Fibre Bundles and Physics

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Abstract

These are my personal notes on fibre bundles and their usage in physics. They are a work in progress, will have mistakes, and will always be lacking in many ways. The main texts I have learnt the topic from are [1], [2] and [3] and I have aimed to synthesise but also augment these considerably with the fruits of my own labours. Where I am particularly indebted to a reference it will be listed.

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1 Fibre Bundles

1.1 Fibre Bundles

1.1.1 Definitions

Definition 1.1. A **smooth fibre bundle**, (E, M, π, F, G), variously abbreviated E, $\pi : E \to M$, etc, consists of:

(i) A smooth manifold E, called the **total space**.

(ii) A smooth manifold M, called the **base space**.

(iii) A smooth manifold F, called the **(typical) fibre**.

(iv) A surjection $\pi : E \to M$, called the **projection**, such that $\pi^{-1}(p) := F_p \cong F$ for all $p \in M$.

(v) A Lie group G, called the **structure group**, with a smooth left-action on F.

(vi) An open covering $\{U_i\}$ of M, with diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ satisfying $\pi \circ \phi_i(p, f) = p$, called **local trivialisations**.

(vii) Maps $t_{ij} : U_i \cap U_j \to G$ on overlaps satisfying $\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$, called **transition functions**.

Proposition 1.2. The transition functions should satisfy

$$\begin{aligned} t_{ii}(p) &= e \\ t_{ij}(p)t_{jk}(p) &= t_{ik}(p) \end{aligned}$$

for all $p \in M$ and $p \in U_i \cap U_j \cap U_k$, respectively. The latter is called the **cocycle condition**.

Proof. These follow from the fact that the local trivialisations are isomorphisms and the invertibility of group elements of G. The first is obvious. For the second, note that on the one hand

$$\begin{aligned} \phi_k(p, f) &= \phi_j(p, t_{jk}(p)f) \\ &= \phi_i(p, t_{ij}(p)t_{jk}(p)f) \end{aligned}$$

and on the other

$$\phi_k(p, f) = \phi_i(p, t_{ik}(p)f)$$

$\square$

Corollary 1.3. *In particular, by the cocycle condition,*

$$t_{ij}(p)t_{ji}(p)t_{ik}(p) = t_{ik}(p)$$

so

$$t_{ij}(p)^{-1} = t_{ji}(p)$$

Definition 1.4. If it is possible to set $t_{ij}(p) = e$ for all $p \in U_i \cap U_j$ for all intersecting charts U_i and U_j, the bundle E is called a **trivial bundle**.

Definition 1.5. A **smooth section** of a bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The set of smooth sections of E is denoted $\Gamma(M, E)$.

Theorem 1.6 (Fibre bundle construction theorem)**.** *Given a base space M with open covering $\{U_i\}$, typical fibre F, structure group G, and transition functions t_{ij} it is possible to construct a unique bundle $\pi : E \to M$.*

Proof. Define

$$X = \bigsqcup_i U_i \times F$$

and an equivalence relation

$$(p, f) \sim (q, f') \quad \text{iff} \quad p = q \text{ and } f' = t_{ji}(p)f$$

if $p \in U_i \cap U_j$. Then we claim

$$E = X/\sim$$

which has elements $[(p, f)]$ which are equivalence classes, is a bundle, with the projection $\pi : E \to M$ defined by

$$\pi : [(p, f)] \mapsto p$$

Firstly, we note that this is well-defined, since every $(p', f') \in [(p, f)]$ has $p' = p$. It is also clearly surjective, since $\{U_i\}$ covers M. Local trivialisations are choices of representatives. Choose some fixed v_i such that $(p, v_i) \in [(p, v)] = u$, and define $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ by

$$\phi_i(p, v_i) = [(p, v)]$$

Certainly

$$\pi \circ \phi_i(p, v_i) = \pi([(p, v)]) = p$$

as required. Then, for each intersecting patch U_j, define a local trivialisation ϕ_j which selects the representative (p, v_j) from $[(p, v)]$. v_j is determined by using the transition functions: we

$$\phi_j(p, v_j) = [(p, v)] = \phi_i(p, v_i)$$

so we are forced to have $v_j = t_{ij}(p)v_i$. That is, after choosing a single local trivialisation all the rest are fixed by the transition functions. This makes the bundle unique. By the cocycle condition on the transition functions this local trivialisation construction can always be done consistently for multiple local trivialisations. $\square$

Remark. Note that, while the use of equivalence classes $([p, f])$ may suggest that the elements $u \in \pi^{-1}(\{p\})$ are not in one-to-one correspondence with the typical fibre F, we can always choose an open cover such that p is not in an overlap, and hence such that there are no transition functions t_{ij} defined on p, in which case equivalence classes are just singletons.

1.1.2 Bundle Maps

Definition 1.7. Let $\pi : E \to M$ and $\pi' : E' \to M'$ be two bundles. A smooth map $\bar{f} : E' \to E$ is called a **bundle map** if it does not mix fibres, i.e. if $\bar{f} \circ \pi' = \pi \circ \bar{f}$. Then $\bar{f}$ naturally induces a smooth map $f : M' \to M$ defined by $f = \pi'^{-1} \circ \bar{f} \circ \pi$.

Definition 1.8. Let $\bar{f}$ be a bundle map from $\pi' : E' \to M'$ to $\pi : E \to M$. It is called a **bundle isomorphism** if both $\bar{f}$ and the induced f are diffeomorphisms.

Definition 1.9. Two bundles $\pi : E \to M$ and $\pi' : E' \to M$ over the same base space are called **equivalent** if there exists a bundle isomorphism $\bar{f} : E' \to E$ which induces the identity map $f = \mathrm{id}_M$ on the base space.

Definition 1.10. Given a bundle $\pi : E \to M$ with fibre F and a smooth map $f : N \to M$, the **pullback bundle** f^*E over N is the space

$$f^*E = \{(p, u) \in N \times E \mid f(p) = \pi(u)\}$$

That is, the fibres F_p of f^*E are identical to those $F_{f(p)}$ of E.

Proposition 1.11. The pullback bundle can in fact be made into a bundle over N by inheriting properties from E.

Proof. From the fibre bundle construction theorem we know that we can construct f^*E as a bundle. We just need an open covering of N, which is given by $\{f^{-1}(U_i)\}$ if $\{U_i\}$ is an open covering of M. Since the fibres F_p of f^*E are just those $F_{f(p)}$ of E, the equivalence relation in the construction of the construction theorem that gives us f^*E is clearly

$$(p, f) \sim (p, f') \quad \text{iff} \quad f' = t_{ji}(f(p))f$$

That is, we set transition functions

$$t^*_{ij}(p) = t_{ij}(f(p))$$

The projection $\pi^* : f^*E \to N$ is defined by $\pi^* : (p, u) \mapsto p$ which is clearly surjective. Now, suppose the local trivialisations on E are $\phi_i^{-1}(u) = (\pi(u), f_i)$ then at each $(p, u) \in f^*E$ (and automatically $f(p) = \pi(u)$), define $\psi_i^{-1} : (\pi^*)^{-1}(f^{-1}(U_i)) \to f^{-1}(U_i) \times F$ by

$$\psi_i^{-1}(p, u) = (p, f_i)$$

That is,

$$\psi_i(p, f_i) = (p, \phi_i(f(p), f_i))$$

Then $\pi^* \circ \psi_i(p, f_i) = p$, so ψ_i are indeed local trivialisations. We also confirm that

$$\begin{aligned}
\psi_j(p, f_j) &= (p, \phi_j(f(p), f_j)) \\
&= (p, \phi_i(f(p), t_{ij}(f(p))f_j)) \\
&= (p, \phi_i(f(p), t^*_{ij}(p)f_j)) \\
&= \psi_i(p, t^*_{ij}(p)f_j)
\end{aligned}$$

Lastly, the transition functions t^*_{ij} obviously satisfy the requirements since the t_{ij} do. Thus we have made f^*E into a bundle. □

Theorem 1.12 (Homotopy axiom). *If $\pi : E \to M$ is a bundle and $f, g : N \to M$ are homotopic maps, f^*E is equivalent to g^*E.*

Proof. Proving this will take us a fair bit off course, and anyway we have only introduced it for the purpose of the following handy corollary: □

Corollary 1.13. *Any bundle over a simply connected base space is trivial.*

Proof. Let M be simply connected and E be any bundle over M. If $g : M \to M$ is a constant map $g(p) = p_0$, it is homotopic to the identity, so g^*E is equivalent to E. But g^*E is the pullback of $\{p_0\} \times F$, where F is the typical fibre of E, so is trivial. So E is trivial. □

1.2 Vector Bundles

1.2.1 Definitions

Definition 1.14. A fibre bundle $\pi : E \to M$ whose typical fibre $F = V$ is a vector space is called a **vector bundle**. If $V = \mathbb{R}^k$ or $\mathbb{C}^k$, k is called the **fibre dimension**. A vector bundle with $k = 1$ is called a **line bundle**. The action of the structure group G on V is given by a representation $\rho : G \to GL(V)$, which we will generally leave implicit.

Proposition 1.15. If $\pi : E \to M$ is a vector bundle with fibre V, $\Gamma(M, E)$ naturally inherits vector space structure from V.

Proof. Let s, t be any elements of $\Gamma(M, E)$, p any point in M, and $\alpha \in \mathcal{F}$, the field over which V is defined. Then give $\Gamma(M, E)$ vector space structure by setting

$$(s + t)(p) = s(p) + t(p)$$
$$(\alpha s)(p) = \alpha s(p)$$

It is then trivial to check that $\Gamma(M, E)$ is indeed a vector space. $\square$

Corollary 1.16. *We can further make* $\Gamma(M, E)$ $C^\infty(M)$*-linear by defining*

$$(fs)(p) = f(p)s(p)$$

for any $f \in C^\infty(M)$ *and* $s \in \Gamma(M, E)$.

Definition 1.17. It is therefore possible to locally choose k linearly independent sections, $\{e_i\}$, defining a **frame** over a local neighbourhood U of M. Then a generic local section can be written as $s = s^i e_i$, where $s_i \in C^\infty(M)$. Any two frames $\{e_i\}$ and $\{\tilde{e}_j\}$ are related by some $G \in GL(V)$ by $\tilde{e}_j = G^i_j e_i$.

Definition 1.18. A **vector bundle isomorphism** $f : E \to E'$ is a bundle map of vector bundles whose restriction to each fibre is a linear isomorphism, i.e. if $f\,|_{\pi^{-1}(p)}: V_p \to V'_{f(p)}$ is a linear isomorphism.

Theorem 1.19. *A vector bundle has a global frame iff there exists a vector isomorphism to a trivial vector bundle.*

Proof. Suppose $\pi : E \to M$ has a global frame $\{e_i\}$ for $i = 1, ..., k$. Then we can define a map from the trivial $\mathbb{R}^k$ (or $\mathbb{C}^k$) bundle over M to E by

$$f : (p, v) \mapsto v^i e_i(p)$$

Firstly, this is clearly a bundle map, since both sides are projected down to p. Furthermore, this is clearly a vector bundle isomorphism.
This argument also runs the other way. $\square$

Corollary 1.20. *Recall that any bundle over a simply connected base space is trivial. Then we have that any vector bundle over a simply connected base space has a global frame.*

Corollary 1.21. *From this construction, we can see that given a local frame $\{e_i\}$ over a patch U, f is a canonical local trivialisation. That is, we define $\phi : U \times V \to \pi^{-1}(U)$ by*

$$\phi : (p, v) \mapsto v^i e_i(p)$$

As observed, this satisfies $\pi \circ \phi(p, v) = p$. Note that if ϕ and $\tilde{\phi}$ are canonical local trivialisations associated with frames $\{e_i\}$ and $\{\tilde{e}_j\}$, we have

$$\begin{aligned} \tilde{\phi}(p, v) &= v^i \tilde{e}_j(p) \\ &= G^i_j(p) v^j e_i(p) \\ &= \phi(p, G^i_j(p) e_i(p)) \end{aligned}$$

That is, the transition functions $t(p)$ are mapped by the representation $\rho : G \mapsto GL(V)$ to the matrices $G(p)$ relating frames.

1.2.2 New Vector Bundles from Old Ones

Definition 1.22. Let $\pi : E \to M$ be a vector bundle with typical fibre V. Then its **dual bundle** $\pi^* : E^* \to M$ is the bundle whose fibres are V^*. A local frame $\{e_i\}$ of E over $U \subset M$ induces a basis $\{e_i(p)\}$ on V_p for each $p \in U$, each of which induces a dual basis on V_p^*, $\{e^i(p)\}$, which can be extended to a local frame $\{e^i\}$ of E^*. This makes E^* isomorphic to E.

Definition 1.23. Let $\pi : E \to M$ and $\pi' : E' \to M'$ be vector bundles with fibres V and V' and structure groups G and G'. Their **product bundle** is $\pi \times \pi' : E \times E' \to M \times M'$ and has fibre $V \oplus V'$ and structure group $G \oplus G'$ with the obvious inherited left-action.

Proposition 1.24. This is indeed a vector bundle.

Proof. Let $\{U_i\}$ and $\{U_\alpha\}$ be open covers of M and M' respectively. Then $\{U_i \times U_\alpha\}$ is an open cover of $M \times M'$. Denote local trivialisations of E and E' ϕ_i and ϕ'_α respectively, and transition functions t_{ij} and $t'_{\alpha\beta}$ respectively. Then define $\psi_{i\alpha} : U_i \times U_\alpha \times (V \oplus V')$ by

$$\psi_{i\alpha} : (p, q, (v, w)) \mapsto (\phi_i(p, v), \phi'_\alpha(q, w))$$

Then

$$\begin{aligned}(\pi \times \pi') \circ \psi_{i\alpha}(p, q, (v, w)) &= (\pi \times \pi')(\phi_i(p, v), \phi'_\alpha(q, w)) \\ &= (\pi \circ \phi_i(p, v), \pi' \circ \phi'_\alpha(q, w)) \\ &= (p, q)\end{aligned}$$

as desired, so $\psi_{i\alpha}$ is indeed a local trivialisation. Also define transition functions $T_{ij,\alpha\beta} : (U_i \cap U_j) \times (U_\alpha \times U_\beta) \to G \oplus G'$ (again leaving representations implicit) by

$$T_{ij,\alpha\beta}(p, q) = \begin{pmatrix} t_{ij}(p) & \\ & t'_{\alpha\beta}(q) \end{pmatrix}$$

This naturally inherits the requirements on transition functions from the original ones. Then notice that indeed

$$\begin{aligned}\psi_{j\beta}(p, q, (v, w)) &= (\phi_j(p, v), \phi'_\beta(q, w)) \\ &= (\phi_i(p, t_{ij}v), \phi'_\alpha(q, t'_{\alpha\beta}w)) \\ &= \psi_{i\alpha}(p, q, T_{ij,\alpha\beta}(p, q))\end{aligned}$$

as expected. $E \times E'$ is now a bundle; that it is a vector bundle is obvious. □

Proposition 1.25. Given three bundles E, E', E'', the product $E \times E' \times E''$ is well-defined, i.e.

$$(E \times E') \times E'' = E \times (E' \times E'')$$

(up to canonical isomorphisms).

Proof. This is fairly clear from the same statement about the fibres but I won't show it explicitly. □

Definition 1.26. Let $\pi : E \to M$ and $\pi' : E' \to M$ be vector bundles over M with fibres V and V' respectively, and $f : M \to M \times M$ be the diagonal map $f(p) = (p, p)$. Then the **Whitney sum bundle** of E and E' is

$$E \oplus E' = f^*(E \times E')$$

That is,

$$E \oplus E' = \{(u, u') \in E \times E' \mid (\pi \times \pi')(u, u') = (p, p)\}$$

(The qualifier may also be written $\pi(u) = \pi'(u') = p$.) This makes the Whitney sum bundle the natural notion of a bundle whose fibre at p is the direct sum of those of two other bundles, also at p.

Proposition 1.27. This is indeed a vector bundle.

Proof. Showing this is exactly the same as showing that $E \times E'$ is a vector bundle. □

Proposition 1.28. Given three bundles E, E', E'', the sum $E \oplus E' \oplus E''$ is well-defined, i.e.

$$(E \oplus E') \oplus E'' = E \oplus (E' \oplus E'')$$

(up to canonical isomorphisms).

Proof. This follows from the same result for product bundles. □

Corollary 1.29. *Given a collection of bundles E_i,*

$$\bigoplus_i E_i$$

is well-defined.

Definition 1.30. If $s \in \Gamma(M, E)$ and $s' \in \Gamma(M, E')$, we define $(s, s') \in \Gamma(M, E \oplus E')$ satisfying

$$(s, s')(p) = (s(p), s'(p))$$

for all p on which s and s' are defined.

Proposition 1.31. Any section $t \in \Gamma(M, E \oplus E')$ can be written as $t = (s, s')$ for unique s, s'. That is,

$$\Gamma(M, E \oplus E') = \Gamma(M, E) \oplus \Gamma(M, E')$$

Proof. Since $t(p) \in V \oplus V'$, it is fairly clear that we can find s, s' such that $t(p) = (s(p), s'(p))$. If $(s(p), s'(p)) = (\tilde{s}(p), \tilde{s}'(p))$ for all p on which t is defined, clearly $s = \tilde{s}$ and $s' = \tilde{s}'$, making this unique. □

Definition 1.32. Let $\pi : E \to M$ and $\pi' : E' \to M$ be vector bundles over M with fibres V and V', and transition functions t_{ij} and t'_{ij}, respectively. Then the **tensor product bundle** of E and E' is a bundle over M with fibres $V \otimes V'$ and transition functions $t_{ij}(p) \otimes t'_{ij}(p)$.

Proposition 1.33. This is indeed a vector bundle.

Proof. We will follow the fibre bundle construction theorem. Initially we have bundles

$$E = \bigsqcup_i U_i \times V / \sim$$
$$E' = \bigsqcup_i U_i \times V' / \sim'$$

where $\sim$ and $\sim'$ are due to transition functions t_{ij} and t'_{ij} respectively. The projections are $\pi([(p,v)]) = p$ and $\pi'([(p,v')]) = p$, and define local trivialisations $\phi_i : (p, v_i) \mapsto [(p,v)]$ and $\phi'_i : (p_i, v') \mapsto [(p,v')]$ over U_i. Remember that defining a local trivialisation is a choice of representative of the equivalence classes which constitute the bundle. From these bundles form the tensor product bundle

$$E \otimes E' = \bigsqcup_i U_i \times (V \otimes V') / \sim''$$

where $\sim''$ is due to transition functions T_{ij}. Its elements are equivalence classes

$$[(p, v \otimes v')] = \{(p, w \otimes w') \mid w \otimes w' = T_{ji}(p)(v \otimes v')\}$$

where, by hypothesis, $T_{ji}(p) = t_{ji}(p) \otimes t'_{ji}(p)$. The projection $(\pi \otimes \pi') : E \otimes E' \to M$ maps $[(p, v \otimes v')] \mapsto p$. Now, given local trivialisations on the constituent bundles, we can define a local trivialisation $\psi_i : U \times (V \otimes V') \to (\pi \otimes \pi')^{-1}(U_i)$ over U_i which picks out precisely the representative $(p, v_i \otimes v'_i)$, i.e.

$$\psi_i(p, v_i \otimes v'_i) = [(p, v \otimes v')]$$

Then, as required,

$$(\pi \otimes \pi') \circ \psi_i(p, v_i \otimes v'_i) = \pi([(p, v \otimes v')])$$

This, along with $T_{ij}(p)$ then fixes the rest of our local trivialisations, by requiring

$$\psi_j(p, v_j \otimes v'_j) = \psi_i(p, v_i \otimes v'_i)$$

In fact, this construction is well defined precisely because $T_{ij}(p) = t_{ij}(p) \otimes t'_{ij}(p)$, i.e. since we have

$$\begin{aligned} T_{ij}(p)(v_i \otimes v'_i) &= (t_{ij}(p) \otimes t'_{ij}(p))(v_i \otimes v'_i) \\ &= t_{ij}(p)v_i \otimes t'_{ij}(p)v'_i \\ &= v_j \otimes v'_j \end{aligned}$$

We can then happily start with any local trivialisation ψ_i and construct the others consistently. Thus we have made $E \otimes E'$ into a fibre bundle, and of course not only that but a vector bundle. □

Remark. The construction of the new local trivialisations here is just $\psi = \phi \otimes \phi'$, where

Definition 1.34. The tensor product $\otimes : E \times E' \to E \otimes E'$ is defined fibrewise by

$$[(p, v)] \otimes [(p, v')] = [(p, v \otimes v')]$$

Proposition 1.35. Given three bundles E, E', E'', the tensor product, $E \otimes E' \otimes E''$ is well-defined, i.e.

$$(E \otimes E') \otimes E'' = E \otimes (E' \otimes E'')$$

(up to canonical isomorphisms).

Proof. Again, this follows from the same statement about fibres. □

Corollary 1.36. *Given a collection of bundles E_i,*

$$\bigotimes_i E_i$$

is well-defined.

Definition 1.37. Given $s \in \Gamma(M, E)$ and $s' \in \Gamma(M, E')$, we define $s \otimes s' \in \Gamma(M, E \otimes E')$ by

$$(s \otimes s')(p) = s(p) \otimes s'(p)$$

for all p on which s and s' are defined.

Remark. Although this is clearly well-defined, we cannot go the other way. That is, there is always ambiguity in decomposing $t \in \Gamma(M, E \otimes E')$ into s and s'. To see this, use local frames $\{e_i\}$ and $\{e'_j\}$, and write $t = t^{ij} e_i \otimes e'_j$ and $s = s^i e_i$, $s' = s'^j e'_j$. Then clearly $t = s \otimes s'$ for non-unique choices of s^i and s'^j. For instance, I can double all the s^i and halve all the s'^j without affecting t.

1.2.3 Vector Bundle Algebra

Definition 1.38. A vector bundle $\pi : E \to M$ generates a **tensor algebra**

$$T(E) = \bigoplus_{k=0}^{\infty} \bigotimes^{k} E$$

under the tensor product

$$(u_1 \otimes ... \otimes u_k) \otimes (u_{k+1} \otimes ... \otimes u_{k+l}) = u_1 \otimes ... \otimes u_k \otimes u_{k+1} \otimes ... \otimes u_{k+l}$$

Definition 1.39. Given a vector bundle E, let the symmetric group S_k act on $\bigotimes^k E$ by

$$u_1 \otimes ... \otimes u_k \mapsto u_{\sigma(1)} \otimes ... \otimes u_{\sigma(k)}$$

Then the k^{th} **exterior power** of E is the bundle

$$\bigwedge^{k} E = \{a \in \bigotimes^{k} E \mid \sigma : a \mapsto \operatorname{sgn}(\sigma) a \ \forall \ \sigma \in S_k\}$$

E generates the **exterior algebra**

$$\bigwedge(E) = \bigoplus_{k=0}^{\infty} \bigwedge^{k} E$$

with the **wedge product** $\wedge$, which is the fully antisymmetrised tensor product.

Definition 1.40. Sections $s \in \Gamma(M, E)$ and $\lambda \in \Gamma(M, E^*)$ naturally define a smooth function $\lambda(s) \in C^\infty(M)$ defined by

$$\lambda(s) : p \mapsto \lambda(p)(s(p))$$

In dual local frames $\{e_i\}$ and $\{e^i\}$, this function is just

$$\lambda(s) : p \mapsto \lambda_i(p) s^i(p)$$

Proposition 1.41. Such a function $\lambda(s)$ is $C^\infty(M)$-linear in λ and s.

Proof. Let $f, g \in C^\infty(M)$, $\lambda, \mu \in \Gamma(M, E^*)$ and $s, t \in \Gamma(M, E)$. Firstly,

$$\begin{aligned}((f\lambda + g\mu)(s))(p) &= (f\lambda + g\mu)(p)s(p) \\ &= f(p)\lambda(p)s(p) + g(p)\mu(p)s(p) \\ &= (f(\lambda(s)))(p) + (g(\mu(s)))(p)\end{aligned}$$

Secondly,

$$\begin{aligned}(\lambda(fs + gt))(p) &= \lambda(p)(fs + gt)(p) \\ &= \lambda(p)f(p)s(p) + \lambda(p)g(p)t(p) \\ &= (\lambda(fs))(p) + (\lambda(gt))(p)\end{aligned}$$

□

Definition 1.42. Let E be a vector bundle. Then denote by $\mathrm{End}(E)$ a bundle whose sections define fibrewise linear functions from E to itself. That is, if $S \in \Gamma(M, \mathrm{End}(E))$, then

$$S(p) : [(p, v)] \mapsto [(p, s(p)v)]$$

for an endomorphism $s(p) \in \mathrm{End}(V_p)$. But since $\mathrm{End}(V) \cong V \otimes V^*$, we can identify $\mathrm{End}(E)$ as the tensor product bundle $E \otimes E^*$. Then in a local frame $\{e_i\}$ for E, a section S of $\mathrm{End}(E)$ can be written $S^i_j e_i \otimes e^j$.

Definition 1.43. Given $s \in \Gamma(M, E)$ and $T \in \Gamma(M, \mathrm{End}(E))$, we define $Ts \in \Gamma(M, E)$ by

$$(Ts)(p) = T(p)s(p)$$

for all p on which s and T are defined. With a local frame $\{e_i\}$ on E, this new section is

$$Ts = T^i_j s^j e_i$$

Similarly, given $S, T \in \Gamma(M, \mathrm{End}(E))$, define $ST \in \Gamma(M, \mathrm{End}(E))$ by

$$(ST)(p) = S(p)T(p)$$

Locally,

$$ST = S^i_k T^k_j e_i \otimes e^j$$

Proposition 1.44. These are $C^\infty(M)$-linear in S, T and s.

Proof. This is straightforward and much the same as the similar proof for $\lambda(s)$ earlier. $\square$

Definition 1.45. A **differential form** is a section of the bundle $\bigwedge(T^*M)$. A **vector-valued differential form** is a section of a bundle $E \otimes \bigwedge(T^*M)$, where E is a vector bundle. We say that such a form takes values in V, the typical fibre of E. It can be decomposed (though not necessarily uniquely) as $s \otimes \omega$, where $s \in \Gamma(M, E)$ and ω is a differential form. (On T^*M see the appendix.)

Definition 1.46. We can make the following extensions to the wedge product:

(i) $\wedge : \Gamma(M, E \otimes \bigwedge(T^*M)) \times \Gamma(M, \bigwedge(T^*M)) \to \Gamma(M, E \otimes \bigwedge(T^*M))$ by

$$(s \otimes \omega) \wedge \mu = s \otimes (\omega \wedge \mu)$$

(ii) $\wedge : \Gamma(M, \mathrm{End}(E) \otimes \bigwedge(T^*M)) \times \Gamma(M, E \otimes \bigwedge(T^*M)) \to \Gamma(M, E \otimes \bigwedge(T^*M))$ by

$$(T \otimes \omega) \wedge (s \otimes \mu) = (Ts) \otimes (\omega \wedge \mu)$$

(iii) $\wedge : \Gamma(M, \mathrm{End}(E) \otimes \bigwedge(T^*M)) \times \Gamma(M, \mathrm{End}(E) \otimes \bigwedge(T^*M)) \to \Gamma(M, \mathrm{End}(E) \otimes \bigwedge(T^*M))$ by

$$(S \otimes \omega) \wedge (T \otimes \mu) = (ST) \otimes (\omega \wedge \mu)$$

Proposition 1.47. These all have the appropriate $C^\infty(M)$-linearity.

Proof. Again, this is straightforward and similar to other proofs. $\square$

Definition 1.48. Let $\omega \in \Gamma(M, \mathrm{End}(E) \otimes \bigwedge^p T^*M)$ and $\mu \in \Gamma(M, \mathrm{End}(E) \otimes \bigwedge^q T^*M)$. Then their **graded commutator** is

$$[\omega, \mu] = \omega \wedge \mu - (-1)^{pq} \mu \wedge \omega$$

Proposition 1.49. The graded commutator satisfies

(i) graded antisymmetry,

$$[\omega, \mu] = -(-1)^{pq} [\mu, \omega]$$

(ii) the graded Jacobi identity

$$[\omega, [\mu, \eta]] + (-1)^{p(q+r)}[\mu, [\eta, \omega]] + (-1)^{r(p+q)}[\eta, [\omega, \mu]] = 0$$

where additionally $\eta \in \Gamma(M, \mathrm{End}(E) \otimes \bigwedge^r T^*M)$.

Proof.

(i) We have

$$\begin{aligned} -(-1)^{pq}[\mu, \omega] &= -(-1)^{pq}\mu \wedge \omega - (-1)^{2pq+1}\omega \wedge \mu \\ &= \omega \wedge \mu - (-1)^{pq}\mu \wedge \omega \\ &= [\omega, \mu] \end{aligned}$$

(ii) Consider

$$\begin{aligned} [\omega, [\mu, \eta]] &= [\omega, \mu \wedge \eta - (-1)^{qr}\eta \wedge \mu] \\ &= \omega \wedge \mu \wedge \eta - (-1)^{p(q+r)}\mu \wedge \eta \wedge \omega - (-1)^{qr}\omega \wedge \eta \wedge \mu + (-1)^{qr+p(q+r)}\eta \wedge \mu \wedge \omega \end{aligned}$$

Then similarly

$$\begin{aligned} (-1)^{p(q+r)}[\mu, [\eta, \omega]] &= (-1)^{p(q+r)}(\mu \wedge \eta \wedge \omega - (-1)^{q(r+p)}\eta \wedge \omega \wedge \mu \\ &\quad - (-1)^{rp}\mu \wedge \omega \wedge \eta + (-1)^{rp+q(r+p)}\omega \wedge \eta \wedge \mu) \\ &= (-1)^{p(q+r)}\mu \wedge \eta \wedge \omega - (-1)^{r(p+q)}\eta \wedge \omega \wedge \mu \\ &\quad - (-1)^{pq}\mu \wedge \omega \wedge \eta + (-1)^{qr}\omega \wedge \eta \wedge \mu \end{aligned}$$

So

$$\begin{aligned} [\omega, [\mu, \eta]] + (-1)^{p(q+r)}[\mu, [\eta, \omega]] &= \omega \wedge \mu \wedge \eta - (-1)^{pq}\mu \wedge \omega \wedge \eta \\ &\quad - (-1)^{r(p+q)}\eta \wedge \omega \wedge \mu + (-1)^{qr+p(q+r)}\eta \wedge \mu \wedge \omega \end{aligned}$$

Also,

$$\begin{aligned} (-1)^{r(p+q)}[\eta, [\omega, \mu]] &= (-1)^{r(p+q)}(\eta \wedge \omega \wedge \mu - (-1)^{r(p+q)}\omega \wedge \mu \wedge \eta \\ &\quad - (-1)^{pq}\eta \wedge \mu \wedge \omega + (-1)^{pq+r(p+q)}\mu \wedge \omega \wedge \eta) \\ &= (-1)^{r(p+q)}\eta \wedge \omega \wedge \mu - \omega \wedge \mu \wedge \eta \\ &\quad - (-1)^{r(p+q)+pq}\eta \wedge \mu \wedge \omega + (-1)^{pq}\mu \wedge \omega \wedge \eta \end{aligned}$$

Combining everything, we get the result.

□

1.3 Principal Bundles

1.3.1 Definitions

Definition 1.50. A **principal fibre bundle** is a fibre bundle with total space P whose typical fibre is just its structure group G. We denote such a principal G-bundle over M $P(M, G)$.

Definition 1.51. Given a local neighbourhood U_i of M with local trivialisation ϕ_i over U_i, we define a right-action of G on itself by maps $R_a : \pi^{-1}(U_i) \to \pi^{-1}(U_i)$ for each $a \in G$ defined by

$$R_a(u) = ua = \phi_i(p, g_i a)$$

where

$$u = \phi_i(p, g_i)$$

Proposition 1.52. This is well-defined.

Proof. Given two local trivialisations ϕ_i and ϕ_j over overlapping neighbourhoods, we have

$$\begin{aligned} ua &= \phi_i(p, g_i a) \\ &= \phi_j(p, t_{ji}(p) g_i a) \\ &= \phi_j(p, g_j a) \\ &= ua \end{aligned}$$

where $g_j = t_{ji}(p) g_i$ by definition. That is, R_a is compatible with the way different local trivialisations choose representatives of equivalence classes, since left-action commutes with right-action. $\square$

Corollary 1.53. *This allows us to extend R_a to all of P for all $a \in G$.*

Proposition 1.54. This right-action is transitive and free.

Proof. Given any $g, g' \in G$, there is always some $a \in G$ such that $ga = g'$, since the usual right-multiplication of groups is transitive. Similarly, if for all $g \in G$, $ga = g$, then $a = e$. These properties of right-multiplication of groups also hold here, since each fibre $\pi^{-1}(p)$ is isomorphic to the group itself. $\square$

Corollary 1.55. *Given any $u \in \pi^{-1}(p)$, the whole fibre can be reconstructed as*

$$\pi^{-1}(p) = \{ua \mid a \in G\}$$

Definition 1.56. Given a local section s_i over U_i, there is a **canonical local trivialisation** $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ defined by

$$\phi_i(p, e) = s_i(p)$$

for all $p \in U_i$. For any $u \in \pi^{-1}(\{p\})$ there exists (since the right-action is transitive) a unique (since it is free) element $g_u \in G$ such that $u = s_i(p)g_u$, allowing us to extend ϕ by

$$\phi_i(p, g) = \phi_i(p, e)g = s_i(p)g$$

Proposition 1.57. This is well-defined, and given a collection of local sections $\{s_i\}$ transition functions are defined by

$$s_j(p) = s_i(p)t_{ij}(p)$$

Proof. Firstly,

$$\pi \circ \phi_i(p, g) = \pi \circ (s_i(p)g) = p$$

right-multiplication by g operates purely in the fibre. Next, we have

$$\begin{aligned} s_j(p) &= \phi_j(p, e) \\ &= \phi_i(p, t_{ij}(p)e) \\ &= \phi_i(p, t)t_{ij}(p) \\ &= s_i(p)t_{ij}(p) \end{aligned}$$

$\square$

Proposition 1.58. A local trivialisation $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ induces a local section s_i over U_i.

Proof. Given a local trivialisation, we can define a local section with regard to which it is the canonical local trivialisation just from the definition of that concept:

$$s_i(p) = \phi_i(p, e)$$

$\square$

Corollary 1.59. *In this way local trivialisations and local sections have a natural one-to-one correspondence, which proves the following theorem:*

Theorem 1.60. *A principal bundle is trivial iff it admits a global section.*

1.3.2 Associated Bundles

Definition 1.61. Let $P(M, G)$ be a principal bundle, and F a manifold on which G has a faithful left-action which we just denote $g : f \mapsto gf$ for $f \in F$ and $g \in G$. Then define an action on $P \times F$ by

$$g : (u, f) \mapsto (ug, g^{-1}f)$$

In this way we define $E(P) = (P \times F)/G$, called the **associated fibre bundle**. In the case that $F = V$ is a vector space, use the notation $E = P \times_\rho V = (P \times V)/G$, where the action of G on V is that of the representation ρ, is called an **associated vector bundle**.

Proposition 1.62. *E is a well-defined bundle over M with typical fibre F, projection $\pi_E : [(u, f)] \mapsto \pi(u)$, and transition functions t_{ij}, the same as those of P.*

Proof. First observe that the projection π_E is well-defined, since $\pi(ug) = \pi(u)$. Now, fix some $p \in M$. The fibre above p, $\pi_E^{-1}(\{p\})$, consists of equivalence classes $[(u, f)]$ such that $\pi(u) = p$. To show that this fibre is isomorphic to F, we just need to establish a one-to-one correspondence between the two. To do this, make an arbitrary choice $u_0 \in \pi^{-1}(\{p\})$. Then given any $[(u, f)]$, there exists an $f' \in F$ such that $[(u, f)] = [(u_0 g, f)] = [(u_0, gf)] = [(u_0, f')]$, which moreover is unique, since the action of G on P is free (making g unique given u), and on F is faithful (making f' unique given g). On the other hand, given any $f' \in F$, there is a unique equivalence class $[(u_0, f')]$. Thus our one-to-one correspondence is established, and hence the fibres $\pi_E^{-1}(\{p\}) \cong F$. Now, if local trivialisations on P satisfy

$$\phi_j(p, g) = \phi_i(p, t_{ij}(p)g)$$

then by hypothesis our local trivialisations Φ_i on E should satisfy

$$\Phi_j(p, f) = \Phi_i(p, t_{ij}(p)f)$$

We just need to check that such local trivialisations exist. Suppose we have used local sections s_i and s_j to define ϕ_i, ϕ_j and t_{ij} on P. Then define $\Phi_i : U_i \times F \to E$ by

$$\Phi_i(p, f) = [(s_i(p), f)]$$

Then we have

$$\begin{aligned}\Phi_j(p, f) &= [(s_j(p), f)] \\ &= [(s_i(p)t_{ij}(p), f)] \\ &= [(s_i(p), t_{ij}(p)f)] \\ &= \Phi_i(p, t_{ij}(p)f)\end{aligned}$$

as desired. It just remains to check that

$$\begin{aligned}\pi_E \circ \Phi_i(p, f) &= \pi_E([(s_i(p), f)]) \\ &= \pi(s_i(p)) \\ &= p\end{aligned}$$

So Φ_i really is a local trivialisation, and we are done. □

Proposition 1.63. If $\pi : E \to M$ is a vector bundle with fibre V, it is the associated vector bundle with fibre V of a principal bundle $P(E) = P(M, GL(V))$.

Proof. We know that both E and $E(P(M, GL(V)))$ have the same fibre V and base space. All we really need to do is ask if it is possible that E have the same transition functions as $P(M, GL(V))$. But we have already seen that the transition functions of a vector bundle are mapped by the appropriate representation to $GL(V)$, so this certainly is possible. □

Proposition 1.64. Any vector bundle E admits at least one global section (the null section). However, we have that E is trivial iff $P(E)$ admits a global section.

Proof. $P(E)$ admits a global section iff it is trivial iff all of its transition functions may be set to the identity. But these are the transition functions of E as well, so this is iff all of E's transition functions may be set to the identity, iff E is trivial. □

Definition 1.65. A local section of an associated vector bundle $E(P)$ s may be written with the aid of a local section σ of the principal bundle P:

$$s(p) = [(\sigma(p), v(p))]$$

We can then make $\Gamma(M, E(P))$ into a $C^\infty(M)$- linear vector space by defining

$$f(p)[(\sigma(p), v(p))] + g(p)[(\sigma(p), w(p))] = [(\sigma(p), f(p)v(p) + g(p)w(p))]$$

Proposition 1.66. This is indeed well-defined and allows addition of any two sections of $\Gamma(M, E(P))$.

Proof. Although it naively looks from the definition like we can only add sections with the same P part, we can in fact have

$$\begin{aligned}[(\sigma(p), v(p))] + [(\tau(p), w(p))] &= [(\sigma(p), v(p))] + [(\sigma(p)g(p), w(p))] \\ &= [(\sigma(p), v(p))] + [(\sigma(p), g(p)w(p))] \\ &= [(\sigma(p), v(p) + g(p)w(p))]\end{aligned}$$

for some $g : M \to G$. $\square$

Proposition 1.67. Any non-vanishing section $s \in \Gamma(M, E(P))$ may be written with a constant V part.

Proof. Suppose

$$s(p) = [(\sigma(p), v(p))]$$

Then choose some fixed $v \in V$, and define $g(p)$ by $g(p)v = v(p)$. This is possible as long as $v(p)$ is never zero. Then we have

$$s(p) = [(\sigma(p)g(p), v)]$$

$\square$

Corollary 1.68. *Given a local section σ of P above U and a basis $\{\tilde{e}_i\}$ of V, we have a local basis of sections of $E(P)$ above U given by*

$$e_i(p) = [(\sigma(p), \tilde{e}_i)]$$

2 Connections on Fibre Bundles

A Riemannian Geometry from the Tangent Bundle

Remark. Here we will apply some of the above theory to the tangent bundle TM and related bundles T^*M, $\bigwedge(T^*M)$, and so on.

Definition A.1. The **tangent bundle** TM of a manifold M is

$$TM = \bigcup_{p \in M} T_pM$$

Proposition A.2. The tangent bundle is a vector bundle.

Proof. First we need to show that TM is a manifold. Let (U, φ) be a local chart on M, defining coordinates $\varphi(p) = x^\mu$, where $\mu = 1, ..., \dim M = m$. φ also induces a basis $\{\partial_\mu\}$ for all T_pM with $p \in U$. We can then define a homeomorphism $TU_i \mapsto \mathbb{R}^{2m}$ by

$$(p, v) \mapsto (x^\mu, v^\mu)$$

In this way we can build an atlas on TM to $\mathbb{R}^{2m}$, making TM a manifold. Now, to regard it as a vector bundle, obviously the base space is M and the fibre $\mathbb{R}^m$. The projection locally is $\pi : (p, v) \mapsto p$ and local trivialisations are canonically induced by charts, i.e. $\phi : U_i \times \mathbb{R}^m \mapsto \pi^{-1}(U_i)$ can be defined by

$$\phi : (p, v) \mapsto (p, v^\mu)$$

Now, if we change coordinate systems to y^ν

$$\tilde{v}^\mu = \frac{\partial y^\mu}{\partial x^\nu} v^\nu$$

The only restriction is that $(\partial y^\mu / \partial x^\nu)^m_{\mu,\nu=1}$ be non-singular. That is, the structure group is $GL(\mathbb{R}^m)$, and transition functions are just these Jacobians, which clearly satisfy the transition function requirements. Thus TM is a vector bundle. $\square$

Remark. Note that we could also have started from the base space M and required the typical fibre to be $\mathbb{R}^m$ and transition functions to be coordinate transformations in $GL(\mathbb{R}^m)$, using the fibre bundle construction theorem to find TM, the projection and local trivialisations.

Definition A.3. Sections of the tangent bundle are called **vector fields**.

Definition A.4. Given the tangent bundle TM of a manifold M, we immediately have the **cotangent bundle** T^*M, as well as its exterior powers, $\bigwedge^k T^*M$. Sections of $\bigwedge^k T^*M$ are called **differential** k**-forms**.

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