Fibre Bundles and Physics

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Abstract

These are my personal notes on fibre bundles and their usage in physics. They are a work in progress, will have mistakes, and will always be lacking in many ways. The main texts I have learnt the topic from are [1], [2] and [3] and I have aimed to synthesise but also augment these considerably with the fruits of my own labours. Where I am particularly indebted to another reference it will be listed. I will be adding sections on characteristic classes and spin structures in the future, and more to Section 3 and Appendix A.

Contents

1 Fibre Bundles

1.1 Fibre Bundles

1.1.1 Definitions

Definition 1.1. A **smooth fibre bundle**, (E, M, π, F, G), variously abbreviated E, $\pi : E \to M$, etc, consists of:

(i) A smooth manifold E, called the **total space**.

(ii) A smooth manifold M, called the **base space**.

(iii) A smooth manifold F, called the **(typical) fibre**.

(iv) A surjection $\pi : E \to M$, called the **projection**, such that $\pi^{-1}(p) := F_p \cong F$ for all $p \in M$.

(v) A Lie group G, called the **structure group**, with a smooth left-action on F.

(vi) An open covering $\{U_i\}$ of M, with diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ satisfying $\pi \circ \phi_i(p, f) = p$, called **local trivialisations**.

(vii) Maps $t_{ij} : U_i \cap U_j \to G$ on overlaps satisfying $\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$, called **transition functions**.

Proposition 1.2. The transition functions should satisfy

$$\begin{aligned} t_{ii}(p) &= e \\ t_{ij}(p)t_{jk}(p) &= t_{ik}(p) \end{aligned}$$

for all $p \in M$ and $p \in U_i \cap U_j \cap U_k$, respectively. The latter is called the **cocycle condition**.

Proof. These follow from the fact that the local trivialisations are isomorphisms and the invertibility of group elements of G. The first is obvious. For the second, note that on the one hand

$$\begin{aligned} \phi_k(p, f) &= \phi_j(p, t_{jk}(p)f) \\ &= \phi_i(p, t_{ij}(p)t_{jk}(p)f) \end{aligned}$$

and on the other

$$\phi_k(p, f) = \phi_i(p, t_{ik}(p)f)$$

□

Corollary 1.3. *In particular, by the cocycle condition,*

$$t_{ij}(p)t_{ji}(p)t_{ik}(p) = t_{ik}(p)$$

so

$$t_{ij}(p)^{-1} = t_{ji}(p)$$

Definition 1.4. If it is possible to set $t_{ij}(p) = e$ for all $p \in U_i \cap U_j$ for all intersecting charts U_i and U_j, the bundle E is called a **trivial bundle**.

Definition 1.5. A **smooth section** of a bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The set of smooth sections of E is denoted $\Gamma(E)$.

Theorem 1.6 (Fibre bundle construction theorem)**.** *Given a base space M with open covering $\{U_i\}$, typical fibre F, structure group G, and transition functions t_{ij} it is possible to construct a unique bundle $\pi : E \to M$.*

Proof. Define

$$X = \bigsqcup_i U_i \times F$$

and an equivalence relation

$$(p, f) \sim (q, f') \quad \text{iff} \quad p = q \text{ and } f' = t_{ji}(p)f$$

if $p \in U_i \cap U_j$. Then we claim

$$E = X/\sim$$

which has elements $[(p, f)]$ which are equivalence classes, is a bundle, with the projection $\pi : E \to M$ defined by

$$\pi : [(p, f)] \mapsto p$$

Firstly, we note that this is well-defined, since every $(p', f') \in [(p, f)]$ has $p' = p$. It is also clearly surjective, since $\{U_i\}$ covers M. Local trivialisations are choices of representatives. Choose some fixed v_i such that $(p, v_i) \in [(p, v)] = u$, and define $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ by

$$\phi_i(p, v_i) = [(p, v)]$$

Certainly

$$\pi \circ \phi_i(p, v_i) = \pi([(p, v)]) = p$$

as required. Then, for each intersecting patch U_j, define a local trivialisation ϕ_j which selects the representative (p, v_j) from $[(p, v)]$. v_j is determined by using the transition functions: we

$$\phi_j(p, v_j) = [(p, v)] = \phi_i(p, v_i)$$

so we are forced to have $v_j = t_{ij}(p)v_i$. That is, after choosing a single local trivialisation all the rest are fixed by the transition functions. This makes the bundle unique. By the cocycle condition on the transition functions this local trivialisation construction can always be done consistently for multiple local trivialisations. $\square$

Remark. Note that, while the use of equivalence classes ($[p, f]$) may suggest that the elements $u \in \pi^{-1}(\{p\})$ are not in one-to-one correspondence with the typical fibre F, we can always choose an open cover such that p is not in an overlap, and hence such that there are no transition functions t_{ij} defined on p, in which case equivalence classes are just singletons.

1.1.2 Bundle Maps

Definition 1.7. Let $\pi : E \to M$ and $\pi' : E' \to M'$ be two bundles. A smooth map $\bar{f} : E' \to E$ is called a **bundle map** if it does not mix fibres, i.e. if $\bar{f} \circ \pi' = \pi \circ \bar{f}$. Then $\bar{f}$ naturally induces a smooth map $f : M' \to M$ defined by $f = \pi'^{-1} \circ \bar{f} \circ \pi$.

Definition 1.8. Let $\bar{f}$ be a bundle map from $\pi' : E' \to M'$ to $\pi : E \to M$. It is called a **bundle isomorphism** if both $\bar{f}$ and the induced f are diffeomorphisms.

Definition 1.9. Two bundles $\pi : E \to M$ and $\pi' : E' \to M$ over the same base space are called **equivalent** if there exists a bundle isomorphism $\bar{f} : E' \to E$ which induces the identity map $f = \mathrm{id}_M$ on the base space.

Definition 1.10. Given a bundle $\pi : E \to M$ with fibre F and a smooth map $f : N \to M$, the **pullback bundle** f^*E over N is the space

$$f^*E = \{(p, u) \in N \times E \mid f(p) = \pi(u)\}$$

That is, the fibres F_p of f^*E are identical to those $F_{f(p)}$ of E.

Proposition 1.11. The pullback bundle can in fact be made into a bundle over N by inheriting properties from E.

Proof. From the fibre bundle construction theorem we know that we can construct f^*E as a bundle. We just need an open covering of N, which is given by $\{f^{-1}(U_i)\}$ if $\{U_i\}$ is an open covering of M. Since the fibres F_p of f^*E are just those $F_{f(p)}$ of E, the equivalence relation in the construction of the construction theorem that gives us f^*E is clearly

$$(p, f) \sim (p, f') \quad \text{iff} \quad f' = t_{ji}(f(p))f$$

That is, we set transition functions

$$t^*_{ij}(p) = t_{ij}(f(p))$$

The projection $\pi^* : f^*E \to N$ is defined by $\pi^* : (p, u) \mapsto p$ which is clearly surjective. Now, suppose the local trivialisations on E are $\phi_i^{-1}(u) = (\pi(u), f_i)$ then at each $(p, u) \in f^*E$ (and automatically $f(p) = \pi(u)$), define $\psi_i^{-1} : (\pi^*)^{-1}(f^{-1}(U_i)) \to f^{-1}(U_i) \times F$ by

$$\psi_i^{-1}(p, u) = (p, f_i)$$

That is,

$$\psi_i(p, f_i) = (p, \phi_i(f(p), f_i))$$

Then $\pi^* \circ \psi_i(p, f_i) = p$, so ψ_i are indeed local trivialisations. We also confirm that

$$\begin{aligned}
\psi_j(p, f_j) &= (p, \phi_j(f(p), f_j)) \\
&= (p, \phi_i(f(p), t_{ij}(f(p))f_j)) \\
&= (p, \phi_i(f(p), t^*_{ij}(p)f_j)) \\
&= \psi_i(p, t^*_{ij}(p)f_j)
\end{aligned}$$

Lastly, the transition functions t^*_{ij} obviously satisfy the requirements since the t_{ij} do. Thus we have made f^*E into a bundle. □

Theorem 1.12 (Homotopy axiom). *If $\pi : E \to M$ is a bundle and $f, g : N \to M$ are homotopic maps, f^*E is equivalent to g^*E.*

Proof. Proving this will take us a fair bit off course, and anyway we have only introduced it for the purpose of the following handy corollary: □

Corollary 1.13. *Any bundle over a simply connected base space is trivial.*

Proof. Let M be simply connected and E be any bundle over M. If $g : M \to M$ is a constant map $g(p) = p_0$, it is homotopic to the identity, so g^*E is equivalent to E. But g^*E is the pullback of $\{p_0\} \times F$, where F is the typical fibre of E, so is trivial. So E is trivial. □

1.2 Vector Bundles

1.2.1 Definitions

Definition 1.14. A fibre bundle $\pi : E \to M$ whose typical fibre $F = V$ is a vector space is called a **vector bundle**. If $V = \mathbb{R}^k$ or $\mathbb{C}^k$, k is called the **fibre dimension**. A vector bundle with $k = 1$ is called a **line bundle**. The action of the structure group G on V is given by a representation $\rho : G \to GL(V)$, which we will generally leave implicit.

Proposition 1.15. If $\pi : E \to M$ is a vector bundle with fibre V, $\Gamma(E)$ naturally inherits vector space structure from V.

Proof. Let s, t be any elements of $\Gamma(E)$, p any point in M, and $\alpha \in \mathcal{F}$, the field over which V is defined. Then give $\Gamma(E)$ vector space structure by setting

$$(s+t)(p) = s(p) + t(p)$$
$$(\alpha s)(p) = \alpha s(p)$$

It is then trivial to check that $\Gamma(E)$ is indeed a vector space. □

Corollary 1.16. *We can further make* $\Gamma(E)$ $C^\infty(M)$*-linear by defining*

$$(fs)(p) = f(p)s(p)$$

for any $f \in C^\infty(M)$ *and* $s \in \Gamma(E)$.

Definition 1.17. It is therefore possible to locally choose k linearly independent sections, $\{e_i\}$, defining a **frame** over a local neighbourhood U of M. Then a generic local section can be written as $s = s^i e_i$, where $s_i \in C^\infty(M)$. Any two frames $\{e_i\}$ and $\{e'_j\}$ are related by some $G \in GL(V)$ by $e'_j = G^i_j e_i$.

Definition 1.18. A **vector bundle isomorphism** $f : E \to E'$ is a bundle map of vector bundles whose restriction to each fibre is a linear isomorphism, i.e. if $f\,|_{\pi^{-1}(p)} : V_p \to V'_{f(p)}$ is a linear isomorphism.

Theorem 1.19. *A vector bundle has a global frame iff there exists a vector isomorphism to a trivial vector bundle.*

Proof. Suppose $\pi : E \to M$ has a global frame $\{e_i\}$ for $i = 1, ..., k$. Then we can define a map from the trivial $\mathbb{R}^k$ (or $\mathbb{C}^k$) bundle over M to E by

$$f : (p, v) \mapsto v^i e_i(p)$$

Firstly, this is clearly a bundle map, since both sides are projected down to p. Furthermore, this is clearly a vector bundle isomorphism.
This argument also runs the other way. □

Corollary 1.20. *Recall that any bundle over a simply connected base space is trivial. Then we have that any vector bundle over a simply connected base space has a global frame.*

Corollary 1.21. *From this construction, we can see that given a local frame $\{e_i\}$ over a patch U, f is a canonical local trivialisation. That is, we define $\phi : U \times V \to \pi^{-1}(U)$ by*

$$\phi : (p, v) \mapsto v^i e_i(p)$$

As observed, this satisfies $\pi \circ \phi(p, v) = p$. Note that if ϕ and $\tilde{\phi}$ are canonical local trivialisations associated with frames $\{e_i\}$ and $\{\tilde{e}_j\}$, we have

$$\begin{aligned}\tilde{\phi}(p, v) &= v^i \tilde{e}_j(p) \\ &= G^i_j(p) v^j e_i(p) \\ &= \phi(p, G^i_j(p) e_i(p))\end{aligned}$$

That is, the transition functions $t(p)$ are mapped by the representation $\rho : G \mapsto GL(V)$ to the matrices $G(p)$ relating frames.

1.2.2 New Vector Bundles from Old Ones

Definition 1.22. Let $\pi : E \to M$ be a vector bundle with typical fibre V. Then its **dual bundle** $\pi^* : E^* \to M$ is the bundle whose fibres are V^*. A local frame $\{e_i\}$ of E over $U \subset M$ induces a basis $\{e_i(p)\}$ on V_p for each $p \in U$, each of which induces a dual basis on V_p^*, $\{e^i(p)\}$, which can be extended to a local frame $\{e^i\}$ of E^*. This makes E^* isomorphic to E.

Definition 1.23. Let $\pi : E \to M$ and $\pi' : E' \to M'$ be vector bundles with fibres V and V' and structure groups G and G'. Their **product bundle** is $\pi \times \pi' : E \times E' \to M \times M'$ and has fibre $V \oplus V'$ and structure group $G \oplus G'$ with the obvious inherited left-action.

Proposition 1.24. This is indeed a vector bundle.

Proof. Let $\{U_i\}$ and $\{U_\alpha\}$ be open covers of M and M' respectively. Then $\{U_i \times U_\alpha\}$ is an open cover of $M \times M'$. Denote local trivialisations of E and E' ϕ_i and ϕ'_α respectively, and transition functions t_{ij} and $t'_{\alpha\beta}$ respectively. Then define $\psi_{i\alpha} : U_i \times U_\alpha \times (V \oplus V')$ by

$$\psi_{i\alpha} : (p, q, (v, w)) \mapsto (\phi_i(p, v), \phi'_\alpha(q, w))$$

Then

$$\begin{aligned}(\pi \times \pi') \circ \psi_{i\alpha}(p, q, (v, w)) &= (\pi \times \pi')(\phi_i(p, v), \phi'_\alpha(q, w)) \\ &= (\pi \circ \phi_i(p, v), \pi' \circ \phi'_\alpha(q, w)) \\ &= (p, q)\end{aligned}$$

as desired, so $\psi_{i\alpha}$ is indeed a local trivialisation. Also define transition functions $T_{ij,\alpha\beta}$: $(U_i \cap U_j) \times (U_\alpha \times U_\beta) \to G \oplus G'$ (again leaving representations implicit) by

$$T_{ij,\alpha\beta}(p, q) = \begin{pmatrix} t_{ij}(p) & \\ & t'_{\alpha\beta}(q) \end{pmatrix}$$

This naturally inherits the requirements on transition functions from the original ones. Then notice that indeed

$$\begin{aligned}\psi_{j\beta}(p, q, (v, w)) &= (\phi_j(p, v), \phi'_\beta(q, w)) \\ &= (\phi_i(p, t_{ij}v), \phi'_\alpha(q, t'_{\alpha\beta}w)) \\ &= \psi_{i\alpha}(p, q, T_{ij,\alpha\beta}(p, q))\end{aligned}$$

as expected. $E \times E'$ is now a bundle; that it is a vector bundle is obvious. □

Proposition 1.25. Given three bundles E, E', E'', the product $E \times E' \times E''$ is well-defined, i.e.

$$(E \times E') \times E'' = E \times (E' \times E'')$$

(up to canonical isomorphisms).

Proof. This is fairly clear from the same statement about the fibres but I won't show it explicitly. □

Definition 1.26. Let $\pi : E \to M$ and $\pi' : E' \to M$ be vector bundles over M with fibres V and V' respectively, and $f : M \to M \times M$ be the diagonal map $f(p) = (p, p)$. Then the **Whitney sum bundle** of E and E' is

$$E \oplus E' = f^*(E \times E')$$

That is,

$$E \oplus E' = \{(u, u') \in E \times E' \mid (\pi \times \pi')(u, u') = (p, p)\}$$

(The qualifier may also be written $\pi(u) = \pi'(u') = p$.) This makes the Whitney sum bundle the natural notion of a bundle whose fibre at p is the direct sum of those of two other bundles, also at p.

Proposition 1.27. This is indeed a vector bundle.

Proof. Showing this is exactly the same as showing that $E \times E'$ is a vector bundle. □

Proposition 1.28. Given three bundles E, E', E'', the sum $E \oplus E' \oplus E''$ is well-defined, i.e.

$$(E \oplus E') \oplus E'' = E \oplus (E' \oplus E'')$$

(up to canonical isomorphisms).

Proof. This follows from the same result for product bundles. □

Corollary 1.29. *Given a collection of bundles E_i,*

$$\bigoplus_i E_i$$

is well-defined.

Definition 1.30. If $s \in \Gamma(E)$ and $s' \in \Gamma(E')$, we define $(s, s') \in \Gamma(E \oplus E')$ satisfying

$$(s, s')(p) = (s(p), s'(p))$$

for all p on which s and s' are defined.

Proposition 1.31. Any section $t \in \Gamma(E \oplus E')$ can be written as $t = (s, s')$ for unique s, s'. That is,

$$\Gamma(E \oplus E') = \Gamma(E) \oplus \Gamma(E')$$

Proof. Since $t(p) \in V \oplus V'$, it is fairly clear that we can find s, s' such that $t(p) = (s(p), s'(p))$. If $(s(p), s'(p)) = (\tilde{s}(p), \tilde{s}'(p))$ for all p on which t is defined, clearly $s = \tilde{s}$ and $s' = \tilde{s}'$, making this unique. □

Definition 1.32. Let $\pi : E \to M$ and $\pi' : E' \to M$ be vector bundles over M with fibres V and V', and transition functions t_{ij} and t'_{ij}, respectively. Then the **tensor product bundle** of E and E' is a bundle over M with fibres $V \otimes V'$ and transition functions $t_{ij}(p) \otimes t'_{ij}(p)$.

Proposition 1.33. This is indeed a vector bundle.

Proof. We will follow the fibre bundle construction theorem. Initially we have bundles

$$E = \bigsqcup_i U_i \times V / \sim$$
$$E' = \bigsqcup_i U_i \times V' / \sim'$$

where $\sim$ and $\sim'$ are due to transition functions t_{ij} and t'_{ij} respectively. The projections are $\pi([(p, v)]) = p$ and $\pi'([(p, v')]) = p$, and define local trivialisations $\phi_i : (p, v_i) \mapsto [(p, v)]$ and $\phi'_i : (p_i, v') \mapsto [(p, v')]$ over U_i. Remember that defining a local trivialisation is a choice of representative of the equivalence classes which constitute the bundle. From these bundles form the tensor product bundle

$$E \otimes E' = \bigsqcup_i U_i \times (V \otimes V') / \sim''$$

where $\sim''$ is due to transition functions T_{ij}. Its elements are equivalence classes

$$[(p, v \otimes v')] = \{(p, w \otimes w') \mid w \otimes w' = T_{ji}(p)(v \otimes v')\}$$

where, by hypothesis, $T_{ji}(p) = t_{ji}(p) \otimes t'_{ji}(p)$. The projection $(\pi \otimes \pi') : E \otimes E' \to M$ maps $[(p, v \otimes v')] \mapsto p$. Now, given local trivialisations on the constituent bundles, we can define a local trivialisation $\psi_i : U \times (V \otimes V') \to (\pi \otimes \pi')^{-1}(U_i)$ over U_i which picks out precisely the representative $(p, v_i \otimes v'_i)$, i.e.

$$\psi_i(p, v_i \otimes v'_i) = [(p, v \otimes v')]$$

Then, as required,

$$(\pi \otimes \pi') \circ \psi_i(p, v_i \otimes v'_i) = \pi([(p, v \otimes v')])$$

This, along with $T_{ij}(p)$ then fixes the rest of our local trivialisations, by requiring

$$\psi_j(p, v_j \otimes v'_j) = \psi_i(p, v_i \otimes v'_i)$$

In fact, this construction is well defined precisely because $T_{ij}(p) = t_{ij}(p) \otimes t'_{ij}(p)$, i.e. since we have

$$\begin{aligned} T_{ij}(p)(v_i \otimes v'_i) &= (t_{ij}(p) \otimes t'_{ij}(p))(v_i \otimes v'_i) \\ &= t_{ij}(p)v_i \otimes t'_{ij}(p)v'_i \\ &= v_j \otimes v'_j \end{aligned}$$

We can then happily start with any local trivialisation ψ_i and construct the others consistently. Thus we have made $E \otimes E'$ into a fibre bundle, and of course not only that but a vector bundle. $\square$

Remark. The construction of the new local trivialisations here is just $\psi = \phi \otimes \phi'$, where

Definition 1.34. The tensor product $\otimes : E \times E' \to E \otimes E'$ is defined fibrewise by

$$[(p, v)] \otimes [(p, v')] = [(p, v \otimes v')]$$

Proposition 1.35. Given three bundles E, E', E'', the tensor product, $E \otimes E' \otimes E''$ is well-defined, i.e.

$$(E \otimes E') \otimes E'' = E \otimes (E' \otimes E'')$$

(up to canonical isomorphisms).

Proof. Again, this follows from the same statement about fibres. $\square$

Corollary 1.36. *Given a collection of bundles E_i,*

$$\bigotimes_i E_i$$

is well-defined.

Definition 1.37. Given $s \in \Gamma(E)$ and $s' \in \Gamma(E')$, we define $s \otimes s' \in \Gamma(E \otimes E')$ by

$$(s \otimes s')(p) = s(p) \otimes s'(p)$$

for all p on which s and s' are defined.

Remark. Although this is clearly well-defined, we cannot go the other way. That is, there is always ambiguity in decomposing $t \in \Gamma(E \otimes E')$ into s and s'. To see this, use local frames $\{e_i\}$ and $\{e'_j\}$, and write $t = t^{ij} e_i \otimes e'_j$ and $s = s^i e_i$, $s' = s'^j e'_j$. Then clearly $t = s \otimes s'$ for non-unique choices of s^i and s'^j. For instance, I can double all the s^i and halve all the s'^j without affecting t.

1.2.3 Vector Bundle Algebra

Definition 1.38. A vector bundle $\pi : E \to M$ generates a **tensor algebra**

$$T(E) = \bigoplus_{k=0}^{\infty} \bigotimes^{k} E$$

under the tensor product

$$(u_1 \otimes ... \otimes u_k) \otimes (u_{k+1} \otimes ... \otimes u_{k+l}) = u_1 \otimes ... \otimes u_k \otimes u_{k+1} \otimes ... \otimes u_{k+l}$$

Definition 1.39. Given a vector bundle E, let the symmetric group S_k act on $\bigotimes^k E$ by

$$u_1 \otimes ... \otimes u_k \mapsto u_{\sigma(1)} \otimes ... \otimes u_{\sigma(k)}$$

Then the $k^{\mathbf{th}}$ **exterior power** of E is the bundle

$$\bigwedge^{k} E = \{a \in \bigotimes^{k} E \mid \sigma : a \mapsto \mathrm{sgn}(\sigma) a \ \forall\ \sigma \in S_k\}$$

E generates the **exterior algebra**

$$\bigwedge(E) = \bigoplus_{k=0}^{\infty} \bigwedge^{k} E$$

with the **wedge product** $\wedge$, which is the fully antisymmetrised tensor product.

Definition 1.40. Sections $s \in \Gamma(E)$ and $\lambda \in \Gamma(E^*)$ naturally define a smooth function $\lambda(s) \in C^\infty(M)$ defined by

$$\lambda(s) : p \mapsto \lambda(p)(s(p))$$

In dual local frames $\{e_i\}$ and $\{e^i\}$, this function is just

$$\lambda(s) : p \mapsto \lambda_i(p) s^i(p)$$

Proposition 1.41. Such a function $\lambda(s)$ is $C^\infty(M)$-linear in λ and s.

Proof. Let $f, g \in C^\infty(M)$, $\lambda, \mu \in \Gamma(E^*)$ and $s, t \in \Gamma(E)$. Firstly,

$$\begin{aligned}((f\lambda + g\mu)(s))(p) &= (f\lambda + g\mu)(p)s(p) \\ &= f(p)\lambda(p)s(p) + g(p)\mu(p)s(p) \\ &= (f(\lambda(s)))(p) + (g(\mu(s)))(p)\end{aligned}$$

Secondly,

$$\begin{aligned}(\lambda(fs + gt))(p) &= \lambda(p)(fs + gt)(p) \\ &= \lambda(p)f(p)s(p) + \lambda(p)g(p)t(p) \\ &= (\lambda(fs))(p) + (\lambda(gt))(p)\end{aligned}$$

□

Definition 1.42. Let E be a vector bundle. Then denote by $\mathrm{End}(E)$ a bundle whose sections define fibrewise linear functions from E to itself. That is, if $S \in \Gamma(\mathrm{End}(E))$, then

$$S(p) : [(p, v)] \mapsto [(p, s(p)v)]$$

for an endomorphism $s(p) \in \mathrm{End}(V_p)$. But since $\mathrm{End}(V) \cong V \otimes V^*$, we can identify $\mathrm{End}(E)$ as the tensor product bundle $E \otimes E^*$. Then in a local frame $\{e_i\}$ for E, a section S of $\mathrm{End}(E)$ can be written $S^i_j e_i \otimes e^j$.

Definition 1.43. Given $s \in \Gamma(E)$ and $T \in \Gamma(\mathrm{End}(E))$, we define $Ts \in \Gamma(E)$ by

$$(Ts)(p) = T(p)s(p)$$

for all p on which s and T are defined. With a local frame $\{e_i\}$ on E, this new section is

$$Ts = T^i_j s^j e_i$$

Similarly, given $S, T \in \Gamma(\mathrm{End}(E))$, define $ST \in \Gamma(\mathrm{End}(E))$ by

$$(ST)(p) = S(p)T(p)$$

Locally,

$$ST = S^i_k T^k_j e_i \otimes e^j$$

Proposition 1.44. These are $C^\infty(M)$-linear in S, T and s.

Proof. This is straightforward and much the same as the similar proof for $\lambda(s)$ earlier. □

Definition 1.45. A **differential form** is a section of the bundle $\bigwedge(T^*M)$. More generally, a **vector-valued differential form** is a section of a bundle $\bigwedge(T^*M)\otimes E$, where $E = M \times V$ is a trivial vector bundle. We say that such a form takes values in V. Sections can be decomposed (though not necessarily uniquely) as $\omega \otimes s$, where ω is a differential form and $s \in \Gamma(E)$ can be thought of as a V-valued function on M since E is trivial. There is, however, a canonical decomposition given a basis $\{e_i\}$ for V, namely that of the form $\omega = \omega^i \otimes e_i$. We introduce the notation $\Omega^p(M) = \Gamma(\bigwedge^p T^*M)$ and $\Omega^p(M,V) = \Gamma(\bigwedge^p T^*M \otimes E)$. (On T^*M see Appendix A.)

Definition 1.46. We can make the following extensions to the wedge product:

(i) $\wedge : \Omega^p(M,V) \times \Omega^q(M) \to \Omega^{p+q}(M,V)$ by

$$(\omega \otimes s) \wedge \mu = (\omega \wedge \mu) \otimes s$$

(ii) $\wedge : \Omega^p(M, \operatorname{End} V) \times \Omega^q(M,V) \to \Omega^{p+q}(M,V)$ by

$$(\omega \otimes T) \wedge (\mu \otimes s) = (\omega \wedge \mu) \otimes (Ts)$$

(iii) $\wedge : \Omega^p(M,V) \times \Omega^q(M,V) \to \Omega^{p+q}(M,V)$, where V is an algebra (in particular, $\operatorname{End} V$ or $\mathfrak{g}$), by

$$(\omega \otimes S) \wedge (\mu \otimes T) = (\omega \wedge \mu) \otimes (ST)$$

Proposition 1.47. These all have the appropriate $C^\infty(M)$-linearity.

Proof. Again, this is straightforward and similar to other proofs. □

Definition 1.48. Let $\omega \in \Omega^p(M,V)$ and $\mu \in \Omega^q(M,V)$, and V be an algebra (in particular, $\operatorname{End} V$ or $\mathfrak{g}$). Then their **graded commutator** is

$$[\omega, \mu] = \omega \wedge \mu - (-1)^{pq} \mu \wedge \omega$$

The graded commutator may also be defined in the same way for $\mathfrak{g}$-valued forms. Note that for normal differential forms the RHS vanishes. Indeed, we have:

Proposition 1.49. The graded commutator measures non-commutativity of the algebra (i.e. $\operatorname{End} V$ or $\mathfrak{g}$) factors.

Proof. By direct calculation,

$$[S \otimes \omega, T \otimes \mu] = (ST - TS) \otimes (\omega \wedge \mu)$$

In particular, on $\mathfrak{g}$-valued forms

$$[S \otimes \omega, T \otimes \mu] = [S, T] \otimes (\omega \wedge \mu)$$

So in a basis $\{T_\alpha\}$ for $\mathfrak{g}$, in which the structure constants are $f^\gamma_{\alpha\beta}$, we can expand our $\mathfrak{g}$-valued forms as $\omega = T_\alpha \otimes \omega^\alpha$ and $\mu = T_\alpha \otimes \mu^\alpha$, giving

$$[\omega, \mu] = f^\gamma_{\alpha\beta} T_\gamma \otimes \omega^\alpha \wedge \mu^\beta$$

□

Proposition 1.50. The graded commutator satisfies

(i) graded antisymmetry,
$$[\omega, \mu] = -(-1)^{pq}[\mu, \omega]$$

(ii) the graded Jacobi identity
$$[\omega, [\mu, \eta]] + (-1)^{p(q+r)}[\mu, [\eta, \omega]] + (-1)^{r(p+q)}[\eta, [\omega, \mu]] = 0$$
where additionally $\eta \in \Omega^r(M, \operatorname{End} V)$.

Proof.

(i) We have
$$\begin{aligned} -(-1)^{pq}[\mu, \omega] &= -(-1)^{pq}\mu \wedge \omega - (-1)^{2pq+1}\omega \wedge \mu \\ &= \omega \wedge \mu - (-1)^{pq}\mu \wedge \omega \\ &= [\omega, \mu] \end{aligned}$$

(ii) Consider
$$\begin{aligned} [\omega, [\mu, \eta]] &= [\omega, \mu \wedge \eta - (-1)^{qr}\eta \wedge \mu] \\ &= \omega \wedge \mu \wedge \eta - (-1)^{p(q+r)}\mu \wedge \eta \wedge \omega - (-1)^{qr}\omega \wedge \eta \wedge \mu + (-1)^{qr+p(q+r)}\eta \wedge \mu \wedge \omega \end{aligned}$$

Then similarly

$$
\begin{aligned}
(-1)^{p(q+r)}[\mu,[\eta,\omega]] &= (-1)^{p(q+r)}(\mu\wedge\eta\wedge\omega - (-1)^{q(r+p)}\eta\wedge\omega\wedge\mu \\
&\qquad - (-1)^{rp}\mu\wedge\omega\wedge\eta + (-1)^{rp+q(r+p)}\omega\wedge\eta\wedge\mu) \\
&= (-1)^{p(q+r)}\mu\wedge\eta\wedge\omega - (-1)^{r(p+q)}\eta\wedge\omega\wedge\mu \\
&\qquad - (-1)^{pq}\mu\wedge\omega\wedge\eta + (-1)^{qr}\omega\wedge\eta\wedge\mu
\end{aligned}
$$

So

$$
\begin{aligned}
[\omega,[\mu,\eta]] + (-1)^{p(q+r)}[\mu,[\eta,\omega]] = \omega\wedge\mu\wedge\eta - (-1)^{pq}\mu\wedge\omega\wedge\eta \\
- (-1)^{r(p+q)}\eta\wedge\omega\wedge\mu + (-1)^{qr+p(q+r)}\eta\wedge\mu\wedge\omega
\end{aligned}
$$

Also,

$$
\begin{aligned}
(-1)^{r(p+q)}[\eta,[\omega,\mu]] &= (-1)^{r(p+q)}(\eta\wedge\omega\wedge\mu - (-1)^{r(p+q)}\omega\wedge\mu\wedge\eta \\
&\qquad - (-1)^{pq}\eta\wedge\mu\wedge\omega + (-1)^{pq+r(p+q)}\mu\wedge\omega\wedge\eta) \\
&= (-1)^{r(p+q)}\eta\wedge\omega\wedge\mu - \omega\wedge\mu\wedge\eta \\
&\qquad - (-1)^{r(p+q)+pq}\eta\wedge\mu\wedge\omega + (-1)^{pq}\mu\wedge\omega\wedge\eta
\end{aligned}
$$

Combining everything, we get the result.

□

1.2.4 Exterior Calculus of Vector-Valued Differential Forms

Definition 1.51. We extend the usual **exterior derivative** $d : \Omega^p(M) \to \Omega^{p+1}(M)$ to vector-valued differential forms by $d : \Omega^p(M,V) \to \Omega^{p+1}(M,V)$ by expanding any $\omega \in \Omega^p(M,V)$ in a basis $\{v_i\}$ of V as $\omega^i \otimes v_i$ and defining

$$d\omega = d\omega^i \otimes v_i$$

where the d on the RHS is the usual exterior derivative on forms. It immediately follows that $d^2 = 0$ as usual.

Remark. Note that this works because the E in $\bigwedge^p T^*M \otimes E$ is trivial, which allows us to regard the vector fibre as a single fixed object. Then when we change basis, $v'_j = G^i_j v_i$ for some $G \in GL(V)$, this G is independent of p, so

$$d\omega'^i \otimes v'_i = d((G^{-1})^i_j \omega^j) G^k_i v_k$$

and our derivative is well-defined.

Proposition 1.52. The exterior derivative on vector-valued forms exhibits the usual Leibniz rule. That is, if $\omega \in \Omega^p(M, V)$ (or $\Omega^p(M, \operatorname{End} V)$) and $\mu \in \Omega^q(M, V)$,

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu$$

Proof. This can be proved in the usual way for normal differential forms. $\square$

Proposition 1.53. If $\omega \in \Omega^p(M, \operatorname{End} V)$ and $\mu \in \Omega^q(M, \operatorname{End} V)$,

$$d[\omega, \mu] = [d\omega, \mu] + (-1)^p [\omega, d\mu]$$

Proof. We have

$$\begin{aligned}
d[\omega, \mu] &= d(\omega \wedge \mu) - (-1)^{pq} d(\mu \wedge \omega) \\
&= d\omega \wedge \mu + (-1)^p \omega \wedge d\mu - (-1)^{pq}(d\mu \wedge \omega + (-1)^q \mu \wedge d\omega) \\
&= d\omega \wedge \mu - (-1)^{(p+1)q} \mu \wedge d\omega + (-1)^p (\omega \wedge d\mu - (-1)^{p(q+1)} d\mu \wedge \omega) \\
&= [d\omega, \mu] + (-1)^p [\omega, d\mu]
\end{aligned}$$

$\square$

1.3 Principal Bundles

1.3.1 Definitions

Definition 1.54. A **principal fibre bundle** is a fibre bundle with total space P whose typical fibre is just its structure group G. We denote such a principal G-bundle over M $P(M, G)$.

Definition 1.55. Given a local neighbourhood U_i of M with local trivialisation ϕ_i over U_i, we define a right-action of G on itself by maps $R_g : \pi^{-1}(U_i) \to \pi^{-1}(U_i)$ for each $g \in G$ defined by

$$R_g(u) = ug = \phi_i(p, g_i g)$$

where

$$u = \phi_i(p, g_i)$$

Proposition 1.56. This is well-defined.

Proof. Given two local trivialisations ϕ_i and ϕ_j over overlapping neighbourhoods, we have

$$\begin{aligned} ug &= \phi_i(p, g_i g) \\ &= \phi_j(p, t_{ji}(p) g_i g) \\ &= \phi_j(p, g_j g) \\ &= ug \end{aligned}$$

where $g_j = t_{ji}(p) g_i$ by definition. That is, R_g is compatible with the way different local trivialisations choose representatives of equivalence classes, since left-action commutes with right-action. □

Corollary 1.57. *This allows us to extend R_g to all of P for all $g \in G$.*

Proposition 1.58. This right-action is transitive and free.

Proof. Given any $g, g' \in G$, there is always some $h \in G$ such that $gh = g'$, since the usual right-multiplication of groups is transitive. Similarly, if for all $g \in G$, $gh = g$, then $a = e$. These properties of right-multiplication of groups also hold here, since each fibre $\pi^{-1}(p)$ is isomorphic to the group itself. □

Corollary 1.59. *Given any $u \in \pi^{-1}(p)$, the whole fibre can be reconstructed as*

$$\pi^{-1}(p) = \{ug \mid g \in G\}$$

Definition 1.60. Given a local section σ_i over U_i, there is a **canonical local trivialisation** $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ defined by

$$\phi_i(p, e) = \sigma_i(p)$$

for all $p \in U_i$. For any $u \in \pi^{-1}(\{p\})$ there exists (since the right-action is transitive) a unique (since it is free) element $g_u \in G$ such that $u = \sigma_i(p) g_u$, allowing us to extend ϕ by

$$\phi_i(p, g) = \phi_i(p, e) g = \sigma_i(p) g$$

Proposition 1.61. This is well-defined, and given a collection of local sections $\{\sigma_i\}$ transition functions are defined by

$$\sigma_j(p) = \sigma_i(p) t_{ij}(p)$$

Proof. Firstly,

$$\pi \circ \phi_i(p, g) = \pi \circ (\sigma_i(p)g) = p$$

right-multiplication by g operates purely in the fibre. Next, we have

$$\begin{aligned}\sigma_j(p) &= \phi_j(p, e) \\ &= \phi_i(p, t_{ij}(p)e) \\ &= \phi_i(p, t)t_{ij}(p) \\ &= \sigma_i(p)t_{ij}(p)\end{aligned}$$

□

Proposition 1.62. A local trivialisation $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ induces a local section σ_i over U_i.

Proof. Given a local trivialisation, we can define a local section with regard to which it is the canonical local trivialisation just from the definition of that concept:

$$\sigma_i(p) = \phi_i(p, e)$$

□

Corollary 1.63. *In this way local trivialisations and local sections have a natural one-to-one correspondence, which proves the following theorem:*

Theorem 1.64. *A principal bundle is trivial iff it admits a global section.*

1.3.2 Associated Bundles

Definition 1.65. Let $P(M, G)$ be a principal bundle, and F a manifold on which G has a faithful left-action which we just denote $g : f \mapsto gf$ for $f \in F$ and $g \in G$. Then define an action on $P \times F$ by

$$g : (u, f) \mapsto (ug, g^{-1}f)$$

In this way we define $E(P) = (P \times F)/G$, called the **associated fibre bundle**. In the case that $F = V$ is a vector space, use the notation $E = P \times_\rho V = (P \times V)/G$, where the action of G on V is that of the representation ρ, is called an **associated vector bundle**.

Proposition 1.66. E is a well-defined bundle over M with typical fibre F, projection $\pi_E : [(u, f)] \mapsto \pi(u)$, and transition functions t_{ij}, the same as those of P.

Proof. First observe that the projection π_E is well-defined, since $\pi(ug) = \pi(u)$. Now, fix some $p \in M$. The fibre above p, $\pi_E^{-1}(\{p\})$, consists of equivalence classes $[(u, f)]$ such that $\pi(u) = p$. To show that this fibre is isomorphic to F, we just need to establish a one-to-one correspondence between the two. To do this, make an arbitrary choice $u_0 \in \pi^{-1}(\{p\})$. Then given any $[(u, f)]$, there exists an $f' \in F$ such that $[(u, f)] = [(u_0 g, f)] = [(u_0, gf)] = [(u_0, f')]$, which moreover is unique, since the action of G on P is free (making g unique given u), and on F is faithful (making f' unique given g). On the other hand, given any $f' \in F$, there is a unique equivalence class $[(u_0, f')]$. Thus our one-to-one correspondence is established, and hence the fibres $\pi_E^{-1}(\{p\}) \cong F$. Now, if local trivialisations on P satisfy

$$\phi_j(p, g) = \phi_i(p, t_{ij}(p)g)$$

then by hypothesis our local trivialisations Φ_i on E should satisfy

$$\Phi_j(p, f) = \Phi_i(p, t_{ij}(p)f)$$

We just need to check that such local trivialisations exist. Suppose we have used local sections σ_i and σ_j to define ϕ_i, ϕ_j and t_{ij} on P. Then define $\Phi_i : U_i \times F \to E$ by

$$\Phi_i(p, f) = [(\sigma_i(p), f)]$$

Then we have

$$\begin{aligned} \Phi_j(p, f) &= [(\sigma_j(p), f)] \\ &= [(\sigma_i(p)t_{ij}(p), f)] \\ &= [(\sigma_i(p), t_{ij}(p)f)] \\ &= \Phi_i(p, t_{ij}(p)f) \end{aligned}$$

as desired. It just remains to check that

$$\begin{aligned} \pi_E \circ \Phi_i(p, f) &= \pi_E([(\sigma_i(p), f)]) \\ &= \pi(\sigma_i(p)) \\ &= p \end{aligned}$$

So Φ_i really is a local trivialisation, and we are done. □

Proposition 1.67. If $\pi : E \to M$ is a vector bundle with fibre V, it is the associated vector bundle with fibre V of a principal bundle $P(E) = P(M, GL(V))$.

Proof. We know that both E and $E(P(M, GL(V)))$ have the same fibre V and base space. All we really need to do is ask if it is possible that E have the same transition functions as $P(M, GL(V))$. But we have already seen that the transition functions of a vector bundle are mapped by the appropriate representation to $GL(V)$, so this certainly is possible. $\square$

Proposition 1.68. Any vector bundle E admits at least one global section (the null section). However, we have that E is trivial iff $P(E)$ admits a global section.

Proof. $P(E)$ admits a global section iff it is trivial iff all of its transition functions may be set to the identity. But these are the transition functions of E as well, so this is iff all of E's transition functions may be set to the identity, iff E is trivial. $\square$

Definition 1.69. A local section of an associated vector bundle $E(P)$ s may be written with the aid of a local section σ of the principal bundle P:

$$s(p) = [(\sigma(p), v(p))]$$

We can then make $\Gamma(M, E(P))$ into a $C^\infty(M)$- linear vector space by defining

$$f(p)[(\sigma(p), v(p))] + g(p)[(\sigma(p), w(p))] = [(\sigma(p), f(p)v(p) + g(p)w(p))]$$

Proposition 1.70. This is indeed well-defined and allows addition of any two sections of $\Gamma(M, E(P))$.

Proof. Although it naively looks from the definition like we can only add sections with the same P part, we can in fact have

$$\begin{aligned}[(\sigma(p), v(p))] + [(\tau(p), w(p))] &= [(\sigma(p), v(p))] + [(\sigma(p)g(p), w(p))] \\ &= [(\sigma(p), v(p))] + [(\sigma(p), g(p)w(p))] \\ &= [(\sigma(p), v(p) + g(p)w(p))]\end{aligned}$$

for some $g : M \to G$. $\square$

Proposition 1.71. Any non-vanishing section $s \in \Gamma(E(P))$ may be written with a constant V part.

Proof. Suppose
$$s(p) = [(\sigma(p), v(p))]$$
Then choose some fixed $v \in V$, and define $g(p)$ by $g(p)v = v(p)$. This is possible as long as $v(p)$ is never zero. Then we have
$$s(p) = [(\sigma(p)g(p), v)]$$
□

Corollary 1.72. *Given a local section σ of P above U and a basis $\{e^0_\alpha\}$ of V, we have a local basis of sections $\{e_\alpha\}$ of $E(P)$ above U given by*
$$e_\alpha(p) = [(\sigma(p), e^0_\alpha)]$$

2 Connections on Fibre Bundles

2.1 Connections on Principal Bundles

Definition 2.1 (Connection definition 1)**.** Define the **vertical subspace** $V_uP \subset T_uP$ of the tangent space of a principal bundle $P(M,G)$ by

$$V_uP = \ker \pi_*$$

where $\pi_* : T_uP \to T_{\pi(u)}M$ is the pushforward by the projection. Then a **connection** on $P(M,G)$ is a choice of **horizontal subspace** $H_uP \subset T_uP$ such that

$$T_uP = V_uP \oplus H_uP$$

such that

$$R_{g*}(H_uP) = H_{ug}P$$

for all $g \in G$, $u \in P$ and that H_uP varies smoothly with u, i.e. that there exist $m = \dim M$ vector fields which span the H_uP in some neighbourhood of u.

Remark. Notice that while it may seem that H_uP is automatically specified by V_uP, this would only be so if we had an inner product on T_uP, which we do not. So a connection can be regarded as a choice of inner product, compatible with right-action and varying smoothly with u.

Definition 2.2. Given a connection in the above sense and any form $\phi \in \Omega^p(P)$ (or more generally in $\Omega^p(P,V)$), we define $\phi^H \in \Omega^p(P)$ by

$$\phi^H(X_1, ..., X_p) = \phi(X_1^H, ..., X_p^H)$$

Definition 2.3. Let $A \in \mathfrak{g}$, the Lie algebra of the structure group G. Then define a vector field $A^\#$ on G by

$$A^\#_g = L_{g*}A = \frac{d}{dt}(g\exp(tA)) \mid_{t=0}$$

$A^\#$ is called the **fundamental field generated by** A. In this way we can define a field on P by

$$A^\#_u = \frac{d}{dt}(u\exp(tA)) \mid_{t=0}$$

Proposition 2.4. The map $A \mapsto A^\#_u$ provides a vector space isomorphism $\mathfrak{g} \cong V_uP$.

Proof. First we need to confirm that $A_u^\# \in V_uP$. To see this, notice that the curve $u\exp(tA)$, being defined purely by right action on some starting element u, is fixed under π. Therefore π_* must map any tangent to it to zero. So $A_u^\# \in V_uP$ indeed. Now, $A \mapsto A_u^\#$ is injective since the action of G on P is free, and furthermore from the definition we must have $\dim V_uP = \dim P - \dim M = \dim G = \dim \mathfrak{g}$. So we have an isomorphism. □

Definition 2.5 (Connection definition 2)**.** A **connection** on a principal fibre bundle $P(M,G)$ is a $\mathfrak{g}$-valued 1-form ω on P, i.e. $\omega \in \Omega^1(P,\mathfrak{g})$ satisfying

$$\omega(A^\#) = A$$

for any $A \in \mathfrak{g}$ and its fundamental field A, and

$$R_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$$

for any $g \in G$, where $\mathrm{Ad}_{g^{-1}}\,\omega$ is defined by

$$(\mathrm{Ad}_{g^{-1}}\,\omega)(X) = \mathrm{Ad}_{g^{-1}}(\omega(X))$$

Theorem 2.6. *Definitions 2.1 and 2.5 are equivalent.*

Proof. Start from the second definition, and let $H_uP = \ker \omega_u$. We will show that this is a legitimate choice of horizontal subspaces. If $X_u \in V_uP$, we can write $X_u = A_u^\#$ for some $A \in \mathfrak{g}$, and then $\omega(X) = A$. So $V_uP \cap H_uP = \{0\}$ only, i.e. H_uP is a complement to V_uP, so we have $T_uP = V_uP \oplus H_uP$. Further, $\omega(X) = 0$ implies $R_g^*\omega(X) = \omega(R_{g*}X) = 0$, so if $X_u \in H_uP$, $R_{g*}X_u \in H_{ug}P$, so $R_{g*}(H_uP) \subset H_{ug}P$. But since R_{g*} is invertible, we must be able to write any $Y_{ug} \in H_{ug}P$ as $Y_{ug} = R_{g*}X_u$ for some $X_u \in H_uP$. So indeed $R_{g*}(H_uP) = H_{ug}P$ and we have a connection in the first sense.
On the other hand, if we start from the first definition, decompose any $X_u \in T_uP$ as $X_u = A_u^\# + X_u^H$ and define $\omega \in \Omega^1(P,\mathfrak{g})$ by

$$\omega_u(A_u^\# + X_u^H) = A_u$$

Then immediately $\omega(A^\#) = A$. Furthermore, we have

$$\begin{aligned}
R_g^*\omega_u(A_u^\# + X_u^H) &= \omega_{ug}(R_{g*}A_u^\# + R_{g*}X_u^H) \\
&= \omega_{ug}(R_{g*}A_u^\#) \\
&= \omega_{ug}\left(R_{g*}\frac{d}{dt}u\exp(tA)|_{t=0}\right) \\
&= \omega_{ug}\left(\frac{d}{dt}u\exp(tA)g|_{t=0}\right) \\
&= \omega_{ug}\left(\frac{d}{dt}ugg^{-1}\exp(tA)g|_{t=0}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \omega_{ug}\left(\frac{d}{dt} ug\phi_{g^{-1}} \exp(tA)|_{t=0}\right) \\
&= \omega_{ug}(ug \operatorname{Ad}_{g^{-1}} A) \\
&= \omega_{ug}\left(\frac{d}{dt} ug \exp\big(t \operatorname{Ad}_{g^{-1}} A\big)|_{t=0}\right) \\
&= \omega_{ug}((\operatorname{Ad}_{g^{-1}} A)^{\#}_{ug}) \\
&= \operatorname{Ad}_{g^{-1}} A \\
&= \operatorname{Ad}_{g^{-1}}(\omega_u(A^{\#}_u))
\end{aligned}
$$

where we have used $R_{g*}(H_uP) = H_{ug}P$ to get the second equality, and $\phi_g : h \mapsto ghg^{-1}$ is the inner automorphism (see Appendix B). So indeed

$$R_g^*\omega = \operatorname{Ad}_{g^{-1}} \omega$$

Thus we have defined a connection in the second sense. □

2.2 Curvature

2.2.1 Definitions

Definition 2.7. Let $P(M,G)$ be a principal bundle equipped with a connection ω. We define the **covariant derivative** with respect to ω $D^\omega : \Omega^p(P,V) \to \Omega^{p+1}(P,V)$ for any vector space V by

$$D^\omega\phi = (d\phi)^H$$

The superscript ω will often be omitted where unambiguous.

Definition 2.8. The **curvature 2-form** $\Omega^\omega \in \Omega^2(P,\mathfrak{g})$ is the covariant derivative of the connection 1-form with respect to itself,

$$\Omega^\omega = D^\omega\omega$$

That is,

$$\Omega(X,Y) = d\omega(X^H, Y^H)$$

Again, the superscript may be omitted where unambiguous.

Proposition 2.9. $R_g^*\Omega = \operatorname{Ad}_{g^{-1}} \Omega$ for all $g \in G$.

Proof. We have

$$\begin{aligned} R_g^*\Omega(X,Y) &= \Omega(R_{g*}X, R_{g*}Y) \\ &= d\omega((R_{g*}X)^H, (R_{g*}Y)^H) \end{aligned}$$

Now, since $R_{g*}(H_uP) = H_{ug}P$, $(R_{g*}X)^H = R_{g*}X^H$, so

$$\begin{aligned} R_g^*\Omega(X,Y) &= d\omega(R_{g*}X^H, R_{g*}Y^H) \\ &= R_g^* d\omega(X^H, Y^H) \\ &= dR_g^*\omega(X^H, Y^H) \\ &= d(\mathrm{Ad}_{g^{-1}}\,\omega)(X^H, Y^H) \\ &= g^{-1} d\omega(X^H, Y^H) g \\ &= \mathrm{Ad}_{g^{-1}}\,\Omega(X,Y) \end{aligned}$$

□

2.2.2 The Cartan Structure Equation and Bianchi Identity

Lemma 2.10. *Let $X_u \in H_uP$ and $Y_u \in V_uP$. Then $[X,Y]_u \in H_uP$.*

Proof. Let $\phi(t)$ be the one-parameter group of diffeomorphisms generated by Y. Then we have

$$[X,Y]_u = -[Y,X]_u = -\mathcal{L}_Y X_u = -\lim_{t\to 0} \frac{1}{t}(R_{\phi(t)*}X_u - X_u)$$

Then

$$\omega_u([X,Y]_u) = -\lim_{t\to 0} \frac{1}{t}(R_{\phi(t)}^*\omega_u(X_u) - \omega_u(X_u)) = 0$$

since $X_u \in H_uP$. □

Theorem 2.11 (Cartan structure equation)**.** *The curvature 2-form is given by*

$$\Omega = d\omega + \omega \wedge \omega$$

Alternatively,

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)]$$

for all $X, Y \in \Gamma(TP)$.

Proof. We will prove the latter statement then show it is equivalent to the former. By linearity and antisymmetry, we need only consider the following three cases:

(i) $X_u, Y_u \in H_uP$. Then $\omega(X) = \omega(Y) = 0$, so both sides vanish and the statement holds.

(ii) $X_u \in H_uP$ and $Y_u \in V_uP$. Then $\omega(X) = 0$ and $Y^H = 0$, so second term on the RHS and the LHS vanish respectively. This just leaves

$$\begin{aligned} d\omega(X,Y) &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ &= X(\omega(Y)) - \omega([X,Y]) \end{aligned}$$

where the first line is a standard result. We can write $Y = A^{\#}$ for some $A \in \mathfrak{g}$. Then $\omega(Y) = A$ is constant, so the first term vanishes. Further, by Lemma 2.10 $[X,Y]$ is horizontal, so the second term vanishes too. Thus everything is zero, and the statement holds.

(iii) $X_u, Y_u \in V_uP$. Then $X^H = Y^H = 0$, so the LHS vanishes. If we write $X = A^{\#}$ and $Y = B^{\#}$, on the RHS the first term is

$$\begin{aligned} d\omega(X,Y) &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ &= X(A) - Y(B) - \omega([A,B]^{\#}) \\ &= -[A,B] \end{aligned}$$

where we have used Proposition B.4 to get the second line. But the second term is

$$[\omega(X), \omega(Y)] = [A,B]$$

so these cancel, and the statement holds.

Thus we have shown that

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)]$$

for any $X, Y \in \Gamma(TP)$. We now want to prove that the first statement follows from this. To do this, first note that, employing a basis $\{T_\alpha\}$ of $\mathfrak{g}$, in which we can write $\omega = T_\alpha \otimes \omega^\alpha$,

$$\begin{aligned} [\omega,\omega](X,Y) &= [T_\alpha, T_\beta] \otimes \omega^\alpha \wedge \omega^\beta(X,Y) \\ &= [T_\alpha, T_\beta](\omega^\alpha(X)\omega^\beta(Y) - \omega^\alpha(Y)\omega^\beta(X)) \\ &= [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] \\ &= 2[\omega(X), \omega(Y)] \\ 2\omega \wedge \omega(X,Y) &= 2[\omega(X), \omega(Y)] \end{aligned}$$

where we have started from a result in the proof of Proposition 1.49. Thus we have the first statement. $\square$

Theorem 2.12 (Bianchi identity)**.** *The curvature 2-form is covariantly conserved:*

$$D\Omega = 0$$

Proof. We have

$$D\Omega(X,Y,Z) = d\Omega(X^H, Y^H, Z^H)$$

The Cartan structure equation,

$$\Omega = d\omega + \omega \wedge \omega$$

can be written using a basis $\{T_\alpha\}$ for $\mathfrak{g}$ as

$$\begin{aligned} T_\alpha \otimes \Omega^\alpha &= T_\alpha \otimes d\omega^\alpha + [T_\beta, T_\gamma] \otimes \omega^\beta \wedge \omega^\gamma \\ \Omega^\alpha &= d\omega^\alpha + f^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma \end{aligned}$$

Then we have

$$d\Omega^\alpha = f^\alpha_{\beta\gamma}(d\omega^\beta \wedge \omega^\gamma - \omega^\beta \wedge d\omega^\gamma)$$

But then

$$D\Omega^\alpha(X,Y,Z) = d\omega^\alpha(X^H, Y^H, Z^H) = 0$$

so

$$D\Omega = 0$$

□

2.3 Working Locally

2.3.1 Forms on the Base Space

Definition 2.13. Let $P(M,G)$ be a principal bundle with connection ω, and $U_i \subset M$ a local neighbourhood. Then a local section σ (or equivalently a local trivialisation ϕ) of P over U induces a **local connection 1-form** or **local gauge connection** $\mathcal{A} \in \Omega^1(U, \mathfrak{g})$ by

$$\mathcal{A} = \sigma^*\omega$$

Proposition 2.14. Given a local connection $\mathcal{A}_i$ on U_i and a local section σ_i over U_i,

$$\omega_i = g_i^{-1}\pi^*\mathcal{A}_i g_i + g_i^{-1} d_P g_i$$

is a connection form on P, where, if ϕ_i is the canonical local trivialisation induced by σ_i,

$$\phi_i(p, g_i) = \sigma_i(p) g_i = u$$

Proof. First we must show that this definition is consistent, i.e. that $\mathcal{A}_i = \sigma_i^*\omega_i$. Let $X_p \in T_pM$. Then we have

$$\begin{aligned}\sigma_i^*\omega_{i,p}(X_p) &= \omega_{i,\sigma_i(p)}(\sigma_{i*}X_p)\\ &= g_{i,\sigma_i(p)}^{-1}\mathcal{A}_{i,\pi(\sigma_i(p))}(\pi_*\sigma_{i*}X_p)g_{i,\sigma_i(p)} + g_{i,\sigma_i(p)}d_P g_{i,\sigma_i(p)}(\sigma_{i*}X_p)\end{aligned}$$

Now, by definition $g_{i,\sigma_i(p)} = e$, so this is just

$$\begin{aligned}\sigma_i^*\omega_{i,p}(X_p) &= \mathcal{A}_{i,p}(\pi_*\sigma_{i*}X_p)\\ &= \mathcal{A}_{i,p}((\pi\circ\sigma)_*X_p)\\ &= \mathcal{A}_{i,p}(X_p)\end{aligned}$$

So indeed

$$\sigma_i^*\omega_i = \mathcal{A}_i$$

Now we need to show that ω_i is indeed a connection form on P. Consider $A_u^\# \in V_uP$. By definition, $\pi_*A_u^\# = 0$, so the first term in $\omega_i(A^\#)$ vanishes. Then we just have

$$\begin{aligned}\omega_{i,u}(A_u^\#) &= g_{i,u}^{-1}d_P g_{i,u}(A_u^\#)\\ &= g_{i,u}^{-1}\frac{d}{dt}g_{i,u}u\exp(tA)|_{t=0}\\ &= g_{i,u}^{-1}g_{i,u}\frac{d}{dt}u\exp(tA)|_{t=0}\\ &= A\end{aligned}$$

Secondly, for any $h \in G$, $X_u \in T_uP$

$$\begin{aligned}R_h^*\omega_{i,u}(X_u) &= \omega_{i,uh}(R_{h*}X_u)\\ &= g_{i,uh}^{-1}\mathcal{A}_i(\pi_*R_{h*}X_u)g_{i,uh} + g_{i,uh}^{-1}d_P g_{i,uh}(R_{h*}X_u)\end{aligned}$$

Now, using $g_{i,uh} = g_{i,u}h$ and $\pi\circ R_h = \pi$,

$$R_h^*\omega_{i,u}(X_u) = h^{-1}g_{i,u}\mathcal{A}_i(\pi_*X_u)g_{i,u}h + h^{-1}g_{i,u}^{-1}d_P g_{i,uh}(R_{h*}X_u)$$

Let $\gamma : [0,1] \to P$ be a curve in P with $\gamma(0) = u$ and $\gamma'(0) = X_u$. Then we have

$$\begin{aligned}d_P g_{i,uh}(R_{h*}X_u) &= \frac{d}{dt}g_{i,\gamma(t)h}|_{t=0}\\ &= \frac{d}{dt}g_{i,\gamma(t)}|_{t=0}h\\ &= d_P g_{i,u}(X_u)h\end{aligned}$$

So finally

$$\begin{aligned}R_h^*\omega_{i,u}(X_u) &= h^{-1}g_{i,u}\mathcal{A}_i(\pi_*X_u)g_{i,u}h + h^{-1}g_{i,u}^{-1}d_P g_{i,u}(X_u)h\\ &= \mathrm{Ad}_{h^{-1}}\,\omega_{i,u}(X_u)\end{aligned}$$

Thus we have confirmed that ω_i is a connection 1-form. $\square$

Lemma 2.15. *Let σ_i and σ_j be local sections on U_i and U_j, defining canonical local trivialisations ϕ_i and ϕ_j, with transition function t_{ij}. Then for any $X \in TM$, on $U_i \cap U_j$,*

$$\sigma_{j*}X = R_{t_{ij}*}(\sigma_{i*}X) + (t_{ij}^{-1}dt_{ij}(X))^{\#}$$

Proof. Let $\gamma : [0,1] \to M$ be a curve in M with $\gamma(0) = p$ and $\gamma'(0) = X_p$. Then we have

$$\sigma_{j*}X_p = \frac{d}{dt}\sigma_j(\gamma(t))|_{t=0}$$

Recall from Proposition 1.61

$$\sigma_j(p) = \sigma_i(p)t_{ij}(p)$$

so

$$\begin{aligned}
\sigma_{j*}X_p &= \frac{d}{dt}\sigma_i(\gamma(t))t_{ij}(\gamma(t))|_{t=0} \\
&= \sigma_{i*}X_p t_{ij}(p) + \sigma_i(p)\frac{d}{dt}t_{ij}(\gamma(t))|_{t=0} \\
&= R_{t_{ij}(p)*}(\sigma_{i*}X_p) + \sigma_j(p)t_{ij}(p)^{-1}\frac{d}{dt}t_{ij}(\gamma(t))|_{t=0} \\
&= R_{t_{ij}(p)*}(\sigma_{i*}X_p) + \sigma_j(p)t_{ij}(p)^{-1}dt_{ij}(X_p) \\
&= R_{t_{ij}(p)*}(\sigma_{i*}X_p) + (t_{ij}(p)^{-1}dt_{ij}(X))^{\#}_{\sigma_j(p)}
\end{aligned}$$

□

Proposition 2.16. Given an open cover $\{U_i\}$ of M and the corresponding local connections $\{\mathcal{A}_i\}$, the induced connection 1-forms ω_i agree if

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}dt_{ij}$$

Then we can write $\omega_i = \omega \mid_{U_i}$.

Proof. For the $\{\omega_i\}$ to agree, they must be equal on any intersections. If we set $\omega_i = \omega_j$ above $U_i \cap U_j$ as restrictions of ω, we see that, for any $X \in \Gamma(TM)$ over $U_i \cap U_j$,

$$\begin{aligned}
\mathcal{A}_j(X) &= \omega_j(\sigma_{j*}X) \\
&= \omega(\sigma_{j*}X) \\
&= \omega\left(R_{t_{ij}*}(\sigma_{i*}X) + (t_{ij}^{-1}d_{tij}(X))^{\#}\right) \\
&= R_{t_{ij}}^*\omega(\sigma_{i*}X) + t_{ij}^{-1}dt_{ij}(X) \\
&= t_{ij}^{-1}\omega(\sigma_{i*}X)t_{ij} + t_{ij}^{-1}dt_{ij}(X) \\
&= t_{ij}^{-1}\mathcal{A}_i(X)t_{ij} + t_{ij}^{-1}dt_{ij}(X)
\end{aligned}$$

So we must have

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij} dt_{ij}$$

□

Definition 2.17. Let $P(M, G)$ be equipped with a connection ω with curvature Ω, and let σ be a local section of P over U. Then the **local curvature 2-form** or **Yang-Mills field strength** $\mathcal{F} \in \Omega^2(M, \mathfrak{g})$ is defined by

$$\mathcal{F} = \sigma^*\Omega$$

2.3.2 Local Results on the Base Space

Theorem 2.18. *The local connection and curvature forms on M satisfy the same Cartan structure equation as the global forms on P,*

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

or alternatively

$$\mathcal{F}_i(X, Y) = d\mathcal{A}_i(X, Y) + [\mathcal{A}_i(X), \mathcal{A}_i(Y)]$$

for any $X, Y \in \Gamma(TM)$.

Proof. This follows just by taking the pullback of the Cartan structure equation by the local section σ_i used to define $\mathcal{F}_i$ and $\mathcal{A}_i$:

$$\begin{aligned} \Omega &= d\omega + \omega \wedge \omega \\ \sigma_i^*\Omega &= d\sigma_i^*\omega + \sigma_i^*\omega \wedge \sigma_i{*}^{\omega} \\ \mathcal{F}_i &= d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \end{aligned}$$

□

Proposition 2.19. The YM field strengths on overlapping charts U_i and U_j are related by

$$\mathcal{F}_j = \mathrm{Ad}_{t_{ij}^{-1}}\, \mathcal{F}_i$$

where t_{ij} is the transition function between the two local trivialisations in which the field strengths are defined.

Proof. We omit the subscripts on t for simplicity. By the Cartan structure equation and Proposition 2.16,

$$\begin{aligned}
\mathcal{F}_j &= d(t^{-1}\mathcal{A}_i t + t^{-1}dt) + (t^{-1}\mathcal{A}t + t^{-1}dt) \wedge (t^{-1}\mathcal{A}t + t^{-1}dt) \\
&= dt^{-1} \wedge \mathcal{A}_i t + t^{-1}d\mathcal{A}_i t - t^{-1}\mathcal{A}_i \wedge dt + dt^{-1} \wedge dt \\
&\qquad + t^{-1}\mathcal{A}_i \wedge \mathcal{A}_i t + t^{-1}\mathcal{A}_i \wedge dt + t^{-1}dt \wedge t^{-1}\mathcal{A}_i t + t^{-1}dt \wedge t^{-1}dt
\end{aligned}$$

Now,

$$0 = d(t^{-1}t) = dt^{-1}t + t^{-1}dt$$

So

$$dt^{-1} = -t^{-1}dtt^{-1}$$

So we have

$$\begin{aligned}
\mathcal{F}_j &= t^{-1}\big(- dtt^{-1} \wedge \mathcal{A}_i t + d\mathcal{A}_i t - \mathcal{A}_i \wedge dt - dtt^{-1} \wedge dt \\
&\qquad + \mathcal{A}_i \wedge \mathcal{A}_i t + \mathcal{A}_i \wedge dt + dt \wedge t^{-1}\mathcal{A}_i t + dt \wedge t^{-1}dt\big) \\
&= t^{-1}(d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i)t \\
&= \mathrm{Ad}_{t^{-1}} \mathcal{F}_i
\end{aligned}$$

□

Definition 2.20. Given a local connection $\mathcal{A}$ over U, define the covariant derivative $\mathcal{D} : \Omega^p(M, \mathfrak{g}) \to \Omega^{p+1}(M, \mathfrak{g})$ by

$$\mathcal{D}\alpha = d\alpha + [\mathcal{A}, \alpha]$$

Theorem 2.21. *The Yang-Mills field strength satisfies the local Bianchi identity*

$$\mathcal{D}\mathcal{F} = 0$$

Proof. Start with the Cartan structure equation on the principal bundle

$$\begin{aligned}
\Omega &= d\omega + \omega \wedge \omega \\
d\Omega &= d\omega \wedge \omega - \omega \wedge d\omega \\
d\mathcal{F} &= d\mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge d\mathcal{A}
\end{aligned}$$

where we have taken σ^* to get the last line. Now, notice that

$$\begin{aligned}
\mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} &= (d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}) \wedge \mathcal{A} - \mathcal{A} \wedge (d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}) \\
&= d\mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge d\mathcal{A}
\end{aligned}$$

So

$$\begin{aligned} d\mathcal{F} &= \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} \\ &= [\mathcal{F}, \mathcal{A}] \end{aligned}$$

Thus

$$\mathcal{D}\mathcal{F} = 0$$

□

2.4 Horizontal Lifts and Holonomy

2.4.1 Horizontal Lifts

Definition 2.22. Let $\gamma : [0, 1] \to M$ be a curve in M and $P(M, G)$ a principal bundle with a connection. Then a curve $\tilde{\gamma} : [0, 1] \to P$ is called a **horizontal lift** of γ if $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}P$ for all $t \in [0, 1]$.

Theorem 2.23. *Let γ be based at $p \in M$, and $u_0 \in \pi^{-1}(p)$. Then there is a unique horizontal lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = u_0$.*

Proof. We assume WLOG that γ lies completely within a local neighbourhood U, over which we use a local section σ to locally trivialise:

$$\phi(p, g) = \sigma(p)g$$

We can use this to define g such that

$$\tilde{\gamma}(t) = \sigma(t)g(t)$$

where the argument t is shorthand for $\gamma(t)$ on the RHS. Then if we find $g(t)$ we have found $\tilde{\gamma}(t)$. WLOG we can choose σ such that $\sigma(0) = \tilde{\gamma}(0)$, i.e. $g(0) = e$. Let $X = \gamma'(0) \in T_pM$ and $\tilde{X} = \tilde{\gamma}'(0)$. We have

$$\begin{aligned} \tilde{X}_{u_0} &= \frac{d}{dt}\gamma(t)|_{t=0} \\ &= \frac{d}{dt}\sigma(t)g(t)|_{t=0} \end{aligned}$$

This is of the same form as Lemma 2.15, so we can just state the result of the computation:

$$\tilde{X} = g(t)^{-1}\sigma_* X g(t) + (g(t)^{-1}dg(X))^{\#}$$

Now, since it is tangent to a horizontal lift, by definition $\tilde{X}_{u_0} \in H_{u_0}P$, so ω annihilates it. Therefore we must have

$$\begin{aligned}
0 &= g(t)^{-1}\omega(\sigma_* X)g(t) + g(t)^{-1}dg(X) \\
&= \omega(\sigma_* X)g(t) + \frac{dg}{dt} \\
&= \mathcal{A}(X)g(t) + \frac{dg}{dt}
\end{aligned}$$

where $\mathcal{A}$ is the local connection 1-form induced by σ. This is an ODE for $g(t)$, which by existence and uniqueness theorems has a unique solution. Then it follows that there exists a unique $\tilde{\gamma}$. $\square$

Corollary 2.24. *Given $\tilde{\gamma}$, the horizontal lift of γ starting at u_0, we can define $\tilde{\gamma}_g = R_g\tilde{\gamma}$. Since $\pi \circ R_g = \pi$ and $R_{g*}(H_uP) = H_{ug}P$, this is also a horizontal lift of γ, and it is based at u_0g. But since right-multiplication is transitive. So every horizontal lift of a curve can be straightforwardly deduced from a single one.*

Definition 2.25. Let $\gamma : [0,1] \to M$ be a curve in M with $\gamma(0) = p_0$ and $\gamma(1) = p_1$. This induces the isomorphism $\Gamma(\gamma) : \pi^{-1}(p_0) \to \pi^{-1}(p_1)$ by $\Gamma(\gamma)(u_0) = u_1$, where if $\tilde{\gamma}$ is the unique horizontal lift of γ based at u_0 then $u_1 = \tilde{\gamma}(1)$. We say that u_1 is the **parallel transport** of u_0 along $\tilde{\gamma}$. When the curve in the base space is unambiguous we will just denote the map Γ.

Proposition 2.26. The parallel transport map commutes with right-action,

$$R_g\Gamma = \Gamma R_g$$

Equivalently, $\Gamma(ug) = \Gamma(u)g$.

Proof. Let $\Gamma(u_0) = u_1$. Then $R_g\Gamma(u_0) = u_1g$. Also $\Gamma R_g(u_0) = \Gamma(u_0g)$. Now, $\Gamma(u_0g) = u_1g'$ for some g'. But by the uniqueness of horizontal lifts through given points in the fibre, $g' = g$. $\square$

Proposition 2.27. $\Gamma(\gamma^{-1}) = \Gamma(\gamma)^{-1}$, where γ^{-1} is defined by $\gamma^{-1}(t) = \gamma(1-t)$.

Proof. Both $\Gamma(\gamma)$ and $\Gamma(\gamma)^{-1}$ are isomorphisms, and by the uniqueness of horizontal lifts make the same identifications. Of course they go in opposite directions, so they are inverses. $\square$

Proposition 2.28. Let $\alpha, \beta : [0,1] \to M$ be curves in M with $\alpha(1) = \beta(0)$, and their product $\alpha * \beta : [0,1] \to M$ the curve

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Then

$$\Gamma(\alpha * \beta) = \Gamma(\beta) \circ \Gamma(\alpha)$$

Proof. Finding $\Gamma(\alpha * \beta)$ amounts to solving two differential equations, corresponding to $\Gamma(\alpha)$ and $\Gamma(\beta)$ respectively, with boundary conditions matched, so by the uniqueness of horizontal lifts $\Gamma(\alpha * \beta)$ must be fully determined by $\Gamma(\alpha)$ and $\Gamma(\beta)$. The matching of boundary conditions is equivalent to making the composition $\Gamma(\beta) \circ \Gamma(\alpha)$. □

2.4.2 Holonomy Groups

Definition 2.29. Let $\gamma : [0,1] \to M$ be a loop in M based at p. Then we have an isomorphism $\Gamma : \pi^{-1}(p) \to \pi^{-1}(p)$, of the form $u \mapsto ug$ for some $g \in G$. Let $C(p)$ be the set of all loops in M at p. Then we define the **holonomy group** at u as the subgroup

$$\Phi_u = \{g \in G \mid \Gamma(\gamma)(u) = ug,\ \gamma \in C(p)\}$$

Proposition 2.30. Φ_u is a group.

Proof. Firstly, we have closure, since if g and g' correspond to loops γ and γ', gg' corresponds to the loop $\gamma * \gamma'$. Second, we have associativity since the product on loops is clearly associative. Then, $e \in \Phi_u$ by taking γ as the constant loop, $\gamma(t) = p$ for all t. Lastly, as we have seen, $\Gamma(\gamma)^{-1} = \Gamma(\tilde{\gamma})$, so we have inverses. □

Proposition 2.31. All holonomy groups in the same fibre are isomorphic, $\Phi_u \cong \Phi_{ug}$.

Proof. We have

$$\Phi_u = \{h \in G \mid \Gamma(\gamma)(u) = uh,\ \gamma \in C(p)\}$$
$$\Phi_{ug} = \{h \in G \mid \Gamma(\gamma)(ug) = ugh,\ \gamma \in C(p)\}$$

Suppose that $\Gamma(\gamma)(u) = uh_0$ for some $h_0 \in \Phi_u$. Then we have

$$\begin{aligned}\Gamma(\gamma)(ug) &= \Gamma(\gamma)(u)g \\ ugh &= uh_0g\end{aligned}$$

for some $h \in \Phi_{ug}$. But then $gh = h_0g$, so $h = g^{-1}h_0g$. This establishes a one-to-one correspondence between the holonomy groups, so

$$\begin{aligned}\Phi_{ug} &= g^{-1}\Phi_u g \\ &\cong \Phi_u\end{aligned}$$

□

Proposition 2.32. All holonomy groups on the same horizontal lift are isomorphic, $\Phi_u \cong \Phi_{u'}$.

Proof. Let $p = \pi(u)$ and $p' = \pi(u')$. Then

$$\begin{aligned}\Phi_u &= \{g \in G \mid \Gamma(\alpha)(u) = ug,\ \alpha \in C(p)\} \\ \Phi_{u'} &= \{h \in G \mid \Gamma(\beta)(ug) = ugh,\ \beta \in C(p')\}\end{aligned}$$

Let γ be a curve in M from p to p'. Then for every $\alpha \in C(p)$, we have a $\beta = \gamma^{-1} * \alpha * \gamma \in C(p')$, and on the other hand for every $\beta \in C(p')$ we have $\alpha = \gamma * \beta * \gamma^{-1} \in C(p)$. Furthermore since

$$\Gamma(\gamma * \alpha * \gamma^{-1}) = \Gamma(\gamma) \circ \Gamma(\alpha) \circ \Gamma(\gamma)^{-1}$$

if we write $\Gamma(\gamma * \alpha * \gamma^{-1}) = u'h_\beta$ for $h_\beta \in \Phi_{u'}$, $\Gamma(\gamma)(u') = u'g_\gamma$, and $\Gamma(\alpha)(u) = ug_\alpha$, then

$$\begin{aligned}\Gamma(\gamma * \alpha * \gamma^{-1})(u') &= \Gamma(\gamma) \circ \Gamma(\alpha) \circ \Gamma(\gamma)^{-1}(u') \\ u'h_\beta &= \Gamma(\gamma) \circ \Gamma(\alpha)(ug_\gamma^{-1}) \\ &= \Gamma(\gamma)(ug_\gamma^{-1}g_\alpha) \\ &= u'g_\gamma^{-1}g_\alpha g_\gamma\end{aligned}$$

Thus $h_\beta = g_\gamma^{-1}g_\alpha g_\gamma$, so

$$\begin{aligned}\Phi_{u'} &= g_\gamma^{-1}\Phi_u g_\gamma \\ &\cong \Phi_u\end{aligned}$$

□

Theorem 2.33. *Holonomy groups at all $u, u' \in P$ such that $\pi(u)$ and $\pi(u')$ are in the same connected component of M are isomorphic.*

Proof. For any two such u and u', there exists a curve γ from $\pi(u)$ to $\pi(u')$. Starting at u, follow a horizontal lift to $\pi^{-1}(\pi(u'))$. Generically this will bring us to $u'g$ for some g. Then the previous two propositions give us

$$\Phi_u \cong \Phi_{u'g} \cong \Phi_{u'}$$

□

Theorem 2.34. *The curvature 2-form Ω measures the non-triviality of holonomy groups.*

Proof. Recall from the proof of the Cartan structure equation that if $X, Y \in H_uP$ then

$$\Omega(X, Y) = -\omega([X, Y])$$

Now, $[X, Y]$ must be in V_uP, by the following logic: π_*X and π_*Y can be used to define an infinitesimal parallelogram in M; then a horizontal lift $\tilde{\gamma}$ will not necessarily close, but its failure to close will be measured by $[X, Y]$; however, it must start and end in the same fibre, since γ is closed, so $[X, Y]$ must be vertical. Then we can write $[X, Y] = A^{\#}$ for some $A \in \mathfrak{g}$, and regard this A as measuring the failure of the horizontal lift to close. But

$$\Omega(X, Y) = -\omega([X, Y]) = -A$$

So in fact $\Omega(X, Y)$ measures this too. □

2.5 Covariant Derivatives on Associated Vector Bundles

2.5.1 Parallel Transport and the Covariant Derivative

Definition 2.35. Let $P(M, G)$ be a principal bundle and E a vector bundle associated to it. We say a section $s \in \Gamma(E)$ is **parallel transported** along a curve $\gamma : [0, 1] \to M$ in M if, parameterising as

$$s(t) = [(\tilde{\gamma}(t), \eta(t))]$$

(where the argument t means $\gamma(t)$ where relevant) for any horizontal lift $\tilde{\gamma}$ of γ, η is constant.

Proposition 2.36. This is well-defined, i.e. independent of the choice of horizontal lift $\tilde{\gamma}$.

Proof. Recall that any other horizontal lift of γ can be written as $R_g\tilde{\gamma}$ for some g. Then if we use this new horizontal lift, we have

$$\begin{aligned} s(t) &= [(\tilde{\gamma}(t)g, \mu(t))] \\ &= [(\tilde{\gamma}(t), g^{-1}\mu(t))] \end{aligned}$$

So we must have $g^{-1}\mu(t) = \eta(t)$, so $\mu(t)$ is constant iff $\eta(t)$ is constant. Then indeed parallel transport is well-defined. □

Definition 2.37. Let $s \in \Gamma(E)$ be parameterised along $\gamma : [0,1] \to M$ in the same way. Then we define the **covariant derivative** of s along γ at $\gamma(0)$, $\nabla_{\gamma'(0)}s \in E$, by

$$\nabla_{\gamma'(0)}s = \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\eta(t)|_{t=0}\right)\right]$$

We then define the **covariant derivative** with respect to $X \in \Gamma(TM)$, the map $\nabla_X : \Gamma(E) \to \Gamma(E)$, by

$$\nabla_X s\ |_p = \nabla_{X|_p} s$$

Further we can define $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1(M)$ by

$$\nabla s(X) = \nabla_X s$$

Proposition 2.38. This is well-defined, i.e. independent of the choice of horizontal lift $\tilde{\gamma}$.

Proof. If we choose another horizontal lift $\tilde{\gamma}(t)g$, we can write

$$\nabla_{\gamma'(0)}s = [(\tilde{\gamma}(t)g, \mu(t))]$$

where $\mu = g\eta$ again. Then with this parameterisation,

$$\begin{aligned} \nabla_{\gamma'(0)}s &= \left[\left(\tilde{\gamma}(0)g, \frac{d}{dt}\mu(t)|_{t=0}\right)\right] \\ &= \left[\left(\tilde{\gamma}(0), g^{-1}\frac{d}{dt}g\eta(t)|_{t=0}\right)\right] \\ &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\eta(t)|_{t=0}\right)\right] \end{aligned}$$

in agreement with the $\tilde{\gamma}$ parameterisation. □

Proposition 2.39. If s is parallel transported along γ, an integral curve of X, then

$$\nabla_X s = 0$$

Proof. Trivial. $\square$

Proposition 2.40. Let $a, b \in \mathbb{R}$, $s, s' \in \Gamma(E)$, $f \in C^\infty(M)$, and $X, Y \in \Gamma(TM)$. Then ∇ satisfies:

(i) $\nabla_X(as + bs') = a\nabla_X s + b\nabla_X s'$

(ii) $\nabla(as + bs') = a\nabla s + b\nabla s'$

(iii) $\nabla_{aX+bY} s = a\nabla_X s + b\nabla_Y s$

(iv) $\nabla_X(fs) = X(f)s + f\nabla_X s$

(v) $\nabla(fs) = (df)s + f\nabla s$

(vi) $\nabla_{fX} s = f\nabla_X s$

(i), (iii), (iv) and (vi) comprise the **covariant derivative axioms**.

Proof. Let γ and δ be curves in M with $\gamma(0) = p$ and $\gamma'(0) = X_p$. Parameterise s and s' along this as

$$\begin{aligned} s(t) &= [(\tilde{\gamma}(t), \eta(t))] \\ s'(t) &= [(\tilde{\gamma}(t), \mu(t))] \end{aligned}$$

(i) At p,

$$\begin{aligned} \nabla_{X_p}(as + bs') &= \nabla_{X_p}[(\tilde{\gamma}(t), a\eta(t) + b\mu(t))] \\ &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(a\eta(t) + b\mu(t))|_{t=0}\right)\right] \\ &= \left[\left(\tilde{\gamma}(0), a\frac{d}{dt}\eta(t)|_{t=0}\right)\right] + \left[\left(\tilde{\gamma}(0), b\frac{d}{dt}\mu(t)|_{t=0}\right)\right] \\ &= a\nabla_{X_p} s + b\nabla_{X_p} s' \end{aligned}$$

This holds at all p.

(ii) It then follows that

$$\begin{aligned} \nabla(as + bs')(X) &= a\nabla_X s + b\nabla_X s' \\ &= a\nabla s(X) + b\nabla s'(X) \\ &= (a\nabla s + b\nabla s')(X) \end{aligned}$$

for all X.

(iii) Use some different definitions here: let $\gamma, \gamma_1, \gamma_2$ be based at p, with $\gamma_1'(0) = X_{1p}$, $\gamma_2'(0) = X_{2p}$, and $\gamma'(0) = aX_{1p} + bX_{2p}$. Then

$$\frac{d}{dt}\eta(\gamma(t)) = \frac{d}{dt}(a\eta(\gamma_1(t)) + b\eta(\gamma_2(t)))$$

holds. So at p we have

$$\begin{aligned}
\nabla_{aX_{1p}+bX_{2p}} s &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\eta(\gamma(t))|_{t=0}\right)\right] \\
&= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(a\eta(\gamma_1(t)) + b\eta(\gamma_2(t)))|_{t=0}\right)\right] \\
&= \left[\left(\tilde{\gamma}(0), a\frac{d}{dt}\eta(\gamma_1(t))|_{t=0}\right)\right] + \left[\left(\tilde{\gamma}(0), b\frac{d}{dt}\eta(\gamma_2(t))|_{t=0}\right)\right] \\
&= a\nabla_{X_{1p}} s + b\nabla_{X_{2p}} s
\end{aligned}$$

This holds at all p.

(iv) Return to the previous definitions. We have

$$\begin{aligned}
\nabla_{X_p}(fs) &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(f(t)\eta(t)|_{t=0}\right)\right] \\
&= \left[\left(\tilde{\gamma}(0), \left(\frac{d}{dt}f(t)\right)\Big|_{t=0}\eta(p)\right)\right] + \left[\left(\tilde{\gamma}(0), f(p)\frac{d}{dt}\eta(t)|_{t=0}\right)\right] \\
&= X(f)(p)s(p) + f(p)\nabla_{X_p} s
\end{aligned}$$

This holds for all p.

(v) Then it follows that

$$\begin{aligned}
\nabla(fs)(X) &= X(f)s + f\nabla_X s \\
&= (df)(X)s + f\nabla s(X) \\
&= ((df)s + f\nabla s)(X)
\end{aligned}$$

for all X.

(vi) Make some new definitions: $\gamma(0) = \gamma_X(0) = p$, $\gamma'(0) = f(p)X_p$, $\gamma_X'(0) = X_p$. Then

$$\frac{d}{dt}(\gamma(t))|_{t=0} = f(p)\frac{d}{dt}\eta(\gamma_X(t))|_{t=0}$$

holds. So we have

$$\begin{aligned}\nabla_{f(p)X_p s} &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\eta(\gamma(t))|_{t=0}\right)\right] \\ &= \left[\left(\tilde{\gamma}(0), f(p)\frac{d}{dt}\eta(\gamma_X(t))|_{t=0}\right)\right] \\ &= f(p)\nabla_X s\end{aligned}$$

This holds for all p.

□

Corollary 2.41. *Given a local basis of sections $\{e_\alpha\}$, an arbitrary local section $s = f^\alpha e_\alpha$ has covariant derivative*

$$\nabla_X(f^\alpha e_\alpha) = X(f^\alpha)e_\alpha + f^\alpha \nabla_X e_\alpha$$

2.5.2 Local Form of the Covariant Derivative

Proposition 2.42. Let $\{e^0_\alpha\}$ be a basis for V, defining a local basis of sections $\{e_\alpha\}$ over U (in a canonical trivialisation defined by a section σ) by

$$e_\alpha(p) = [(\sigma(p), e^0_\alpha)]$$

(see Corollary 1.72) and let $\mathcal{A}$ be a local gauge connection. Let $\rho : G \to GL(V)$ be the representation of the associated bundle, and $\rho' : \mathfrak{g} \to \mathfrak{GL}(V)$ the induced Lie algebra representation. (We will use these explicitly here for clarity.) Then, writing $\rho'(\mathcal{A}) = A$, and expanding as $A = A^\beta_\alpha e^0_\alpha \otimes e^{0\beta}$, the covariant derivative of the basis sections is given by

$$\nabla_X e_\alpha|_p = \nabla_{X_p} e_\alpha = A^\beta_\alpha(X_p)e_\beta$$

for any X, p.

Proof. Let $\gamma : [0,1] \to U$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = X_p$, which for simplicity is purely within U. We can write any horizontal lift $\tilde{\gamma}$ of γ as $\tilde{\gamma}(t) = \sigma(t)g(t)$, where $g(t)$ can be solved according to Theorem 2.23. Then our basis sections are parameterised along γ as

$$\begin{aligned}e_\alpha(t) &= [(\tilde{\gamma}(t)g(t)^{-1}, e^0_\alpha)] \\ &= [(\tilde{\gamma}(t), \rho(g(t)^{-1})^\beta_\alpha e^0_\beta)]\end{aligned}$$

Thus we calculate

$$\begin{aligned}
\nabla_{X_p} e_\alpha &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\rho(g(t)^{-1})^\beta_\alpha|_{t=0} e^0_\beta\right)\right] \\
&= \left[\left(\tilde{\gamma}(0), -\rho(g(0)^{-1})^\beta_\alpha \frac{d}{dt}\rho(g(t))^\gamma_\beta|_{t=0}\rho(g(0)^{-1})^\delta_\gamma e^0_\delta\right)\right] \\
&= \left[\left(\tilde{\gamma}(0)g(0)^{-1}, -\frac{d}{dt}\rho(g(t))^\beta_\alpha|_{t=0}\rho(g(0)^{-1})^\gamma_\beta e^0_\gamma\right)\right]
\end{aligned}$$

Now, recall from Theorem 2.23 that

$$g(t)^{-1}\frac{dg}{dt} = -g(t)^{-1}\mathcal{A}(X)g(t)$$

Applying ρ' to this and using matrix multiplication, we can write

$$\begin{aligned}
\rho(g(t)^{-1})^\alpha_\beta \frac{d}{dt}\rho(g(t))^\beta_\gamma &= -\rho(g(t)^{-1})^\alpha_\beta \rho'(\mathcal{A}(X))^\beta_\delta \rho(g(t))^\delta_\gamma \\
\frac{d}{dt}\rho(g(t))^\alpha_\beta &= -A^\alpha_\gamma(X)\rho(g(t))^\gamma_\beta
\end{aligned}$$

So we have

$$\begin{aligned}
\nabla_{X_p} e_\alpha &= \left[\left(\tilde{\gamma}(0)g(0)^{-1}, A^\beta_\alpha(X_p)\rho(g(0))^\gamma_\beta \rho(g(0)^{-1})^\delta_\gamma e^0_\delta\right)\right] \\
&= \left[\left(\tilde{\gamma}(0)g(0)^{-1}, A^\beta_\alpha(X_p) e^0_\beta\right)\right] \\
&= \left[\left(\sigma(0), A^\beta_\alpha(X_p) e^0_\beta\right)\right] \\
&= A^\beta_\alpha(X_p) e_\beta(p)
\end{aligned}$$

and so

$$\nabla_X e_\alpha = A^\beta_\alpha(X) e_\beta$$

□

Corollary 2.43. *This is equivalent to*

$$\nabla_\mu e_\alpha = A^\beta_{\mu\alpha} e_\beta$$

or

$$\nabla e_\alpha = A^\beta_\alpha e_\beta$$

Corollary 2.44. *For a generic local section $f^\alpha e_\alpha$,*

$$\nabla_X(f^\alpha e_\alpha) = X(f^\alpha)e_\alpha + f^\alpha A^\beta_\alpha(X)e_\beta$$

or

$$\nabla_\mu(f^\alpha e_\alpha) = (\partial_\mu f^\alpha)e_\alpha + f^\alpha A^\beta_{\mu\alpha} e_\beta$$

2.5.3 Local Covariant Derivatives on Related Vector Bundles

Proposition 2.45. Let E be an associated vector bundle of a principal bundle $P(M, G)$, on which a covariant derivative ∇ is defined. Then there is a naturally induced covariant derivative ∇^* on the dual bundle E^* such that, for all $s \in \Gamma(E)$, $\lambda \in \Gamma(E^*)$ and $X \in \Gamma(TM)$,

$$(\nabla^*_X \lambda)(s) = X(\lambda(s)) - \lambda(\nabla_X s)$$

We will refer to ∇^* on the dual bundle as simply ∇ hereafter.

Proof. We need to check the covariant derivative axioms hold for ∇^*. Let $a, b \in \mathbb{R}$, $\lambda, \mu \in \Gamma(E^*)$, $s \in \Gamma(E)$ $X \in \Gamma(TM)$ and $f \in C^\infty(M)$. First,

$$\begin{aligned}(\nabla^*_X(a\lambda + b\mu))(s) &= X((a\lambda + b\mu)(s)) - (a\lambda + b\mu)(\nabla_X s)\\ &= aX(\lambda(s)) + bX(\mu(s)) - a\lambda(\nabla_X s) - b\mu(\nabla_X s)\\ &= a(\nabla^*_X \lambda)(s) + b(\nabla^*_X \mu)(s)\end{aligned}$$

so

$$\nabla^*_X(a\lambda + b\mu) = a\nabla^*_X\lambda + b\nabla^*_X\mu$$

Next,

$$\begin{aligned}(\nabla^*_{aX+bY}\lambda)(s) &= (aX + bY)(\lambda(s)) - \lambda(\nabla_{aX+bY} s)\\ &= aX(\lambda(s)) + bY(\lambda(s)) - \lambda(a\nabla_X s + b\nabla_Y s)\\ &= a(\nabla^*_X\lambda)(s) + b(\nabla^*_Y\lambda)(s)\end{aligned}$$

so

$$\nabla^*_{aX+bY}\lambda = a\nabla^*_X\lambda + b\nabla^*_Y\lambda$$

Then

$$\begin{aligned}(\nabla^*_X(f\lambda))(s) &= X(f\lambda(s)) - f\lambda(\nabla_X s)\\ &= X(f)\lambda(s) + fX(\lambda(s)) - f\lambda(\nabla_X s)\\ &= X(f)\lambda(s) + f(\nabla^*_X\lambda)(s)\end{aligned}$$

so

$$\nabla^*_X(f\lambda) = X(f)\lambda + f\nabla^*_X\lambda$$

Lastly,

$$\begin{aligned}(\nabla^*_{fX}\lambda)(s) &= fX(\lambda(s)) - \lambda(\nabla_{fX} s)\\ &= fX(\lambda(s)) - \lambda(f\nabla_X s)\\ &= fX(\lambda(s)) - f\lambda(\nabla_X s)\\ &= f(\nabla^*_X\lambda)(s)\end{aligned}$$

so

$$\nabla^*_{fX}\lambda = f\nabla^*_X\lambda$$

□

Corollary 2.46. *Then on basis sections,*

$$\begin{aligned}(\nabla_X e^\alpha)(e_\beta) &= X(\delta^\alpha_\beta) - e^\alpha(\nabla_X e_\beta)\\ &= -e^\alpha(A^\gamma_\beta(X)e_\gamma)\\ &= -A^\alpha_\beta(X)\end{aligned}$$

So

$$\nabla_X e^\alpha = -A^\alpha_\beta(X)e^\beta$$

Alternatively,

$$\nabla_\mu e^\alpha = -A^\alpha_{\mu\beta}e^\beta$$

Then more generally,

$$\nabla_X(\lambda_\alpha e^\alpha) = X(\lambda_\alpha)e^\alpha - \lambda_\alpha A^\alpha_\beta(X)e^\beta$$

or

$$\nabla_\mu(\lambda_\alpha e^\alpha) = (\partial_\mu\lambda_\alpha)e^\alpha - \lambda_\alpha A^\alpha_{\mu\beta}e^\beta$$

Proposition 2.47. Let E and E' be associated vector bundles of a principal bundle $P(M, G)$, on which covariant derivatives ∇ and ∇' are defined. Then there is a naturally induced covariant derivative $\nabla\otimes\nabla'$ on the tensor product bundle $E\otimes E'$ such that, for any $s\otimes s' \in \Gamma(E\otimes E')$,

$$(\nabla\otimes\nabla')_X(s\otimes s') = \nabla_X s\otimes s' + s\otimes\nabla'_X s'$$

Proof. First, note that since decomposition of sections of $E\otimes E'$ into the form $s\otimes s'$ is not unique, we need to check that this is well-defined. Denote a section of $E\otimes E'$ by $s^{\alpha\beta}e_\alpha\otimes e'_\beta$. Then we see that

$$\begin{aligned}(\nabla\otimes\nabla')_X(s^{\alpha\beta}e_\alpha\otimes e'_\beta) &= \nabla_X(s^{\alpha\beta}e_\alpha)\otimes e_\beta + s^{\alpha\beta}e_\alpha\otimes\nabla'_X e_\beta\\ &= X(s^{\alpha\beta})e_\alpha\otimes e'_\beta + s^\alpha\nabla_X e_\alpha\otimes e'_\beta + s^{\alpha\beta}\\ &= \nabla_X e_\alpha\otimes s^{\alpha\beta}e'_\beta + e_\alpha\otimes\nabla'_X(s^{\alpha\beta}e'_\beta)\\ &= (\nabla\otimes\nabla')_X(e_\alpha\otimes s^{\alpha\beta}e'_\beta)\end{aligned}$$

This shows that $\nabla\otimes\nabla'$ is in fact well-defined. Now we have to show it satisfies the covariant derivative axioms. Let $a, b \in \mathbb{R}$, $s, t \in \Gamma(E)$, $s', t' \in \Gamma(E')$, $X \in \Gamma(TM)$ and $f \in C^\infty(M)$. First,

$$\begin{aligned}(\nabla \otimes \nabla')_X(as \otimes s' + bt \otimes t') &= (\nabla \otimes \nabla')_X((as^{\alpha\beta} + bt^{\alpha\beta})e_\alpha \otimes e'_\beta) \\ &= \nabla_X((as^{\alpha\beta} + bt^{\alpha\beta})e_\alpha) \otimes e'_\beta + (as^{\alpha\beta} + bt^{\alpha\beta})e_\alpha \otimes \nabla'_X e'_\beta \\ &= a\nabla_X(s^{\alpha\beta}e_\alpha) \otimes e'_\beta + b\nabla_X(t^{\alpha\beta}e_\alpha) \otimes e'_\beta \\ &\quad + as^{\alpha\beta}e_\alpha \otimes \nabla'_X e'_\beta + bt^{\alpha\beta}e_\alpha \otimes \nabla'_X e'_\beta \\ &= a(\nabla \otimes \nabla')_X(s^{\alpha\beta}e_\alpha \otimes e'_\beta) + b(\nabla \otimes \nabla')_X(t^{\alpha\beta}e_\alpha \otimes e'_\beta)\end{aligned}$$

Next,

$$\begin{aligned}(\nabla \otimes \nabla')_{aX+bY}(s \otimes s') &= \nabla_{aX+bY}s \otimes s' + s \otimes \nabla'_{aX+bY}s' \\ &= s\nabla_X s \otimes s' + b\nabla_Y s \otimes s' + a\nabla_X s \otimes s' + b\nabla_Y s \otimes s' \\ &= a(\nabla \otimes \nabla')_X(s \otimes s') + b(\nabla \otimes \nabla')_Y(s \otimes s')\end{aligned}$$

Then,

$$\begin{aligned}(\nabla \otimes \nabla')_X(fs \otimes s') &= \nabla_X(fs) \otimes s' + fs \otimes \nabla'_X s' \\ &= X(f)s \otimes s' + f\nabla_X s \otimes s' + fs \otimes \nabla'_X s \\ &= X(f)s \otimes s' + f(\nabla \otimes \nabla')_X(s \otimes s')\end{aligned}$$

Lastly,

$$\begin{aligned}(\nabla \otimes \nabla')_{fX}(s \otimes s') &= \nabla_{fX}s \otimes s' + s \otimes \nabla'_{fX}s' \\ &= f\nabla_X s \otimes s' + s \otimes f\nabla'_X s' \\ &= f(\nabla \otimes \nabla')_X(s \otimes s')\end{aligned}$$

□

Corollary 2.48. *Then, if A and A' are the local gauge connections on E and E',*

$$(\nabla \otimes \nabla')_X(e_\alpha \otimes e'_\beta) = A^\gamma_\alpha(X)e_\gamma \otimes e'_\beta + A'^\gamma_\alpha e_\alpha \otimes e'_\gamma$$

Equivalently,

$$(\nabla \otimes \nabla')_\mu(e_\alpha \otimes e'_\beta) = A^\gamma_{\mu\alpha}e_\gamma \otimes e'_\beta + A'^\gamma_{\mu\alpha}e_\alpha \otimes e'_\gamma$$

Then more generally,

$$(\nabla \otimes \nabla')_X(s^{\alpha\beta}e_\alpha \otimes e'_\beta) = X(s^{\alpha\beta})e_\alpha \otimes e'_\beta + s^{\alpha\beta}\left(A^\gamma_\alpha(X)e_\gamma \otimes e'_\beta + A'^\gamma_\alpha(X)e_\alpha \otimes e'_\gamma\right)$$

or

$$(\nabla \otimes \nabla')_\mu(s^{\alpha\beta}e_\alpha \otimes e'_\beta) = (\partial_\mu s^{\alpha\beta})e_\alpha \otimes e'_\beta + s^{\alpha\beta}\left(A_{\mu\alpha}\gamma e_\gamma \otimes e'_\beta + A'^\gamma_{\mu\alpha}e_\alpha \otimes e'_\gamma\right)$$

Corollary 2.49. *Now given a covariant derivative ∇ on E, we have induced covariant derivatives on E^*, $E \otimes E$, $E \otimes E^*$, etc, i.e. on all type-(p,q) E-tensor bundles $\bigotimes^p E \otimes \bigotimes^q E^*$, which we all denote ∇, and such that, if we write basis sections $e^{\alpha_1 \dots \alpha_q}_{\beta_1 \dots \beta_p}$,*

$$\nabla_\mu e^{\alpha_1 \dots \alpha_q}_{\beta_1 \dots \beta_p} = A^\gamma_{\mu\beta_1} e^{\alpha_1 \dots \alpha_q}_{\gamma\beta_2 \dots \beta_p} + \dots + A^\gamma_{\mu\beta_p} e^{\alpha_1 \dots \alpha_q}_{\beta_1 \dots \beta_{p-1}\gamma} - A^{\alpha_1}_{\mu\gamma} e^{\gamma\alpha_2 \dots \alpha_q}_{\beta_1 \dots \beta_p} - \dots - A^{\alpha_q}_{\mu\gamma} e^{\alpha_1 \dots \alpha_{q-1}\gamma}_{\beta_1 \dots \beta_p}$$

3 Applications to Physics

3.1 Topological Aspects of Gauge Theories

3.1.1 The Dirac Monopole

Example 3.1. Consider a $U(1)$ bundle over $M = \mathbb{R}^3\backslash\{0\}$, $P(M, U(1))$, with an atlas $\{U_N, U_S\}$ on M. The intersection $U_N \cap U_S$ will be homotopy equivalent to a circle S^1. The possible transition functions $t_{NS} : U_N \cap U_S \to U(1)$ can therefore be classified by $\pi_1(U(1)) \cong \mathbb{Z}$. Obviously if $t_{NS} \in 0 \in \pi_1(U(1))$ we have a trivial bundle. Dirac magnetic monopoles exist for non-trivial bundles. More specifically, if we treat t_{NS} as a map just from the equatorial S^1 for simplicity, and write it in the form

$$t_{NS}(\phi) = e^{if(\phi)}$$

for some function $f : S^1 \to \mathbb{R}$, so $f(\phi) = f(\phi + 2n\pi)$ necessarily. Then if we have local gauge connections $\mathcal{A}_N$ and $\mathcal{A}_S$,

$$\begin{aligned} \mathcal{A}_N &= t_{NS}^{-1}\mathcal{A}_S t_{NS} + t_{NS}^{-1} dt_{NS} \\ &= \mathcal{A}_S + id\phi \end{aligned}$$

Suppose that these are of the form

$$\mathcal{A}_N = ig(1 - \cos\theta)d\phi \quad \mathcal{A}_S = -ig(1 + \cos\theta)d\phi$$

in polar coordinates, for magnetic charge g

$$df = -i(\mathcal{A}_N - \mathcal{A}_S) = 2gd\phi$$

Then

$$\begin{aligned} f(2\pi) - f(0) &= \int_{S^1} df = 2g \int_0^{2\pi} d\phi \\ &= 4\pi g \end{aligned}$$

So if f is properly defined, we must have $2g \in \mathbb{Z}$. Thus magnetic charge is quantised. It is also clear that this $2g$ is the integer classifying t_{NS}, and hence the non-triviality of the bundle.

3.2 Coupling to Gauge Fields

3.2.1 Scalar Fields

Definition 3.2. A **particle field** is a section of a vector bundle. If this vector bundle is associated to a principle G-bundle, we say that the particle field is **coupled** to G-gauge theory, and in the representation ρ of G that defines the vector bundle.

Example 3.3. The theory of a complex scalar field coupled to $U(1)$ in the fundamental representation is described by a principal bundle $P(M, U(1))$ and associated vector bundle $E = P \times_\rho \mathbb{C}$, where $\rho : U(1) \to GL(\mathbb{C})$ is the fundamental representation. Regarded as a vector space over the complex numbers, $\mathbb{C}$ is one-dimensional, so has bases consisting of single elements. Then a local basis of sections of E just consists of a single element, which we denote e. Then given a local section used to define $\mathcal{A}$, and its induced representation A on $\mathbb{C}$, the covariant derivative of a generic section $\phi = \Phi e$ is given by

$$\begin{aligned}\nabla_\mu \phi &= (\partial_\mu \Phi) e + A_\mu \Phi e \\ &= (\partial_\mu + A_\mu)\phi\end{aligned}$$

Recall that $\mathfrak{u}(1)$ is just the imaginary numbers; in physics literature the i is factored out of A to make it real. A charge factor is also usually factored out. (Alternatively, we can define a charge-q representation ρ_q by $\rho_q(g) = q\rho(g)$ and use this.) If we additionally have the complex scalar's conjugate, ϕ^*, this must be in the anti-fundamental representation $\bar{\rho}$, which for $U(1)$ is just given by $\bar{\rho}(g) = -\rho(g)$. (Alternatively, $\bar{\rho} = \rho_{-q}$.) Then we will have

$$\nabla_\mu \phi^* = (\partial_\mu - A_\mu)\phi^*$$

Example 3.4. Suppose instead a complex scalar is in the fundamental of $SU(n)$ Yang-Mills. Then it is described by $P(M, SU(n))$ and $E = P \times_\rho \mathbb{C}^n$, where $\rho : SU(n) \to \mathbb{C}^n$ is the fundamental representation. Then, making appropriate definitions,

$$\nabla_\mu \phi = (\partial_\mu \Phi^\alpha + A^\alpha_{\mu\beta} \Phi^\beta) e_\alpha$$

3.2.2 Gauge Bosons

Example 3.5. Gauge bosons are in the adjoint representation of the gauge field, so described by $P(M, G)$ and $E = P \times_{\mathrm{Ad}} \mathfrak{g}$, where $\mathrm{Ad} : G \to GL(\mathfrak{g})$ induces the Lie algebra representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{GL}(\mathfrak{g})$. For clarity we will retrace the proof of Proposition 2.42 for this case. Take a basis $\{T_\alpha\}$ for $\mathfrak{g}$ and write a generic element as $V = V^\alpha T_\alpha$. Then use a local section of P to define a section

$$s(p) = [(\sigma(p), V(p))]$$

of E. If we take a curve γ with $\gamma(0) = p$ and $\gamma'(0) = X_p$, and write its horizontal lift as $\tilde{\gamma}(t) = \sigma(t) g(t)$, along γ,

$$\begin{aligned}s(t) &= [(\tilde{\gamma}(t) g(t)^{-1}, V(t))] \\ &= [(\tilde{\gamma}(t), \mathrm{Ad}_{g(t)^{-1}} V(t))]\end{aligned}$$

Then

$$\begin{aligned}\nabla_X s\mid_p = \nabla_{X_p} s &= \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\operatorname{Ad}_{g(t)^{-1}} V(t)|_{t=0}\right)\right] \\ &= \left[\left(\sigma(0), \operatorname{Ad}_{g(0)} \frac{d}{dt}\operatorname{Ad}_{g(t)^{-1}} V(t)|_{t=0}\right)\right]\end{aligned}$$

Now, we have

$$\begin{aligned}\frac{d}{dt}\operatorname{Ad}_{g(t)^{-1}} V(t)|_{t=0} &= \frac{d}{dt} g(t)^{-1}V(t)g(t)|_{t=0} \\ &= \left[-g(t)^{-1}\frac{dg}{dt}g(t)^{-1}V(t)g(t) + g(t)^{-1}\frac{dV}{dt}g(t) + g(t)^{-1}V(t)\frac{dg}{dt}\right]\Big|_{t=0} \\ &= g(0)^{-1}\left[-\frac{dg}{dt}\Big|_{t=0} g(0)^{-1}V(0) + \frac{dV}{dt}\Big|_{t=0} + V(0)\frac{dg}{dt}\Big|_{t=0} g(0)^{-1}\right] g(0) \\ \operatorname{Ad}_{g(0)}\frac{d}{dt}\operatorname{Ad}_{g(t)^{-1}} V(t)|_{t=0} &= -\frac{dg}{dt}\Big|_{t=0} g(0)^{-1}V(0) + \frac{dV}{dt}\Big|_{t=0} + V(0)\frac{dg}{dt}\Big|_{t=0} g(0)^{-1} \\ &= \frac{dV}{dt}\Big|_{t=0} - \left[\frac{dg}{dt}g(t)^{-1}, V(t)\right]\Big|_{t=0} \\ &= \frac{dV}{dt}\Big|_{t=0} + [\mathcal{A}(X), V(t)]|_{t=0} \\ &= \frac{dV}{dt}\Big|_{t=0} + \operatorname{ad}(\mathcal{A}(X))(V(t))|_{t=0}\end{aligned}$$

where to get the second last line we have used the definition of $\mathcal{A}$, and in the last line we have put this in the more familiar language of action of the induced Lie algebra representation. Expanding this, we get

$$\begin{aligned}\frac{dV}{dt}\Big|_{t=0} + [\mathcal{A}(X), V(t)]|_{t=0} &= X_p^\mu(\partial_\mu V_p^\alpha)T_\alpha + X_p^\mu[\mathcal{A}_\mu^\alpha T_\alpha, V_p^\beta T_\beta] \\ &= X_p^\mu(\partial_\mu V_p^\alpha)T_\alpha + X_p^\mu \mathcal{A}_\mu^\alpha V_p^\beta f_{\alpha\beta}^\gamma T_\gamma \\ &= X_p^\mu(\partial_\mu V_p^\alpha + f_{\beta\gamma}^\alpha \mathcal{A}_\mu^\beta V^\gamma)T_\alpha\end{aligned}$$

So, if $\{T_\alpha\}$ is used to define a basis of sections $\{e_\alpha\}$,

$$\nabla_\mu(V^\alpha e_\alpha) = (\partial_\mu V^\alpha + f_{\beta\gamma}^\alpha \mathcal{A}_\mu^\beta V^\gamma)e_\alpha$$

A Riemannian Geometry from the Tangent Bundle

Remark. Here we will apply some of the above theory to the tangent bundle TM and related bundles T^*M, $\bigwedge(T^*M)$, and so on.

Definition A.1. The **tangent bundle** TM of a manifold M is

$$TM = \bigcup_{p \in M} T_p M$$

Proposition A.2. The tangent bundle is a vector bundle.

Proof. First we need to show that TM is a manifold. Let (U, φ) be a local chart on M, defining coordinates $\varphi(p) = x^\mu$, where $\mu = 1, ..., \dim M = m$. φ also induces a basis $\{\partial_\mu\}$ for all $T_p M$ with $p \in U$. We can then define a homeomorphism $TU_i \mapsto \mathbb{R}^{2m}$ by

$$(p, v) \mapsto (x^\mu, v^\mu)$$

In this way we can build an atlas on TM to $\mathbb{R}^{2m}$, making TM a manifold. Now, to regard it as a vector bundle, obviously the base space is M and the fibre $\mathbb{R}^m$. The projection locally is $\pi : (p, v) \mapsto p$ and local trivialisations are canonically induced by charts, i.e. $\phi : U_i \times \mathbb{R}^m \mapsto \pi^{-1}(U_i)$ can be defined by

$$\phi : (p, v) \mapsto (p, v^\mu)$$

Now, if we change coordinate systems to y^ν

$$\tilde{v}^\mu = \frac{\partial y^\mu}{\partial x^\nu} v^\nu$$

The only restriction is that $(\partial y^\mu / \partial x^\nu)^m_{\mu,\nu=1}$ be non-singular. That is, the structure group is $GL(\mathbb{R}^m)$, and transition functions are just these Jacobians, which clearly satisfy the transition function requirements. Thus TM is a vector bundle. $\square$

Remark. Note that we could also have started from the base space M and required the typical fibre to be $\mathbb{R}^m$ and transition functions to be coordinate transformations in $GL(\mathbb{R}^m)$, using the fibre bundle construction theorem to find TM, the projection and local trivialisations.

Definition A.3. Sections of the tangent bundle are called **vector fields**.

Definition A.4. Given the tangent bundle TM of a manifold M, we immediately have the **cotangent bundle** T^*M, as well as its tensor and exterior powers, $\bigotimes^k T^*M$ and $\bigwedge^k T^*M$ respectively. Sections of $\bigotimes^p TM \otimes \bigotimes^q T^*M$ are called **type-(p,q) tensors** (of TM), and sections of $\bigwedge^k T^*M$ are called **differential k-forms**.

Definition A.5. The **frame bundle** LM over a manifold M is the principal $GL(\mathbb{R}^m)$-bundle to which the tangent bundle is associated.

Proposition A.6. Given a connection ω on LM with local connection $\mathcal{A}$, with an induced Lie algebra representation ρ' on $\mathfrak{GL}(\mathbb{R}^m)$, write $\rho'(\mathcal{A}) = \Gamma$. Then, if e_α is a basis of vector fields, e^α the dual basis of 1-forms, e_α^β the induced basis of type-(1,1) tensors, etc, we have covariant derivatives:

$$\nabla_\mu e_\alpha = \Gamma^\beta_{\mu\alpha} e_\beta$$

on vector fields, by Corollary 2.43;

$$\nabla_\mu e^\alpha = -\Gamma^\alpha_{\mu\beta} e^\beta$$

on 1-forms, by Corollary 2.46; and more generally

$$\nabla_\mu e^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_p} = \Gamma^\gamma_{\mu\beta_1} e^{\alpha_1 \ldots \alpha_q}_{\gamma\beta_2 \ldots \beta_p} + \ldots + \Gamma^\gamma_{\mu\beta_p} e^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_{p-1}\gamma} - \Gamma^{\alpha_1}_{\mu\gamma} e^{\gamma\alpha_2 \ldots \alpha_q}_{\beta_1 \ldots \beta_p} - \ldots - \Gamma^{\alpha_q}_{\mu\gamma} e^{\alpha_1 \ldots \alpha_{q-1}\gamma}_{\beta_1 \ldots \beta_p}$$

on type-(p,q) tensors, by Corollary 2.49.

Remark. Note that this notation makes it clear that not all of the indices on Γ are the same, even though in standard Riemannian geometry notation this appears to be so. Thus it is not surprising that Γ does not transform as a type-(1,2) tensor of TM.

B Lie Groups

Remark. Here we establish some definitions and basic results regarding Lie groups relevant for our study of principal bundles.

Definition B.1. Let $A \in \mathfrak{g}$, $A^{\#} \in \Gamma(TG)$ be the fundamental vector field associated with it, i.e. $\bar{A}_g = L_{g*}A$, ϕ_t the one-parameter group of diffeomorphisms generated by $A^{\#}$, and $\gamma : [0,1] \to G$ the curve defined by $\gamma(t) = \phi_t(e)$. Then we define the **exponential map** $\exp : \mathfrak{g} \to G$ by $\exp(A) = \gamma(1)$. Then

$$\begin{aligned} \gamma(t) &= \exp(tA) \\ \phi_t(g) = g\gamma(t) &= g \exp(tA) \end{aligned}$$

Definition B.2. Let G be a Lie group. For each $g \in G$ there is an **inner automorphism** $\phi_g : G \to G$ given by $\phi_g h = ghg^{-1}$. This induces a Lie algebra automorphism $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{Ad}_g = \phi_{g*} |_e$, with the usual identification $T_eG \cong \mathfrak{g}$. Then the **adjoint group representation** $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is defined by $\mathrm{Ad}(g) = \mathrm{Ad}_g$. This in turn induces the **adjoint Lie algebra representation** $\mathrm{ad} : \mathfrak{g} \to \mathfrak{GL}(\mathfrak{g})$ by $\mathrm{ad} = \mathrm{Ad}_* |_e$.

Lemma B.3. *The adjoint Lie algebra representation is given by the Lie bracket:*

$$\mathrm{ad}(A)(B) = [A, B]$$

Proof. Let $A^{\#}$ and $B^{\#}$ be the fundamental vector fields associated with A and B, and ϕ_t the one-parameter group of diffeomorphisms generated by $A^{\#}$. We have

$$\begin{aligned} [A, B] &= [A^{\#}, B^{\#}]_e \\ &= \frac{d}{dt}\phi_{-t*}(B^{\#}_{\phi_t(e)})|_{t=0} \\ &= \frac{d}{dt}\phi_{-t*}\left(\frac{d}{ds}\phi_t(e)\exp(sB)\right)\Big|_{s=t=0} \\ &= \frac{d}{ds}\frac{d}{dt}\phi_{-t}(\phi_t(e)\exp(sB))|_{s=t=0} \\ &= \frac{d}{ds}\frac{d}{dt}\phi_{-t}(\exp(tA)\exp(sB))|_{s=t=0} \\ &= \frac{d}{ds}\frac{d}{dt}\exp(tA)\exp(sB)\exp(-tA)|_{s=t=0} \end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dt} \exp(tA) B \exp(-tA)|_{t=0} \\
&= \frac{d}{dt} \operatorname{Ad}(\exp(tA))(B)|_{t=0} \\
&= \operatorname{Ad}_* |_e \left(\frac{d}{dt} \exp(tA)|_{t=0} \right) (B) \\
&= \operatorname{ad}(A)(B)
\end{aligned}
$$

where the first equality is the definition of the Lie algebra bracket. $\square$

Proposition B.4. The map $A \mapsto A^{\#}$ is naturally compatible with the Lie bracket,

$$[A, B]^{\#} = [A^{\#}, B^{\#}]$$

Proof. Use the same notation as the previous lemma. We have

$$
\begin{aligned}
[A^{\#}, B^{\#}]_g &= \frac{d}{dt} \phi_{-t*}(B^{\#}_{\phi_t(g)})|_{t=0} \\
&= \frac{d}{dt} \phi_{-t*} \left(\frac{d}{ds} \phi_t(g) \exp(sB)|_{s=0} \right) \Big|_{t=0} \\
&= \frac{d}{dt} \frac{d}{ds} \phi_{-t}(\phi_t(g) \exp(sB))|_{t=s=0} \\
&= \frac{d}{dt} \frac{d}{ds} \phi_{-t}(g \exp(tA) \exp(sB))|_{t=s=0} \\
&= \frac{d}{dt} \frac{d}{ds} g \exp(tA) \exp(sB) \exp(-tA)|_{t=s=0} \\
&= \frac{d}{dt} g \exp(tA) B \exp(-tA)|_{t=0} \\
&= \frac{d}{dt} g \operatorname{Ad}_{\exp(tA)} B|_{t=0} \\
&= \frac{d}{dt} \frac{d}{ds} g \exp\big(s \operatorname{Ad}_{\exp(tA)} B\big)|_{t=s=0} \\
&= \frac{d}{ds} \left[g \left(\frac{d}{dt} s \operatorname{Ad}_{\exp(tA)} B \right) \exp\big(s \operatorname{Ad}_{\exp(tA)} B\big) \right] \Big|_{t=0} \\
&= \frac{d}{ds} \left[g \left(\frac{d}{dt} s \operatorname{Ad}_{\exp(tA)} B \right) \Big|_{t=0} \exp(sB) \right] \Big|_{s=0} \\
&= \left[\frac{d}{ds} \frac{d}{dt} g s \operatorname{Ad}_{\exp(tA)} B|_{t=0} \right] \exp(sB) \Big|_{s=0} + g \frac{d}{dt} s \operatorname{Ad}_{\exp(tA)} B|_{t=0} \frac{d}{ds} \exp(sB)|_{s=0} \\
&= \frac{d}{dt} g \operatorname{Ad}_{\exp(tA)} B|_{t=0}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d}{ds} g \exp\left(s \frac{d}{dt} \operatorname{Ad}_{\exp(tA)} B \right) \Big|_{s=t=0} \\
&= \frac{d}{ds} g \exp\left(s \frac{d}{dt} \operatorname{Ad}(\exp(tA))(B) \right) \Big|_{s=t=0} \\
&= \frac{d}{ds} g \exp(s \operatorname{ad}(A)(B))|_{s=0} \\
&= \frac{d}{ds} g \exp(s[A, B])|_{s=0} \\
&= [A, B]_g^{\#}
\end{aligned}
$$

where we have used the previous lemma for the second to last equality. $\square$

References

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[3] Mikio Nakahara. *Geometry, Topology and Physics*. CRC Press, 2003.