Chapter 3

Multivariable Calculus : Integration

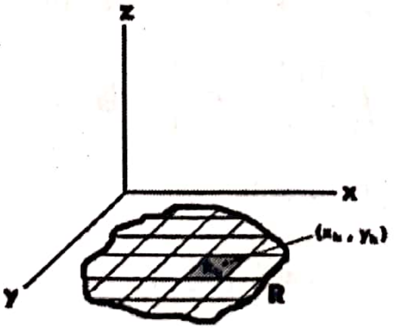
In this section we develop notions needed to extend the definite integral defined for functions of one variable to functions of two or more number of variables.

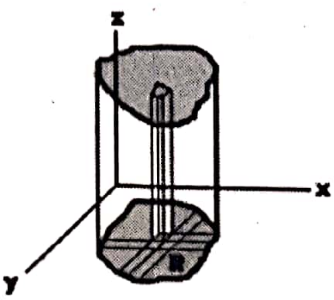
3.1 Multiple integral

Let **R** be a bounded region in the XY-plane in the sense that **R** can be enclosed within some suitably large rectangle (or circle with centre and radius chosen suitably) with sides parallel to the coordinate axes so that the region **R** does not extend indefinitely in any direction. Let $z = f(x, y)$ be a non-negative and continuous function defined at each point in the region **R**. Divide the region **R** into sub-rectangles using lines parallel to the coordinate axes. Let $\Delta\mathbf{R}_i$, $i = 1, 2, \ldots, n$ denote the sub-rectangles which are properly contained in the region **R** and ΔA_i be their areas and let (x_i^*, y_i^*) be an arbitrary interior point of the sub-rectangle $\Delta\mathbf{R}_i$ and let $z_i = f(x_i^*, y_i^*)$. The double integral of $z = f(x, y)$ over the region **R** is defined as

$$\int_R\int f(x,y)dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i = \lim_{n\to\infty}\sum_{i=1}^{n} z_i\Delta A_i$$

Geometrically, the quantity $f(x_i^*, y_i^*)\Delta A_i$ measures the volume of the rectangular box with base area ΔA_i and height $z_i = f(x_i^*, y_i^*)$. As the number of divisions of **R** into sub-rectangle increases, the limiting value approach the exact volume of the region enclosed by the surface $z = f(x, y)$ with base **R**. So the value of the above double integral is volume of the enclosed region with base **R**, $V = \int\int_R f(x, y)dA$.





If $z = f(x,y)$ takes both positive and negative values, the value of the above double integral gives the volume enclosed by surface $z = f(x,y)$ above the base **R** minus the volume enclosed by the surface $z = f(x,y)$ below **R**. In this case positive value means the volume above the region **R** is bigger and a negative value means the volume below the region **R** is bigger. So when we want to find the exact volume, we compute the sum of the absolute values of volumes above and below the region **R**.

Computing double integral as a limit is very complicated except in some simple cases. Analogous to partial differentiation, we perform partial integration two times to compute the double integral. One variable is kept constant and the function $f(x,y)$ is integrated with respect to the other variable and then the resulting function is integrated with respect to the variable which is kept constant in the first partial integration. That is,

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

The following theorem shows that the above two integrals are the same.

Fubini's Theorem

Let **R** be a rectangle defined by the inequalities $a \leq x \leq b$, $c \leq x \leq d$ and the function $z = f(x,y)$ be continuous on the rectangle, then

$$\int\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Example 3.1.1. *Consider* $\int\int_R (x^2+y^2)\,dxdy$ *where* R *is the rectangular region defined by* $-1 \leq x \leq 2$ *and* $0 \leq y \leq 2$.

$$\begin{aligned}
\int\int_R (x^2+y^2)\,dxdy &= \int_0^2 \left(\int_{-1}^2 (x^2+y^2)dx\right)dy \\
&= \int_0^2 \left[\frac{x^3}{3} + y^2 x\right]_{-1}^2 dy \\
&= \int_0^2 (3+3y^2)\,dy \\
&= \left[3y + 3\frac{y^3}{3}\right]_0^2 = 14
\end{aligned}$$

Also
$$\begin{aligned}
\int\int_R (x^2+y^2)\,dxdy &= \int_{-1}^2 \left(\int_0^2 (x^2+y^2)dy\right)dx \\
&= \int_{-1}^2 \left[x^2 y + \frac{y^3}{3}\right]_0^2 dx \\
&= \int_{-1}^2 \left[2x^2 + \frac{8}{3}\right] dx \\
&= \left[2\frac{x^3}{3} + \frac{8}{3}x\right]_{-1}^2 \\
&= 14
\end{aligned}$$

Problem 3.1.2. *Evaluate* $\int_0^1 \int_0^1 \frac{dxdy}{1+x+y}$

Solution:

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{dxdy}{1+x+y} &= \int_0^1 \left(\int_0^1 \frac{1}{1+x+y}dx\right) dy \\
&= \int_0^1 [\log(1+x+y)]_0^1\, dy \\
&= \int_0^1 [\log(2+y) - \log(1+y)]\, dy \\
&= \int_0^1 \log(2+y)\, dy - \int_0^1 \log(1+y)]\, dy \\
&= [(\log(2+y))\,(y)]_0^1 - \int_0^1 \frac{1}{2+y}\,(y)\, dy \\
&\quad - \left\{[(\log(1+y))\,(y)]_0^1 - \int_0^1 \frac{1}{1+y}\,(y)\,dy\right\} \\
&= \log 3 - \int_0^1 \left[1 - \frac{2}{2+y}\right] dy - \left\{\log 2 - \int_0^1 \left[1 - \frac{1}{1+y}\right] dy\right\} \\
&= \log 3 - \int_0^1 dy + \int_0^1 \frac{2}{2+y}dy - \left\{\log 2 - \int_0^1 dy + \int_0^1 \frac{1}{1+y}dy\right\} \\
&= \log 3 - [y]_0^1 + [2\log(2+y)]_0^1 - \{\log 2 - [y]_0^1 + [\log(1+y)]_0^1\} \\
&= \log 3 - [1-0] + [2\log 3 - 2\log 2] - \{\log 2 - [1-0] + [\log 2 - \log 1]\} \\
&= 3\log 3 - 4\log 2
\end{aligned}$$

Remark 1: Suppose that **R** is a rectangle defined by the inequalities $a \leq x \leq b$, $c \leq x \leq d$ and the function $z = f(x,y)$ be continuous on the rectangle such that $z = f(x,y) = g(x)h(y)$, then

$$\int\int_R f(x,y)dA = \int_a^b \int_c^d g(x)\,h(y)\,dydx$$

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$$= \int_a^b g(x)\left(\int_c^d h(y)dy\right)dx$$
$$= \left(\int_c^d h(y)dy\right)\int_a^b g(x)dx$$
$$= \int_a^b g(x)dx \times \int_c^d h(y)dy$$

Example 3.1.3. *Evaluate* $\int_0^1\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}dxdy$ [KTU. DEC 2016]

$$\int_0^1\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}dxdy = \int_0^1\left(\int_0^1 \frac{1}{\sqrt{(1-x^2)}} \times \frac{1}{\sqrt{(1-y^2)}}\,dx\right)dy$$
$$= \int_0^1 \frac{1}{\sqrt{(1-x^2)}}dx \times \int_0^1 \frac{1}{\sqrt{(1-y^2)}}dy$$
$$= \left[\sin^{-1}x\right]_0^1 \times \left[\sin^{-1}y\right]_0^1$$
$$= \left[\sin^{-1}(1)-\sin^{-1}(0)\right]\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$$
$$= \left[\frac{\pi}{2}-0\right]\left[\frac{\pi}{2}-0\right]$$
$$= \frac{\pi^2}{4}$$

Example 3.1.4. *Evaluate* $\int_0^a\int_0^b \frac{1}{\sqrt{(a^2-x^2)(b^2-y^2)}}dydx$

$$\int_0^a\int_0^b \frac{1}{\sqrt{(a^2-x^2)(b^2-y^2)}}dydx = \int_0^a\left(\int_0^b \frac{1}{\sqrt{(a^2-x^2)}} \times \frac{1}{\sqrt{(b^2-y^2)}}\,dy\right)dx$$
$$= \int_0^a \frac{1}{\sqrt{(a^2-x^2)}}dx \times \int_0^b \frac{1}{\sqrt{(b^2-y^2)}}dy$$
$$= \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a \times \left[\sin^{-1}\left(\frac{y}{b}\right)\right]_0^b$$
$$= \left[\sin^{-1}(1)-\sin^{-1}(0)\right]\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$$
$$= \left[\frac{\pi}{2}-0\right]\left[\frac{\pi}{2}-0\right]$$
$$= \frac{\pi^2}{4}$$

Example 3.1.5. *Evaluate* $\int_1^b\int_1^a \frac{dxdy}{xy}$

$$\int_1^b\int_1^a \frac{dxdy}{xy} = \int_1^b\left(\int_1^a \frac{1}{x}\times\frac{1}{y}\,dx\right)dy$$

$$= \int_1^b \frac{1}{y}\,dy \times \int_1^a \frac{1}{x}\,dx$$
$$= [\log y]_1^b \times [\log x]_1^a$$
$$= [\log b - \log 1][\log a - \log 1]$$
$$= \log a\ \log b \qquad [\because \log 1 = 0]$$

Remark 2: Many of the properties of definite integrals of functions of one variable are true for double integrals.

Let α, β be scalars and $f(x,y)$ and $g(x,y)$ be two continuous functions of two variables x and y defined on a region **R**. Then

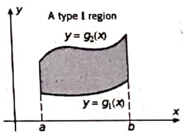
$$\int_R\int (\alpha f(x,y) + \beta g(x,y))\,dA = \alpha\int_R\int f(x,y)dA + \beta\int_R\int g(x,y)dA$$

and if **R** is the union of two disjoint regions $\mathbf{R_1}$ and $\mathbf{R_2}$, we have

$$\int_R\int f(x,y)dA = \int_{R_1}\int f(x,y)dA + \int_{R_2}\int f(x,y)dA$$

3.2 Double integral over non-rectangular regions

In the previous discussions, we considered only rectangular regions. Now we extend double integration over non-rectangular regions namely regions bounded on the left and right by $x = a$ and $x = b$ and bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$. For reference and convenience, we call this region type I region.



The type II region which we refer is a region bounded below and above by $y = c$ and $y = d$ and bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$.

The following theorem will enable us to compute double integrals over the above two types of regions.

Let $f(x,y)$ be a continuous function on a region **R**, then the double integrals for the above two type of regions can be computed using the following formulae. We use

$$\int_R\int f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

when limits of x are constants and we use

$$\int_R\int f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

when limits of y are constants.

Example 3.2.1. *Evaluate* $\int_1^2\int_0^x \frac{dydx}{x^2+y^2}$

$$\int_1^2\int_0^x \frac{dydx}{x^2+y^2} = \int_1^2\left(\int_0^x \frac{1}{x^2+y^2}dy\right)dx$$
$$= \int_1^2\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$
$$= \int_1^2 \frac{1}{x}\left[\tan^{-1}1 - \tan^{-1}0\right] dx$$
$$= \frac{\pi}{4}\int_1^2 \frac{1}{x}dx$$
$$= \frac{\pi}{4}[\log x]_1^2 = \frac{\pi}{4}[\log 2 - \log 1]$$
$$= \frac{\pi}{4}\log 2$$

Example 3.2.2. *Evaluate* $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

$$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1\left(\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}dy\right)dx$$
$$= \int_0^1\left(\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}dy\right)dx$$
$$= \int_0^1\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$
$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left\{\left[\tan^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)\right] - \left[\tan^{-1}(0)\right]\right\} dx$$
$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\frac{\pi}{4}dx$$
$$= \frac{\pi}{4}\left[\log[x+\sqrt{1+x^2}]\right]_0^1 = \frac{\pi}{4}\left[\log[1+\sqrt{2}] - \log 1\right]$$
$$= \frac{\pi}{4}\log[1+\sqrt{2}]$$

Example 3.2.3. *Evaluate* $\int_0^4\int_0^{x^2} e^{\frac{y}{x}}dydx$

$$\int_0^4\int_0^{x^2} e^{\frac{y}{x}}dydx = \int_0^4\left(\int_0^{x^2} e^{\frac{1}{x}y}dy\right)dx$$
$$= \int_0^4\left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^{x^2} dx = \int_0^4\left[xe^{\frac{y}{x}}\right]_0^{x^2} dx$$
$$= \int_0^4\left\{\left[xe^{\frac{x^2}{x}}\right] - [xe^0]\right\} dx = \int_0^4 (xe^x - x)\,dx$$
$$= \left[xe^x - e^x - \frac{x^2}{2}\right]_0^4$$
$$= 3e^4 - 7$$

Example 3.2.4. *Evaluate* $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{y}{\sqrt{x^2+y^2}}dydx$

$$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{y}{\sqrt{x^2+y^2}}dydx = \int_0^1\left(\int_x^{\sqrt{2-x^2}} \frac{y}{\sqrt{x^2+y^2}}dy\right)dx$$
$$= \int_0^1\left(\frac{1}{2}\int_x^{\sqrt{2-x^2}} \frac{2y}{\sqrt{x^2+y^2}}dy\right)dx$$
$$= \int_0^1 \frac{1}{2}\left[2\sqrt{x^2+y^2}\right]_x^{\sqrt{2-x^2}} dx$$
$$= \int_0^1 \left\{[\sqrt{x^2+(2-x^2)}] - [\sqrt{x^2+x^2}]\right\} dx = \int_0^1 (\sqrt{2} - \sqrt{2}\,x)\,dx$$
$$= \sqrt{2}\int_0^1 (1-x)\,dx$$
$$= \sqrt{2}\left[x - \frac{x^2}{2}\right]_0^1$$
$$= \sqrt{2}\left[1 - \frac{1}{2}\right]$$
$$= \frac{1}{\sqrt{2}}$$

Example 3.2.5. *Evaluate* $\int_0^1\int_0^{\sqrt{1-x^2}} xye^{x^2}dydx$

$$\int_0^1\int_0^{\sqrt{1-x^2}} xye^{x^2}dydx = \int_0^1\left(\int_0^{\sqrt{1-x^2}} xye^{x^2}dy\right)dx$$

$$= \int_0^1 x e^{x^2} \left(\int_0^{\sqrt{1-x^2}} y\, dy \right) dx$$
$$= \int_0^1 x e^{x^2} \left[\frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} dx$$
$$= \frac{1}{2} \int_0^1 x e^{x^2} (1 - x^2)\, dx$$
$$= \frac{1}{2} \int_0^1 (x - x^3)\, e^{x^2}\, dx$$
$$= \frac{1}{2} \left[\int_0^1 x e^{x^2}\, dx - \int_0^1 x^3 e^{x^2}\, dx \right]$$
$$= \frac{1}{2} \left[\frac{1}{2} \int_0^1 e^u\, du - \frac{1}{2} \int_0^1 u e^u\, du \right] \qquad [\text{ Take } u = x^2]$$
$$= \frac{1}{4} \{ [e^u]_0^1 - [(u)(e^u) - (1)(e^u)]_0^1 \}$$
$$= \frac{1}{4} \{ [e - 1] - [(e - e) - (0 - 1)] \}$$
$$= \frac{1}{4}(e - 2)$$

Example 3.2.6. *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$

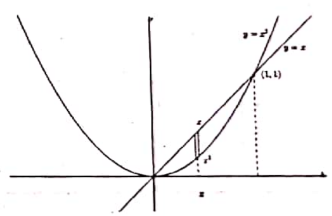
$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2 - x^2})^2 - y^2}\, dy \right) dx$$
$$= \int_0^a \left\{ \frac{y}{2} \sqrt{(\sqrt{a^2 - x^2})^2 - y^2} + \frac{(\sqrt{a^2 - x^2})^2}{2} \sin^{-1}\left(\frac{y}{\sqrt{a^2 - x^2}} \right) \right\}_0^{\sqrt{a^2-x^2}} dx$$
$$= \int_0^a \left\{ \left[0 + \frac{a^2 - x^2}{2} \sin^{-1}(1) \right] - [0 + 0] \right\} dx$$
$$= \frac{1}{2} \times \frac{\pi}{2} \int_0^a (a^2 - x^2)\, dx$$
$$= \frac{\pi}{4} \left[a^2 x - \frac{x^3}{3} \right]_0^a$$
$$= \frac{\pi a^3}{6}$$

Method of finding limits

First make a rough sketch of the region (not necessary). When the limits of x are constants (type I region), we take a typical fixed value of x within the given range and draw a line parallel to the y-axis through $(x, 0)$. Clearly along this line x remains constant. This line intersect the curves $y = g_1(x)$ and $y = g_2(x)$ giving the lower and upper limits of y, which are the inner limits of the double integral. Then imagine the line through $(x, 0)$ moving over the region from left to right. The left most point of intersection and the right most point of intersection with the region gives the lower limit and upper limit which are $x = a$ and $x = b$

When the limits of y are constants, we take a typical fixed value of y within the given range and draw a line parallel to the x-axis through $(0, y)$. Clearly along this line y remains constant. This line intersect the curves $x = h_1(y)$ and $x = h_2(y)$ giving the lower and upper limits of x, which are the inner limits of the double integral. Then imagine the line through $(0, y)$ moving over the region from bottom to top. The bottom most point of intersection and the top most point of intersection with the region gives the lower limit and upper limit which are $y = c$ and $y = d$

Example 3.2.7. *Evaluate* $\int\int_R xy(x + y)dxdy$ *where R is the region between $y = x^2$ and $y = x$*

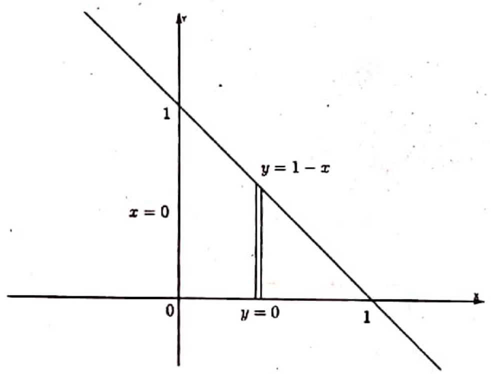


To find the point of intersection of the given curves we have to solve the equations $y = x$ and $y = x^2$
$x^2 = x \Longrightarrow x^2 - x = 0 \Longrightarrow x(x - 1) = 0 \Longrightarrow x = 0, 1$
So the curves intersects at the points $(0, 0)$ and $(1, 1)$ Now we draw a line parallel to the y-axis through the point $(x, 0)$ where x is fixed typical value between 0 and 1. Clearly along this which is parallel to the y-axis value of x-coordinate remains the same. That is, x. This line segment meets the two bounding curves and the lower limit for y-coordinate is x^2 and its upper limit is x. Imagine that we move the line drawn through the point $(x, 0)$ parallel to the y-axis over the region formed by the curves $y = x^2$ and $y = x$. Then x-coordinate varies from 0 to 1, which are the upper and lower limits for x. Now we can

compute

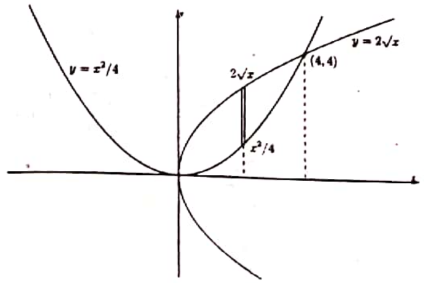
$$\iint_R xy(x+y)dxdy = \int_0^1 \left(\int_{x^2}^{x} x^2y + xy^2\, dy\right) dx$$
$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$
$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3}\right] - \left[\frac{x^6}{2} + \frac{x^7}{3}\right] dx$$
$$= \int_0^1 \frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} dx$$
$$= \left[\frac{5x^5}{6\times 5} - \frac{x^7}{2\times 7} - \frac{x^8}{3\times 8}\right]_0^1$$
$$= \frac{3}{56}$$

Example 3.2.8. *Evaluate* $\iint (x^2+y^2)\, dxdy$ *over the region in the positive quadrant for which* $x+y \le 1$



$$\iint_R (x^2+y^2)\, dxdy = \int_0^1 \left(\int_0^{1-x} (x^2+y^2)\, dy\right) dx$$
$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx$$
$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$
$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\frac{(1-x)^4}{(-1)4}\right]_0^1$$
$$= \frac{1}{6}$$

Example 3.2.9. *Evaluate* $\iint_R y^2 dxdy$ *where* R *is the region bounded by the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$



To find the point of intersection of the given curves we have to solve the equations $y = 2\sqrt{x}$ and $y = \frac{x^2}{4}$

$2\sqrt{x} = \frac{x^2}{4} \Longrightarrow 64x = x^4 \Longrightarrow x^4 - 64x = 0 \Longrightarrow x(x^3-64) = 0 \Longrightarrow x = 0, x = 4$

Therefore the curves intersects at the points $(0,0)$ and $(4,4)$

$$\iint_R y^2 dxdy = \int_0^4 \left(\int_{\frac{x^2}{4}}^{2\sqrt{x}} y^2\, dy\right) dx$$
$$= \int_0^4 \left[\frac{y^3}{3}\right]_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$
$$= \int_0^4 \left(\frac{8\,x^{3/2}}{3} - \frac{x^6}{3\times 64}\right) dx$$
$$= \left[\frac{8\,x^{5/2}}{3} \times \frac{2}{5} - \frac{x^7}{1344}\right]_0^4$$
$$= \frac{768}{35}$$

Example 3.2.10. *Evaluate* $\iint_R y\, dxdy$ *where* R *is the region bounded by the parabolas* $y^2 = 4ax$ *and* $x^2 = 4ay$

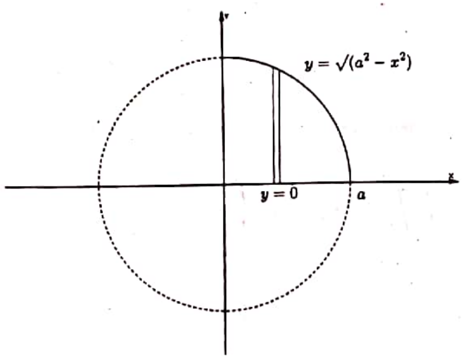
To find the point of intersection of the given curves we have to solve the equations $y = 2\sqrt{a}\sqrt{x}$ and $y = \frac{x^2}{4a}$

$2\sqrt{a}\sqrt{x} = \frac{x^2}{4a} \Longrightarrow 64a^3x = x^4 \Longrightarrow x^4 - 64a^3x = 0 \Longrightarrow x(x^3 - 64a^3) = 0 \Longrightarrow x = 0, x = 4a$

Therefore the curves intersects at the points $(0,0)$ and $(4a,4a)$

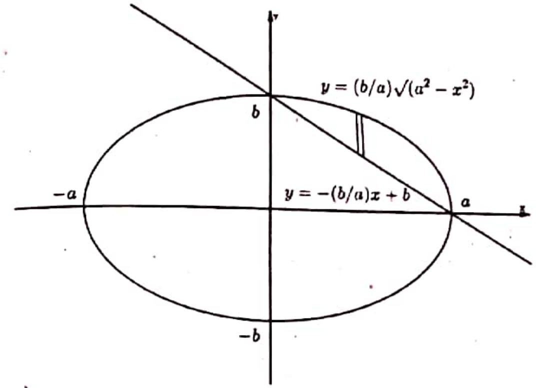
$$\begin{aligned}
\iint_R y\,dxdy &= \int_0^{4a}\left(\int_{\frac{x^2}{4a}}^{2\sqrt{a}\sqrt{x}} y\,dy\right)dx \\
&= \int_0^{4a}\left[\frac{y^2}{2}\right]_{\frac{x^2}{4a}}^{2\sqrt{a}\sqrt{x}} dx \\
&= \int_0^{4a}\left[\frac{4ax}{2} - \frac{x^4}{2\times 16a^2}\right]dx \\
&= \left[\frac{2ax^2}{2} - \frac{x^5}{160a^2}\right]_0^{4a} \\
&= \frac{48a^3}{5}
\end{aligned}$$

Example 3.2.11. *Find* $\iint xy\,dxdy$ *over the positive quadrant of the circle* $x^2+y^2=a^2$



$$\begin{aligned}
\iint xydxdy &= \int_0^a\left(\int_0^{\sqrt{a^2-x^2}} xy\,dy\right)dx \\
&= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx \\
&= \frac{1}{2}\int_0^a x(a^2-x^2)\,dx \\
&= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a \\
&= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] \\
&= \frac{a^4}{8}
\end{aligned}$$

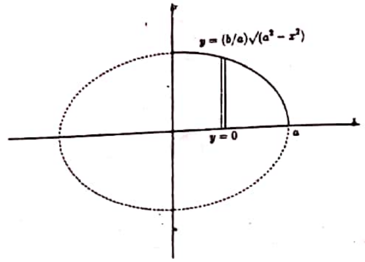
Example 3.2.12. *Evaluate* $\iint_R x\,dA$ *where R is the region in the first quadrant included between* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and* $\frac{x}{a}+\frac{y}{b}=1$



Clearly the curves intersects at the points $(a,0)$ and $(0,b)$.

$$\begin{aligned}
\iint_R xdA &= \int_0^a\left(\int_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} x\,dy\right)dx \\
&= \int_0^a x\left(\int_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\right)dx \\
&= \int_0^a x\,[\,y\,]_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx \\
&= \int_0^a x\left[\frac{b}{a}\sqrt{a^2-x^2} - \frac{b}{a}(a-x)\right]dx \\
&= \frac{b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx - \frac{b}{a}\int_0^a (ax-x^2)\,dx \\
&= \frac{b}{a}\times\frac{-1}{2}\left[(a^2-x^2)^{3/2}\times\frac{2}{3}\right]_0^a - \frac{b}{a}\left[\frac{ax^2}{2}-\frac{x^3}{3}\right]_0^a \\
&= \frac{ba^2}{3} - \frac{b}{a}\left[\frac{a^3}{2}-\frac{a^3}{3}\right] \\
&= \frac{ba^2}{3} - \frac{ba^2}{6} \\
&= \frac{a^2b}{6}
\end{aligned}$$

Example 3.2.13. *Evaluate* $\iint_R xy\,dA$ *where R is the region bounded by* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and lying in the first quadrant.*



$$\iint_R xy\,dA = \int_0^a \left(\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dy \right) dx$$

$$= \int_0^a x \left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\,\frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{b^2}{2a^2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{a^2b^2}{8}$$

Example 3.2.14. *Evaluate* $\iint (x+y)^2 dxdy$ *over the area bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\iint (x+y)^2\,dxdy = \int_{-a}^{a}\left(\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+2xy+y^2)\,dy\right) dx$$

$$= \int_{-a}^{a}\left\{x^2\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy + 2x\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy + \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2\,dy\right\} dx$$

$$= \int_{-a}^{a}\left\{x^2\,[y]_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} + 0 + 2\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y^2\,dy\right\} dx$$

$$= \int_{-a}^{a}\left\{2x^2\frac{b}{a}\sqrt{a^2-x^2} + 2\left[\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\right\} dx$$

$$= \frac{2b}{a}\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx + \frac{2b^3}{3a^3}\int_{-a}^{a}(a^2-x^2)^{3/2}\,dx$$

$$= \frac{4b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx + \frac{4b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2}\,dx$$

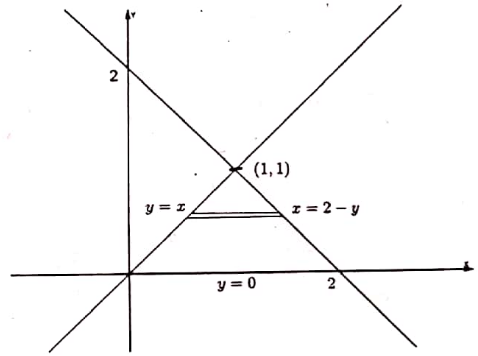
Put $x = a\cos\theta$

$$= \frac{4b}{a}\int_0^{\pi/2} a^4\sin^2\theta\cos^2\theta\,d\theta + \frac{4b^3}{3a^3}\int_0^{\pi/2} a^4\sin^4\theta\,d\theta$$

$$= \frac{4ba^4}{a}\times\frac{1\times 1}{4\times 2}\times\frac{\pi}{2} + \frac{4b^3a^4}{3a^3}\times\frac{3\times 1}{4\times 2}\times\frac{\pi}{2}$$

$$= \frac{\pi ab(a^2+b^2)}{4}$$

Example 3.2.15. *Evaluate* $\iint_R xy^2 dA$ *where R is the triangular region bounded by* $y=0$, $x=y$ *and* $x+y=2$



$$\iint_R xy^2\,dA = \int_0^1\left(\int_y^{2-y} xy^2\,dx\right) dy$$

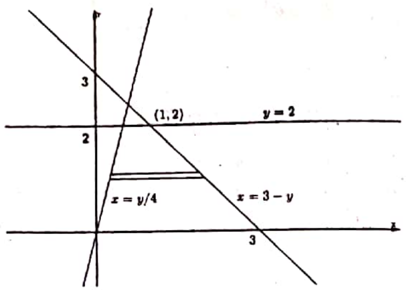
$$= \int_0^1 y^2\left[\frac{x^2}{2}\right]_y^{2-y} dy = \frac{1}{2}\int_0^1 y^2\left[(2-y)^2-y^2\right] dy$$

$$= \frac{1}{2}\int_0^1 y^2[4-4y]\,dy = \frac{1}{2}\int_0^1 [4y^2-4y^3]\,dy$$

$$= \frac{1}{2}\left[\frac{4y^3}{3}-\frac{4y^4}{4}\right]_0^1$$

$$= \frac{1}{6}$$

Example 3.2.16. *Evaluate* $\int\int(x^2+y^2)dxdy$ *throughout the area enclosed by the curves* $y=4x, x+y=3, y=0$ *and* $y=2$.



$$\int\int(x^2+y^2)\,dxdy = \int_0^2\left(\int_{y/4}^{3-y}(x^2+y^2)\,dx\right)dy$$
$$= \int_0^2\left[\frac{x^3}{3}+y^2x\right]_{y/4}^{3-y}dy$$
$$= \int_0^2\left\{\left[\frac{(3-y)^3}{3}+y^2(3-y)\right]-\left[\frac{y^3}{3\times 64}+\frac{y^3}{4}\right]\right\}dy$$
$$= \int_0^2\left[\frac{(3-y)^3}{3}+3y^2-y^3-\frac{y^3}{192}-\frac{y^3}{4}\right]dy$$
$$= \left[\frac{(3-y)^4}{(-1)4\times 3}+3\frac{y^3}{3}-\frac{y^4}{4}-\frac{y^4}{4\times 192}-\frac{y^4}{4\times 4}\right]_0^2$$
$$= \frac{463}{48}$$

3.2.1 Method of changing the order of integration

When the lower and upper limits of x and y are constants, by Fubini's theorem, we have

$$\int_R\int f(x,y)dA = \int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy$$

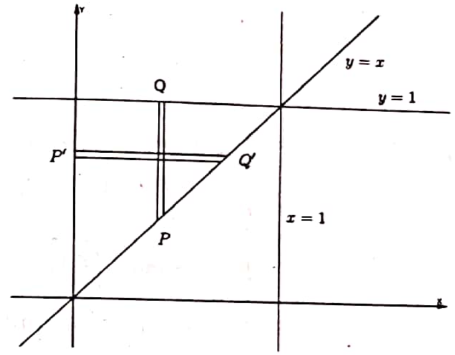
That is, the value of the double integral is invariant under the change of order of integration and this result is true even if the inner limits are variables. The advantage of changing the order of integration is that some complicated integrals can be computed easily by changing the order of integration.

When we want to compute an integral of the form $\int_R\int f(x,y)dxdy$, we draw a rough sketch of the region (not necessary). We first keep y fixed, say $y=y_1$ and draw a line parallel to the x-axis in the region $\mathbf{R}$ through the point $(0,y_1)$. This line intersect the left bounding curve and right bounding curve of the region giving the lower and upper limits for x in terms of y or constants. Now move this line parallel to the x-axis from bottom to top over the region which will give the lower and upper limit of the variable y.

When we want to compute an integral of the form $\int_R\int f(x,y)dydx$, we draw a rough sketch of the region (not ncessary). We first keep x fixed, say $x=x_1$ and draw a line parallel to the y-axis in the region $\mathbf{R}$ through the point $(x_1,0)$. This line intersect the bottom bounding curve and top bounding curve of the region giving the lower and upper limits for y in terms of x or constants. Now move this line parallel to the y-axis from left to right over the region which will give the lower and upper limits of the variable x.

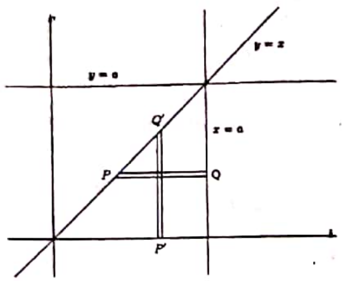
Example 3.2.17. *Change the order of integration and hence evaluate*

$$\int_0^1\int_x^1\frac{x}{x^2+y^2}\,dydx$$



$$\int_0^1\int_x^1\frac{x}{x^2+y^2}\,dydx = \int_0^1\left(\int_0^y\frac{x}{x^2+y^2}\,dx\right)dy$$
$$= \int_0^1\frac{1}{2}\left(\int_0^y\frac{2x}{x^2+y^2}\,dx\right)dy$$
$$= \frac{1}{2}\int_0^1\left[\log(x^2+y^2)\right]_0^y dy$$
$$= \frac{1}{2}\int_0^1\left[\log(y^2+y^2)-\log(0+y^2)\right]dy$$
$$= \frac{1}{2}\int_0^1\left[\log(2y^2)-\log(y^2)\right]dy$$
$$= \frac{1}{2}\int_0^1\log\left(\frac{2y^2}{y^2}\right)dy$$
$$= \frac{1}{2}\log 2\int_0^1 dy$$
$$= \frac{1}{2}\log 2\,[y]_0^1$$
$$= \frac{1}{2}\log 2$$

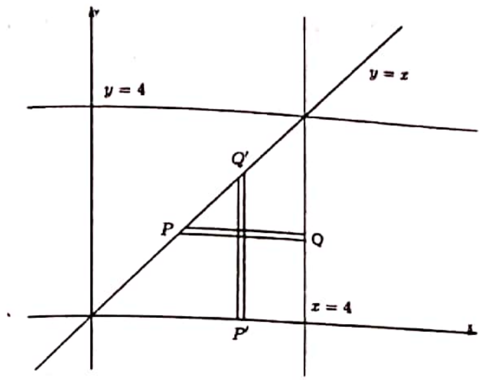
Example 3.2.18. *Change the order of integration and hence evaluate* $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy$



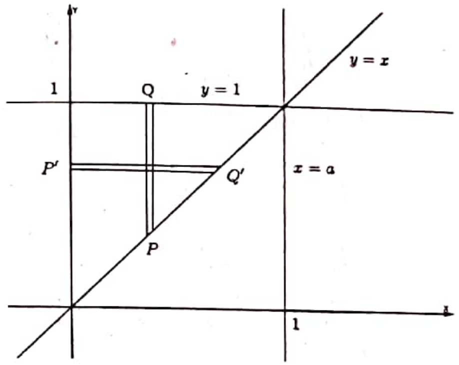
$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dxdy = \int_0^a \left(\int_0^x \frac{x}{x^2+y^2}\, dy \right) dx$$
$$= \int_0^a x \left(\int_0^x \frac{1}{x^2+y^2}\, dy \right) dx$$
$$= \int_0^a x \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x dx$$
$$= \int_0^a \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$
$$= \int_0^a \frac{\pi}{4}\, dx$$
$$= \frac{\pi}{4} \int_0^a dx$$
$$= \frac{\pi}{4} [x]_0^a = \frac{\pi a}{4}$$

Example 3.2.19. *Evaluate the integral* $\int_0^4 \int_y^4 \frac{x}{x^2+y^2}\, dx\, dy$ *by reversing the order of integration.* *[KTU. DEC 2016]*

$$\int_0^4 \int_y^4 \frac{x}{x^2+y^2}\, dx\, dy = \int_0^4 \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$
$$= \int_0^4 x \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x dx$$
$$= \int_0^4 \frac{\pi}{4}\, dx$$
$$= \frac{\pi}{4} [x]_0^4 = \pi$$

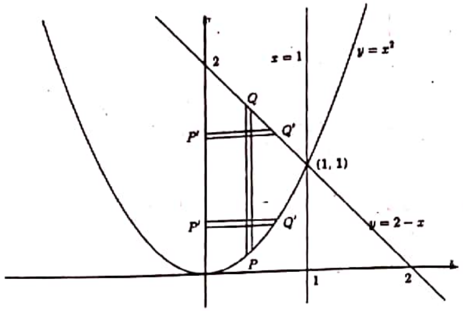


Example 3.2.20. *Change the order of integration and hence evaluate* $\int_0^1 \int_x^1 \sin(y^2)\, dydx$



$$\int_0^1 \int_x^1 \sin(y^2)\, dydx = \int_0^1 \left(\int_0^y \sin(y^2)\, dx \right) dy$$
$$= \int_0^1 \sin(y^2) \left(\int_0^y dx \right) dy$$
$$= \int_0^1 \sin(y^2)\, [x]_0^y\, dy$$
$$= \int_0^1 y \sin(y^2)\, dy$$
$$= \frac{1}{2} \int_0^1 2y \sin(y^2)\, dy$$
$$= -\frac{1}{2} [\cos(y^2)]_0^1$$
$$= \frac{1}{2}[1 - \cos 1]$$

Example 3.2.21. *Change the order of integration and hence evaluate* $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$



$$\int_0^1 \int_{x^2}^{2-x} xy\, dydx = \int_0^1 \left(\int_0^{\sqrt{y}} xy\, dx \right) dy + \int_1^2 \left(\int_0^{2-y} xy\, dx \right) dy$$
$$= \int_0^1 y \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} dy + \int_1^2 y \left[\frac{x^2}{2} \right]_0^{2-y} dy$$
$$= \frac{1}{2} \int_0^1 y[(\sqrt{y})^2 - 0]\, dy + \frac{1}{2} \int_1^2 y[(2-y)^2 - 0]\, dy$$
$$= \frac{1}{2} \int_0^1 y^2\, dy + \frac{1}{2} \int_1^2 (4y - 4y^2 + y^3)\, dy$$
$$= \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1 + \frac{1}{2} \left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4} \right]_1^2$$
$$= \frac{3}{8}$$

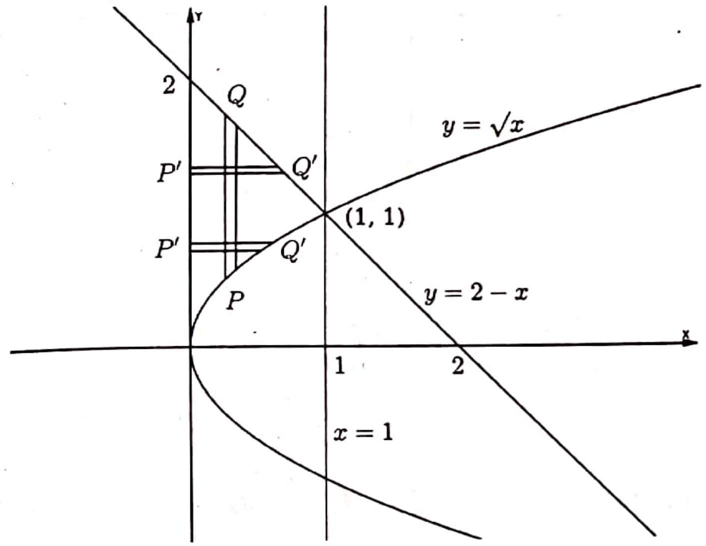
Example 3.2.22. *Change the order of integration and hence evaluate*

$$\int_0^a \int_{x^2/a}^{2a-x} xy\, dydx$$

$$\int_0^a \int_{x^2/a}^{2a-x} xy\, dydx = \int_0^a \left(\int_0^{\sqrt{a}\sqrt{y}} xy\, dx \right) dy + \int_a^{2a} \left(\int_0^{2a-y} xy\, dx \right) dy$$
$$= \int_0^a y \left[\frac{x^2}{2} \right]_0^{\sqrt{a}\sqrt{y}} dy + \int_a^{2a} y \left[\frac{x^2}{2} \right]_0^{2a-y} dy$$
$$= \frac{1}{2} \int_0^a y[ay]\, dy + \frac{1}{2} \int_a^{2a} y(2a-y)^2\, dy$$

$$= \frac{a}{2} \int_0^a y^2\, dy + \frac{1}{2} \int_a^{2a} (4a^2y - 4ay^2 + y^3)\, dy$$
$$= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a + \left[4a^2\frac{y^2}{2} - 4a\frac{y^3}{3} + \frac{y^4}{4} \right]_a^{2a}$$
$$= \frac{a^4}{6} + \left[8a^4 - 32\frac{a^4}{3} + 4a^4 \right] - \left[2a^4 - \frac{4a^4}{3} + \frac{a^4}{4} \right]$$
$$= \frac{3a^4}{8}$$

Example 3.2.23. *Change the order of integration* $\int_0^1 \int_{\sqrt{x}}^{2-x} f(x,y)dydx$



$$\int_0^1 \int_{\sqrt{x}}^{2-x} f(x,y)\, dydx = \int_0^1 \left(\int_0^{y^2} f(x,y)\, dx \right) dy + \int_1^2 \left(\int_0^{2-y} f(x,y)\, dx \right) dx$$

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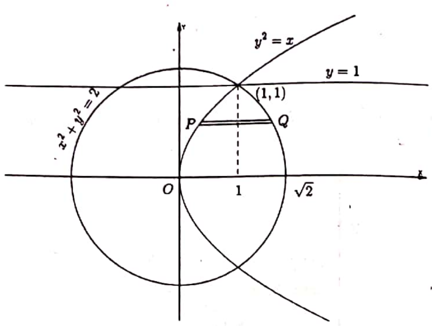
Example 3.2.24. *Change the order of integration in* $\int_0^1 \int_{y^2}^{\sqrt{2-y^2}} f(x,y)\, dx\, dy$

[KTU. FEB 2017]

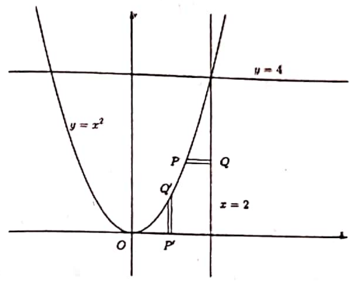


$$\int_0^1 \int_{y^2}^{\sqrt{2-y^2}} f(x,y)\, dx\, dy = \int_0^1 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} f(x,y)\, dy\, dx$$

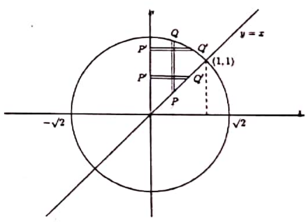
Example 3.2.25. *Evaluate the integral* $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$ *by reversing the order of integration.*

[KTU. FEB 2017]

$$\begin{aligned}
\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy &= \int_0^2 \int_0^{x^2} e^{x^3}\, dy\, dx \\
&= \int_0^2 e^{x^3} [y]_0^{x^2}\, dx \\
&= \int_0^2 x^2 e^{x^3}\, dx \\
&= \frac{1}{3}[e^{x^3}]_0^2 \\
&= \frac{1}{3}(e^8 - 1)
\end{aligned}$$



Example 3.2.26. *Change the order of integration and hence evaluate* $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dydx$



$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dydx = \int_0^1 \left(\int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dx \right) dy + \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx \right) dy$$

$$= \int_0^1 \frac{1}{2}\left(\int_0^y \frac{2x}{\sqrt{x^2+y^2}}\,dx\right) dy + \int_1^{\sqrt{2}} \frac{1}{2}\left(\int_0^{\sqrt{2-y^2}} \frac{2x}{\sqrt{x^2+y^2}}\,dx\right) dy$$

$$= \int_0^1 \frac{1}{2}\left[2\sqrt{x^2+y^2}\right]_0^y dy + \int_1^{\sqrt{2}} \frac{1}{2}\left[2\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left\{\left[\sqrt{y^2+y^2}\right] - \left[\sqrt{0+y^2}\right]\right\} dy + \int_1^{\sqrt{2}} \left\{\left[\sqrt{(2-y^2)+y^2}\right] - \left[\sqrt{0+y^2}\right]\right\} dy$$

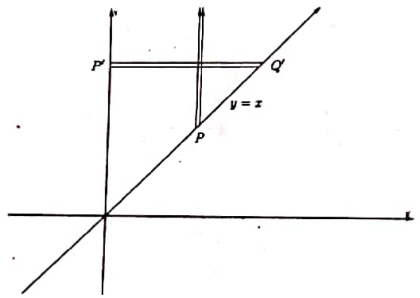
$$= \int_0^1 \left(\sqrt{2}\,y - y\right) dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y\right) dy$$

$$= \left[\sqrt{2}\,\frac{y^2}{2} - \frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \left[\frac{1}{\sqrt{2}} - \frac{1}{2}\right] + [2-1] - \left[\sqrt{2} - \frac{1}{2}\right]$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}}$$

Example 3.2.27. *Change the order of integration and hence evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$



$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx = \int_0^\infty \left(\int_0^y \frac{e^{-y}}{y}\,dx\right) dy = \int_0^\infty \frac{e^{-y}}{y}\,[x]_0^y\,dy$$

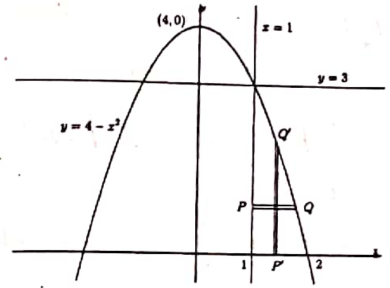
$$= \int_0^\infty \frac{e^{-y}}{y}\,[y-0]\,dy$$

$$= \int_0^\infty e^{-y}\,dy = \left[\frac{e^{-y}}{(-1)}\right]_0^\infty = [0] - [(-1)]$$

$$= 1$$

Example 3.2.28. *Change the order of integration and hence evaluate*

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dxdy$$



$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dxdy = \int_1^2 \left(\int_0^{4-x^2} (x+y)\,dy\right) dx$$

$$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2}\right] dx$$

$$= \int_1^2 \left[4x - x^3 + \frac{1}{2}(16 - 8x^2 + x^4)\right] dx$$

$$= \left[4\frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{2}\left(16x - 8\frac{x^3}{3} + \frac{x^5}{5}\right)\right]_1^2$$

$$= \frac{241}{60}$$

Example 3.2.29. *Change the order of integration and hence evaluate* $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}}\,dydx$

$$\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}}\,dydx = \int_0^\infty \left(\int_y^\infty x e^{-\frac{x^2}{y}}\,dx\right) dy$$

$$= \int_0^\infty \left(\frac{-y}{2}\right)\left[\int_y^\infty \left(\frac{-2x}{y}\right) e^{\frac{-x^2}{y}}\,dx\right] dy = -\frac{1}{2}\int_0^\infty y \left[e^{\frac{-x^2}{y}}\right]_y^\infty dy$$

$$= -\frac{1}{2}\int_0^\infty y\,[0 - e^{-y}]\,dy$$

$$= \frac{1}{2}\int_0^\infty y e^{-y} dy$$

$$= \frac{1}{2}\left[(y)\left(\frac{e^{-y}}{(-1)}\right) - (1)\left(\frac{e^{-y}}{(-1)(-1)}\right)\right]_0^\infty = \frac{1}{2}\left[-ye^{-y} - e^{-y}\right]_0^\infty$$

$$= \frac{1}{2}$$

3.2.2 Computing area as a double integral

The volume bounded by a surface $z = f(x,y)$ with the region **R** in the XY-plane is given by

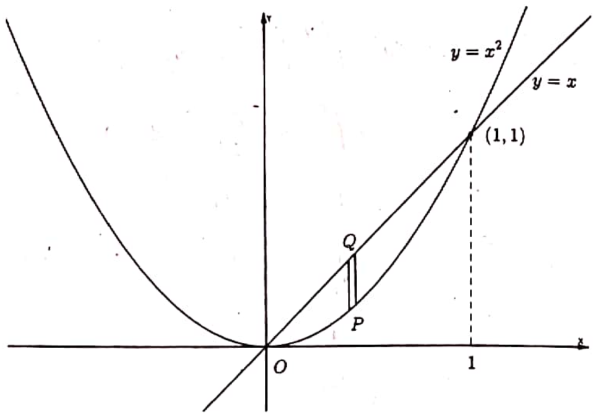
$$\int\int_R f(x,y)dA = \int\int_R f(x,y)dxdy$$

Choose $f(x,y) = 1$, then the double integral becomes

$$\int\int_R f(x,y)dA = \int\int_R f(x,y)dxdy = \int\int_R 1\,dxdy = \int\int_R dxdy$$

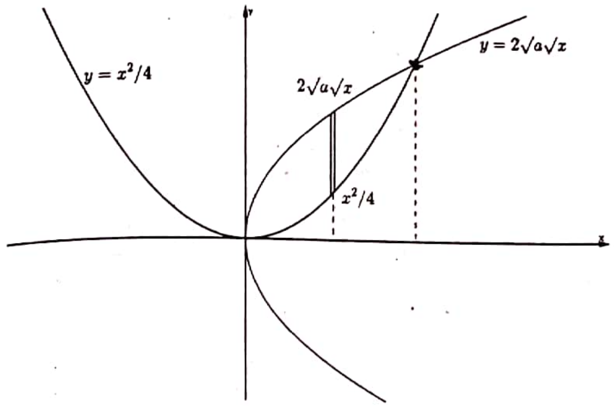
which is the area of the region **R**

Example 3.2.30. *Using double integral find the area lying between the parabola $y = x^2$ and the line $y = x$* [KTU. FEB 2017]



$$\text{Area} = \int\int_R dxdy$$
$$= \int_0^1 \int_{x^2}^{x} dydx$$
$$= \int_0^1 [\, y\,]_{x^2}^{x}\; dx$$
$$= \int_0^1 (x - x^2)\; dx$$
$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$= \left[\frac{1}{2} - \frac{1}{3}\right]_0^1$$
$$= \frac{1}{6}$$

Example 3.2.31. *Using double integral find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$*



$$\text{Area} = \int\int_R dxdy$$
$$= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{a}\sqrt{x}} dydx$$
$$= \int_0^{4a} \left[2\sqrt{a}\sqrt{x} - \frac{x^2}{4a}\right] dx$$
$$= \left[2\sqrt{a} \times \frac{2}{3}x^{3/2} - \frac{1}{4a} \times \frac{x^3}{3}\right]_0^{4a}$$
$$= \frac{16a^2}{3}$$

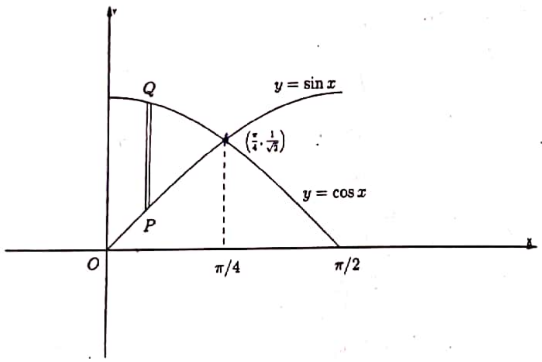
Example 3.2.32. *Using double integral find the area enclosed by the lines $x = 0$, $y = 0$ and $\frac{x}{a} + \frac{y}{b} = 1$*

$$\text{Area} = \int\int_R dxdy$$
$$= \int_0^{a} \int_0^{\frac{b}{a}(a-x)} dy\; dx$$
$$= \int_0^{a} \frac{b}{a}(a - x)\; dx$$
$$= \frac{b}{a}\left[ax - \frac{x^2}{2}\right]_0^a = \frac{ab}{2}$$

Example 3.2.33. *Using double integral find the area of a plate in the form of a quadrant of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

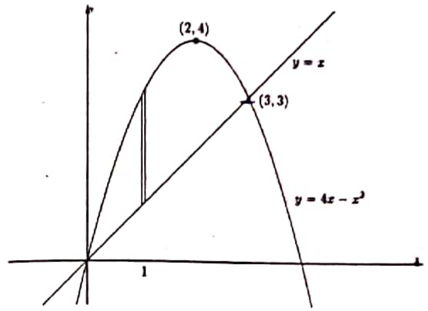
$$\begin{aligned}
\text{Area} &= \int\int_R dxdy \\
&= \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx \\
&= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx \\
&= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a \\
&= \frac{b}{a}\left(\frac{a^2}{2}\times\frac{\pi}{2}\right) \\
&= \frac{\pi ab}{4}
\end{aligned}$$

Example 3.2.34. *Use double integration to find the area of the plane region enclosed by the given curves* $y=\sin x$ *and* $y=\cos x$ *for* $0\le x\le\frac{\pi}{4}$. *[KTU. DEC 2016]*



$$\begin{aligned}
\text{Area} &= \int\int_R dxdy \\
&= \int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx \\
&= \int_0^{\pi/4}(\cos x-\sin x)\,dx \\
&= [\sin x+\cos x]_0^{\pi/4} \\
&= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-(0+1)=\sqrt{2}-1
\end{aligned}$$

Example 3.2.35. *Using double integral find the area lying between the parabola* $y=4x-x^2$ *and the line* $y=x$

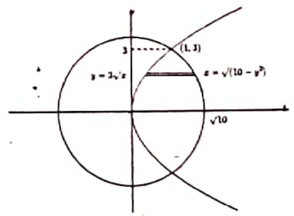


$$y=4x-x^2\implies(x-2)^2=-(y-4)$$

The points of intersection of the curves $y=4x-x^2$ and $y=x$ are $(0,0)$ and $(3,3)$.

$$\begin{aligned}
\text{Area} &= \int\int_R dxdy \\
&= \int_0^3\int_x^{4x-x^2} dy\,dx \\
&= \int_0^3\left[(4x-x^2)-x\right]\,dx \\
&= \left[3\frac{x^2}{2}-\frac{x^3}{3}\right]_0^3=\frac{9}{2}
\end{aligned}$$

Example 3.2.36. *Using double integral find the area in the first quadrant bounded by x-axis and the curves* $x^2+y^2=10$ *and* $y^2=9x$

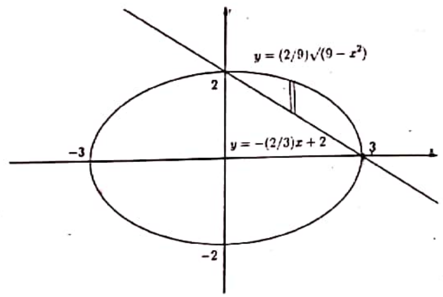


Solving the equations $x^2+y^2=10$ and $y^2=9x$
$x^2+9x=10\implies(x+10)(x-1)=0\implies x=-10,1$
$x=-10\implies y^2=-90$, which is not possible.

$x = 1 \Longrightarrow y^2 = 9 \Longrightarrow y = \pm 3$

$$\begin{aligned}
\text{Area} &= \iint_R dxdy \\
&= \int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dx\, dy \\
&= \int_0^3 \sqrt{10-y^2} - \frac{y^2}{9}\, dy \\
&= \left[\frac{y}{2}\sqrt{10-y^2} + \frac{10}{2}\sin^{-1}\left(\frac{y}{\sqrt{10}}\right)\right]_0^3 - \left[\frac{y^3}{27}\right]_0^3 \\
&= \frac{1}{2} + 5\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)
\end{aligned}$$

Example 3.2.37. *Using double integral find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$*



$\int\int$

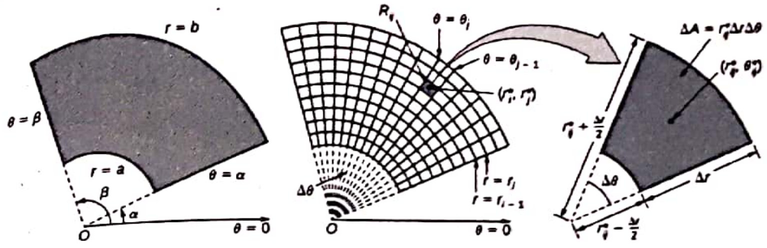
$$4x^2 + 9y^2 = 36 \Longrightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$
$$2x + 3y = 6 \Longrightarrow \frac{x}{3} + \frac{y}{2} = 1$$

$$\begin{aligned}
\text{Area} &= \iint_R dxdy \\
&= \int_0^3 \int_{\frac{2}{3}(3-x)}^{\frac{2}{3}\sqrt{9-x^2}} dy\, dx \\
&= \int_0^3 \frac{2}{3}\sqrt{9-x^2} - \frac{2}{3}(3-x)\, dx \\
&= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 - \frac{2}{3}\left[3x - \frac{x^2}{2}\right]_0^3 \\
&= \frac{2}{3}\left[\frac{9}{2} \times \frac{\pi}{2}\right] - \frac{2}{3} \times \frac{9}{2} \\
&= \frac{3\pi}{2} - 3
\end{aligned}$$

3.3 Double integral in Polar coordinates

Some classes of double integrals can be evaluated easily by converting them from Cartesian coordinates to polar coordinates. This is specifically true when the equation of curves bounding the region becomes simpler in polar coordinates.

For convenience, we call a region bounded by two rays $\theta = \alpha$ and $\theta = \beta$ and two continuous polar curves $r = r_1(\theta)$ and $r = r_2(\theta)$ such that $\alpha \le \beta$, $\beta - \alpha \le 2\pi$, $0 \le r_1(\theta) \le r_2(\theta)$ as a **simple polar region**. If the bounding polar curves of the polar region are circular arcs, it is called **polar rectangle**.



(a) (b) (c)

To compute the volume bounded by a continuous surface $f(r,\theta)$ in a polar region, we subdivide the region $\mathbf{R}$ into polar rectangles ΔR_k each of area ΔA_k, $k = 1, 2, \ldots, n$. Let (r_k^*, θ_k^*) be an arbitrary interior point in ΔR_k. The value of the product $f(r_k^*, \theta_k^*)\Delta A_k$ gives the volume of the region with base ΔA_k and height $f(r_k^*, \theta_k^*)$. The volume of the region with base $\mathbf{R}$ is given by $V = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$. Symbollically, this is expressed as

$$\iint_R f(r,\theta)dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k$$

provided the limit exits (finite and unique). For continuous functions the above limit is always unique. For sufficiently small polar rectangles we get $\Delta A_k = r_k \Delta r_k \Delta\theta_k$, so that the above integral takes the form

$$\iint_R f(r,\theta)dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k = \iint_R f(r,\theta) r\, dr\, d\theta$$

Let $f(r,\theta)$ be a continuous function defined at each point in a simple polar region $\mathbf{R}$ with boundary $\theta = \alpha$ and $\theta = \beta$, $\alpha < \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ then,

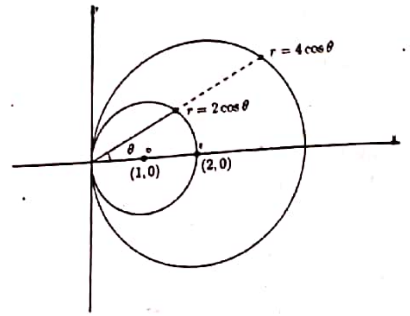
$$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta) r\, dr\, d\theta$$

Example 3.3.1. *Use polar coordinates to evaluate the double integral*

$$\iint_R (x^2 + y^2)\, dA$$

where R is the region between the circles $x^2 + y^2 = 2x$ and $x^2 + y^2 = 4x$.

Solution:



Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2+y^2 = r^2, dA = r\,dr d\theta$.

$$x^2+y^2 = 2x \implies r^2 = 2r\cos\theta \implies r = 2\cos\theta$$
$$x^2+y^2 = 4x \implies r^2 = 4r\cos\theta \implies r = 4\cos\theta$$

Therefore in the region R, r varies form $2\cos\theta$ to $4\cos\theta$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\therefore \quad \int\int_R (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^2\, r\,drd\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$
$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2}[256\cos^4\theta - 16\cos\theta]d\theta$$
$$= 60\int_{-\pi/2}^{\pi/2}\cos^4\theta d\theta$$
$$= 120\int_0^{\pi/2}\cos^4\theta d\theta$$
$$= 120\frac{3\times 1}{4\times 2}\frac{\pi}{2}$$
$$= \frac{45\pi}{2}$$

Method of finding limits

To find the limits in polar coordinates, first we keep θ fixed, say $\theta = \theta_1$ (along a ray). Then we vary r along the ray such that we move over the region **R** and the first and the last point common to the ray and the region **R** give the lower and upper limit for r. Rotating the ray $\theta = \theta_1$ over the region in the clockwise direction gives lower and upper limit for θ

Alternately, we fix the value of $r = r_1$ and rotate the radial line over the region **R** in the anticlockwise direction, we get the lower and upper limits for θ and then vary $r = r_1$ over the region we can find the limits for r.

When we take $f(r,\theta) = 1$, the above formula gives the area of the region **R**. Note that to convert problems in Cartesian (rectangular) coordinates system into polar coordinates, we use the transformation

$$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$$

Example ~~3~~.3.2. *Use polar coordinates to evaluate the double integral*

$$\int\int_R \sqrt{16-x^2-y^2}\,dA$$

where R is the region in the first quadrant within the circle $x^2+y^2=16$.

Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2+y^2 = r^2, dA = r\,dr d\theta$.

$$x^2+y^2 = 16 \implies r^2 = 16 \implies r = 4$$

Therefore in the region R, r varies form 0 to 4 and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \quad \int\int_R \sqrt{16-x^2-y^2}\,dA = \int_0^{\pi/2}\int_0^4 \sqrt{16-r^2}\, r\,drd\theta$$
$$= \int_0^{\pi/2}\left(-\frac{1}{2}\int_0^4\sqrt{16-r^2}\,(-2r)\,dr\right)d\theta$$
$$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{(16-r^2)^{3/2}}{(3/2)}\right]_0^4 d\theta$$
$$= -\frac{1}{2}\times\frac{2}{3}[0-64]\left[\theta\right]_0^{\pi/2}$$
$$= \frac{32\pi}{3}$$

Example ~~3~~.3.3. *Use polar coordinates to evaluate the double integral*

$$\int\int_R e^{-(x^2+y^2)}\,dA$$

where R is the region enclosed by the circles $x^2+y^2=1$.

Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2+y^2 = r^2, dA = r\,dr d\theta$.

$$x^2+y^2 = 1 \implies r^2 = 1 \implies r = 1$$

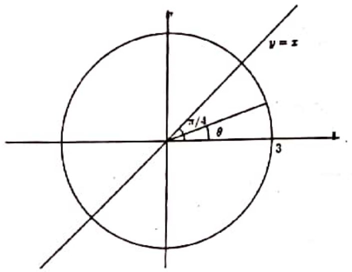
Therefore in the region R, r varies form 0 to 1 and θ varies from 0 to 2π.

$$\begin{aligned}
\therefore \quad \iint_R e^{-(x^2+y^2)}\,dA &= \int_0^{2\pi}\int_0^1 e^{-r^2} r\,dr d\theta \\
&= \int_0^{2\pi}\left(-\frac{1}{2}\int_0^1 e^{-r^2}(-2r)\,dr\right)d\theta \\
&= -\frac{1}{2}\int_0^{2\pi}\left(e^{-r^2}\right)_0^1 d\theta \\
&= -\frac{1}{2}[e^{-1}-1][\theta]_0^{2\pi} \\
&= \pi(1-e^{-1})
\end{aligned}$$

Example 3.3.4. *Use polar coordinates to evaluate the double integral*

$$\iint_R \frac{1}{1+x^2+y^2}\,dA$$

where R is the sector in the first quadrant bounded by $y=0, y=x$ and $x^2+y^2=9$.



Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2+y^2 = r^2, dA = r\,drd\theta$.

$$y = x \implies y = r\cos\theta$$
$$x^2+y^2 = 9 \implies r^2 = 9 \implies r = 3$$

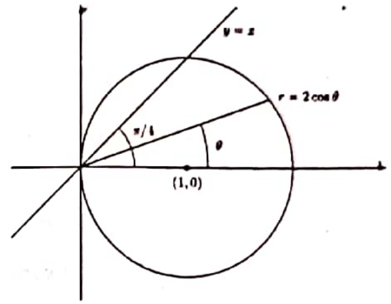
Therefore in the region R, r varies form 0 to 3 and θ varies from 0 to $\pi/4$.

$$\begin{aligned}
\therefore \quad \iint_R \frac{1}{1+x^2+y^2}\,dA &= \int_0^{\pi/4}\int_0^3 \frac{1}{1+r^2} r\,drd\theta \\
&= \int_0^{\pi/4}\left(\frac{1}{2}\int_0^3 \frac{1}{1+r^2}(2r)\,dr\right)d\theta \\
&= \frac{1}{2}\int_0^{\pi/4}\left(\log(1+r^2)\right)_0^3 d\theta \\
&= \frac{1}{2}\log(10)[\theta]_0^{\pi/4} \\
&= \frac{\pi}{8}\log 10
\end{aligned}$$

Example 3.3.5. *Use polar coordinates to evaluate the double integral*

$$\iint_R 2y\,dA$$

where R is the region in the first quadrant bounded above by the circle $(x-1)^2+y^2 = 1$ and below by the line $y = x$.



Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2+y^2 = r^2, dA = r\,drd\theta$.

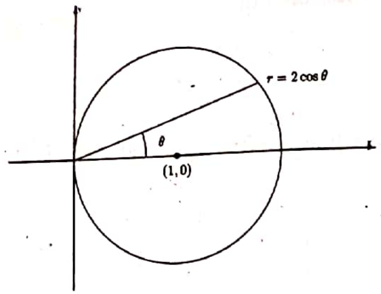
$$(x-1)^2+y^2 = 1 \implies r^2 - 2r\cos\theta + 1 = 1 \implies r = 2\cos\theta$$

Therefore in the region R, r varies form 0 to $2\cos\theta$ and θ varies from 0 to $\pi/4$.

$$\begin{aligned}
\therefore \quad \iint_R 2y\,dA &= \int_0^{\pi/4}\int_0^{2\cos\theta} 2r\sin\theta\; r\,drd\theta \\
&= \int_0^{\pi/4} 2\sin\theta\left(\frac{r^3}{3}\right)_0^{2\cos\theta} d\theta \\
&= \frac{16}{3}\int_0^{\pi/4}\cos^3\theta\sin\theta\,d\theta \\
&= \frac{16}{3}\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/4} \\
&= -\frac{4}{3}\left[\frac{1}{4}-1\right] \\
&= 1
\end{aligned}$$

Example 3.3.6. *Transform to polar co-ordinates and hence evaluate*

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dydx$$



Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2+y^2 = r^2, dA = r\,dr d\theta$. From the limit of integration, we observe that the region of integration is bounded by $y = 0, y = \sqrt{2x-x^2}, x = 0, x = 2$.

$$y = \sqrt{2x-x^2} \implies y^2 = 2x - x^2 \implies r^2 = 2r\cos\theta \implies r = 2\cos\theta$$

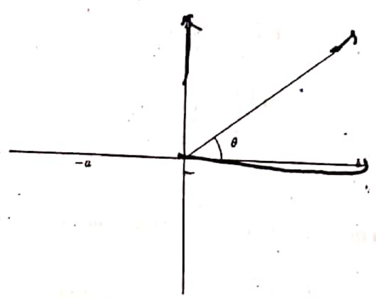
Therefore in the region R, r varies form 0 to $2\cos\theta$ and θ varies from 0 to $\pi/2$.

$$\begin{aligned}\therefore \quad \int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dydx &= \int_0^{\pi/2}\int_0^{2\cos\theta} r\;r\,drd\theta \\ &= \int_0^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2\cos\theta} d\theta \\ &= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta \\ &= \frac{8}{3}\times\frac{2}{3} \\ &= \frac{16}{9}\end{aligned}$$

$\cos^3\theta =$

Example 3.3.7. *Transform to polar co-ordinates and hence evaluate*

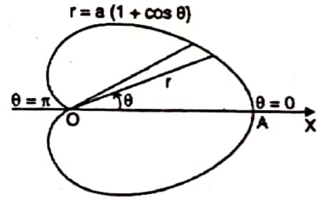
$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dydx$$



Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2+y^2 = r^2, dA = r\,drd\theta$. From the limit of integration, we observe that the region of integration is the first quadrant. Therefore in the region R, r varies form 0 to ∞ and θ varies from 0 to $\pi/2$.

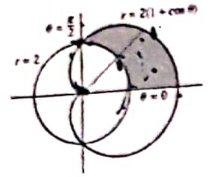
$$\begin{aligned}\therefore \quad \iint_R e^{-(x^2+y^2)}\,dA &= \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,drd\theta \\ &= \int_0^{\pi/2}\left(-\frac{1}{2}\int_0^\infty e^{-r^2}(-2r)\,dr\right)d\theta \\ &= -\frac{1}{2}\int_0^{\pi/2}\left(e^{-r^2}\right)_0^\infty d\theta \\ &= -\frac{1}{2}[0-1][\theta]_0^{\pi/2} \\ &= \frac{\pi}{4}\end{aligned}$$

Example 3.3.8. *Evaluate* $\iint r^2\sin\theta\,drd\theta$ *over the area of the cardioid* $r = a(1+\cos\theta)$ *above the initial line.*



$$\begin{aligned}\iint r^2\sin\theta\,drd\theta &= \int_0^\pi\int_0^{a(1+\cos\theta)} r^2\sin\theta\,drd\theta \\ &= \int_0^\pi \sin\theta\left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)} d\theta \\ &= \frac{1}{3}\int_0^\pi a^3(1+\cos\theta)^3\sin\theta d\theta \\ &= \frac{a^3}{3}\left[-\frac{(1+\cos\theta)^4}{4}\right]_0^\pi \\ &= -\frac{a^3}{12}[0-2^4] \\ &= \frac{4a^3}{3}\end{aligned}$$

Example 3.3.9. *Evaluate* $\iint \sin\theta \, dA$ *where* R *is the region in the first quadrant that is outside the circle* $r = 2$ *and inside the cardioid* $r = 2(1 + \cos\theta)$.



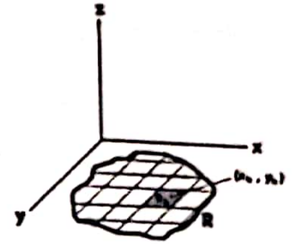
$$\iint \sin\theta \, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta \, r \, dr d\theta$$
$$= \int_0^{\pi/2} \sin\theta \left(\frac{r^2}{2}\right)_2^{2(1+\cos\theta)} d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} [4(1+\cos\theta)^2 - 4] \sin\theta d\theta$$
$$= 2\left[-\frac{(1+\cos\theta)^3}{3} + \cos\theta\right]_0^{\pi/2}$$
$$= 2\left[-\frac{1}{3} - \left(-\frac{8}{3}\right) + (0-1)\right]$$
$$= \frac{8}{3}$$

3.4 Centre of Mass using Double integrals

When a system is in rest or in motion in space, at every instant of time, there is always a unique point (location) which is the average position of the mass of the system. This point is called **the centre of mass or centre of gravity** of the system.

In many practical instances, one of the important tasks of many real engineers is to find the centre of mass of a complex machine . Knowing the location of the centre of mass of a car, for example, is enough to estimate whether it can be tipped over by maneuvers on level ground. The centre of mass of a boat must be low enough for the boat to be stable. Any propulsive force acting on a space craft must be directed towards the centre of mass otherwise it will induce rotational motion. Tracking the trajectory of the centre of mass of an exploding plane can determine whether or not it was hit by a massive object. Any rotating piece of machinery must have its center of mass on the axis of rotation and if the centre of mass is not on the axis of rotation it will cause much vibration. Also, many calculations in mechanics are greatly simplified by making use of a centre of mass of the system. In particular, the whole complicated distribution of nearearth gravity forces on a body is equivalent to a single force at the body's centre of mass.

In this section, we would like to compute the centre of mass of an inhomogeneous lamina. A **lamina** can be understood as a thin two-dimensional plane region. A lamina is said to be homogeneous if the distribution of materials is uniform throughout the region



The density ρ of the lamina at each point is defined as the mass per unit area surrounding the point. Since the distribution of the materials is uniform, density of homogeneous lamina having mass M and area A will be the same at every point and is given by $\rho = \frac{M}{A}$.

But when the lamina is inhomogeneous, that is , the distribution of materials is not uniform, the lamina will be having variable density. The density will depend on the point at which it is measured and it will be a function of the location called density function denoted by $\rho(x, y)$.

We divide the region R having area A and mass M into small rectangular portions R_k by lines parallel to the coordinate axes and let this rectangular portion be centred around the point (x_k, y_k). Let ΔA_k be the area and M_k be the mass of the small rectangular portion R_k. The point density function $\rho(x, y)$ in this typical rectangular portion can be defined as

$$\rho(x_k, y_k) = \lim_{\Delta A_k \to 0} \frac{\Delta M_k}{\Delta A_k}$$

From this relation we get

$$\Delta M_k \approx \rho(x_k, y_k)\, \Delta A_k$$

Assuming that the point density function is continuous at each point on the inhomogeneous lamina, we get that the total mass of the lamina is

$$M = \iint_R \rho(x, y) dA$$

Let there be n number of small rectangular portions and assume that the mass of the rectangular portions R_k is concentrated about the centre (x_k, y_k). Let $(\overline{x}, \overline{y})$ be the point at which the entire mass of the whole region R is concentrated. So the lamina balances on the lines $x = \overline{x}$ and $y = \overline{y}$. Hence the sum of the moments of the rectangular pieces about these lines should be zero, theoretically. Symbolically, we can express this statement in the following form as

$$\lim_{n\to\infty} \sum_{k=1}^{n} (x_k - \overline{x})\Delta M_k = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k - \overline{x})\rho(x_k, y_k)\Delta A_k = 0$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} (y_k - \overline{y})\Delta M_k = \lim_{n\to\infty} \sum_{k=1}^{n} (y_k - \overline{y})\rho(x_k, y_k)\Delta A_k = 0$$

Since we assume that the point density function is continuous throughout the region, it is easy to see that

$$\int\int_R (x-\overline{x})\rho(x,y)dA = 0 \quad \text{and} \quad \int\int_R (y-\overline{y})\rho(x,y)dA = 0$$

We solve for $\overline{x}$ and $\overline{y}$ from the above two equations and get

$$\overline{x} = \frac{\int\int_R x\rho(x,y)dA}{\int\int_R \rho(x,y)dA} = \frac{1}{M}\int\int_R x\rho(x,y)dA$$

and

$$\overline{y} = \frac{\int\int_R y\rho(x,y)dA}{\int\int_R \rho(x,y)dA} = \frac{1}{M}\int\int_R y\rho(x,y)dA$$

Example 3.4.1. *Find the mass and centre of mass of a triangular lamina with vertices $(0,0),(1,0)$ and $(0,2)$ if the density function is $\rho(x,y) = 1+3x+y$*

Let $A(0,0), B(1,0)$ and $C(0,2)$ be the vertices. Then equation of BC is

$$\frac{y-0}{0-2} = \frac{x-1}{1-0} \implies y = 2-2x$$

The mass of the lamina is

$$\begin{aligned}
M &= \int\int_D \rho(x,y)\,dA \\
&= \int_0^1\int_0^{2-2x} (1+3x+y)\,dy\,dx \\
&= \int_0^1 \left[y+3xy+\frac{y^2}{2}\right]_0^{2-2x} dx \\
&= 4\int_0^1 (1-x^2)\,dx \\
&= 4\left[x-\frac{x^3}{3}\right]_0^1 \\
&= \frac{8}{3}
\end{aligned}$$

$M = \iint \rho\, dA$

$\bar{x} = \frac{1}{M}\iint x\,\rho(x)\,dA$

$\bar{x} = \dfrac{\iint x\,\rho\, dA}{\iint \rho\, dA}$

$$\begin{aligned}
\overline{x} &= \frac{1}{M}\int\int_D x\,\rho(x,y)\,dA \\
&= \frac{3}{8}\int_0^1\int_0^{2-2x} (x+3x^2+xy)\,dy\,dx \\
&= \frac{3}{8}\int_0^1 \left[xy+3x^2y+x\frac{y^2}{2}\right]_0^{2-2x} dx \\
&= \frac{3}{2}\int_0^1 (x-x^3)\,dx \\
&= \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 \\
&= \frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
\overline{y} &= \frac{1}{M}\int\int_D y\,\rho(x,y)\,dA \\
&= \frac{3}{8}\int_0^1\int_0^{2-2x} (y+3xy+y^2)\,dy\,dx \\
&= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2}+3x\frac{y^2}{2}+\frac{y^3}{3}\right]_0^{2-2x} dx \\
&= \frac{1}{4}\int_0^1 (7-9x-3x^2+5x^3)\,dx \\
&= \frac{1}{4}\left[7x-9\frac{x^2}{2}-x^3+5\frac{x^4}{4}\right]_0^1 \\
&= \frac{11}{16}
\end{aligned}$$

The centre of mass is $\left(\frac{3}{8},\frac{11}{16}\right)$.

Example 3.4.2. *Find the total mass and centre of gravity of a triangular lamina with vertices $(0,0),(0,1)$ and $(1,0)$ if the density function is $\rho(x,y) = xy$.*

Let $A(0,0), B(0,1)$ and $C(1,0)$ be the vertices. Then equation of BC is

$$\frac{y-1}{1-0} = \frac{x-0}{0-1} \implies y = 1-x$$

The mass of the lamina is

$M = \iint_D \rho(x,y)\,dA$

$$\begin{aligned}
M &= \int\int_D \rho(x,y)\,dA \\
&= \int_0^1\int_0^{1-x} (xy)\,dy\,dx \\
&= \int_0^1 \left[x\frac{y^2}{2}\right]_0^{1-x} dx \\
&= \frac{1}{2}\int_0^1 (x^3-2x^2+x)\,dx \\
&= \frac{1}{2}\left[\frac{x^4}{4}-2\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 \\
&= \frac{1}{24}
\end{aligned}$$

$$
\begin{aligned}
\bar{x} &= \frac{1}{M}\int\int_D x\,\rho(x,y)\,dA \\
&= 24\int_0^1\int_0^{1-x}(x^2y)\,dy\,dx \\
&= 24\int_0^1\left[x^2\frac{y^2}{2}\right]_0^{1-x} dx \\
&= 24\times\frac{1}{2}\int_0^1\left(x^4-2x^3+x^2\right)\,dx \\
&= 12\left[\frac{x^5}{5}-2\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1 \\
&= \frac{2}{5}
\end{aligned}
$$

$$
\begin{aligned}
\bar{y} &= \frac{1}{M}\int\int_D y\,\rho(x,y)\,dA \\
&= 24\int_0^1\int_0^{1-x}(xy^2)\,dy\,dx \\
&= 24\int_0^1\left[x\frac{y^3}{3}\right]_0^{1-x} dx \\
&= 24\times\frac{1}{3}\int_0^1\left(-x^4+3x^3-3x^2+x\right)\,dx \\
&= 8\left[-\frac{x^5}{5}+3\frac{x^4}{4}-3\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 \\
&= \frac{2}{5}
\end{aligned}
$$

The centre of mass is $\left(\frac{2}{5},\frac{2}{5}\right)$.

Example 3.4.3. *The density at any point on a semicircular lamina is proportional to the distance from the centre of the circle. Find the centre of mass of the lamina.*

Let the lamina be the upper half of the circle $x^2+y^2=a^2$. Then the distance from a point (x,y) to the centre of the circle is $\sqrt{x^2+y^2}$. Therefore the density function is

$$\rho(x,y)=k\sqrt{x^2+y^2}$$

where k is some constant.

$$
\begin{aligned}
M &= \int\int_D \rho(x,y)\,dA \\
&= \int\int_D k\sqrt{x^2+y^2}\,dA
\end{aligned}
$$

Taking $x=r\cos\theta, y=r\sin\theta$. Then $x^2+y^2=r^2, dA=r\,dr\,d\theta$. Also in the region D, r varies form 0 to a and θ varies from 0 to π

$$
\begin{aligned}
M &= \int_0^\pi\int_0^a (kr)\,r\,dr\,d\theta \\
&= k\int_0^\pi\left[\frac{r^3}{3}\right]_0^a d\theta \\
&= \frac{ka^3}{3}\int_0^\pi d\theta \\
&= \frac{ka^3\pi}{3}
\end{aligned}
$$

Since the lamina and the density function are symmetric with respect tp $y-$axis, $\bar{x}=0$.

$$
\begin{aligned}
\bar{y} &= \frac{1}{M}\int\int_D y\,\rho(x,y)\,dA \\
&= \frac{3}{ka^3\pi}\int_0^\pi\int_0^a (r\sin\theta)(kr)\,r\,dr\,d\theta \\
&= \frac{3k}{ka^3\pi}\int_0^\pi\int_0^a r^3\sin\theta\,dr\,d\theta \\
&= \frac{3}{a^3\pi}\int_0^\pi \sin\theta\left[\frac{r^4}{4}\right]_0^a d\theta \\
&= \frac{3}{a^3\pi}\frac{a^4}{4}\int_0^\pi \sin\theta\,d\theta \\
&= \frac{3a}{4\pi}[-\cos\theta]_0^\pi \\
&= \frac{3a}{4\pi}[2] \\
&= \frac{3a}{2\pi}
\end{aligned}
$$

The centre of mass is $\left(0,\frac{3a}{2\pi}\right)$.

Example 3.4.4. *Find the mass and center of mass of a triangular lamina with vertices* $(0,0),(1,0)$ *and* $(0,2)$ *if the density function is* $\rho(x,y)=1+3x+y$

Let $A(0,0), B(1,0)$ and $C(0,2)$ be the vertices. Then equation of BC is

$$\frac{y-0}{0-2}=\frac{x-1}{1-0}\implies y=2-2x$$

The mass of the lamina is

$$\begin{aligned}
M &= \iint_D \rho(x,y)\,dA \\
&= \int_0^1 \int_0^{2-2x} (1+3x+y)\,dy\,dx \\
&= \int_0^1 \left[y + 3xy + \frac{y^2}{2}\right]_0^{2-2x} dx \\
&= 4\int_0^1 (1-x^2)\,dx \\
&= 4\left[x - \frac{x^3}{3}\right]_0^1 \\
&= \frac{8}{3}
\end{aligned}$$

$$\begin{aligned}
\overline{x} &= \frac{1}{M}\iint_D x\,\rho(x,y)\,dA \\
&= \frac{3}{8}\int_0^1 \int_0^{2-2x} (x+3x^2+xy)\,dy\,dx \\
&= \frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2}\right]_0^{2-2x} dx \\
&= \frac{3}{2}\int_0^1 (x-x^3)\,dx \\
&= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 \\
&= \frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
\overline{y} &= \frac{1}{M}\iint_D y\,\rho(x,y)\,dA \\
&= \frac{3}{8}\int_0^1 \int_0^{2-2x} (y+3xy+y^2)\,dy\,dx \\
&= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_0^{2-2x} dx \\
&= \frac{1}{4}\int_0^1 (7-9x-3x^2+5x^3)\,dx \\
&= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 \\
&= \frac{11}{16}
\end{aligned}$$

The centre of mass is $\left(\frac{3}{8}, \frac{11}{16}\right)$.

Example 3.4.5. *Find the total mass and center of gravity of a triangular lamina with vertices $(0,0)$, $(0,1)$ and $(1,0)$ if the density function is $\rho(x,y) = xy$.*

Let $A(0,0)$, $B(0,1)$ and $C(1,0)$ be the vertices. Then equation of BC is

$$\frac{y-1}{1-0} = \frac{x-0}{0-1} \implies y = 1-x$$

$\rho(x,y) = xy$

$\frac{y-1}{1} = \frac{x-0}{-1}$

$y = 1-x$

The mass of the lamina is

$$\begin{aligned}
M &= \iint_D \rho(x,y)\,dA \\
&= \int_0^1 \int_0^{1-x} (xy)\,dy\,dx \\
&= \int_0^1 \left[x\frac{y^2}{2}\right]_0^{1-x} dx \\
&= \frac{1}{2}\int_0^1 (x^3-2x^2+x)\,dx \\
&= \frac{1}{2}\left[\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 \\
&= \frac{1}{24}
\end{aligned}$$

$$\begin{aligned}
\overline{x} &= \frac{1}{M}\iint_D x\,\rho(x,y)\,dA \\
&= 24\int_0^1 \int_0^{1-x} (x^2y)\,dy\,dx \\
&= 24\int_0^1 \left[x^2\frac{y^2}{2}\right]_0^{1-x} dx \\
&= 24 \times \frac{1}{2}\int_0^1 (x^4-2x^3+x^2)\,dx \\
&= 12\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 \\
&= \frac{2}{5}
\end{aligned}$$

$$\bar{y} = \frac{1}{M}\int\int_D y\,\rho(x,y)\,dA$$
$$= 24\int_0^1\int_0^{1-x} (xy^2)\,dy\,dx$$
$$= 24\int_0^1 \left[x\frac{y^3}{3}\right]_0^{1-x} dx$$
$$= 24 \times \frac{1}{3}\int_0^1 (-x^4 + 3x^3 - 3x^2 + x)\ dx$$
$$= 8\left[-\frac{x^5}{5} + 3\frac{x^4}{4} - 3\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$$
$$= \frac{2}{5}$$

The centre of mass is $\left(\frac{2}{5}, \frac{2}{5}\right)$.

Example 3.4.6. *The density at any point on a semicircular lamina is proportional to the distance from the centre of the circle. Find the centre of mass of the lamina.*

Let the lamina be the upper half of the circle $x^2 + y^2 = a^2$. Then the distance from a point (x, y) to the centre of the circle is $\sqrt{x^2 + y^2}$. Therefore the density function is

$$\rho(x, y) = k\sqrt{x^2 + y^2}$$

where k is some constant.

$$M = \int\int_D \rho(x, y)\,dA$$
$$= \int\int_D k\sqrt{x^2 + y^2}\,dA$$

Taking $x = r\cos\theta, y = r\sin\theta$. Then $x^2 + y^2 = r^2, dA = r\,dr\,d\theta$. Also in the region D, r varies form 0 to a and θ varies from 0 to π

$$M = \int_0^{\pi}\int_0^a (kr)\,r\,dr\,d\theta$$
$$= k\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta$$
$$= \frac{ka^3}{3}\int_0^{\pi} d\theta$$
$$= \frac{ka^3\pi}{3}$$

Since the lamina and the density function are symmetric with respect tp y-axis, $\bar{x} = 0$.

$$\bar{y} = \frac{1}{M}\int\int_D y\,\rho(x, y)\,dA$$
$$= \frac{3}{ka^3\pi}\int_0^{\pi}\int_0^a (r\sin\theta)(kr)\,r\,dr\,d\theta$$
$$= \frac{3k}{ka^3\pi}\int_0^{\pi}\int_0^a r^3\sin\theta\,dr\,d\theta$$
$$= \frac{3}{a^3\pi}\int_0^{\pi}\sin\theta\left[\frac{r^4}{4}\right]_0^a d\theta$$
$$= \frac{3}{a^3\pi}\frac{a^4}{4}\int_0^{\pi}\sin\theta\,d\theta$$
$$= \frac{3a}{4\pi}[-\cos\theta]_0^{\pi}$$
$$= \frac{3a}{4\pi}[2]$$
$$= \frac{3a}{2\pi}$$

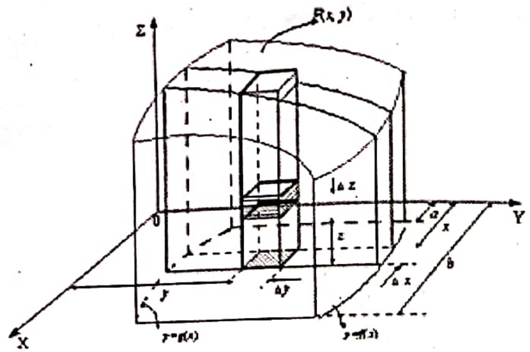
The centre of mass is $\left(0, \frac{3a}{2\pi}\right)$.

3.5 Triple integral

To define triple integral, we consider a solid region $\mathbf{H}$ that can be enclosed by a box whose sides are parallel to the coordinate planes in three dimensional space. Let $f(x, y, z)$ be a continuous functions defined at each point in the solid region $\mathbf{H}$ having volume V. Analogous to the subdivision of region in the plane using lines parallel to the coordinate axes, we divide the solid region into small cubes ΔH_k having volume ΔV_k using planes parallel to the coordinate planes. We take only those cubes which are fully contained in the solid region for the computation.

Let (x_k^*, y_k^*, z_k^*) be an arbitrary interior point in the cube ΔH_k. Then the triple integral of the continuous function $f(x, y, z)$ over the sold region $\mathbf{H}$ is denoted by $\int\int_H\int f(x, y, z)dv$ and is defined as

$$\int\int_H\int f(x, y, z)dV = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$



We make use of the following properties of triple integral to solve problems:

Let $f(x,y,z)$ and $g(x,y,z)$ be two continuous functions in three variables x, y, and z defined at all points in a sold region H having volume V and α and β be scalars then,

$$\int\int_H\int (\alpha f(x,y,z)+\beta g(x,y,z))\,dV = \alpha \int\int_H\int f(x,y,z)dV + \beta\int\int_H\int g(x,y,z)dV$$

If the region H is divided into two disjoint sub-regions H_1 and H_2 then,

$$\int\int_H\int f(x,y,z)dV = \int\int_{H_1}\int f(x,y,z)dV + \int\int_{H_2}\int f(x,y,z)dV$$

Fubini's theorem on triple integral

Let $f(x,y,z)$ be a continuous function at all points in a rectangular box H defined by $a\le x\le b$, $c\le y\le d$, $e\le z\le f$. Then,

$$\int\int_H\int f(x,y,z)dV = \int_e^f\int_c^d\int_a^b f(x,y,z)dx\,dy\,dz$$

Also the value of triple integral is invariant under change of order of integration. That is, the value will not change even if we change the order of integration.

Example 3.5.1. *Consider* $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)dzdydx$

$$\begin{aligned}\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dz\,dy\,dx &= \int_0^a\int_0^b \left[x^2z+y^2z+\frac{z^3}{3}\right]_0^c dy\,dx\\ &= \int_0^a\int_0^b\left[x^2c+y^2c+\frac{c^3}{3}\right]dy\,dx\\ &= \int_0^a\left[cx^2y+c\frac{y^3}{3}+\frac{c^3}{3}y\right]_0^b dx\end{aligned}$$

$$\begin{aligned}&= \int_0^a\left[bcx^2+\frac{cb^3}{3}+\frac{bc^3}{3}\right]dx\\ &= \left[bc\frac{x^3}{3}+\frac{b^3c}{3}x+\frac{bc^3}{3}x\right]_0^a\\ &= \frac{a^3bc}{3}+\frac{ab^3c}{3}+\frac{abc^3}{3}\\ &= \frac{abc}{3}(a^2+b^2+c^2)\end{aligned}$$

To compute the triple integral we use the following procedure:

Let H be the solid region with lower surface $z=g_1(x,y)$ and upper surface $z=g_2(x,y)$ and let R be the projection of H on the XY-plane. Let $f(x,y,z)$ be a continuous functions defined at all points in the region H. Then,

$$\int\int_H\int f(x,y,z)dV = \int_R\int\left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\right)dA$$

In the above formula, we projected the solid region on a plane parallel to the xy-plane. We compute the triple integral by taking the projection on yz-plane or zx-plane. The formula for computing the triple integral takes the form

$$\begin{aligned}\int\int_H\int f(x,y,z)dV &= \int_R\int\left(\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)dx\right)dA\\ &= \int_R\int\left(\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)dy\right)dA\end{aligned}$$

Example 3.5.2. *Evaluate* $\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz$

$$\begin{aligned}\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz &= \int_0^3 x\,dx\times\int_0^2 y\,dy\times\int_0^1 z\,dz\\ &= \left[\frac{x^2}{2}\right]_0^3\times\left[\frac{y^2}{2}\right]_0^2\times\left[\frac{z^2}{2}\right]_0^1\\ &= \frac{9}{2}\times\frac{4}{2}\times\frac{1}{2}\\ &= \frac{9}{2}\end{aligned}$$

Example 3.5.3. *Evaluate* $\int_0^1\int_0^1\int_{z=\sqrt{x^2+y^2}}^2 xyz\,dzdydx$

$$\int_0^1\int_0^1\int_{z=\sqrt{x^2+y^2}}^2 xyz\,dzdydx = \int_0^1\int_0^1\left[\frac{z^2}{2}\right]_{z=\sqrt{x^2+y^2}}^2 xy\,dy\,dx$$

$$= \int_0^1 \int_0^1 \left[2 - \frac{(x^2+y^2)}{2}\right] xy\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 \int_0^1 \left[4xy - x^3y - xy^3\right] dy\,dx$$
$$= \frac{1}{2}\int_0^1 \left[4x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^1 dx$$
$$= \frac{1}{2}\int_0^1 \left[2x - \frac{x^3}{2} - \frac{x}{4}\right] dx$$
$$= \frac{1}{2}\left[x^2 - \frac{x^4}{8} - \frac{x^2}{8}\right]_0^1$$
$$= \frac{3}{8}$$

Example 3.5.4. *Evaluate* $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dydxdz$

$$\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dydxdz = \int_0^4 \int_0^{2\sqrt{z}} [y]_0^{\sqrt{4z-x^2}}\,dx\,dz$$
$$= \int_0^4 \int_0^{2\sqrt{z}} \sqrt{4z-x^2}\,dx\,dz$$
$$= \int_0^4 \left[\frac{x}{2}\sqrt{4z-x^2} + \frac{4z}{2}\sin^{-1}\left(\frac{x}{2\sqrt{z}}\right)\right]_0^{2\sqrt{z}} dz$$
$$= \int_0^4 2z \times \frac{\pi}{2}\,dz$$
$$= \pi\left[\frac{z^2}{2}\right]_0^4$$
$$= 8\pi$$

Example 3.5.5. *Show that* $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\,dydxdz = 0$

$$\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\,dydxdz = \int_{-1}^1 \int_0^z \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\,dz$$
$$= \int_{-1}^1 \int_0^z 4xz + 2z^2\,dx\,dz$$
$$= \int_{-1}^1 \left[4z\frac{x^2}{2} + 2z^2x\right]_0^z dz$$
$$= \int_{-1}^1 4z^3\,dz$$
$$= 0$$

Example 3.5.6. *Evaluate* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyzdxdydz$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dxdydz = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$
$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy - x^3y - xy^3\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$
$$= \frac{1}{2}\int_0^1 \left[\frac{x}{4} - \frac{x^3}{2} + \frac{x^5}{4}\right] dx$$
$$= \frac{1}{2}\left[\frac{x^2}{8} - \frac{x^4}{8} + \frac{x^6}{24}\right]_0^1 = \frac{1}{48}$$

Example 3.5.7. *Show that* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$

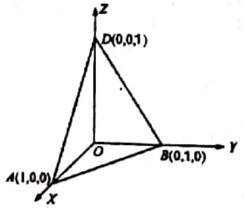
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$$
$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$
$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\,dy\,dx$$
$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\,dx$$
$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$
$$= \frac{\pi}{2} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi^2}{8}$$

Example 3.5.8. *Evaluate* $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dzdydx$

$$\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dzdydx = \int_1^e \int_1^{\log y} [z\log z - z]_1^{e^x}\,dy\,dx$$

$$= \int_1^e \int_1^{\log y} [xe^x - e^x] - [0-1]\, dy\, dx$$
$$= \int_1^e \int_1^{\log y} (xe^x - e^x + 1)\, dx\, dy$$
$$= \int_1^e \{[(x)(e^x) - (1)(e^x)] - e^x + x\}_1^{\log y}\, dy$$
$$= \int_1^e [y\log y - 2y + \log y] - [e - 2e + 1]\, dy$$
$$= \int_1^e [y\log y - 2y + \log y + e - 1]\, dy$$
$$= \left[(\log y)\left(\frac{y^2}{2}\right) - \int \left(\frac{1}{y}\right)\left(\frac{y^2}{2}\right) dy\right]_1^e$$
$$- 2\left[\frac{y^2}{2}\right]_1^e + [y\log y - y]_1^e + (e-1)[y]_1^e$$
$$= \left[\frac{e^2}{2} - \frac{e^2}{4}\right] - \left[0 - \frac{1}{4}\right] - [e^2 - 1] + [e - e] - [0 - 1] + (e-1)(e-1)$$
$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$
$$= \frac{e^2 - 8e + 13}{4}$$

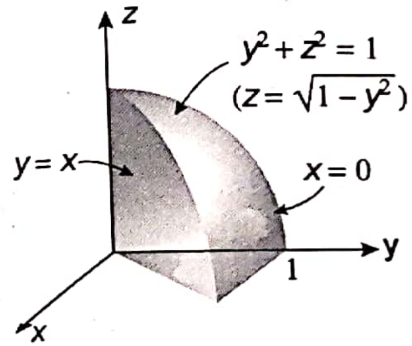
Example 3.5.9. *Evaluate* $\iiint (x+y+z)\,dxdydz$ *over the tetrahedron* $x \geq 0, y \geq 0, z \geq 0$ *and* $x + y + z \leq 1$

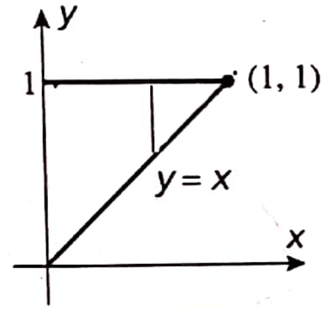


$$\iiint (x+y+z)\,dxdydz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\,dz\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} \left[xz + yz + \frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$
$$= \int_0^1 \int_0^{1-x} \left[-\frac{x^2}{2} - \frac{y^2}{2} - xy + \frac{1}{2}\right] dy\,dx$$

$$= \int_0^1 \left[-\frac{x^2 y}{2} - \frac{y^3}{6} - \frac{xy^2}{2} + \frac{y}{2}\right]_0^{1-x} dx$$
$$= \int_0^1 \left[-\frac{x^2}{2} + \frac{x^3}{2} - \frac{(1-x)^3}{6} - \frac{x}{2} + x^2 - \frac{x^3}{2} + \frac{(1-x)}{2}\right] dx$$
$$= \int_0^1 \left[\frac{x^2}{2} - \frac{(1-x)^3}{6} - x + \frac{1}{2}\right] dx$$
$$= \left[\frac{x^3}{6} + \frac{(1-x)^4}{24} - \frac{x^2}{2} + \frac{x}{2}\right]_0^1$$
$$= \frac{1}{8}$$

Example 3.5.10. *Evaluate* $\iiint_V z\,dV$ *where V be the wedge in the first octant that is cut from the cylindrical solid* $y^2 + z^2 \leq 1$ *by the planes* $y = x$ *and* $x = 0$





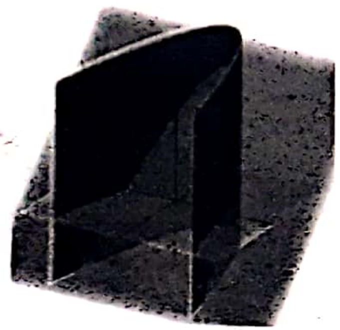
$$\iiint_V z\,dV = \int_0^1 \int_x^1 \int_0^{\sqrt{1-y^2}} z\,dz\,dy\,dx$$
$$= \int_0^1 \int_x^1 \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}} dy\,dx$$
$$= \frac{1}{2}\int_0^1 \int_x^1 (1-y^2)\,dy\,dx$$
$$= \frac{1}{2}\int_0^1 \left[y - \frac{y^3}{3}\right]_x^1 dx$$
$$= \frac{1}{2}\int_0^1 \left[\frac{2}{3} - x + \frac{x^3}{3}\right] dx$$
$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1$$
$$= \frac{1}{8}$$

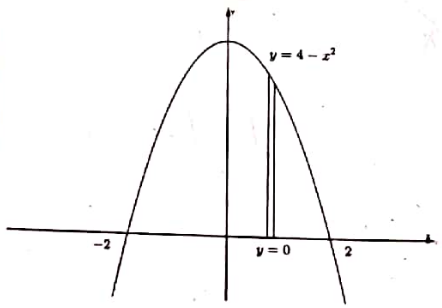
How take the parms.

Example 3.5.11. *Evaluate* $\iiint_V xyz\,dV$ *where V is the solid in the first octant that is bounded by the parabolic cylinder* $z = 3 - x^2$ *and the planes* $z = 0, y = x$, *and* $y = 0$.

$$\begin{aligned}
\iiint_V xyz\,dV &= \int_0^{\sqrt{3}}\int_0^{x}\int_0^{3-x^2} xyz\,dz\,dy\,dx\\
&= \int_0^{\sqrt{3}}\int_0^{x} xy\left[\frac{z^2}{2}\right]_0^{3-x^2} dy\,dx\\
&= \frac{1}{2}\int_0^{\sqrt{3}}\int_0^{x} xy[9-6x^2+x^4]\,dy\,dx\\
&= \frac{1}{2}\int_0^{\sqrt{3}}[9x-6x^3+x^5]\left[\frac{y^2}{2}\right]_0^{x} dx\\
&= \frac{1}{4}\int_0^{\sqrt{3}}[9x^3-6x^5+x^7]\,dx\\
&= \frac{1}{4}\left[\frac{9}{4}x^4-6\frac{x^6}{6}+\frac{x^8}{8}\right]_0^{\sqrt{3}}\\
&= \frac{27}{32}
\end{aligned}$$

Example 3.5.12. *Evaluate* $\iiint_V y\,dV$ *where V is the solid enclosed by the plane* $z = y$, *the xy-plane, and the parabolic cylinder* $y = 4 - x^2$.

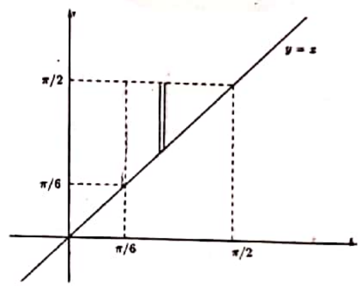




$$\begin{aligned}
\iiint_V y\,dV &= \int_{-2}^{2}\int_0^{4-x^2}\int_0^{y} y\,dz\,dy\,dx\\
&= \int_{-2}^{2}\int_0^{4-x^2} y(y-0)\,dy\,dx\\
&= \int_{-2}^{2}\left[\frac{y^3}{3}\right]_0^{4-x^2} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3}\int_{-2}^{2}[64-48x^2+12x^4-x^6]\,dx\\
&= \frac{1}{3}\times 2\left[64x-48\frac{x^3}{3}+12\frac{x^5}{5}-\frac{x^7}{7}\right]_0^{2}\\
&= \frac{4096}{105}
\end{aligned}$$

Example 3.5.13. *Evaluate* $\iiint_V \cos(z/y)\,dV$ *where V is the solid defined by the inequalities* $\pi/6 \le y \le \pi/2$, $y \le x \le \pi/2$, $0 \le z \le xy$.



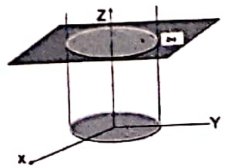
$$\begin{aligned}
\iiint_V \cos(z/y)\,dV &= \int_{\pi/6}^{\pi/2}\int_{\pi/6}^{x}\int_0^{xy} \cos(z/y)\,dz\,dy\,dx\\
&= \int_{\pi/6}^{\pi/2}\int_{\pi/6}^{x}\left[\frac{\sin(z/y)}{(1/y)}\right]_0^{xy} dy\,dx\\
&= \int_{\pi/6}^{\pi/2}\int_{\pi/6}^{x} y\sin x\,dy\,dx\\
&= \int_{\pi/6}^{\pi/2}\sin x\left[\frac{y^2}{2}\right]_{\pi/6}^{x} dx\\
&= \frac{1}{2}\int_{\pi/6}^{\pi/2}(x^2-\pi^2/6)\sin x\,dx\\
&= \frac{1}{2}\Big[(x^2-\pi^2/6)(-\cos x)-(2x)(-\sin x)+(2)(\cos x)\Big]_{\pi/6}^{\pi/2}\\
&= \frac{1}{2}\left[\pi-\frac{\pi}{6}-\sqrt{3}\right.\\
&= \frac{5\pi}{12}-\frac{\sqrt{3}}{2}
\end{aligned}$$

3.5.1 Computing volume as a triple integral

Using triple integral we can compute the volume V of the region H by taking $f(x,y,z)=1$. That is,

$$\text{Volume of a region } \; H = \iiint_H dV$$

Example 3.5.14. *Find the volume of the cylinder $x^2+y^2=4$ bounded by the planes $z=0$ and $z=3$ plane.*



$$\begin{aligned}
\text{Volume} &= \iiint dxdydz \\
&= \int_0^{2\pi}\int_0^{2}\int_0^{3} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{2} r\,[z]_0^3\,dr\,d\theta \\
&= 3\int_0^{2\pi}\int_0^{2}\cdot\left[\frac{r^2}{2}\right]_0^2 d\theta \\
&= 6\int_0^{2\pi} d\theta \\
&= 12\pi
\end{aligned}$$

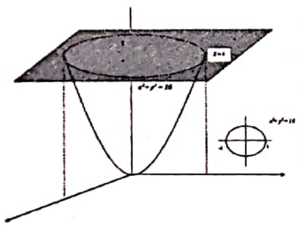
Example 3.5.15. *Find the volume of the paraboloid of revolution $x^2+y^2=4z$ cut off by the plane $z=4$*

$x^2+y^2=r^2=4z$

$z=r^2/4$

$r=4$

$\iiint dzdydx$ $r^2/4$

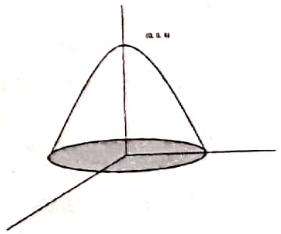


$$\begin{aligned}
\text{Volume} &= \iiint dxdydz \\
&= \int_0^{2\pi}\int_0^{4}\int_{\frac{r^2}{4}}^{4} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{4} r\,[z]_{\frac{r^2}{4}}^{4}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{4} r\left[4-\frac{r^2}{4}\right] dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{4} \left[4r-\frac{r^3}{4}\right] dr\,d\theta \\
&= \int_0^{2\pi} \left[2r^2-\frac{r^4}{16}\right]_0^4 d\theta \\
&= 16\int_0^{2\pi} d\theta \\
&= 32\pi
\end{aligned}$$

Example 3.5.16. *Find the volume of the region bounded by the surface $y=x^2, x=y^2$ and the planes $z=0, z=3$.*

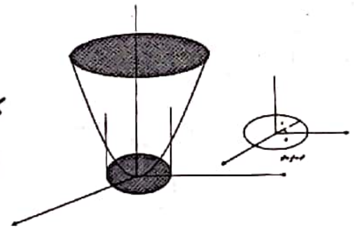
$$\begin{aligned}
\text{Volume} &= \iiint dxdydz \\
&= \int_0^{1}\int_{x^2}^{\sqrt{x}}\int_0^{3} dzdydx \\
&= 3\int_0^{1}\int_{x^2}^{\sqrt{x}} dydx \\
&= 3\int_0^{1} [\sqrt{x}-x^2]\,dx \\
&= 3\left[\frac{2}{3}x^{3/2}-\frac{x^3}{3}\right]_0^1 \\
&= 1
\end{aligned}$$

Example 3.5.17. *Find the volume of the region bounded by $z=4-x^2-y^2$ and the xy–plane*



$$\text{Volume} = \iiint dxdydz$$
$$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 r(4-r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (4r-r^3)\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta$$
$$= 4\int_0^{2\pi} d\theta$$
$$= 8\pi$$

Example 3.5.18. *Find the volume of the region above the $x-y$ plane bounded by the paraboloid $x^2+y^2=az$ and the cylinder $x^2+y^2=r^2$*



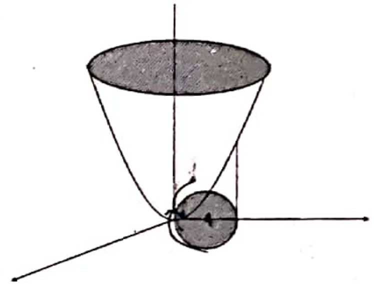
$az = r^2$

$z = \frac{r^2}{a}$

$$\text{Volume} = \iiint dxdydz$$
$$= \int_0^{2\pi}\int_0^r\int_0^{\frac{R^2}{a}} R\,dz\,dR\,d\theta$$
$$= \int_0^{2\pi}\int_0^r R\,[z]_0^{\frac{R^2}{a}}\,dR\,d\theta$$
$$= \int_0^{2\pi}\int_0^r R\frac{R^2}{a}\,dR\,d\theta$$
$$= \frac{1}{a}\int_0^{2\pi}\int_0^r R^3\,dR\,d\theta$$

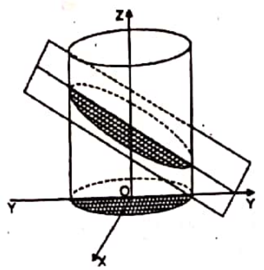
$$= \frac{1}{a}\int_0^{2\pi}\left[\frac{R^4}{4}\right]_0^r d\theta$$
$$= \frac{r^4}{4a}\int_0^{2\pi} d\theta$$
$$= \frac{r^4}{4a}\times 2\pi$$
$$= \frac{\pi r^4}{2a}$$

Example 3.5.19. *Find the volume bounded by the paraboloid $x^2+y^2=az$, the cylinder $x^2+y^2=2ay$ and the plane $z=0$*



$$\text{Volume} = \iiint dxdydz$$
$$= \int_0^{\pi}\int_0^{2a\sin\theta}\int_0^{\frac{r^2}{a}} r\,dz\,dr\,d\theta$$
$$= \int_0^{\pi}\int_0^{2a\sin\theta} r\,[z]_0^{\frac{r^2}{a}}\,dr\,d\theta$$
$$= \frac{1}{a}\int_0^{\pi}\int_0^{2a\sin\theta} r^3\,dr\,d\theta$$
$$= \frac{1}{a}\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta$$
$$= 4a^3\int_0^{\pi}\sin^4\theta\,d\theta$$
$$= 4a^3\times 2\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta$$
$$= 8a^3\times\frac{3\times 1}{4\times 2}\frac{\pi}{2}$$
$$= \frac{3\pi a^3}{2}$$

Example 3.5.20. *Find the volume bounded by the cylinder $x^2+y^2=4$ and the plane $y+z=4$.*

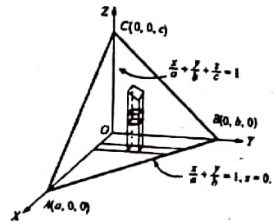


$$\text{Volume} = \iiint dxdydz$$
$$= \int_0^{2\pi}\int_0^2\int_0^{4-r\sin\theta} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 r\,[4-r\sin\theta]\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[4\frac{r^2}{2}-(\sin\theta)\frac{r^3}{3}\right]_0^2 d\theta$$
$$= \int_0^{2\pi}\left[8-\frac{8}{3}\sin\theta\right] d\theta$$
$$= \left[8\theta-\frac{8}{3}(-\cos\theta)\right]_0^{2\pi}$$
$$= 16\pi$$

Example 3.5.21. *Find the volume bounded by the xy-plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$.*

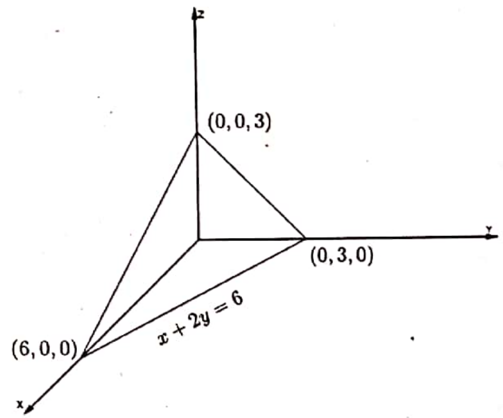
$$\text{Volume} = \iiint dxdydz$$
$$= \int_0^{2\pi}\int_0^1\int_0^{1-r\cos\theta-r\sin\theta} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 r[1-r\cos\theta-r\sin\theta]\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^3}{3}\cos\theta-\frac{r^3}{3}\sin\theta\right]_0^1 d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{2}-\frac{1}{3}\cos\theta-\frac{1}{3}\sin\theta\right] d\theta$$
$$= \left[\frac{\theta}{2}-\frac{1}{3}\sin\theta-\frac{1}{3}(-\cos\theta)\right]_0^{2\pi}$$
$$= \pi$$

Example 3.5.22. *Find the volume bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$*



$$\text{Volume} = \iiint dxdydz$$
$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$
$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left[1-\frac{x}{a}-\frac{y}{b}\right] dy\,dx$$
$$= c\int_0^a\left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$
$$= c\int_0^a\left[b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right] dx$$
$$= bc\int_0^a\left[1-\frac{x}{a}-\frac{x}{a}+\frac{x^2}{a^2}-\frac{b}{2}+\frac{bx}{a}+\frac{bx^2}{2a^2}\right] dx = \frac{abc}{6}$$

Example 3.5.23. *Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x+2y+z=6$* *[KTU. FEB 2017]*



$$\begin{aligned}
\text{Volume} &= \iiint dxdydz \\
&= \int_0^6 \int_0^{1/2(6-x)} \int_0^{6-x-2y} dz\,dy\,dx \\
&= \int_0^6 \int_0^{\frac{(6-x)}{2}} (6-x-2y)\,dy\,dx \\
&= \int_0^6 (6y - xy - y^2)_0^{\frac{(6-x)}{2}}\,dx \\
&= \int_0^6 \left(9 - 3x + \frac{x^2}{4}\right) dx \\
&= \left[9x - \frac{3x^2}{2} + \frac{x^3}{12}\right]_0^6 \\
&= 18
\end{aligned}$$

Example 3.5.24. *Find the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\begin{aligned}
\text{Volume} &= \iiint dxdydz \\
&= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx \\
&= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx \\
&= 8bc\int_0^a \left[\frac{y}{2b}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} + \frac{\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y/b}{\sqrt{1-\frac{x^2}{a^2}}}\right)\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx \\
&= 8bc\int_0^a \left[\frac{\left(1-\frac{x^2}{a^2}\right)}{2} \times \frac{\pi}{2}\right] dx \\
&= 2\pi bc\int_0^a \left[1-\frac{x^2}{a^2}\right] dx \\
&= 2\pi bc\left[x - \frac{x^3}{3a^2}\right]_0^a \\
&= \frac{4\pi abc}{3}
\end{aligned}$$

Example 3.5.25. *Find the volume of the region common to the intersecting cylinders* $x^2+y^2 = a^2$ *and* $x^2 + z^2 = a^2$

$$\begin{aligned}
\text{Volume} &= \iiint dxdydz \\
&= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx \\
&= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx \\
&= 8\int_0^a (a^2-x^2)\,dx \\
&= 8\left[a^2x - \frac{x^3}{3}\right]_0^a \\
&= \frac{16a^3}{3}
\end{aligned}$$

Example 3.5.26. *Find the volume included between the sphere* $x^2+y^2+z^2 = a^2$ *and the cone* $x^2+y^2 = z^2$ *and lying above the xy- plane.*

$$\begin{aligned}
\text{Volume} &= \iiint dxdydz \\
&= \int_0^{2\pi} \int_0^{a/\sqrt{2}} \int_r^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^{a/\sqrt{2}} r\left[\sqrt{a^2-r^2} - r\right] dr\,d\theta \\
&= \int_0^{2\pi} \int_0^{a/\sqrt{2}} \left[-\frac{1}{2}(-2r)\sqrt{a^2-r^2} - r^2\right] dr\,d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{2} \times \left(\frac{2}{3}(a^2-r^2)^{3/2}\right) - \frac{r^3}{3}\right]_0^{a/\sqrt{2}} d\theta \\
&= \int_0^{2\pi} \left[-\frac{a^3}{6\sqrt{2}} - \frac{a^3}{6\sqrt{2}} + \frac{a^3}{3}\right] d\theta \\
&= \frac{\pi a^3(2-\sqrt{2})}{3}
\end{aligned}$$

Example 3.5.27. *Find the volume the solid bounded by the surface* $y = x^2$ *and the planes* $y+z=4$ *and* $z=0$.

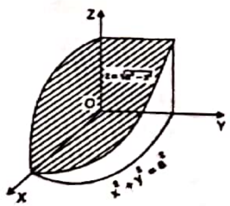
$$\begin{aligned}
\text{Volume} &= \iiint dV \\
&= \int_{-2}^{2} \int_{x^2}^{4} \int_0^{4-y} dz\,dy\,dx
\end{aligned}$$

$$= \int_{-2}^{2}\int_{x^2}^{4}(4-y)\,dy\,dx$$
$$= \int_{-2}^{2}\left[4y-\frac{y^2}{2}\right]_{x^2}^{4}dx$$
$$= \int_{-2}^{2}\left[8-4x^2+\frac{x^4}{2}\right]dx$$
$$= 2\left[8x-4\frac{x^3}{3}+\frac{x^5}{10}\right]_0^2$$
$$= \frac{256}{15}$$

Example 3.5.28. *Find the volume of the solid bounded by the graphs of $z = 4-y^2$, $x+z=4$, $x=0$ and $z=0$.*

$$\text{Volume} = \iiint dV$$
$$= \int_{-2}^{2}\int_{0}^{4-y^2}\int_{0}^{4-z} dx\,dz\,dy$$
$$= \int_{-2}^{2}\int_{0}^{4-y^2}(4-z)\,dz\,dy$$
$$= \int_{-2}^{2}\left[4z-\frac{z^2}{2}\right]_0^{4-y^2}dy$$
$$= \int_{-2}^{2}\left[4(4-y^2)-\frac{1}{2}(4-y^2)^2\right]dy$$
$$= \frac{1}{2}\int_{-2}^{2}(16-y^4)$$
$$= 2\left[16y-\frac{y^5}{5}\right]_0^2$$
$$= \frac{128}{5}$$

Example 3.5.29. *Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$*



$$\text{Volume} = \iiint dV$$
$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$$
$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy\,dx$$
$$= 8\int_0^a\sqrt{a^2-x^2}\Big[y\Big]_0^{\sqrt{a^2-x^2}}dx$$
$$= 8\int_0^a(a^2-x^2)\,dx$$
$$= 8\left[a^2x-\frac{x^3}{3}\right]_0^a$$
$$= \frac{16a^3}{6}$$

Example 3.5.30. *Evaluate $\iiint_V y\,dV$ where V is the region that lies below the plane $z=x+2$ above the $xy-$plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.*

Take $x=r\cos\theta, y=r\sin\theta, z=z$. Then $dV = r\,dz\,dr\,d\theta$

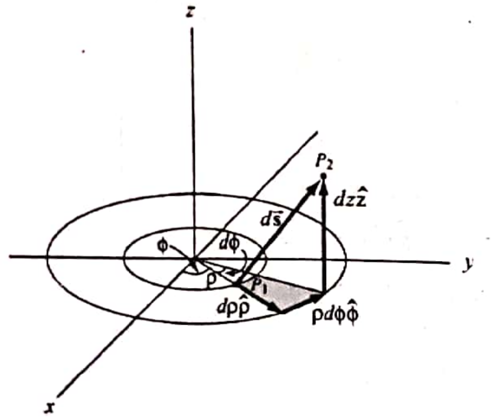
$$0\le z\le x+2 \implies 0\le z\le r\cos\theta+2$$
$$x^2+y^2=1 \implies r=1$$
$$x^2+y^2=4 \implies r=2$$
$$\therefore 0\le\theta\le 2\pi,\quad 1\le r\le 2$$

$$\iiint_V y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2}(r\sin\theta)r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta[z]_0^{r\cos\theta+2}dr\,d\theta$$
$$= \int_0^{2\pi}\int_1^2(r^3\cos\theta\sin\theta+2r^2\sin\theta)\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\cos\theta\sin\theta\frac{r^4}{4}+2\sin\theta\frac{r^3}{3}\right]_1^2d\theta$$
$$= \int_0^{2\pi}\left[\frac{15}{4}\cos\theta\sin\theta+\frac{14}{3}\sin\theta\right]d\theta$$
$$= \int_0^{2\pi}\left[\frac{15}{8}\sin 2\theta+\frac{14}{3}\sin\theta\right]d\theta$$
$$= \left[-\frac{15}{16}\cos 2\theta-\frac{14}{3}\cos\theta\right]_0^{2\pi}$$
$$= 0$$

3.6 Triple integration in Cylindrical and Spherical Coordinates

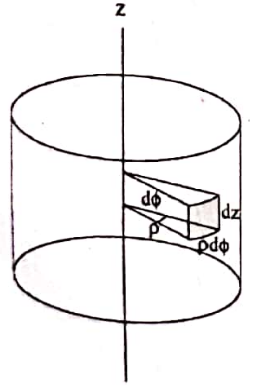
Cylindrical Coordinates

Cylindrical coordinates is a generalization of polar coordinates to three dimensions by superimposing a height (z axis). A point in this system is represented by the triplet (ρ, ϕ, z) or (r, θ, z), where ρ or r, the radial distance, ϕ or θ, the circumferential distance and z, axial distance. The following figure is a schematic diagram representing the displacement of an object when it moves from the position P_1 to the position P_2.



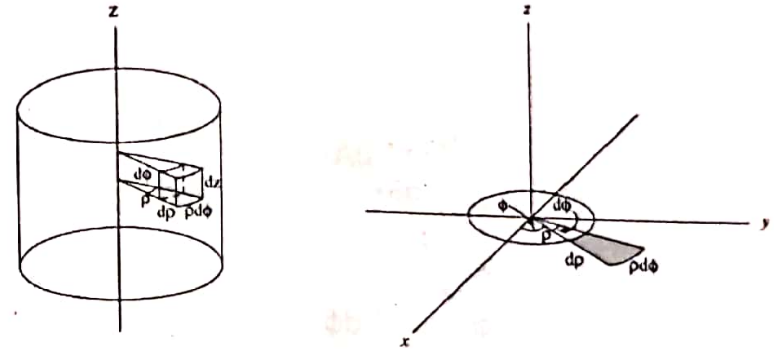
In the following diagram, an infinitesimal area element is schematically represented on the surface of a cylinder. From this diagram, it is easily seen that the area of the area element is

$$dA = \rho\, d\phi\, dz$$



An infinitesimal volume element is schematically represented in the following diagram. The volume of the infinitesimal volume element can be computed using

$$dV = \text{area of the sector shaped region} \times \text{height} = (\rho \times d\phi\, d\rho) \times dz = \rho\, d\phi\, d\rho\, dz$$

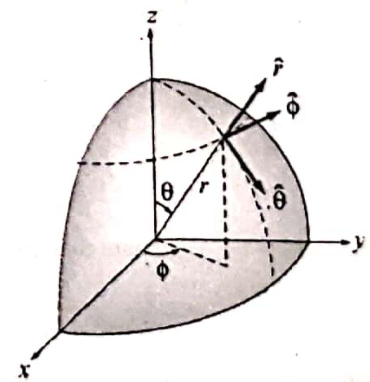


The transformation which effects the coordinate changes from cylindrical to Cartesian coordinates is given by

$$x = \rho Cos\phi,\ \ y = \rho Sin\phi,\ \ z = z,\ \ \rho \geq 0,\ 0 \leq \phi \leq 2\pi$$

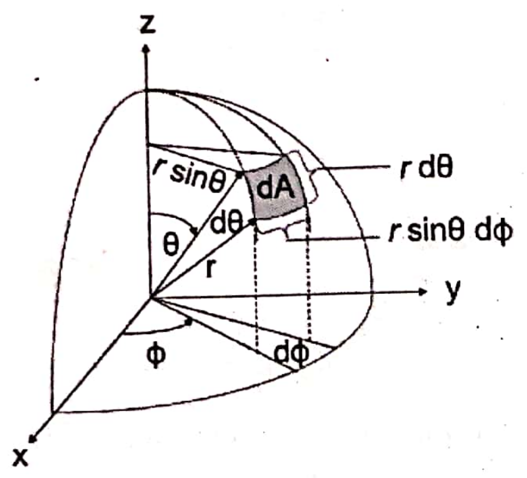
Spherical Coordinates

In this coordinate system, we use three numbers to specify a point in a three dimensional space. Usually, we use a triplet (r, θ, ϕ) to represent point in space. Here r is called the radial distance of the point from the fixed origin, θ is the polar angle measured from a fixed zenith direction and ϕ is the azimuth angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith direction (z-axis). A schematic diagram of spherical coordinates is given below



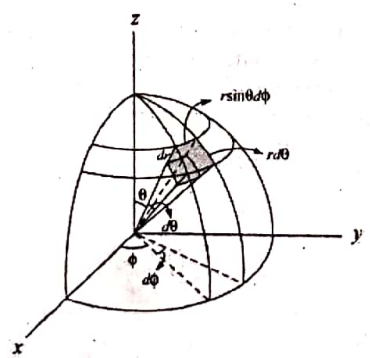
In the following diagram, an infinitesimal area element is schematically represented on the surface of a sphere. From this diagram, it is easily seen that the area of the area element is

$$dA = (r\,d\theta) \times (r\,Sin\theta\,d\phi) = r^2\,Sin\theta\,d\theta\,d\phi$$



An infinitesimal volume element is schematically represented in the following diagram. The volume of the infinitesimal volume element can be computed using

$$dV = (r^2\,Sin\theta\,d\theta d\phi) \times dr = r^2\,Sin\theta\,d\theta\,d\phi\,dr$$



The transformation which effects the coordinate changes from cylindrical to Cartesian coordinates is given by

$$x = r\,Sin\theta\,Cos\phi,\;\; y = r\,Sin\theta\,Sin\phi,\;\; z = r\,Cos\theta,\; r \geq 0,\;\; 0 \leq \theta \leq \pi,\;\; 0 \leq \phi \leq 2\pi$$

Example 3.6.1. *Convert the point $(-1, 1, -\sqrt{2})$ from Cartesian to spherical coordinates.*

Relation between Cartesian coordinates and spherical coordinates is given by

$$x = \rho \sin\theta \cos\phi, y = \rho \sin\theta \sin\phi, z = \rho\cos\theta$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+1+2} = 2$$

$$z = \rho\cos\theta \implies \cos\theta = \frac{z}{\rho} = \frac{-\sqrt{2}}{2} \implies \theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

$$y = \rho\sin\theta\sin\phi \implies \sin\phi = \frac{y}{\sin\theta} = \frac{1}{\sin\left(\frac{3\pi}{4}\right)} = \frac{1}{\sqrt{2}} \implies \phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Since $x = -1$ and $y = 1$ lies in the second quadrant, ϕ must in the second quadrant. Therefore $\phi = \frac{3\pi}{4}$. So the spherical coordinates of the given point is $\left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$

Example 3.6.2. *Evaluate $\iiint_V y\,dV$ where V is the region that lies below the plane $z = x + 2$ above the $xy-$plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.*

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$0 \leq z \leq x + 2 \implies 0 \leq z \leq r\cos\theta + 2$$
$$x^2 + y^2 = 1 \implies r = 1$$
$$x^2 + y^2 = 4 \implies r = 2$$
$$\therefore 0 \leq \theta \leq 2\pi, \quad 1 \leq r \leq 2$$

$$\begin{aligned}
\iiint_V y\,dV &= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} (r\sin\theta) r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 r^2\sin\theta[z]_0^{r\cos\theta+2} dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 (r^3\cos\theta\sin\theta + 2r^2\sin\theta)\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\cos\theta\sin\theta\frac{r^4}{4} + 2\sin\theta\frac{r^3}{3}\right]_1^2 d\theta \\
&= \int_0^{2\pi}\left[\frac{15}{4}\cos\theta\sin\theta + \frac{14}{3}\sin\theta\right] d\theta \\
&= \int_0^{2\pi}\left[\frac{15}{8}\sin 2\theta + \frac{14}{3}\sin\theta\right] d\theta \\
&= \left[-\frac{15}{16}\cos 2\theta - \frac{14}{3}\cos\theta\right]_0^{2\pi} \\
&= 0
\end{aligned}$$

Example 3.6.3. *Using cylidrical coordinates, evaluate $\iiint_V z\,dV$ where V is the region enclosed by the cylinder $x^2 + y^2 = 4$, bounded by the paraboloid $z = x^2 + y^2$, and bounded by the $xy-$plane.*

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$x^2+y^2 = 4 \implies r = 2$$
$$z = x^2+y^2 \implies z = r^2$$
$$\therefore 0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad 0 \le z \le r^2$$

$$\begin{aligned}\iiint_V z\,dV &= \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 r\left[\frac{z^2}{2}\right]_0^{r^2} dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 \left[\frac{r^5}{2}\right] dr\,d\theta\\ &= \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta\\ &= \frac{16}{3}\int_0^{2\pi} d\theta\\ &= \frac{32\pi}{3}\end{aligned}$$

Example 3.6.4. *Using cylidrical coordinates, evaluate* $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dzdydx$

From the limit of integration, we observe that

$$-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2$$

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$x^2+y^2 = 4 \implies r = 2$$
$$z = x^2+y^2 \implies z = r^2$$
$$\therefore 0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad r \le z \le 2$$

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dzdydx &= \int_0^{2\pi}\int_0^2\int_r^2 (r^2)r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 (r^3)[z]_r^2\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 (r^3)(2-r)\,dr\,d\theta\\ &= \int_0^{2\pi}\left[2\frac{r^4}{4}-\frac{r^5}{5}\right]_0^2 d\theta\\ &= \int_0^{2\pi}\frac{8}{5}\,d\theta\\ &= \frac{16\pi}{5}\end{aligned}$$

Example 3.6.5. *Use cylidrical coordinates to evaluate* $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\,dzdydx$

From the limit of integration, we observe that

$$-3 \le x \le 3, -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}, 0 \le z \le 9-x^2-y^2$$

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$x^2+y^2 = 9 \implies r = 3$$
$$z = 9-x^2-y^2 \implies z = 9-r^2$$
$$\therefore 0 \le \theta \le 2\pi, \quad 0 \le r \le 3, \quad 0 \le z \le 9-r^2$$

$$\begin{aligned}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\,dzdydx &= \int_0^{2\pi}\int_0^3\int_0^{9-r^2}(r^2\cos^2\theta)r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^3 (r^3\cos^2\theta)[z]_0^{9-r^2}\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^3 (9r^3-r^5)\cos^2\theta\,dr\,d\theta\\ &= \int_0^{2\pi}\cos^2\theta\left[9\frac{r^4}{4}-\frac{r^6}{6}\right]_0^3 d\theta\\ &= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta\\ &= \frac{243\pi}{4}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{243\pi}{8}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}\\ &= \frac{243\pi}{4}\end{aligned}$$

Example 3.6.6. *Use cylidrical coordinates to evaluate* $\iiint_V \sqrt{x^2+y^2}\,dV$ *where* V *is the region lies within the cylinder* $x^2+y^2=1$*, below the plane* $z=4$ *and above the paraboloid* $z = 1-x^2-y^2$.

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$x^2+y^2 = 1 \implies r = 1$$
$$z = 1-x^2-y^2 \implies z = 1-r^2$$
$$\therefore 0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad 1-r^2 \le z \le 4$$

$$\iiint_V \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (r)r\, dz\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 (r^2)[z]_{1-r^2}^4\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 r^2(3+r^2)\, dr\, d\theta$$
$$= \int_0^{2\pi}\left[3\frac{r^3}{3}+\frac{r^5}{5}\right]_0^1 d\theta$$
$$= \frac{6}{5}\int_0^{2\pi} d\theta$$
$$= \frac{12\pi}{5}$$

Example 3.6.7. *Using cylidrical coordinates find the volume of the solid that is bounded above by the hemisphere $z=\sqrt{25-x^2-y^2}$, below by the $xy-$plane, and laterally by the cylinder $x^2+y^2=9$.*

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\, dz\, dr\, d\theta$

$$x^2+y^2=9 \implies r=3$$
$$z=\sqrt{25-x^2-y^2} \implies z=\sqrt{25-r^2}$$
$$\therefore 0\le\theta\le 2\pi,\quad 0\le r\le 3,\quad 0\le z\le\sqrt{25-r^2}$$

$$Volume = \iiint_V dV$$
$$= \int_0^{2\pi}\int_0^3\int_0^{\sqrt{25-r^2}} r\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^3 r\sqrt{25-r^2}\, dr\, d\theta$$
$$= -\frac{1}{2}\int_0^{2\pi}\int_0^3 (-2r)\sqrt{25-r^2}\, dr\, d\theta$$
$$= -\frac{1}{2}\int_0^{2\pi}\left[\frac{(25-r^2)^{3/2}}{3/2}\right]_0^3 d\theta$$
$$= \int_0^{2\pi}\frac{61}{3} d\theta$$
$$= \frac{122\pi}{3}$$

Example 3.6.8. *Use spherical coordinates to evaluate*

$$\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dzdxdy$$

Take $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. Then $x^2+y^2+z^2=\rho^2$, and $dV = \rho^2\sin\phi\; d\rho\, d\phi\, d\theta$ From the limit of integration, we observe that the upper surface is the hemisphere $z=\sqrt{4-x^2-y^2}$ and the lower surface is the circular region $x^2+y^2=4$ in the $xy-$plane. Therefore

$$0\le\theta\le 2\pi,\quad 0\le\phi\le\pi/2,\quad 0\le\rho\le 2$$

$$\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dzdxdy$$
$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (\rho^2\cos^2\phi)\rho\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/2}\left[\frac{\rho^6}{6}\right]_0^2 \cos^2\phi\sin\phi\; d\phi\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/2}\frac{32}{3}\cos^2\phi\sin\phi\; d\phi\, d\theta$$
$$= \frac{32}{3}\int_0^{2\pi}\left[-\frac{\cos^3\phi}{2}\right]_0^{\pi/2}$$
$$= \frac{32}{9}\int_0^{2\pi} d\theta$$
$$= \frac{64\pi}{9}$$

Example 3.6.9. *Use spherical coordinates to evaluate $\iiint_V e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where V is the unit ball $x^2+y^2+z^2\le 1$.*

Take $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. Then $x^2+y^2+z^2=\rho^2$, and $dV = \rho^2\sin\phi\; d\rho\, d\phi\, d\theta$

$$0\le\theta\le 2\pi,\quad 0\le\phi\le\pi,\quad 0\le\rho\le 1$$

$$\iiint_V e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{\rho^3}\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\left[e^{\rho^3}\right]_0^1 \sin\phi\; d\phi\, d\theta$$
$$= \frac{(e-1)}{3}\int_0^{2\pi}(-\cos\phi)_0^{\pi}\, d\theta$$
$$= \frac{2(e-1)}{3}\int_0^{2\pi} d\theta$$
$$= \frac{4\pi(e-1)}{3}$$

Example 3.6.10. *Use spherical coordinates to find the volume of the solid bounded above by the sphere $x^2+y^2+z^2=16$ and below by the cone $z=\sqrt{x^2+y^2}$.*

Take $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. Then $x^2+y^2+z^2 = \rho^2$, and $dV = \rho^2\sin\phi\; d\rho\, d\phi\, d\theta$

$$\begin{aligned} x^2+y^2+z^2 = 16 &\implies \rho = 4 \\ z = \sqrt{x^2+y^2} &\implies \rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2} \\ &\implies \rho\cos\phi = \rho\sin\phi \implies \phi = \pi/4 \end{aligned}$$

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/4, \quad 0 \le \rho \le 4$$

$$\begin{aligned} Volume &= \iiint_V dV \\ &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \rho^2\sin\phi\; d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^4 \sin\phi\; d\phi\, d\theta \\ &= \frac{64}{3}\int_0^{2\pi}(-\cos\phi)_0^{\pi/4}\, d\theta \\ &= \frac{64}{3}\left[1-\frac{1}{\sqrt{2}}\right]\int_0^{2\pi} d\theta \\ &= \frac{128\pi}{3}\left[1-\frac{1}{\sqrt{2}}\right] \end{aligned}$$

Example 3.6.11. *Using cylidrical coordinates, evaluate $\iiint_V z\; dV$ where V is the region enclosed by the cylinder $x^2+y^2=4$, bounded by the paraboloid $z=x^2+y^2$, and bounded by the xy−plane.*

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$\begin{aligned} x^2+y^2 = 4 &\implies r = 2 \\ z = x^2+y^2 &\implies z = r^2 \end{aligned}$$

$$\therefore 0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad 0 \le z \le r^2$$

$$\begin{aligned} \iiint_V z\; dV &= \int_0^{2\pi}\int_0^2\int_0^{r^2} z\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^2 r\left[\frac{z^2}{2}\right]_0^{r^2} dr\, d\theta \\ &= \int_0^{2\pi}\int_0^2 \left[\frac{r^5}{2}\right] dr\, d\theta \\ &= \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta \\ &= \frac{16}{3}\int_0^{2\pi} d\theta \\ &= \frac{32\pi}{3} \end{aligned}$$

Example 3.6.12. *Using cylidrical coordinates, evaluate* $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\; dzdydx$

From the limit of integration, we observe that

$$-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2$$

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$\begin{aligned} x^2+y^2 = 4 &\implies r = 2 \\ z = x^2+y^2 &\implies z = r^2 \end{aligned}$$

$$\therefore 0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad r \le z \le 2$$

$$\begin{aligned} \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\; dzdydx &= \int_0^{2\pi}\int_0^2\int_r^2 (r^2)r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^2 (r^3)[z]_r^2\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^2 (r^3)(2-r)\, dr\, d\theta \\ &= \int_0^{2\pi}\left[2\frac{r^4}{4}-\frac{r^5}{5}\right]_0^2 d\theta \\ &= \int_0^{2\pi}\frac{8}{5}\, d\theta \\ &= \frac{16\pi}{5} \end{aligned}$$

Example 3.6.13. *Use cylidrical coordinates to evaluate* $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\; dzdydx$

From the limit of integration, we observe that

$$-3 \le x \le 3, -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}, 0 \le z \le 9-x^2-y^2$$

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$x^2+y^2=9 \implies r=3$$
$$z = 9-x^2-y^2 \implies z = 9-r^2$$
$$\therefore 0\le\theta\le 2\pi,\quad 0\le r\le 3,\quad 0\le z\le 9-r^2$$

$$\begin{aligned}
\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\,dzdydx &= \int_0^{2\pi}\int_0^3\int_0^{9-r^2}(r^2\cos^2\theta)r\,dz\,dr\,d\theta\\
&= \int_0^{2\pi}\int_0^3 (r^3\cos^2\theta)[z]_0^{9-r^2}\,dr\,d\theta\\
&= \int_0^{2\pi}\int_0^3 (9r^3-r^5)\cos^2\theta\,dr\,d\theta\\
&= \int_0^{2\pi}\cos^2\theta\left[9\frac{r^4}{4}-\frac{r^6}{6}\right]_0^3 d\theta\\
&= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta\\
&= \frac{243\pi}{4}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta\\
&= \frac{243\pi}{8}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}\\
&= \frac{243\pi}{4}
\end{aligned}$$

Example 3.6.14. *Use cylidrical coordinates to evaluate* $\iiint_V \sqrt{x^2+y^2}\,dV$ *where V is the region lies within the cylinder* $x^2+y^2=1$*, below the plane* $z=4$ *and above the paraboloid* $z=1-x^2-y^2$.

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$x^2+y^2=1 \implies r=1$$
$$z = 1-x^2-y^2 \implies z = 1-r^2$$
$$\therefore 0\le\theta\le 2\pi,\quad 0\le r\le 1,\quad 1-r^2\le z\le 4$$

$$\begin{aligned}
\iiint_V \sqrt{x^2+y^2}\,dV &= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (r)r\,dz\,dr\,d\theta\\
&= \int_0^{2\pi}\int_0^1 (r^2)[z]_{1-r^2}^4\,dr\,d\theta\\
&= \int_0^{2\pi}\int_0^1 r^2(3+r^2)\,dr\,d\theta\\
&= \int_0^{2\pi}\left[3\frac{r^3}{3}+\frac{r^5}{5}\right]_0^1 d\theta\\
&= \frac{6}{5}\int_0^{2\pi}d\theta\\
&= \frac{12\pi}{5}
\end{aligned}$$

Example 3.6.15. *Using cylidrical coordinates find the volume of the solid that is bounded above by the hemisphere* $z=\sqrt{25-x^2-y^2}$*, below by the* $xy-$*plane, and laterally by the cylinder* $x^2+y^2=9$.

Take $x = r\cos\theta, y = r\sin\theta, z = z$. Then $dV = r\,dz\,dr\,d\theta$

$$x^2+y^2=9 \implies r=3$$
$$z=\sqrt{25-x^2-y^2} \implies z=\sqrt{25-r^2}$$
$$\therefore 0\le\theta\le 2\pi,\quad 0\le r\le 3,\quad 0\le z\le\sqrt{25-r^2}$$

$$\begin{aligned}
Volume &= \iiint_V dV\\
&= \int_0^{2\pi}\int_0^3\int_0^{\sqrt{25-r^2}} r\,dr\,d\theta\\
&= \int_0^{2\pi}\int_0^3 r\sqrt{25-r^2}\,dr\,d\theta\\
&= -\frac{1}{2}\int_0^{2\pi}\int_0^3(-2r)\sqrt{25-r^2}\,dr\,d\theta\\
&= -\frac{1}{2}\int_0^{2\pi}\left[\frac{(25-r^2)^{3/2}}{3/2}\right]_0^3 d\theta\\
&= \int_0^{2\pi}\frac{61}{3}\,d\theta\\
&= \frac{122\pi}{3}
\end{aligned}$$

3.7 Exercise

1. Evaluate $\int_0^3 \int_1^2 xy(x^2+y^2)dxdy$ **Solution:** $\frac{378}{3}$

2. Evaluate $\int_0^{\log 3} \int_0^{\log 2} e^{x+2y}dydx$
   Solution: $\int_0^{\log 3} \int_0^{\log 2} e^{x+2y}dydx = \int_0^{\log 3} e^x dx \times \int_0^{\log 2} e^{2y}dy = 3$

3. Evaluate $\int_2^4 \int_1^2 \frac{1}{xy}dxdy$ **Solution:** $(\log 2)^2$

4. Evaluate $\int_0^2 \int_0^1 \frac{x}{(1+xy)^2}dydx$
   Solution: $\int_0^2 \int_0^1 \frac{x}{(1+xy)^2}dydx = \int_0^2 \left[-\frac{1}{(1+xy)}\right]_0^1 dx = 2 - \log 3$

5. Evaluate $\int_5^7 \int_1^2 \frac{1}{(1+xy)^2}dydx$
   Solution: $\int_5^7 \int_1^2 \frac{1}{(1+xy)^2}dydx = \int_5^7 \left[-\frac{1}{x(1+xy)}\right]_1^2 dx$

6. Evaluate $\int_{\pi/2}^{\pi} \int_1^2 x\sin(xy)dydx$
   Solution: $\int_{\pi/2}^{\pi} \int_1^2 x\sin(xy)dydx = \int_{\pi/2}^{\pi} x\left[-\frac{\cos(xy)}{x}\right]_1^2 dx$

7. Evaluate $\int_0^{\log 2} \int_0^1 xye^{xy^2}dydx$ **Solution:** $\frac{1-\log 2}{2}$

8. Evaluate $\int\int_R x\sqrt{1-x^2}dA$ where R is the rectangular region defined by $R = \{(x,y): 0 \le x \le 1, 2 \le y \le 3\}$
   Solution: $\int\int_R x\sqrt{1-x^2}dA = \int_2^3 \left(-\frac{1}{2}\int_0^1 \sqrt{1-x^2}(-2x)\,dx\right)dy = \frac{1}{3}$

9. Evaluate $\int\int e^{2x+y}dydx$ over the triangle bounded by $x=0$, $y=0$ and $x+y=2$.
   Solution: $\frac{(e^2-1)^2}{2}$

10. Using double integration find the area between the parabolas $y^2 = x$ and $x^2 = y$.
    Solution: $\frac{1}{3}$

11. Transform to polar co-ordinates and hence evaluate $\int\int \sqrt{a^2-x^2-y^2}dxdy$ over the semi circle $x^2+y^2 = ax$ in the positive quadrant.

Solution: Hint : Polar representation of $x^2+y^2 = ax$ and $x^2+y^2 = 2ax$ are $r = a\cos\theta$ and $r = 2a\cos\theta$ respectively. r varies form $a\cos\theta$ to $2a\cos\theta$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

12. Transform to polar co-ordinates and hence evaluate $\int\int y^2 dxdy$ over region between the circles $x^2+y^2-ax=0$ and $x^2+y^2-2ax=0$

13. Transform to polar co-ordinates and hence evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}dydx$

14. Transform to polar co-ordinates and hence evaluate $\int_0^a \int_0^x \frac{x}{x^2+y^2}dydx$
    Solution: $\frac{\pi a}{4}$

15. Transform to polar co-ordinates and hence evaluate $\int\int \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy$ over the region in the positive quadrant for which $x^2+y^2=1$

16. Find the mass and center of gravity of a triangular lamina with vertices $(0,0),(2,1),(0,3)$ if the density function is $\rho(x,y) = x+y$.

17. Find the mass and center of gravity of a lamina bounded by the x−axis, the line $x=1$ and the curve $y=\sqrt{x}$ if the density function is $\rho(x,y) = x+y$.

18. Find the mass and center of gravity of a lamina with density $\rho(x,y) = y$ bounded by x−axis and the arch $y = \sin x, 0 \le x \le \pi$.

19. Find the mass and center of gravity of a lamina with density $\rho(x,y) = xy$ in the first quadrant bounded by the circle $x^2+y^2=a^2$ and the coordinate axes.

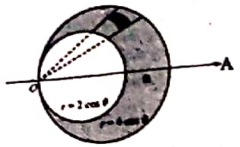
20. Find the mass and center of gravity of a triangular lamina with vertices $(0,0),(2,1),(0,3)$ if the density function is $\rho(x,y) = x+y$.

21. Find the mass and center of gravity of a lamina bounded by the x−axis, the line $x=1$ and the curve $y=\sqrt{x}$ if the density function is $\rho(x,y) = x+y$.

22. Find the mass and center of gravity of a lamina with density $\rho(x,y) = y$ bounded by x−axis and the arch $y = \sin x, 0 \le x \le \pi$.

23. Find the mass and center of gravity of a lamina with density $\rho(x,y) = xy$ in the first quadrant bounded by the circle $x^2+y^2=a^2$ and the coordinate axes.

24. Evaluate $\int\int r^2 drd\theta$ over the area between $r = 2\cos\theta$ and $r = 4\cos\theta$



25. Evaluate $\int\int r^3\,drd\theta$ over the area included between the circles $r = 2sin\theta$ and $r = 4sin\theta$.
Solution: Hint : Polar representation of $x^2+y^2 = 2y$ and $x^2+y^2 = 4y$ are $r = 2\sin\theta$ and $r = 4\sin\theta$ respectively. r varies form $2\sin\theta$ to $4\sin\theta$ and θ varies from 0 to π **Answer:** $\frac{45\pi}{2}$

26. Evaluate $\int\int\int 2xe^y \sin z\,dV$ where V is defined by $1 \le x \le 2$, $0 \le y \le 1$ and $0 \le z \le \pi$ **Solution:** $6(e-1)$

27. Evaluate $\int\int\int 6xy\,dV$ where V is the tetrahedron bounded by the planes $x = 0, y = 0$, $z = 0$ and $2x+y+z = 4$ **Solution:** $\frac{64}{5}$

28. Evaluate $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dxdydz$ **Solution:** $\frac{1}{2}$

29. Find the volume the solid bounded by the surface $z = \sqrt{y}$ and the planes $x+y = 1$, $x = 0$, and $z = 0$.

30. Use spherical coordinates to evaluate $\int\int\int_V (x^+y^2)\,dV$ where V is the region between the spheres $x^2+y^2+z^2 = 4$ and $x^2+y^2+z^2 = 9$.

31. Evaluate the integral by changing to spherical coordinates $\int_0^2\int_0^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2\,dzdx$

32. Use spherical coordinates to evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dzdydx$.

33. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

34. Use cylidrical coordinates to evaluate $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dzdxdy$.

35. Use cylidrical coordinates to evaluate $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2}\,dzdydx$.

3.8 Additional Exercises

1. Change the order of integration in $\int_0^1\int_{y^2}^{\sqrt{2-y^2}} f(x,y)\,dx\,dy$ [KTU FEB 2017]

2. Find the area of the region enclosed by $y = x^2$ and $y = x$. [KTU FEB 2017]

3. Evaluate the integral $\int_0^4\int_{\sqrt{y}}^{2} e^{x^3}\,dx\,dy$ by reversing the order of integration. [KTU FEB 2017]

4. Evaluate $\int_0^1\int_0^{y^2}\int_{-1}^{z} z\,dx\,dz\,dy$ [KTU FEB 2017]

5. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x+2y+z = 6$. [KTU FEB 2017]

6. Evaluate $\int_0^1\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dx\,dy$ [KTU DEC 2016, JUNE 2017]

7. Use double integration to find the area of the plane region enclosed by the given curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{4}$. [KTU DEC 2016]

8. Evaluate the integral $\int_0^4\int_y^4 \frac{x}{x^2+y^2}\,dx\,dy$ by reversing the order of integration. [KTU DEC 2016]

9. Evaluate $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy$ [KTU DEC 2016]

10. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x+y+z = 1$. [KTU DEC 2016]

11. Evaluate $\int_0^3\int_0^{\sqrt{9-y^2}} 2y\,dx\,dy$ [KTU DEC 2016]

12. Find the area of the region R between the parabola $y = \frac{x^2}{2}$ and the line $y = 2x$. [KTU DEC 2016, DEC 2017]

13. Evaluate $\int\int_R y\,dA$ where R is the region in the first quadrant enclosed between the circle $x^2+y^2 = 25$ and the line $x+y = 5$. [KTU DEC 2016]

14. Change the order of integration and hence evaluate $\int_1^2\int_y^{y^2} y^2\,dxdy$ [KTU DEC 2016]

15. Find the volume bounded by the cylinder $x^2+y^2 = 4$, the planes $y+z = 3$ and $z = 0$. [KTU DEC 2016]

16. Evaluate $\int_1^a \int_1^b x^2y\,dx\,dy$ [KTU FEB 2017]

17. Find the area of the region R enclosed by $y = 1, y = 2, x = 0, x = y$ [KTU FEB 2017]

18. Evaluate the integral by converting into polar co-ordinates $\int_0^2 \int_0^{\sqrt{4-x^2}} x^2 + y^2\,dydx$ [KTU FEB 2017]

19. Using triple integral to find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 3$ and $z = 0$. [KTU FEB 2017]

20. Change the order of integration and hence evaluate $\int_0^1 \int_x^1 \frac{x}{x^2 + y^2}dydx$ [KTU FEB 2017, APL 2018]

21. Evaluate $\int\int_R \frac{\sin x}{x}\,dxdy$ where R is the triangular region bounded by the x−axis , $y = x$ and $x = 1$. [KTU JUNE 2017]

22. Change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dydx$ [KTU JUNE 2017]

23. Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} y(x^2 + y^2)\,dydx$ using polar coordinates. [KTU JUNE 2017]

24. Find the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut off by the plane $z = 4$. [KTU JUNE 2017]

25. Evaluate $\int_0^{\log 3} \int_0^{\log 2} e^{x+2y}\,dydx$ [KTU DEC 2017]

26. Evaluate $\int\int_R xydA$ where R is the region bounded by the curves $y = x^2$ and $x = y^2$ [KTU DEC 2017]

27. Change the order of integration and hence evaluate $\int_0^1 \int_{4x}^4 e^{-y^2}\,dydx$ [KTU DEC 2017]

28. Using triple integral to find the volume bounded by the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$. [KTU DEC 2017]

29. Using double integral find the area enclosed by the lines $x = 0$,$y = 0$ and $\frac{x}{a} + \frac{y}{b} = 1$ [KTU APL 2018] 017]

30. Evaluate $\int_{-1}^2 \int_0^2 \int_0^1 (x^2 + y^2 + z^2)\,dxdydz$ [KTU APL 2018]

31. If R is the region bounded by the parabolas $y = x^2$ and $y^2 = x$ in the first quadrant, evaluate $\int\int_R (x + y)dA$ [KTU APL 2018]

32. Using triple integral to find the solid bounded by the surface $y = x^2$ and the planes $y + z = 4, z = 0$. [KTU APL 2018]