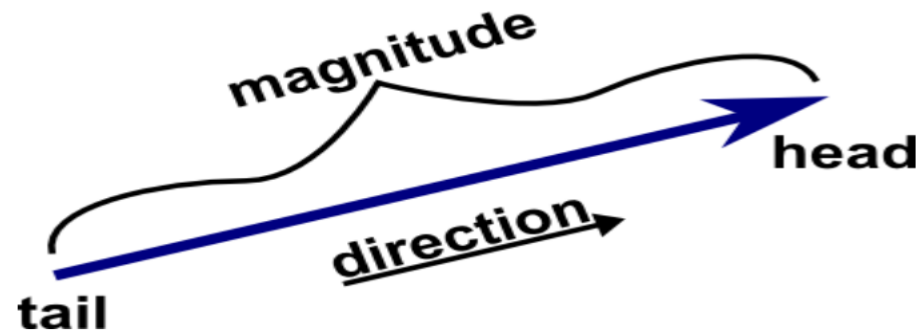


ARTIFICIAL INTELLIGENCE & MACHINE LEARNING

SESSION NO :13

Definition of a vector

A vector is an object that has both a magnitude and a direction. Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector is from its tail to its head.



Vector operations

Addition $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

Subtraction $(a_1, b_1) - (a_2, b_2) = (a_1 - a_2, b_1 - b_2)$

Scalar multiplication $k \cdot (a, b) = (k \cdot a, k \cdot b)$

Matrices

- Rectangular display of vectors in rows and columns
- Can inform about the same vector intensity at different times or different voxels at the same time
- Vector is just a $n \times 1$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix}$$

Square (3 x 3)

$$\mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 8 \end{bmatrix}$$

Rectangular (3 x 2)

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

d_{ij} : i^{th} row, j^{th} column

Defined as rows x columns (R x C)

Matrix Operations

- **Matrix Operations** are basic calculations performed on matrices to solve problems or manipulate their structure. Common operations include:
 - **Addition:** Add two matrices of the same size.
 - **Subtraction:** Subtract two matrices of the same size.
 - **Scalar Multiplication:** Multiply each element of a matrix by a constant.
 - **Matrix Multiplication:** Multiply two matrices to create a new matrix.
 - **Transpose:** Flip the rows and columns of a matrix.
 - **Inverse:** Find the inverse of a Matrix

Properties of Matrix Addition

There are various properties associated with matrix addition that are, for matrices A, B, and C of the same order, then,

- **Commutative Law:** $A + B = B + A$

- **Associative Law:** $(A + B) + C = A + (B + C)$

- **Identity of Matrix:** $A + O = O + A = A$, where O is a zero matrix, which is the Additive Identity of the Matrix

- **Additive Inverse:** $A + (-A) = O = (-A) + A$, where (-A) is obtained by changing the sign of every element of A, which is the additive inverse of the matrix

Matrix Operations

- Matrix operations mainly include four basic algebraic operations namely, the addition of matrices, subtraction of matrices, and multiplication of matrices and division of matrices. We all know that Matrix is an array of numbers or expressions arranged in rows (horizontal array) and columns (vertical array).
- The order of the matrix is the number of rows and columns that a given matrix is having or in other words, the order can be defined as the dimension of the matrix. In any matrix, the number of rows is listed first and is followed by the number of columns. Thus, if any matrix has the order (or dimension) as 3×4 , it means that it has 3 rows and 4 columns.

Matrix Addition

- Addition of Matrices If two matrices $A[a_{ij}]_{m \times n}$ and $B[b_{ij}]_{m \times n}$ are of the same order then their sum $A + B$ is a matrix of the order $m \times n$. Each element of the resultant matrix is the sum of the corresponding elements of the matrices A and B, i.e., $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Properties of Matrix Addition

- Properties of Matrix Addition: If A, B, and C are three given matrices of same the order, then,
- Commutative Law: Matrix addition is commutative, $A + B = B + A$.
- Associative Law: Matrix addition is associative, $(A + B) + C = A + (B + C)$.
- Identity Matrix: Matrix addition follows identity rule, i.e., $O + A = A = A + O$, where O is zero matrix which is additive identity of the matrix.
- Additive Inverse: $A + (-A) = 0 = (-A) + A$, where (-A) is obtained by changing the sign of every element of A which is additive inverse of the matrix.

Subtraction of Matrices

- two matrices $A[a_{ij}]_{m \times n}$ and $B[b_{ij}]_{m \times n}$ are of the same order then their difference $A - B$ is defined as a matrix of order $m \times n$ and is found by adding a negative of the corresponding elements of the second matrix to the first one. $A - B = A + (-B) = [a_{ij} - b_{ij}]_{m \times n}$

Multiplication of Matrices

- Multiplication of two matrices is defined only when the number of columns of the 1st matrix is equal to the number of rows of the 2nd matrix and the resultant or product will have the same number of rows as the 1st matrix, and the same number of columns as the 2nd matrix.

Properties of Matrix Multiplication

- Associative Law: Matrix multiplication follows associative law, i.e., for any three given matrices A, B, and C, $(AB)C = A(BC)$, whenever both sides of the multiplication or equality are defined.
- Distributive Law: Matrix multiplication follows distributive law for three matrices, i.e., if A, B, and C are three matrices then, $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$, whenever both sides of the equality are defined.
- Existence of Multiplicative Identity: Matrix multiplication follows multiplicative identity, i.e., for every square matrix A, there exists an identity matrix of the same order such that $IA = AI = A$.
- Properties of Scalar Multiplication of Matrices: If A and B are two matrices and m and n be any scalar: $m(A + B) = mA + mB$, $(m + n)A = mA + nA$, $m(nA) = mnA = n(mA)$

Division of Matrices

- Technically there is nothing like division of matrix, i.e., $A \div B$ is undefined instead we write $A \times B^{-1}$ i.e product of A and B-1.

Theory of Eigen Decomposition

- Eigen decomposition separates a matrix into its eigenvalues and eigenvectors. Mathematically, for a square matrix A , if there exists a scalar λ (eigenvalue) and a non-zero vector v (eigenvector) such that:

- $Av = \lambda v$

Where:

A is the matrix.

λ is the eigenvalue.

v is the eigenvector.

Perform Eigen decomposition

- Step 1: Find the Eigenvalues:
- Step 2: Find the Eigenvectors:
- Step 3: Construct the Eigenvector Matrix V:
- Step 4 Form the Diagonal Matrix Λ :
- Step 5: Calculate the Inverse of V:
- Step 6: Verify $A = V\Lambda V^{-1}$

Eigen decomposition (Cont...I)

Example: Let Matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Find eigenvalues and eigenvectors of A using Eigen decomposition.

Step 1: Find the Eigenvalues:

$$(A - \lambda I) = 0$$
$$\begin{vmatrix} 4 - \lambda & 1 - \lambda \\ 2 - \lambda & 3 - \lambda \end{vmatrix} = 0;$$

$$(4 - \lambda) * (3 - \lambda) - 2 = 0;$$

$$\lambda^2 - 7\lambda + 10 = 0;$$

Eigen values are $\lambda_1 = 5, \lambda_2 = 2$

Eigen decomposition (Cont...II)

Step 2: Find the Eigenvectors:

Eigen vector for $\lambda_1=5$:

$$(A - 5I)v = 0; \rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0; \rightarrow x = y;$$

Therefore Eigen vector for $\lambda_1=5$ is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen vector for $\lambda_2=2$:

$$(A - 2I)v = 0; \rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0; \rightarrow y = -2x,$$

There fore Eigen vector for $\lambda_2=2$ is $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Eigen decomposition (Cont...III)

Step 3: Construct the Eigenvector Matrix V:

$$\text{Eigen Matrix } V = [v_1 \quad v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Step 4 Form the Diagonal Matrix Λ :

$$\text{Diagonal Matrix } \Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Step 5: Calculate the Inverse of V:

$$\text{Inverse of V} = V^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Conclusion: $A = V\Lambda V^{-1}$

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} * \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} * \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Hence verified.

Importance of Eigen decomposition

Eigen decomposition is widely used because it makes complex tasks simpler:

- **Simplifying Matrix Powers:** It helps in easily calculating powers of matrices, which is useful in solving equations and modeling systems.
- **Data Simplification:** It is used in techniques like PCA to reduce large datasets into fewer dimensions, making them easier to analyze.
- **Image Processing:** It is used in tasks like image compression and enhancement, making handling images more efficient.

Singular Value Decomposition (SVD)

- Singular Value Decomposition (SVD) is a factorization method in linear algebra that decomposes a matrix into three other matrices, providing a way to represent data in terms of its singular values.
- SVD helps you split that table into three parts:
- **U**: This part tells you about the people (like their general preferences).
- **Σ** : This part shows how important each factor is (how much each rating matters).
- **V^T** : This part tells you about the products (how similar they are to each other)

Steps for Singular Value Decomposition (SVD)

Step1: Find the Eigen values of $A^T A$

Step2: Find the Eigen vectors of $A^T A$ & Right singular vectors (transpose of orthogonal matrix V)

Step3: Find the Left singular vectors (orthogonal matrix) $U = [u_1 \quad u_2]$

Step4: Find the Diagonal matrix with singular values (non-negative real numbers) **S**

Step5: Verify $A = USV^T$

Singular Value Decomposition (SVD) (Cont..I)

Example: Find the SVD of a matrix $A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$

Step1: Find the Eigen values of $A^T A$

$$A^T A = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} * \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 32 \\ 32 & 65 \end{bmatrix}$$

Eigen values of $A^T A = \begin{vmatrix} 17 - \lambda & 32 \\ 32 & 65 - \lambda \end{vmatrix} = 0$

Singular Value Decomposition (SVD)

(Cont..II)

Step2: Find the Eigen vectors of $A^T A$ & Right singular vectors (transpose of orthogonal matrix V)

Eigen vector for $\lambda_1 = 81$ is:
$$\begin{bmatrix} 17 - \lambda & 32 \\ 32 & 65 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} -64 & 32 \\ 32 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$2x_1 = x_2$$

Therefore Eigen vector for $\lambda_1 = 81$ is:
$$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigen vector for $\lambda_2 = 1$ is:

$$\begin{bmatrix} 17 - \lambda & 32 \\ 32 & 65 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 16 & 32 \\ 32 & 64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = -2x_2$$

Therefore Eigen vector for $\lambda_2 = 1$ is:
$$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Therefore the Eigen Matrix of $A^T A = V \Lambda V^T = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \Lambda \begin{bmatrix} v_1^T & v_2^T \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \Lambda \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Singular Value Decomposition (SVD)

(Cont..III)

Step3: Find the Left singular vectors (orthogonal matrix) $U = [u_1 \ u_2]$

We know that, $u_i = \frac{Av_i}{\sigma_i}$; where $\sigma_i = \sqrt{\lambda_i}$

$$u_1 = \frac{Av_1}{\sigma_1} = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} * \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} * \frac{1}{9} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} * \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} * \frac{1}{1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore the matrix= $U = [u_1 \ u_2] = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$

Step4: Find the Diagonal matrix with singular values (non-negative real numbers) $S = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$

Step5: Verify $A = USV^T \Rightarrow$ $\begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} * \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} * \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$

Applications of Singular Value Decomposition (SVD)

- SVD has some fascinating algebraic characteristics and conveys relevant geometrical and theoretical insights regarding linear transformations.
- SVD has some critical applications in data science too.
- Mathematical applications of the SVD involve calculating the matrix approximation, rank of a matrix and so on.
- The SVD is also greatly useful in science and engineering.
- It has some applications of statistics, for example, least-squares fitting of data and process control.

Thank You