APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

STUDY MATERIALS

KTU
ASSIST

CS201: DISCRETE COMPUTATIONAL STRUCTURES	**Semester III**
Module VI	
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Syllabus: ***Predicate Logic:***- *Predicates - Variables - Free and bound variables - Universal and Existential Quantifiers - Universe of discourse . Logical equivalences and implications for quantified statements - Theory of inference : Validity of arguments.*

Proof techniques: *Mathematical induction and its variants - Proof by Contradiction - Proof by Counter Example - Proof by Contra positive.*

Federal Institute of Science And Technology (FISAT)

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6.1 Predicate Logic

Definition 6.1 *Logic based upon the analysis of predicates in any statement is called predicate logic.*

Definition 6.2 *A predicate is any incomplete (English) phrase with specified gaps such that when the gaps are filled with names of things the phrase becomes a proposition.*

In any proposition that contains names we can obtain a predicate by deleting one or more of the names. Or we can say that a predicate becomes a prop osition when specific values are assigned to the variables. The other way to turn a predicate into a proposition is to add a quantifier like “all” or “some” that indicates the number of values for which the predicate is true.

6.1.1 Variables

Definition 6.3 *We call a letter used for generality a variable.*

If we have a predicate 'x is a dog' and wish to treat *x* as a variable to express generality and not simply as a place-holder, we can make a proposition by saying 'There is a thing, x , such that x is a dog'.

6.1.1.1 Free and bound variables

Definition 6.4 *A particular occurrence of x in A is bound in A if it immediately follows an occurrence of the symbol* $\forall$ *or* $\exists$ *or lies within the scope of an occurrence of* $\forall$ *or* $\exists$*. If an occurrence of x in A is not bound, it is free in A .*

A free variable represents a **genuine unknown**, whose value must be specified before the given statement's truth or falsity can be determined. A bound variable is really a "dummy variable," like the variable of integration in a definite integral.

For example suppose you are asked whether the equation $x + 5 = 3$ is true. Here x is free, so you would want to know the value of x. This is free means that if we want to know the truth or falsity we need to find the value of x.

Now suppose you are asked whether the statement $\exists x(x + 5 = 3)$ is true. This time x is bound, so it makes no sense to ask the value of x. We know that this is true for x and it is evident by the existence of quantifier $\exists$. In this case we need to find the domain of x.

6.1.2 Universal and Existential Quantifiers

There are two types of quantifiers

1. Universal

2. Existential

The universal quantifier corresponds to the words "for all," "for every," "for any," or "for each" and is represented by the symbol $\forall$. The existential quantifier corresponds to the words "there exists" or "for some" or "there is a" (meaning that there is at least one), and is represented by the symbol $\exists$.

The grammatical rules for the use of quantifiers are simple: if P is any statement, and x is any mathematical variable, then $\forall xP$ and $\exists xP$ are also statements.

6.1.3 Universe of discourse

Definition 6.5 *The domain or universe or universe of discourse for a predicate variable is the set of values that may be assigned to the variable.*

6.1.4 Logical equivalences and implications for quantified statements

For a prescribed universe and any open statements $p(x), q(x)$ in the variable x, the important equivalence and implication formulas are

$\neg(\forall(x)A(x)) \Leftrightarrow \exists(x)\neg A(x)$
$\neg(\exists(x)A(x)) \Leftrightarrow \forall(x)\neg A(x)$
$\exists(x)(A(x) \vee B(x)) \Leftrightarrow \exists A(x) \vee \exists B(x)$
$\forall(x)(A(x) \wedge B(x)) \Leftrightarrow \forall A(x) \wedge \forall B(x)$
$\forall(x)A(x) \vee \forall(x)B(x) \Rightarrow \forall(A(x) \vee B(x))$
$\exists(x)(A(x) \wedge B(x)) \Rightarrow \exists A(x) \wedge \exists B(x)$

6.1.5 Rules of Ineference for Predicate Calculus

Universal Specification: From $\forall(x)A(x)$ we can conclude $A(y)$

Existential Specification: Form $\exists(x)A(x)$ we can conclude $A(y)$ provided y is not free in any given premise and also not free any prior step of derivation

Universal Generalization: From $A(x)$ we can conclude $\forall(y)A(y)$ provided x is not free any given premise.

Existential generalization: From $A(x)$ we can conclude $\exists(y)A(y)$

6.2 Proof Techniques

6.2.1 Mathematical induction

Theorem 6.6 ***Principle of Mathematical Induction:*** *Let S(n) denote a mathematical statement that involves one or more occurrences of of the variable n, which represents a positive integer*

1. *If S(1) is true; and*
2. *If whenever S(k) is true (where $k \in Z^+$); then S(k+1) is true*

Then S(n) is true for all $n \in Z^+$

First condition is known as *basis step* and second condition is known as *inductive step*. For the first condition, the choice of 1 is not mandatory.

Example: For any $n \in Z^+, \Sigma_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

We have $S(n) : \Sigma_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Basis step: $S(1) : \Sigma_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$. So S(1) is true.

Assume S(n) be true for some $k \in Z^+$.

We need to show that $\Sigma_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

We proceed as follows

$\Sigma_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

So by the principle of induction S(n) is true for all $n \in Z^+$

6.2.2 Proof by Contradiction

Definition 6.7 *Proof by contradiction is a form of proof that establishes the truth or validity of a proposition by showing that the proposition's being false would imply a contradiction.*

If P is the proposition to be proved from Q (That is $P \to Q$) , the steps for proving are

1. Assume P is true and Q is not true. That is P and $\neg Q$
2. Arrive at a contradiction. That is $\neg P$

Example: Prove that if $3n + 2$ is odd then n is odd

This is of the form $P \to Q$

Assume n is even ($\neg Q$). Therefore

$n = 2k$

Therefore

$$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

This shows $3n+2$ is even ($\neg P$). This is a contradiction. Therefore we can conclude using contradiction that "if $3n+2$ is odd then n is odd "

6.2.3 Proof by Counter Example

Definition 6.8 *Proof by Counter example allows to prove that a property is not true by providing an example where it does not hold.*

Example: Prove that not all triangles are obtuse

Counter example: the equilateral triangle having all angles equal to sixty.

6.2.4 Proof by Contra positive

Definition 6.9 *In logic, the contrapositive of a conditional statement is formed by negating both terms and reversing the direction of inference. That is since*

$$P \to Q \Leftrightarrow \neg Q \to \neg P$$

We can use $\neg Q \to \neg P$ to prove sentences of the form $P \to Q$

Example: If $3n + 2$ is odd, then n is odd.

This is of the form $P \to Q$. This can be written in the form $\neg Q \to \neg P$. That is we are proving

"If n is even, then $3n + 2$ is even"

We can write $n = 2k$. Therefore

$$\begin{aligned} 3n+2 &= 3(2k)+2 \\ &= 6k+2 \\ &= 2(3k+1) \end{aligned}$$

Since 2 times a number will always even, we can conclude that $3n+2$ is even. Therefore "If n is even, then $3n+2$ is even". This in turn means "If n is odd, then $3n+2$ is odd"