is defined on cohomology, but the second, $d * F = J$, is not.

Proof. Clearly $d(F + dA) = dF$, whereas $d * (F + dA) \neq d * F$ in general. Specifically, this would require $d * dA = 0$. ☐

Definition 3.16. If $F \in 0 \in H^2(M)$ is exact, we write $F = dA$, where A is called the **potential 1-form**. When A is defined, $dF = d^2 A = 0$ is trivial, and we just have to solve

$$d * dA = J$$

We immediately have that

Proposition 3.17. A potential A exists if $H^2(M) = 0$, and more generally if

$$(c_i, F) = \int_{c_i} F = 0$$

for each basic 2-cycle c_i, $i = 1, ..., b_2(M)$.

Definition 3.18. A transformation $A \mapsto A + \alpha$, where α is a closed 1-form, is called a **gauge transformation**, and leaves Maxwell's equations invariant. The corresponding degrees of freedom in the potential are called **gauge degrees of freedom**

Proposition 3.19. The potential has $b_1(M)$ gauge degrees of freedom (modulo cohomology).

Proof. That is, we regard gauge transformations $A \to A + d\phi$ as trivial. Then there are as many non-trivial gauge degrees of freedom as dimensions of $H^1(M)$, i.e. $b_1(M)$. $\square$

Corollary 3.20. *If M is simply connected, $b_1 = 0$, and A only has trivial gauge degrees of freedom, i.e. all the possible gauge transformations are of the form $A \to A + d\phi$.*

3.3 Electrostatics and Magnetostatics

In this section we focus on electromagnetic solutions which are time-independent. We will therefore use M to mean our space, rather than spacetime.

Proposition 3.21. Maxwell's equations for time-independent fields are

$$\begin{aligned} dB &= 0 \\ dE &= 0 \\ d * E &= \rho \\ d * B &= *j \end{aligned}$$

where now

$$\begin{aligned} \rho &= \tilde{\rho} dx \wedge dy \wedge dz \\ j &= j_x dx + j_y dy + j_z dz \end{aligned}$$

and $\tilde{\rho}$ is the scalar charge density, and $\boldsymbol{j}$ the vector current. The continuity equation is

$$d * j = 0$$

Proof. We have

$$\begin{aligned} dB &= d(B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy) \\ &= (\partial_x B_x + \partial_y B_y + \partial_z B_z) dx \wedge dy \wedge dz \end{aligned}$$

so

$$dB = 0 \quad \Leftrightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Next,

$$\begin{aligned} dE &= d(E_x dx + E_y dy + E_z dz) \\ &= (\partial_x E_y - \partial_y E_x) dx \wedge dy + (\partial_y E_z - \partial_z E_y) dy \wedge dz + (\partial_z E_x - \partial_x E_z) dz \wedge dx \end{aligned}$$

so

$$dE \quad \Leftrightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{E} = 0$$

For the second pair,

$$\begin{aligned} d * E &= d(E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy) \\ &= (\partial_x E_x + \partial_y E_y + \partial_z E_z) dx \wedge dy \wedge dz \end{aligned}$$

so

$$d * E = \rho \quad \Leftrightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = \tilde{\rho}$$

Lastly

$$\begin{aligned} d * B &= d(B_x dx + B_y dy + B_z dz) \\ &= (\partial_x B_y - \partial_y B_x) dx \wedge dy + (\partial_y B_z - \partial_z B_y) dy \wedge dz + (\partial_z B_x - \partial_x B_z) dz \wedge dx \end{aligned}$$

so

$$d * B = j \quad \Leftrightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j}$$

That the continuity equation is $d * j = 0$ follows from the full time-dependent case, or working it out just like $d * E$. $\square$

Remark. We still have electromagnetic duality $B \to E$ and $E \to -B$ in vacuum, and the obstruction out of vacuum is the absence of magnetic charge.

Corollary 3.22. *In vacuum, E and B are both closed and coclosed. Then if M is compact and orientable, or equivalently if M is orientable and we demand fields vanish at infinity, they are both harmonic. Then there are $b_1(M)$ and $b_2(M)$ independent E and B fields, respectively.*

3.3.1 Monopoles

Proposition 3.23. The general solutions to the first pair of Maxwell equations ($dB = dE = 0$) in n dimensions are

- If $b_1 = 0$
$$E = -dV$$
(where the sign is conventional).
- If $b_1 \neq 0$
$$E = E_m - dV$$
- If $b_{n-1} = 0$
$$B = dA$$
- If $b_{n-1} \neq 0$
$$B = B_m + dA$$

Here, E_m and B_m are closed.

Proof. This is just the statement that B and E are closed. The only remark to make is that B should be an $(n-1)$-form in general, of the form

$$B = B_i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

which can be motivated by the possibility of electromagnetic duality, $B = *E$. $\square$

Definition 3.24. In the previous proposition, B_m is called a **magnetic monopole**. In n dimensions, the **magnetic charge** of a magnetic monopole by

$$m = \int_{S^{n-1}} B$$

(where the choice of sphere is appropriately made). Similarly, the **electric charge** associated with E is given by the integral

$$q = \int_{S^{n-1}} *E$$

Remark. Specifically, the S^{n-1} should be a non-trivial $(n-1)$-cycle - i.e. not the boundary of a submanifold of M.

Proposition 3.25. These charges are defined on cohomology.

Proof. Consider the charge of a magnetic monopole, and shift the B field by an exact form dA:

$$\int_{S^{n-1}} (B + dA) = \int_{S^{n-1}} B + \int_{\partial S^{n-1}} A = \int_{S^{n-1}} B$$

The same is true of $*E$ (not E!). □

Definition 3.26. Let $\rho = 0$. Then $d * E = 0$ is closed. For $b_{n-1} \neq 0$, this is solved by

$$*E = D - dU$$

where D is closed, and called an **electric monopole**.

Remark. The fact that monopoles are due to non-trivial cohomology classes fits intuitively with the notion that they are generally not perturbative objects.

Proposition 3.27. We now have that b_{n-1} counts the number of magnetic and electric monopoles.

Example 3.28. Consider $M = \mathbb{R}^3 \backslash \{0\}$ with the usual metric. We have $b_1 = 0$ and $b_2 = 1$, so E is exact and we have monopoles. We want to find the non-trivial cohomology. In fact this is a classic example, and the form

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

is closed but not exact. (Notice that it is not defined on all of $\mathbb{R}^3$.) With the appropriate normalisation,

$$\begin{aligned} B &= \frac{m}{4\pi}\omega \\ &= \frac{m}{4\pi} \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \end{aligned}$$

where m is the charge of the magnetic monopole. Then we have

$$\begin{aligned}
*B &= \frac{m}{4\pi}\frac{xdx + ydy + zdz}{(x^2+y^2+z^2)^{3/2}} \\
d*B &= \frac{3m}{4\pi}\frac{(x-y)dx\wedge dy + (y-z)dy\wedge dz + (z-x)dz\wedge dx}{(x^2+y^2+z^2)^{5/2}} \\
\Rightarrow j &= \frac{3m}{4\pi}\frac{(y-z)dx + (z-x)dy + (x-y)dz}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}$$

Alternatively, if we have an electric monopole we want

$$*E = \frac{q}{4\pi}\omega$$

where q is the charge of the electric monopole. Then

$$E = \frac{q}{4\pi}\frac{xdx + ydy + zdz}{(x^2+y^2+z^2)^{3/2}}$$

Indeed, we then have $E = -dV$ with

$$V = \frac{q}{4\pi}\frac{1}{(x^2+y^2+z^2)^{1/2}}$$

Example 3.29. Consider a universe with two asymptotically flat parts, connected by a wormhole. This can be described by taking $\mathbb{R}\times S^2$ and giving it a metric

$$g = dr\otimes dr + f(r)^2(d\phi\otimes d\phi + \sin^2\phi d\theta\otimes d\theta)$$

where $f(r)^2 > 0$ for all r, so we can choose $f(r) > 0$ for all r, and we want $f(r)\to r$ at large $|r|$. The two parts of the universe are described by $r > r_0$ and $r < -r_0$, where we regard $[-r_0, r-0]$ to be the wormhole. Using the Künneth formula,

$$\begin{aligned}
b_1(\mathbb{R}\times S^2) &= b_1(S^2) = 0 \\
b_2(\mathbb{R}\times S^2) &= b_2(S^2) = 1
\end{aligned}$$

Therefore E is exact and we have a monopole configuration. In particular, since the non-trivial aspect of the cohomology of this manifold is due to $b_2(S^2)$, we take the volume form of this factor:

$$\omega = \sin\phi d\theta\wedge d\phi$$

is a closed but not exact 2-form. Then we set

$$B = \frac{m}{4\pi}\sin\phi d\theta\wedge d\phi$$

where m is the charge of the monopole. Now, we have

$$*dr = \frac{1}{2!} f(r)^2 \sin\phi \varepsilon^r{}_{ij} dx^i \wedge dx^j$$
$$= f(r)^2 \sin\phi d\theta \wedge d\phi$$

so

$$*B = \frac{m}{4\pi} \frac{dr}{f(r)^2}$$
$$d * B = 0$$
$$\Rightarrow j = 0$$

Alternatively, for the electric monopole configuration,

$$*E = \frac{q}{4\pi} \sin\phi d\theta \wedge d\phi$$

where q is the charge of the monopole. Then

$$E = \frac{q}{4\pi} \frac{dr}{f(r)^2}$$

Indeed then we have $E = -dV$, where

$$V(r) = -\frac{q}{4\pi} \int_0^r \frac{ds}{f(s)^2}$$

is a function of r only. In particular, notice at large $|r|$ that

$$E \to \frac{q}{4\pi} \frac{dr}{r^2}$$
$$V \to \frac{q}{4\pi} \frac{1}{r}$$

In other words, far away from the wormhole, it looks like a charged particle of charge q. Note that observers on the two different parts of the universe will see opposite charges - the sign will come from the fact that for consistency the orientation on S^2 must be switched (since the sign of r flips).

Example 3.30. Consider the flat torus, $M = T^2$ with the metric

$$g = d\theta \otimes d\theta + d\phi \otimes d\phi$$

We have $b_1 = 2$, which since $\dim M = 2$ means that we have two independent monopole configurations, but no electric potential. The closed but not exact forms on T^2 are $d\theta$ and $d\phi$ (neither θ nor ϕ can be both globally defined and smooth). We therefore want

$$B = kd\theta, \quad B = kd\phi$$

for some constant k. To find k, we need to integrate over a non-trivial 1-cycle. Specifically, we want to integrate over the cycle parameterised by θ or ϕ, depending on which configuration we are considering. Since the flat metric does not discriminate between θ and ϕ, the result is the same:

$$m = \int_0^{2\pi} B d\theta = \int_0^{2\pi} k d\theta = 2\pi k d\theta$$

so $k = m/2\pi$.

$$B = \frac{m}{2\pi} d\theta, \quad B = \frac{m}{2\pi} d\phi$$

The Hodge star with the flat metric is given by $*d\theta = d\phi$, $*d\phi = -d\theta$, so we immediately have

$$E = \frac{q}{2\pi} d\theta, \quad E = \frac{q}{2\pi} d\phi$$

3.3.2 The Bohm Aharanov Effect

Example 3.31. Consider $\mathbb{R}^3$ with a solenoid along the z-axis with radius R. The magnetic field will be parallel to the z axis inside the solenoid, and vanish outside it. That is,

$$*B = f(r)dz$$

where $f(r)$ is some function which is constant for $r < R$ and zero for $r > R$. Then

$$B = f(r) r dr \wedge d\theta$$

Now, the interior of a cylinder clearly has trivial cohomology, so in particular $b_2 = 0$ and we can introduce a potential A:

$$A = \left(\int_0^r f(s) s ds \right) d\theta$$

Define the constant

$$\begin{aligned} \Phi &= \int_{D^1} f(r) r dr \wedge d\theta \\ &= 2\pi \int_0^R f(r) r dr \end{aligned}$$

where the D^1 is a cross-sectional disc of the solenoid. Then outside the solenoid we have

$$A = \frac{\Phi}{2\pi} d\theta$$

(even though $B = 0$ here).
Now consider a quantum treatment. We want to consider the problem of a charged particle

moving from a to b in $\mathbb{R}^3$ along a path γ, the set of which we denote $\mathcal{P}$. The particle starts at a in a state ψ and arrives at b in a state

$$\phi(b) = \int_{\mathcal{P}} \mathcal{D}\gamma \ \psi(a) e^{iS[\gamma]}$$

Turning on the solenoid affects this in two ways: firstly, we shift

$$S[\gamma] \to S[\gamma] - q \int_{\gamma} A$$

where q is the charge of our particle; secondly, the solenoid excludes our particle, so $\mathcal{P}$ is a set of paths not in $\mathbb{R}^3$, but in $\mathbb{R}^3 \backslash (\mathbb{R} \times S^1)$. The space through which our particle can propagate is no longer simply connected, $b_1 \neq 0$. Suppose $R < 1$ and consider two homotopically distinct paths γ_0 and γ_1 from $(1, 0, 0)$ to $(-1, 0, 0)$. They pick up phases due to the solenoid of

$$\exp\left\{ -iq \int_{\gamma} A \right\} = \exp\left\{ \pm \frac{iq\Phi}{2} \right\}$$

Therefore, if we set $\Phi = 2\pi/q$, these phases are $\pm i$ - i.e. the two paths are completely out of phase. In this way we can match pairs of homotopically distinct paths to get total destructive interference.

This section was written with the help of [2] and [7].

4 BRST Quantisation

4.1 Lie Algebra Cohomology

4.2 Yang-Mills Theory

Definition 4.1. Yang-Mills theory is a field theory with gauge symmetry given by a compact, reductive Lie algebra. Typically we think of $SU(n)$ Yang-Mills. The gauge-fixed Lagrangian for Yang-Mills theory is

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}_i(i\not{D} - m)_{ij}\psi_j - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + (\partial_\mu \bar{c}^a)(D^\mu c^a)$$

Here, a is the adjoint gauge group index (so $a = 1, ..., n^2 - 1$ for $SU(n)$), i, j are flavour indices, running from 1 to n_f, the number of flavours, A^a_μ are the **gauge fields**, $F^a_{\mu\nu}$ is the field strength, given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu$$

where f^{abc} are the structure constants of the Lie algebra, defined by

$$[T^a_R, T^b_R] = i f^{abc} T^c_R$$

where T^a_R are the generators in any representation R. Further, the ψ_i are quarks, and have mass m, $\not{D} = \gamma^\mu D_\mu$, where γ^μ are the Dirac matrices and

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu$$

is the **covariant derivative**, where g is the coupling constant. Finally, c^a are **ghost fields**, which are Grassmann-valued, and ξ is a **gauge-fixing parameter**.

Proposition 4.2. The un-gauge-fixed Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}_i(i\not{D} - m)_{ij}\psi_j$$

exhibits gauge (i.e. local) symmetry of the relevant Lie group.

Proposition 4.3. Gauge-fixing (e.g. by Faddeev-Popov) introduces the gauge-fixing and ghost terms, and breaks the local $SU(n)$ symmetry to a subgroup whose parameters θ^a satisfy $\partial^2\theta^a = 0$.

Proof. The equations of motion for the ghosts in the gauge-fixed theory are

$$\partial^2 c = \partial^2 \bar{c} = 0$$

$\square$

Corollary 4.4. *If θ is a constant Grassmann parameter, θc^a can be regarded as a gauge group parameter satisfying $\partial^2 \theta c^a = 0$.*

Definition 4.5. The **BRST transformations** are gauge transformations on the fields in the theory parameterised by gauge parameters θc^a.

Proposition 4.6. The BRST transformations are

$$\begin{aligned} \psi_i &\to \psi_i + i\theta c^a T^a_{ij}\psi_j \\ A^a_\mu &\to A^a_\mu + \frac{1}{g}\theta D_\mu c^a \\ c^a &\to c^a - \frac{1}{2}\theta f^{abc} c^b c^c \\ \bar{c}^a &\to \bar{c}^a - \frac{1}{g}\theta \frac{1}{\xi}\partial^\mu A^a_\mu \end{aligned}$$

Proposition 4.7. The BRST transformations are an exact symmetry of the gauge-fixed Lagrangian.

Definition 4.8. Define the **BRST operator** Q by its action on fields such that the BRST transformations are of the form

$$X \to X + \theta Q X$$

for any field X.

Proposition 4.9. $Q^2 = 0$ on any product of fields.

4.3 The Bosonic String

This section was written with the help of [5] and [6] and

5 Kähler and Calabi-Yau Manifolds

5.1 Complex Manifolds

5.1.1 Complex Manifolds and the Hermitian Metric

Definition 5.1. A metric g on a complex manifold with complex structure J is a metric satisfying
$$g(JX, JY) = g(X, Y)$$
for any vector fields X, Y.

Proposition 5.2. Any complex manifold admits a Hermitian metric.

Proof. Any manifold admits a Riemannian metric. Let g be any Riemannian metric on a complex manifold, and define $\tilde{g}$ by
$$\tilde{g}(X, Y) = \frac{1}{2}(g(X, Y) + g(JX, JY))$$
Then $\tilde{g}$ is a metric and is moreover Hermitian. □

Proposition 5.3. The pure components of a Hermitian metric vanish.

Proof. We have
$$\begin{aligned} g_{\mu\nu} = g(\partial_\mu, \partial_\nu) &= g(J\partial_\mu, J\partial_\nu) \\ &= -g_{\mu\nu} \end{aligned}$$
since J squares to -1. Thus $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$. □

Definition 5.4. Given a Hermitian metric g and complex structure J, we define a 2-form called the **Kähler 2-form** by
$$\Omega = \frac{1}{2}J_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu = ig_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu$$
That is,
$$\Omega(X, Y) = g(JX, Y)$$
for any vector fields X, Y.

Proposition 5.5. Any complex manifold is orientable.

Proof. Any complex manifold M admits a Hermitian metric g, and therefore we have a Kähler form Ω. If $\dim_{\mathbb{C}} M = n$, consider $\Omega^n = \Omega \wedge ... \wedge \Omega$. If we act this on n holomorphic vector fields $\{e_i\}$ and n antiholomorphic vector fields $\{Je_i\}$, we get, up to permutations and combinatorics,

$$g(e_1, Je_1)...g(e_n, Je_n)$$

This is clearly positive-definite, so Ω^n is a volume form and M is orientable. □

Corollary 5.6. *Volume forms are closed but not exact, so for any complex manifold M with $\dim_{\mathbb{C}} M = n$, $b_{2n} \geq 1$.*

Definition 5.7. A connection on a complex manifold is called **Hermitian** if it is metric-compatible and pure in its indices.

Proposition 5.8. The Hermitian connection exists on any manifold and is uniquely given by

$$\Gamma^{\lambda}{}_{\mu\nu} = g^{\lambda\bar{\rho}} \partial_\mu g_{\nu\bar{\rho}}$$

and conjugate.

Proof. Metric compatibility for a pure connection reads

$$\partial_\mu g_{\nu\bar{\rho}} - g_{\lambda\bar{\rho}} \Gamma^{\lambda}{}_{\mu\nu}$$

and complex conjugate. Then we have the above form uniquely. □

Definition 5.9. A p-form ω on a complex manifold is of **bidegree** (r, s) (where $r + s = p$), or called an (r, s)-form, if it has local form

$$\omega = \frac{1}{r!s!} \omega_{\mu_1 ... \mu_r \bar{\nu}_1 ... \bar{\nu}_s} dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}$$

The space of such forms is denoted $\Omega^{r,s}(M)$. Clearly

$$\Omega^p(M) = \bigoplus_{r+s=p} \Omega^{r,s}(M)$$

That is, any p-form can be written as a sum of forms of definite bidegree.

5.1.2 Dolbeault Operators and Dolbeault Cohomology

Definition 5.10. Let $\pi^{(r,s)} : \Omega^{r+s}(M) \to \Omega^{r,s}(M)$ be the natural projection. Then define the maps

$$\partial = \pi^{(r+1,s)} \circ d : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M)$$
$$\bar{\partial} = \pi^{(r,s+1)} \circ d : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M)$$

These are called **Dolbeault operators**. Then $d = \partial + \bar{\partial}$.

Proposition 5.11. $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$

Proof. We have
$$(\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0$$
If we act on an (r,s)-form ω, $\partial^2\omega$ is an $(r+2,s)$-form, $\bar{\partial}^2\omega$ is an $(r,s+2)$-form, and $(\partial\bar{\partial} + \bar{\partial}\partial)\omega$ is an $(r+1,s+1)$-form. These must all vanish separately. □

Corollary 5.12. *Then we can form exact sequences using ∂ and $\bar{\partial}$.*

Definition 5.13. The $(r,s)^{\text{th}}$ **Dolbeault cohomology group** of a complex manifold M is
$$H^{r,s}_{\bar{\partial}}(M) = Z^{r,s}_{\bar{\partial}}(M)/B^{r,s}_{\bar{\partial}}(M)$$
where $Z_{\bar{\partial}}(M)$ is the set of $\bar{\partial}$-closed (r,s)-forms, and $B_{\bar{\partial}}(M)$ the set of $\bar{\partial}$-exact (r,s)-forms. The $(r,s)^{\text{th}}$ **Hodge number** is the complex dimension of this space.

$$b_{r,s}(M) = \dim_{\mathbb{C}} H^{r,s}_{\bar{\partial}}(M)$$

5.1.3 Basics of Hodge Theory on Complex Manifolds

Proposition 5.14. The Hodge star on a complex manifold with a Hermitian metric maps (r,s)-forms to $(n-s, n-r)$-forms.

Proof. Consider action on a basis element of $\Omega^{r,s}(M)$. The only non-zero metric components are mixed, so we have

$$*dz^{\mu_1}\wedge ...\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge ...\wedge d\bar{z}^{\nu_s} \propto g^{\mu_1\bar{\lambda}_1}...g^{\mu_r\bar{\lambda}_r}\varepsilon_{\bar{\lambda}_1...\bar{\lambda}_r}g^{\bar{\nu}_1\kappa_1}...g^{\bar{\nu}_s\kappa_s}\varepsilon_{\kappa_1...\kappa_s}dz^{\kappa_{s+1}}\wedge ...\wedge dz^{\kappa_n}\wedge d\bar{z}^{\lambda_{r+1}}\wedge ...\wedge d\bar{z}^{\lambda_n}$$

This is an $(n-s, n-r)$-form. □

Proposition 5.15. $*^2 = (-1)^{(r+s)^2}$ on a Riemannian manifold, with an extra factor of $(-1)^s$ on a pseudo-Riemannian manifold with signature $(n-s, s)$.

Proof. Using real coordinates, we have $*^2 = (-1)^{p(n-p)}$ in the Lorentzian case. Here, n is even and $p = r + s$. $\square$

Corollary 5.16. $*$ *is invertible. Specifically,*

$$*^{-1} = (-1)^{(r+s)^2} *$$

Definition 5.17. Given a compact orientable Riemannian complex manifold without boundary M, we can define an inner product on the space of (r, s)-forms by

$$(\alpha, \beta) = \int \alpha \wedge *\bar{\beta}$$

Definition 5.18. Given this inner product we define **adjoint Dolbeault operators** $\partial^\dagger$ and $\bar{\partial}^\dagger$ as the adjoint operators of ∂ and $\bar{\partial}$:

$$(\partial\alpha, \beta) = (\alpha, \partial^\dagger\beta)$$
$$(\bar{\partial}\alpha, \beta) = (\alpha, \bar{\partial}^\dagger\beta)$$

That is, $d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$.

Proposition 5.19. These adjoint Dolbeault operators are given by

$$\partial^\dagger = - * \bar{\partial} *$$
$$\bar{\partial}^\dagger = - * \partial *$$

Proof. On real manifolds, we have

$$d^\dagger = (-1)^{n(p+1)+1} * d *$$

Here, this means

$$\partial^\dagger + \bar{\partial}^\dagger = - * (\partial + \bar{\partial}) *$$

Act with both sides on an (r, s)-form:

$$\partial^\dagger\omega + \bar{\partial}^\dagger\omega = - * \partial * \omega - *\bar{\partial} * \omega$$

On the LHS we have an $(r-1,s)$-form and an $(r,s-1)$-form. On the RHS, $*\omega \in \Omega^{n-s,n-r}(M)$, so $\partial * \omega \in \Omega^{n-s+1,n-r}(M)$, so $*\partial * \omega \in \Omega^{r,s-1}(M)$ and similarly $*\bar{\partial} * \omega \in \Omega^{r-1,s}(M)$. Therefore equating forms of same bidegree, we have the proposition. $\square$

Corollary 5.20. $\partial^{\dagger 2} = \bar{\partial}^{\dagger 2} = 0$.

Proposition 5.21. If

$$\omega = \frac{1}{r!s!}\omega_{\mu_1...\mu_r\bar{\nu}_1...\bar{\nu}_s} dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}$$

Then

$$\partial^\dagger \omega = -\frac{1}{(r-1)!s!}\nabla^{\mu_1}\omega_{\mu_1...\mu_r\bar{\nu}_1...\bar{\nu}_s} dz^{\mu_2} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}$$
$$\bar{\partial}^\dagger \omega = -\frac{1}{r!(s-1)!}\nabla^{\bar{\nu}_1}\omega\mu_1...\mu_r\bar{\nu}_1...\bar{\nu}_s dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_2} \wedge ... \wedge d\bar{z}^{\nu_s}$$

where ∇ is the Levi-Civita connection.

Proof. This follows from adapting the form of $d^\dagger$ on a real manifold, from Proposition 1.18. $\square$

Definition 5.22. We define the ∂- and $\bar{\partial}$-**Laplacians** by

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial$$
$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$$

If $\Delta_\partial \omega = 0$, we say ω is ∂-**harmonic**. Similarly if $\Delta_{\bar{\partial}}\omega = 0$, we say ω is $\bar{\partial}$-**harmonic**. The associated spaces are denoted $\mathrm{Harm}^{r,s}_\partial(M)$ and $\mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$.

Proposition 5.23. A form on a compact orientable Riemannian complex manifold is ∂-harmonic iff it is both ∂-closed and ∂-coclosed. Similarly it is only $\bar{\partial}$-harmonic iff it is both $\bar{\partial}$-closed and $\bar{\partial}$-coclosed.

Proof. The proofs are entirely analogous to the proof for the same property of d (Proposition 1.22). $\square$

Theorem 5.24 (Hodge's decomposition theorem for complex manifolds)**.** *The space of (r, s)-forms on a compact orientable Riemannian complex manifold without boundary uniquely decomposes as*

$$\Omega^{r,s}(M) = \bar{\partial}\Omega^{r,s-1}(M) \oplus \bar{\partial}^\dagger\Omega^{r,s+1}(M) \oplus \mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$$

That is, any (r, s)-form ω can be uniquely written in the form

$$\omega = \bar{\partial}\alpha + \bar{\partial}^\dagger\beta + \gamma$$

where γ is $\bar{\partial}$-harmonic.

Proof. The proof runs just as that for Hodge's decomposition theorem for real manifolds, Theorem 1.24. □

Proposition 5.25. A $\bar{\partial}$-cohomology class has a unique $\bar{\partial}$-harmonic representative.

Proof. This follows from the decomposition theorem just as Proposition 1.25. □

Theorem 5.26 (Hodge's theorem for complex manifolds)**.** *If M is a compact orientable Riemannian complex manifold without boundary,*

$$H^{r,s}_{\bar{\partial}}(M) \cong \mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$$

Proof. See Theorem 1.26. □

Theorem 5.27 (Poincaré duality for complex manifolds)**.** *For a compact orientable Riemannian complex n-manifold, $H^{r,s}_{\partial}(M) \cong H^{n-s,n-r,M}_{\bar{\partial}}()$.*

Proof. We have $\partial^\dagger = - * \bar{\partial} *$ and $\bar{\partial}^\dagger = - * \partial *$. So if ω is ∂-closed, then $*\omega$ is $\bar{\partial}$-coclosed, and if ω is ∂-coclosed, $*\omega$ is $\bar{\partial}$-closed, and vice versa. Therefore $*$ establishes an isomorphism

$$\mathrm{Harm}^{r,s}_{\partial}(M) \cong \mathrm{Harm}^{n-s,n-r}_{\bar{\partial}}(M)$$

and hence

$$H^{r,s}_{\partial}(M) \cong H^{n-s,n-r}_{\bar{\partial}}(M)$$

□

Corollary 5.28. *In terms of Hodge numbers,*

$$b_{r,s} = b_{n-s,n-r}$$

5.2 Kähler Manifolds

Definition 5.29. If $d\Omega = 0$, we say the manifold is **Kähler**.

Proposition 5.30. On a Kähler manifold,

$$\partial_\lambda g_{\mu\bar{\nu}} = \partial_\mu g_{\lambda\bar{\nu}}$$

and conjugate.

Proof. We have

$$\begin{aligned} d\Omega &= (\partial + \bar{\partial})\Omega \\ &= i\partial_\lambda g_{\mu\bar{\nu}} dz^\lambda \wedge dz^\mu \wedge d\bar{z}^\nu - i\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\lambda \wedge d\bar{z}^\nu \end{aligned}$$

On a Kähler manifold this is zero; the two terms are of different bidegree so must vanish separately. The vanishing of the first term requires

$$\partial_\lambda g_{\mu\bar{\nu}} = \partial_\mu g_{\lambda\bar{\nu}}$$

and the second its complex conjugate. □

Corollary 5.31. *Locally, there exists some function ϕ, called the **Kähler potential**, such that*

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \phi$$

Proposition 5.32. The Hermitian connection on a Kähler manifold is the Levi-Civita connection.

Proof. The Levi-Civita connection on a manifold is the unique metric-compatible torsion-free connection. The Hermitian connection is metric-compatible. Its non-zero components are

$$\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\bar{\kappa}} \partial_\mu g_{\nu\bar{\kappa}}$$

and its conjugate. But since

$$\partial_\mu g_{\nu\bar{\kappa}} = \partial_\nu g_{\mu\bar{\kappa}}$$

this is symmetric in its lower indices, i.e. torsion-free. Therefore it is the Levi-Civita connection. □

Corollary 5.33. *We then have propositions earlier established for Levi-Civita connections on Kähler manifolds. In particular, if*

$$\omega = \frac{1}{p!}\omega_{m_1 \ldots m_p} dx^{m_1} \wedge \ldots \wedge dx^{m_p}$$

(where the Latin m_i may be real, holomorphic or antiholomorphic) then

$$d\omega = \frac{1}{p!}\nabla_n \omega_{m_1 \ldots m_p} dx^n \wedge dx^{m_1} \wedge \ldots \wedge dx^{m_p}$$
$$d^\dagger \omega = -\frac{1}{(p-1)!}\nabla^{m_1}\omega_{m_1 \ldots m_p} dx^{m_2} \wedge \ldots \wedge dx^{m_p}$$

So covariantly conservation of components implies both closure and coclosure of a form, and for forms of less than maximal degree, this is an iff relation. Further, if a Kähler manifold is also compact, covariant conservation of components of a form mean it is harmonic. (Recall from Proposition 5.5 that all complex manifolds are orientable.)

Corollary 5.34. *In particular, this gives us an alternative definition of a Kähler manifold: one on which Ω is covariantly conserved. Then in fact Ω is both closed and coclosed, and hence on a compact Kähler manifold harmonic. Thus in this case $b_2 \geq 1$.*

Theorem 5.35. *On a Kähler manifold,*

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

Proof. This can be done by recalling the component form of $\Delta\omega$, as mentioned in that case, this proof is long and I haven't got it exactly, but some of it is in my solutions to Candelas' Lectures on Complex Manifolds. □

Corollary 5.36. *On a Kähler manifold form is d-harmonic iff it is $\bar{\partial}$-harmonic. Therefore we have*

$$\mathrm{Harm}^p(M) \cong \bigoplus_{r+s=p} \mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$$

This then implies

$$H^p(M) \cong \bigoplus_{r+s=p} H^{r,s}_{\bar{\partial}}(M)$$

So

$$b_p(M) = \sum_{r+s=p} b_{r,s}(M)$$

Corollary 5.37. *Any complex manifold is orientable, so we have $b_{2n} \geq 1$. Then*

$$\sum_{r+s=2n} b_{r,s} \geq 1$$

However, $r, s \leq n$, so in fact this is

$$b_{n,n} \geq 1$$

If our manifold is also compact, we can say that $b_{n,n} = b_{2n} = b_0$ is the number of connected components.

Corollary 5.38. *Poincaré duality for the complex manifold implies Poincaré duality for the underlying real manifold. This is perhaps obvious, but useful to confirm it follows. On a compact Kähler manifold*

$$\begin{aligned} b_p &= \sum_{r+s=p} b_{r,s} \\ &= \sum_{r+s=p} b_{n-s,n-r} \\ &= \sum_{r'+s'=2n-p} b_{r',s'} \\ &= b_{2n-p} \end{aligned}$$

Corollary 5.39. *Suppose ω is harmonic, so $\Delta_\partial \omega = \bar{\partial}\Delta\omega = 0$. Then $\Delta_{\bar{\partial}}\bar{\omega} = \overline{\Delta_\partial \omega} = \overline{\Delta_{\bar{\partial}}\omega}$ where in the last equality we used $\Delta_\partial = \Delta_{\bar{\partial}}$ for Kähler manifolds. Therefore ω is harmonic iff $\bar{\omega}$ is, setting up an isomorphism*

$$\mathrm{Harm}^{r,s}_{\bar{\partial}}(M) \cong \mathrm{Harm}^{s,r}_{\bar{\partial}}(M)$$

And hence

$$H^{r,s}_{\bar{\partial}}(M) \cong H^{s,r}_{\bar{\partial}}(M)$$

So

$$b_{r,s} = b_{s,r}$$

Corollary 5.40. *If a Kähler manifold is simply connected, $b_1 = b_{1,0} + b_{0,1} = 2b_{1,0} = 0$.*

Corollary 5.41. *The Kähler form is harmonic on a Kähler manifold, so* $b_2 \geq 1$*. Thus*

$$b_{0,2} + b_{1,1} + b_{2,0} \geq 1$$

But furthermore $b_{r,s} = b_{s,r}$*, so*

$$2b_{0,2} + b_{1,1} \geq 1$$

Definition 5.42. Hodge numbers can be arranged into a **Hodge diamond**:

$$\begin{array}{ccccccccc}
 & & & & b_{m,m} & & & & \\
 & & & b_{m,m-1} & & b_{m-1,m} & & & \\
 & & . & & & & . & & \\
b_{m,0} & b_{m,1} & & & & & & b_{1,m-1} & b_{0,m} \\
 & & . & & & & . & & \\
 & & & b_{1,0} & & b_{0,1} & & & \\
 & & & & b_{0,0} & & & &
\end{array}$$

Assuming compactness, by Corollary 5.28 this is symmetric along the horizontal, and by Corollary 5.39 also along the vertical. This reduces the total $(n+1)^2$ Hodge numbers to only $(n/2+1)^2$ if n is even, or $(n+1)(n+3)/4$ if n is odd.

Proposition 5.43. On a Kähler manifold

(i) b_{2p-1} is even

(ii) $b_{2p} \geq 1$

for any $p = 1, ..., n$.

Proof.

(i) We have

$$\begin{aligned}
b_{2p-1} &= \sum_{r+s=2p-1} b_{r,s} \\
&= \sum_{r+s=2p-1, r>s} b_{r,s} + \sum_{r+s=2p-1, s>r} b_{r,s} \\
&= \sum_{r+s=2p-1, r>s} b_{r,s} + \sum_{r+s=2p-1, s>r} b_{s,r} \\
&= 2 \sum_{r+s=2p-1, r>s} b_{r,s}
\end{aligned}$$

(ii) Consider Ω^p. It is closed on a Kähler manifold since Ω is. Suppose it is exact, and write $\Omega^p = d\alpha$. Then we can write

$$\begin{aligned}\Omega^n = \Omega^{n-p} \wedge \Omega^p &= \Omega^{n-p} \wedge d\alpha \\ &= d(\Omega^{n-p} \wedge \alpha)\end{aligned}$$

Then, integrating over all M, this vanishes. But Ω^n is a volume element, so this cannot be so. Therefore Ω^p is a representative of a non-trivial cohomology class and $b_{2p} \geq 1$.

□

Remark. To summarise, for compact Kähler manifolds, we have

- Relation between de Rham and Dolbeault cohomology: $b_p = \sum_{r+s=p} b_{r,s}$
- $b_0 = b_{0,0}$ is the number of connected components.
- Orientability of complex manifolds: $b_{2n} = b_{n,n} \geq 1$.
- Poincaré duality for the complex manifold: $b_{r,s} = b_{n-s,n-r}$, so in particular $b_{2n} = b_{n,n} = b_{0,0} = b_0$ - there is a unique volume form (up to scaling) for each connected component.
- Poincaré duality for the underlying real manifold: $b_p = b_{2n-p}$.
- Harmonicity of the Kähler form: $b_2 \geq 1$, or alternatively $2b_{0,2} + b_{1,1} \geq 1$.
- b_{2p-1} even for any $p = 1, ..., n$.
- $b_{2p} \geq 1$ for any $p = 1, ..., n$.
- If simply connected: $b_1 = 2b_{1,0} = 0$.

5.3 Calabi-Yau Manifolds

Definition 5.44. The **Ricci 2-form** is

$$\mathcal{R} = iR^{\bar{\lambda}}{}_{\bar{\lambda}\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$$

Remark. On a Kähler manifold, though we will not prove this, $R^k{}_{lmn} = R^k{}_{mln}$, so in fact the components of $\mathcal{R}$ are just those of the Ricci tensor, with an i.

Proposition 5.45. The Ricci form is given by

$$\mathcal{R} = -i\partial\bar{\partial}\ln g$$

where g is the determinant of the metric.

Proof. We have

$$R^{\bar{\lambda}}{}_{\bar{\lambda}\mu\bar{\nu}} = -\partial_\nu(g^{\rho\bar{\lambda}}\partial_{\bar{\mu}}g_{\rho\bar{\lambda}})$$

Now, we can use the matrix identity

$$\ln\det M = \operatorname{tr}\ln M$$

to see that

$$\ln g = \operatorname{tr}\ln g_{\mu\bar{\nu}}$$
$$\partial_{\bar{\mu}}\ln g = g^{\rho\bar{\lambda}}\partial_{\bar{\mu}}g_{\rho\bar{\lambda}}$$

Thus

$$R^{\bar{\lambda}}{}_{\bar{\lambda}\mu\bar{\nu}} = -\partial_\nu\partial_{\bar{\mu}}\ln g$$
$$\mathcal{R} = -i\partial\bar{\partial}\ln g$$

□

Corollary 5.46. *Using $d = (\partial + \bar{\partial})$ and $\partial^2 = \bar{\partial}^2 = 0$, the Ricci form is closed. However, in general, $\partial\bar{\partial}\ln g$ is not a globally defined function, so even though*

$$\partial\bar{\partial} = -\frac{1}{2}d(\partial - \bar{\partial})$$

$\mathcal{R}$ is not generally exact.

Definition 5.47. The cohomology class $[\frac{1}{2\pi}\mathcal{R}] \in H^2(M)$ is called the **first Chern class**.

Definition 5.48. A **Calabi-Yau** manifold is a Kähler manifold with vanishing first Chern class.

This section was written with the help of [3] and [4].

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