Chapter 1

Complex Variable

1.1 Introduction

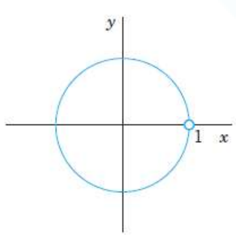
A complex number z is an ordered pair (x, y) of real numbers and is written as $z = x + iy$ where $i = \sqrt{-1}$. The real numbers x and y are called the real and imaginary parts of z. In the complex plane, the complex number z is represented by the point $P(x, y)$.

Let (r, θ) be the polar co-ordinates of the point P, then $r = \sqrt{x^2 + y^2}$ is called the modulus of z and is denoted by $|z|$. Also $\tan^{-1}\left(\frac{y}{x}\right) = \theta$ is called the **argument** of z and is denoted by $arg(z)$ or amplitude of z, i.e; $amp(z)$. Every non zero complex number z can be expressed as

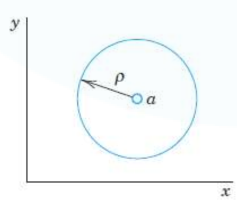
$$z = r(\cos\theta + isin\theta) = re^{i\theta}$$

If $z = x + iy$, then the complex number $x - iy$ is called the **conjugate** of the complex number z and is denoted by $\bar{z}$.

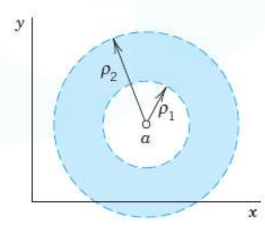
1.1.1 Circles, Disks and Half planes



Unit circle



Circle in the complex plane



Annulus in the complex plane

- A unit circle can be represented as $|z| = 1$
- General circle of centre at a and radius ρ is $|z - a| = \rho$
- An open circular disk is $|z - a| < \rho$. It is also called neighborhood of the point a.

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- A closed circular disk is $|z-a| \leq \rho$
- $\rho_1 < |z-a| < \rho_2$ is called an open annulus and $\rho_1 \leq |z-a| \leq \rho_2$ is called a closed annulus.
- The set of all points $z = x + iy$ such that
  - $y > 0$ is called Upper half plane
  - $y < 0$ is called Lower half plane
  - $x > 0$ is called Right half plane
  - $x < 0$ is called Left half plane

1.2 Function of a Complex Variable

If x and y are real variables, then $z = x + iy$ is called a complex variable. Let $S \subseteq C$. A complex function defined on S is a rule that assigns to every $z \in S$, a complex number w.

$$\text{i.e; } w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

eg. $w = z^2$
Let $z = x+iy, w = u+iv = f(z)$, then $u+iv = (x+iy)^2 = x^2 - y^2 + 2ixy$. Separating the real and imaginary parts, $u = x^2 - y^2, v = 2xy$. Thus u, v - the real and imaginary parts of w are functions of the real variables x and y. Thus

$$w = f(z) = u + iv = u(x,y) + iv(x,y)$$

[NOTE: Notation for real and imaginary parts of $f(z) = u+iv$ are $u = Re(f)$ and $v = Im(f)$]
Problems:

1. Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1+3i$.
   Ans: Let $w = u + iv, z = x + iy$. Substitute in the given equation,

   $$\begin{aligned} u + iv &= (x+iy)^2 + 3(x+iy) \\ &= x^2 - y^2 + 3x + 2ixy + i3y \end{aligned}$$

   Separating the real and imaginary parts, we get

   $$u = x^2 - y^2 + 3x, v = 2xy + 3y$$

   Also

   $$\begin{aligned} f(1+3i) &= (1+3i)^2 + 3(1+3i) \\ &= 1 + 6i - 9 + 3 + 9i = -5 + 15i \end{aligned}$$

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which shows that $u(1,3) = -5, v(1,3) = 15$.

2. Let $w = f(z) = 2iz + 6\,\bar{z}$. Find u, v and the value of f at $z = \frac{1}{2} + 4i$.
   Ans: Let $w = u + iv, z = x + iy$. Therefore the given equation becomes $u + iv = 2i(x + iy) + 6(x - iy) = 2ix - 2y + 6x - 6iy$ so that $u = 6x - 2y, v = 2x - 6y$ and

$$f\left(\frac{1}{2} + 4i\right) = 2i\left(\frac{1}{2} + 4i\right) + 6\left(\frac{1}{2} - 4i\right)$$
$$= i - 8 + 3 - 24i = -5 - 23i.$$

HOMEWORK:

Find the values of $Re(f)$ and $Im(f)$ at the indicated points

1. $f = z^2 + 2z + 2$ at $1 - i$
   Ans:$Re(f) = 4, Im(f) = -4$

2. $f = \dfrac{1}{1 - z}$ at 7+2i
   Ans: $Re(f) = \dfrac{-3}{20}, Im(f) = \dfrac{1}{20}$

3. $f = \dfrac{z - 2}{z + 2}$ at $4i$
   Ans: $Re(f) = \dfrac{3}{5}, Im(f) = \dfrac{4}{5}$

1.2.1 Single Valued and Multi Valued functions

If to each value of z, there corresponds one and only one value of w, then w is called a *single valued function* of z. For example $f(z) = \frac{1}{z}$ is a single valued function.

If to each value of z, there corresponds more than one value of w, then w is called a *multi valued function* of z. For example, $w = \sqrt{z}$ is a multivalued function of z. In this example w takes two values for each value of z except at $z = 0$.

1.2.2 Limit and Continuity of $f(z)$

A function $f(z)$ is said to have the **limit** l as z approaches z_0, written as $\lim\limits_{z \to z_0} f(z) = l$ if for every positive real ϵ, we can find a positive real δ such that for all $z \neq z_0$ in the disk $|z - z_0| < \delta$, we have $|f(z) - l < \epsilon|$.

Here z can approach z_0 from any direction in the complex plane. Also, if a limit exists it is unique.

A function $f(z)$ is said to be **continuous** at $z = z_0$ if $f(z_0)$ is defined and $\lim\limits_{z \to z_0} f(z) = f(z_0)$. A function $f(z)$ is said to be continuous in a region R of the z - plane if it is continuous at every point of the region.

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1.2.3 Derivative of $f(z)$

The derivative of a complex function $w = f(z)$ at a point z_0 is defined as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists. Then $f(z)$ is differentiable at z_0.
If $\Delta z = z - z_0$, then $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$
eg: Let $f(z) = z^2$. Then $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim\limits_{\Delta z \to 0} (2z + \Delta z) = 2z$
NOTE: Let $z = x + iy$. Then $z + \Delta z = x + \Delta x + i(y + \Delta y)$. Therefore $Q(z + \Delta z) = Q(x + \Delta x, y + \Delta y)$. Let $P(z)$ is fixed and $Q(z + \Delta z)$ is a neighboring point. The point Q may approach P along any straight line or curved path in the region. That is $\Delta z \to 0$ in any manner, i.e; either along a straight line or a curved path.

1.2.4 Analyticity of $f(z)$

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.
$f(z)$ is said to be analytic at $z = z_0$ if $f(z)$ is analytic in a neighborhood of z_0. Another name for analytic function is holomorphic function. A point at which the function ceases to be analytic is called a singular point.
NOTE: $\lim_{z \to 0} f(z)$ exists only when $f(z)$ approaches the same value for all the different paths in which $z \to 0$.
PROBLEMS:

1. If $f(z) = \dfrac{(x + y)^2}{x^2 + y^2}$, show that $\lim_{z \to 0} f(z)$ does not exists.
   Ans: In order to show that $\lim_{z \to 0} f(z)$ exists, it is necessary to show that $f(z)$ approaches the same value along all paths leading to the origin. Now

   $$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{(x + y)^2}{x^2 + y^2}$$

   Let $z \to 0$ along the line $y = mx$. Then

   $$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{(x + mx)^2}{x^2 + m^2x^2} = lim_{x \to 0} \frac{(1 + m)^2}{1 + m^2}$$

   Hence the limiting value depends on m which is not fixed. That is $f(z)$ approaches different values along different radial lines and hence no limit exists.

2. Show that $f(z) = \dfrac{(x + y)^2}{x^2 + y^2}$ is discontinuous at the origin, given that $f(0) = 0$.
   Ans: Let $z \to 0$ along the line $y = mx$. Then

   $$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{(x + mx)^2}{x^2 + m^2x^2} = lim_{x \to 0} \frac{(1 + m)^2}{1 + m^2}$$

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provided this limit exists. Then $f(z)$ is differentiable at z_0.
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NOTE: $\lim_{z \to 0} f(z)$ exists only when $f(z)$ approaches the same value for all the different paths in which $z \to 0$.
PROBLEMS:

1. If $f(z) = \dfrac{(x + y)^2}{x^2 + y^2}$, show that $\lim_{z \to 0} f(z)$ does not exists.
   Ans: In order to show that $\lim_{z \to 0} f(z)$ exists, it is necessary to show that $f(z)$ approaches the same value along all paths leading to the origin. Now

   $$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{(x + y)^2}{x^2 + y^2}$$

   Let $z \to 0$ along the line $y = mx$. Then

   $$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{(x + mx)^2}{x^2 + m^2x^2} = lim_{x \to 0} \frac{(1 + m)^2}{1 + m^2}$$

   Hence the limiting value depends on m which is not fixed. That is $f(z)$ approaches different values along different radial lines and hence no limit exists.

2. Show that $f(z) = \dfrac{(x + y)^2}{x^2 + y^2}$ is discontinuous at the origin, given that $f(0) = 0$.
   Ans: Let $z \to 0$ along the line $y = mx$. Then

   $$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{(x + mx)^2}{x^2 + m^2x^2} = lim_{x \to 0} \frac{(1 + m)^2}{1 + m^2}$$

which is not unique for different values of m. In other words the limit does not exist. Hence the limit cannot be equal to $f(0)$. So the function is not continuous at the origin.

3. Given that $f(z) = \dfrac{x^3 - y^3}{x^3 + y^3}$ and $f(0) = 0$. Show that $f(z)$ is not continuous at the origin.
   Ans: $\lim_{z\to 0} f(z) = \lim_{z\to 0} \dfrac{x^3 - y^3}{x^3 + y^3}$
   Let $z \to 0$ along the line $y = mx$. Then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{1 - m^3}{1 + m^3}$$

   which is not unique for different values of m. Therefore the limit does not exist and hence the function is discontinuous at the origin $z = 0$.

4. Show that the function $f(z) = \dfrac{x^2 y}{x^4 + y^2}$ is discontinuous at the origin and given $f(0) = 0$.
   Ans: $\lim_{z\to 0} f(z) = \lim_{z\to 0} \dfrac{x^2 y}{x^4 + y^2}$
   Let $z \to 0$ along the line $y = mx$. Then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{mx^3}{x^4 + m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2 + m^2} = 0 \text{ ——-(1)}$$

   Along every straight line the function approaches the same limit. Allow z to approach origin along the circle $y = x^2$. Then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \text{ ——(2)}$$

   (1) and (2) are different, so the limit does not exist and hence the function is discontinuous at the origin.

5. Show that the function $f(z) = \dfrac{xy(x - y)}{x^2 + y^2}$ is continuous at the origin, given that $f(0) = 0$.
   Ans: $\lim_{z\to 0} f(z) = \lim_{z\to 0} \dfrac{xy(x - y)}{x^2 + y^2}$
   Let $z \to 0$ along the line $y = mx$. Then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{mx^2(x - mx)}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{mx^3(1 - m)}{x^2(1 + m^2)} = 0 \text{ ——(1)}$$

   Let $z \to 0$ along the circle $y = x^2$. Then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x^3(x - x^2)}{x^2 + x^4} = 0 \text{ —-(2)}$$

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Equations (1) and (2) same. So $\lim_{z\to 0} f(z) = 0 = f(0)$. Hence the function is continuous at the origin.

6. Prove that $f(z) = \overline{z}$ is nowhere differentiable at any point, but continuous at $z = 0$.
   Ans: $f(z) = \overline{z}, f(z+\delta z) = \overline{z+\delta z} = \overline{z} + \overline{\delta z}$
   By the definition of derivative,

$$f'(z) = \lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$
$$= \lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$
$$= \lim_{\delta z\to 0} \frac{\overline{z} + \overline{\delta z} - \overline{z}}{\delta z} = \lim_{\delta z\to 0} \frac{\overline{\delta z}}{\delta z}$$

   $[\delta z = \delta x + i\delta y, \overline{\delta z} = \delta x - i\delta y]$
   Let $\delta z \to 0$ along the real axis, i.e; along a line parallel to x-axis so that $\delta z = \delta x, \overline{\delta z} = \delta x$. Then equation (1) becomes

$$f'(z) = \lim_{\delta x\to 0} \frac{\delta x}{\delta x} = 1 \text{ —(2)}$$

   Let $\delta z \to 0$ along the y-axis. Then $\delta z = i\delta y, \overline{\delta z} = -i\delta y$ and equation (1) becomes

$$f'(z) = \lim_{\delta y\to 0} \frac{i\delta y}{-i\delta y} = -1 \text{ —(3)}$$

   (2) and (3) are different. Since $f'(z)$ is not unique, $f(z) = \overline{z}$ is not differentiable at any point and hence not analytic at any point. To test the continuity at $z = 0$, consider

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}(x - iy)$$

   Let $z \to 0$ along the line $y = mx$ so that $\lim_{z\to 0} f(z) = \lim_{x\to 0}(x - imx) = 0$
   Let $z \to 0$ along the circle $y = x^2$ so that $\lim_{z\to 0} f(z) = \lim_{x\to 0}(x - ix^2) = 0$
   These two limits are equal and so $\lim_{z\to 0} f(z) = 0 = f(0)$. Therefore $f(z) = \overline{z}$ is continuous at $z = 0$.

7. Prove that $f(z) = |z|^2 = z\overline{z}$ is differentiable only at $z = 0$ and no where else.
   Ans: $f'(z) = \lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ —-(1)
   Here $f(z) = |z|^2 = z\overline{z}$, $f(z+\delta z) = (z+\delta z)(\overline{z+\delta z})$. Then

$$f(z+\delta z) - f(z) = z\overline{z} + z\overline{\delta z} + \delta z\overline{z} + \delta z\overline{\delta z} - z\overline{z}$$
$$= z\overline{\delta z} + \delta z\overline{z} + \delta z\overline{\delta z}$$

   Equation (1) becomes

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$$f'(z) = \lim_{\delta z \to 0} \frac{z\overline{\delta z} + \delta z\overline{z} + \delta z\overline{\delta z}}{\delta z}$$
$$= \lim_{\delta z \to 0}\left(z\frac{\overline{\delta z}}{\delta z} + \overline{z} + \overline{\delta z}\right) \text{ ---(2)}$$

Let $\delta z \to 0$ along the real axis so that $\delta z = \delta x . \overline{\delta z} = \delta x$

$$f'(z) = \lim_{\delta x \to 0}\left(z\frac{\delta x}{\delta x} + \overline{z} + \delta x\right) = z + \overline{z} \text{ ---(3)}$$

Now let $\delta z \to 0$ along the imaginary axis so that $\delta z = i\delta y . \overline{\delta z} = -i\delta y$ and

$$f'(z) = \lim_{\delta y \to 0}\left(z\frac{-i\delta y}{i\delta y} + \overline{z} - i\delta y\right) = \overline{z} - z \text{ ---(4)}$$

Since (3) and (4) are not equal, $f'(z)$ does not exist. Therefore $f(z)$ is not differentiable at $z \neq 0$.

For $z = 0$, the values of $f'(z)$, i.e; equation (3) and (4) are same. Therefore it is differentiable at $z = 0$.

NOTE: Though $f(z)$ is differentiable at $z = 0$, it is not analytic at at $z = 0$, since it is not differentiable at any point in the neighbourhood of $z = 0$. Therefore $f(z)$ is not an analytic function.

8. Find out whether $f(z)$ is continuous at $z = 0$ if $f(z) = \dfrac{Rez^2}{|z|}, z \neq 0, f(0) = 0$.

Ans: $z = x + iy \implies z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$

$Rez^2 = x^2 - y^2$, $|z| = \sqrt{x^2 + y^2}$ and so $f(z) = \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$. Now

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \text{ ----(1)}$$

Let $z \to 0$ along the line $y = mx$. Then equation (1) becomes

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{x^2 - m^2x^2}{\sqrt{x^2 + m^2x^2}} = \lim_{x \to 0} \frac{x(1 - m^2)}{\sqrt{1 + m^2}} = 0 \text{ ---(2)}$$

Let $z \to 0$ along the line $y = x^2$. Then equation (1) becomes

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{x^2 - x^4}{\sqrt{x^2 + x^4}} = \lim_{x \to 0} \frac{x(1 - x^2)}{\sqrt{1 + x^2}} = 0 \text{ ---(3)}$$

Values of (2) and (3) are the same. Therefore $\lim_{z \to 0} f(z) = 0 = f(0)$ and so continuous at $z = 0$.

9. Show that $f(z) = \begin{matrix} \dfrac{zRez}{|z|} & , & z \neq 0 \\ 0 & , & z = 0 \end{matrix}$ is not differentiable at $z = 0$.

Ans: $f(z) = \dfrac{(x + iy)x}{\sqrt{x^2 + y^2}} = \dfrac{x^2 + ixy}{\sqrt{x^2 + y^2}}$

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$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{x^2+ixy-0}{(x+iy)\sqrt{x^2+y^2}}$$

Let $z \to 0$ along $y = mx$. Then

$$f'(0) = \lim_{x\to 0}\frac{x^2+imx^2}{(x+imx)\sqrt{x^2+m^2x^2}}$$
$$= \lim_{x\to 0}\frac{1+im}{(i+im)\sqrt{1+m^2}} = \frac{1}{\sqrt{1+m^2}}$$

which is not unique. Therefore $f(z)$ is not differentiable at $z = 0$.

Homework

1. Find out whether $f(z)$ is continuous at $z = 0$ if

   i. $f(z) = \begin{matrix} \dfrac{Im(z)}{|z|} & , & z \neq 0 \\ 0 & , & z = 0 \end{matrix}$

   Ans: Not Continuous

   ii. $f(z) = \begin{matrix} \dfrac{Re(z)}{1+|z|} & , & z \neq 0 \\ 0 & , & z = 0 \end{matrix}$

   Ans: Continuous

2. Show that the function $f(z)$ is discontinuous at the origin, given

   $f(z) = \begin{matrix} \dfrac{xy(x-y)}{x^3+y^3} & , & z \neq 0 \\ 0 & , & z = 0 \end{matrix}$

   Ans: Discontinuous

1.2.5 Cauchy-Riemann Equations

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then at that point, the first order partial derivatives of u and v exist and satisfy the Cauchy - Riemann equations

$$u_x = v_y, u_y = -v_x \text{ ——-(1)}$$

Hence if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) at all points of D.

Proof:

By assumption, the derivative $f'(z)$ at z exists and

$$f'(z) = \lim_{\Delta z \longrightarrow 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

Let $\Delta z = \Delta x + i\Delta y$. So

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$$f'(z) = \lim_{\Delta z \longrightarrow 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

Suppose $\Delta z \longrightarrow 0$ along $x-$ axis. Then $\Delta y = 0$ and $\Delta z \longrightarrow 0$ becomes $\Delta x \longrightarrow 0$. So

$$f'(z) = \lim_{\Delta x \longrightarrow 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right] + i \lim_{\Delta x \longrightarrow 0} \left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right] = u_x + iv_x$$

Now suppose $\Delta z \longrightarrow 0$ along $y-$ axis. Then $\Delta x = 0$ and $\Delta z \longrightarrow 0$ becomes $\Delta y \longrightarrow 0$. So

$$f'(z) = \lim_{\Delta y \longrightarrow 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y}\right] + i \lim_{\Delta y \longrightarrow 0} \left[\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}\right] = -iu_y + v_y$$

Since $f'(z)$ exists, these two must be equal. So we get

$$u_x + iv_x = -iu_y + v_y \implies u_x = v_y, u_y = -v_x$$

Theorem: If two real valued continuous functions $u(x,y)$ and $v(x,y)$ of two real variables x and y have continuous first partial derivatives that satisfy the CR equations in some domain D, then the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in D.

NOTE:

- If $f(z)$ is analytic, then $f'(z)$ must be continuous since differentiability implies continuity.
- All polynomials, trigonometric, hyperbolic and exponential functions are continuous.

PROBLEMS:

1. Prove that the function $\sinh z$ is analytic and hence find its derivative.

   Ans: Let

$$\begin{aligned} f(z) = \sinh z &= \frac{1}{i} \sin iz \\ &= \frac{1}{i} \sin i(x+iy) = \frac{1}{i} \sin(ix - y) \\ &= \frac{1}{i}(\sin ix \cos y - \cos ix \sin y) \\ &= \frac{1}{i}(i \sinh x \cos y - \cosh x \sin y) \\ &= \sinh x \cos y + i \cosh x \sin y \end{aligned}$$

   So $u + iv = \sinh x \cos y + i \cosh x \sin y \implies u = \sinh x \cos y$ and $v = \cosh x \sin y$

   Then

$$u_x = \cosh x \cos y, u_y = -\sinh x siny$$
$$v_x = \sinh x \sin y, v_y = \cosh x \cos y$$

   Therefore $u_x = v_y$ and $u_y = -v_x$. Partial derivatives of u and v being product of trigonometric and hyperbolic functions, they are continuous. So u_x, u_y, v_x, v_y are continuous. Hence $f(z)$ is analytic everywhere.

$$\begin{aligned} f'(z) = u_x + iv_x &= \cosh x \cos y + i \sinh x \sin y = \cos ix \cos y + \sin ix \sin y \\ &= \cos(ix - y) = \cos i(x+iy) = \cos iz = \cosh z \end{aligned}$$

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$$f'(z) = \lim_{\Delta z \longrightarrow 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

Suppose $\Delta z \longrightarrow 0$ along $x-$ axis. Then $\Delta y = 0$ and $\Delta z \longrightarrow 0$ becomes $\Delta x \longrightarrow 0$. So

$$f'(z) = \lim_{\Delta x \longrightarrow 0} \left[\frac{u(x + \Delta x, y) - u(x,y)}{\Delta x}\right] + i \lim_{\Delta x \longrightarrow 0} \left[\frac{v(x + \Delta x, y) - v(x,y)}{\Delta x}\right] = u_x + iv_x$$

Now suppose $\Delta z \longrightarrow 0$ along $y-$ axis. Then $\Delta x = 0$ and $\Delta z \longrightarrow 0$ becomes $\Delta y \longrightarrow 0$. So

$$f'(z) = \lim_{\Delta y \longrightarrow 0} \left[\frac{u(x, y + \Delta y) - u(x,y)}{i\Delta y}\right] + i \lim_{\Delta y \longrightarrow 0} \left[\frac{v(x, y + \Delta y) - v(x,y)}{i\Delta y}\right] = -iu_y + v_y$$

Since $f'(z)$ exists, these two must be equal. So we get

$$u_x + iv_x = -iu_y + v_y \implies u_x = v_y, u_y = -v_x$$

Theorem: If two real valued continuous functions $u(x,y)$ and $v(x,y)$ of two real variables x and y have continuous first partial derivatives that satisfy the CR equations in some domain D, then the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in D.

NOTE:

- If $f(z)$ is analytic, then $f'(z)$ must be continuous since differentiability implies continuity.
- All polynomials, trigonometric, hyperbolic and exponential functions are continuous.

In polar form $z = r(\cos\theta + i\sin\theta), f(z) = u(r,\theta) + iv(r,\theta)$. Then the Cauchy-Riemann equations are

$$u_r = \frac{1}{r}v_\theta, v_r = -\frac{1}{r}u_\theta$$

PROBLEMS:

1. Prove that the function $\sinh z$ is analytic and hence find its derivative.
   Ans: Let

$$\begin{aligned} f(z) = \sinh z &= \frac{1}{i}\sin iz \\ &= \frac{1}{i}\sin i(x+iy) = \frac{1}{i}\sin(ix - y) \\ &= \frac{1}{i}(\sin ix \cos y - \cos ix \sin y) \\ &= \frac{1}{i}(i\sinh x \cos y - \cosh x \sin y) \\ &= \sinh x \cos y + i\cosh x \sin y \end{aligned}$$

   So $u + iv = \sinh x \cos y + i\cosh x \sin y \implies u = \sinh x \cos y$ and $v = \cosh x \sin y$
   Then

$$u_x = \cosh x \cos y, u_y = -\sinh x siny$$
$$v_x = \sinh x \sin y, v_y = \cosh x \cos y$$

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Therefore $u_x = v_y$ and $u_y = -v_x$. Partial derivatives of u and v being product of trigonometric and hyperbolic functions, they are continuous. So u_x, u_y, v_x, v_y are continuous. Hence $f(z)$ is analytic everywhere. Now

$$f'(z) = u_x + iv_x = \cosh x \cos y + i \sinh x \sin y = \cos ix \cos y + \sin ix \sin y$$
$$= \cos(ix - y) = \cos i(x + iy) = \cos iz = \cosh z$$

2. If $f(z) = z^2$, show that $f(z)$ is analytic and that $f'(z) = 2z$.
   Ans: Let $z = x + iy$, then $f(z) = z^2 = x^2 - y^2 + i2xy$. Therefore $u = x^2 - y^2$ and $v = 2xy$.

$$u_x = 2x, u_y = -2y, v_x = 2y, v_y = 2x \implies u_x = v_y, u_y = -v_x$$

   Hence CR equations are satisfied and u_x, u_y, v_x, v_y are continuous. So the function is analytic. Now

$$f'(z) = u_x + iv_x = 2x + i2y = 2(x + iy) = 2z$$

3. If $f(z) = e^z$, show that $f(z)$ is analytic.
   Ans: $f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$
   Then $u = e^x \cos y, v = e^x \sin y$
   $u_x = e^x \cos y, u_y = -e^x \sin y, v_x = e^x \sin y, v_y = e^x \cos y$
   Therefore $u_x = v_y, u_y = -v_x$, CR equations are satisfied and u_x, u_y, v_x, v_y are continuous. So the function is analytic.

4. Show that the function $f(z) = xy + iy$ is everywhere continuous, but is not analytic.
   Ans: $f(z) = xy + iy \implies u = xy, v = y$. Since u and v are continuous everywhere(being polynomial functions), $f(z)$ is also everywhere continuous. Now

$$u_x = y, u_y = x, v_x = 0, v_y = 1 \implies u_x \neq v_y, u_y \neq -vx$$

   CR equations are not satisfied, and so the function is not analytic.

5. Show that the function $w = \log z$ is analytic everywhere in the complex plane except at the origin and that its derivative is $\frac{1}{z}$.
   Ans: $f(z) = \log z = \log(re^{i\theta}) = \log r + i\theta = \log(\sqrt{x^2 + y^2}) + i \tan^{-1}\left(\frac{y}{x}\right)$
   So $u = \frac{1}{2}\log(x^2 + y^2), v = \tan^{-1}\left(\frac{y}{x}\right)$
   $u_x = \frac{x}{x^2 + y^2}, u_y = \frac{y}{x^2 + y^2}, v_x = \frac{-y}{x^2 + y^2}, v_y = \frac{x}{x^2 + y^2} \implies u_x = v_y, u_y = -v_x$
   Hence CR equations are satisfied and u_x, u_y, v_x, v_y are continuous and defined except at origin. So the function is analytic everywhere except at the origin($\because$ the function is not defined at the origin)
   $f'(z) = u_x + iv_x = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$

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6. Show that $f(z) = e^{-x}\cos y - ie^{-x}\sin y$ is differentiable everywhere and find its derivative.
   Ans: Here $u = e^{-x}\cos y, v = -e^{-x}\sin y$
   $u_x = -e^{-x}\cos y, u_y = -e^{-x}\sin y, v_x = e^{-x}\sin y, v_y = -e^{-x}\cos y \implies u_x = v_y, u_y = -v_x$
   Hence CR equations are satisfied and u_x, u_y, v_x, v_y are continuous. So the function is analytic. Therefore the function is differentiable everywhere.
   $f'(z) = u_x + iv_x = -e^{-x}\cos y + ie^{-x}\sin y = -e^{-x}(\cos y - i\sin y) = e^{-x-iy} = e^{-z}$

Homework:

Determine which of the following functions are analytic

1. $\sin z$ [Ans: Analytic]
2. $sin\bar{z}$ [Ans: Not analytic]
3. $\cos\bar{z}$ [Ans: Not analytic]
4. $z + \bar{z}$ [Ans: Not analytic]
5. $|z|^2 = z\bar{z}$ [Ans: Not analytic]

NOTE:

- Every analytic function can be expressed as a function of z alone. The above results show that $\bar{z}, z\bar{z}, \cos\bar{z}$ etc are not analytic
- By definition

$$\frac{\partial u}{\partial x} = \lim_{\delta x \longrightarrow 0} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x} \text{ and}$$
$$\frac{\partial u}{\partial y} = \lim_{\delta y \longrightarrow 0} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta y}$$

  Then

$$\frac{\partial u}{\partial x}_{(0,0)} = \lim_{x \longrightarrow 0} \frac{u(x,0) - u(0,0)}{\delta x} \text{ [by taking } \delta x = x, x = 0 = y]$$
$$\frac{\partial u}{\partial y}_{(0,0)} = \lim_{y \longrightarrow 0} \frac{u(0,y) - u(0,0)}{\delta y} \text{ [by taking } \delta y = y, x = 0 = y]$$

1. Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin although the CR equations are satisfied at the origin.
   Ans: Let $f(z) = \sqrt{|xy|} \implies u = \sqrt{|xy|}, v = 0$
   At the origin $(0,0)$,

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$$u_x = \frac{\partial u}{\partial x} = \lim_{x \longrightarrow 0} \frac{u(x,0) - u(0,0)}{\delta x} = \lim_{x \longrightarrow 0} \frac{0-0}{x} = 0$$
$$u_y = \frac{\partial u}{\partial y} = \lim_{y \longrightarrow 0} \frac{u(0,y) - u(0,0)}{\delta y} = \lim_{y \longrightarrow 0} \frac{0-0}{y} = 0$$
$$v_x = \frac{\partial v}{\partial x} = \lim_{x \longrightarrow 0} \frac{v(x,0) - v(0,0)}{\delta x} = \lim_{x \longrightarrow 0} \frac{0-0}{x} = 0$$
$$v_y = \frac{\partial v}{\partial y} = \lim_{y \longrightarrow 0} \frac{v(0,y) - v(0,0)}{\delta y} = \lim_{y \longrightarrow 0} \frac{0-0}{y} = 0$$

Hence CR equations are satisfied at the origin. Now

$$f'(z) = \lim_{z \longrightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{z \longrightarrow 0} \frac{\sqrt{|xy|}}{x+iy}$$

Let $z \longrightarrow 0$ along $y = mx$. Then

$$f'(z) = \lim_{x \longrightarrow 0} \frac{\sqrt{|mx^2|}}{x+imx} = \lim_{x \longrightarrow 0} \frac{\sqrt{m}}{1+im}$$

This limit is not unique as it depends on the value of m. Therefore $f'(0)$ does not exist. Hence $f(z)$ is not regular at the origin.

2. Prove that the function $f(z)$ defined by $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases}$ is continuous and that the CR equations are satisfied at the origin but $f'(0)$ does not exist.

Ans: $f(z) = \dfrac{x^3 - y^3}{x^2+y^2} + i\dfrac{x^3+y^3}{x^2+y^2}$

$u = \dfrac{x^3 - y^3}{x^2+y^2}, v = \dfrac{x^3+y^3}{x^2+y^2}$

Numerator and denominator of u, v are polynomials in x and y. So u, v and hence $f(z)$ is continuous when $z \neq 0$. To test the continuity at $xz = 0$, we have to prove that $\lim_{z \longrightarrow 0} f(z) = 0 = f(0)$. Let $z \longrightarrow 0$ along $y = mx$.Then

$$\lim_{z \longrightarrow 0} f(z) = \lim_{x \longrightarrow 0} \frac{x^3 - m^3x^3}{x^2+m^2x^2} + i\frac{x^3+m^3y^3}{x^2+m^2x^2}$$
$$= \lim_{x \longrightarrow 0} \frac{x[1 - m^3 + i(1+m^3)]}{1+m^2} = 0 \text{ —-(1)}$$

Let $z \longrightarrow 0$ along $y = x^2$.Then

$$\lim_{z \longrightarrow 0} f(z) = \lim_{x \longrightarrow 0} \frac{x^3 - x^6}{x^2+x^4} + i\frac{x^3+x^6}{x^2+x^4}$$
$$= \lim_{x \longrightarrow 0} \frac{x[1 - x^3 + i(1+x^3)]}{1+x^2} = 0 \text{ —-(2)}$$

(1) and (2) are the same. Therefore $\lim_{z \longrightarrow 0} f(z) = 0 = f(0)$ and hence $f(z)$ is continuous at $z = 0$. So $f(z)$ is continuous for all values of z. At the origin $(0,0)$,

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$$u_x = \frac{\partial u}{\partial x} = \lim_{x \longrightarrow 0} \frac{u(x,0) - u(0,0)}{\delta x} = \lim_{x \longrightarrow 0} \frac{x - 0}{x} = 1$$
$$u_y = \frac{\partial u}{\partial y} = \lim_{y \longrightarrow 0} \frac{u(0,y) - u(0,0)}{\delta y} = \lim_{y \longrightarrow 0} \frac{-y - 0}{y} = -1$$
$$v_x = \frac{\partial v}{\partial x} = \lim_{x \longrightarrow 0} \frac{v(x,0) - v(0,0)}{\delta x} = \lim_{x \longrightarrow 0} \frac{x - 0}{x} = 1$$
$$v_y = \frac{\partial v}{\partial y} = \lim_{y \longrightarrow 0} \frac{v(0,y) - v(0,0)}{\delta y} = \lim_{y \longrightarrow 0} \frac{y - 0}{y} = 1$$

Hence CR equations are satisfied at the origin. Now

$$f'(0) = \lim_{z \longrightarrow 0} \frac{f(z) - f(0)}{z} = \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

Let $z \longrightarrow 0$ along $y = mx$. Then

$$f'(0) = \lim_{x \longrightarrow 0} \frac{x^3 - m^3x^3 + i(x^3 + m^3x^3)}{(x^2 + m^2x^2)(x + imx)}$$
$$= \lim_{x \longrightarrow 0} \frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)}$$

which is not unique as it depends on the value of m. Therefore $f'(0)$ does not exist.

Laplace's Equation:

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \text{ and } \nabla^2 v = v_{xx} + v_{yy} = 0$$

respectively in D and have continuous second partial derivatives in D.

Proof: Since f is analytic, $u_x = v_y$ and $u_y = -v_x$. So $u_{xx} = v_{yx}$ and $u_{yy} = v_{xy}$ and so $u_{xx} + u_{yy} = 0$. Similarly $v_{xx} = -u_{yx}, v_{yy} = u_{xy}$ and so $v_{xx} + v_{yy} = 0$.

Solution of Laplace's equation having continuous second order partial derivatives are called **Harmonic Functions**. If two harmonic functions u and v satisfy the CR equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a *conjugate harmonic function* of u in D.

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$$
\begin{aligned}
u_x &= \frac{\partial u}{\partial x} = \lim_{x \longrightarrow 0} \frac{u(x,0)-u(0,0)}{\delta x} = \lim_{x \longrightarrow 0} \frac{x-0}{x} = 1 \\
u_y &= \frac{\partial u}{\partial y} = \lim_{y \longrightarrow 0} \frac{u(0,y)-u(0,0)}{\delta y} = \lim_{y \longrightarrow 0} \frac{-y-0}{y} = -1 \\
v_x &= \frac{\partial v}{\partial x} = \lim_{x \longrightarrow 0} \frac{v(x,0)-v(0,0)}{\delta x} = \lim_{x \longrightarrow 0} \frac{x-0}{x} = 1 \\
v_y &= \frac{\partial v}{\partial y} = \lim_{y \longrightarrow 0} \frac{v(0,y)-v(0,0)}{\delta y} = \lim_{y \longrightarrow 0} \frac{y-0}{y} = 1
\end{aligned}
$$

Hence CR equations are satisfied at the origin. Now

$$
f'(0) = \lim_{z \longrightarrow 0} \frac{f(z)-f(0)}{z} = \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}
$$

Let $z \longrightarrow 0$ along $y = mx$. Then

$$
\begin{aligned}
f'(0) &= \lim_{x \longrightarrow 0} \frac{x^3-m^3x^3+i(x^3+m^3x^3)}{(x^2+m^2x^2)(x+imx)} \\
&= \lim_{x \longrightarrow 0} \frac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}
\end{aligned}
$$

which is not unique as it depends on the value of m. Therefore $f'(0)$ does not exist.

Laplace's Equation:

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace's equation

$$
\nabla^2 u = u_{xx} + u_{yy} = 0 \text{ and } \nabla^2 v = v_{xx} + v_{yy} = 0
$$

respectively in D and have continuous second partial derivatives in D.

Proof: Since f is analytic, $u_x = v_y$ and $u_y = -v_x$. So $u_{xx} = v_{yx}$ and $u_{yy} = v_{xy}$ and so $u_{xx} + u_{yy} = 0$. Similarly $v_{xx} = -u_{yx}, v_{yy} = u_{xy}$ and so $v_{xx} + v_{yy} = 0$.

Solution of Laplace's equation having continuous second order partial derivatives are called **Harmonic Functions**. If two harmonic functions u and v satisfy the CR equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a *conjugate harmonic function* of u in D.

Properties of an Analytic Function:

Property 1: If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic.

Proof: Let $f(z) = u + iv$ be an analytic function in some region of the z-plane. Then u and v satisfy the CR equations

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \text{ ——-(1)} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \text{ ....(2)}
\end{aligned}
$$

Differentiating (1) partially w.r.t x and (2) w.r.t (y), we get

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$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$
$$\implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ ——-(3)}$$

Similarly differentiating (1) partially w.r.t y and (2) w.r.t (x), we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$$
$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$
$$\implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ —-(4)}$$

Equations (4) and (5) show that the real and imaginary parts u and v of an analytic function $f(z)$ satisfies the Laplace's equation and hence u and v are harmonic.

Orthogonal Curves: Two curves are said to be orthogonal to each other if they intersect at right angles at each point of their intersection.

Property 2: If $w = f(z) = u + iv$ is an analytic function, then show that the curves $u = c_1, v = c_2$ represented on the plane intersect at right angles.

Proof: Consider the curves

$$u(x, y) = c_1 \text{ ——(1) \& } v(x, y) = c_2 \text{ ————(2)}$$
$$\implies du = 0$$

Since u is a function of two independent variables x and y,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \implies \frac{dy}{dx} = \frac{\frac{-\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1$$

Similarly from (2), $dv = 0$. Then

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0 \implies \frac{dy}{dx} = \frac{\frac{-\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2 \text{ —-(3)}$$

Product of the slopes $= m_1 m_2 = \frac{\frac{-\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{\frac{-\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$

Since $f(z)$ is analytic, u, v satisfy the CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Milne-Thomson Method:

Suppose $f(z) = u + iv$ is analytic and u is known. We need to find value of v. For that we know that

$$f'(z) = u_x + iv_x = u_x - iu_y$$

by CR equations. i.e; $f'(z) = \phi_1(x, y) - i\phi_2(x, y)$

In Milne Thomson method, we replace x by z and y by 0 so that $f'(z)$ will be a function of z. i.e; $f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$. Integrating we get

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$$f(z) = \int(\phi_1(z,0) - i\phi_2(z,0))dz + c$$

Writing the imaginary part, we get value of v.

Problems:

1. Prove that the function $u = x^3 - 3xy^2$ is harmonic and find its conjugate harmonic function and the corresponding analytic function $f(z)$ in terms of z.
   Ans:$u = x^3 - 3xy^2$
   $u_x = 3x^2 - 3y^2, u_y = -6xy, u_{xx} = 6x, u_{yy} = -6x$
   Therefore $u_{xx} + u_{yy} = 0$ and so u is harmonic. We know that
   $$f'(z) = u_x + iv_x = u_x - iu_y = 3x^2 - 3y^2 + i6xy$$
   Replace x by z and y by 0, using Milne Thomson method we get $f'(z) = 3z^2$. Integrating, $f(z) = z^3 + ic$. i.e;
   $$f(z) = (x + iy)^3 + ic = x^3 - 3xy^2 + i(3x^2y - y^3 + c)$$
   Therefore $v = 3x^2y - y^3 + c$

2. Show that the function $u = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find a function v such that $u + iv$ is an analytic function.
   Ans: $u = 3x^2y + x^2 - y^3 - y^2$. Then
   $u_x = 6xy + 2x, u_{xx} = 6y + 2, u_y = 3x^2 - 3y^2 - 2y, u_{yy} = -6y - 2$
   Therefore $u_{xx} + u_{yy} = 0$ and so u is harmonic. We know that
   $$f'(z) = u_x + iv_x = u_x - iu_y = 6xy + 2x - i(3x^2 - 3y^2 - 2y)$$
   Replace x by z and y by 0, using Milne Thomson method we get $f'(z) = 2z - i3z^2$. Then
   $$\begin{gathered} f(z) = z^2 - iz^3 + ic \\ =(x+iy)^2 - i(x+iy)^3 \\ =x^2 - y^2 + i2xy - i(x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3) + ic \\ =x^2 - y^2 + 3x^2y - y^3 + i(2xy - x^3 + 3xy^2 + c) \end{gathered}$$
   Therefore $v = 2xy - x^3 + 3xy^2 + c$.
   NOTE: If we are finding v from $f(z)$, write the constant in $f(z)$ as ic and if we are finding u from $f(z)$ write the constant in $f(z)$ as c.

3. Show that $v(x,y) = \log(x^2 + y^2)$ is a harmonic function. Find a function $u(x,y)$ such that $u + iv$ is an analytic function.
   Ans: Given $v(x,y) = \log(x^2 + y^2)$. Then

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$$v_x = \frac{2x}{x^2+y^2}, v_y = \frac{2y}{x^2+y^2}$$
$$v_{xx} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}, v_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$
$$\implies v_{xx}+v_{yy}=0$$

Therefore v is harmonic. Now

$$f'(z) = u_x + iv_x = v_y + iv_x = \frac{2y}{x^2+y^2} + i\frac{2x}{x^2+y^2}$$

Replace x by z and y by 0, using Milne Thomson method we get $f'(z) = \frac{2i}{z} \implies$ $f(z) = 2i\log z + c$.
i.e; $f(z) = 2i\log(re^{i\theta}) + c = 2i(\log r + i\theta) + c = 2i\log r + c - 2\theta$
Therefore $u = c - 2\theta = c - 2\tan^{-1}\left(\frac{y}{x}\right)$

4. If $u = (x-1)^3 - 3xy^2 + 3y^2$, find the analytic function $f(z) = u + iv$.
Ans:

$$f'(z) = u_x + iv_x = u_x - iu_y = 3(x-1)^2 - 3y^2 - i(-6xy + 6y)$$
$$\implies f'(z) = 3(z-1)^2 \implies f(z) = (z-1)^3 + c$$

5. Find an analytic function $f(z) = u+iv$ of which the real part is $u = e^x(x\cos y - y\sin y)$.
Ans: $f'(z) = u_x+iv_x = u_x-iu_y = e^x(x\cos y - y\sin y) + e^x\cos y - i\left(e^x(-x\sin y - y\cos y - \sin y)\right)$
Replace x by z and y by 0, using Milne Thomson method we get

$$f'(z) = ze^z + e^z = (z+1)e^z$$
$$\implies f(z) = \int (z+1)e^z dz + c = ze^z + c$$

6. If $w = f(z) = u + iv$ and $u - v = e^x(\cos y - \sin y)$, find w in terms of z.
Ans: $u - v = e^x(\cos y - \sin y)$. Then

$$u_x - v_x = e^x(\cos y - \sin y) \text{ ---(1)}$$
$$u_y - v_y = e^x(-\sin y - \cos y) \implies -v_x - u_x = e^x(-\sin y - \cos y), \text{ or}$$
$$u_x + v_x = e^x(\cos y + \sin y) \text{ ----(2)}$$
$$(1)+(2) \implies 2u_x = 2e^x\cos y \text{ or } u_x = e^x\cos y$$
$$(1)-(2) \implies v_x = e^x\sin y$$
$$\text{Then } f'(z) = u_x + iv_x = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y) = e^{x+iy} = e^z$$
$$\implies f(z) = e^z + c$$

7. Is there any analytic function $f(z) = u + iv$ for which $v = e^{\frac{y}{x}}$?
Ans: $v = e^{\frac{y}{x}} \implies v_x = e^{\frac{y}{x}} \times \frac{-y}{x^2}$ and $v_{xx} = \frac{y}{x^4}e^{\frac{y}{x}}(y+2x)$
$v_y = e^{\frac{y}{x}} \times \frac{1}{x}$ and $v_{yy} = \frac{1}{x^2}e^{\frac{y}{x}}$
Here $v_{xx} + v_{yy} \neq 0$ and so there cannot be an analytic function $f = u + iv$ for which $v = e^{\frac{y}{x}}$.

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8. Find the relation between the real constants a, b, c if the function $f(z) = x + ay + i(bx + cy)$ is analytic.
   Ans:$Here u = x + ay$ and $v = bx + cy$.
   $u_x = 1, u_y = a, v_x = b.v_y = c$ Since f is analytic, it satisfy the CR equations $u_x = v_y, u_y = -v_x$. Therefore we get $a = -b, c = 1$.

9. Show that an analytic function with constant real part is constant.
   Ans: Let $f(z) = u + iv$ and given $u = c$. Then $u_x = 0, u_y = 0$. But since $f(z)$ is analytic, $u_x = v_y = 0, u_y = -v_x = 0 \implies v = constant$. Then $f(z) = u + iv$ is a constant.

10. Prove that an analytic function of constant imaginary part is constant.

11. Prove that an analytic function with constant modulus is constant.
    Ans: $|f(z) = k \implies u^2 + v^2 = k$. Then

$$uu_x + vv_x = 0 \text{ ---(1)}$$
$$uu_y + vv_y = 0 \text{ ---(2)}$$

Since f is analytic, it satisfy the CR equations. Use $v_x = -u_y$ in (1) and $v_y = u_x$ in (2), we get

$$uu_x - vu_y = 0, uu_y + vu_x = 0$$
$$\implies (u^2 + v^2)u_x = 0 \text{ and } (u^2 + v^2)u_y = 0$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$ and hence $f = u + iv = 0$ a constant.
If $k \neq 0$ then $u_x = 0, u_y = 0 \implies u = constant$ and by CR equations we get $v_x = 0, v_y = 0 \implies v = constant$. Hence $f = constant$.

12. Prove that an analytic function $f(z)$ is constant if $arg f(z)$ is constant.
    Ans: Let $f(z) = u + iv$. Then $arg f(z) = \tan^{-1} \frac{v}{u} = c$, c a constant.
    $\frac{v}{u} = \tan c$ and $\frac{u}{v} = \cot c = k$. Therefore $u - kv = 0$ ---(1)
    Consider

$$(1 + ik)f(z) = (1 + ik)(u + iv) = u - kv + i(v + ku)$$
$$(1 + ik)f(z) = 0 + i(v + ku) \text{ by equation (1)}$$

Hence real part of $(1 + ik)f(z)$ is 0, which can be considered as a constant. But we know that an analytic function with constant real part is constant and so $(1 + ik)f(z)$ is constant, which implies $f(z)$ is constant.

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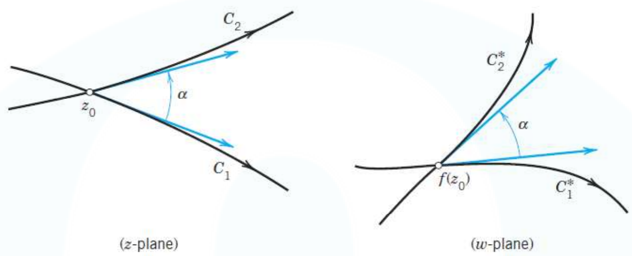
1.3 Conformal Mapping

For a complex function $w = f(z) = u(x, y) + iv(x, y)$, we need two planes, the z-plane in which we plot values of z and w plane in which we plot the corresponding function values $w = f(z)$. In this way, a given function f assigns to each point z in its domain of definition D, the corresponding point $w = f(z)$ in the w-plane. Then we say that f defines a mapping of D into the w plane.

A mapping $w = f(z)$ is called conformal if it preserves angles between oriented curves in magnitude as well as in sense.



Curves C_1 and C_2 and their respective images C_1^* and C_2^* under a conformal mapping $w = f(z)$

The figure shows the conformal mapping $w = f(z)$ for the curves C_1 and C_2 and their respective images C_1^* and C_2^*. The angle $\alpha(0 \leq \alpha \leq \pi)$ between two intersecting curvesC_1 and C_2 is defined to be the angle between their oriented tangents at the intersection point z_0. And conformality means that the images of C_1^* and C_2^* of C_1 and C_2 make the same angle as the curves themselves in both magnitude and direction.

The mapping $w = f(z)$ by an analytic function f is conformal, except at **critical points**, that is, points at which the derivative f' is zero.

1.3.1 The mapping $w = f(z) = z^2$

Let $w = z^2$. In polar form, $z = r(\cos\theta + i\sin\theta)$ and $w = R(\cos\phi + i\sin\phi) = r^2(\cos 2\theta + i\sin 2\theta)$. So circles with radius $r = r_0$ are mapped onto circles with radius $R = r_0^2$ in the w-plane and angles $\theta = \theta_0$ onto angles $\phi = 2\theta_0$.

Then the region $1 \leq |z| \leq \frac{3}{2}, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ in z-plane is mapped on to the region $1 \leq |w| \leq \frac{9}{4}, \frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$ in the w-plane.

In cartesian coordinates, we have $z = x + iy$. Then $w = z^2 = x^2 - y^2 + i2xy$. Therefore $u = x^2 - y^2, v = 2xy$.

Then the vertical lines $x = c$ are mapped onto

$$u = c^2 - y^2, v = 2cy$$
$$\implies y^2 = c^2 - u, v^2 = 4c^2y^2$$
$$\implies v^2 = 4c^2(c^2 - u), \text{ parabolas open to the left.}$$

Similarly horizontal lines $y = k$ are mapped onto

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$$u = x^2 - k^2, v = 2xk$$
$$\implies x^2 = k^2 + u, v^2 = 4x^2k^2$$
$$\implies v^2 = 4k^2(k^2 + u), \text{ parabolas open to the right.}$$

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Hence real part of $(1+ik)f(z)$ is 0, which can be considered as a constant. But we know that an analytic function with constant real part is constant and so $(1+ik)f(z)$ is constant, which implies $f(z)$ is constant.

Homework:
Determine whether the following functions are harmonic. If so find its conjugate harmonic function and the corresponding analytic function $f(z)$ in terms of z

1. $u = xy$
2. $v = xy$
3. $u = \sin x\ coshy$
4. $v = (2x+1)y$
5. $u = x^3 - 3xy^2$
6. $v = e^x \sin 2y$

Determine a and b so that the given function is harmonic and find a harmonic conjugate.

1. $u = ax^3 + bxy$
   Ans: $u_x = 3ax^2 + by, u_y = bx$ and $u_{xx} = 6ax, u_{yy} = 0$
   Since u is harmonic, $u_{xx} + u_{yy} = 0 \implies 6ax = 0 \implies a = 0$
   According to CR equations, $u_x = v_y \implies v_y = 3ax^2 + by = by$
   $\implies v = \int bydy + \phi(x) = \dfrac{by^2}{2} + \phi(x)$
   Then $v_x = \phi'(x) =$. But $v_x = -u_y = -bx \implies \phi(x) = \int -bxdx = \dfrac{-bx^2}{2} + c$
   Therefore $v = \dfrac{b(y^2 - x^2)}{2} + c.$
2. $u = e^{\pi x} \cos av$
3. $u = \cos ax \cosh 2y$
4. $u = \cosh ax \cos y$

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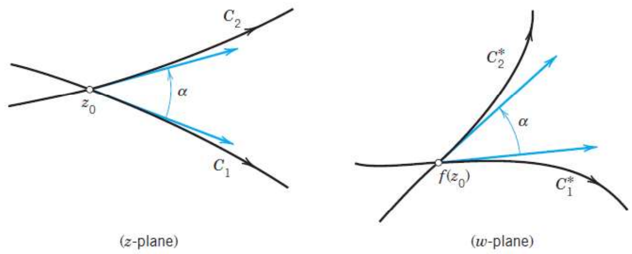

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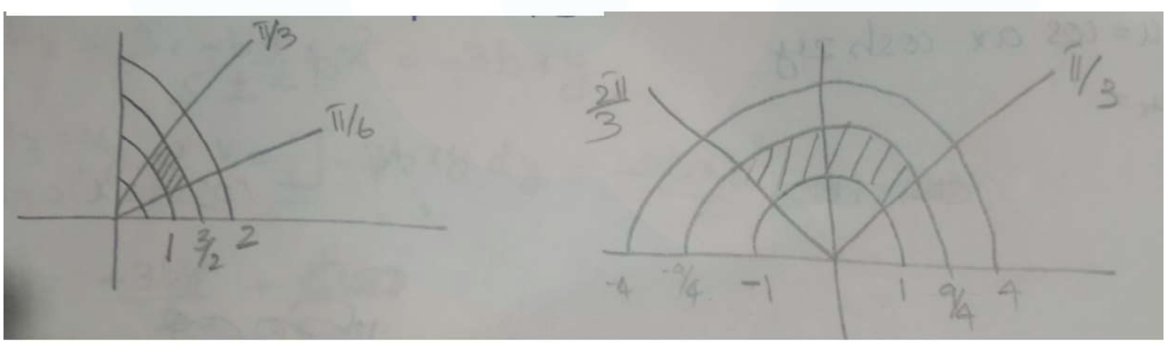

In cartesian coordinates, we have $z = x + iy$. Then $w = z^2 = x^2 - y^2 + i2xy$. Therefore $u = x^2 - y^2, v = 2xy$.

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Similarly horizontal lines $y = k$ are mapped onto

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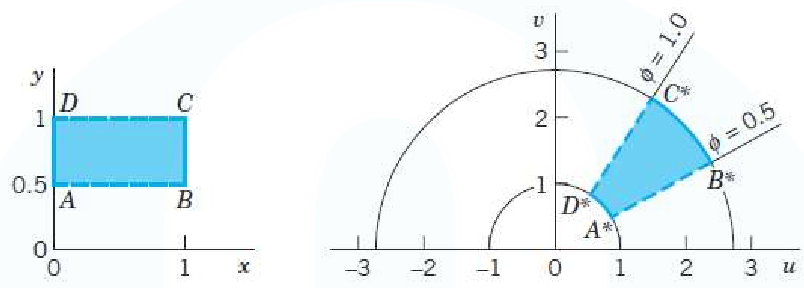
$$u = x^2 - k^2, v = 2xk$$
$$\implies x^2 = k^2 + u, v^2 = 4x^2k^2$$
$$\implies v^2 = 4k^2(k^2 + u), \text{ parabolas open to the right.}$$

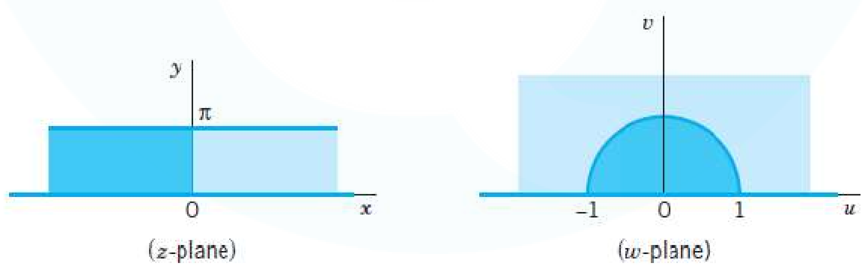
1.3.2 The mapping $w = e^z$

We have $w = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$. Then $|w| = |e^z| = e^x = e^{Rez}$ and arg $w = y = Im(z)$. Hence e^z maps a vertical straight line $x = x_0 = constant$ onto the circle $|w| = e^{x_0}$ and a horizontal straight line $y = y_0 = constant$ onto the ray arg$w = y_0$. The rectangle in figure is mapped onto a region bounded by circles and rays as shown below.



Mapping by $w = e^z$

The fundamental region $-\pi < argz \leq \pi$ of e^z in the z-plane is mapped bijectively and conformally onto the entire w-plane without the origin $w = 0$ (because $e^z = 0$ for no z). Following figure shows that the upper half of the fundamental region $0 < y \leq \pi$ is mapped onto the upper half-plane $0 < argw \leq \pi$, the left half $(0 < y \leq \pi, x \leq 0)$being mapped inside the unit disk $|w| \leq 1$ and the right half (when $x > 0$) is mapped onto the outside of $|w| > 1$.



Mapping by $w = e^z$

1.3.3 Linear Fractional Transformation

Linear fractional transformations (or Mobius transformations) are mappings

$$w = \frac{az + b}{cz + d}, (ad - bc \neq 0) \text{ ——(1)}$$

where a, b, c, d are complex or real numbers. Now

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$$w' = \frac{ad - bc}{(cz + d)^2}$$

This motivates our requirement $ad - bc \neq 0$. It implies conformality for all z and excludes the totally uninteresting case $w' = 0$ once and for all. Special cases of (1) are

- $w = z + b \longleftarrow$ Translations
- $w = az$ with $|a| = 1 \longleftarrow$ Rotations
- $w = az + b \longleftarrow$ Linear Transformations
- $w = \frac{1}{z} \longleftarrow$ Inversion in the unit circle.

1.3.4 The mapping $w = \frac{1}{z}$

In polar coordinates, $z = re^{i\theta}$ and $w = Re^{i\phi}$. Then

$$Re^{i\phi} = \frac{1}{re^{i\theta}} \implies R = \frac{1}{r}, \phi = -\theta$$

Hence the unit circle $|z| = r = 1$ is mapped on the unit circle $|w| = R = 1$.

This transformation maps $|z| < 1$ (the interior of the circle $|z| = 1$) is mapped on to $|w| > 1$ (the exterior of the circle $|w| = 1$) and $|z| > 1$ (the exterior of the circle $|z| = 1$) is mapped on to $|w| < 1$ (the interior of the circle $|w| = 1$)
The origin $z = 0$ is mapped onto the point at infinity $w = \infty$.
In cartesian coordinates, $z = x + iy$ and $w = u + iv$. Then

$$\begin{aligned} u + iv &= \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \\ \implies u &= \frac{x}{x^2 + y^2} \text{ and } v = \frac{-y}{x^2 + y^2} \text{ or} \\ \implies x &= \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2} \end{aligned}$$

Now consider the general equation of a circle in the z-plane

$$\begin{aligned} x^2 + y^2 + 2gx + 2fy + c = 0 \text{ ---(1)} \\ \implies \frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + 2g\frac{u}{u^2 + v^2} - 2f\frac{v}{u^2 + v^2} + c = 0 \\ \implies \frac{1}{(u^2 + v^2)^2} + 2g\frac{u}{u^2 + v^2} - 2f\frac{v}{u^2 + v^2} + c = 0 \\ \implies c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \text{ ---(2)} \end{aligned}$$

If $c \neq= 0$, the circle (1) does not pass through the origin and (2) represents a circle in the -plane. If $c = 0$, the circle in (1) passes through the origin and (2) reduces to $2gu - 2fv + 1 = 0$ which is a straight line in the w plane.

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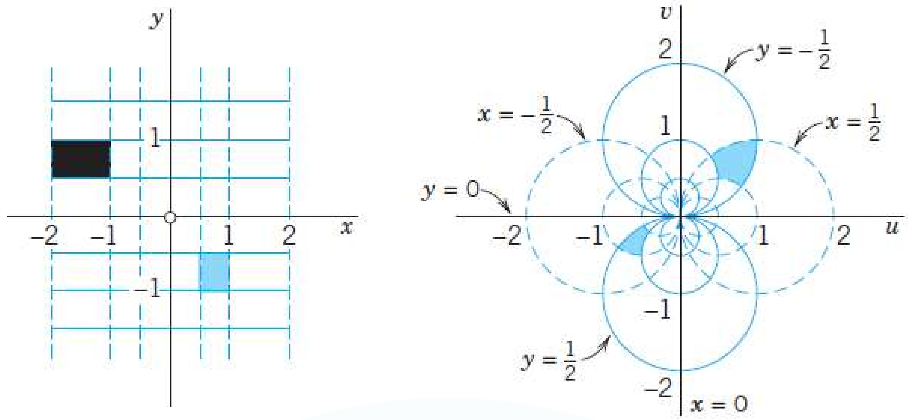

Mapping (Inversion) $w = 1/z$

$w = \dfrac{1}{z}$ maps every straight line or circle onto a circle or straight line.

1.3.5 The mapping $w = \sin z$

Let $z = x + iy$ and $w = u + iv$. Then

$$u + iv = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$
$$\implies u = sinx \cosh y, v = \cos x \sinh y$$

Since $\sin z$ is periodic with period 2π, we restrict z to the vertical strip $\dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$. The mapping is conformal at all points except at

$$f'(z) = 0 \implies \cos z = 0 \implies z = \pm\frac{\pi}{2}$$

Case 1: When $x = c$, a constant

$u = sinc \cosh y, v = \cos c \sinh y$. Then

$$\frac{u}{\sin c} = \cosh y \text{ and } \frac{v}{\cos c} = \sinh y$$
$$\cosh^2 y - \sinh^2 y = 1 \implies \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^c} = 1$$

Therefore the straight lines $x = c$ are mapped on to hyperbolas.

Case 2: When $y = c$, a constant

$u = sinx \cosh c, v = \cos x \sinh c$. Then

$$\frac{u}{\cosh c} = \sin x \text{ and } \frac{v}{\sinh c} = \cos x$$
$$\sin^2 x + \cos^2 x = 1 \implies \frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

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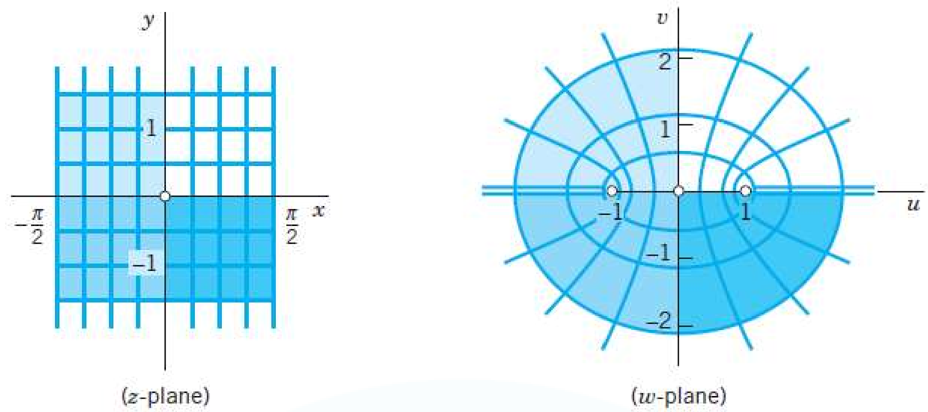
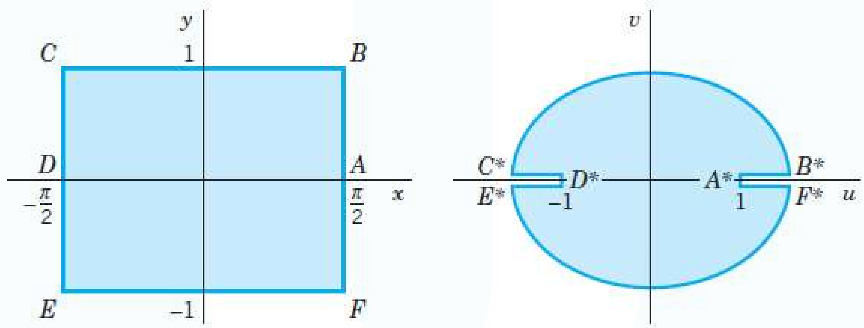


Mapping $w = u + iv = \sin z$

Therefore the straight lines $y = c$ are mapped on to ellipses.

Case 3: When $x = \dfrac{-\pi}{2}, u = -\cosh y$ and $v = 0$ and hence the image is u axis such that $u \leq 1$.

Case 4: When $x = \dfrac{\pi}{2}, u = \cosh y$ and $v = 0$ and hence the image is u axis such that $u \geq 1$.



Mapping by $w = \sin z$

PROBLEMS:

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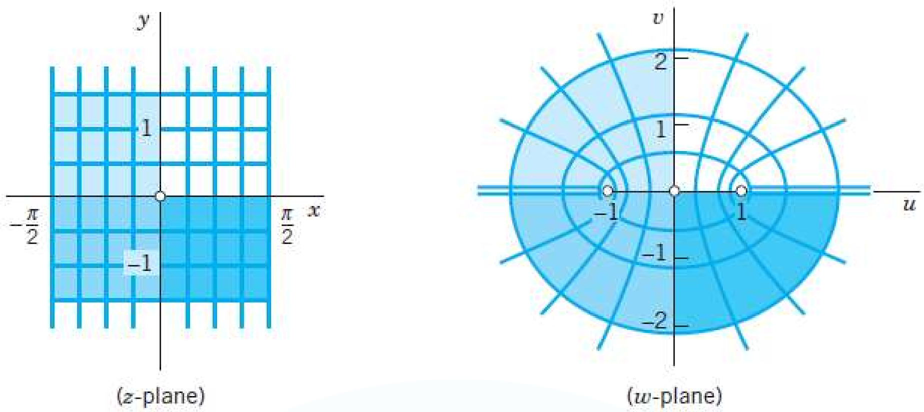

Mapping $w = u + iv = \sin z$

Therefore the straight lines $y = c$ are mapped on to ellipses.

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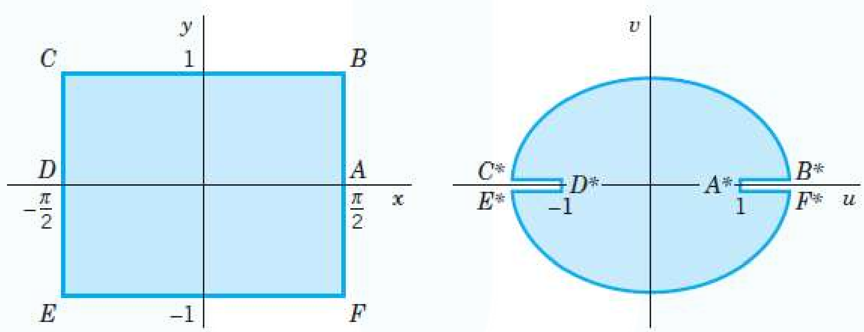

Mapping by $w = \sin z$

PROBLEMS:

1. Find the image of the circle $|z-3| = 5$ under the transformation $w = \dfrac{1}{z}$.

   Ans: $w = u + iv = \dfrac{1}{x+iy}$ or $x + iy = \dfrac{1}{u+iv} \implies x = \dfrac{u}{u^2+v^2}, y = \dfrac{-v}{u^2+v^2}$

$$
\begin{aligned}
|z-3| = 5 &\implies |x+iy-3| = 5 \\
\implies (x-3)^2 + y^2 = 25 &\implies x^{+}y^2 - 6x - 16 = 0 \\
\text{i.e; } \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} &- 6\frac{u}{u^2+v^2} - 16 = 0 \\
\implies 16(u^2+v^2) &+ 6u - 1 = 0
\end{aligned}
$$

   which represents a circle in the w-plane. So a circle in the z-plane is mapped onto a circle in the w-plane under the given transformation.

2. Find the image of the circle $|z-2i|=2$ under the transformation $w=\frac{1}{z}$.

   Ans: $w=u+iv=\frac{1}{x+iy}$ or $x+iy=\frac{1}{u+iv} \implies x=\frac{u}{u^2+v^2}, y=\frac{-v}{u^2+v^2}$

$$|z-2i|=2 \implies |x+i(y-2)=2 \implies x^2+y^2-4y=0, \text{ a circle in the z-plane}$$
$$\text{Then } \frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{4v}{u^2+v^2}=0$$
$$\implies \frac{1}{u^2+v^2}+\frac{4v}{u^2+v^2}=0 \implies 1+4v=0$$

   which is a straight line in the w-plane.

3. Determine the image of the region $\frac{1}{2} \le x \le 1, \frac{1}{2} \le y \le 1$ under the transformation $w=z^2$.

   Ans: Let $z=x+iy$ and $w=u+iv$. Then $u+iv=x^2-y^2+i2xy$ and so $u=x^2-y^2, v=2xy$.

   When $x=\frac{1}{2}, u=\frac{1}{4}-y^2, v=y$ so that $u=\frac{1}{4}-v^2$ or $v^2=-\left(u-\frac{1}{4}\right)$, which is a parabola open to the left with vertex at $\left(\frac{1}{4},0\right)$.

   When $x=1, u=1-y^2, v=2y$ so that $u=1-\frac{v^2}{4}$ or $v^2=-4(u-1)$ which is a parabola open to the left with vertex at (1,0).

   When $y=\frac{1}{2}, u=x^2-\frac{1}{4}, v=x$. Then $u=v^2-\frac{1}{4}$ or $v^2=u+\frac{1}{4}$ which is a parabola open to the right with vertex at $\left(\frac{-1}{4},0\right)$.

   When $y=1, u=x^2-1, y=2x$. Then $v^2=4(u+1)$ which is a parabola open to the right with vertex at $(-1,0)$.

   Thus the rectangular region bounded by the straight lines $x=\frac{1}{2}, x=1, y=\frac{1}{2}, y=1$ is mapped onto the region bounded by the parabolas $v^2=-\left(u-\frac{1}{4}\right), v^2=-4(u-1), v^2=u+\frac{1}{4}, v^2=4(u+1)$.

4. Find the image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w=\frac{1}{z}$.

   Ans: $w=\frac{1}{z} \implies z=\frac{1}{w}$ and so $x=\frac{u}{u^2+v^2}, y=\frac{-v}{x^2+y^2}$.

   When $y=\frac{1}{4}, \frac{-v}{x^2+y^2}=\frac{1}{4} \implies u^2+v^2+4v=0$ or which is a circle centered at $(0,-2)$ and radius 2.

   When $y=\frac{1}{2}, \frac{-v}{x^2+y^2}=\frac{1}{2} \implies u^2+v^2+2v=0$ or $u^2+(v+1)^2=1$ which is a circle centered at $(0,-1)$ and radius 1.

   Hence the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ is mapped onto the region between the circles $u^2+(v+2)^2=4$ and $u^2+(v+1)^2=1$.

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5. Show that the transformation $w = z^2$ maps the circle $|z-1| = 1$ into the centroid $\rho = 2(1+\cos\phi)$ where $w = \rho e^{i\phi}$ in the w-plane.
Ans: Let $z = re^{i\theta}$. Then $\rho = r^2, \phi = 2\theta$. Now

$$\begin{gathered}|z-1| = 1 \implies |x-1+iy| = 1 \text{ or } (x-1)^2+y^2 = 1 \\ \implies x^2+y^2-2x = 0 \text{ or } r^2 - 2r\cos\theta = 0 \\ r = 2\cos\theta \text{ or } r^2 = 4\cos^2\theta = 2(1+\cos 2\theta) \\ \text{i.e; } \rho = 2(1+\cos\phi)\end{gathered}$$

Therefore the circle $|z-1| = 1$ in z-plane is mapped into the centroid $\rho = 2(1+\cos\phi)$ in w-plane.

6. Find the image of $0 \le x \le 1, \frac{1}{2} \le y \le 1$ under the mapping $w = e^z$.
Ans: Let $w = Re^{i\phi}$ and $z = x+iy$. Then $Re^{i\phi} = e^x e^{iy} \implies R = e^x, \phi = y$.
When $x = 0, R = 1$. Therefore the line $x = 0$ is mapped onto a circle with radius .
When $x = 1, R = e$. So the line $x = 1$ is mapped onto a circle with radius e in the w-plane.
When $y = \frac{1}{2}$, $\phi = \frac{1}{2}$ and so the line $y = \frac{1}{2}$ is mapped on to a radial line making an angle of $\frac{1}{2}$ radian with u-axis. Similarly the line $y = 1$ is mapped on to a radial line making an angle of 1 radian with u-axis.
Therefore the image of the given ractangle in z-plane is mapped onto the region included between two circles and two rays.

7. Find the image of the triangle bounded by the lines $x+y = 2, x = 0, y = 0$ under the mapping $w = z^2$
Ans: Let $z = z+iy$ and $w = u+iv$. Then $u+iv = (x+iy)^2 = x^2-y^2+i2xy \implies u = x^2-y^2, v = 2xy$.
When $x = 0$, we get $u = -y^2, v = 0$ where $0 \le y \le 2$. Therefore the imaginary axis $x = 0$ is mapped onto $-4 \le u \le 0$.
When $y = 0, u = x^2 \& v = 0$, where $0 \le x \le 2$. Therefore the real axis $y = 0$ is mapped onto $0 \le u \le 4$.
Given $x+y = 2$. Then $y = 2-x$ and $u = x^2-(2-x)^2 \implies u = 4x-4$ or $x = \frac{1}{4}(u+4)$.
Then $y = \frac{1}{4}(4-u)$.
Therefore $v = 2xy$ becomes $v = \frac{1}{8}(16-u^2)$. Then $u^2 = -8(v-2)$ which is a parabola with vertex at $(2,0)$.
Thus the rectangular region is mapped onto the region bounded by the parabola and the u- axis where $-4 \le u \le 4$.

8. Find the image of the infinite strip $-1 \le x \le 1$ under the mapping $w = e^z$.
Ans: Let $w = Re^{i\phi}$ and $z = x+iy$. Then $R = e^x, \phi = y$. When $x = -1, R = e^-1$.

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Therefore the line $x = -1$ in the z-plane is mapped onto a circle with radius $R = e^{-1}$. When $x = 1, R = e$. Then the line $x = 1$ in the z-plane is mapped onto a circle with radius $R = e$. Thus the infinite strip $-1 \leq x \leq 1$ is mapped on to the region between the two circles with radii $R = e$ and $R = e^{-1}$.

HOMEWORK:

1. Find the image of the circle $|z| = a$ under the mapping $w = z^2$.
2. Find the image of the line $y - x + 1 = 0$ under the mapping $w = \frac{1}{z}$.
   Ans: $u^2 + v^2 - v - u = 0$ which is a circle in the w-plane.
3. Determine the region of the w-plane into which the first quadrant of the z-plane is mapped by the transformation $w = z^2$ where $w = \rho e^{i\phi}$.
   Ans: The first quadrant of the z-plane is mapped onto the upper half of the w plane.
4. Find the image of the region $|z - \frac{1}{3}| \leq \frac{1}{3}$ under the transformation $w = \frac{1}{z}$.
   Ans: $u \geq \frac{3}{2}$
5. Find the image of the semicircle $y = \sqrt{4 - x^2}$ under the transformation $w = z^2$.
   Ans: The given semicircle in $z-$ plane is mapped onto a circle with center at $(0, 0)$ and radius 4.

Fixed Points:

Fixed points of a mapping $w = f(z)$ are points that are mapped onto themselves, are kept fixed under the mapping. Thus they are obtained from $w = f(z) = z$.

A linear fractional transformation, not the identity, has at most two fixed points. If a linear fractional transformation is known to have three or more fixed points, it must be the identity mapping $w = z$.

Find the fixed points of:

1. $w = (a + ib)z^2$
   Ans: $w = z \implies (a + ib)z^2 = z \implies z((a + ib)z - 1) = 0$.
   Then the fixed points are $z = 0, \frac{1}{a + ib}$.
2. $w = 16z^5$
   Ans: $w = z \implies 16z^5 = z\ impliesz(16z^4 - 1) = 0$.
   Then the fixed points are $z = 0, \pm\frac{1}{2}, \pm\frac{i}{2}$
3. $w = \frac{iz + 4}{2z - 5i}$
   Ans: $w = z \implies \frac{iz + 4}{2z - 5i} = z \implies z^2 - 3iz - 2 = 0 \implies (z - i)(z - 2i) = 0$
   Then the fixed points are $z = i, 2i$.

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