ARTIFICIAL INTELLIGENCE & MACHINE LEARNING

Calculus & Optimization: Convexity, Taylor expansion, Optimization objectives

Session – 15

AIM OF THE SESSION

To familiarize students with the basic concept of Convexity, Taylor expansion, and Optimization.

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Explain Convexity, Taylor expansion, and Optimization.

LEARNING OUTCOMES

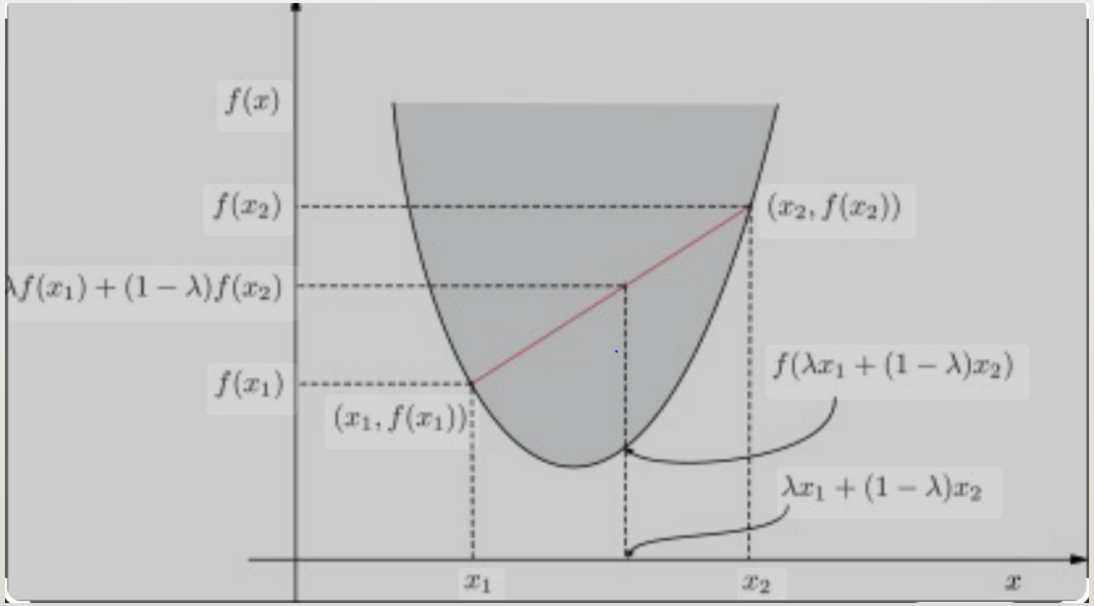
At the end of this session, you should be able to:

1. Define the concepts of Convexity, Taylor expansion, and Optimization.

Convexity

- A function $f: R^n \to \mathrm{R}$ is called **convex** if for all $x_1, x_2 \in$ domain of f and for all $\lambda \in [0,1]$, the following inequality holds:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$



Conditions for Convexity

❖ A function $f(x)$ with **one variable** is convex on an interval if: $\boldsymbol{f''(x) \geq 0}$ for all x in the interval

Example:

$$f(x) = x^2$$
$$f'(x) = 2x$$
$$f''(x) = 2 \geq 0$$

Therefore $f(x)$ is convex function.

❖ A function f(x,y) with **two variables** is convex if its **Hessian matrix H** is **positive semi-definite:**

i.e., $\boldsymbol{det(H) \geq 0}$ (or) The eigen values of H are positive. where $\boldsymbol{H = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} \end{bmatrix}}$

Example:

$$f(x) = x^2 + y^2, \ |H| = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \geq 0$$

Therefore $f(x)$ is convex function.

Taylor expansion

Definition: Taylor expansion is the representation of a function as an infinite sum of terms derived from its derivatives at a single point.
For a function f(x) that is infinitely differentiable at a point a, the Taylor series expansion about a is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots\ldots\ldots\ldots\ldots$$

(or)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

Purpose:
Machine learning: Optimization and gradient descent
Computer Science: Function approximation in algorithms

Taylor expansion

Example-1: Taylor expansion of ***sin(x)*** about a=0

$$f(x) = \sin(x), f'(x) = \cos(x), f''(x) = -\sin(x), \ldots\ldots\ldots\ldots\ldots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots\ldots$$

Approximation:

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots\ldots\ldots\ldots$$

Example-2: Taylor expansion of $\boldsymbol{e^x}$ about a=0

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \quad -\infty < x < \infty$$

Optimization

The Optimization Problem is: Find values of the variables that minimize or maximize the objective function while satisfying the constraints.

Optimization problems consist of three basic components.

1. **Objective function:** Which we want to minimize or maximize
2. **Variables:** A set of unknown values that affect the value of the objective functions.
3. **Constraints:** A set of conditions that allow the variables to take on certain values but exclude others.

Optimization Objectives

Example:
A factory produces two products: A and B.
- Profit per unit A: $3
- Profit per unit B: $5

Each product requires labor and material:
  - A: 1 hour labor, 2 units material
  - B: 2 hours labor, 1 unit material

The factory has:
8 hours of labor available
6 units of material available
How many units of A and B should be produced to **maximize profit**?

Optimization Objectives

Formulation

Let:

x = number of units of product A

y = number of units of product B

1. Objective Function (to maximize profit):

$$\max z = 3x + 5y$$

2. Variables: x, y

3. Subject to Constraints:

Labor: $x + 2y \leq 8$

Material: $2x + y \leq 6$

Non-negativity: $x \geq 0, y \geq 0$

Reference Books:

1. Russel and Norvig, ‘Artificial Intelligence’, third edition, Pearson Education, PHI, (2015)

2. Elaine Rich & Kevin Knight, ‘Artificial Intelligence’, 3nd Edition, Tata Mc Graw Hill Edition, Reprint(2008)

Sites and Web links:

1. https://www.virtusa.com/digital-themes/heuristic-search-techniques
2. https://towardsdatascience.com/a-star-a-search-algorithm-eb495fb156bb
3. https://www.tutorialandexample.com/local-search-algorithms-and-optimization-problem/
4. http://cgi.di.uoa.gr/~ys02/siteAI2008/local-search-2spp.pdf
5. http://aima.eecs.berkeley.edu/slides-pdf/chapter04b.pdf