Module - 4

Complex Integration

Chapter-1 Line Integrals on a complex plain

Line Integrals or Contour Integrals

Complex definite integrals are called line integrals and it is denoted by $\int_C f(z)\,dz$ where $dz = dx + i\,dy$.

c is called the path of integration

Evaluation of line integrals

Method 1: First evaluation method

Theorem :-

Let $f(z)$ be analytic in a domain D, with $F'(z) = f(z)$. Then,

$$\int_{z_0}^{z_1} f(z)\,dz = F(z)\Big]_{z_0}^{z_1} = F(z_1) - F(z_0)$$

Q) $\int_0^{1+i} z^2\,dz$

z^2 is analytic

$$\int_0^{1+i} z^2 dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3}$$

$$= \frac{1+3i-3-i}{3}$$

$$= \frac{-2+2i}{3}$$

Q) $\displaystyle\int_{-\pi i}^{\pi i} \cos z\, dz$

$\cos z$ is analytic

$$\Rightarrow \int_{-\pi i}^{\pi i} \cos z\, dz = \left[\sin z\right]_{-\pi i}^{\pi i} = \sin(\pi i) - \sin(-\pi i)$$

$$= i\sinh\pi + i\sinh\pi$$

$$= 2i\sinh\pi$$

Q) $\displaystyle\int_{8+\pi i}^{8-3\pi i} e^{z/2}\, dz$

$\Rightarrow e^{z/2}$ is analytic

$$\Rightarrow \int_{8+\pi i}^{8-3\pi i} e^{z/2}\, dz = 2\left[e^{z/2}\right]_{8+\pi i}^{8-3\pi i}$$

$$= 2\left[e^{\frac{8-3\pi i}{2}} - e^{\frac{8+\pi i}{2}}\right]$$

$$= 2e^4\left[e^{\frac{-3\pi i}{2}} - e^{\frac{\pi i}{2}}\right]$$

$$= 2e^4\left[\cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2} - \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right]$$

$$= 2e^4\left[0 + i - 0 - i\right]$$

$$= \underline{\underline{0}}$$

Q) $\int_{-i}^{i} \frac{dz}{z}$, $z \neq 0$

$\frac{1}{z}$ is analytic

$$\Rightarrow \int_{-i}^{i} \frac{dz}{z} = \left[\log z\right]_{-i}^{i} = \log i - \log(-i)$$

We know that $\log z = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \log i = \frac{1}{2}\log 1 + i\tan^{-1}\infty$$

$$\log i = \frac{i\pi}{2}$$

$$\log(-i) = \frac{1}{2}\log 1 + i\tan^{-1}\left(\frac{-1}{0}\right)$$

$$\log(-i) = -\frac{i\pi}{2}$$

$$\therefore \int_{-i}^{i} \frac{dz}{z} = \frac{i\pi}{2} + \frac{i\pi}{2} = \underline{\underline{i\pi}}$$

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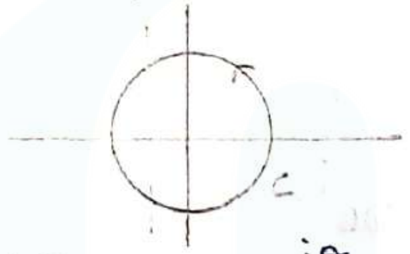
II - Second Evaluation method

This method is not restricted to analytic functions.

1) Evaluate $\oint_C \frac{dz}{z}$ where c is the unit circle with centre at the origin in the anticlockwise direction.



We have $z = re^{i\theta}$

here $r = 1$

$\therefore z = e^{i\theta}, \quad 0 \le \theta \le 2\pi$

$dz = ie^{i\theta} d\theta$

$$\therefore \oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta$$

$$= \int_0^{2\pi} i \, d\theta$$

$$= i\left[\theta\right]_0^{2\pi}$$

$$\oint_C \frac{dz}{z} = \underline{\underline{2\pi i}}$$

2) Evaluate $\oint_C (z-z_0)^m dz$ where c is the circle of radius r with centre z_0 in the anticlockwise direction and m is any integer.

$$z - z_0 = re^{i\theta}$$

$$z = z_0 + re^{i\theta}$$

$$dz = re^{i\theta} \cdot i\,d\theta \quad , 0 \le \theta \le 2\pi$$

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} (re^{i\theta})^m re^{i\theta} \cdot i\,d\theta$$

$$= r^{m+1} i \int_0^{2\pi} e^{(m+1)i\theta} d\theta$$

$$= r^{m+1} i \cdot \left[\frac{e^{(m+1)i\theta}}{(m+1)i} \right]_0^{2\pi}$$

$$= \frac{r^{m+1}}{m+1} \left[e^{i(m+1)2\pi} - 1 \right]$$

$$= \frac{r^{m+1}}{m+1}\left[\cos 2\pi(m+1) + i\sin(m+1)2\pi - 1\right]$$

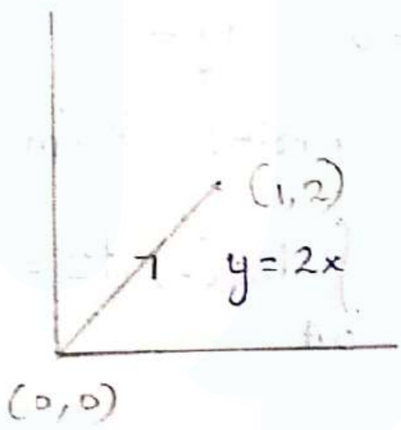
$$= \frac{r^{m+1}}{m+1}\left[1 + 0 - 1\right]$$

$$= \underline{\underline{0}}.$$

3) Evaluate $\int_0^{1+2i} Re(z)\, dz$

$$\int_0^{1+2i} Re(z)dz = \int_0^{1+2i} x(dx + i\,dy)$$

$$=$$



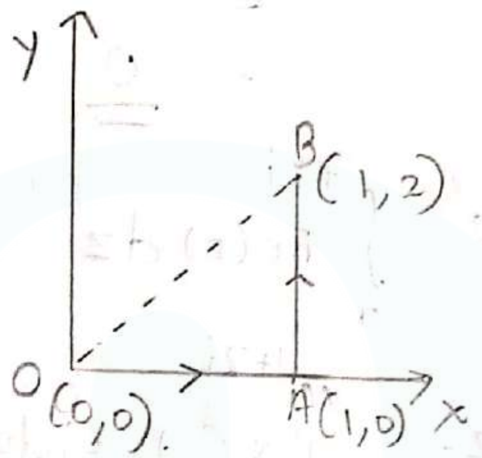
here $dy = 2dx$

$$= \int_{x=0}^{1} x(dx + i\,2dx)$$

$$= (1+2i)\int_0^1 x\, dx$$

$$(1+2i)\left[\frac{x^2}{2}\right]_0^1 = \frac{(1+2i)}{2}$$

4) Integrate $f(z) = Re(z)$ along the real axis from 0 to 1 and then along a straight line parallel to the imaginary axis from 1 to $1+2i$



Along OA

$y = 0$, $dy = 0$

x varies from 0 to 1

$$\therefore \int_{OA} Re(z)\,dz = \int_{OA} x\,(dx + i\,dy)$$

$$= \int_0^1 x\,dx$$

$$\int_{OA} Re(z)\,dz = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Along AB

$x = 1$, $dx = 0$

y varies from 0 to 2

$$\therefore \int_{AB} Re(z)dz = \int_{AB} x(dx+idy)$$

$$= \int_0^2 1(0+idy)$$

$$= \int_0^2 idy$$

$$= i[y]_0^2$$

$$\int_{AB} Re(z)dz = \underline{\underline{2i}}$$

$$\therefore \int_C Re(z)dz = \frac{1}{2} + 2i$$

$$\int_C Re(z)dz = \frac{1+4i}{2}$$

5) Evaluate $\int_C Re(z)dz$ where C is the parabola $y = 1+\frac{(x-1)^2}{2}$ from $(1+i)$ to $(3+3i)$

$$\int_C Re(z)dz = \int_{1+i}^{3+3i} x(dx+idy)$$

Given $y = 1+\frac{(x-1)^2}{2}$

$$\therefore dy = \frac{2(x-1)dx}{2} = (x-1)dx$$

$$dy = (x-1)dx$$

$$\therefore \int_C Re(z)dz = \int_1^3 x(dx + i(x-1)dx)$$

$$= \int_1^3 x\,dx\,(1 + i(x-1))$$

$$= \int_1^3 x + i(x^2 - x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^3 + i\left[\left[\frac{x^3}{3}\right]_1^3 - \left[\frac{x^2}{2}\right]_1^3\right]$$

$$= \frac{9-1}{2} + i\left[\frac{27-1}{3} - \frac{8}{2}\right]$$

$$= 4 + i\left[\frac{26}{3} - 4\right]$$

$$= 4 + i\left[\frac{26-12}{3}\right]$$

$$= 4 + i\left(\frac{14}{3}\right)$$

$$\int_C Re(z)dz = \frac{12+14i}{3}$$

6) Evaluate $\oint_c z^2 dz$ where c is the

(i) line $x = 2y$ from $z=0$ to $z = 2+i$

(ii) line along the real axis from $x=0$ to $x=2$ and then vertical to $2+i$

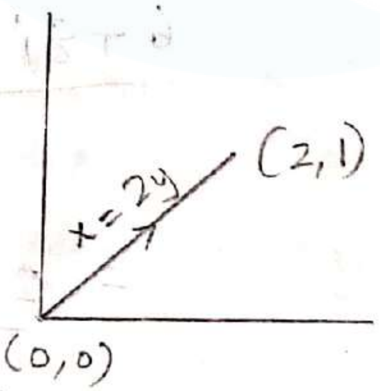
(iii) along the imaginary axis from $z=0$ to $z=i$ and then horizontally to $2+i$

$$\oint_c z^2 dz = \int_c (x+iy)^2 (dx+idy)$$

$$= \int_c (x^2 + 2xyi - y^2)(dx+idy)$$

$$\oint_c z^2 dz = \int_c (x^2 - y^2 + 2xyi)(dx+idy) \quad \text{---(1)}$$

(i) $x = 2y$



$x = 2y$

$dx = 2dy$

$\therefore$ eq ① implies

$$\oint_c z^2 dz = \int_0^1 ((2y)^2 - y^2 + 2i \times 2y \times y)(2\,dy + i\,dy)$$

$$= \int_0^1 (4y^2 - y^2 + 4y^2 i)\,(2+i)\,dy$$

$$= (2+i) \int_0^1 (3y^2 + 4y^2 i)\,dy$$

$$(2+i)\left[3\left[\frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^3}{3}\right]_0^1 i\right]$$

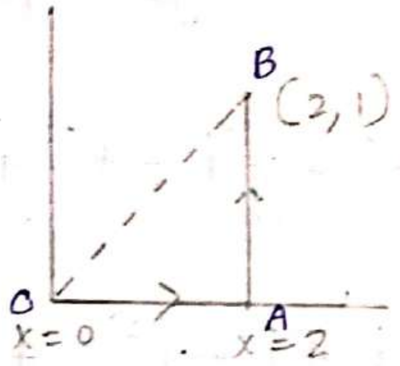
$$= (2+i)\left[1 + \frac{4}{3}i\right]$$

$$(2+i) \times \left[\frac{3+4i}{3}\right]$$

$$\frac{6+8i+3i+4i^2}{3}$$

$$= \frac{2+11i}{3}$$

(ii)



along OA

$y=0 \quad dy=0$

x varies from 0 to 2

eq ① becomes

$$\int_{OA} z^2 dz = \int_0^2 (x^2 - 0 + 0)(dx)$$

$$= \int_0^2 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$$\int_{OA} z^2 dz = \frac{8}{3}$$

along AB

$x=2 \quad dx=0$

y varies from 0 to 1

∴ eq ① becomes

$$\oint_{AB} z^2 dz = \int_0^1 (4-y^2+4iy)\, i\, dy$$

$$= \int_0^1 (4i - y^2 i + 4i^2 y) dy$$

$$= \int_0^1 (4i - y^2 i - 4y)\, dy$$

$$= 4i[y]_0^1 - i\left[\frac{y^3}{3}\right]_0^1 - 4\left[\frac{y^2}{2}\right]_0^1$$

$$= 4i - \frac{i}{3} - 2$$

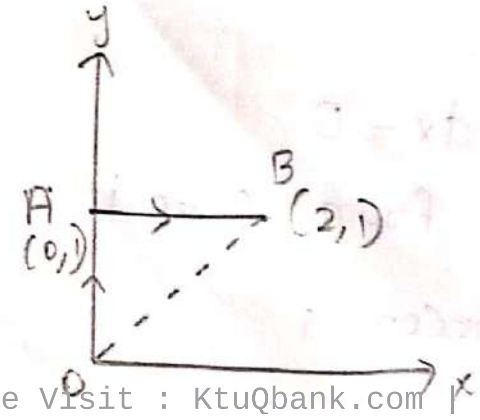
$$\oint_{AB} z^2 dz = \frac{11i-6}{3}$$

$$\therefore \oint_C z^2 dz = \frac{8}{3} + \frac{11i-6}{3}$$

$$= \frac{11i+2}{3}$$

$$= \frac{2+11i}{3}$$

(iii)



along OA

$x = 0, \; dx = 0$

y varies from 0 to 1

eq ① implies

$$\int_{OA} z^2 dz = \int_0^1 (-y^2)\, i (dy)$$

$$= -i\left[\frac{y^3}{3}\right]_0^1$$

$$\int_{OA} z^2 dz = \frac{-i}{3}$$

along AB

$y = 1, \; dy = 0$

x varies from 0 to 2

eq ① implies

$$\int_{AB} z^2 dz = \int_0^2 (x^2 - 1 + 2xi)\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 - [x]_0^2 + 2i\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{8}{3} - 2 + 4i$$

$$\int_{AB} z^2 dz = \frac{2+12i}{3}$$

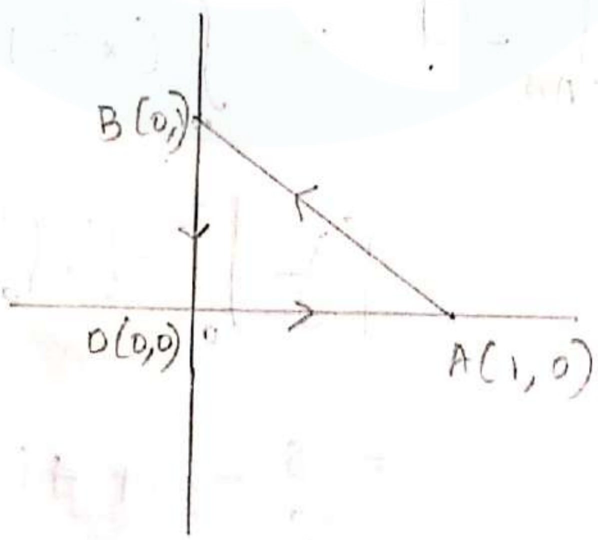
$$\oint_C z^2 dz = \frac{-i}{3} + \frac{2+12i}{3}$$

$$= \frac{2+11i}{3}$$

Remark:- Since z^2 is analytic $\int_C z^2 dz$ is the same for all paths from origin to $2+i$, ie, Integral of analytic functions are independent of path

7) Evaluate $\int_C \text{Im}(z^2)dz$ counterclockwise around the triangle with vertices $(0,1,i)$

$$\int_C \text{Im}(z^2)dz = \int_C 2xy(dx+idy)$$



along OA

$y=0, \; dy=0$

x varies from 0 to 1

$$\int_{OA} \operatorname{Im}(z^2)\,dz = \int_0^1 0\,dx$$

$$= \underline{\underline{0}}$$

along AB

$x+y=1$

$y = 1-x$

$dy = -dx$, x varies from 1 to 0

$$\oint_{AB} \operatorname{Im}(z^2)\,dz = \int_1^0 2x(1-x)\big(dx+i(-dx)\big)$$

$$= (1-i)\cdot 2\int_0^1 (x-x^2)$$

$$(1-i)\times 2\left[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right]$$

$$(1-i)\times 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$(1-i)\times 2\times \frac{1}{6}$$

$$= \underline{\frac{(1-i)}{3}}$$

along BO

$x=0, dx=0$

y varies from 1 to 0

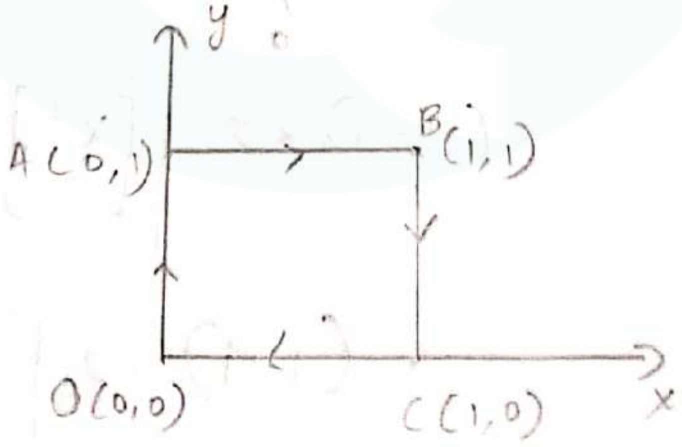
$$\oint_{BO} Im(z^2)dz = \int_1^0 0(idy)$$

$$= \underline{\underline{0}}$$

$$\therefore \int_C Im(z^2)dz = 0 + \frac{(1-i)}{3} + 0$$

$$\int_C Im(z^2)dz = \underline{\underline{\frac{1-i}{3}}}$$

8) Evaluate $\oint_C Re(z^2)dz$ clockwise along the boundary of the square with vertices $(0, i, (1+i), 1)$



$$\oint_C Re(z^2)dz = \oint_C (x^2-y^2) \quad (dx+idy) \quad —①$$

along OA

$x = 0$, $dx = 0$

y varies from 0 to 1

eq① becomes

$$\int_{OA} Re(z^2)\,dz = \int_0^1 (-y^2)(i\,dy)$$

$$= -i \int_0^1 y^2\,dy$$

$$= -i \left[\frac{y^3}{3}\right]_0^1$$

$$\int_{OA} Re(z^2)\,dz = \frac{-i}{3}$$

along AB :

$y = 1$, $dy = 0$

x varies from 0 to 1

eq ① becomes

$$\int_{AB} Re(z^2)\,dz = \int_0^1 (x^2 - 1)(dx)$$

$$= \left[\frac{x^3}{3}\right]_0^1 - [x]_0^1$$

$$\int_{AB} Re(z^2)\,dz = \frac{1}{3} - 1 = \frac{-2}{3}$$

along BC

$x = 1$, $dx = 0$

y varies from 1 to 0.

eq ① becomes

$$\int_{BC} Re(z^2)\,dz = \int_1^0 (1-y^2)(i\,dy)$$

$$= i\int_1^0 (1-y^2)\,dy$$

$$= i\left[[y]_1^0 - \left[\frac{y^3}{3}\right]_1^0\right]$$

$$= i\left[-1 - \frac{[-1]}{3}\right]$$

$$= i\left[-1 + \frac{1}{3}\right]$$

$$\int_{BC} Re(z^2)\,dz = i\left[\frac{-3+1}{3}\right] = \frac{-2}{3}i$$

along CO

$y = 0$, $dy = 0$

x varies from 1 to 0

eq ① becomes

$$\int_{C_0} Re(z^2)\,dz = \int_1^0 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^0$$

$$\int_{C_0} Re(z^2)\,dz = \frac{-1}{3}$$

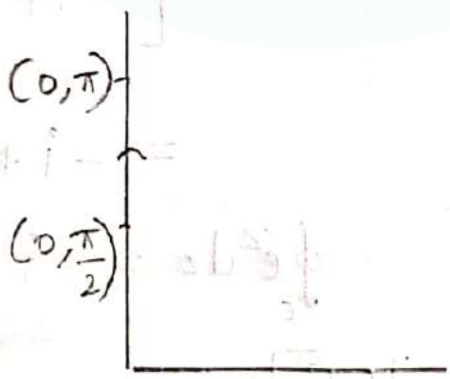
$$\therefore \oint_C Re(z^2)\,dz = \frac{-i}{3} - \frac{2}{3} - \frac{2i}{3} - \frac{1}{3}$$

$$= \frac{-3i-3}{3}$$

$$\oint_C Re(z^2)\,dz = -i-1$$

Homework

1) Evaluate $\oint_C e^z\,dz$ where c is the shortest path from $\frac{\pi i}{2}$ to πi



$$\oint_C e^z\,dz = \oint_C e^{(x+iy)}\,(dx+i\,dy) \quad -(1)$$

$x = 0, \quad dx = 0$

y varies from $\frac{\pi}{2}$ to π.

eq ① becomes

$$\oint_c e^z dz = \int_{\pi/2}^{\pi} e^{(iy)} (i\,dy)$$

$$= i \int_{\pi/2}^{\pi} e^{iy}\, dy$$

$$= i \int_{\pi/2}^{\pi} (\cos y + i \sin y)\, dy$$

$$= i \left[[\sin y]_{\pi/2}^{\pi} + i \left[-\cos y\right]_{\pi/2}^{\pi} \right]$$

$$= i \left[0 - 1 - i \left[(-1) - 0\right] \right]$$

$$= i \left[-i + i \right]$$

$$= -i + i^2$$

$$\oint_c e^z dz = \underline{\underline{-i - 1}}$$

$\boxed{\text{OR}}$

$x = 0$, $dx = 0$

y varies from $\frac{\pi}{2}$ to π

eq ① becomes

$$\oint_C e^z dz = \int_{\pi/2}^{\pi} (e^{iy})(i\,dy)$$

$$= i \int_{\pi/2}^{\pi} e^{iy} dy$$

$$= i \left[\frac{e^{iy}}{i} \right]_{\pi/2}^{\pi}$$

$$= \left[e^{iy} \right]_{\pi/2}^{\pi}$$

$$= \left[e^{i\pi} - e^{i\frac{\pi}{2}} \right]$$

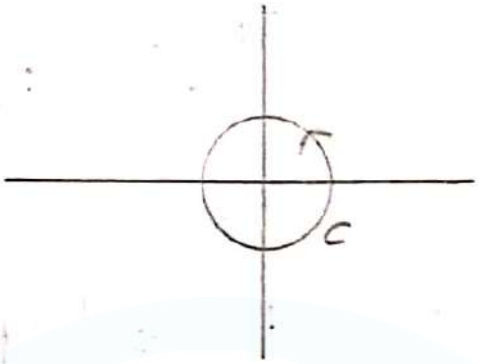
$$= \left[\cos\pi + i\sin\pi - \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \right] \right]$$

$$= \left[(-1) + 0 - \left[0 + i \right] \right]$$

$$= -1 + (-i)$$

$$\oint_C e^z dz = \underline{\underline{-i-1}}$$

2) Evaluate $\oint_C \left(z+\frac{1}{z}\right)dz$ where c is the unit circle counter clockwise.



Put $z = re^{i\theta}$

here $r = 1$

$\therefore z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$, θ varies from 0 to 2π

$$\oint_C \left(z+\frac{1}{z}\right)dz = \int_0^{2\pi}\left(e^{i\theta}+\frac{1}{e^{i\theta}}\right)ie^{i\theta}d\theta$$

$$= i\int_0^{2\pi}\left(e^{i2\theta}+1\right)d\theta$$

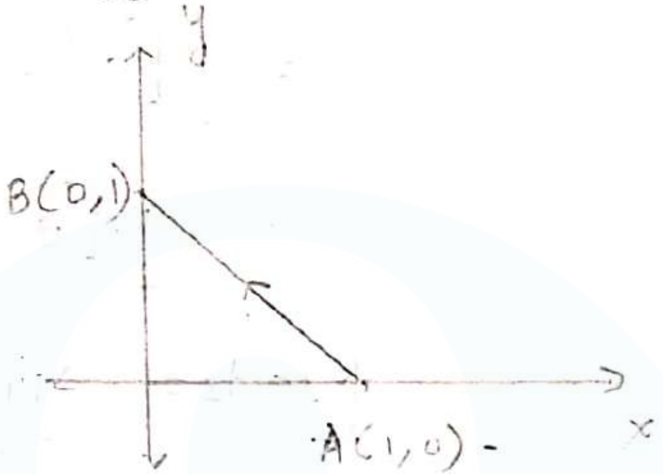
$$= i\left[\left[\frac{e^{2i\theta}}{2i}\right]_0^{2\pi} + \left[\theta\right]_0^{2\pi}\right]$$

$$= i\left[\frac{e^{4\pi i}-1}{2i} + 2\pi\right]$$

$$\oint_c \left(z+\frac{1}{z}\right)dz = \frac{e^{4\pi i}-1}{2} + 2\pi i = \frac{\cos 4\pi + i\sin 4\pi - 1}{2} + 2\pi i$$

$$= 0 + 2\pi i = \underline{\underline{2\pi i}}$$

3) Evaluate $\oint_C z\exp(z^2)\,dz$ from 1 to i.



$$\oint_C z\exp(z^2)\,dz = \oint_C z e^{z^2}\,dz$$

Put $z^2 = t$

$2z\,dz = dt$

$z\,dz = \frac{dt}{2}$

z varies from 1 to i

when $z = 1$

$t = z^2 = 1^2 = 1$

when $z = i$

$t = i^2 = -1$

$\therefore$ t varies from 1 to -1

$$\therefore \oint_C z e^{z^2} dz = \int_1^{-1} e^t \frac{dt}{2}$$

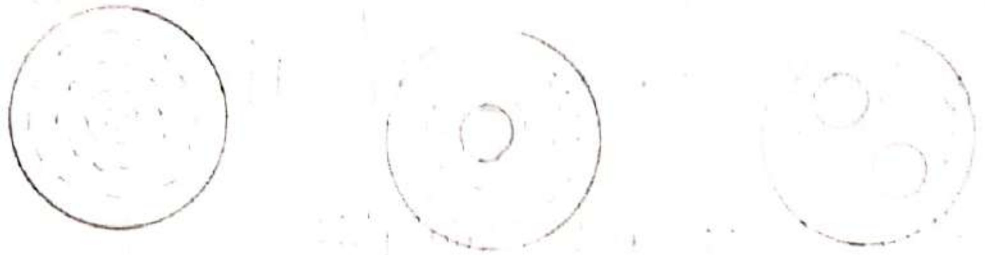
$$\frac{1}{2}\left[e^t\right]_1^{-1}$$

$$= \frac{1}{2}\left[e^{-1} - e\right]$$

$$= -\frac{1}{2} \times \left[e - e^{-1}\right]$$

$$\oint_C z e^{z^2} dz = \underline{\underline{-\sinh 1}}$$

A domain that is not simply connected is called a multi-connected domain and it will have holes in it.



Simply connected

doubly connected

triply connected

*** **Cauchy's Integral Theorem / Cauchy-Goursat Theorem**

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path, c, the line integral

$$\oint_c f(z)\,dz = 0$$

15/09/20 **Remark** ÷ In general

(1) $\oint_c e^z\,dz = 0$

(2) $\oint_c \sin z\,dz = 0$

(3) $\oint_c \cos z\,dz = 0$

(4) $\oint_c z^n\,dz = 0$

(5) $\oint_c \sinh z\,dz = 0$

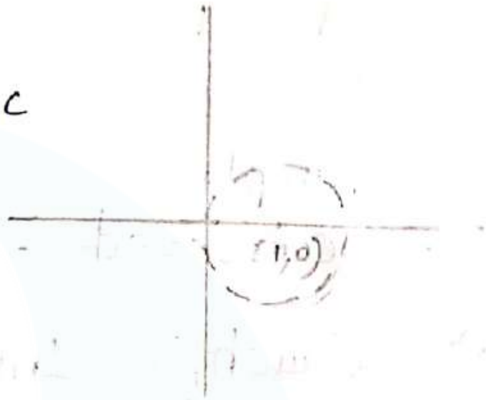
(6) $\oint_C \cosh z \, dz = 0$

as these functions are entire functions.

Problem

1) Evaluate the following line integrals

① $\oint_C \frac{1}{z} dz$ where $C : |z-1| < 1$



$f(z) = \frac{1}{z}$ is not analytic at $z=0$. Here $z=0$ is outside C, ie, $f(z)$ is analytic in C. Hence by Cauchy's integral theorem,

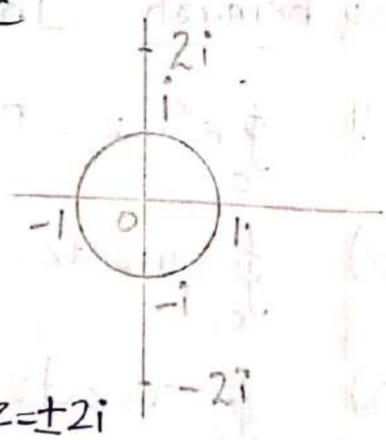
$$\oint_{|z-1|<1} \frac{1}{z} dz = 0$$

② $\oint_C \frac{dz}{z^2+4}$, $C : |z| = 1$

$f(z) = \frac{1}{z^2+4}$ is not analytic when $z^2+4=0$

$z^2 = -4$

$z = \pm 2i$



Now the singular points $z = \pm 2i$ lie outside C. Therefore $f(z)$ is analytic in C. Hence by Cauchy's integral theorem

$$\oint_{|z|=1} \frac{dz}{z^2+4} = 0$$

(3) $\oint_C \sec z\, dz$, $C : |z| = 1$

Here $f(z) = \frac{1}{\cos z}$ which is not analytic when $\cos z = 0$

ie, $z = (2n+1)\frac{\pi}{2}$

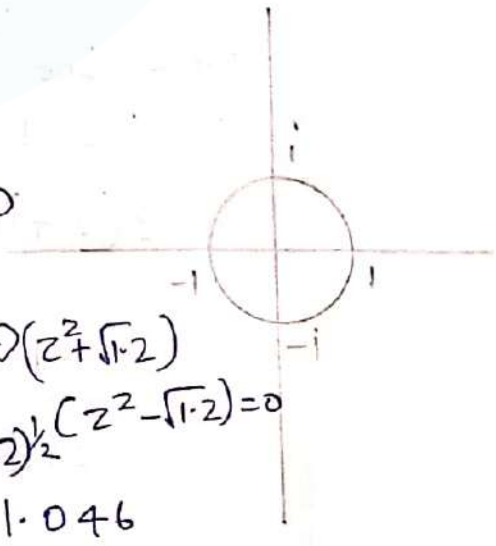
where n is any integer

(ie, $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$)

These singular points lie outside the circle $|z| = 1$. Hence by Cauchy's integral theorem $\oint_{|z|=1} \sec z\, dz = 0$.

(4) $\oint_C \frac{1}{z^4 - 1.2}\, dz$, $C : |z| = 1$

Here $f(z) = \frac{1}{z^4 - 1.2}$ which is not analytic when $z^4 - 1.2 = 0$



$z^4 = 1.2 \Rightarrow z^4 - 1.2 = 0 \Rightarrow (z^2 + \sqrt{1.2})(z^2 - \sqrt{1.2}) = 0$

$z^2 = (1.2)^{1/2}$, $z^2 = -(1.2)^{1/2}$

$z = \pm 1.046$, $\pm i\, 1.046$

Now the singular points $z = \pm 1.046$, $z = \pm i 1.046$ lie outside C. Therefore $f(z)$ is analytic in

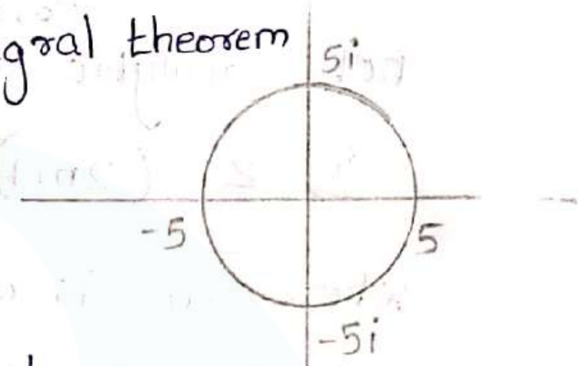
C. Hence by cauchy's integral theorem

$$\oint_{|z|=1} \frac{1}{z^4-1\cdot 2}\,dz = 0$$

5) $\oint_C (x^2-y^2+i2xy)\,dz$, C: $|z|=5$

Here $x^2-y^2+i2xy = z^2$ which is an entire function

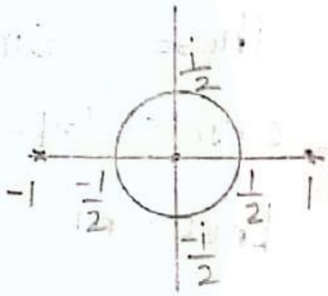
$\therefore$ By cauchy's integral theorem



$$\oint_{|z|=5} z^2\,dz = 0$$

6) $\oint_C \frac{3z-1}{z^3-z}\,dz$, C: $|z|=\frac{1}{2}$

Here $f(z)=\frac{3z-1}{z^3-z}$ is not analytic when $z^3-z=0$



$z(z^2-1)=0$

$z=0$, $z^2=1$

$z=0$, $z=\pm 1$

Now the singular point (0) lies inside the region.

$$\therefore \frac{3z-1}{z(z+1)(z-1)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-1}$$

$$\frac{3z-1}{z(z+1)(z-1)} = \frac{A(z+1)(z-1) + Bz(z-1) + C(z+1)z}{z(z+1)(z-1)}$$

$$\frac{3z-1}{z(z+1)(z-1)} = \frac{A(z^2-1)+B(z^2-z)+C(z^2+z)}{z(z+1)(z-1)}$$

equating coefficients of z, z^2 and constants on both sides we get

$$0 = A+B+C \quad -(1)$$

$$3 = -B+C \quad -(2)$$

$$-1 = -A \Rightarrow A = 1$$

$$\therefore B+C = -1 \quad -(3)$$

$$(2)+(3) \Rightarrow 2C = 2 \Rightarrow C = 1$$

$$\therefore B = \underline{\underline{-2}}$$

$$\therefore \oint_C \frac{3z-1}{z^3-z} = \oint_C \left(\frac{1}{z} - \frac{2}{z+1} + \frac{1}{z-1}\right) dz$$

$\therefore$

$$= \oint_C \left(\frac{1}{z} + \left(\frac{-2}{z+1}\right) + \frac{1}{z-1}\right) dz$$

$z = \pm 1$, lies outsides $|z| = \frac{1}{2}$

$$\therefore \oint_C \left(\frac{-2\,dz}{z+1}\right) = \oint_C \frac{1}{z-1}\,dz = 0$$

$$\therefore \oint_C \frac{3z-1}{z^3-z}\,dz = \oint_C \frac{1}{z}\,dz$$

Put $z = \frac{1}{2} e^{i\theta}$

$$dz = \frac{1}{2} i e^{i\theta} d\theta$$

θ varies from 0 to 2π

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{\frac{1}{2} i e^{i\theta} d\theta}{\frac{1}{2} e^{i\theta}}$$

$$= \int_0^{2\pi} i \, d\theta$$

$$= i \int_0^{2\pi} d\theta$$

$$= i \left[\theta\right]_0^{2\pi}$$

$$\oint_C \frac{1}{z} dz = 2\pi i$$

Homework

I Is Cauchy's Integral Theorem applicable? If not find the integrals using evaluation Theorems.

(1) $\oint_C \bar{z} \, dz$, $c : |z| = 1$

(2) $\oint_C \frac{dz}{z^2}$, $c : |z| = 1$

What can you infer?

Ans) (1) $\oint_C \bar{z} \, dz$, $c : |z| = 1$

Here $f(z) = \bar{z}$ is not analytic

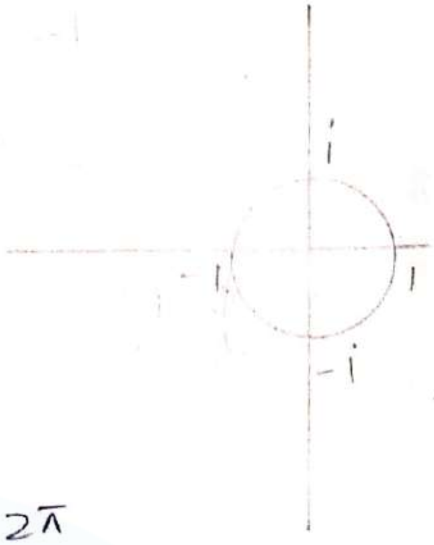
$\therefore$ Cauchy's Integral theorem is not applicable

Put $z = re^{i\theta}$

here $r = 1$

$\therefore z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$



$\therefore \bar{z} = e^{-i\theta}, \quad 0 \leq \theta \leq 2\pi$

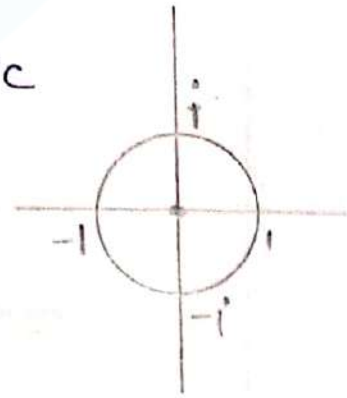
$$\therefore \oint_c \bar{z}\, dz = \int_0^{2\pi} e^{-i\theta} \times ie^{i\theta} d\theta = \int_0^{2\pi} i\, d\theta$$

$$= i[\theta]_0^{2\pi}$$

$$\oint_c \bar{z}\, dz = \underline{\underline{2\pi i}}$$

(2) $\oint_c \frac{dz}{z^2}, \quad c: |z| = 1$

Here $f(z) = \frac{1}{z^2}$ is not analytic at $z=0$. Here $z=0$ is inside c.



$\therefore f(z)$ is not analytic in c

$\therefore$ Cauchy's Integral theorem is not applicable.

Put $z = re^{i\theta}$

here $r = 1$

$$\therefore z = e^{i\theta}$$

$$dz = ie^{i\theta}d\theta$$

$$0 \le \theta \le 2\pi$$

$$\therefore \oint_C \frac{dz}{z^2} = \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{(e^{i\theta})^2}$$

$$= \int_0^{2\pi} ie^{-i\theta}d\theta$$

$$= i\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$$

$$= -\left[e^{-i\theta}\right]_0^{2\pi}$$

$$= -\left[e^{-2\pi i} - 1\right]$$

$$= -\left[\cos 2\pi - i\sin 2\pi - 1\right]$$

$$\oint_C \frac{dz}{z^2} = -\left[1 - 0 - 1\right] = \underline{\underline{0}}$$

$$\therefore \oint_C \frac{dz}{z^2} = \underline{\underline{0}}$$

This shows that converse of Cauchy's Integral theorem is not true, in general ie, $\oint_c f(z)dz = 0$ doesnot imply that $f(z)$ is analytic.

HW II

Verify Cauchy's Theorem for $f(z) = z$ in $c : |z| = 1$

Since z is analytic everywhere, by Cauchy's Integral theorem

$$\int_c z\,dz = 0.$$

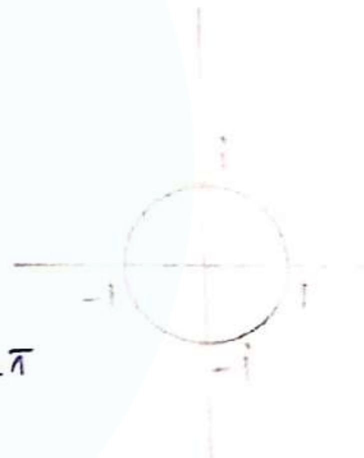
<u>Verification:</u>

Put $z = re^{i\theta}$

here $r = 1$

$\therefore z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$, $0 \leq \theta \leq 2\pi$



$$\int_c z\,dz = \int_0^{2\pi} e^{i\theta} \times ie^{i\theta}d\theta$$

$$= i\int_0^{2\pi} e^{2i\theta}d\theta$$

$$= i\left[\frac{e^{2i\theta}}{2i}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[e^{4\pi i} - 1\right]$$

$$= \frac{1}{2}\left[\cos 4\pi + i\sin 4\pi - 1\right]$$

$$= \frac{1}{2}\left[1 + 0 - 1\right]$$

$$\therefore \int_C z\,dz = \underline{\underline{0}}$$

$$= \frac{1}{2}\left[\cos 4\pi + i\sin 4\pi - 1\right]$$

$$= \frac{1}{2}\left[1+0-1\right]$$

$$\therefore \int_C z\,dz = \underline{\underline{0}}$$

Cauchy's Integral Formula

Integral of the form $\oint_C \frac{f(z)}{z-z_0}\,dz$.

Let $f(z)$ be analytic in a simply connected domain D and z_0 is a point in D. Then

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

Where C is a simple closed path in D in the anticlockwise direction.

If C is in the clockwise direction

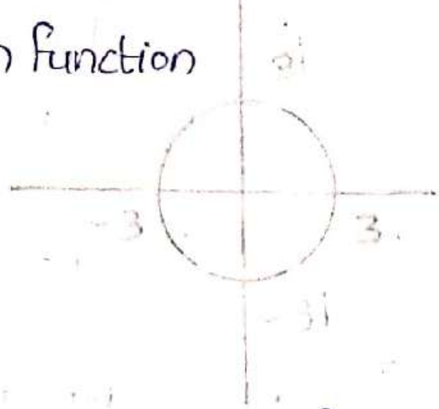
$$\oint_C \frac{f(z)}{z-z_0}\,dz = -2\pi i\, f(z_0)$$

Problem

Evaluate the following integral

(1) $\oint_C \frac{z^2}{z-2}\,dz$; $C: |z|=3$

The singular point $z_0 = 2$ is inside c. Therefore given function is not analytic in c.



$f(z) = z^2$ is analytic

$f(z_0) = f(2) = 4$

Therefore by Cauchy's integral formula

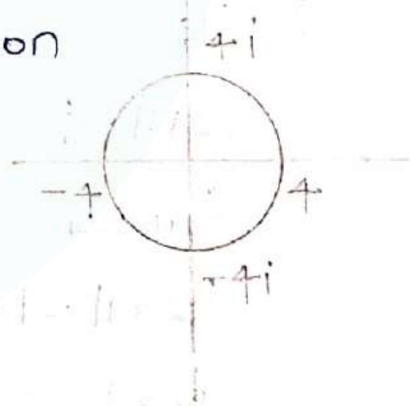
$$\oint_c \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

$$\oint_c \frac{z^2}{z-2}\,dz = 2\pi i \times 4$$

$$= \underline{\underline{8\pi i}}$$

(2) $\oint_c \dfrac{dz}{z-3}$; $c: |z| = 4$

The singular point $z_0 = 3$ is inside c. Therefore the given function is not analytic in c.



$f(z) = 1$

$f(z_0) = f(3) = 1$

Therefore by Cauchy's Integral formula

$$\oint_c \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

$$\therefore \oint_c \frac{dz}{z-3} = 2\pi i \cdot (1)$$

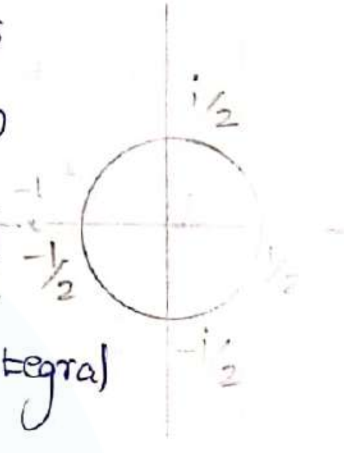
$$= \underline{\underline{2\pi i}}$$

$$\oint_C \frac{3z^2+7z+1}{z+1}\,dz$$

① $C : |z| = \frac{1}{2}$

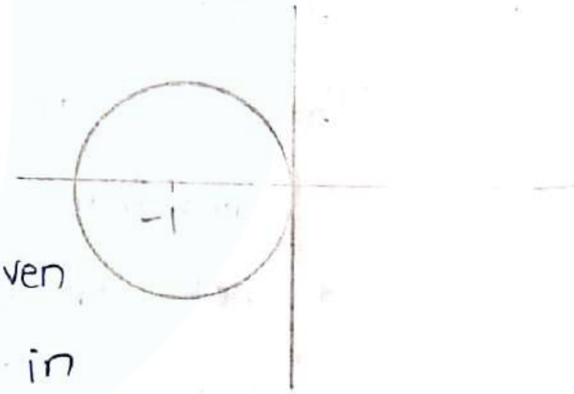
② $C : |z+1| = 1$

Ans) ① The singular point $z_0 = -1$ is outside $|z| = \frac{1}{2}$. Therefore the given function is analytic in $|z| = \frac{1}{2}$. Therefore by Cauchys Integral theorem,



$$\oint_{|z|=1/2} \left(\frac{3z^2+7z+1}{z+1}\right) dz = 0$$

② Here $z_0 = -1$ is inside $|z+1| = 1$. Therefore the given function is not analytic in $|z+1| = 1$.



$$f(z) = 3z^2+7z+1$$

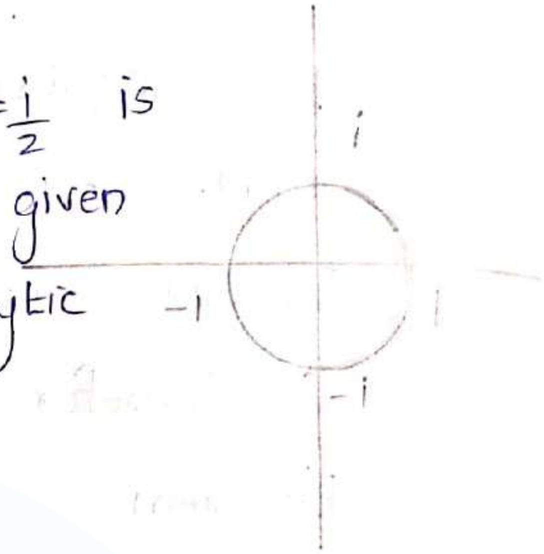
$$f(z_0) = f(-1) = -3$$

Therefore by Cauchys Integral form

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\therefore \oint_{|z+1|=1} \frac{3z^2+7z+1}{z+1}\,dz = 2\pi i(-3) = -6\pi i$$

(4) $\oint_C \frac{z^3-6}{2z-i}\,dz$, $C : |z| = 1$



The singular point $z_0 = \frac{i}{2}$ is inside c. Therefore the given function is not analytic in c.

$$\therefore f(z) = \frac{z^3-6}{2}$$

$$f(z_0) = f\left(\frac{i}{2}\right) = \frac{-\frac{i}{8}-6}{2} = \frac{-i-48}{16}$$

Therefore by Cauchys Integral formula

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\therefore \oint_C \frac{z^3-6}{2z-i}\,dz = 2\pi i\left(\frac{-i-48}{16}\right) = \frac{-\pi i^2-48\pi i}{8}$$

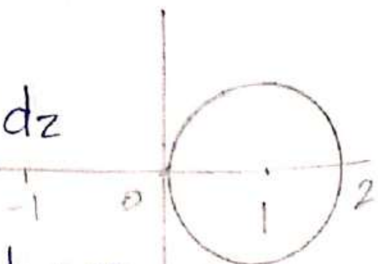
$$= \frac{\pi-48\pi i}{8}$$

★ (5) $\oint_C \frac{z^2+1}{z^2-1}\,dz$ where C is clockwise around

(i) $|z-1| = 1$ (ii) $|z+1| = 1$ (iii) $|z-i| = 1$

(1) $z = \pm 1$

$$\oint_C \frac{z^2+1}{(z-1)(z+1)}\,dz = \oint_C \frac{(z^2+1)/(z+1)}{z-1}\,dz$$



The singular point $z_0 = 1$ lies inside $|z-1| = 1$. Therefore given function is not analytic at $|z-1|=1$

$$f(z) = \frac{z^2+1}{z+1}$$

$$f(1) = \frac{1+1}{1+1} = \frac{2}{2} = 1$$

$\therefore$ By Cauchys integral formula,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = -2\pi i f(z_0)$$

$$\therefore \oint_{|z-1|=1} \frac{(z^2+1)/(z+1)}{z-1}\,dz = -2\pi i$$

(ii) Singular point $z_0 = \pm 1$

$$\therefore \oint_C \frac{z^2+1}{(z-1)(z+1)}\,dz = \oint_C \frac{(z^2+1)/z-1}{z+1}\,dz$$

The singular point $z_0=-1$ lies inside $|z+1|=1$

Therefore given fn is not analytic at $|z+1|=1$

$$f(z) = \frac{z^2+1}{z-1}$$

$$f(z_0) = f(-1) = \frac{1+1}{-1-1} = -1$$

By Cauchys Integral formula

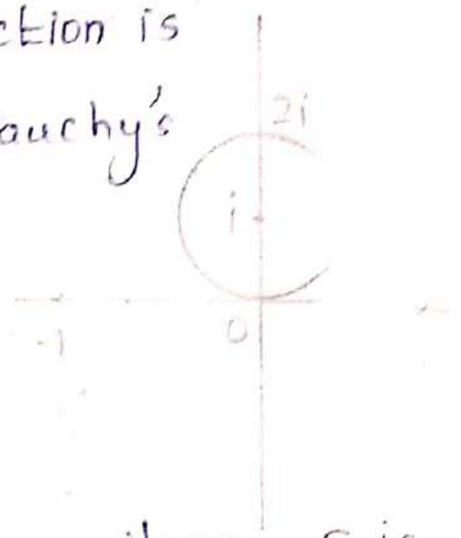
$$\therefore \oint_C \frac{f(z)}{z-z_0}\,dz = -2\pi i f(z_0)$$

$$= \oint_{|z+1|=1} \frac{(z^2+1)/z+1}{(z-1)}\,dz = -2\pi i(-1)$$

$$= 2\pi i$$

(iii) Singular point $z_0 = \pm 1$ both lie outside

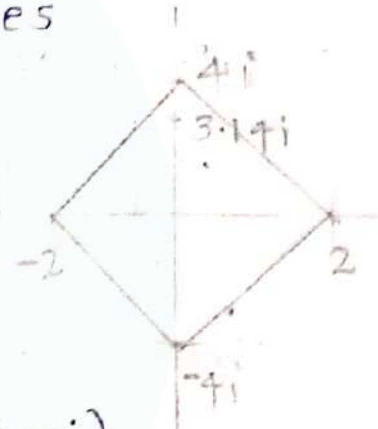
C. Therefore the given function is analytic in C. Hence by Cauchy's Integral theorem



$$\oint_C \frac{z^2+1}{z^2-1}\, dz = 0$$

6) Evaluate $\oint_C \frac{\cosh(z^2-\pi i)}{z-\pi i}\, dz$ where C is the rectangle with vertices $\pm 2, \pm 4i$

The singular point $z_0 = \pi i$ lies inside C. Therefore the function is not analytic in C



$$\therefore f(z) = \cosh(z^2-\pi i)$$

$$f(z_0) = f(\pi i) = \cosh((\pi i)^2 - \pi i)$$

$$f(z_0) = \cosh(-\pi^2-\pi i)$$

Therefore by Cauchy's Integral formula

$$\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0)$$

$$\therefore \oint_C \frac{\cosh(z^2-\pi i)}{z-\pi i}\, dz = 2\pi i \cosh(\pi^2+\pi i)$$

we have $\cosh\theta = \cos i\theta$

$$\therefore \cosh(\pi^2+\pi i) = \cos i(\pi^2+\pi i)$$

$$= \cos(i\pi^2 - \pi)$$

$$= \cos(i\pi^2)\cos\pi + \sin(i\pi^2)\sin\pi$$

$$= \cos(i\pi^2) \cdot (-1)$$

$$\therefore \cosh(\pi^2 + \pi i) = -\cosh\pi^2$$

$$\therefore \oint_C \frac{\cosh(z^2 - \pi i)}{z - \pi i} dz = -2\pi i \cosh\pi^2$$

7) Evaluate $\oint_C \frac{\sin z}{4z^2 - 8iz} dz$ where C is the square with vertices

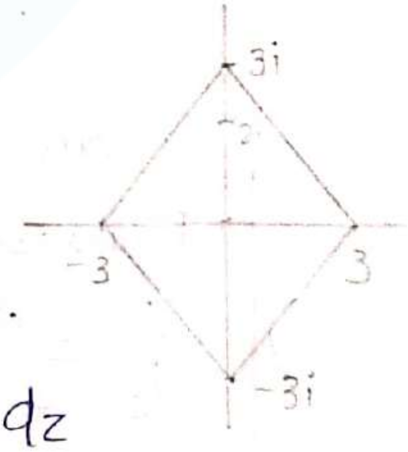
a) $(\pm 3, \pm 3i)$

b) $(\pm 1, \pm i)$

Ans) Singular points are given by

$$4z^2 - 8iz = 0 \Rightarrow 4z(z - 2i) = 0$$

$$\Rightarrow z = 0,\ z = 2i$$



a) Both the singular points lies inside C. Therefore the given function is not analytic in C.

$$\therefore \oint_C \frac{\sin z}{4z^2 - 8iz} dz = \oint_C \frac{\sin z}{4z(z - 2i)} dz$$

$$= \frac{1}{4}\left[\oint_C \frac{\sin z/(z - 2i)}{z} dz + \oint_C \frac{\sin z / z}{z - 2i} dz\right]$$

$$= 2\pi i \left[\frac{1}{4} \times 0 + \frac{1}{4} \times \frac{\sin 2i}{2i} \right]$$

$$= \frac{i\pi \sinh 2}{4}$$

b) The singular point $z_0 = 0$ lies inside C. Therefore the given function is not analytic at C.

$$f(z) = \frac{\sin z}{z - 2i}$$

$$f(z_0) = f(0) = 0$$

By Cauchys Integral formula

$$\therefore \oint_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0)$$

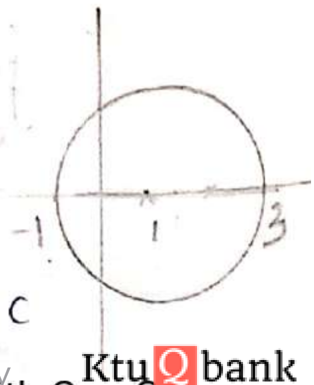
$$\therefore \oint_C \frac{\sin z/(z-2i)}{z} = 2\pi i \times 0$$

$$= 0$$

Homework

Evaluate the following integrals.

1) $\oint_C \frac{z+2}{z-2}\,dz$, $C : |z-1| = 2$



Here the singular point $z_0 = 2$ lies inside C. Therefore the given function is not analytic in C

$f(z) = z+2$

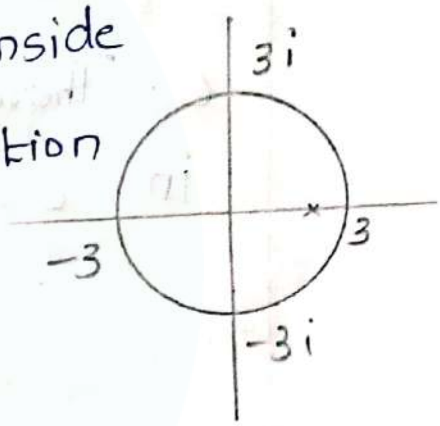
$f(z_0) = f(2) = 2+2 = \underline{4}$

$\therefore$ By Cauchy's Integral Formula

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0)$$

$$\therefore \oint_C \frac{z+2}{z-2}\,dz = 2\pi i \times 4 = \underline{8\pi i}$$

2) $\oint_C \frac{e^z}{z-2}\,dz$ (i) $c: |z|=3$ (ii) $|z|=1$



Ans) (i) Singular point $z_0 = 2$ lies inside $|z|=3$. Therefore the given function is not analytic at $|z|=3$.

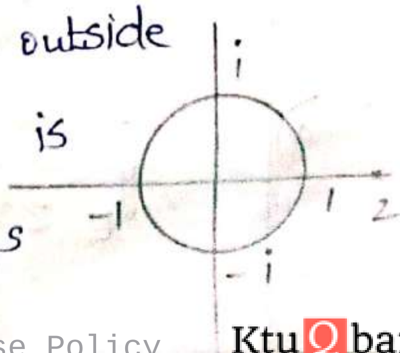
$\therefore f(z) = e^z$

$f(z_0) = f(2) = \underline{e^2}$

$\therefore$ By Cauchys Integral formula

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0)$$

$$\oint_{|z|=3} \frac{e^z}{z-2}\,dz = \underline{2\pi i e^2}$$



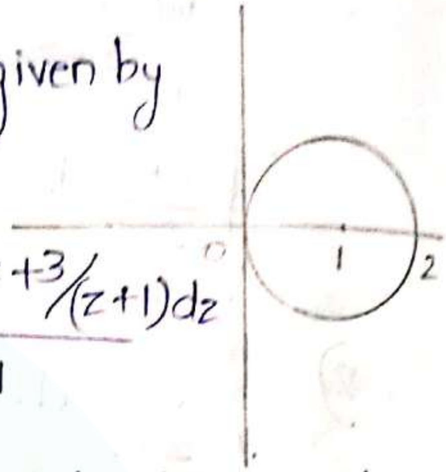
(2) The singular point $z_0 = 2$ lies outside $|z|=1$. Therefore the function is analytic in $|z|=1$. By Cauchys Integral theorem

$$\oint_{|z|=1} \frac{e^z}{z-2}\,dz = 0$$

3) $\oint_C \frac{z^2+2z+3}{z^2-1}\,dz$; $C: |z-1| = 1$ clockwise

The singular points are given by

$z^2-1=0$, $z=\pm 1$



$$\therefore \oint_C \frac{z^2+2z+3}{(z+1)(z-1)}\,dz = \oint_C \frac{z^2+2z+3/(z+1)\,dz}{z-1}$$

$\therefore$ The singular point $z_0 = 1$ lies inside C. Therefore the function is not analytic in C.

$$\therefore f(z) = \frac{z^2+2z+3}{z+1}$$

$$f(z_0) = f(1) = \frac{1+2+3}{1+1} = \frac{6}{2} = 3$$

$\therefore$ By Cauchys Integral formula,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = -2\pi i f(z_0)$$

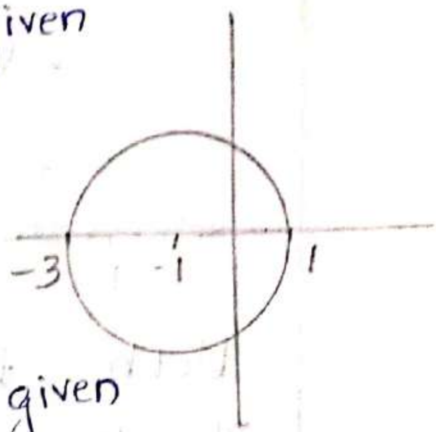
$$\therefore \oint_{|z-1|=1} \frac{z^2+2z+3}{z^2-1}\,dz = -2\pi i \times 3$$

$$= -6\pi i$$

4) $\oint_C \frac{z\,dz}{z^2+4z+3}$, $C: |z+1| = 2$

The singular points are given by $z^2+4z+3=0$

$z = -1, -3$



Both the singular points lie inside C. Therefore the given function is not analytic in C.

$$\therefore \oint_C \frac{z\,dz}{z^2+4z+3} = \oint_C \frac{z\,dz}{(z+1)(z+3)}$$

By Cauchy's Integral Formula,

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0)$$

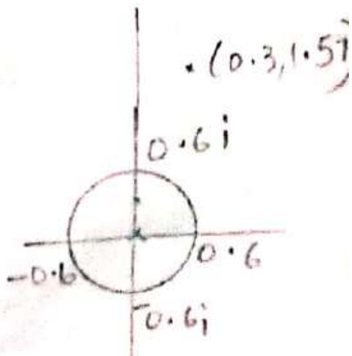
$$\therefore \oint_C \frac{z\,dz}{(z+1)(z+3)} = \oint_C \frac{z\,dz/z+1}{z+3} + \oint_C \frac{z\,dz/z+3}{z+1}$$

$$= 2\pi i\left[\frac{-3}{-2} + \frac{(-1)}{2}\right]$$

$$= 2\pi i\left[(3-1)/2\right]$$

$$\therefore \oint_{|z+1|=2} \frac{z\,dz}{z^2+4z+3} = \underline{\underline{2\pi i}}$$

5) $\oint_C \dfrac{e^z\,dz}{ze^2-2iz}$, $C: |z| = 0.6$



The singular points are given by $ze^z - 2iz = 0$

$$z(e^z - 2i) = 0$$

$$z = 0 \quad , \quad e^z = 2i$$

$$z = \log 2i = \frac{1}{2}\log 4 + i\frac{\pi}{2}$$

$$z = 0.3 + i(1.57)$$

$\therefore$ The singular point $z_0 = 0$ lies inside C. Therefore the given function is not analytic in C.

$$\therefore \oint_C \frac{e^z dz}{ze^z - 2iz} = \oint_C \frac{e^z dz / e^z - 2i}{z}$$

By Cauchy's Integral Formula

$$\oint_C \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0)$$

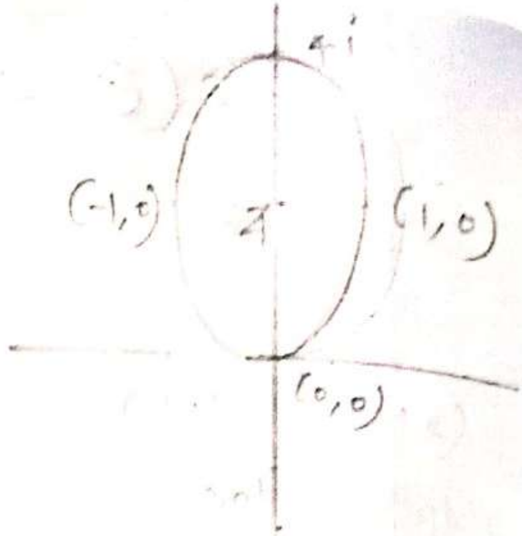
$$\therefore \oint_C \frac{e^z dz / e^z - 2i}{z} = 2\pi i \cdot \left[\frac{e^0}{e^0 - 2i}\right]$$

$$= 2\pi i \left[\frac{1}{1 - 2i}\right]$$

$$\oint_{|z| = 0.6} \frac{e^z dz}{ze^z - 2iz} = \frac{2\pi i}{1 - 2i}$$

Questions

① $\oint_C \frac{dz}{z^2 + 4}$, $C: 4x^2 + (y-2)^2 = 4$



$$4x^2+(y-2)^2=4$$

$$4\left(x^2+\frac{(y-2)^2}{4}\right)=4$$

$$x^2+\frac{(y-2)^2}{4}=1$$

The singular point is given by

$z^2+4=0 \Rightarrow z^2=-4 \Rightarrow z=\pm 2i$

Here the singular point $z=2i$ lies inside the ~~eli~~ c. Therefore the function is not analytic at c.

$$\therefore \oint_c \frac{dz}{z^2+4} = \oint_c \frac{dz}{(z-2i)(z+2i)}$$

By Cauchy's Integration formula,

$$\oint_c \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0)$$

$f(z)=\frac{1}{z+2i}$ $\quad f(2i)=\frac{1}{4i}$

$$\therefore \oint_c \frac{dz/z+2i}{z-2i} = 2\pi i\left[\frac{1}{4i}\right]$$

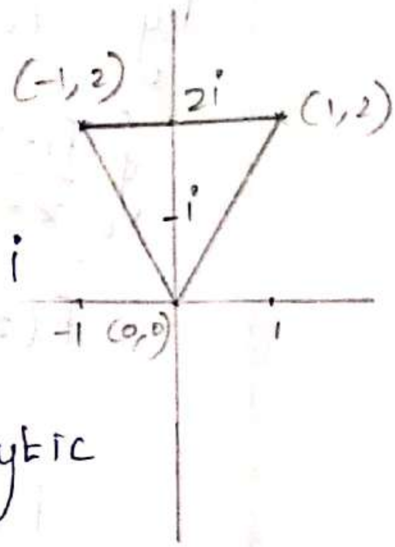
$$\therefore \oint_c \frac{dz}{z^2+4} = \frac{\pi}{2}$$

② $\oint_c \frac{\tan z}{z-i}\,dz$, c: triangle $(0,(1+2i),(-1+2i))$

Singular points are given by

$z - i = 0$

$z = i$



Here the singular point $z_0 = i$ lies inside c. Therefore the given function is not analytic in c.

By Cauchys Integral formula,

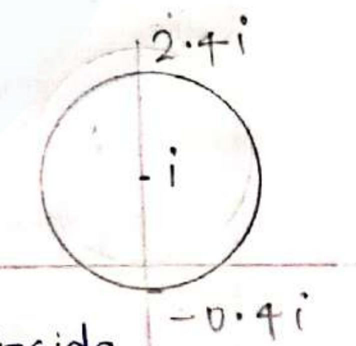
$$\oint_c \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0)$$

$f(z) = \tan z$ $\quad f(i) = \tan i = i \tanh 1$

$$\therefore \oint_c \frac{\tan z\,dz}{z - i} = 2\pi i\,(i \tanh 1)$$

$$\oint_c \frac{\tan z\,dz}{z - i} = -2\pi \tanh 1$$

③ $\displaystyle\oint_c \frac{\log(z+1)}{z^2+1}\,dz$, $c : |z - i| = 1.4$



Singular points are given by

$z^2 + 1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$

The singular point $z_0 = i$ lies inside c. Therefore f is not. not analytic in c.

$$\oint_c \frac{\log(z+1)}{(z+i)(z-i)}\,dz = \oint_c \frac{\log(z+1)\,dz / z+i}{z - i}$$

By Cauchy's integral theorem,

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0)$$

$$f(z) = \frac{\log(z+1)}{z+i}$$

$$f(z_0) = f(i) = \frac{\log(i+1)}{2i} = \frac{\left[\frac{1}{2}\log 2 + \frac{i\pi}{4}\right]}{2i}$$

$$f(i) = \frac{\log 2}{4i} + \frac{\pi}{8}$$

$$f(i) = \frac{-\log 2}{4} + \frac{\pi}{8}$$

$$\therefore \oint_{|z-i|=1\cdot 4} \frac{\log(z+1)}{z^2+1}\,dz = 2\pi i\left[\frac{-\log 2}{4} + \frac{\pi}{8}\right]$$

$$= \frac{\pi \log 2}{2} + \left[\frac{-\pi^2 i}{4}\right]$$

$$\therefore \oint_{|z-i|=1\cdot 4} \frac{\log(z+1)}{z^2+1}\,dz = \frac{\pi \log 2}{2} \quad \frac{-\pi^2 i}{4}$$

Chapter-4 Cauchy's Integral formula for higher derivatives

Let $f(z)$ be analytic in a simply connected domain D and z_0 is a point in D. Then

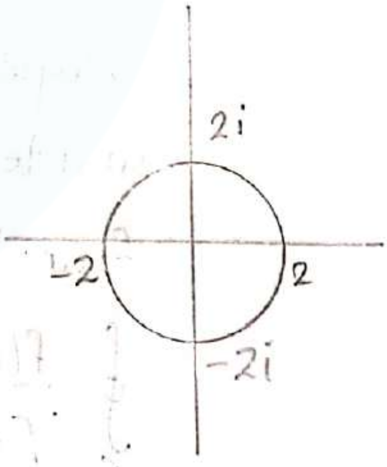
$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$$ where c is a closed path in D in the counter clockwise direction.

1) $$\oint_{|z|=2} \frac{z^4-3z^2+6}{(z+i)^3}\,dz$$

Singular points are given by

$$(z+i)^3 = 0$$

$$z = -i$$



The singular point $z=-i$ lies inside $|z|=2$.

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = z^4-3z^2+6$$

$$f'(z) = 4z^3-6z$$

$$f''(z) = 12z^2-6$$

$f''(i) = -12-6 = -18$

$$\oint_{|z|=2} \frac{z^4-3z^2+6}{(z+i)^3}\,dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$\therefore \oint_{|z|=2} \frac{z^4-3z^2+6}{(z+i)^3} = \frac{2\pi i \times (-18)}{2!}$$

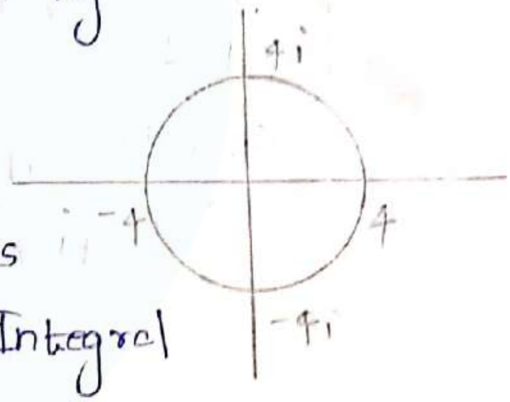
$$= -18\pi i$$

2) $\displaystyle\oint_{|z|=4} \frac{\cos z}{(z-\pi i)^2}\,dz$

Singular points are given by

$z = \pi i$

$z = 3.14 i$



$\therefore$ Singular point $z_0 = \pi i$ lies inside C. By cauchys Integral formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$f(z) = \cos z$

$f'(z) = -\sin z$

$\therefore f'(\pi i) = -\sin(\pi i)$

$f'(\pi i) = -i\sinh\pi$

$$\oint_{|z|=4} \frac{\cos z}{(z-\pi i)^2}\,dz = \frac{2\pi i}{1!} \times (-i\sinh\pi)$$

$$= \underline{\underline{2\pi \sinh\pi}}$$

3) $$\oint_{|z|=1} \frac{z+1}{z^2}\,dz$$

Singular point $z_0 = 0$ lies outside C. By Cauchys Integral Formula

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = z+1$$

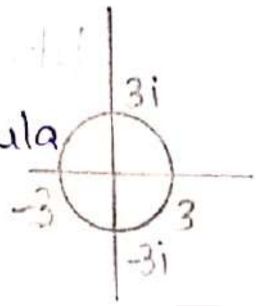
$$f'(z) = 1$$

$$f'(0) = 1$$

$$\therefore \oint_{|z|=1} \frac{z+1}{z^2}\,dz = \frac{2\pi i}{1!} \cdot (1) = \underline{\underline{2\pi i}}$$

4) $$\oint_{|z|=3} \frac{z^2+5}{(z-2)^3}\,dz$$

Singular point $z_0 = 2$ lies inside $|z| = 3$. By Cauchy's Integral Formula



$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = z^2+5$$

$$f'(z) = 2z$$

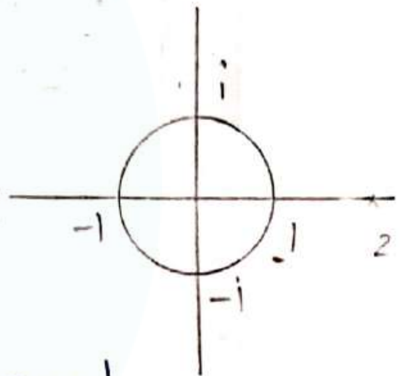
$$f''(z) = 2$$

$$f''(2) = \underline{\underline{2}}$$

$$\oint_C \frac{z^2+5}{(z-2)^3}\, dz = \frac{2\pi i}{2!} \times 2$$

$$= \underline{\underline{2\pi i}}$$

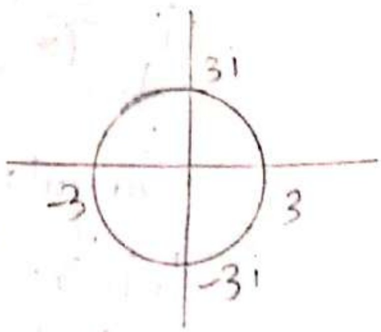
* 5) $\displaystyle\oint_{|z|=1} \frac{z^2+5}{(z-2)^3}\, dz$



Singular point $z_0 = 2$ lies outside $|z|=1$. Therefore the given function is analytic in $|z|=1$. By Cauchys Integral theorem

$$\oint_{|z|=1} \frac{z^2+5}{(z-2)^3}\, dz = 0$$

6) $\displaystyle\oint_{|z|=3} \frac{2z^2-z-2}{(z-2)^3}\, dz$



Singular point $z_0 = 2$ lies inside $|z|=3$. By Cauchy s

Integral Formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = 2z^2 - z - 2$$

$$f'(z) = 4z - 1$$

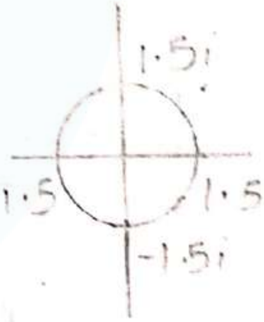
$$f''(z) = 4$$

$$f''(2) = 4$$

$$\therefore \oint_{|z|=3} \frac{2z^2 - z - 2}{(z-2)^3}\,dz = \frac{2\pi i}{2!} \times 4$$

$$= \underline{\underline{4\pi i}}$$

* 7) $\displaystyle\oint_{|z|=1.5} \frac{e^z}{(z+1)^2(z^2+4)}\,dz$



Singular points are given by

$z = 1, 1, \pm 2i$

$$\oint_C \frac{e^z/(z^2+4)}{(z-1)^2}\,dz$$

Here singular $z_0 = 1$ lies inside C

By Cauchys Integral formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = \frac{e^z}{z^2+4}$$

$$f'(z) = \frac{(z^2+4)e^z - e^z \times 2z}{(z^2+4)^2}$$

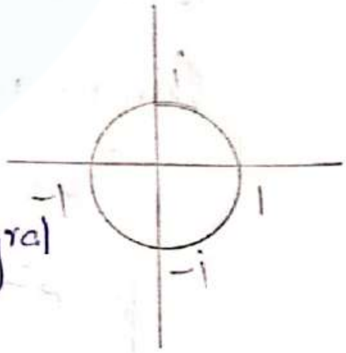
$$f'(1) = \frac{5 \times e - e \times 2}{(5)^2}$$

$$f'(1) = \frac{3e}{25}$$

$$\oint\limits_{|z|=1.5} \frac{e^z}{(z-1)^2(z^2+4)} dz = \frac{2\pi i}{1!} \times \frac{3e}{25}$$

$$= \frac{6\pi i e}{25}$$

8) $\oint\limits_{|z|=1} \frac{e^z}{z^5} dz$



Singular point $z_0 = 0$ lies inside C. By Cauchys Integral formula

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = e^z$$

$$f'(z) = e^z$$

$$f^{IV}(z) = e^z$$

$$f^{IV}(0) = e^0 = \underline{\underline{1}}$$

$$\therefore \oint_{|z|=1} \frac{e^z}{z^5} dz = \frac{2\pi i}{4!} \times 1$$

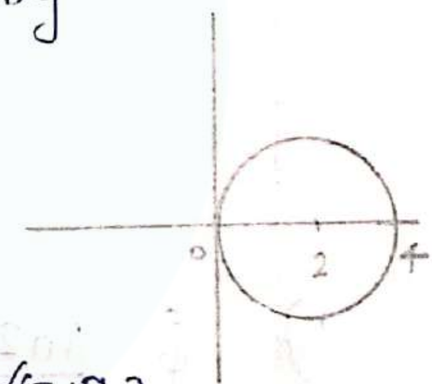
$$= \frac{2\pi i}{\cancel{16}\ 24}$$

$$= \underline{\underline{\frac{\pi i}{12}}}$$

9) $\oint_{|z-2|=2} \frac{z^2}{(z+2)(z-1)^2} dz$ clockwise

Singular points are given by

$z = 1, 1, -2$



Singular points $z_0 = 1$ lies inside $|z-2| = 2$

$$\therefore \oint_{|z-2|=2} \frac{z^2}{(z+2)(z-1)^2} dz = \oint_{|z-2|=2} \frac{z^2 dz/(z+2)}{(z-1)^2}$$

$\therefore$ By Cauchys Integral Formula,

$$\oint_c \frac{f(z)dz}{(z-z_0)^{n+1}} = -\frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = \frac{z^2}{z+2}$$

$$f'(z) = \frac{(z+2)\cdot 2z - z^2(1)}{(z+2)^2}$$

$$= \frac{2z^2+4z-z^2}{(z+2)^2}$$

$$f'(z) = \frac{z^2+4z}{(z+2)^2}$$

$$f'(1) = \frac{1+4}{(1+2)^2} = \frac{5}{9}$$

$$\therefore \oint_{|z-2|=2} \frac{z^2}{(z+2)(z-1)^2}\,dz = -\frac{2\pi i}{1!} \times \frac{5}{9}$$

$$= -\frac{10\pi i}{9}$$

10) $\oint_{|z|=1} \frac{\sin 2z}{z^4}\,dz.$

Singular point $z_0=0$ lies inside $|z|=1$. By Cauchys Integral formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = \sin 2z$$

$f'(z) = 2\cos 2z$

$f''(z) = -4\sin 2z$

$f'''(z) = -8\cos 2z$

$f'''(0) = \underline{\underline{-8}}$

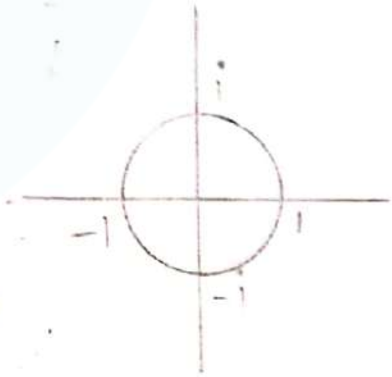
$$\oint_{|z|=1} \frac{\sin 2z}{z^4}\,dz = \frac{2\pi i}{3!} \times (-8)$$

$$= \frac{2\pi i}{6} \times -8$$

$$= \underline{\underline{-\frac{8\pi i}{3}}}$$

ii) $\oint_{|z|=1} \frac{\sinh 2z}{(z-1/2)^4}\,dz$

Singular point $z_0 = \frac{1}{2}$ lies inside $|z|=1$. By Cauchys Integral formula,



$$\oint_c \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$f(z) = \sinh 2z$

$f'(z) = 2\cosh 2z$

$f''(z) = 4\sinh 2z$

$f'''(z) = 8\cosh 2z$

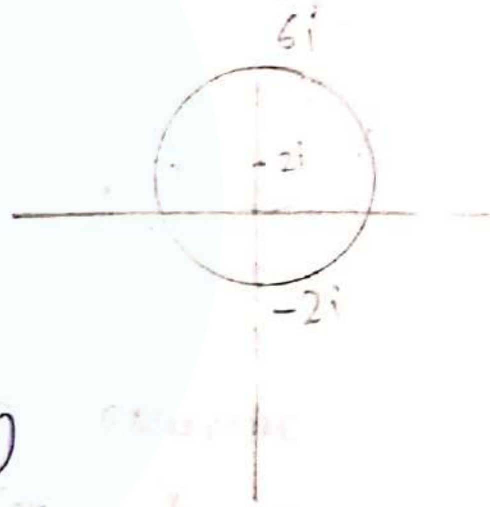
$$f'''\left(\tfrac{1}{2}\right) = \; = 8\cosh 1$$

$$\oint_{|z|=1} \frac{\sinh 2z}{\left(z-\frac{1}{2}\right)^4}\,dz = \frac{2\pi i}{3!} \times 8\cosh 1$$

$$= \frac{8\pi i \cosh 1}{3}$$

12) $\displaystyle\oint_{|z-2i|=4} \left(\frac{5}{z-2i} - \frac{6}{(z-2i)^2}\right) dz$ Clockwise

Singular point $z_0 = 2i$ lies inside $|z-2i| = 4$.



$\therefore$ By Cauchys Integral ~~theorem~~ formula,

$$\oint_C \frac{f(z)\,dz}{z-z_0} = -2\pi i f(z_0)$$

$$\oint_{|z-2i|=4} \frac{5}{z-2i}\,dz = -2\pi i \times 5 = -10\pi i$$

By Cauchys Integral formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{-2\pi i}{n!} f^n(z_0)$$

$$f(z) = 6$$

$$f'(z) = 0 \qquad f'(2i) = 0$$

$$\oint_{|z-2i|=4} \frac{6}{(z-2i)^2} dz = -\frac{2\pi i}{1!} \times 0$$

$$= \underline{\underline{0}}$$

$$\therefore \oint_{|z-2i|=4} \left(\frac{5}{z-2i} - \frac{6}{(z-2i)^2} \right) dz = -10\pi i$$

13) $\oint_{|z|=2} \frac{\sin z}{\left(z-\frac{\pi}{2}\right)^3} dz$

Singular point $z_0 = \frac{\pi}{2}$ lies inside $|z|=2$. By Cauchys Integral formula

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = \sin z$$

$$f'(z) = \cos z$$

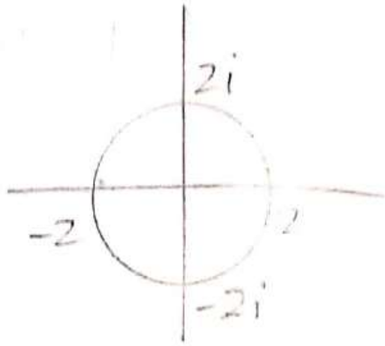
$$f''(z) = -\sin z$$

$$f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = \underline{\underline{-1}}$$

$$\therefore \oint_{|z|=2} \frac{\sin z}{\left(z-\frac{\pi}{2}\right)^3} dz = \frac{2\pi i}{2!} \times (-1)$$

$$= \underline{\underline{-\pi i}}$$

14) $$\oint_{|z|=2} \frac{dz}{z^2(z-3)} = \oint_{|z|=2} \frac{dz/z-3}{z^2}$$



Singular point $z_0 = 0$ lies inside $|z|=2$. By Cauchys Integral formula

$$\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = \frac{1}{z-3}$$

$$f'(z) = \frac{-1}{(z-3)^2}$$

$$f'(0) = \frac{-1}{9}$$

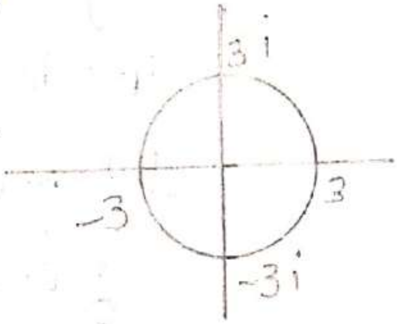
$$\therefore \oint_{|z|=2} \frac{dz}{z^2(z-3)} = \frac{2\pi i}{1!} \times \left(-\frac{1}{9}\right)$$

$$= -\frac{2\pi i}{9}$$

15) $$\oint_C \frac{2z^3-z^2-2}{(z-2)^4} dz$$ (a) $|z|=3$ (b) $|z|=1$

(a) Singular point $z_0 = 2$ lies inside $|z| = 3$. By Cauchys Integral formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$



$$f(z) = 2z^3 - z^2 - 2$$

$$f'(z) = 6z^2 - 2z$$

$$f''(z) = 12z - 2$$

$$f'''(z) = 12$$

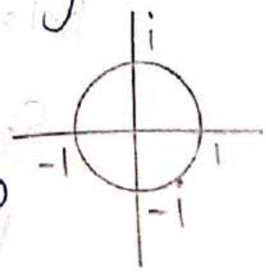
$$f'''(2) = 12$$

$$\therefore \oint_{|z|=3} \frac{2z^3 - z^2 - 2}{(z-2)^4}\,dz = \frac{2\pi i}{3!} \times 12$$

$$= \frac{2\pi i}{6} \times 12$$

$$= \underline{\underline{4\pi i}}$$

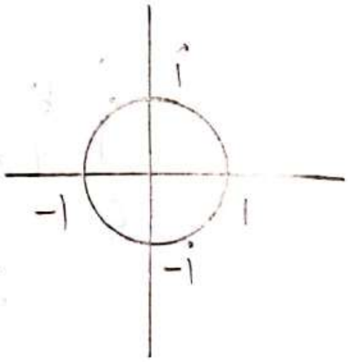
(b) Singular point $z_0 = 2$ lies outside $|z| = 1$. Therefore the given function is analytic in $|z| = 1$. By Cauchys Integral theorem,



$$\oint \frac{\cancel{f} 2z^3 - z^2 - 2}{(z-2)^4}\,dz = 0$$

16) $$\oint_{|z|=1} \frac{z^3+1}{(3z+1)^3}\,dz = \frac{1}{27}\oint_{|z|=1} \frac{z^3+1}{\left(z+\frac{1}{3}\right)^3}\,dz$$

Singular point $z_0 = \frac{-1}{3}$ lies inside $|z|=1$. By Cauchys Integral formula,



$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} \times f^n(z_0)$$

$$f(z) = z^3+1$$

$$f'(z) = 3z^2$$

$$f''(z) = 6z$$

$$f''\left(\frac{-1}{3}\right) = 6 \times \frac{-1}{3} = -2$$

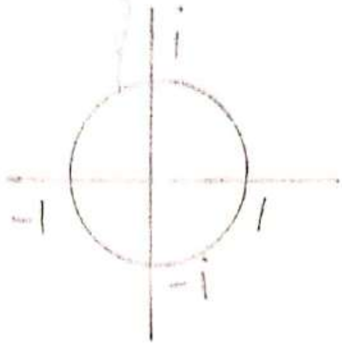
$$\therefore \oint_{|z|=1} \frac{z^3+1}{(3z+1)^3}\,dz = \frac{1}{27} \times \frac{2\pi i}{2!} \times -2$$

$$= \frac{-2\pi i}{27}$$

17) $$\oint_{|z|=1} \frac{z^6}{(2z-1)^6}\,dz = \frac{1}{64}\oint_{|z|=1} \frac{z^6}{\left(z-\frac{1}{2}\right)^6}\,dz$$

Singular point $z_0 = \frac{1}{2}$ lies inside $|z|=1$. By Cauchys Integral formula

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$



$$f(z) = z^6$$

$$f'(z) = 6z^5$$

$$f''(z) = 30z^4$$

$$f'''(z) = 120z^3$$

$$f^{IV}(z) = 360z^2$$

$$f^{V}(z) = 720z$$

$$f^{V}\left(\tfrac{1}{2}\right) = 720 \times \frac{1}{2}$$

$$= \underline{\underline{360}}$$

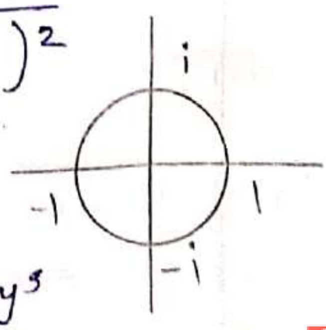
$$\therefore \oint_{|z|=1} \frac{z^6}{\left(z-\frac{1}{2}\right)^6}\,dz = \frac{2\pi i}{5!} \times 360 \times \frac{1}{64}$$

$$\frac{2\pi i}{120} \times 360 \times \frac{1}{64}$$

$$= 2\pi i \times 3 \times \frac{1}{64}$$

$$= \underline{\underline{6\pi i}} \times \frac{1}{64} = \underline{\underline{\frac{3\pi i}{32}}}$$

18) $$\oint_{|z|=1} \frac{dz}{(z-2i)^2\left(z-\frac{i}{2}\right)^2} = \oint_{|z|=1} \frac{dz/(z-2i)^2}{\left(z-\frac{i}{2}\right)^2}$$



Singular point $z_0 = \frac{i}{2}$ lies inside ~~outside~~ $|z|=1$. By Cauchy's

Integral formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i f^n(z_0)}{n!}$$

$$f(z) = \frac{1}{(z-2i)^2}$$

$$f'(z) = \frac{-2}{(z-2i)^3}$$

$$f'\left(\frac{i}{2}\right) = \frac{-2}{\left(\frac{i}{2}-2i\right)^3}$$

$$= \frac{-2}{\left(\frac{i-4i}{2}\right)^3}$$

$$= \frac{-2}{\left(\frac{-3i}{2}\right)^3}$$

$$= \frac{-2}{\frac{27i}{8}}$$

$$= \frac{-16i^3}{27}$$

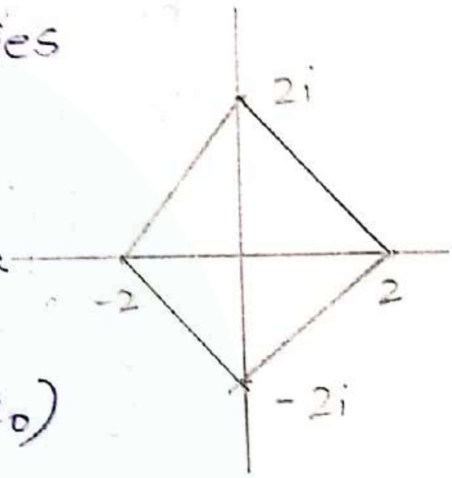
$$= \frac{16i}{27}$$

$$\therefore \oint_{|z|=1} \frac{dz\,(z-2i)^2}{\left(z-\frac{i}{2}\right)^2} = \frac{2\pi i}{1!} \times \frac{16i}{27}$$

$$= \frac{-32\pi}{27}$$

17) $\oint_C \frac{z^3+\sin z}{(z-i)^3}\,dz$, C: square $(\pm 2, \pm 2i)$

The singular point $z_0 = i$ lies inside the square. By Cauchys Integral formula



$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = z^3 + \sin z$$

$$f'(z) = 3z^2 + \cos z$$

$$f''(z) = 6z - \sin z$$

$$f''(i) = 6i - \sin i = 6i - i\sinh 1$$

$$\therefore \oint_C \frac{z^3+\sin z}{(z-i)^3}\,dz = \frac{2\pi i}{2!} \times (6i - i\sinh 1)$$

$$= \pi i(6i - i\sinh 1)$$

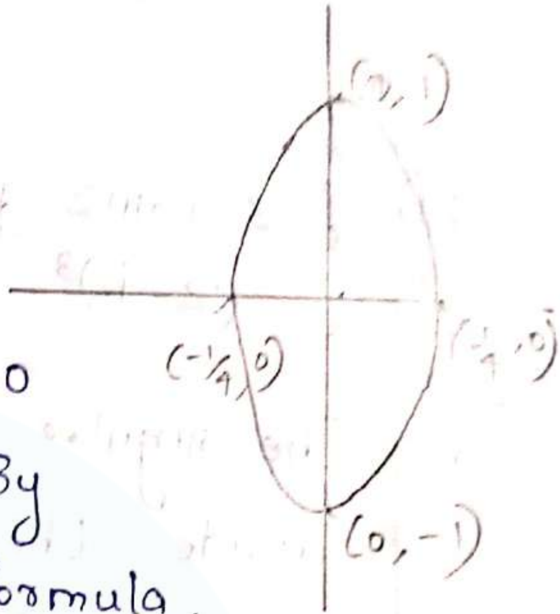
$$= 6\pi i^2 - \pi i^2 \sinh 1$$

$$= -6\pi + \pi \sinh 1$$

20) $\oint_C \frac{\tan \pi z}{z^2} dz \qquad C: 16x^2+y^2=1$

$16x^2+y^2=1$

$\frac{x^2}{\left(\frac{1}{4}\right)^2}+y^2=1$



The singular point $z_0=0$ lies inside the ellipse. By Cauchy's Integral formula,

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$f(z) = \tan \pi z$$

$$f'(z) = \pi \sec^2 \pi z$$

$$f'(0) = \pi \times \frac{1}{\cos^2 0}$$

$$f'(0) = \pi$$

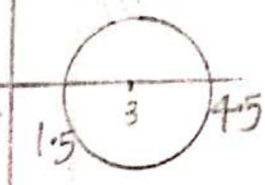
$$\therefore \oint_C \frac{\tan \pi z}{z^2} dz = \frac{2\pi i}{1!} \times (\pi)$$

$$= 2\pi^2 i$$

Hw

21) $\oint \frac{e^{-z} - \sin z}{(z-4.6)^3} dz \qquad C: |z-3| = \frac{3}{2}$ Clockwise.

The singular point $z_0 = 4.6$ lies outside the path $|z-3| = \frac{3}{2}$. Therefore the given function is analytic in $|z-3| = \frac{3}{2}$. By Cauchys Integral theorem

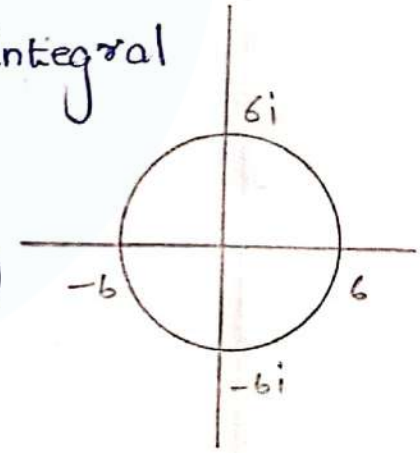


$$\oint_{|z-3|=\frac{3}{2}} \frac{e^{-z} - \sin z}{(z-4.6)^3}\, dz = 0$$

HW

22) $\oint_C \frac{\cosh 4z}{(z-4)^3}$ (a) $|z|=6$ positive (anticlockwise) (b) $|z-3|=2$ negative (clockwise)

(a) The singular point $z_0 = 4$ lies inside $|z|=6$. By Cauchys Integral theorem



$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{n}(z_0)$$

$f(z) = \cosh 4z$

$f'(z) = 4\sinh 4z$

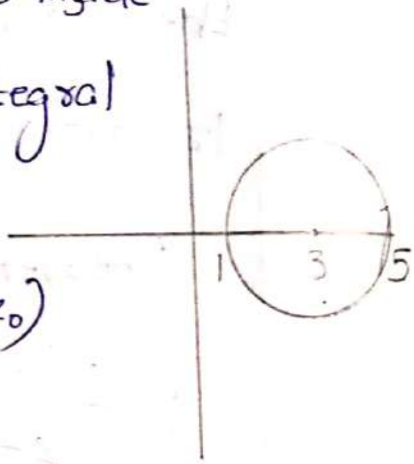
$f''(z) = 16\cosh 4z$

$f''(4) = 16\cosh 16$

$$\therefore \oint_C \frac{\cosh 4z}{(z-4)^3} = \frac{2\pi i}{2!} \times 16\cosh 16$$

$= 16\pi i \cosh 16$

b) The singular point $z_0 = 4$ lies inside $|z-3| = 2$. By Cauchy's Integral theorem



$$\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = -\frac{2\pi i}{n!} \times f^n(z_0)$$

$$f(z) = \cosh 4z$$

$$f'(z) = 4\sinh 4z$$

$$f''(z) = 16\cosh 4z$$

$$f''(4) = 16\cosh 16$$

$$\therefore \oint_C \frac{\cosh 4z\,dz}{(z-4)^3} = -\frac{2\pi i}{2!} \times 16\cosh 16$$

$$= -16\pi i \cosh 16$$

* POWER SERIES *

There exists a power series representation for every analytic function in complex analysis.

The series representation in power of $(z-z_0)$ of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots\dots$$

where z is a complex variable is called a power series. $a_0, a_1, \dots$ are called the coefficients of the series and 'z_0' is a complex number called the centre of the series.

If $z_0 = 0$, $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots\dots$

consider the smallest circle with centre z_0 that includes all points at which the above power series converge. Let R denotes its radius. The circle $|z-z_0| = R$ is called circle of convergence and 'R' is called the Radius of convergence.

Maclaurin series

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots\cdots$$

Q* Find the maclaurin series of the following.

1* e^z

$f(z) = e^z$

$f'(z) = f''(z) = f'''(z) = e^z$

$f'(0) = f''(0) = f'''(0) = 1$

maclaurin series is given by

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots$$

$\Rightarrow$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots$$

2* $\cos z$

$f(z) = \cos z$, $f(0) = 1$

$f'(z) = -\sin z$, $f'(0) = 0$

$f''(z) = -\cos z$, $f''(0) = -1$

$f'''(z) = \sin z$, $f'''(0) = 0$

$f^{iv}(z) = \cos z$ $f^{iv}(0) = 1$

Maclaurin series is given by

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots$$

$\therefore$

$$\cos z = 1 + \frac{z}{1!} \times 0 + \frac{z^2}{2!} \times (-1) + \frac{z^3}{3!} \times 0 + \frac{z^4}{4!} \times (1) + \dots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\dots$$

(3) $\sin z$

$f(z) = \sin z \qquad f(0) = 0$

$f'(z) = \cos z \qquad f'(0) = 1$

$f''(z) = -\sin z \qquad f''(0) = 0$

$f'''(z) = -\cos z \qquad f'''(0) = -1$

$f^{iv}(z) = \sin z \qquad f^{iv}(0) = 0$

$f^{v}(z) = \cos z \qquad f^{v}(0) = 1$

Maclaurin series is given by

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \dots$$

$\therefore$

$$\sin z = \frac{z}{1!} \times 1 + 0 + \frac{z^3}{3!} \times (-1) + 0 + \frac{z^5}{5!} \times 1$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\dots\dots$$

(4) $\cosh z$

$f(z) = \cosh z \qquad f(0) = 1$

$f'(z) = \sinh z \qquad f'(0) = 0$

$f''(z) = \cosh z \qquad f''(0) = 1$

$f'''(z) = \sinh z \qquad f'''(0) = 0$

$f^{iv}(z) = \cosh z \qquad f^{iv}(0) = 1$

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots$$

$$\therefore \cosh z = 1 + z \times 0 + \frac{z^2}{2!} \times 1 + \frac{z^3}{3!} \times 0 + \frac{z^4}{4!} \times 1 + \cdots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

(5) $\sinh z$

$f(z) = \sinh z \qquad f(0) = 0$

$f'(z) = \cosh z \qquad f'(0) = 1$

$f''(z) = \sinh z \qquad f''(0) = 0$

$f'''(z) = \cosh z \qquad f'''(0) = 1$

$f^{iv}(z) = \sinh z \qquad f^{iv}(0) = 0$

$f^{v}(z) = \cosh z \qquad f^{v}(0) = 1$

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots$$

$$\therefore \sinh z = 0 + z + 0 + \frac{z^3}{3!} + 0 + \frac{z^5}{5!} + \cdots$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

(6) $\log(1+z)$

$f(z) = \log(1+z)$, $f(0) = 0$

$f'(z) = \frac{1}{1+z}$, $f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2}$, $f''(0) = -1$

$4\times3\times3$

$f'''(z) = \frac{2}{(1+z)^3}$, $f'''(0) = 2$ -6

$f(z) = f(z_0) + \frac{z}{1!}f'(z_0) + \frac{z^2}{2!}f''(z_0) + \cdots$

$\therefore$

$\log(1+z) = 0 + \frac{z}{1} + \frac{z^2}{2!}\times -1 + \frac{z^3}{3!}\times 2 + \cdots$

$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\cdots$

(7) $-\log(1-z)$

$f(z) = -\log(1-z)$ $f(0) = 0$

$f'(z) = -\frac{1}{1-z}\times -1 = \frac{1}{1-z}$ $f'(0) = 1$

$f''(z) = \frac{-1}{(1-z)^2}\times -1 = \frac{1}{(1-z)^2}$ $f''(0) = 1$

$f'''(z) = \frac{2}{(1-z)^3}$ $f'''(0) = 2$

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots,$$

$$\therefore -\log(1-z) = 0 + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

(8) $\frac{1}{1+z^2}$

$$f(z) = \frac{1}{1+z^2}$$

$$f'(z) = \frac{-1}{1+z^2} \times 2z = \frac{-2z}{1+z^2}$$

$$f''(z) = \frac{-2}{(1+z^2)^2}\left[(1+z^2) - z(2z)\right]$$

$$= \frac{-2(1-z^2)}{(1+z^2)^2}$$

$$f'''(z) = -2\left[\frac{(1+z^2)^2 \times (-2z) - (1-z^2)2(1+z^2)\cdot 2z}{(1+z^2)^4}\right]$$

$$= -\frac{2(1+z^2)}{(1+z^2)^4}\left[-2z - 2z^3 - 4z + 4z^3\right]$$

$$= \frac{-2\left[2z^3 - 6z\right]}{(1+z^2)^3}$$

$$f^{ive}(z) = \frac{-2\left[(1+z^2)^3(6z^2-6) - (2z^3-6z)3(1+z^2)^2\cdot 2z\right.}{(1+z^2)^6}$$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = -2$

$f'''(0) = 0$

$f^{iv}(0) = -12$

$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots$

$\frac{1}{1+z^2} = 1 + 0 + \frac{z^2}{2!} \times -2 + 0 + \frac{z^4}{4!} \times +12$

$= \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots\cdots$

9

$\frac{z+2}{1-z^2}$

let $\frac{z+2}{1-z^2} = \frac{A}{1+z} + \frac{B}{1-z}$

$2 = A + B$

$A = 2 - B$

$A = 2 - 1 = 1$

$z+2 = A(1-z) + B(1+z)$

For $z = 1$

$3 = 2B$; $B = \frac{3}{2}$

For $z = -1$

$1 = 2A$; $A = \frac{1}{2}$

$$\frac{z+2}{1-z^2} = \frac{1}{2(1+z)} + \frac{3}{2(1-z)}$$

$$f(z) = \frac{1}{2(1+z)} + \frac{3}{2(1-z)}$$

$$f'(z) = \frac{-1}{2(1+z)^2} + \frac{3}{2(1-z)^2}$$

$$f''(z) = \frac{1}{(1+z)^3} + \frac{3}{(1-z)^3}$$

$$f'''(z) = \frac{-3}{(1+z)^4} + \frac{9}{(1-z)^4}$$

$$f(0) = \frac{3}{2} + \frac{1}{2} = \underline{\underline{2}}$$

$$f'(0) = -\frac{1}{2} + \frac{3}{2} = \underline{\underline{1}}$$

$$f''(0) = 1 + 3 = \underline{\underline{4}}$$

$$f'''(0) = -3 + 9 = \underline{\underline{6}}$$

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots$$

$$\frac{z+2}{1-z^2} = 2 + z + \frac{z^2}{2} \times 4 + \frac{z^3}{3 \times 2} \times 6$$

$$\frac{z+2}{1-z^2} = 2 + z + \underline{\underline{2z^2 + z^3}} + \cdots$$

10* $\sin(2z^2)$

Maclaurin series of $\sin z$ is given by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots \quad (1)$$

Here for $\sin(2z^2)$

z in (1) is replaced by $2z^2$ so

we get.

$$\sin 2z^2 = 2z^2 - \frac{(2z^2)^3}{3!} + \frac{(2z^2)^5}{5!} - \dots$$

$$= 2z^2 - \frac{4}{3}z^6 + \frac{4}{15} \times z^{10} - \dots$$

10* $\sin(2z^2)$

Maclaurin series of $\sin z$ is given by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots \quad (1)$$

Here for $\sin(2z^2)$

z in (1) is replaced by $2z^2$ so we get.

$$\sin 2z^2 = 2z^2 - \frac{(2z^2)^3}{3!} + \frac{(2z^2)^5}{5!} - \cdots$$

$$= 2z^2 - \frac{4}{3} z^6 + \frac{4}{15} \times z^{10} - \cdots$$

11* $f(z) = \frac{e^{z^2} - 1}{z^2}$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$f(z) = \frac{1}{z^2}\left[1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \frac{z^6}{3!} + \cdots - 1\right]$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{3!} + \cdots$$

12* $f(z) = 2\sin^2\frac{z}{2}$ $\cos 2\theta = 1 - 2\sin^2\frac{z}{2}$

$2\sin^2\theta = 1 - \cos 2\theta$ $2\sin^2$

$f(z) = 1 - \cos z$

$$f(z) = 1 - \left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]$$

$$= \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\cdots$$

$\left[e^{-t^2} \cdot -2t\right]_0^2$

13* $\int_0^z \exp(-t^2)\,dt$

$$f(z) = \int_0^z e^{-t^2}\,dt$$

$\left(e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\right)$

$$= \int_0^z 1 + \frac{(-t^2)}{1!} + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots \quad dt$$

$$= \int_0^z 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots \quad dt$$

$$= \left[t - \frac{t^3}{3} + \frac{t^5}{10} - \frac{t^7}{7 \times 3!} + \cdots\right]_0^z$$

$$= z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots$$

$J \times 4 \times 3 \times 2$

a

zeda

14* $f(z) = \int_0^z \sin t^2 \, dt$

we know that $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\dots$

$\therefore$ ~~sin~~

$f(z) = \int_0^z t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \dots\dots \, dt$

$= \left[\frac{t^3}{3} - \frac{t^7}{42} + \frac{t^{11}}{120 \times 11} - \dots \right]_0^z$

$= \frac{z^3}{3} - \frac{z^7}{42} + \frac{z^{11}}{1320} - \dots\dots$

Chapter-6

Taylor Series

let $f(z)$ be analytic in a circle $|z-z_0| = r$ then for any 'z' with in the circle $|z-z_0| < r$ we can represent the function $f(z)$ in the form $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

where $a_n = \frac{1}{n!} f^n(z_0)$

This expansion is called Taylor series of $f(z)$ about $z = z_0$

Q* Expand the following as Taylor series, also find the region of validity if any.

1* $f(z) = \frac{1}{z+1}$ about a) $z = 3$
b) $z = -2$

$$\left.\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ a_n &= \frac{1}{n!} f^n(z_0) \end{aligned}\right\} \text{Taylor series expansion}$$

a)

$$f(z) = \frac{1}{z+1} = \frac{1}{(z-3)+4}$$

$$= \frac{1}{4\left[1 + \frac{(z-3)}{4}\right]}$$

$$= \frac{1}{4}\left[1 + \left[\frac{z-3}{4}\right]\right]^{-1}$$

we know that

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots$$

$\therefore$

$$f(z) = \frac{1}{4}\left[1 - \frac{z-3}{4} + \left(\frac{z-3}{4}\right)^2 - \left(\frac{z-3}{4}\right)^3 + \dots\right]\dots$$

The region of validity is $\left|\frac{z-3}{4}\right| < 1$

or $|z-3| < 4$

b) $f(z) = \frac{1}{z+1} = \frac{1}{(z+2)-1} = \frac{-1}{[1-(z+2)]}$

$$= -[1-(z+2)]^{-1}$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$\therefore$

$f(z) = -[1-(z+2)]^{-1} = -[1 + (z+2) + (z+2)^2 + (z+2)^3 + \dots]$

$= -1 - (z+2) - (z+2)^2 - (z+2)^3 + \dots$

The region of validity is $|z+2| < 1$ which is an open disc with centre -2 and radius 1

2\) $f(z) = \frac{1}{z^2}$ about $z=2$

$f(z) = \frac{1}{z^2} = \frac{1}{((z-2)+2)^2} = \frac{1}{2^2\left(\frac{z-2}{2}+1\right)^2}$

$$= \frac{1}{4}\left[1 + \frac{z-2}{2}\right]^{-2}$$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

$\therefore f(z) = \frac{1}{4}\left[1 - 2\left(\frac{z-2}{2}\right) + 3\left(\frac{z-2}{2}\right)^2 - 4\left(\frac{z-2}{2}\right)^3 + \dots\right]$

Region of validity is $|z-2| < 2$ ~~which~~ which is an open disc with centre 2 and radius 2

 Chapter 6: (7mks)

Taylor Series:

Let $f(z)$ be analytic in a circle $|z-z_0| = r$ then for any z within the circle $|z-z_0| < r$ we can represent the fn: $f(z)$ in the form;

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where;}$$

$$a_n = \frac{1}{n!} f^n(z_0)$$

This expansion is called Taylor Series of $f(z)$ about $z = z_0$

Expand the following as Taylor Series Also find the region of validity if any: (7mks)

75. $f(z) = \frac{1}{z+1}$. about (a) $z = 3$

(b) $z = -2$.

(a) About $z = 3$;

$$f(z) = \frac{1}{z+1} = \frac{1}{(z-3)+4}$$

$$\boxed{\begin{aligned}(1+x)^{-1} &= 1-x+x^2-x^3 \quad |x|<1\\ (1-x)^{-1} &= 1+x+x^2+x^3 \quad |x|<1\end{aligned}}$$

$$= \frac{1}{4}\left[\frac{1}{1+\frac{(z-3)}{4}}\right]$$

$$= \frac{1}{4}\left[1+\left(\frac{z-3}{4}\right)\right]^{-1}$$

$$= \frac{1}{4}\left[1 - \left(\frac{z-3}{4}\right) + \left(\frac{z-3}{4}\right)^2 - \left(\frac{z-3}{4}\right)^3 + \dots\right]$$

The region of validity is $\left|\frac{z-3}{4}\right| < 1$

$|z-3| < 4$ (open disc)

Interior of a circle with centre 3 & radius 4

(b) About $z = -2$.

$$f(z) = \frac{1}{z+1}$$

$$f(z) = \frac{1}{(z+2) - 1}$$

$$\cancel{= \frac{1}{3}\left[\frac{1}{1+\left(\frac{z-2}{3}\right)}\right]}$$

$$\cancel{= \frac{1}{3}\left[1+\left(\frac{z-2}{3}\right)\right]^{-1}}$$

$$\cancel{= \frac{1}{3}\Big[}$$

$$f(z) = \frac{1}{(z+2)-1} = -1\left[1-(z+2)\right]^{-1}$$

$$= -1\left[1+(z+2)+(z+2)^2+(z+2)^3+\dots\right]$$

Region of validity : $|z+2|<1$

Open disc with $c(-2)$ & $r(1)$.

Interior of circle.

76. $f(z) = \dfrac{1}{z^2}$ about $z = 2$.

$$= \frac{1}{(z-2+2)^2} = \frac{1}{(z-2+2)^2} = \frac{1}{((z-2)+2)^2}$$

$$= \frac{1}{4}\left[\frac{1}{\left(1+\frac{z-2}{2}\right)^2}\right]$$

$$= \frac{1}{4\left[1+\frac{z-2}{2}\right]^2}$$

$$= \frac{1}{4}\left(1+\frac{z-2}{2}\right)^{-2}$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\ldots$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots$$ M. Series.

$$= \frac{1}{4}\left[1-2\left(\frac{z-2}{2}\right)+3\left(\frac{z-2}{2}\right)^2 - \frac{4(z-2)^3}{2^3}+\ldots\right]$$

region of Validity ; $\left|\frac{z-2}{2}\right| < 1$ Interior of circle.

$\Rightarrow |z-2| < 2$.

$c(2)$ & $r(2)$ open disc.

24/9/2020

7). $e^{z(z-2)}$ $z=1$

e^{z^2-2z}

$\frac{e^{z^2}}{e^{2z}}$

$$e^{z^2-2z} = e^{(z-1)^2 - 2(z+1)\cancel{+(2z-1)}-2}$$

$\Rightarrow e^{(z-1)^2-1}$

$\Rightarrow e^{-1}\, e^{(z-1)^2}$

$$= \frac{1}{e}\left[1+\frac{(z-1)^2}{1!}+\frac{(z-1)^4}{2!}+\ldots\right]$$

78. $\sin(z)$ $z = \pi/4$

$\sin\left((z-\pi/4)+\pi/4\right)$

$\sin(z-\pi/4)\cos\pi/4 + \sin\pi/4\cos(z-\pi/4)$

$\sin z \Rightarrow \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

$$\frac{1}{\sqrt{2}}\left[\frac{z-\pi/4}{1!} - \frac{(z-\pi/4)^3}{3!} + \frac{(z-\pi/4)^5}{5!} + \cdots + 1 - \frac{(z-\pi/4)^2}{2!} + \frac{(z-\pi/4)^4}{4!} - \frac{(z-\pi/4)^6}{6!} + \cdots\right]$$

79. $\cosh(z-\pi i)$ about $z=\pi i$

$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$

$\cosh(z-\pi i) = 1 + \frac{(z-\pi i)^2}{2!} + \frac{(z-\pi i)^4}{4!} + \cdots$

$= 1 + \frac{(z-\pi i)^2}{2!} + \frac{(z-\pi i)^4}{4!} + \cdots$

80 $\cos(z)$ about $z=\pi$.

$\cos(z-\pi+\pi)$

$\cos(z-\pi)\cos\pi - \sin(z-\pi)\sin\pi$.

$-\cos(z-\pi)$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\Rightarrow \cancel{-1 - \frac{z^2}{2!} + z^4}$$

$$\Rightarrow -\left[1 - \frac{(z-\pi)^2}{2!} + \frac{(z-\pi)^4}{4!} - \frac{(z-\pi)^6}{6!} + \cdots\right].$$

$$= -1 + \frac{(z-\pi)^2}{2!} - \frac{(z-\pi)^4}{4!} + \frac{(z-\pi)^6}{6!} - \cdots$$

81 $\frac{\sin z}{z-\pi}$, $z=\pi$

$$\frac{\sin(z-\pi+\pi)}{(z-\pi)}$$

$$\sin((z-\pi)+\pi) = \sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi$$

$$\frac{-\sin(z-\pi)}{z-\pi}.$$

$$\frac{-1}{z-\pi}\left[\frac{z-\pi}{1!} - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \cdots\right]$$

$$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \cdots$$

82. $\frac{1}{1+z}$ about $z=-i$.

$$\frac{1}{1+z+i-i}$$

$$\frac{1}{(z+i)+(1-i)} \Rightarrow \frac{1}{(1-i)\left[1+\frac{(z+i)}{1-i}\right]}$$

$$\frac{1}{(1-i)}\left[1+\left(\frac{z+i}{1-i}\right)\right]^{-1}$$

$$\frac{1}{(1-i)}\left[1-\left(\frac{z+i}{1-i}\right)+\left(\frac{z+i}{1-i}\right)^2-\left(\frac{z+i}{1-i}\right)^3+\dots\right]$$

$$\frac{1+i}{2}\left[1-\frac{(z+i)(1+i)}{2}+\left(\frac{(z+i)(1+i)}{2}\right)^2-\dots\right]$$

$$\left[\frac{1+i}{2}-\frac{(z+i)\times \cancel{2}i}{\cancel{4}\,2}+\frac{(z+i)^2(i-1)\times\cancel{2}}{\cancel{8}\,4}+\dots\right]$$

$$\frac{1}{2}\left[(1+i)-(z+i)i-\frac{(1-i)(z+i)^2}{2}+\dots\right]$$

Taylor Series → Analytic coz no terms of z in the denominator
defined circle

analytic part & principal part