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Search

# Heuristic Search

CSE 415: Introduction to Artificial Intelligence
University of Washington
Winter, 2020

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# Outline

Motivation, Definition

A* Algorithm

Admissibility and Consistency

Heuristics for the 8 Puzzle

Designing Heuristics

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# Motivation

- Blind search can waste time and space. (due to the combinatorial explosion.)
- Additional knowledge MAY be available -- how could it help?
  - E.g., finding a shortest route from Wash. U. to U. Wash. (St. Louis to Seattle).
  - Blind search considers all directions equally, including towards Wash. D.C.
  - Additional knowledge: we need to head northwest.

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# Starting from St. Louis--Blind

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# Starting from St. Louis--Informed

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# Definition

A *heuristic function* (or simply heuristic) is a function $h$: $\Sigma \rightarrow \Re$, that takes a state as its argument and returns a real number that is an estimate of the distance (or cost) from that state to the closest (or having lowest-cost path) goal state.

$$h(s) = r$$

The function h is typically based on *partial information* about the relationship between each state s and the closest goal state $\gamma$ to s.

For example, if each state has an $(x,y)$ location, then knowing only $x_s$ and $x_\gamma$, we could estimate the distance between s and g as $| x_s - x_\gamma |$.

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# A* Algorithm

Given a state space $\Sigma$ having a distance (or cost) function on moves (graph edges): $d(s_i, s_j)$, the A* algorithm searches for a shortest (lowest-cost) path from the initial state $s_0$ to a goal state $\gamma$.

The following algorithm gives the general control structure for A*. It omits a few details:

1. Back pointers for backtracing a path when a goal state is reached.
2. Details of computing g. (done in a manner similar to that in Dijkstra's algorithm, i.e., Uniform Cost Search).
3. Details of implementing the OPEN list and its methods for inserting, finding, and removing.

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# A* Algorithm

1. For the start state $s_0$, compute $f(s_0) = g(s_0)+h(s_0) = h(s_0)$ and put $[s_0,f(s_0)]$ on a list (priority queue) OPEN.
2. If OPEN is empty, output "DONE" and stop.
3. Find and remove the item [s,p] on OPEN having **lowest** p. Break ties arbitrarily

   Put [s,p] on CLOSED.

   If s is a goal state: output its description (and backtrace a path), and

   if h is known to be admissible, halt.
4. Generate the list L of [s',f(s')] pairs where the s' are the successors

   of s and their f values are computed using f(s') = g(s')+h(s').

   Consider each [s',f(s')].

   If there is already a pair [s', q] on CLOSED (for any value q):

   if f(s') > q, then remove [s,f(s')] from L.

   If $f(s') \leq q$, then remove [s',q] from CLOSED.

   Else if there is already a pair [s', q] on OPEN (for any value q):

   if f(s') > q, then remove [s,f(s')] from L.

   If $f(s') \leq q$, then remove [s',q] from OPEN.
5. Insert all members of L onto OPEN.
6. Go to Step 2.

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# A* Algorithm Behavior

During the search A* gives highest priority to that as-yet unexplored state (except in cases where some previously explored state needs to be re-examined) that has the lowest sum of distance from the initial state plus estimated distance to a goal.

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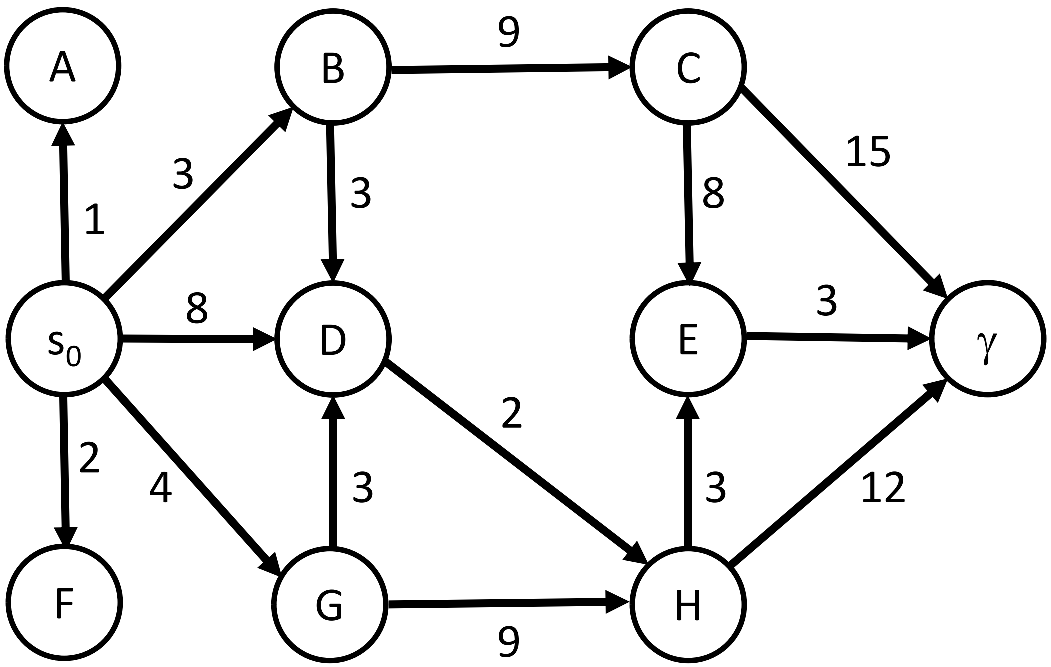
# Example




| state s: | $s_0$ | A | B | C | D | E | F | G | H | γ |
|---|---|---|---|---|---|---|---|---|---|---|
| heuristic h(s): | 14 | 15 | 4 | 10 | 3 | 2 | 16 | 10 | 5 | 0 |

We show newly enqueued [s,p] pairs in green, and updated [s,p] pairs in red.

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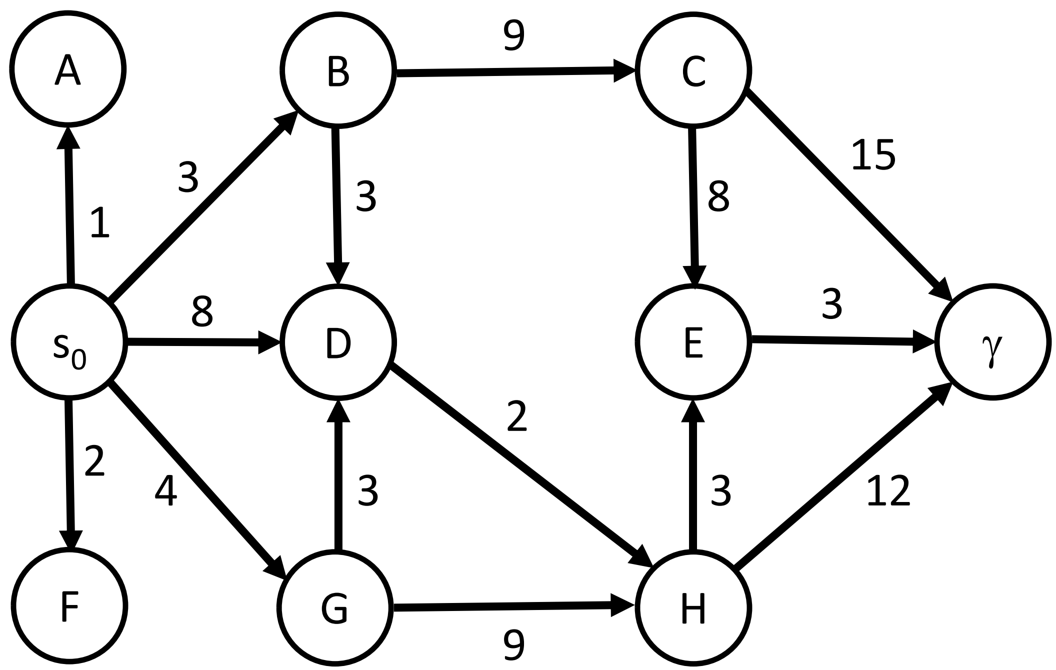
# Example

OPEN: [$s_0$,14]




CLOSED:

| state s: | $s_0$ | A | B | C | D | E | F | G | H | $\gamma$ |
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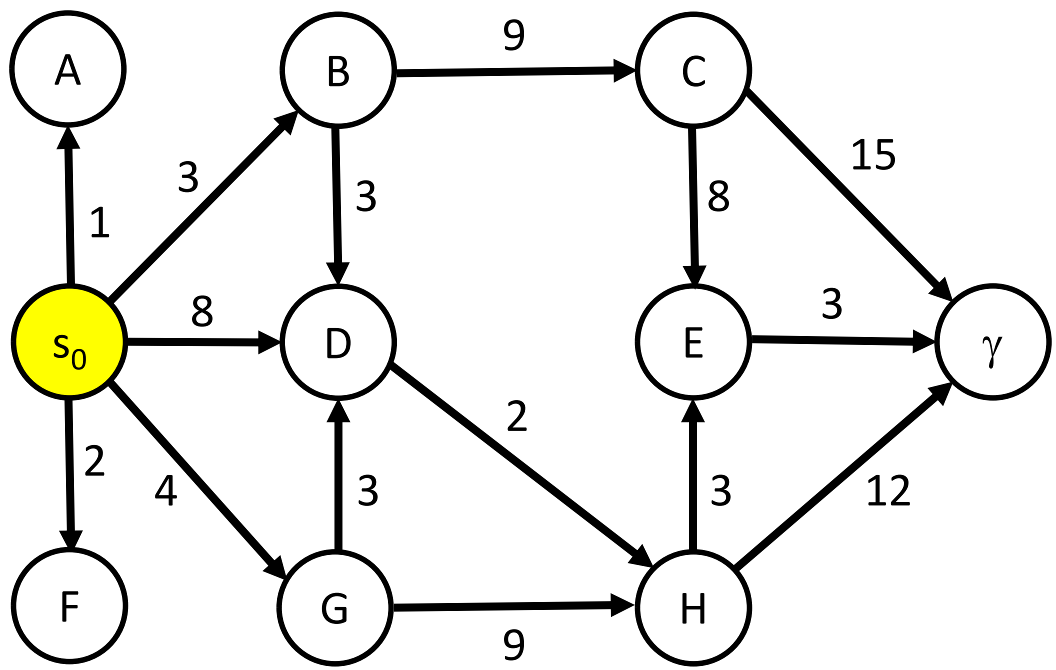


OPEN: ~~[$s_0$,14]~~
[B,7], [D,11], [G,14], [A,16], [F,18]

CLOSED: [$s_0$,14]

| state s: | $s_0$ | A | B | C | D | E | F | G | H | $\gamma$ |
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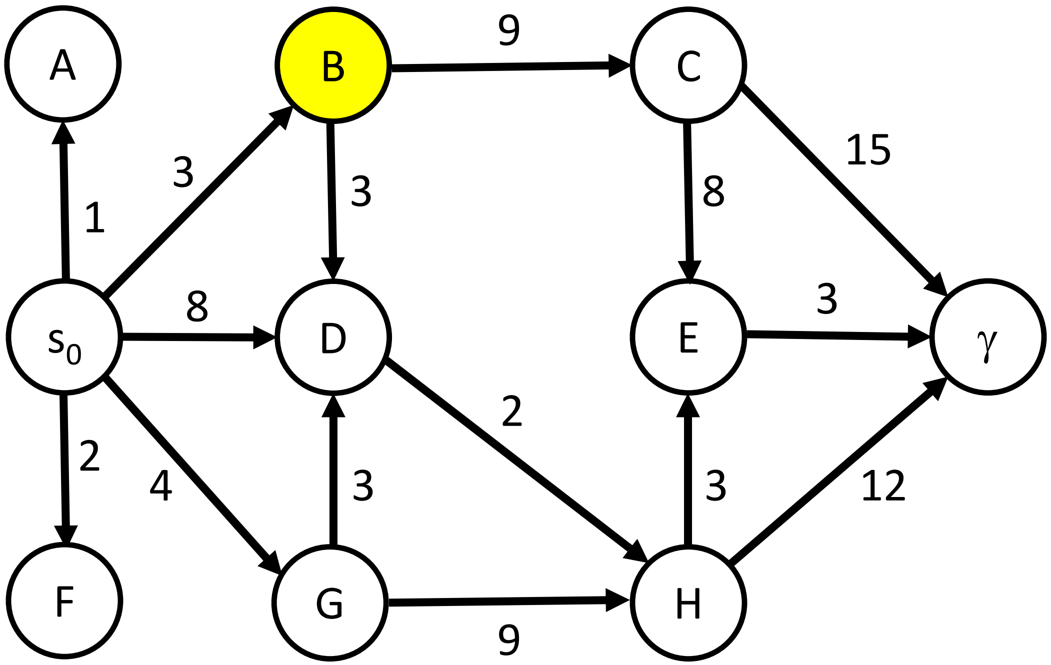


OPEN: ~~[$s_0$,14]~~
~~[B,7], [D,11], [G,14], [A,16], [F,18]~~
[D,9], [G,14], [A,16], [F,18], [C,22]

CLOSED: [$s_0$,14], [B,7]

| state s: | $s_0$ | A | B | C | D | E | F | G | H | γ |
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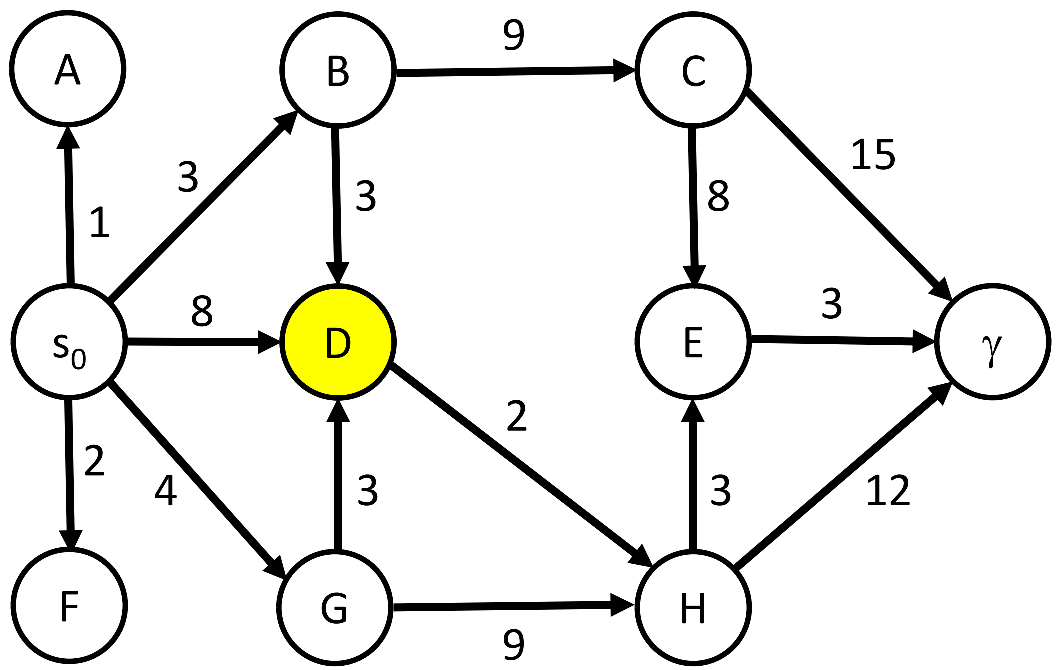


OPEN: ~~[$s_0$,14]~~
~~[B,7], [D,11], [G,14], [A,16], [F,18]~~
~~[D,9], [G,14], [A,16], [F,18], [C,22]~~
[H,13], [G,14], [A,16], [F,18], [C,22]

CLOSED: [$s_0$,14], [B,7], [D,9]

| state s: | $s_0$ | A | B | C | D | E | F | G | H | $\gamma$ |
|---|---|---|---|---|---|---|---|---|---|---|
| heuristic h(s): | 14 | 15 | 4 | 10 | 5 | 2 | 16 | 8 | 5 | 0 |

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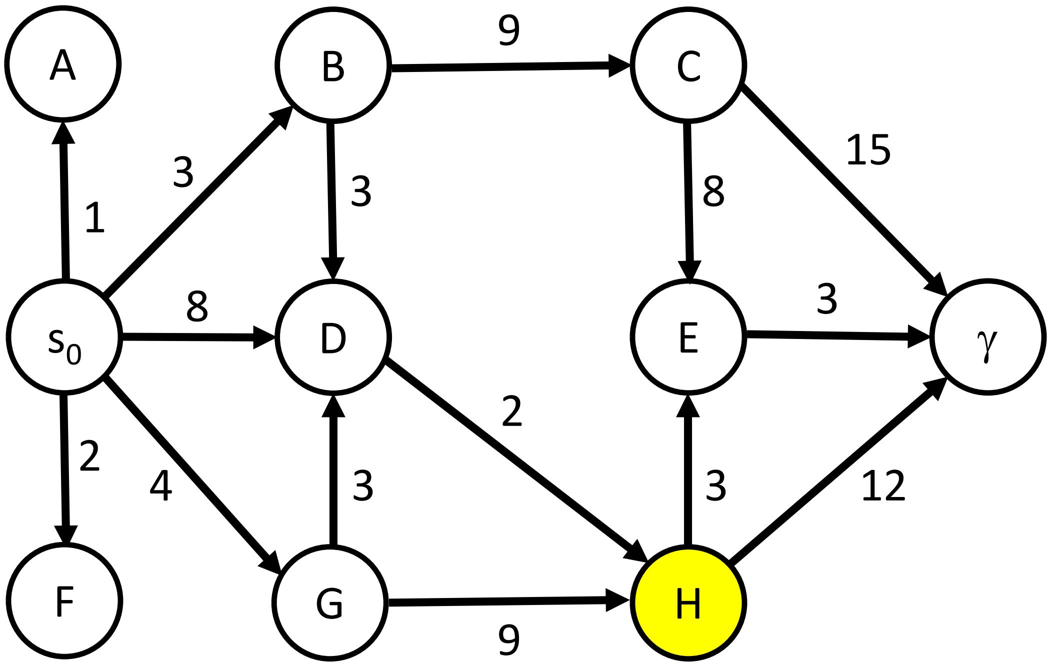


OPEN: ~~[$s_0$,14]~~
~~[B,7], [D,11], [G,14], [A,16], [F,18]~~
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[E,13], [G,14], [A,16], [F,18], [γ,20], [C,22]

CLOSED: [$s_0$,14], [B,7], [D,9], [H,13]

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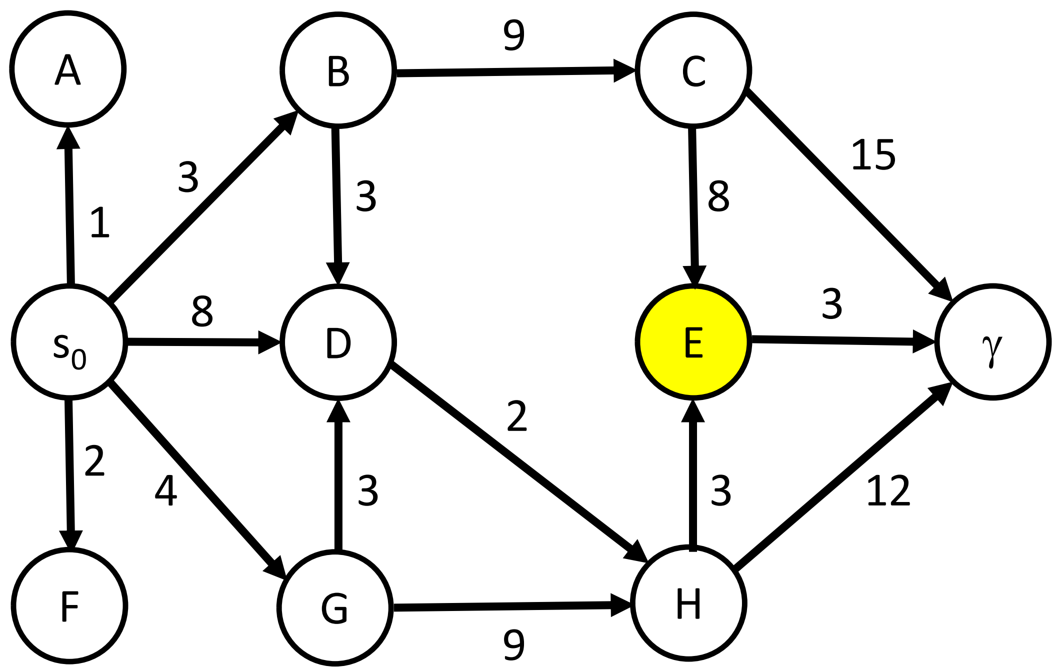


OPEN: ~~[$s_0$,14]~~
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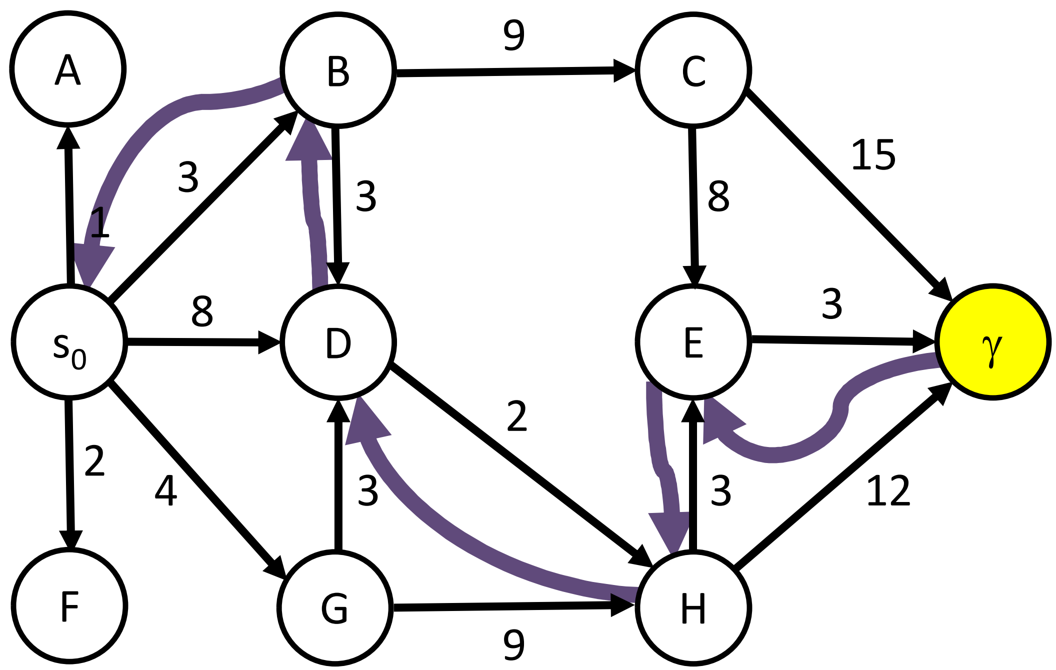


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~~[E,13], [G,14], [A,16], [F,18], [$\gamma$,20], [C,22]~~
~~[$\gamma$,14], [G,14], [A,16], [F,18], [C,22]~~
[G,14], [A,16], [F,18], [C,22]

Predecessor links that mark the shortest path

CLOSED: [$s_0$,14], [B,7], [D,9], [H,13], [E,13], [$\gamma$,14]

| state s: | $s_0$ | A | B | C | D | E | F | G | H | $\gamma$ |
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# Notes on the Example

The path found is the same as that found by UCS (Dijkstra).
However, fewer nodes are expanded; e.g., A and F are never expanded.

The heuristic *h* is admissible, so as soon as $\gamma$ becomes the current state, we can stop.

But *h* is not consistent: $h(G) - h(D) > d(G,D)$.
This fact doesn't cause trouble here, fortunately.
(If *h* is consistent, no state will ever have to be expanded a second time, i.e., never have to be moved from CLOSED back to OPEN.)

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# Admissibility and Consistency

Heuristic $h$ is admissible if and only if

$$(\forall s \in \Sigma)(\forall \gamma \in \Gamma)\ h(s) \leq d(s, \gamma)$$

Here $d(s, \gamma)$ is the length (cost) of the shortest (lowest-cost) path from $s$ to $\gamma$.

I.e., $h$ estimates but never overestimates the distance from $s$ to the closest goal.

If $h$ is admissible, then A*, using $h$, finds a shortest path (optimal path) as soon as it expands any goal $\gamma$.

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# Admissibility and Consistency

Heuristic $h$ is *consistent* if and only if

For each edge $(s_i, s_j)$ in the problem-space graph,

$$h(s_i) - h(s_j) \leq d(s_i, s_j)$$

Here $d(s_i, s_j)$ is the length (cost) of the edge from $s_i$ to $s_j$.

If $h$ is consistent, then along any shortest path from a node (state) $s$ to its closest goal, then $h$ values will be monotonically non-increasing along the path.

If $h$ is consistent, then A*, using $h$, never has to re-expand a node.

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# Admissibility of A*

A* is *admissible*: Provided its heuristic is admissible, and a path exists from $s_0$ to $\gamma$, then A* will find a shortest path and stop.

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# Heuristics for the Eight Puzzle

$h_0(s) = 0$ (uninformed; blind search)

$h_1(s)$ = number of tiles out of place. ("Hamming") $0 \leq h_1(s) \leq 8$.

$\gamma =$

| | | |
|---|---|---|
| | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |

$s_a =$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 7 | 6 | 5 |
| 4 | | 8 |

$h_1(s_a)$ = 4.

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# Heuristics for the Eight Puzzle

$h_2(s)$ = number of rows tile 7 is away from its place.

$\gamma =$

| | | |
|---|---|---|
| | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |

$s_a =$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 7 | 6 | 5 |
| 4 | | 8 |

$h_2(s_a)$ = 1.

Using tile 7 is arbitrary here -- just an example of a heuristic based on a single, particular tile.

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# Heuristics for the Eight Puzzle

$h_3(s)$ = number of rows and columns tile 7 is away from its place.

$\gamma =$

| | | |
|---|---|---|
| | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |

$s_a =$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 7 | 6 | 5 |
| 4 | | 8 |

$h_3(s_a)$ = 2.

This sum is known as the *Manhattan distance* (for a single tile).

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# Heuristics for the Eight Puzzle

$h_4(s)$ = sum of Manhattan distances for all 8 tiles.

$\gamma =$

| | | |
|---|---|---|
| | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |

$s_a =$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 7 | 6 | 5 |
| 4 | | 8 |

$h_4(s_a)$ = 7.

This is called the Manhattan distance heuristic.
In this example $h_4(s_a) = h(s_a)$ (the actual shortest distance).

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# Heuristics for the Eight Puzzle

$h_5(s)$ = sum of Euclidean distances for all 8 tiles.

$$= \sum_{i=1}^{8} \sqrt{dx_i^2 + dyi^2}$$

$\gamma =$

| | | |
|---|---|---|
| | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |

$s_a =$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 7 | 6 | 5 |
| 4 | | 8 |

$h_5(s_a) = 1 + 3\sqrt{2} \approx 5.2326$

This is called the Euclidean distance heuristic.
In (at least) this example $h_5(s_a) < h_4(s_a)$.
Euclidean is not as good as Manhattan.

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# Heuristic Domination

If $(\forall s \in \Sigma)\ h_i(s) \geq h_j(s)$, then we say $h_i$ dominates $h_j(s)$. However, we assume both heuristics are admissible.

If $h_i$ dominates $h_j$, then it we call $h_i$ "more informed" than $h_j$. Having a highly informed heuristic is good for limiting a search to relevant parts of the state space.

However, one has to trade off this off against the higher computational cost that usually goes with more informed heuristics.

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# Designing Heuristics

A common approach to defining heuristics is to create a simpler type of problem.

E.g., For the 8 Puzzle, allow tiles to be removed an put back anywhere, and "charge" a cost of 0.5 for each removal and 0.5 for each putting back. That leads to the Hamming heuristic.

Or: allow tiles to be piled up on top of one another, thus making it easier to move each tile (still one square at a time) to its destination. This leads to the Manhattan heuristic.

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# Designing Heuristics (cont)

Another way to simplify: change some of the tiles into "blanks" (like the blank tile in Scrabble). The new goal is to get only the non-blank tiles into their proper positions; the blanks are "don't-care" tiles that still take up space, but whose relative ordering is not important. For example:

$\gamma' =$

| | | |
|---|---|---|
| | 1 | 2 |
| 3 | 4 | 5 |
| / | / | / |

$s_a' =$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| / | / | 5 |
| 4 | | / |

To compute $h(s_a')$, we transform $s_a$ into $s_a'$, and solve the simplified problem, getting a path length $d(s_a', \gamma')$, which we use as the value of $h$. If the reduced problem is easy enough, then we can precompute a table of $d(s_a', \gamma')$ values to speed up computing $h$ during the search. Such a table is called a *pattern database*.

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Completeness and Optimality

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*done!*