Homework 8

Problem 1: Accuracy and Stability

Consider the ordinary differential equation

$$\dot{x}(t) = a\sin(x), \quad \text{with} \quad x(0) = \frac{\pi}{4} \tag{1}$$

where a is a constant. If you are particularly good at trigonometry, you can check that the solution to this equation is

$$x(t) = 2\arctan\left(\frac{e^{at}}{1+\sqrt{2}}\right). \tag{2}$$

We will use this equation to explore the concepts of accuracy and stability for the forward and backward Euler methods. Throughout this problem, we will use $a = 5$.

It is always a good idea to plot your solutions alongside the true solution to check your work, but make sure that you comment out any plot commands before submitting to scorelator.

(a) Use the forward Euler method to solve equation (1) from $t = 0$ to $t = 1$ with a time step of $\Delta t = 0.1$. You should obtain a vector `x` with 11 entries such that $x_1 = x(0)$, $x_2 \approx x(0.1)$, $x_3 \approx x(0.2)$, $\ldots$, $x_{11} \approx x(1)$. Save the approximation you found for $x(1)$ in `A1.dat`. (That is, save the last x-value you found using forward Euler, not the value you get when you plug $t = 1$ into equation (2).)

With the exception of time 0, the solution you found is only an approximation. The error at time $t = (k-1)\Delta t$ is given by $|x_k - x((k-1)\Delta t)|$. Find the maximum such error and save it in `A2.dat`. (In other words, find the infinity norm of the error between your approximation and the true solution.)

(b) Repeat part (a) with a time step of $\Delta t = 0.01$. Save your approximation for $x(1)$ in `A3.dat` and save the maximum error in `A4.dat`.

Things to think about: Plot your approximations from parts (a) and (b) alongside the true solution. Do these solutions seem fairly accurate? Is that reflected in the maximum error you found? Experiment with other values of Δt.

For a challenge that is mostly unrelated to numerics, can you confirm that (2) is the solution to (1)? If you are familiar with separable differential equations, can you solve (1) directly?

(c) Divide your error from `A2.dat` by your error from `A4.dat` and save the result in `A5.dat`.

Things to think about: What does this ratio tell you about the order of accuracy for the forward Euler method? Can you confirm that the global error is $\mathcal{O}(\Delta t)$? Does this trend continue if you reduce Δt even more? What if you make Δt larger?

(d) Now use the forward Euler method to solve equation (1) from $t = 0$ to $t = 100$ with a time step of $\Delta t = 1$. Save the approximation you found for $x(100)$ in `A6.dat`. You should find that this is nowhere close to the true value of $x(100)$.

Things to think about: Plot your approximation alongside the true solution. (You should do this part even if you don't normally do the extra problems.) Does this look like a good approximation? What happens if you increase the final time? What happens if you change Δt? How does this relate to the concept of stability we discussed in class? Do you think that forward Euler is stable for this problem?

(e) Finally, use the backward Euler method to solve equation (1) from $t = 0$ to $t = 100$ with a time step of $\Delta t = 1$. At each step you will have to solve an equation of the form $x_{k+1} - a\sin(x_{k+1}) - x_k = 0$ for x_{k+1}. You should solve this equation using the `fzero` command with an initial guess of `x0 = 3`. Save the approximation you found for $x(100)$ in `A7.dat`.

Things to think about: Plot your approximation alongside the true solution. (You should do this part even if you don't normally do the extra problems.) Does this look like a good approximation? How does it compare to the forward Euler version? What happens if you increase the final time? What happens if you change Δt?

You should find that the backward Euler method captures the long term behavior of this equation very well, even with a fairly large Δt. How well does it capture the short term behavior? How does it compare to forward Euler with a smaller final time?

Problem 2: Second Order Differential Equation

Consider the second order differential equation

$$\ddot{x} + \eta\dot{x} - \frac{g}{L}x = 0, \text{ with } x(0) = 1 \text{ and } \dot{x}(0) = 0. \tag{3}$$

This equation models the angle x of a pendulum hanging nearly straight down. g is the acceleration due to gravity, L is the length of the pendulum and η is a friction coefficient.

If we let $v = \dot{x}$, then we can rewrite equation (3) as a system of two first order equations of the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}v \\ a_{21}x + a_{22}v \end{pmatrix}, \text{ with } \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4}$$

where the coefficients a_{ij} depend on g, L and η. Find these coefficients by hand.

For the rest of this problem, let $g = -9.8$, $L = 2$ and $\eta = 0.5$.

(a) Use the forward Euler method to solve equation (4) from $t = 0$ to $t = 10$ with a time step of $\Delta t = 0.05$. Save the approximation you found for $x(10)$ in `A8.dat`.

(b) Repeat part (a) using the backward Euler method. (Since equation (4) is linear, there is no need to use `fzero` in this case.) Save the approximation you found for $x(10)$ in `A9.dat`.

(c) Repeat part (a) using `ode45`. Save the approximation you found for $x(10)$ in `A10.dat`.

Things to think about: Plot x vs t for all three parts. If you assume that `ode45` gives the exact solution, how well do forward and backward Euler do? Are their errors related? Can you think of an easy way to improve these methods? (Hint: This is similar to how we derived the central difference scheme and the trapezoidal method.) Plot x vs v. This is called the phase plane for equation (4).

Experiment with other values of g, L and η. How well do forward/backward Euler do? How well does your new method do?

Try solving these systems with a larger final time. Are these numerical methods stable? What happens if you increase Δt?