Polchinski - String Theory

Selected Exercises

1 A First Look at Strings

Exercise 1.3
We have already seen that the variation of the Gauss-Bonnet action term

$$\chi_1 = \frac{1}{4\pi}\int_M d\tau d\sigma\ \sqrt{-\gamma}R$$

under Weyl transformations $\gamma_{ab} \to \exp(2\omega)\gamma_{ab}$ is

$$\begin{aligned}\delta\chi_1 &= -\frac{1}{2\pi}\int_M d\tau\ d\sigma\ \sqrt{-\gamma}\nabla^2\omega \\ &= -\frac{1}{2\pi}\int_M d\tau\ d\sigma\ \partial_a(\sqrt{-\gamma}\partial^a\omega) \\ &= -\frac{1}{2\pi}\int_{\partial M} ds\ n_a\partial^a\omega\end{aligned}$$

where we have used Stokes' theorem to get the last line, and ds is the induced 'volume form' on $\partial\Sigma$. We want a second term χ_2 whose variation cancels this. We consider

$$\chi_2 = \frac{1}{2\pi}\int_{\partial M} ds\ k$$

where

$$k = \pm t^a n_b \nabla_a t^b$$

Here t^a and n^a are unit vectors which are tangent and normal, respectively, to ∂M, and the $\pm$ means Lorentzian (Euclidean) worldsheet signature. Now, since $ds = \sqrt{-\gamma_{\tau\tau}}d\tau$, under Weyl transformations

$$ds \to \sqrt{-e^{2\omega}\gamma_{\tau\tau}}d\tau = e^{\omega}ds$$

If v^a is a unit vector, we have

$$\gamma_{ab}v^a v^b \to e^{2\omega}\gamma_{ab}v'^a v'^b = \pm 1$$

so $v^a \to e^{-\omega} v^a$. So $t^a \to e^{-\omega} t^a$, and $n^a \to e^{-\omega} n^a$, so $n_a \to e^{\omega} n_a$. These two will cancel. Finally we want to consider $\nabla_a t^b$. We will have

$$\begin{aligned}\nabla_a t^b &\to \nabla'_a t'^b \\ &= \partial_a (e^{-\omega} t^b) + \Gamma'^b_{ac} e^{-\omega} t^c \\ &= e^{-\omega} \left(-(\partial_a \omega) t^b + (\partial_a t^b + \Gamma'^b_{ac} t^c) \right)\end{aligned}$$

where Γ' are the new connection coefficients,

$$\begin{aligned}\Gamma'^b_{ac} &= \frac{1}{2} \gamma'^{bd} (\partial_a \gamma'_{cd} + \partial_c \gamma'_{ad} - \partial_d \gamma'_{ac}) \\ &= \frac{1}{2} e^{-2\omega} \gamma^{bd} (\partial_a (e^{2\omega} \gamma_{cd}) + \partial_c (e^{2\omega} \gamma_{ad}) - \partial_d (e^{2\omega} \gamma_{ac})) \\ &= \frac{1}{2} \gamma^{bd} (2(\partial_a \omega) \gamma_{cd} + \partial_a \gamma_{cd} + 2(\partial_c \omega) \gamma_{ad} + \partial_c \gamma_{ad} - 2(\partial_d) \gamma_{ac} - \partial_d \gamma_{ac}) \\ &= \Gamma^b_{ac} + \partial_a \omega \delta^b_c + \partial_c \omega \delta^b_a - (\partial_d \omega) \gamma^{bd} \gamma_{ac}\end{aligned}$$

That is,

$$\begin{aligned}\nabla_a t^b &\to e^{-\omega} \left(-(\partial_a \omega) t^b + \nabla_a t^b + (\partial_a \omega \delta^b_c + \partial_c \omega \delta^b_a - (\partial_d \omega) \gamma^{bd} \gamma_{ac}) t^c \right) \\ &= e^{-\omega} \left(\nabla_a t^b + (\partial_c \omega) t^c \delta^b_a - (\partial_c \omega) \gamma^{bc} t_a \right)\end{aligned}$$

So, we finally have

$$k ds \to \pm t^a n_b \left(\nabla_a t^b + (\partial_c \omega) t^c \delta^b_a - (\partial_c \omega) \gamma^{bc} t_a \right) ds$$

That is,

$$\begin{aligned}\delta \chi_2 &= \pm \frac{1}{2\pi} \int_{\partial M} t^a n_b (\partial_c \omega)(t^c \delta^b_a - \gamma^{bc} t_a) ds \\ &= \pm \frac{1}{2\pi} \int_{\partial M} (\partial_c \omega)(t^b n_b t^c - t^a t_a n^c) ds \\ &= \frac{1}{2\pi} \int_{\partial M} n^a \partial_a \omega\end{aligned}$$

where in the last line we have used that t^a and n^a are orthogonal, and that t^a is a unit vector, which on a timelike curve ∂M means $t^a t_a = -1$ with Lorentzian signature and $+1$ with Euclidean, cancelling the $\pm$. This is indeed $-\partial \chi_1$, so the full Gauss-Bonnet term

$$\chi = \frac{1}{4\pi} \int_M d\tau \ d\sigma \ \sqrt{-\gamma} R + \frac{1}{2\pi} \int_{\partial M} ds \ k$$

is Weyl-invariant.

Exercise 1.4

(i) At the first massive level, $m^2 = 1/\alpha'$, $N = 2$, we have states of the form

$$\alpha^i_{-2} \left|0\right\rangle , \quad \alpha^i_{-1}\alpha^j_{-1} \left|0\right\rangle$$

The first of these is just an $SO(D-2)$ vector representation. The second is a rank-2 tensor representation, which is symmetric since $[\alpha^i_{-1}, \alpha^j_{-1}] = 0$. (Note that it can be decomposed into a traceless symmetric and (trace) singlet to get the irreducible content). Now we want to package all this into some $SO(D-1)$ representation. The obvious choice is the symmetric traceless. Suppose M is a matrix in this representation. Then we can write

$$M = \begin{pmatrix} N & v \\ v^T & c \end{pmatrix}$$

where N is a symmetric rank-2 $SO(D-2)$ matrix and $c = -\operatorname{tr} N$ is not independent. Therefore the symmetric rank-2 $SO(D-1)$ breaks to a symmetric rank-2 $SO(D-1)$ and a vector $SO(D-1)$, as desired.

(ii) At the second massive level, $m^2 = 2/\alpha'$, $N = 3$, we have states of the form

$$\alpha^i_{-3} \left|0\right\rangle , \quad \alpha^i_{-2}\alpha^j_{-1} \left|0\right\rangle , \quad \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1} \left|0\right\rangle$$

That is, the $SO(D-2)$ representation content is a vector, rank-2 tensor and fully symmetric rank-3 tensor. We can decompose the rank-2 tensor into symmetric traceless, antisymmetric and trace (singlet) parts, and the rank-3 tensor becomes an irreducible fully symmetric rank-3 tensor and a vector, by writing

$$\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1} = \begin{cases} \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1} & i \neq j \neq k \\ (\sum \alpha^i_{-1}\alpha^i_{-1})\alpha^j_{-1} \end{cases}$$

Now we try to find this from some $SO(D-1)$ content. First consider the irreducible fully symmetric rank-3 $SO(D-1)$ tensor representation. Denote an object transforming under this $\psi_{(abc)}$, where a, b, c are $SO(D-1)$ indices (and i, j, k are $SO(D-2)$ indices). Breaking to $SO(D-2)$, we have three pieces:

$$\begin{gathered} \psi_{(ijk)} \\ \psi_{(ij)a} = \psi_{(i)a(j)} = \psi_{a(ij)} \\ \psi_{iaa} = \psi_{aia} = \psi_{aai} \end{gathered}$$

where now a is the fixed $(D-1)$th index. So in terms of $SO(D-2)$ we have a fully symmetric rank-3 tensor, a symmetric rank-2 tensor, and a vector. This just leaves the antisymmetric rank-2 tensor and another vector. It's not hard to guess that these will come from an $SO(D-1)$ antisymmetric rank-2 tensor. Indeed, schematically

$$M = \begin{pmatrix} N & v \\ v^T & 0 \end{pmatrix}$$

where N is in the antisymmetric rank-2 representation of $SO(D-2)$ and v is an $SO(D-2)$ vector. Thus the $SO(D-1)$ content is a fully symmetric irreducible rank-3 tensor representation and an antisymmetric rank-2 tensor representation.

Exercise 1.7

We consider open strings with ND boundary conditions:

$$X^{25}(\tau,0)=0, \quad \partial_\sigma X^{25}(\tau,l)=0$$

The zero mode will have to vanish due to the Dirichlet boundary condition, and we will need mode numbers to be half integers. We claim the mode expansion is of the form

$$X^{25}(\tau,\sigma)=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^{25}\exp\left\{-\frac{i\pi nc\tau}{l}\right\}\sin\frac{\pi n\sigma}{l}$$

It is easy to verify that this satisfies the boundary conditions: every term in $X^{25}(\tau,0)$ is proportional to $\sin 0=0$, and every term in $\partial_\sigma X^{25}(\tau,l)$ is proportional to $\cos\pi n$, which since $n\in\mathbb{Z}+\frac{1}{2}$ also vanishes. We note that as usual we have the reality condition $\alpha_{-n}^{25}=(\alpha_n^{25})^\dagger$. Now, the canonical momentum conjugate to X^{25} is

$$\begin{aligned}
\Pi^{25}&=\frac{p^+}{l}\partial_\tau X^{25}\\
&=\frac{p^+}{l}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\left(-\frac{i\pi c}{l}\right)\alpha_n^{25}\exp\left\{-\frac{i\pi nc\tau}{l}\right\}\sin\frac{\pi n\sigma}{l}\\
&=\frac{i}{\sqrt{2\alpha'}l}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_n^{25}\exp\left\{-\frac{i\pi nc\tau}{l}\right\}\sin\frac{\pi n\sigma}{l}
\end{aligned}$$

We then have the canonical relationship

$$\begin{aligned}
[X^{25}(\tau,\sigma),\Pi^{25}(\tau,\sigma')]&=-\frac{i}{l}\sum_{n,n'\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\}\sin\frac{\pi n\sigma}{l}\sin\frac{\pi n'\sigma'}{l}[\alpha_n^{25},\alpha_{n'}^{25}]\\
i\delta(\sigma-\sigma')&=-\frac{i}{l}\sum_{n,n'\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\}\sin\frac{\pi n\sigma}{l}\sin\frac{\pi n'\sigma'}{l}[\alpha_n^{25},\alpha_{n'}^{25}]
\end{aligned}$$

Let $m=n+n'$, and write this as

$$\delta(\sigma-\sigma')=-\frac{1}{l}\sum_{m\in\mathbb{Z}}\exp\left\{-\frac{i\pi mc\tau}{l}\right\}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\sin\frac{\pi n\sigma}{l}\sin\frac{\pi(m-n)\sigma'}{l}[\alpha_n^{25},\alpha_{m-n}^{25}]$$

Now, there is no τ-dependence on the LHS, so the coefficient of the τ term on the RHS must vanish for each $m \neq 0$. So we can write

$$\delta(\sigma-\sigma')\delta_{m,0} = \frac{1}{l}\sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\sin\frac{\pi n\sigma}{l}\sin\frac{\pi(n-m)\sigma'}{l}[\alpha_n^{25},\alpha_{m-n}^{25}]$$

Multiply through by $\sin\pi n'\sigma/l$

$$\sin\frac{\pi n'\sigma}{l}\delta(\sigma-\sigma')\delta_{m,0} = \frac{1}{2l}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\left(\cos\frac{\pi(n'-n)\sigma}{l} - \cos\frac{\pi(n'+n)\sigma}{l}\right)\sin\frac{\pi(n-m)\sigma'}{l}[\alpha_n^{25},\alpha_{m-n}^{25}]$$

and integrate over σ:

$$\sin\frac{\pi n'\sigma'}{l}\delta_{m,0} = \frac{1}{2}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\left(\frac{1}{\pi(n'-n)}\sin\big(\pi(n'-n)\big) - \frac{1}{\pi(n'+n)}\sin\big(\pi(n'+n)\big)\right)$$
$$\sin\frac{\pi(n-m)\sigma'}{l}[\alpha_n^{25},\alpha_{m-n}^{25}]$$

Now, $n'\pm n\in\mathbb{Z}$, so $s\in(\pi(n'\pm n))=0$, and everything vanishes unless the denominators also go to zero. Then, using $\mathrm{sinc}(0)=1$, we have

$$\sin\frac{\pi n'\sigma'}{l}\delta_{m,0} = \frac{1}{2n'}\left(\sin\frac{\pi(n'-m)\sigma'}{l}[\alpha_{n'}^{25},\alpha_{m-n'}^{25}] - \sin\frac{\pi(n'+m)\sigma'}{l}[\alpha_{-n'}^{25},\alpha_{m+n'}^{25}]\right)$$

From this we see that

$$[\alpha_n^{25},\alpha_{m-n}^{25}] = n\delta_{m,0}$$

i.e.

$$[\alpha_n^{25},\alpha_m^{25}] = n\delta_{n+m,0}$$

as in the usual NN case. Now, recall that the Hamiltonian in Equation 1.3.19 is

$$H = \frac{l}{4\pi\alpha' p^+}\int_0^l d\sigma\left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right)$$

where i runs over all the transverse directions. Denote the contribution due to the 25$^{\text{th}}$ direction H^{25}. We have

$$(\Pi^{25})^2 = -\frac{1}{2\alpha' l^2}\sum_{n,n'\in\mathbb{Z}+\frac{1}{2}}\alpha_n^{25}\alpha_{n'}^{25}\exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\}\sin\frac{\pi n'\sigma}{l}\sin\frac{\pi n'\sigma}{l}$$

and

$$\partial_\sigma X^{25} = \frac{\pi}{l}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_n^{25}\alpha_{n'}^{25}\exp\left\{-\frac{i\pi nc\tau}{l}\right\}\cos\frac{\pi n\sigma}{l}$$

so

$$(\partial_\sigma X^{25})^2 = \frac{2\pi^2\alpha'}{l^2} \sum_{n,n'\in\mathbb{Z}+\frac{1}{2}} \alpha_n^{25}\alpha_{n'}^{25} \exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\} \cos\frac{\pi n\sigma}{l}\cos\frac{\pi n'\sigma}{l}$$

Therefore we have

$$\begin{aligned}
H^{25} &= \frac{l}{4\pi\alpha' p^+} \sum_{n,n'\in\mathbb{Z}+\frac{1}{2}} \alpha_n^{25}\alpha_{n'}^{25} \exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\} \frac{\pi}{l^2}\int_0^l d\sigma \left(-\sin\frac{\pi n\sigma}{l}\sin\frac{\pi n'\sigma}{l} + \cos\frac{\pi n\sigma}{l}\cos\frac{\pi n'\sigma}{l}\right) \\
&= \frac{1}{4\alpha' p^+ l} \sum_{n,n'\in\mathbb{Z}+\frac{1}{2}} \alpha_n^{25}\alpha_{n'}^{25} \exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\} \\
&\qquad \frac{1}{2}\int_0^l d\sigma \left(-\cos\frac{\pi(n-n')\sigma}{l} + \cos\frac{\pi(n+n')\sigma}{l} + \cos\frac{\pi(n-n')\sigma}{l} + \cos\frac{\pi(n+n')\sigma}{l}\right) \\
&= \frac{1}{4\alpha' p^+ l} \sum_{n,n'\in\mathbb{Z}+\frac{1}{2}} \alpha_n^{25}\alpha_{n'}^{25} \exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\} \int_0^l d\sigma\ \cos\frac{\pi(n+n')\sigma}{l} \\
&= \frac{1}{4\alpha' p^+ l} \sum_{n,n'\in\mathbb{Z}+\frac{1}{2}} \alpha_n^{25}\alpha_{n'}^{25} \exp\left\{-\frac{i\pi(n+n')c\tau}{l}\right\} \left[\frac{l}{\pi(n+n')}\sin\frac{\pi(n+n')\sigma}{l}\right]_0^l \\
&= \frac{1}{4\alpha' p^+} \sum_{n,\in\mathbb{Z}+\frac{1}{2}} \alpha_n^{25}\alpha_{-n}^{25} \\
&= \frac{1}{4\alpha' p^+} \sum_{n=\frac{1}{2}}^{\infty} \left(\alpha_n^{25}\alpha_{-n}^{25} + \alpha_{-n}^{25}\alpha_n^{25}\right) \\
&= \frac{1}{2\alpha' p^+} \sum_{n=\frac{1}{2}}^{\infty} \left(\alpha_{-n}^{25}\alpha_n^{25} + \frac{n}{2}\right)
\end{aligned}$$

where we have used the sinc observation of earlier again. Now we use

$$\begin{aligned}
\sum_{n=\frac{1}{2}}^{\infty} n = \sum_{n=1}^{\infty}\left(n - \frac{1}{2}\right) &= \frac{1}{24} - \frac{1}{8}(1-1)^2 \\
&= \frac{1}{24}
\end{aligned}$$

to get

$$H^{25} = \frac{1}{2\alpha' p^+}\left(\sum_{n=\frac{1}{2}}^{\infty} \alpha_{-n}^{25}\alpha_n^{25} + \frac{1}{48}\right)$$

To incorporate this into the full H we need to redefine N

$$N = \sum_{i=2}^{24} \sum_{n=1}^{\infty} n N_{in} + \sum_{n=\frac{1}{2}}^{\infty} n N_{25,n}$$

where $N_{\mu n}$ counts the number of level n oscillators in the μ direction in a state, and note that we have a contribution $1/48$ instead of the usual $-1/24$ to the normal ordering constant. So the mass spectrum is given by

$$\begin{aligned} m^2 &= \frac{1}{\alpha'} \left(N - \frac{23}{24} + \frac{1}{48} \right) \\ &= \frac{1}{\alpha'} \left(N - \frac{15}{16} \right) \end{aligned}$$

We can explore the low lying states. Firstly, the ground state, $N = 0$, is still a tachyon, with

$$m^2 = -\frac{15}{16\alpha'}$$

Denote it $|0\rangle$. The first excited state, $N = \frac{1}{2}$, comes only from excitations of the 25^{th} direction, i.e. $N_{25,1/2} = 1$, or $\alpha^{25}_{-1/2} |0\rangle$. This state has mass

$$m^2 = -\frac{7}{16\alpha'}$$

so is still tachyonic. Both states so far are obviously scalars of $SO(23)$ (Lorentz symmetry has been broken from $SO(24)$ due to the D-brane). At $N = 1$ we have two possibilities, $N_{i1} = 1$, so $\alpha^i_{-1} |0\rangle$, or $N_{25,1/2} = 2$, the state $\alpha^{25}_{-1/2}\alpha^{25}_{-1/2} |0\rangle$. This is an $SO(23)$ vector and scalar respectively, both with the same mass

$$m^2 = \frac{1}{16\alpha'}$$

Notice that there are then no massless states in the spectrum. At $N = \frac{3}{2}$, we can have $N_{25,3/2} = 1$, $N_{25,1/2} = 3$, or $N_{25,1/2} = 1$ but $N_{i1} = 1$, corresponding to states $\alpha^{25}_{-3/2} |0\rangle$, $\alpha^{25}_{-1/2}\alpha^{25}_{-1/2}\alpha^{25}_{-1/2} |0\rangle$ and $\alpha^i_{-1}\alpha^{25}_{-1/2} |0\rangle$. These are two scalars and a vector, all with mass

$$m^2 = \frac{9}{16\alpha'}$$

etc.

Exercise 1.8 We consider closed strings on $\mathbb{R}^{25} \times S^1$, satisfying $X^{25} \sim X^{25} + 2\pi R$. Introduction of this periodicity means two things. Firstly, $p^{25} = q/R$ is quantised, $q \in \mathbb{Z}$.

Secondly, we can relax the $X^{25}(\tau, \sigma + l) = X^{25}(\tau, \sigma)$ periodicity to hold modulo integer multiples of $2\pi R$. That is,

$$X^{25}(\tau, \sigma + l) = X^{25}(\tau, \sigma) + 2\pi w R$$

for $w \in \mathbb{Z}$. w is called the winding number. The general solution is the same as usual, but with $p^{25} = q/R$ and the winding term $2\pi w R \sigma / l$:

$$\begin{aligned} X^{25}(\tau, \sigma) = x^{25} + &\frac{q}{p^+ R}\tau + \frac{2\pi w R}{l}\sigma \\ &+ i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^{25}}{n} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} + \frac{\tilde{\alpha}_n^{25}}{n} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right) \end{aligned}$$

We have

$$\begin{aligned} \Pi^{25} &= \frac{p^+}{l} \partial_\tau X^{25} \\ &= \frac{q}{lR} + i\frac{p^+}{l}\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(-\frac{2\pi i c}{l} \right) \left(\alpha_n^{25} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} + \tilde{\alpha}_n^{25} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right) \\ &= \frac{q}{lR} + \frac{1}{l}\frac{1}{\sqrt{2\alpha'}} \sum_{n \neq 0} \left(\alpha_n^{25} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} + \tilde{\alpha}_n^{25} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right) \end{aligned}$$

and

$$\begin{aligned} \partial_\sigma X^{25} &= \frac{2\pi w R}{l} + i\sqrt{\frac{\alpha'}{2}} \left(-\frac{2\pi i}{l} \right) \sum_{n \neq 0} \left(\alpha_n^{25} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} - \tilde{\alpha}_n^{25} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right) \\ &= \frac{2\pi w R}{l} + \frac{\pi}{l}\sqrt{2\alpha'} \sum_{n \neq 0} \left(\alpha_n^{25} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} - \tilde{\alpha}_n^{25} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right) \end{aligned}$$

Thus, if we make different definitions for the zero mode oscillators,

$$\begin{aligned} \alpha_0^{25} &= \sqrt{\frac{\alpha'}{2}}\frac{q}{R} + \frac{wR}{\sqrt{2\alpha'}} \\ \tilde{\alpha}_0^{25} &= \sqrt{\frac{\alpha'}{2}}\frac{q}{R} - \frac{wR}{\sqrt{2\alpha'}} \end{aligned}$$

we have the usual

$$\begin{aligned} \Pi^{25} &= \frac{1}{l}\frac{1}{\sqrt{2\alpha'}} \sum_{n \in \mathbb{Z}} \left(\alpha_n^{25} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} + \tilde{\alpha}_n^{25} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right) \\ \partial_\sigma X^{25} &= \frac{\pi}{l}\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \left(\alpha_n^{25} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} - \tilde{\alpha}_n^{25} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right) \end{aligned}$$

The oscillator algebra is clearly unchanged by these shifts. The computation then largely proceeds as usual, with spectrum given by

$$m^2 = 2p^+ H - p^i p^i$$

However, we now want $i \neq 25$ here. This amounts to shifting m^2 by

$$\begin{aligned}\Delta m^2 &= \frac{1}{\alpha'}\left((\alpha_0^{25})^2 + (\tilde{\alpha}_0^{25})^2\right) \\ &= \frac{1}{\alpha'}\left(\alpha' \frac{q^2}{R^2} + \frac{w^2 R^2}{\alpha'}\right) \\ &= \frac{q^2}{R^2} + \frac{w^2 R^2}{\alpha'^2}\end{aligned}$$

i.e.

$$m^2 = \frac{q^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

The last term is just the usual mass contribution since nothing crucial has changed in the oscillator algebra, but note that N and $\tilde{N}$ are occupation numbers due to oscillators α_{-n}^i and $\tilde{\alpha}_{-1}^i$ where again $i \neq 25$. Before exploring this spectrum, we should check the constraint coming from σ-translations. Now, in the second line of Equation 1.4.10 the sum runs from 1 to infinity. If we go through the computation, this comes from a factor like

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \left(\alpha_n^i \alpha_{-n}^i - \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i\right) \\ &\quad = \sum_{n=1}^{\infty} \left(\alpha_n^i \alpha_{-n}^i + \alpha_{-n}^i \alpha_n^i - \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i - \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i\right) + \alpha_0^i \alpha_0^i - \tilde{\alpha}_0^i \tilde{\alpha}_0^i \\ &\quad = 2\left[\sum_{n=1}^{\infty} \left(\alpha_{-n}^i \alpha_n^i - \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i\right) + A - \tilde{A}\right] + \alpha_0^i \alpha_0^i - \tilde{\alpha}_0^i \tilde{\alpha}_0^i\end{aligned}$$

Now, in the standard case, the zero mode contributions cancel out. Here they do not in the 25^{th} direction. So, remembering that $A = \tilde{A}$ will still hold since nothing substantial has changed about the oscillator algebra, we modify the result (the third line of Equation 1.4.10) to get

$$P = -\frac{2\pi}{l}\left(\frac{1}{2}(\alpha_0^i \alpha_0^i - \tilde{\alpha}_0^{25} \tilde{\alpha}_0^{25}) + N - \tilde{N}\right)$$

where now N and $\tilde{N}$ are the $i \neq 25$ levels. Therefore our level matching condition is shifted by

$$\begin{aligned}\frac{1}{2}\left((\alpha_0^{25})^2 - (\tilde{\alpha}_0^{25})^2\right) &= \frac{1}{2} 4 \sqrt{\frac{\alpha'}{2}} \frac{q}{R} \frac{wR}{\sqrt{2\alpha'}} \\ &= qw\end{aligned}$$

That is, we require

$$N - \tilde{N} + qw = 0$$

Now we can explore the spectrum. Denote states $\left|N, \tilde{N}; q, w\right\rangle$. At level $N = \tilde{N} = 0$, we must have $q = 0$ and/or $w = 0$. That is, we have two infinite towers of particles, $|0,0;q,0\rangle$ and $|0,0;0,q\rangle$. These have masses

$$m^2_{0,0;q,0} = \frac{q^2}{R^2} - \frac{4}{\alpha'} \quad m^2_{0,0;0,w} = \frac{w^2 R^2}{\alpha'^2} - \frac{4}{\alpha'}$$

So the lowest lying q-state is tachyonic if $R^2 > \alpha'/4$, and the lowest lying w-state is tachyonic if $R^2 < 4\alpha'$. The next level we can consider is $N = 1$, $\tilde{N} = 0$, which forces $qw = -1$, i.e. $q = 1$ and $w = -1$, or vice versa. That is, $|1,0;1,-1\rangle$ and $|1,0;-1,1\rangle$. These are vectors of $SO(23)$ since they are produced by α^i_{-1}. They have masses

$$m^2_{1,0;1,-1} = m^2_{1,0;-1,1} = \frac{1}{R^2} + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}$$

Since these are vectors, we are interested in where they become massless. We see that this happens at $R^2 = \alpha'$. (Note: this is the first hint of T-duality, but we will not pursue this any further.) Lastly, it is worth checking $N = \tilde{N} = 1$, since this is where the graviton occurs in the usual non-compactified theory. We must have $qw = 0$, so like in the $N = \tilde{N} = 0$ case we have two infinite towers $|1,1;q,0\rangle$ and $|1,1;0,w\rangle$. We have masses

$$m^2_{1,1;q,0} = \frac{q^2}{R^2} \quad m^2_{1,1;0,w} = \frac{w^2 R^2}{\alpha'^2}$$

(Note, in another hint of T-duality, that if $R^2 = \alpha'$ these coincide up to $q \leftrightarrow w$.) Our massless spin-2 therefore comes from $|1,1;0,0\rangle$.

Exercise 1.9 We consider a twisted closed string in orbifold compactification,

$$X^{25}(\tau, \sigma + l) = -X^{25}(\tau, \sigma)$$

There can be no zero mode, and we will have to have half-odd-integer modes. We claim

$$X^{25}(\tau,\sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}+\frac{1}{2}} \left(\frac{\alpha^{25}_n}{n} \exp\left\{ -\frac{2\pi i n(\sigma + c\tau)}{l} \right\} + \frac{\tilde{\alpha}^{25}_n}{n} \exp\left\{ \frac{2\pi i n(\sigma - c\tau)}{l} \right\} \right)$$

Indeed, adding l to σ adds phases which are -1 precisely since n is half-odd-integer. Now, we will have the usual oscillator algebra (recall from Exercise 1.7 that having half-odd-integer modes does not affect this), and if we define N as in Exercise 1.7,

$$N = \sum_{i=2}^{24} \sum_{n=1}^{\infty} n N_{in} + \sum_{n=1/2}^{\infty} n N_{25,n}$$

then in the usual way we get

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} + A + \tilde{A})$$

and

$$P = -\frac{2\pi}{l}(N - \tilde{N} + A - \tilde{A})$$

Now, the normal ordering constants will be just those found in Exercise 1.7, since n was half-odd-integer there too. So

$$A = \tilde{A} = -\frac{15}{16}$$

whence

$$m^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - \frac{15}{8}\right)$$

and the level-matching condition is again

$$N = \tilde{N}$$

The ground state is tachyonic as usual, with mass

$$m^2 = -\frac{15}{4\alpha'}$$

At the first excited level, $N = \tilde{N} = \frac{1}{2}$, due to $N_{25,1/2} = 1$, we have a tachyonic scalar with mass

$$m^2 = -\frac{7}{4\alpha'}$$

At $N = \tilde{N} = 1$, we can have either $N_{i,1} = 1$ or $N_{25,1/2} = 2$, and $\tilde{N}_{i,1} = 1$ or $\tilde{N}_{25,1/2}$. We therefore have an $SO(23)$ rank-2 tensor, two vectors, and a scalar. These all have masses

$$m^2 = \frac{1}{4\alpha'}$$

In particular, there are no massless vectors or rank-2 tensors in the theory. (In fact this is clear from the equation for the spectrum, since $N + \tilde{N}$ can never be 15/8.)