CS201: DISCRETE COMPUTATIONAL STRUCTURES **Semester III**

Module I

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Syllabus: ***Review of elementary set theory*** *: Algebra of sets - Ordered pairs and Cartesian products - Countable and Uncountable sets* ***Relations*** *:- Relations on sets -Types of relations and their properties – Relational matrix and the graph of a relation - Partitions - Equivalence relations - Partial ordering- Posets - Hasse diagrams - Meet and Join - Infimum and Supremum* ***Functions*** *:- Injective, Surjective and Bijective functions - Inverse of a function- Composition*

Federal Institute of Science And Technology (FISAT)

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1.1 Sets

Definition 1.1 *A set is any well-defined collection of objects called the elements or members of the set.*

Examples include collection of real numbers between zero and one, collection of students with marks greater than 50%, collection of black dogs .... *Well-defined* means that it is possible to decide if a given object belongs to the collection or not. A set is represented by listing elements between braces. For example set of all positive integers that are less than 4 can be written as {1,2,3}

- Order in which elements of a set are lsited is not important.
- Repeated elements of a set can be ignored. For example {1,2,3} and {1,2,3,2,3,1} are same representations
- Uppercase letters are used to denote set and lower case letters denote the members of the set.

x is an element of set A is rpresented as $x \in A$. x is **not** an element of set A is rpresented as $x \notin A$

1.1.1 Algebra of sets

The algebra of sets is the set-theoretic analogue of the algebra of numbers.

Definition 1.2 *If A and B are sets, their* ***union*** *is defined as the set consisting of all elements that belong to A or B and denote it by* $A \cup B$

Let A={a,b,c,d}, B={d,e,f}, then $A \cup B$ is {a,b,c,d,e,f}.

Definition 1.3 *If A and B are sets, their* ***intersection*** *is defined as the set consisting of all elements that belong to both A and B and denote it by* $A \cap B$

Let A={a,b,c,d}, B={d,e,f}, then $A \cap B$ is {d}.

Definition 1.4 *If A and B are sets, then the* ***complement of B with respect to A*** *is the set of all elements that belong to A but not to B. We denote it by A-B.*

Let A={a,b,c,d}, B={d,e,f}, then $A - B$ is {a,b,c}.

Definition 1.5 *If A and B are sets, then the* ***symmetric difference*** *is the set of all elements that belong to A or to B, but not to both A and B. We denote it by* $A \oplus B$.

Let A={a,b,c,d}, B={d,e,f}, then $A \oplus B$ is {a,b,c,e,f}.

Also $A \oplus B = (A - B) \cup (B - A)$

1.1.1.1 Algebraic properties of set operations

Commutative Property:

$A \cup B = B \cup A$
$A \cap B = B \cap A$

Associative Property:

$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

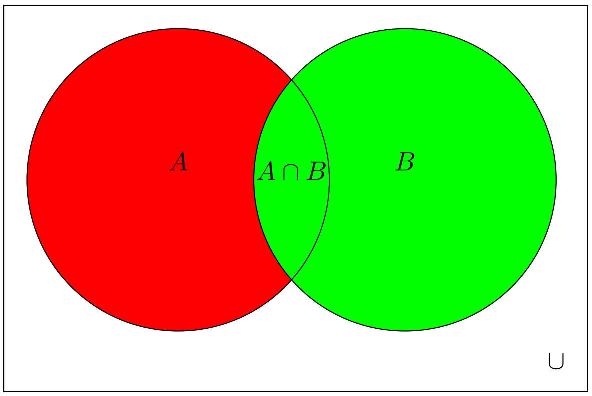
Distributive Property:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Complement Laws:

$A \cup \overline{A} = U$
$A \cap \overline{A} = \emptyset$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$
$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Diagrams which are used to show relationships between sets are called **Venn diagram**.



The colored portion shows $A \cup B$.

1.1.2 Ordered pairs

An *ordered pair* consists of two objects in a given **fixed order**.

- It is not a set consisting of two elements
- Ordering of two objects is important
- Two objects need not be distinct
- Ordered pair x, y is denoted by (x, y)

1.1.3 Cartesian products

Definition 1.6 *For two sets A and B; the set of all ordered pairs such that first member of the ordered pair is an element of A and the second member is an element of B is called the cartesian product of A and B.*

Cartesian product of A and B is written as AXB.

Let A=$\{\alpha, \beta\}$ and B=$\{$1,2$\}$. Then AXB is

$\{(\alpha, 1), (\beta, 1), (\alpha, 2), (\beta, 2)\}$

If $A = \phi$ $B = \{1, 2, 3\}$. Then $AXB = \phi$

For any two finite non empty sets $|AXB| = |A| * |B|$

1.1.4 Countable and Uncountable sets

Definition 1.7 *A set is called countable if it is finite or denumerable. A set is called uncountable if it is infinite* ***and*** *not denumerable.*

Any set which is equivalent to the set of natural numbers is called *denumerable*. A countable set is either a finite set or a countably infinite set. A set is countably infinite if it has one-to-one correspondence with the natural number set, N. Cantor proved that the set of real numbers is uncountable, thus showing that not all infinite sets are countable.

1.1.4.1 Diagonalization Principle

Cantor's diagonal argument, also called the diagonalisation argument is a mathematical proof that there are infinite sets which cannot be put into one-to-one correspondence with the infinite set of natural numbers. Such sets are known as uncountable sets.

Proof: Real numbers uncountable

Assume the set of all reals $0.a_1a_2a_3\ldots$ are countable. Then we could form something like

$$\begin{aligned} d_1 &= 0.a_1a_2a_3\ldots \\ d_2 &= 0.b_1b_2b_3\ldots \\ d_3 &= 0.c_1c_2c_3\ldots \\ &. \\ &. \\ &. \end{aligned}$$

Since we assumed that this set is countable each of this number must appear in this list. But we can construct a real number not in the list by changing each digit of this list. For example construct a new real number $0.x_1x_2x_3\ldots$ where x_1 is 1 if $a_1 = 2$, otherwise x_1 is 2. x_2 is 1 if $b_2 = 2$, otherwise x_2 is 2. Follow this process for each number. The resulting number is an infinite number containing 1's and 2's, but differs from any number we have constructed. This is a contradiction, since we assumed that the set is countable. Therefore it is uncountable.

1.2 Relations

A relation R between sets X and Y is a subset of XxY. A relation is a set of pairs. Relations are denoted by special symbols. The relation $>$ is

$>$ = $\{(x, y)|x, y$ are real numbers and $x > y\}$

1.2.1 Properties

Definition 1.8

- *R is reflexive if xRx holds for all x in X.*
- *R is symmetric if xRy implies yRx for all x and y in X.*
- *R is antisymmetric ifxRy and yRx together imply that $x = y$ for all x and y in X.*
- *R is transitive if xRy and yRz together imply that xRz holds for all x, y, and z in X.*

1.2.2 Relational matrix and the graph of a relation

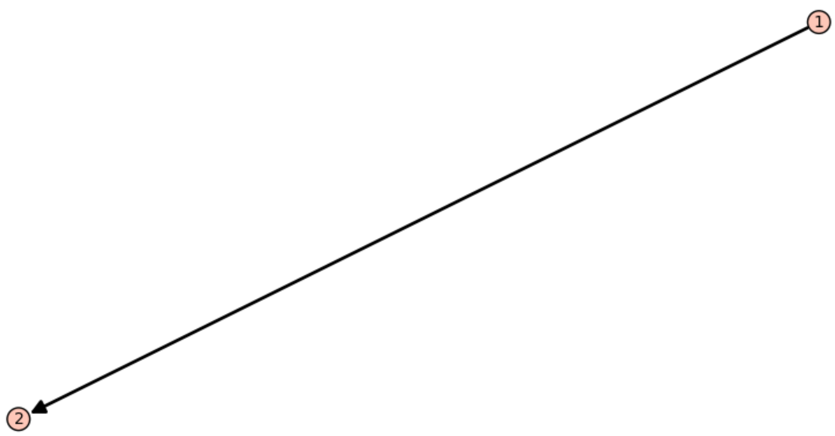
Any relation from A to B can be represented by a matrix. The element in the j^{th} row and k^{th} column is 1 if a_j R b_k; else it is 0.

For example if A=B={1,2,3} and R={(1,1),(1,2)}; then the relation is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In the relation we have a_1 R b_1 and a_1 R b_2; so we have 1 in first and second column of first row.

When representing as a graph, an arrow is drawn from a_j to b_k if a_j R b_k. Graph for the following example is represented with directed arrow element 1 to 2.



1.2.3 Partitions

Definition 1.9 *A* ***partition*** *of a non empty set A is a collection P on non empty subsets of A such that*

1. *Each element of A belongs to one of the sets in P*
2. *If A_1 and A_2 are distinct elements in P, then $A_1 \cap A_2 = \phi$*

Let A={a,b,c,d}. Then A_1={a,c} and A_2={b,d} are partitions of A.

1.2.4 Equivalence Relations

Definition 1.10 *A relation R on a set A is called an* ***equivalence relation*** *if it is reflexive, symmetric and transitive.*

For example let A={1,2,3,4} and R={(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)} is an equivalence relation.

1.2.5 Partial Order

Definition 1.11 *A relation R on a set A is called a partial order if R is reflexive, antisymmetric and transitive. The set A together with the partial order R is called a partially ordered set or simply a* ***poset****.*

For example Z^+ be the set of positive integers. The relation less than or equal to is a partial order on Z^+ as is greater than or equal to.

Example: Let S be a set and $L = P(S)$. $\subseteq$, containment is a partial order on L. For example $S = \{1, 2, 3\}$; then $P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. Then we can see that the set L together with the relation $\subseteq$ is a partial order. For example if transitive relation is considered, $\{1\} \subseteq \{1, 2\}$ and $\{1, 2\} \subseteq \{1, 2, 3\}$ means $\{1\} \subseteq \{1, 2, 3\}$. This is also true for reflexive and antisymmetric relations.

1.2.6 Hasse Diagram

Hasse diagram is a diagramatic representation of partial order. Hasse diagram can be constructed from a directed graph by applying following rules

- Eliminate all edges that are implied by transitive property
- Delete all cycles
- All edges points upwards;so arrows may be omitted from edges
- Vertices are represented by dots

1.3 Infimum and Supremum

Definition 1.12 *Let $(P, \leq)$ be a poset and let $A \subseteq P$. An element $x \in P$ is a Least Upper Bound (LUB) or **supremum** for A if x is an upper bound for A and $x \leq y$ where y is an upper bound for A. Greatest Lower Bound (GLB) or **infimum** for A is an element $x \in P$ such that x is a lower bound and $y \leq x$ for all lower bounds y.*

Least Upper Bound (LUB) of a subset (a,b) is denoted by $a \vee b$ and call its as join of *a* and *b*. Greatest Lower Bound (GLB) of subset (a,b) is denoted by $a \wedge b$ and call its as meet of *a* and *b*.

1.4 Functions

Definition 1.13 *Let A and B be nonempty sets. A **function** f from A to B, which is denoted by $f : A \longrightarrow B$, is a relation from A to B such that for every $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$*

The two conditions for a relation to act as a function are

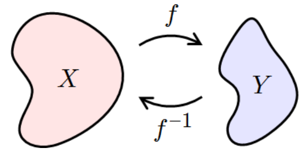
1. **Every** $a \in A$ must be related to some $b \in B$
2. Uniqueness is the second requirement, which says that for a function there should be a unique $b \in B$

1.4.1 Types of functions

1.4.1.1 Injective function

Definition 1.14 *Let f be a function defined on a set A and taking values in a set B. Then f is said to be an injection (or injective map, or embedding) if, whenever $f(x) = f(y)$, it must be the case that $x = y$.*

Also called as one-to-one function. f is an injection if it maps distinct objects to distinct objects.



Source: `https://commons.wikimedia.org/wiki/File:Inverse_Functions_Domain_and_Range.png`

Figure 1.1: If f maps X to Y, then f^{-1} maps Y back to X.

1.4.1.2 Surjective function

Definition 1.15 *Let f be a function defined on a set A and taking values in a set B. Then f is said to be a surjection (or surjective map) if, for any $b \in B$, there exists an $a \in A$ for which $b = f(a)$.*

A surjective function is also called as onto function.

1.4.1.3 Bijective function

Definition 1.16 *A transformation which is one-to-one and a surjection (i.e., "onto").*

1.4.2 Inverse of a function

Definition 1.17 *If f is one-to-one and onto, ie f is bijective; then converse of f denoted by $\overline{f}$ is a function from Y to X. In such cases, $\overline{f}$ is written as f^{-1} so that $f^{-1} : Y \to X$ and f^{-1} is called the inverse of the function f.*

Then if f^{-1} exists, f is called invertible.

1.4.3 Composition of functions

Definition 1.18 *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The composite relation $g \circ f$ such that*

$g \circ f = \{(x, z)|(x \in X) \wedge (z \in Z) \wedge (\exists y)(y \in Y \wedge y = f(x) \wedge z = g(y))\}$

Example: Let X={1,2,3}, Y= {p,q} and Z={a,b}. Also let $f : X \to Y$ be $f : \{(1, p), (2, p), (3, q)\}$ and $g : Y \to Z$ be given by $g = \{(p, b), (q, b)\}$. Find $g \circ f$

$g \circ f = \{(1, b), (2, b), (3, b)\}$