AMATH 301 FinalB

Sanjeev Janarthanan

TOTAL POINTS

131.5 / 150

QUESTION 1

Question 1 16 pts

1.1 Part a 5 / 8

- **0 pts** Correct
- **0.5 pts** Minor syntax error.
- **1 pts** Syntax error.
- **1.5 pts** Data not loaded in.
- **1.5 pts** Incorrect calculation
- ✓ **- 3 pts** **Incorrect calculation.**
- **5 pts** Incorrect calculation.
- **8 pts** Blank/no credit

1.2 Part b 8 / 8

- ✓ **- 0 pts** **Correct**
- **0.5 pts** Minor syntax error.
- **1.5 pts** Syntax error.
- **1.5 pts** Incorrect calculation
- **3 pts** Incorrect calculation
- **6 pts** Incorrect calculation
- **8 pts** Blank/no credit

QUESTION 2

Question 2 18 pts

2.1 Part a 6 / 6

- ✓ **- 0 pts** **Correct**
- **1.5 pts** Minor problem with nature of fix.
- **1.5 pts** Minor problem with location of fix.
- **3 pts** Major problem with nature of fix.
- **3 pts** Major problem with location of fix.
- **6 pts** Blank/no credit

2.2 Part b 6 / 6

- ✓ **- 0 pts** **Correct**
- **1.5 pts** Minor problem with nature of fix.
- **1.5 pts** Minor problem with location of fix.
- **3 pts** Major problem with nature of fix.
- **3 pts** Major problem with location of fix.
- **6 pts** Blank/no credit

2.3 Part c 3 / 6

- **0 pts** Correct
- **1.5 pts** Minor problem in nature of fix.
- **1.5 pts** Minor problem in location of fix.
- ✓ **- 3 pts** **Major problem in nature of fix.**
- **3 pts** Major problem in location of fix.
- **6 pts** Blank/No credit

QUESTION 3

Question 3 16 pts

3.1 Part a 7 / 7

- ✓ **- 0 pts** **Correct**
- **1 pts** One error
- **3 pts** Indexing errors and/or missing boundary points
- **5 pts** Some code written, but incorrect algorithm and/or output.
- **7 pts** Not an answer

3.2 Part b 2 / 5

- **0 pts** Correct
- **1 pts** One error
- **2 pts** Indexing problems
- ✓ **- 3 pts** **Code uses wrong algorithm and/or will not run.**
- **5 pts** Not an answer

3.3 Part 4 2 / 4

- **0 pts** Correct
- **1 pts** Wrong output variable
- **2 pts** Arguments order in quad()
- ✓ **- 2 pts** **X bounds**
- **3 pts** Incorrect arguments
- **4 pts** No answer/no use of quad()

QUESTION 4

4 Question 4 15 / 15

✓ **- 0 pts Correct**

- 2 pts One mistake in update formula

- 3 pts For loop variable problem

- 5 pts RK4 algorithm incorrect.

- 7 pts RK4 correct, but not in runnable Matlab syntax.

- 10 pts Not a solution to the problem, but some code written.

- 14 pts Used built in function

- 15 pts Not an answer.

QUESTION 5

5 Question 5 4 / 10

- 0 pts Correct

- 1 pts Missed 'econ'

- 3 pts SVD command

✓ **- 3 pts Dimensions for all 3 V, S, and U.**

✓ **- 3 pts Rank 1 approximation line**

- 10 pts No answer

QUESTION 6

6 Question 6 18 / 18

✓ **- 0 pts Correct**

- 3 pts approximates

- 2 pts LET order

- 2 pts LET coefficient

- 2 pts order accuracy

- 18 pts no work or equivalent

- 2 pts error work

- 2 pts initial definitions for f(x+h)

- 1.5 pts taylor series computation for approx

QUESTION 7

7 Question 7 11 / 11

✓ **- 0 pts Correct**

- 4 pts Treated output (K and/or E) as a function

- 2 pts Did not specify answer (x)

- 1 pts Syntax error(s)

- 3 pts Answer included both first and second kind.

- 4 pts Found first kind instead of second

- 4.5 pts Found E correctly, then used something else for x

- 2 pts Used wrong tolerance

- 4 pts Used undefined variable tol

- 7 pts Used integral formula to find m

- 8 pts Did not define m

- 4 pts Used K and/or E as if they were builtin functions

- 11 pts Did not answer the question

QUESTION 8

Question 8 16 pts

8.1 Part a 8 / 8

✓ **- 0 pts Correct**

- 5 pts Substituted v = x', but did not successfully isolate v'

- 3 pts Found correct v' equation, then made an incorrect sytem.

- 2 pts Did not include equation x' = v (but did use this substitution elsewhere)

- 1 pts Correct equations, but still written with x'' or x' instead of v' or v

- 2 pts Used an additional variable/equation for x''

- 2 pts Set x' = constant (but used correct substitution elsewhere)

- 8 pts Did not substitute v = x' (completely wrong approach)

- 2 pts Did not rewrite initial conditions correctly

- 1 pts Correct initial conditions, but not written in terms of v.

- 8 pts Did not answer the question

- 0.5 pts Algebra error

8.2 Part b 7 / 8

- 0 pts Correct

- 0.5 pts Used stiff solver

- 1 pts Used an invalid solver (or no solver)

- 1 pts Did not use a function of the form @(t,v)

- 2 pts Wrong/missing function body

- 1 pts Missing x' = v in function (otherwise correct)

- 2 pts Did not define function in code (but did write equations)

- **1 pts** Used a vector instead of a function
- **1 pts** Wrong/missing time span
- **1 pts** Wrong/missing initial condition
- **1 pts** Wrong output syntax

✓ **- 1 pts Did not find x100 correctly**

- **0.5 pts** Minor syntax error
- **1 pts** Major syntax error
- **8 pts** Did not answer question

QUESTION 9

9 Question 9 15 / 15

✓ **- 0 pts Correct**

- **3 pts** Matrix A incorrect or undefined
- **2 pts** Vector b incorrect or undefined
- **2 pts** doesn't use [LUP]=lu()
- **2 pts** inorrect lu() input
- **2 pts** doesn't track P
- **2 pts** incorrect y=L\(P*b) or equivalent
- **2 pts** incorrect x=U\y or equivalent
- **1 pts** Single moderate syntax error or multiple small syntax errors
- **2 pts** Multiple moderate syntax errors
- **15 pts** no answer or equivalent

QUESTION 10

Question 10 15 pts

10.1 Part a 7.5 / 8

- **0 pts** Correct
- **3 pts** Only calculated one outer product by hand
- **4 pts** Computed inner products instead of outer
- **4 pts** Did not multiply correctly (dimensions make sense)
- **5 pts** Did not multiply correctly (dimensions don't make sense)
- **5 pts** Did not calculate by hand

✓ **- 0.5 pts Minor arithmetic error**

- **1 pts** Arithmetic error (several errors or no work shown)
- **0.5 pts** Missing * symbol in code
- **1 pts** Used .* with row/column
- **0.5 pts** Minor syntax error
- **1 pts** Major syntax error
- **2 pts** Completely wrong code
- **2 pts** No code
- **1 pts** Wrong rank / no rank
- **7 pts** Said the formula does not make sense
- **8 pts** Did not answer question

-1 + -3 = -4

10.2 Part b 7 / 7

✓ **- 0 pts Correct**

- **3 pts** Only calculated one inner product by hand
- **4 pts** Computed outer products instead of inner
- **4 pts** Multiplied incorrectly (dimensions make sense)
- **5 pts** Multiplied incorrectly (dimensions don't make sense)
- **5 pts** Did not calculate by hand
- **0.5 pts** Minor arithmetic error
- **1 pts** Arithmetic error (several errors or no work shown)
- **0.5 pts** Missing * symbol in code
- **1 pts** Used .* with row/column
- **0.5 pts** Minor syntax error
- **1 pts** Major syntax error
- **2 pts** Completely incorrect code
- **2 pts** No code
- **6 pts** Said the formula does not make sense
- **7 pts** Did not answer question

AMATH 301 Final Exam

Name: Sanjeev Janarthanan

Student NetID: 0751196

Question	Points	Score
1	16	
2	18	
3	16	
4	15	
5	10	
6	18	
7	11	
8	16	
9	15	
10	15	
Total:	150	

Please write your name on every page you want us to grade.

Many of these questions ask you to write short segments of code. You should try your best to write functioning Matlab code. That means proper use of arithmetic symbols, parentheses, brackets, builtin function names, semicolons, etc.

Name: Sanjeev Danaraman
NetID: 075116

1. (16 points) For the rest of this problem, assume that you have a file called `data.mat` in the same directory as the rest of your code. This file contains two vectors `x` and `y` of the same length. We will try find parameters a, b and c such that the curve $y = ae^{bx}\sin(cx)$ best fits the data.

   (a) First, complete the following function that will compute the root-mean-square error between the curve $y = a\ln x + be^{cx}$ and the data given in `data.mat`. Note that `a`, `b` and `c` are arguments in the function but `x` and `y` are not.

```
function E = rms_error(a, b, c)
load('data.mat');
E = sqrt((a*log(x) + b*exp(c*x)) - y)^2)
```

```
end
```

   (b) Now write code to calculate the coefficients a, b and c that minimize the root-mean-square error from part (a). Use an initial guess of $a = 1$, $b = -1$ and $c = 0$. You should name your results `a`, `b` and `c`. You can assume that the function from part (a) is saved as `rms_error.m` in the same directory as this code and the file `data.mat`.

```
wrap = @(v)(rms_error(v(1), v(2), v(3)));
[results, fun] = fminsearch(wrap, [1 -1 0]);
a = results(1); b = results(2); c = results(3);
```

Name: Sanjeev Janarthanan
NetID: 2751196

2. (18 points) The following code is meant to implement the section search to find the maximum of the function $f(x) = \cos^2(x/3)$. It is easy to check that the maximum is at $x = 0$. This code is written in two different files, called `section.m` and `fun.m`. These files are saved in the same directory. The code has several errors, which we will fix in turn.

FileName: section.m

```
1    clear all; close all; clc;
2
3    a = -1; b = 2;
4    c = 0.54;
5    tolerance = 1e-8;
6
7    for k = 1:100
8        x = c*a + (1 - c)*b;
9        y = (1 - c)*a + c*b;
10
11       if fun(x) < fun(y)
12           b = y;
13       else
14           a = x;
15       if (b - a) < tolerance
16           break
17       end
18   end
```

Filename: fun.m

```
1    function y = fun(x)
2
3    y == cos(x/3)^2;
4
5    end
```

The numbers on the left are just line numbers, not code. For each of the following parts (starting on the following page), clearly indicate any lines that you would need to change or add and what changes you would make. Be sure to indicate which file(s) need changes. Only fix the problem from the given part, not all problems that you see.

Name: Sanjeev Jayarthana
NetID: 0751196

(a) The code from the previous page does not work. When you run `section.m`, Matlab produces the following error message:

```
Error: File: section.m Line: 7 Column: 1
At least one END is missing: the statement may begin here.
```

What is causing this error, and what would you have to change in order to fix it?

After line 14 there is no end statement after the if and else statements. As such, I would add end after line 14.

(b) After correcting the above error, your code still does not work. When you run `section.m`, Matlab produces the following error message:

```
Undefined function or variable 'y'.

Error in fun (line 3)
y == cos(x/3)^2;

Error in section (line 11)
    if fun(x) < fun(y)
```

What is causing this error, and what would you have to change in order to fix it?

y == cos(x/3)^2 should be

y = cos(x/3)^2.

== doesn't assign value to a variable.

(c) After correcting both of the above errors, your code appears to run successfully and stops after 32 steps. However, the final value of x is extremely close to 2, which is not the correct maximum value. What is causing this problem, and what would you have to change in order to fix it?

On line 15, "if (b-a) < tolerance" should be "if abs(b-a) < tolerance".

Name: Sanjeev Janarthanan
NetID: 0751196

3. (16 points) Suppose that `x` is a 100×1 vector whose entries are strictly increasing and evenly spaced `dx` apart and suppose that `g` is a one-variable function. You can assume that `dx` is small. Throughout the problem, you can assume that `x`, `dx` and `g` are already defined, but you should define any other variables you need. You should not use any x-values except the ones in `x`.

(a) Write Matlab code to approximate the derivative g' at each point of `x`. Your answer should be a vector named `gprime`. Use a central difference scheme whenever possible. You can use any order of accuracy.

```
gprime = zeros(100, 1);
gprime(1) = (g(x(1) + dx) - g(x(1)))/dx;
for k = 2:99
    gprime(k) = (g(x(k) + dx) - 2g(x) + ...
                 g(x(k) - dx))/(2*dx);
end
gprime(end) = (g(x(end)) - g(x(end) - dx))/dx;
```

(b) Write Matlab code to approximate the integral of $g(x)$ (over the entire `x` interval) using the left hand rule. Name your answer `LHR`.

```
LHR = 0;
for k = 1:100
    LHR = LHR + g(x(k) + (k-1)*dx)*dx;
end
```

(c) Write a *single line* of Matlab code to approximate the integral of $g(x)$ (over the entire `x` interval) using the `quad` command. Name your answer `INT`.

```
INT = quad(g, 0, 100*dx);
```

Name: Sanjeev Janarthanan
NetID: 0751196

4. (15 points) Complete the following Matlab code to solve the initial value problem $\dot{x} = x + \cos(t)$ from $t = 0$ to $t = 8$ with a time step of $\Delta t = 0.5$ and the initial condition $x(0) = 1$ using the fourth order Runge-Kutta method. (You should write your own solver, not use a builtin function.) Notice that the first line of the for loop is not complete.

```
f = @(t,x)(x + cos(t));
dt = 0.5; t = 0:dt:8;
x = zeros(1, length(t)); x(1) = 1;

for k = 1:(length(t)-1)                    % <- Your code starts here
    tk = dt*k;
    xk = x(k);
    f1 = f(tk, xk);
    f2 = f(tk + dt/2, xk + (dt/2)*f1);
    f3 = f(tk + dt/2, xk + (dt/2)*f2);
    f4 = f(tk + dt, xk + dt*f3);
    x(k+1) = xk + (dt/6)*(f1 + 2*f2 + 2*f3 + f4);
end
```

5. (10 points) Suppose that A is a 8×9 matrix. Write a single line of Matlab code using the `svd` command to find the reduced singular value decomposition of A. (You can assume that `A` is already defined.) What are the dimensions of the matrices returned by `svd`? Write additional Matlab code to find a rank 1 approximation of A. (In other words, approximate A using only one outer product.)

```
[U, S, V] = svd(A, 'econ');
```

U, S, and V are 8 x 8 matrices

```
approx = U*(S^2)*U^-1;
```

Name: Sanjeev Janarthanan
NetID: 0751196

6. (18 points) Your friend just told you that the following difference scheme is an approximation for $f'(x)$:

$$f'(x) \approx \frac{f(x+2\Delta x) - f(x-2\Delta x)}{4\Delta x}$$

Confirm that this scheme really does approximate $f'(x)$ as Δx goes to zero. Find the leading error term and the order of accuracy of this method. (Write the order using big-oh notation.) Show your work!

Taylor series expansion of $f(x+2\Delta x)$:

$$f(x+2\Delta x) = f(x) + 2\Delta x \frac{df}{dx}(x) + \frac{4\Delta x^2}{2!}\cdot\frac{d^2f}{dx^2} - \frac{8\Delta x^3}{3!}\cdot\frac{d^3f}{dx^3}(x) \ldots$$

Taylor series expansion of $f(x-2\Delta x)$:

$$f(x-2\Delta x) = f(x) - 2\Delta x \frac{df}{dx}(x) + \frac{4\Delta x^2}{2!}\frac{d^2f}{dx^2} - \frac{8\Delta x^3}{3!}\cdot\frac{d^3f}{dx^3}(x)$$

As such,

$$f(x+2\Delta x) - f(x-2\Delta x) = 4\Delta x \frac{df}{dx}(x) + \frac{16\Delta x^3}{3!}\cdot\frac{d^3f}{dx^3}(x)$$

$$\frac{f(x+2\Delta x) - f(x-2\Delta x)}{4\Delta x} = \underbrace{\frac{df}{dx}(x)}_{\text{what we want}} + \underbrace{\frac{4\Delta x^2}{3!}\cdot\frac{d^3f}{dx^3}}_{\text{leading error term.}}$$

This shows that my friend's difference scheme does approximate $f'(x)$ and that the leading error term is $\frac{4\Delta x^2}{3!}\cdot\frac{d^3f}{dx^3}(x)$, which has $O(\Delta x^2)$.

Name: Sanjeev Jayarthanan
NetID: 0751196

7. (11 points) The complete elliptic integrals of the first kind ($K(m)$) and the second kine ($E(m)$) are defined as

$$K(m) = \int_0^1 \left[(1-t^2)(1-mt^2)\right]^{-1/2} \, dt \text{ and } E(m) = \int_0^1 \left(1-t^2\right)^{-1/2} \left(1-mt^2\right)^{1/2} \, dt.$$

Matlab has a builtin command to calculate these integrals. You probably haven't seen this function before, but with access to the help file you should be able to use it in Matlab code. Below is a portion of the help file for `ellipke`:

```
ellipke
Complete elliptic integrals of the first and second kind

Syntax
K = ellipke(M)
[K, E] = ellipke(M)
[K, E] = ellipke(M, tol)

Description
K = ellipke(M) returns the complete elliptic integral of the first kind
for each element in M.
[K, E] = ellipke(M) returns the complete elliptic integral of the first
kind and second kind.
[K, E] = ellipke(M, tol) computes the complete elliptic integral to accuracy
tol.  The default value of tol is eps.  Increase tol for a less accurate
but more quickly computed answer.
```

Use the function `ellipke` to find the complete elliptic integral of the second kind at $m = 0$, $m = 0.25$, $m = 0.5$, $m = 0.75$ and $m = 1$. Use the default tolerance. Your answers should be in a vector named `x`. You should not do any work by hand for this problem. You need to write Matlab code using `ellipke`.

```
x = zeros(5, 1);
[K, x(1)] = ellipke(0);
[K, x(2)] = ellipke(0.25);
[K, x(3)] = ellipke(0.5);
[K, x(4)] = ellipke(0.75);
[K, x(5)] = ellipke(1);
```

Name: Sanjeev Janarthanan
NetID: 075196

8. (16 points) Consider the second order initial value problem

$$\ddot{x} - 3(1 - x^2)\dot{x} + x = 0,$$

where $x(0) = -1$ and $\dot{x}(0) = 1$.

(a) Rewrite this equation (by hand) as a system of first order equations. Be sure to rewrite the initial conditions as well. Show your work!

$V = \dot{x}$

$\dot{V} - 3(1-x^2)V + x = 0$

$\dot{V} = 3(1-x^2)V - x$

$$\begin{cases} \dot{x} = V & V(0) = 1 \\ \dot{V} = 3(1-x^2)V - x & x(0) = -1 \end{cases}$$

(b) This equation is known to *not* be stiff. Write Matlab code using an appropriate builtin solver to solve this system of differential equations from $t = 0$ to $t = 100$. Find the value of x at time $t = 100$ and name it `x100`.

```
wrap = @(t, V) (V(2); 3*(1-V(1)^2)*V(2) - V(1)));
[t, f] = ode45(wrap, 0:1:100, [-1;1]);
x100 = f(1, end);
```

Name: Sanjeev Janarthanan
NetID: 0751196

9. (15 points) Consider the system of equations $A\mathbf{x} = \mathbf{b}$, where

$$A = \frac{1}{(\Delta t)^2}\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} f(\Delta t) - 1/(\Delta t)^2 \\ f(2\Delta t) \\ f(3\Delta t) \\ f(4\Delta t) + 2/(\Delta t)^2 \end{pmatrix},$$

$\Delta t = 0.1$ and $f(t) = e^t$.

Write Matlab code to solve this system using LU decomposition. You should explicitly keep track of the permutation matrix P and you should not assume that any Matlab variables have already been defined. Your final answer should be named $\mathbf{x}$.

```
dt = 0.1;
f = @(t)(exp(t));
A = diag([-2 -2 -2 -2]) + diag([1 1 1], 1) ...
                        + diag([1 1 1], -1);
A = A/(dt^2);
b(1) = f(dt) - (1/(dt^2));
b(2) = f(2*dt);
b(3) = f(3*dt);
b(4) = f(4*dt + (2/(dt^2)));

[L, U, P] = lu(A);
y = L\(P*b);
x = U\y;
```

Name: Sanjeev Janarthanan
NetID: 0751196

10. (15 points) Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Decide if each of the following expressions makes sense. If it does, calculate the result by hand and write a *single line* of Matlab code to compute the result. (You can assume that u, v, x and y have already been defined in Matlab.) If the result is a matrix, say what the rank of the matrix is.

(a) $\mathbf{uv}^T + \mathbf{xy}^T$

$$uv^T = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} * \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$xy^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} * \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & 4 & 2 \\ -3 & 6 & 3 \end{bmatrix}$$

$$uv^T + xy^T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 4 & 7 & 3 \end{bmatrix}$$

rank = 2

u * v.' + x * y.'

(b) $\mathbf{u}^T\mathbf{v} + \mathbf{x}^T\mathbf{y}$

$$u^T v = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 - 2 + 0 = -1$$

$$x^T y = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 + 4 + 3 = 6$$

$$u^T v + x^T y = 5$$

u.' * v + x.' * y