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COUNTING

Dr. Subin P. Joseph
Department of Mathematics
Govt. Engg. College, Wayanad

Fundamental Principles of Counting

The rule of sum

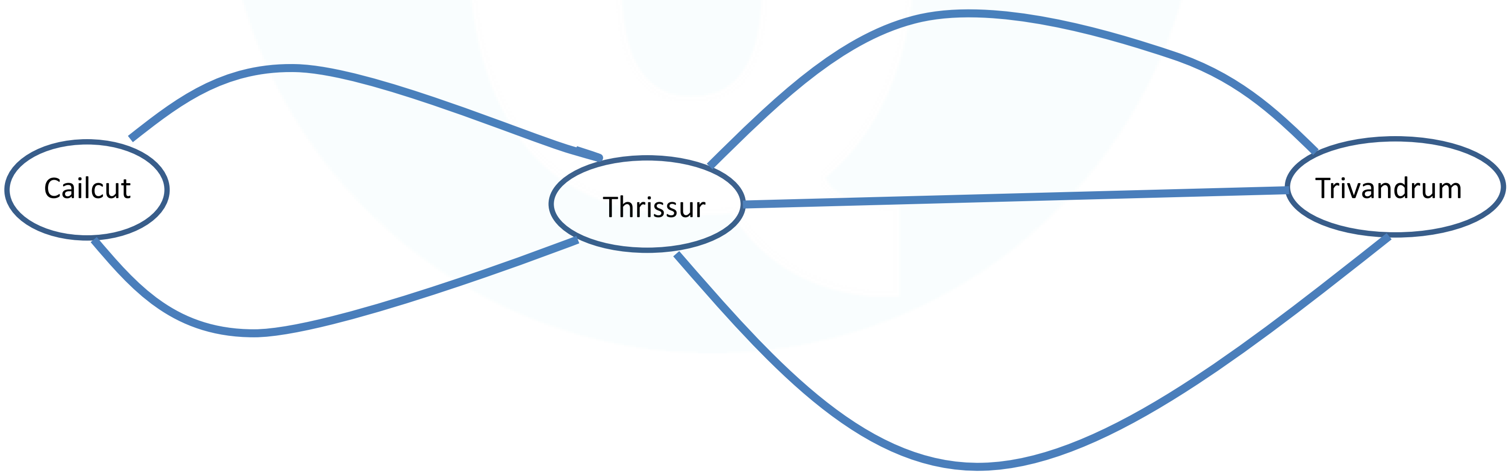
If a first task can be performed in *m* ways, while a second task can be performed in *n* ways, and the tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of *m* + *n* ways.

Total 5 routes from Calicut to Trivandrum

Fundamental Principles of Counting

The rule of product

If a procedure can be broken down into first and second stages, and if there are *m* possible outcomes for the first stage and if, for each of these outcomes, there are *n* possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in *m x n* ways.



Total 6 ways from Calicut to Trivandrum

Permutations and Combinations

Given the set of three letters, {A, B, C}, how many possibilities are there for selecting any two letters where order is important?

(AB, AC, BC, BA, CA, CB)

Given the set of three letters, {A, B, C}, how many possibilities are there for selecting any two letters where order is not important?

(AB, AC, BC).

Permutation: The number of ways in which a subset of objects can be selected from a given set of objects, where order is important.

Combination: The number of ways in which a subset of objects can be selected from a given set of objects, where order is not important.

Permutations and Combinations

Factorial Formula for Permutations

The number of **permutations**, or *arrangements*, of n distinct things taken r at a time, where $r \leq n$, can be calculated as

$$_nP_r = \frac{n!}{(n-r)!}.$$

Factorial Formula for Combinations

The number of **combinations**, or *subsets*, of n distinct things taken r at a time, where $r \leq n$, can be calculated as

$$_nC_r = \frac{_nP_r}{r!} = \frac{n!}{r!(n-r)!}.$$

Permutations and Combinations

How many ways can you select two letters followed by three digits for an ID if repeats are not allowed?

Two parts:

1. Determine the set of two letters.
2. Determine the set of three digits.

$$_{26}P_2 \qquad\qquad _{10}P_3$$

$$\frac{26!}{(26-2)!} \qquad\qquad \frac{10!}{(10-3)!}$$

$$\frac{26!}{24!} \qquad\qquad \frac{10!}{7!}$$

$$26 \cdot 25 \qquad\qquad 10 \cdot 9 \cdot 8$$

$$650 \qquad\qquad 720$$

$$650 \cdot 720$$

$$468{,}000$$

Permutations and Combinations

A common form of poker involves hands (sets) of five cards each, dealt from a deck consisting of 52 different cards. How many different 5-card hands are possible?

Hint: Repetitions are not allowed and order is not important.

$$_{52}C_5$$

$$\frac{52!}{5!\,(52-5)!}$$

$$\frac{52!}{5!\,47!}$$

$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{120}$$

$$\frac{311875200}{120}$$

2,598,960 5-card hands

Permutations and Combinations

Find the number of different subsets of size 3 in the set: $\{m, a, t, h, r, o, c, k, s\}$.

$${}_9C_3$$

$$\frac{9!}{3!\,(9-3)!}$$

$$\frac{9!}{3!\,6!}$$

$$\frac{9 \cdot 8 \cdot 7}{6}$$

$$\frac{504}{6}$$

84 Different subsets

Find the number of arrangements of size 3 in the set: $\{m, a, t, h, r, o, c, k, s\}$.

$${}_9P_3$$

$$\frac{9!}{(9-3)!}$$

$$\frac{9!}{6!}$$

$$9 \cdot 8 \cdot 7$$

504 arrangements

Permutations and Combinations

Guidelines on Which Method to Use

Permutations	**Combinations**
Number of ways of selecting r items out of n items	
Repetitions are not allowed	
Order is important.	Order is not important.
Arrangements of n items taken r at a time	Subsets of n items taken r at a time
${}_nP_r = n!/(n-r)!$	${}_nC_r = n!/[\,r!(n-r)!]$
Clue words: arrangement, schedule, order	Clue words: group, sample, selection

Permutations and Combinations

1. In how many ways can you choose 5 out of 10 friends to invite to a dinner party?

Solution: Does the order of selection matter? If you choose friends in the order A,B,C,D,E or A,C,B,D,E the same set of 5 was chosen, so we conclude that the order of selection does not matter. We will use the formula for combinations since we are concerned with how many **subsets of size 5** we can select from a set of 10.

C(10,5) = $$\frac{P(10,5)}{5!} = \frac{10(9)(8)(7)(6)}{5(4)(3)(2)(1)} = \frac{10(9)(8)(7)}{(5)(4)} = 2(9)(2)(7) = 252$$

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Permutations and Combinations

How many ways can you arrange 10 books on a bookshelf that has space for only 5 books?

Does order matter? The answer is yes since the arrangement ABCDE is a different arrangement of books than BACDE. We will use the formula for permutations. We need to determine the number of arrangements of 10 objects taken 5 at a time.

So we have $P(10,5) = 10(9)(8)(7)(6)=30,240$

Permutations and Combinations

How many bit strings of length 10 contain:

a) exactly four 1's?

- Find the positions of the four 1's
- Does the order of these positions matter?

  Nope!

  Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
- Thus, the answer is $C(10,4) = 210$

b) at most four 1's?

- There can be 0, 1, 2, 3, or 4 occurrences of 1
- Thus, the answer is:

  $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4)$

  $= 1+10+45+120+210$

  $= 386$

Permutations and Combinations

How many bit strings of length 10 contain:

c) at least four 1's?

- There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
  - Thus, the answer is:

$$C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10)$$
$$= 210+252+210+120+45+10+1$$
$$= 848$$

- Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1

d) an equal number of 1's and 0's?

- Thus, there must be five 0's and five 1's
  - Find the positions of the five 1's
  - Thus, the answer is $C(10,5) = 252$

Permutations and Combinations

a) How many ways can a 3-person subcommittee be selected from a committee of a seven people?

b) How many ways con a president vice-president, and secretary be selected from a committee of 7 people?

(A) The number of ways that a three-person subcommittee can be selected from a seven member committee is the number of combinations (since order *is not* important in selecting a subcommittee) of 7 objects 3 at a time. This is:

$$C_{7,3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7\cdot6\cdot5\cdot4!}{3\cdot2\cdot1\cdot4!} = 35$$

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(B) The number of ways a president, vice-president, and secretary can be chosen from a committee of 7 people is the number of permutations (since order *is* important in choosing 3 people for the positions) of 7 objects 3 at a time. This is:

$$P_{7,3} = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7\cdot6\cdot5\cdot4!}{4!} = 7\cdot6\cdot5 = 210$$

From a standard 52-card deck, how many 7-card hands have exactly 5 spades and 3 hearts?

The five spades can be selected in $C_{13,5}$ ways and the two hearts can be selected in $C_{13,2}$ ways. Applying the Multiplication Principle, we have: Total number of hands

$$C_{13,5} \cdot C_{13,2} = \frac{13!}{5!(13-5)!} \cdot \frac{13!}{2!(13-2)!}$$

$$= \frac{13!}{5!8!} \cdot \frac{13!}{2!11!} = 100,386$$

Permutations and Combinations

The English alphabet contains 21 consonants and 5 vowels. How many strings of six lower case letters of the English alphabet contain:

exactly one vowel?

exactly 2 vowels

at least 1 vowel

at least 2 vowels

Permutations and Combinations

The English alphabet contains 21 consonants and 5 vowels. How many strings of six lower case letters of the English alphabet contain:

exactly one vowel?

Note that strings can have repeated letters!

We need to choose the position for the vowel

$C(6,1) = 6!/1!5!$ This can be done 6 ways.

Choose which vowel to use.

This can be done in 5 ways.

Each of the other 5 positions can contain any of the 21 consonants (not distinct).

There are 21^5 ways to fill the rest of the string.

$6*5*21^5$

Permutations and Combinations

The English alphabet contains 21 consonants and 5 vowels. How many strings of six lower case letters of the English alphabet contain:

exactly 2 vowels?

Choose position for the vowels.

$C(6,2) = 6!/2!4! = 15$

Choose the two vowels.

5 choices for each of 2 positions = 5^2

Each of the other 4 positions can contain any of 21 consonants.

21^4

$15*5^2*21^4$

The English alphabet contains 21 consonants and 5 vowels. How many strings of six lower case letters of the English alphabet contain:

at least 1 vowel

Count the number of strings with no vowels and subtract this from the total number of strings.

$26^6 - 21^6$

Permutations and Combinations

The English alphabet contains 21 consonants and 5 vowels. How many strings of six lower case letters of the English alphabet contain:

at least 2 vowels

Compute total number of strings and subtract number of strings with no vowels and the number of strings with exactly 1 vowel.

$26^6 - 21^6 - 6*5*21^5$

Permutations and Combinations

How many committees of 5 people can be chosen from 20 men and 12 women

If exactly 3men must be on each committee?

If at least 4 women must be on each committee?

- *If exactly three men must be on each committee?*
  - We must choose 3 men and 2 women. The choices are not mutually exclusive, we use the product rule

$$\binom{20}{3} \cdot \binom{12}{2}$$

- *If at least 4 women must be on each committee?*
  - We consider 2 cases: 4 women are chosen and 5 women are chosen. Theses choices are mutually exclusive, we use the addition rule:

$$\binom{20}{1} \cdot \binom{12}{4} + \binom{20}{0} \cdot \binom{12}{5} = 10,692$$

Permutations and Combinations

In how many ways can the English letters be arranged so that there are exactly 10 letters between a and z?

- The number of ways is P(24,10)
- Since we can choose either a or z to come first, then there are 2P(24,10) arrangements of the 12-letter block
- For the remaining 14 letters, there are P(15,15)=15! possible arrangements
- In all there are 2P(24,10).15! arrangements

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Permutations and Combinations

How many ways are there for 4 horses to finish if ties are allowed?

Note that order does matter!

Solution by cases

No ties

The number of permutations is P(4,4) = 4! = 24

Two horses tie

There are *C*(4,2) = 6 ways to choose the two horses that tie

There are *P*(3,3) = 6 ways for the "groups" to finish

A "group" is either a single horse or the two tying horses

By the product rule, there are 6*6 = 36 possibilities for this case

Two groups of two horses tie

There are *C*(4,2) = 6 ways to choose the two winning horses

The other two horses tie for second place

Three horses tie with each other

There are *C*(4,3) = 4 ways to choose the two horses that tie

There are *P*(2,2) = 2 ways for the "groups" to finish

By the product rule, there are 4*2 = 8 possibilities for this case

All four horses tie

There is only one combination for this

By the sum rule, the total is 24+36+6+8+1 = 75

If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, . . . , and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + \cdots + n_r = n$, then there are $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$ (linear) arrangements of the given n objects.

The MASSASAUGA is a brown and white venomous snake indigenous to North America. Arranging all of the letters in MASSASAUGA, we find that there are

$$\frac{10!}{4!\, 3!\, 1!\, 1!\, 1!} = 25{,}200$$

possible arrangements. Among these are

$$\frac{7!}{3!\, 1!\, 1!\, 1!\, 1!} = 840$$

in which all four A's are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

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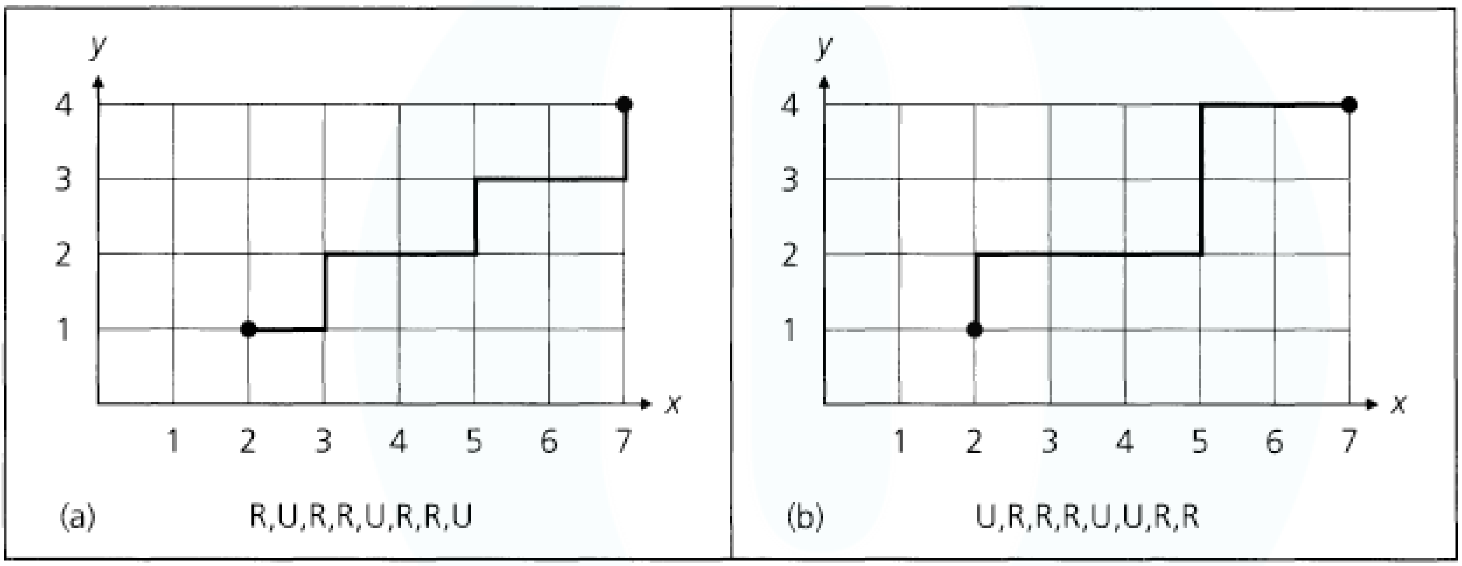
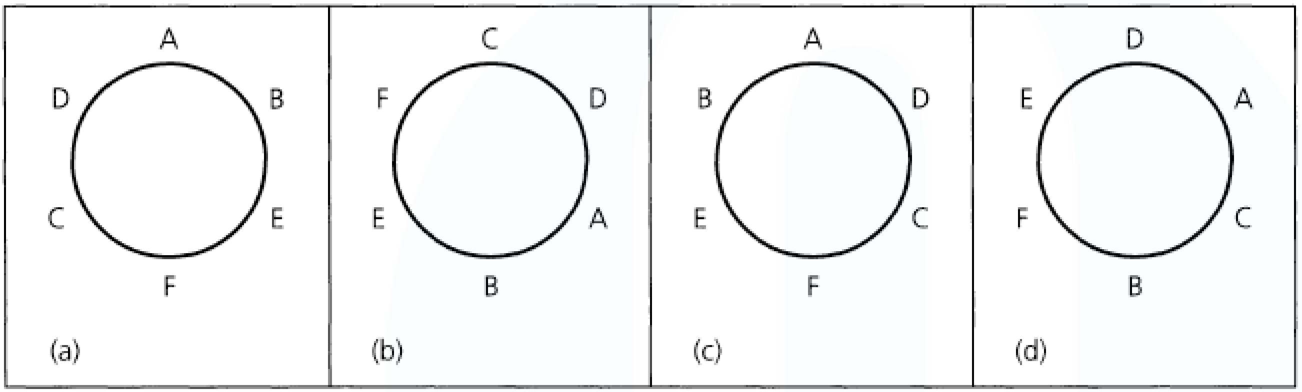
Determine the number of (staircase) paths in the xy-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). The blue lines in Fig. 1.1 show two of these paths.



- Any arrangements of RRRRRUUU, will give a path

$$8!/(5!\,3!) = 56.$$

If six people, designated as A, B, . . . , F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation? [In Fig. 1.2, arrangements (a) and (b) are considered identical, whereas (b), (c), and (d) are three distinct arrangements.]



the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangement. In addition to these two, four other linear arrangements — BEFCDA, DABEFC, EFCDAB, and FCDABE — are found to correspond to the same circular arrangement as in (a) or (b). So inasmuch as each circular arrangement corresponds to six linear arrangements, we have $6 \times$ (Number of circular arrangements of A, B, . . . , F) = (Number of linear arrangements of A, B, . . . , F) = $6!$.

Consequently, there are $6!/6 = 5! = 120$ arrangements of A, B, . . . , F around the circular table.

Suppose now that the six people of Example 1.16 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

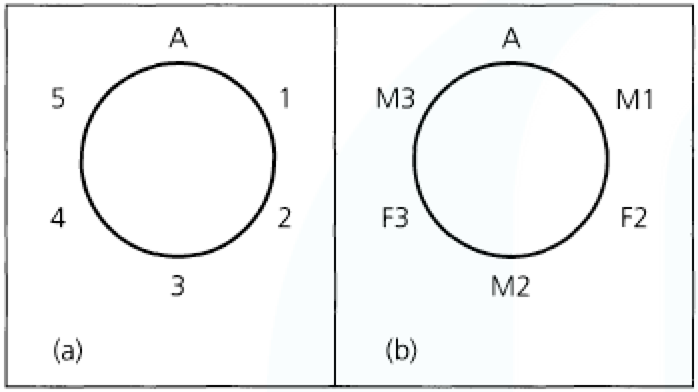


Figure 1.3

To solve the new problem of alternating the sexes, consider the method shown in Fig. 1.3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this manner, by the rule of product, there are $3 \times 2 \times 2 \times 1 \times 1 = 12$ ways in which these six people can be arranged with no two men or women seated next to each other.

21. **a)** How many arrangements are there of all the letters in SOCIOLOGICAL?

b) In how many of the arrangements in part (a) are A and G adjacent?

c) In how many of the arrangements in part (a) are all the vowels adjacent?

22. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

a) How many distinct paths are there from $(-1, 2, 0)$ to $(1, 3, 7)$ in Euclidean three-space if each move is one of the following types?

$$(H): (x, y, z) \to (x + 1, y, z);$$

$$(V): (x, y, z) \to (x, y + 1, z);$$

$$(A): (x, y, z) \to (x, y, z + 1)$$

b) How many such paths are there from $(1, 0, 5)$ to $(8, 1, 7)$?

a) In how many possible ways could a student answer a 10-question true-false test?

b) In how many ways can the student answer the test in part (a) if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?

a) A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here, so the student can answer the examination in

$$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \text{ ways.}$$

b) If the student must answer three questions from the first five and four questions from the last five, three questions can be selected from the first five in $\binom{5}{3} = 10$ ways, and the other four questions can be selected in $\binom{5}{4} = 5$ ways. Hence, by the rule of product, the student can complete the examination in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways.

c) Finally, should the directions on this examination indicate that the student must answer seven of the 10 questions where at least three are selected from the first five, then there are three cases to consider:

- **i)** The student answers three of the first five questions and four of the last five: By the rule of product this can happen in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways, as in part (b).

- **ii)** Four of the first five questions and three of the last five questions are selected by the student: This can come about in $\binom{5}{4}\binom{5}{3} = 5 \times 10 = 50$ ways — again by the rule of product.

- **iii)** The student decides to answer all five of the first five questions and two of the last five: The rule of product tells us that this last case can occur in $\binom{5}{5}\binom{5}{2} = 1 \times 10 = 10$ ways.

Combining the results for cases (i), (ii), and (iii), by the rule of sum we find that the student can make $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 50 + 10 = 110$ selections of seven (out of 10) questions where each selection includes at least three of the first five questions.

The gym teacher of Example 1.21 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four teams? Call the teams A, B, C, and D.

a) To form team A, she can select any nine girls from the 36 enrolled in $\binom{36}{9}$ ways. For team B the selection process yields $\binom{27}{9}$ possibilities. This leaves $\binom{18}{9}$ and $\binom{9}{9}$ possible ways to select teams C and D, respectively. So by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!\,27!}\right)\left(\frac{27!}{9!\,18!}\right)\left(\frac{18!}{9!\,9!}\right)\left(\frac{9!}{9!\,0!}\right)$$

$$= \frac{36!}{9!\,9!\,9!\,9!} \doteq 2.145 \times 10^{19} \text{ ways.}$$

b) For an alternative solution, consider the 36 students lined up as follows:

1st student	2nd student	3rd student	...	35th student	36th student

To select the four teams, we must distribute nine A's, nine B's, nine C's, and nine D's in the 36 spaces. The number of ways in which this can be done is the number of arrangements of 36 letters comprising nine each of A, B, C, and D. This is now the familiar problem of arrangements of nondistinct objects, and the answer is

$$\frac{36!}{9!\,9!\,9!\,9!}, \quad \text{as in part (a).}$$

The number of arrangements of the letters in TALLAHASSEE is

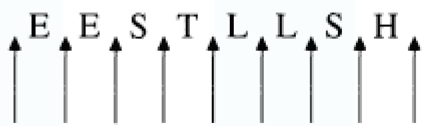
$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600.$$

How many of these arrangements have no adjacent A's?

When we disregard the A's, there are

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.



Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are $5040 \times 84 = 423{,}360$ arrangements of the letters in TALLAHASSEE with no consecutive A's.

13. How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

15. **a)** Fifteen points, no three of which are collinear, are given on a plane. How many lines do they determine?

b) Twenty-five points, no four of which are coplanar, are given in space. How many triangles do they determine?

Binomial theorem

The Binomial Theorem. If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

a) From the binomial theorem it follows that the coefficient of x^5y^2 in the expansion of $(x+y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21$.

b) To obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$, replace $2a$ by x and $-3b$ by y. From the binomial theorem the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$, and $\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$.

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Multinomial theorem

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_t!},$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

a) In the expansion of $(x + y + z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that of x^3z^4 is $\binom{7}{3,0,4} = \frac{7!}{3!\,0!\,4!} = 35$.

b) Suppose we need to know the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$. If we replace a by v, $2b$ by w, $-3c$ by x, $2d$ by y, and 5 by z, then we can apply the multinomial theorem to $(v + w + x + y + z)^{16}$ and determine the coefficient of $v^2w^3x^2y^5z^4$ as $\binom{16}{2,3,2,5,4} = 302{,}702{,}400$. But $\binom{16}{2,3,2,5,4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4 = \binom{16}{2,3,2,5,4}(1)^2(2)^3(-3)^2(2)^5(5)^4(a^2b^3c^2d^5) = 435{,}891{,}456{,}000{,}000\, a^2b^3c^2d^5$.

25. Determine the coefficient of

a) xyz^2 in $(x + y + z)^4$

b) xyz^2 in $(w + x + y + z)^4$

c) xyz^2 in $(2x - y - z)^4$

d) xyz^{-2} in $(x - 2y + 3z^{-1})^4$

26. Find the coefficient of $w^2x^2y^2z^2$ in the expansion of (a) $(w + x + y + z + 1)^{10}$, (b) $(2w - x + 3y + z - 2)^{12}$, and (c) $(v + w - 2x + y + 5z + 3)^{12}$.