Chapter 1

Linear Algebra: Eigenvalue Problem

In the first section we review the concept of matrices which is essential for the forth coming sections.

1.1 Matrices

A system of mn number of elements arranged in m rows and n columns in a definite order and enclosed by bracket ( ) or [ ] is called an $m \times n$ (read m by n) matrix.

We use capital letters A, B, C etc to denote matrices and the elements or numbers are denoted by small letters a, b, c etc. General form of an $m \times n$ matrix A may be expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

In this matrix the elements a_{ij} is in the ith row and jth column. In a more precise and concise notation, the above matrix is expressed as $A = (a_{ij})_{m \times n}$. Note that in a matrix, arrangement of elements is very important. Different arrangement gives different matrices.

1.1.1 Special matrices

(a) **Row and column matrix (vector):** A matrix having only one row is called **row matrix** and a matrix having only one column is called **column matrix.**

Examples of row matrix (vector) of order 1×4 and column matrix (vector) of order 4×1 are $\begin{bmatrix} 2 & 3 & 9 & -1 \end{bmatrix}$ and $\begin{bmatrix} -9 \\ 3 \\ -45 \\ 2 \end{bmatrix}$ respectively.

(b) **Square matrix: Square matrix** is a matrix with number of rows equal to number of columns.

In a square matrix the diagonal from left top corner to right bottom corner is called **principal diagonal** and the diagonal from right top corner to left bottom corner is called **off-diagonal**.

Sum of the elements in the principal diagonal of a square matrix is called **trace** of the matrix.

The matrix $\begin{bmatrix} 2 & -4 & 7 \\ 9 & 0 & -2 \\ -3 & 5 & -4 \end{bmatrix}$ is a square matrix and its trace $= -2$.

(c) **Upper triangular matrix and lower triangular matrix**: A square matrix $A == (a_{ij})_{n \times n}$ is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$ and it is said to be **lower triangular** if $a_{ij} = 0$ for all $i < j$.

The matrix $\begin{bmatrix} 2 & -4 & 7 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \end{bmatrix}$ an upper triangular and $\begin{bmatrix} 2 & 0 & 0 \\ 9 & 4 & 0 \\ -3 & 0 & 0 \end{bmatrix}$ is a lower triangular matrix.

(d) **Diagonal matrix**: A square matrix is said to be **diagonal matrix** if $a_{ij} = 0$ for all $i \neq j$. That is, in a diagonal matrix the entries above and below the principal diagonal are zeros.

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a diagonal matrix.

(d) **Scalar matrix**: A diagonal matrix in which the diagonal elements are equal is called **scalar matrix**.

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a scalar matrix

(e) **Identity matrix**: A diagonal matrix in which all diagonal elements are equal to 1 is called **unit matrix** or **identity matrix**.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

(f) **Zero matrix or null matrix**: A matrix in which all the entries are zeros is called **a zero matrix or null matrix** and is denoted by **0**.

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix.

(f) **Transpose of a matrix** : A matrix B obtained from another matrix A by changing rows into columns is called **Transpose of a matrix** and is denoted by A^T. That is,

$$B = A^T$$

Consider the matrix $A = \begin{bmatrix} 3 & 8 & 6 \\ -5 & -7 & 9 \\ 4 & 0 & 2 \end{bmatrix}$ Its transpose is obtained by changing rows into columns and it is $A^T = \begin{bmatrix} 3 & -5 & 4 \\ 8 & -7 & 0 \\ 6 & 9 & 2 \end{bmatrix}$

(g) **Symmetric and skew-symmetric matrices**: A square matrix A is said to be a symmetric matrix if $A^T = A$ and is said to be skew-symmetric if $A^T = -A$.

$A = \begin{bmatrix} 3 & 8 & 4 \\ 8 & -7 & 9 \\ 4 & 9 & 23 \end{bmatrix}$ is symmetric and $B = \begin{bmatrix} 0 & -2 & 9 \\ 2 & 0 & -5 \\ -9 & 5 & 0 \end{bmatrix}$ is skew-symmetric.

Remark: We can see that in symmetric matrix the elements above the principal diagonal are a mirror reflection of those below it and in a skew-symmetric matrix the elements along the principal diagonal are zeros and the elements above the principal diagonal are negatives of the elements below the principal diagonal.

1.1.2 Operations on matrices

The important operations performed on matrices are scalar multiplication, addition, multiplication, inverse of a matrix and division. Here we describe methods to find inverse of a matrix.

The idea of determinant is useful in finding inverse of a matrix and to divide a matrix by another matrix.

1.1.3 Determinant

Determinant is a number corresponding to a given square matrix. We associate each square matrix with unique number called determinant as follows: The determinant corresponding to a given 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as

$$|A| = ad - bc$$

To find the value called determinant associated with a 3×3 matrix, we need the notion of minor and co-factor of elements in a matrix.

Minor of an element a_{ij} in a 3×3 matrix is the determinant associated with the 2×2 matrix obtained by removing the i th row and j th column from the 3×3 matrix and is denoted by M_{ij}.

The **signed minor** defined by $C_{ij} = (-1)^{i+j} M_{ij}$ is called **co-factor** of the element a_{ij}. The matrix formed by the co-factor C_{ij} in place of elements a_{ij} is called co-factor matrix and transpose of a co-factor matrix is called **adjoint** of a matrix. Adjoint of a matrix A is denoted by adj$A$.

The determinant associated with the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is defined as

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Here we have expanded the determinant associated with the 3×3 matrix A along the first row. We can expand the determinant along any row or column. When we use a row or column to expand the determinant, we have to make use of the corresponding co-factors. The definition can be extended to find the determinant associated with any square matrix of order n.

Remark: Another point of view is to observe determinant as real valued function defined as above on the space of all square matrices (domain space). This view point is highly abstract and we do not need that much abstractness for our purpose, but it is worth mentioning. Note that matrix is not a value (determinant) and determinant is a number (a value).

We can prove that for any square matrix A of order n,

$$\sum_{k=1}^{n} a_{ik}C_{jk} = \delta_{ij}|A|$$

where C_{jk} is the co-factor of of a_{jk} and δ_{ij} is the Kronecker delta function defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

From this result we can easily show that

$$A(adj A) = |A|I_n = (adj A)A$$

where I_n is the unit matrix of order n.

Singular matrix and Intertible matrix

Two square matrices A and B of order n are said to be **inverse** to each other if

$$AB = BA = I$$

Then the matrices A and B are said to be **invertible**. If the matrix is not invertible it is called **singular matrix (non-invertible matrix)**.

Formula for inverse of a matrix: The result $A(adjA) = |A|I_n = (adjA)A$ gives a method to compute the inverse of a given matrix A as

$$\frac{1}{A} = A^{-1} = \frac{adj\, A}{|A|}$$

if $|A| \neq 0$.

This shows that a matrix is invertible (or non-singular) if and only if its determinant is not zero.

Equivalently, **a square matrix is singular if and only if its determinant is zero.**

Note that if A and B are inverse to each other then $|AB| = |A||B| = 1$ and so

$$|A^{-1}| = \frac{1}{|A|}$$

A matrix A is said to be **orthogonal** if $AA^T = I = A^TA$. That is the inverse of orthogonal matrix is its transpose. Also, note that $|A| = |A^T|$ and if A is orthogonal $|A| = \pm 1$.

Now, we shall consider matrices having complex numbers and analogous notions defined for real matrices.

Let H be a matrix having complex numbers as elements. The conjugate of the matrix H denoted by $\overline{H}$ is obtained by replacing each element of H by its conjugate.

A complex matrix H is said to be **Hermitian** if $\overline{H}^T = H$ and is said to be **skew-Hermitian** if $\overline{H}^T = -H$.

That is, **a complex matrix is Hermitian if it is equal to the transposed conjugate of itself and it will be skew-Hermitian if it is equal to negative of its transposed conjugate.**

The matrix

$$\begin{bmatrix} 3 & 2-4i & 5+7i \\ 2+4i & -6 & -7+8i \\ 5-7i & -7-8i & 9 \end{bmatrix}$$

is Hermitian and the matrix

$$\begin{bmatrix} 3i & -2-4i & -5+7i \\ 2+4i & -6i & 7+8i \\ 5-7i & -7-8i & 0 \end{bmatrix}$$

is a skew-Hermitian matrix. Note that in a Hermitian matrix the entries in the Principal diagonal are real and the entries above it are conjugates of those below it. In a skew-Hermitian matrix the principal diagonal entries are either zeros or purely imaginary and the entries above it are those below it with sign of the real part opposite.

A matrix U is said to be **unitary** if $U\overline{U}^T = \overline{U}^TU = I_n$.

That is, **a matrix is unitary if its inverse is equal to its conjugate transpose.** Note that every real symmetric matrix is Hermitian and every orthogonal matrix is unitary. These notions are introduced as preliminary for what we discuss in the following chapters.

1.2 System of linear equations

Consider a system of two linear equations in two variables x and y as

$$2x + 3y = 9$$
$$-4x + 7y = 5$$

From the left side of each equation we get constant row vector (row matrix) and a variable column vector (column matrix) in such a way that we can write each equation as

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 9$$

and

$$\begin{bmatrix} -4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5$$

Combining these two equations, we get a single matrix equation in the form

$$\begin{bmatrix} 2 & 3 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

Generalizing this for a system of m equations in n variables $x_1, x_2, \cdots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

and taking the m constant row vectors, one from each equation, we can express the system of linear equation in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

This system of m linear equations in n variables $x_1, x_2, \ldots, x_n$ is of the form

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Here A is called **coefficient matrix,** X is a column vector of unknowns and B is a column vector of constants.

Definition 1.2.1. *The matrix $[AB]$ obtained by placing B to the right of the coefficient matrix A is called the* ***augmented matrix.***

The system of equations is said to be **non-homogeneous** if $B \neq 0$ and it is said to be **homogeneous** if $B \equiv 0$.

Definition 1.2.2. *The set of values of the variables $x_i, i = 1, 2, \ldots, n$ satisfying $AX = B$ is called a* ***solution.*** *The solution $X = (x_1, x_2, x_3, \ldots, x_n)^T$ for which $x_i = 0$, for all i is called the* ***trivial solution*** *of the system of equations.*

Remark 1: The trivial solution, $X = 0$, is always a solution of the system of homogeneous equations.

Remark 2: Note that the solution $X = (x_1, x_2, x_3, \ldots, x_n)^T$ is expressed as matrix having n rows and one column. That is, it is a column matrix and such type of representation is usually referred to as a **column vector** rather than a column matrix. Analogously a representation by row matrix $(x_1, x_2, x_3, \ldots, x_n)$ is referred to as a **row vector.** In general, a row vector or a column vector is referred to as a **vector.**

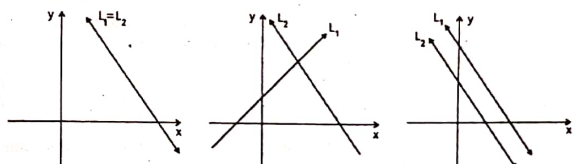
Remark 3: Using the above convention, an $m \times n$ matrix can be thought of as an an entity formed by m row vectors or n column vectors.

When we solve a system of equations to find the unknowns X, there may arise the following situation: the system of equations has solution - may be a unique solution or infinite number of solutions or has no solution at all.

Definition 1.2.3. *A system of equations is said to be* ***consistent*** *if it has a solution (at least one) and it is said to be* ***inconsistent*** *if it has no solution.*

When the system is consistent, the number of solutions may be finite or infinite. Before discussing methods of solution, we give geometrical interpretation of solution of a system of two equations in two unknowns and three equations in three unknowns.

1.2.1 Geometrical interpretation of solutions of system of equations



(a) Infinite Solution (b) Unique Solution © No Solution

We give geometrical interpretation for a system of two equations in two variables and a system of three equations in three variables. When $m = n = 2$, the system becomes

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Note that each equation represents a straight line in a plane. So solution of the system is an ordered pair (x_1, x_2) representing a point through which all the lines pass. When two lines are drawn in a plane, there are three possible cases:

- The two lines intersect at only one point - then the system has only one solution (unique solution).

- The two lines are identical. Then any point on one line satisfies the other line and hence the system will have an infinite number of solutions.

- The two lines are different and parallel - the system has no solution.

In a similar manner we can give geometrical interpretation for a system of three equations in three variables. When m = n = 3 the system of equations becomes

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Each of these equations represents a plane in the three dimensional space. Geometrically we have the following cases:

1. All the three planes intersect at one point - the system of equations has a unique solution.

2. Two of the planes are identical and the third plane intersect the first two planes or all the three planes are identical or the three planes have a common line of intersection - the system of equations have an infinite number of solutions.

3. Two of the planes are different and parallel or All the three planes intersect pairwise and having no common line of intersection - the system of equations have no solution.

In the first two cases the system is consistent and in the third case the system is inconsistent. When the system of equations is homogeneous, the origin will always be a solution of the system and so such system of equations has two cases.

(i) Unique solution if the planes intersect at only one point.

(ii) Infinite number of solutions if the planes are identical or two of the planes are identical and the third plane intersect the two planes along a straight line.

Geometrical interpretation is feasible only up to a system of equations in three variables. In most of the practical problems the number of variables may not be equal to the number of equations and possibly greater than three. So we need a rigorous method of giving all the solutions of a system of m equations in n variables.

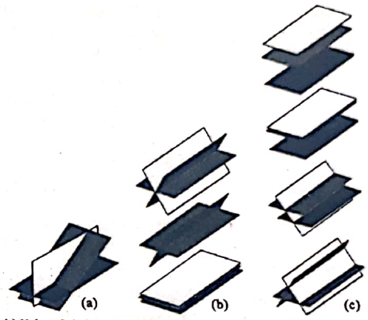
There are different methods for solving a system of equations such as

(a) matrix inverse method (matrix method)

(b) Cramers rule (method of determinant)

(c) Gauss elimination method

(d) method of rank due to Rouche



(a) Unique Solution (b) Infinite number of solution (c) No solution

Matrix inverse method and Cramer's rule are useful only if the number of equations is equal to number of variables and the coefficient matrix is non-singular and if the number of variables is greater than three the computational power needed is very high. So, here we discuss **Gauss elimination method** and **Rouche's method** to solve a system of m linear equations in n variables, which can be applied even when the coefficient matrix is singular or not even a square matrix. Since each equation is associated with a constant row vector (row matrix) we make use of the properties of vectors to devise methods for solving system of equations.

1.2.2 Linear combination of vectors

The idea of linear combination is very much useful when we deal with vectors whether they are space vectors or function space vectors.

In plane, vectors are represented by the expression $a\,i + b\,j$, where a and b are real numbers; i and j are unit vectors along positive directions of x - axis and y - axis respectively. In the frame work of **vector spaces**, the set of vectors in plane is abstractly represented by

$$R^2 = \{(x,\, y) : x,\, y \in \mathbb{R}\}$$

with coordinate wise addition and scalar multiplication. Here we identify the unit vectors i and j as $i = (1,\, 0)$ and $j = (0,\, 1)$. Now observe that

$$a\,i + b\,j = a(1,\, 0) + b(0,\, 1) = (a,\, 0) + (0,\, b) = (a,\, b)$$

This shows that each ordered pair can be identified as a vector in a plane and conversely.

Analogously, each vector $x\,i + y\,j + z\,k$ in space can be identified as an element $(x,\, y,\, z) \in \mathbb{R}^3$ by taking $i = (1,\, 0,\, 0)$, $j = (0,\, 1,\, 0)$ and $k = (0,\, 0,\, 1)$ as

$$\begin{aligned} x\,i + y\,j + z\,k &= x(1,\, 0,\, 0) + y(0,\, 1,\, 0) + z(0,\, 0,\, 1) \\ &= (x,\, 0,\, 0) + (y,\, 1,\, 0) + (z,\, 0,\, 1) = (x,\, y,\, z) \end{aligned}$$

This, in fact, shows that each element of $\mathbb{R}^3$ can be expressed in the form $x\,i + y\,j + z\,k$. This type of expression is called linear combination. This also shows that each element of $\mathbb{R}^3$ can be expressed as a linear combination of the unit vectors i, j, k.

Definition 1.2.4. Let $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_m}\}$ be m vectors of R^n and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be scalars. The expression $\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \cdots + \alpha_m\vec{u_m}$ is called a linear combination of the vectors $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_m}\}$.

Consider the vectors $\vec{u_1} = (3, 3, -1)$, $\vec{u_2} = (-4, -6, 12)$ and $\vec{u_3} = (2, 0, 10)$. Note that

$$2(3,3,-1) + 1(-4,-6,12) = (6,6,-2) + (-4,-6,12) = (2,0,10) = \vec{u_3}$$

Then $\vec{u_3} = 2\vec{u_1} + \vec{u_2}$. We are able to express the vector $\vec{u_3}$ as a linear combination of the vectors $\vec{u_1}$ and $\vec{u_2}$ with non-zero scalars 2 and 1 (at least one of them being non-zero). Then the vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$ are said to be linearly dependent.

Definition 1.2.5. A set of vectors $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_m}\}$ are said to be linearly dependent if one of the vectors can be expressed as a linear combination of the remaining vectors with at least one non-zero scalar. *That is,*

$$\vec{u_r} = \sum_{i=1, i\neq r}^{m} \alpha_i \vec{u_i}$$

with atleast one of the $\alpha_i \neq 0$.

This can be convenient expressed in the form

$$\sum_{i=1}^{m} \alpha_i \vec{u_i} = \vec{O}$$

That is, if the zero vector can be expressed as a linear combination of the given set of vectors with at least one non-zero scalar, then the vectors are linearly dependent.

Definition 1.2.6. *If none of the vectors can be expressed as a linear combination of the remaining vectors with atleat one non-zero scalar, then the vectors are said to be* **linearly independent**.

Alternately, a set of vectors $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_m}\}$ are said to be linearly independent if

$$\sum_{i=1}^{m} \alpha_i u_i = \vec{O} \rightarrow \alpha_i = 0 \quad \text{for all } i$$

Remark:1 Note that the three vectors $\vec{u_1} = (3, 3, -1)$, $\vec{u_2} = (-4, -6, 12)$ and $\vec{u_3} = (2, 0, 10)$ as a set are linearly dependent, but a set having only one vector or any two of the vectors are linearly independent.

Example 1.2.1. *The vectors* (1, 0, 0), (0, 1, 0) *and* (0, 0, 1) *are linearly independent.*

Note that

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0) \implies (c_1, c_2, c_3) = (0, 0, 0)$$

So that $c_1 = c_2 = c_3 = 0$. Hence the given vectors are linearly independent.

Remark:2 In between, we mentioned a term named **vector spaces**. Loosely speaking, vector spaces is a non-empty set V together with a field of scalars F (usually field of real numbers or complex numbers in which addition, subtraction, multiplication and division by non-zero numbers etc., can be done) in which two operations 'vector addition' viz. '+' and 'scalar multiplication' viz. '.' can be done in V such that for any of choice scalars (elements of F) α and β from F and for any choice of vectors $\vec{u}$ and $\vec{v}$ from V, the linear combination $\alpha.\vec{u}+\beta.\vec{v}$ is always an element of V satisfying some basic algebraic operations. The maximum number of linearly independent elements of V is called **dimension** of the vector spaces.

Remark: 3 A subset B of a vector space V is said be a **basis** of the vector space V if B is linearly independent and any element of V can be represented as a linear combination of elements of B. For example, in vector space $R^3 = \{(x, y, z) : x, y, z \in R\}$ with operations coordinate wise addition and scalar multiplication, the set $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ is linearly independent and any element of R^3 can be expressed as a linear combination of elements of B. So B is a basis of R^3. The number of elements in a basis is called **dimension** of the vector space. Note that any element of R^3 can be expressed as linear combination of elements of the basis B. That is, the set of all linear combinations of elements (this is often called **span** and B is the **spanning set**) of the basis B is same as the vector space R^3.

Example 1.2.2. *In* $V = \mathbb{R}^3$ *express the vector* $v = (1, -2, 5)$ *as a linear combination of the vectors* $v_1 = (1, 1, 1), v_2 = (1, 2, 3), v_3 = (2, -1, 1)$

Let $v = av_1 + bv_2 + cv_3$, where $a, b, c \in \mathbb{R}$

$$\begin{aligned} \implies (1, -2, 5) &= a(1,1,1) + b(1,2,3) + c(2,-1,1) \\ &= (a+b+2c, a+2b-c, a+3b+c) \end{aligned}$$

$$\begin{aligned} \implies a+b+2c &= 1 && (1) \\ a+2b-c &= -2 && (2) \\ a+3b+c &= 5 && (3) \end{aligned}$$

Solving the above three equations, we get $a = -6, b = 3, c = 2$

$$\therefore (1, -2, 5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$$

Example 1.2.3. *In* $V = \mathbb{R}^3$ *express the vector* $v = (2, -5, 3)$ *as a linear combination of the vectors* $v_1 = (1, -3, 2), v_2 = (2, -4, -1), v_3 = (1, -5, 7)$

Let $v = av_1 + bv_2 + cv_3$ where $a, b, c \in \mathbb{R}$

$$\text{Consider} \quad (2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$$
$$= (a+2b+c, -3a-4b-5b, 2a-b+7c)$$
$$\Longrightarrow a + 2b + c = 2 \quad (1)$$
$$-3a - 4b - 5b = -5 \quad (2)$$
$$2a - b + 7c = 3 \quad (3)$$
$$3(1) + (2) \Longrightarrow b - c = \frac{1}{2}$$
$$(1) + (2) + (3) \Longrightarrow b - c = 0$$
$$\Longrightarrow 0 = \frac{1}{2}, \text{ which is absurd.}$$

So, it is not possible to express v as a linear combination of the vectors v_1, v_2, v_3

1.2.3 Rank of a system of equations or a matrix

The following notion namely **rank** of a system of equations or of a matrix is an essential tool while solving a system of equations. When we are given a system of m equations in n variables (or equivalently an $m \times n$ matrix or m vectors each having n components), we can form a set having a maximal number of linearly independent set of equations (or row vectors).

Definition 1.2.7. Rank of system of equations (or a matrix) *is defined as the maximum number of linearly independent equations (or row vectors or rows of matrix).*

An alternate definition of rank of a matrix is the following: **Rank of a matrix A is defined as the order of the largest sub-square matrix with non-zero determinant formed from the matrix A.**

To find the rank of a system of equations (or a matrix) we form all possible sub-square matrices using rows and columns of the matrix A and then compute the determinant of each of these sub-square matrices. If we are able to find at least one non-singular sub-square matrix B and any sub-square matrix of higher order formed from A are singular, then rank of A is the order of B.

Then **rank of matrix A is r if there is a non-singular sub-square matrix B formed from r number of rows (or columns) of A and any sub-square matrix of higher order than r formed from A is singular.**

For example, consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 1 & 7 & 3 \end{pmatrix}$$

First we construct all possible 3×3 matrices as there are three rows and four columns. We can construct four square matrices of order three, namley

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 4 & 1 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 4 & 7 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 1 \\ 1 & 7 & 3 \end{pmatrix}$$

Determinant of each of these four matrices can be computed and they are all singular matrices (determinants being 0). So rank of matrix A is not 3.

Then we construct all possible 2×2 matrices from A. There are $3C_2 \times 4C_2 = 3 \times 6 = 18$ square matrices of order 2. So we need to compute the determinants of each of these 18 matrices and we will stop once we get a matrix with non-zero determinant. We can see that the square matrix

$$B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

has determinant $|B| = -3 \neq 0$, So rank of the matrix A is 2.

Computing rank of system of equations (or of a matrix) directly by computing the determinants of each of the sub-square matrices is a tedious and time consuming task when the order of the matrix is large. So we device another method to find the rank of a given system of equations (of a matrix).

1.2.4 Elementary transformations

The following three operations on matrices are called **elementary transformations**:

1. Interchanging two rows (or columns). Symbolically
$$R_i \leftrightarrow R_j \quad \text{or} \quad C_i \leftrightarrow C_j$$
2. Multiply a row or column by a non-zero constant. Symbolically,
$$R_i \to kR_i \quad \text{or} \quad C_i \to kC_i$$
where k is a non-zero constant.
3. Adding a row (or column) to another row (or column). Symbolically,
$$R_i \to R_i + R_j \quad \text{or} \quad C_i \to C_i + C_j$$

The last two operations can be combined to get a single operation,

$$R_i \to \alpha R_i + \beta R_j \quad \text{or} \quad C_i \to \alpha C_i + \beta C_j$$

where α, β are constants. These elementary transformations are helpful to reduce number of steps needed to find the rank of a matrix.

Two system of equations (matrices) are said to be equivalent system of equations (equivalent matrices) if one of them can be obtained from the other by a sequence of elementary transformations.

A matrix obtained from another matrix by using a finite number of elementary row transformations is called **row equivalent form.** This is a special case of equivalent matrices as for forming equivalent matrices we may use either elementary row operations or column operations or both.

When we solve system of equations it is mandatory to use only elementary row operations as interchanging columns will give rise to a new system of equations which are not equivalent to the given system of equations.

As it is convenient to express the theorems and results in terms of matrices rather than system of equations, we may use matrix notation to write the results.

Result: 1 Rank of system of equations (matrix) is invariant under elementary transformation and so equivalent system of equations will have the same rank.

Result: 2 If the rank of a set of vectors is equal to the number of vectors, the set of vectors is linearly independent. If the rank is less than the number of vectors, the vectors are linearly dependent and in this case one of the vectors can be expressed as a linear combination of the remaining vectors.

Elementary matrix

A matrix obtained from a unit matrix by applying an elementary transformation is called an **elementary matrix**.

The matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are elementary matrices.

Result 3: Elementary row (column) transformation can be obtained by pre multiplying (post multiplying) the matrix by elementary matrix.

Definition 1.2.8. *Every $m \times n$ matrix A can transformed into one of the three forms*

$$\begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \text{ or } [I_r \;\; O] \text{ or } \begin{bmatrix} I_r \\ O \end{bmatrix}$$

where I_r is a unit matrix of order r and O represents zero matrix. These forms are called ***normal form*** *of matrix.*

Result 4: Rank of a matrix is the order of the identity matrix in its equivalent normal form.

First non-zero element of a row from the left is called **leading element or pivot** of the row. A matrix is said to be in its **row echelon form or row reduced echelon form** if the leading element in each row (if exits) is 1 and the number of zeros before the leading element in each row is greater than the corresponding number of zeros of the preceding rows. In the above, if the leading element is not 1, then the matrix is called **row reduced form** of the matrix.

Example of row reduced form is

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 3 & -2 & 0 & -4 & 9 \\ 0 & 0 & -9 & 0 & 3 & 5 & -8 & 2 & 1 \\ 0 & 0 & 0 & 2 & -3 & -4 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following matrix is not in row reduced form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 0 & 0 & -4 & 1 \\ 0 & 0 & -9 & 0 & 3 & 5 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 3 & 0 & 2 & -1 \\ 0 & 9 & 0 & 0 & 0 & -2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 5 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Result 5: The rank of a matrix is equal to the number of non-zero rows in its equivalent row reduced form (or row reduced echelon form).

Since elementary row (column) operations can be obtained by pre (post) multiplying by elementary matrices and each of which are non singular matrices, we can find two non singular matrices P and Q for a given matrix A such that PAQ is in the normal form.

Example 1.2.4. *By reducing into row echelon form, find the rank of the matrix*

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{bmatrix}$$

$$A \equiv \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 9R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{bmatrix} R_2 \to R_2 - \frac{1}{4}R_2$$

$$\equiv \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 8R_2 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} R_3 \to (-1)$$

$$\equiv \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 + R_3$$

Rank of the matrix $A = 3$.

Example 1.2.5. *By reducing into row echelon form, find the rank of the matrix*

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$$

$$A \equiv \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 2R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \leftrightarrow R_4$$

$$\equiv \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to \frac{1}{16} R_3$$

Rank of the matrix $A = 3$.

Example 1.2.6. *By reducing into row echelon form, find the rank of the matrix*

$$A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$$

$$A \equiv \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix} \begin{matrix} R_4 \to R_4 - R_3 \\ R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1 \end{matrix}$$

$$A \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 5R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

Rank of the matrix $A = 2$.

1.2.5 Method of solution - Gauss elimination method

In this method the unknowns are eliminated successively and the system is reduced to upper triangular system (upper triangular matrix) from which the unknowns can be found by back substitution. We shall explain this method with the following example.

Example 1.2.7. *Write down the augmented matrix for the following system.*

$$\begin{aligned} 2x + y - z + 1 &= 0 \\ x + 3y + z - 10 &= 0 \\ 4y + z - 11 &= 0 \end{aligned}$$

The given system can be written as

$$\begin{aligned} 2x + y - z &= -1 \\ x + 3y + z &= 10 \\ 4y + z &= 11 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 1 & 3 & 1 & 10 \\ 0 & 4 & 1 & 11 \end{bmatrix}$

Example 1.2.8. *For the given augmented matrix write down the corresponding linear system of equations.* $\begin{bmatrix} 3 & -5 & 7 & -1 & -6 \\ 2 & 1 & 0 & 9 & 7 \\ -4 & 2 & 5 & 11 & -25 \\ 0 & -9 & 2 & 7 & 1 \end{bmatrix}$

The linear system corresponds to the given augmented matrix is

$$\begin{aligned} 3x - 5y + 7z - w &= -6 \\ 2x + y + 9w &= 7 \\ -4x + 2y + 5z + 11w &= -25 \\ -9y + 2z + 7w &= 1 \end{aligned}$$

Example 1.2.9. *Consider the system of equations*

$$\begin{aligned} x + 4y - z &= -5 \\ x + y - 6z &= -12 \\ 3x - y - z &= 4 \end{aligned}$$

The given system of equations are

$$\begin{aligned} x + 4y - z &= -5 && \text{(a)} \\ x + y - 6z &= -12 && \text{(b)} \\ 3x - y - z &= 4 && \text{(c)} \end{aligned}$$

Operate $(b) - (a)$ and $(c) - 3(a)$ to eliminate x,

$$\begin{aligned} -3y - 5z &= -7 && \text{(d)} \\ -13y + 2z &= 19 && \text{(e)} \end{aligned}$$

Operate $(e) - \frac{13}{3}(d)$ to eliminate y, we get

$$\begin{aligned} \frac{71}{3}z &= \frac{148}{3} \\ \implies z &= \frac{148}{71} \end{aligned}$$

By back substituting this value of z in (d) or (e), we get the value of $y = -\frac{81}{71}$. Putting the values of y and z in any one of the given system of equations we get $x = \frac{117}{71}$.

In the new frame work, these sets of operations are, in fact, our elementary row operations which we can perform on the augmented matrix $[AB]$ of the given system of equations and can reduce the augmented matrix to its equivalent upper triangular matrix as follows:

Apply the operations on the augmented matrix $[AB]$ of the given system of equations

$$[AB] = \begin{bmatrix} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -18 \\ 3 & -1 & -1 & -4 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & 0 & \frac{71}{3} & \frac{148}{3} \end{bmatrix} R_3 \to R_3 - \frac{13}{3}R_2$$

This matrix is equivalent to

$$\begin{aligned} x + 4y - z &= -5 \\ -3y - 5yz &= -7 \\ \frac{71}{3}z &= \frac{148}{3} \end{aligned}$$

which can be solved easily to get the solution $z = \frac{148}{71}, y = -\frac{81}{71}$ and $x = \frac{117}{71}$.

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of A = number of variables. Also observe that the system has a unique solution.

Example 1.2.10. *Find the solution set for the linear equation $3x + 2y = 5$.*

$$\begin{aligned} 3x + 2y &= 5 \\ 3x &= 5 - 2y \\ x &= \frac{5}{3} - \frac{2}{3}y \end{aligned}$$

Choose $y = t$, where t is any real number. Then $x = \frac{5}{3} - \frac{2}{3}t$. Thus for every choice of real number t, $x = \frac{5}{3} - \frac{2}{3}t$ and $y = t$ satisfies the equation $3x + 2y = 5$. Since there are infinite number of choices for t there are, in fact, an infinite number of possible solutions to the equation. Here y is a free variable.

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of $A = 2$. But this rank is less than then number of variables. Observe that the system has infinite number of solutions.

Example 1.2.11. *Find the solution set for the linear equation $x + 3y + 5z = 7$.*

$$\begin{aligned} x + 3y + 5z &= 7 \\ 5z &= 7 - x - 3y \\ z &= \frac{7}{5} - \frac{1}{5}x - \frac{3}{5}y \end{aligned}$$

Choose $x = t$ and $y = s$, where t and s are any real numbers. Then

$$z = \frac{7}{5} - \frac{1}{5}t - \frac{3}{5}s$$

Thus the solution set to the linear equation $x + 3y + 5z = 7$ is

$$x = t, y = s, z = \frac{7}{5} - \frac{1}{5}t - \frac{3}{5}s$$

where t and s are arbitrary real numbers. Since there are infinite number of choices for t and s there are, in fact, an infinite number of possible solutions to the equation. Here x and y are free variables.

Example 1.2.12. *Using Gauss elimination method, solve the system of equations*

$$4x - 6y = -11$$
$$-3x + 8y = 10$$

Its augmented matrix is $[AB] = \begin{bmatrix} 4 & -6 & -11 \\ -3 & 8 & 10 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 4 & -6 & -11 \\ 0 & \frac{7}{2} & \frac{7}{4} \end{bmatrix} R_2 \to R_2 + \frac{3}{4}R_1$$

Then the system of equations is equivalent to

$$4x - 6y = -11$$
$$\frac{7y}{2} = \frac{7}{4}$$

Back substitution now yields $y = \frac{1}{2}$ and $x = -2$

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of A = number of variables. Also observe that the system has a unique solution.

Example 1.2.13. *Using Gauss elimination method, find the solution of*

$$x - y = -3$$
$$-2x + 2y = 6$$

Its augmented matrix is $[AB] = \begin{bmatrix} 1 & -1 & -3 \\ -2 & 2 & 6 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 1 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to R_2 + 2R_1$$

Then the system of equations is equivalent to

$$x - y = -3$$
$$0 = 0$$
$$\Longrightarrow x = -3 + y$$

Choose $y = t$, where t is any real number. Then $x = -3 + t$. Since there are infinite number of choices for t there are an infinite number of possible solutions to the system of equations.

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of A = 1. But this rank is less than then number of variables. Observe that the system has infinite number of solutions.

Example 1.2.14. *Using Gauss elimination method, find the solution of*

$$-x - y = 1$$
$$-3x - 3y = 2$$

Its augmented matrix is $[AB] = \begin{bmatrix} -1 & -1 & 1 \\ -3 & -3 & 2 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

Then the system of equations is equivalent to

$$-x - y = 1$$
$$0 = -1$$

The false statement $0 = -1$ shows that the system has no solution.

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB] \neq$ rank of A. Observe that the system has no solution (the system is inconsistent).

Example 1.2.15. *Using Gauss elimination method, find the solution of the system of equations*

$$13x + 12y = -6$$
$$-4x + 7y = -73$$
$$11x - 13y = 157$$

Its augmented matrix is $[AB] = \begin{bmatrix} 13 & 12 & -6 \\ -4 & 7 & -73 \\ 11 & -13 & 157 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 13 & 12 & -6 \\ 0 & \frac{139}{13} & -\frac{973}{13} \\ 0 & -\frac{301}{13} & \frac{2107}{13} \end{bmatrix} \begin{matrix} R_2 \to R_2 + \frac{4}{13}R_1 \\ R_3 \to R_3 - \frac{11}{13}R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 13 & 12 & -6 \\ 0 & \frac{139}{13} & -\frac{973}{13} \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + \frac{301}{13} \times \frac{13}{139} R_2$$

Then the system of equations is equivalent to

$$\begin{aligned} 13x + 12y &= -6 \\ \frac{139}{13}y &= -\frac{973}{13} \\ 0 &= 0 \\ \implies y &= -7, x = 6 \end{aligned}$$

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of A = number of variables. Also observe that the system has a unique solution.

Example 1.2.16. *Using Gauss elimination method, find the solution of the system of equations*

$$\begin{aligned} 2x + 3y &= 9 \\ x - 2y &= -13 \\ x + y &= 7 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 2 & 3 & 9 \\ 1 & -2 & -13 \\ 1 & 1 & 7 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 2 & 3 & 9 \\ 0 & -\frac{7}{2} & -\frac{35}{2} \\ 0 & -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{matrix} R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 2 & 3 & 9 \\ 0 & -\frac{7}{2} & -\frac{35}{2} \\ 0 & 0 & 5 \end{bmatrix} R_3 \to R_3 - \frac{1}{2} \times \frac{2}{7}R_2$$

Then the system of equations is equivalent to

$$\begin{aligned} 2x + 3y &= 9 \\ -\frac{7}{2}y &= -\frac{35}{2} \\ 0 &= 5 \end{aligned}$$

The false statement $0 = 5$ shows that the system has no solution.

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB] \neq$ rank of A. Observe that the system has no solution (the system is inconsistent).

Example 1.2.17. *Using Gauss elimination method, find the solution of the system of equations*

$$\begin{aligned} x + y - z &= 9 \\ 8y + 6z &= -6 \\ -2x + 4y - 6z &= 40 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{bmatrix} R_3 \to R_3 + 2R_1$$

$$\equiv \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & -\frac{25}{2} & \frac{125}{2} \end{bmatrix} R_3 \to R_3 - \frac{6}{8}2R_1$$

Then the system of equations is equivalent to

$$\begin{aligned} x + y - z &= 9 \\ 8y + 6z &= -6 \\ -\frac{25}{2}z &= \frac{125}{2} \\ \implies z &= -5, y = 3, x = 1 \end{aligned}$$

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of A = number of variables. Also observe that the system has a unique solution.

Example 1.2.18. *Using Gauss elimination method, find the solution of the system of equations*

$$\begin{aligned} 2x + z &= 3 \\ x - y - z &= 1 \\ 3x - y &= 4 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 2 & 0 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 - \frac{3}{2}R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

Then the system of equations is equivalent to

$$\begin{aligned} 2x + z &= 3 \\ -y - \frac{3}{2}z &= -\frac{1}{2} \\ 0 &= 0 \end{aligned}$$

Choose $z = t$ where t is any real number. Then $y = \frac{1}{2} - \frac{3}{2}t$ and $x = \frac{3}{2} - \frac{1}{2}t$. Since there are an infinite number of choices for t, we can have an infinite number of possible solutions to the system of equation.

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of $A = 1$. But this rank is less than then number of variables. Observe that the system has infinite number of solutions.

Example 1.2.19. *Using Gauss elimination method, find the solution of the system of equations*

$$\begin{aligned} 4y + 3z &= 8 \\ 2x - z &= 2 \\ 3x + 2y &= 5 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\equiv \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 2 & \frac{3}{2} & 2 \end{bmatrix} R_3 \to R_3 - \frac{3}{2}R_1$$

$$\equiv \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix} R_3 \to R_3 - \frac{2}{4}R_2$$

Then the system of equations is equivalent to

$$\begin{aligned} 2x - z &= 2 \\ 4y + 3z &= 8 \\ 0 &= -2 \end{aligned}$$

The false statement $0 = -2$ shows that the system has no solution.

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB] \neq$ rank of A. Observe that the system has no solution (the system is inconsistent).

Example 1.2.20. *Using Gauss elimination method, find the solution of the system of equations*

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5 \\ 3x - 5y + 5z &= 2 \\ 3x + 9y - z &= 4 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix}$

Now

$$[AB] \equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 11R_2 \\ R_4 \to R_4 + 3R_2 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 + 2R_3$$

The the system of equations is equivalent to

$$\begin{aligned} x + 2y + z &= 3 \\ -y &= -1 \\ 2z &= 4 \\ 0 &= 0 \end{aligned}$$

$$\implies z = 2, y = 1, x = -1$$

Remark: Note that number of non-zero rows of row reduced echelon form (rank of $[AB]$) the augmented matrix $[AB]$ = rank of A = number of variables. Also observe that the system has a unique solution.

From the above examples, we observe that

1. the system of equations $AX = B$ has a unique solution if the number of non-zero rows of the row reduced echelon form of the augmented matrix $[AB]$ = the number of non-zero rows of row reduced echelon form of A = the number of unknowns. That is, the system is consistent and has a unique solution if
rank of $[AB]$ = rank of A = number of variables.

2. the system $AX = B$ will have an infinite number of solutions if the number of non-zero rows of the row reduced echelon form of the augmented matrix $[AB]$ = the number of non-zero rows of row reduced echelon form of A < the number of unknowns. That is, the system is consistent and have an infinite number of solution if
rank of $[AB]$ = rank of A < the number of variables.

3. the system will have no solution if the number of non-zero rows of the row reduced echelon form of the augmented matrix $[AB]$ not equal to the number of non-zero rows of row reduced echelon form of A. That is, the system of equations is inconsistent if rank of $[AB] \neq$ rank of A.

These observations and inferences are the basis for **Rouche's method**.

1.2.6 Method of Solution - Rouche's method

Consider a system of m equations in n variables in the matrix form $AX = B$. Form the augmented matrix $[AB]$. The following result gives conditions under which a system of linear equations is consistent and inconsistent.

1. A system of equations $AX = B$ is consistent if and only if the rank of $[AB]$ is same as the rank of A. The system will have a unique solution if this common rank is equal to the number of variables and the system will have an infinite number of solutions if this common rank r is less than the number of variables n (the system of equations is consistent). In this case, the solutions can be expressed as a linear combination of $n - r$ linearly independent vectors.

2. If rank of A is not equal to rank of $[AB]$, then the system of equations has no solution (the system is inconsistent).

Procedure to solve a system of linear equations
To solve the given system of equations, we use the following procedures:

1. First form the augmented matrix $[AB]$ of the given system of equations $AX = B$ and use elementary row operations to convert it into an equivalent row reduced echelon form.

2. From the echelon form of the augmented matrix, compute ranks of coefficient matrix and augmented matrix.

3. If their ranks are different, the system is inconsistent (no solution).

4. If their ranks are equal, the system is consistent. If the common rank is equal to the number of variables, the system will have unique solution and if the common rank is less than the number of variables, the system will have infinite number of solutions.

Example 1.2.21. *Consider the system of equations*

$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5$$
$$3x - 5y + 5z = 2 \text{ and}$$
$$3x + 9y - z = 4$$

Its augmented matrix is $[AB] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix}$

$$[AB] \equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix} \quad R_2 \to (-1)R_2$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + 11R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -4 & -8 \end{bmatrix} \quad R_3 \to \frac{1}{2}R_3$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + 4R_3$$

Rank of the matrix $[AB]$ =Rank of the matrix A = Number of unknowns= 3. So, the system is consistent and has a unique solution. The given system of equations is equivalent to

$$\begin{aligned} x+2y+z&=3\\ y&=1\\ z&=2\\ \implies x&=-1, y=1, z=2 \end{aligned}$$

Hence the unique solution is $x=-1$, $y=1$, $z=2$.

Example 1.2.22. *Consider the system of equations*

$$\begin{aligned} -x+2y+3z&=-2\\ 2x-5y+z&=2\\ 3x-8y+5z&=2 \text{ and}\\ 5x-12y-z&=6 \end{aligned}$$

Its augmented matrix is $[AB]=\begin{bmatrix} -1 & 2 & 3 & -2\\ 2 & -5 & 1 & 2\\ 3 & -8 & 5 & 2\\ 5 & -12 & -1 & 6 \end{bmatrix}$

$$[AB]\equiv\begin{bmatrix} 1 & -2 & -3 & 2\\ 2 & -5 & 1 & 2\\ 3 & -8 & 5 & 2\\ 5 & -12 & -1 & 6 \end{bmatrix} \quad R_1\to(-1)R_1$$

$$\equiv\begin{bmatrix} 1 & -2 & -3 & 2\\ 0 & -1 & 7 & -2\\ 0 & -2 & 14 & -4\\ 0 & -2 & 14 & -4 \end{bmatrix} \quad \begin{matrix} R_2\to R_2-2R_1\\ R_3\to R_3-3R_1\\ R_4\to R_4-5R_1 \end{matrix}$$

$$\equiv\begin{bmatrix} 1 & -2 & -3 & 2\\ 0 & 1 & -7 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3\to R_3+2R_2\\ R_4\to R_4+2R_2 \end{matrix}$$

Rank of the matrix $[AB]$ =Rank of the matrix $A=2<$ Number of unknowns= Therefore the system is consistent and has infinitely many solutions. We can choose variable as free variable. The given system of equations is equivalent to

$$\begin{aligned} x-2y-3z&=2\\ y-7z&=2 \end{aligned}$$

Choose $z=t$, where t is any real number. Then $y=7t+2$ and $x=17t+6$. As t va we can generate infinitely many solutions.

Example 1.2.23. *Consider the system of equations*

$$\begin{aligned} 2x-2z&=6\\ y+z&=1\\ 2x+y-z&=7 \text{ and}\\ 3y+3z&=0 \end{aligned}$$

Its augmented matrix is $[AB]=\begin{bmatrix} 2 & 0 & -2 & 6\\ 0 & 1 & 1 & 1\\ 2 & 1 & -1 & 7\\ 0 & 1 & 1 & 0 \end{bmatrix}$

$$[AB]\equiv\begin{bmatrix} 1 & 0 & -1 & 3\\ 0 & 1 & 1 & 1\\ 2 & 1 & -1 & 7\\ 0 & 1 & 1 & 0 \end{bmatrix} \quad R_1\to\frac{1}{2}R_1$$

$$\equiv\begin{bmatrix} 1 & 0 & -1 & 3\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 0 \end{bmatrix} \quad R_3\to R_3-2R_1$$

$$\equiv\begin{bmatrix} 1 & 0 & -1 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \begin{matrix} R_3\to R_3-R_2\\ R_4\to R_3-R_2 \end{matrix}$$

$$\equiv\begin{bmatrix} 1 & 0 & -1 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3\longleftrightarrow R_4$$

$$\equiv\begin{bmatrix} 1 & 0 & -1 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3\to(-1)R_3$$

Rank of the matrix $[AB]=3$. Rank of the matrix $A=2$. Here rank of the augmented matrix and coefficient matrix are different and so the system is inconsistent (has no solution).

Example 1.2.24. *Show that the equations*

$$\begin{aligned} x+2y+z&=0\\ 3x+2y+z&=2\\ 2x-y+2z&=5\\ 5x+6y+3z&=2 \text{ and}\\ x+3y-z&=-3 \end{aligned}$$

are consistent and solve the same.

Its augmented matrix is $[AB] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$

$$[AB] \equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & -5 & 0 & 5 \\ 0 & -4 & -2 & 2 \\ 0 & 1 & -2 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 5R_1 \\ R_5 \to R_5 - R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & -5 & 0 & 5 \\ 0 & -4 & -2 & 2 \\ 0 & -4 & -2 & 2 \end{bmatrix} R_2 \leftrightarrow R_5$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & -10 & -10 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 5R_2 \\ R_4 \to R_4 + 4R_2 \\ R_5 \to R_5 + 4R_2 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & -10 & -10 \end{bmatrix} R_3 \to -\frac{1}{10}R_3$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_4 \to R_4 + 10R_3 \\ R_5 \to R_5 + 10R_3 \end{matrix}$$

Rank of the matrix $[AB]$ =Rank of the matrix A = Number of unknowns= 3. Therefore the system is consistent and has a unique solution. The given system of equations is equivalent to

$$\begin{aligned} x + 2y + z &= 0 \\ y - 2z &= -3 \\ z &= 1 \\ \implies x = 1, y &= -1, z = 1 \end{aligned}$$

Example 1.2.25. *Show that the equations*

$$\begin{aligned} y - 3z &= -1 \\ x + z &= 1 \\ 3x + y &= 2 \text{ and} \\ x + y - 2z &= 0 \end{aligned}$$

are consistent and solve the same.

Its augmented matrix is $[AB] = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

$$[AB] \equiv \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

Rank of the matrix $[AB]$ =Rank of the matrix A = 2 < Number of unknowns= 3. Therefore the system is consistent and has infinitely many solutions. We can choose one variable as free variable. The given system of equations is equivalent to

$$\begin{aligned} x + z &= 1 \\ y - 3z &= -1 \end{aligned}$$

Choose $z = t$, where t is any real number. Then $y = 3t - 1$ and $x = 1 - t$

Example 1.2.26. *Test the consistency of the equations*

$$\begin{aligned} 2x - 3y + 7z &= 5 \\ 3x + y - 3z &= 13 \\ 2x + 19y - 47z &= 32 \text{ and} \\ 3y + 3z &= 0 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \\ 0 & 3 & 3 & 0 \end{bmatrix}$

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$$[AB] \equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \\ 0 & 3 & 3 & 0 \end{bmatrix} \quad R_1 \to \frac{1}{2}R_1$$

$$\equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & \frac{11}{2} & -\frac{27}{2} & \frac{11}{2} \\ 0 & 22 & -54 & 27 \\ 0 & 3 & 3 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_2 \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & \frac{11}{2} & -\frac{27}{2} & \frac{11}{2} \\ 0 & 22 & -54 & 27 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad R_4 \to \frac{1}{3}R_4$$

$$\equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 22 & -54 & 27 \\ 0 & \frac{11}{2} & -\frac{27}{2} & \frac{11}{2} \end{bmatrix} \quad R_4 \longleftrightarrow R_2$$

$$\equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -76 & 27 \\ 0 & 0 & -19 & \frac{11}{2} \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 22R_2 \\ R_4 \to R_4 - \frac{11}{2}R_2 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{27}{76} \\ 0 & 0 & -19 & \frac{11}{2} \end{bmatrix} \quad R_3 \to -\frac{1}{76}R_3$$

$$\equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{27}{76} \\ 0 & 0 & 0 & -\frac{5}{4} \end{bmatrix} \quad R_4 \to R_4 + 19R_3$$

$$\equiv \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{27}{76} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \to -\frac{4}{5}R_4$$

Rank of the matrix $[AB] = 4$. Rank of the matrix $A = 3$. Therefore the system is inconsistent.

Example 1.2.27. *Find the values of λ and μ for which the system of equations*

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu \end{aligned}$$

has (i) no solution (ii) a unique solution and (iii) a one-parameter family of solutions. Find the solutions when they exist.

Its augmented matrix is $[AB] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$

$$[AB] \equiv \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix} \quad R_1 \to \frac{1}{2}R_1$$

$$\equiv \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \\ 0 & -\frac{15}{2} & -\frac{39}{2} & -\frac{47}{2} \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 7R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \\ 0 & 1 & \frac{39}{15} & \frac{47}{15} \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix} \quad R_2 \to -\frac{2}{15}R_2$$

(i) The system has no solution if Rank of the matrix $[AB] \neq$Rank of the matrix A. This is possible only if $\lambda - 5 = 0$ and $\mu - 9 \neq 0$. Therefore the system has no solution if $\lambda = 5$ and $\mu \neq 9$.
(ii) The system has an unique solution if Rank of the matrix $[AB]$ =Rank of the matrix A =Number of unknowns= 3 . This is possible only if $\lambda - 5 \neq 0$. Therefore the system has an unique solution if $\lambda \neq 5$ and μ can take any value.
(iii) The system has a one-parameter family of solutions if Rank of the matrix $[AB]$ =Rank of the matrix $A = 2$ <Number of unknowns= 3 . This is possible only if $\lambda - 5 = 0$ and $\mu - 9 = 0$. Therefore the system has a one-parameter family of solutions if $\lambda = 5$ and $\mu = 9$.

(iv) To get a unique solution, take $\lambda = t \neq 5$ and $\mu = s$. Then the system of equations equivalent to

$$\begin{aligned} x + \frac{3}{2}y + \frac{5}{2}z &= \frac{9}{2} \\ y + \frac{39}{15}z &= \frac{47}{15} \\ (t-5)z &= s-9 \end{aligned}$$

$$\implies z = \frac{s-9}{t-5}, y = \frac{47t - 29s + 116}{15(t-5)}, x = \frac{4s - 2t - 116}{10(t-5)}$$

(v) To get one-parameter family of solutions we have to take $\lambda = 5$ and $\mu = 9$. Then the system of equations equivalent to

$$\begin{aligned} x + \frac{3}{2}y + \frac{5}{2}z &= \frac{9}{2} \\ y + \frac{39}{15}z &= \frac{47}{15} \end{aligned}$$

Choose $z = t$. Then $y = \frac{47}{15} - \frac{39}{15}t$ and $x = \frac{14t-2}{10}$.

Example 1.2.28. *(i) Find the values of λ for which the system of equations*

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 4z &= \lambda \\ x + 4y + 10z &= \lambda^2 \end{aligned}$$

will be consistent.

(ii) Show that for each value of λ obtained in part (i) the system has a one-parameter family of solutions, and find these solutions.

Its augmented matrix is $[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$

$$[AB] \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

The system of equations will be consistent only if Rank of the matrix $[AB]$ =Rank of the matrix A. This is possible only if $\lambda^2 - 3\lambda + 2 = 0 \implies \lambda = 1, 2$

Case(i) When $\lambda = 1$. Here the augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Then the system of equations equivalent to

$$\begin{aligned} x + y + z &= 1 \\ y + 3z &= 0 \end{aligned}$$

Choose $z = t$. Then $y = -3t$ and $x = 1 + t$.

Case(ii) When $\lambda = 2$. Here the augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Then the system of equations equivalent to

$$\begin{aligned} x + y + z &= 1 \\ y + 3z &= 1 \end{aligned}$$

Choose $z = t$. Then $y = 1 - 3t$ and $x = 2t$.

Example 1.2.29. *Show that the equations*

$$\begin{aligned} x + y + z &= a \\ 3x + 4y + 5z &= b \\ 2x + 3y + 4z &= c \end{aligned}$$

(i) have no solutions if $a = b = c = 1$ and

(ii) have many solutions if $a = \frac{b}{2} = c = 1$

The augmented matrix of the system of equations is

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & a \\ 3 & 4 & 5 & b \\ 2 & 3 & 4 & c \end{bmatrix}$$

$$[AB] \equiv \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - 3a \\ 0 & 1 & 2 & c - 2a \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - 3a \\ 0 & 0 & 0 & a - b + c \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Case(i) When $a = b = c = 1$.

The augmented matrix of the system is equivalent to $\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Here rank of the matrix $[AB] \neq$ Rank of the matrix A. Therefore the system has no solution.

Case(ii) When $a = \frac{b}{2} = c = 1$

The augmented matrix of the system is equivalent to $\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Here rank of the matrix $[AB]$ = Rank of the matrix $A = 2 <$ Number of unknowns$= 3$. Therefore the system is consistent and has infinitely many solutions. We can choose one variable as free variable. The given system of equations is equivalent to

$$\begin{aligned} x+y+z &= 1 \\ y+2z &= -1 \end{aligned}$$

Choose $z = t$ where t is any real number. Then $y = -1-2t$ and $x = 2+t$.

1.2.7 System of homogeneous linear equations

The system of homogeneous linear equations $AX = 0$ is always consistent, since the rank of coefficient matrix and augmented matrix are equal always. When the number of variables is greater than the number of equations, the system will have infinite number of solutions, because rank of matrix A is always less than or equal to the minimum of m and n, where A is an $m \times n$ matrix. We shall discuss the case of number equations equal to the number of variables separately.

Theorem 1.2.30. Show that a homogeneous system of linear equations $AX = 0$, where A is a square matrix, will have non-trivial solutions if and only if $|A| = 0$

Let a A be a non-singular matrix and so $AA^{-1} = A^{-1}A = I$.
Multiply A^{-1} from the left side of $AX = 0$.

$$\begin{aligned} AX = 0 \implies & A^{-1}(AX) = A^{-1}0 \\ \implies & (A^{-1}A)X = 0 \\ \implies & IX = 0 \implies X = 0 \end{aligned}$$

This observation shows that if A is non-singular, the system of equations

$$AX = 0$$

has only the trivial solution and it has no other solution. So the system will have non-trivial solutions only if $|A| = 0$.

Conversely assume that $|A| = 0$. Then rank of A is less than the number of variables and so by Rouche's theorem the system $AX = 0$ has infinite number of solutions. That is, the system will have non-trival solutions.

Example 1.2.31. *Consider the system of equations*

$$\begin{aligned} 3x+2y+z &= 0 \\ 2x+3z &= 0 \\ x+2y+3z &= 0 \end{aligned}$$

Its augmented matrix is $[AB] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$$\begin{aligned}
[A] &\equiv \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3 \\
&\equiv \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -3 \\ 0 & -4 & -8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \\
&\equiv \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{3}{4} \\ 0 & -4 & -8 \end{bmatrix} \quad R_2 \to -\frac{1}{4}R_2 \\
&\equiv \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & -5 \end{bmatrix} \quad R_3 \to R_3 + 4R_2 \\
&\equiv \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to -\frac{1}{5}R_3
\end{aligned}$$

Rank of the matrix $[A] = 3$ =Number of unknowns. Therefore the system has only trivial solution.

Another method: Here A is a square matrix and $|A| = -20 \neq 0$. Therefore the system has only the trivial solution.

Example 1.2.32. *Consider the system*

$$\begin{aligned} x+3y+2z &= 0 \\ 2x-y+3z &= 0 \\ 3x-5y+4z &= 0 \\ x+17y+4z &= 0 \end{aligned}$$

Its augmented matrix is $[A] = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$

$$[A] \equiv \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{7} \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \quad R_2 \to R_2 - \frac{1}{7}R_1$$

$$\equiv \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + 14R_2 \\ R_4 \to R_4 - 14R_2 \end{matrix}$$

Rank of the matrix $[A] = 2$ <Number of unknowns= 3. So the system has infinite number of solutions. The given system of equations equivalent to

$$x + 3y + 2z = 0$$
$$y + \frac{1}{7}z = 0$$

Choose $n - r = 3 - 2 = 1$ variable arbitrarily. Let $z = t$. Then $y = -\frac{1}{7}t$ and $x = -\frac{11}{7}t$. So the solution of the system of equations is

$$x = -\frac{11}{7}t, \; y = -\frac{1}{7}t, \; z = t$$

where t is a real number.

Example 1.2.33. *Do the equations*

$$x - 3y - 8z = 0$$
$$3x + y = 0 \; \textit{and}$$
$$2x + 5y + 6z = 0$$

have a non-trivial solution? Why?

Its augmented matrix is $[A] = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$

$$[A] \equiv \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 11 & 22 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 \to \frac{1}{11}R_3$$

$$\equiv \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 10 & 24 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\equiv \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \quad R_3 \leftrightarrow R_3 - 10R_2$$

$$\equiv \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \leftrightarrow \frac{1}{4}R_3$$

Rank of the matrix $[A] = 3$ =Number of unknowns. Therefore the system has only trivial solution.

Another method: Here A is a square matrix and $|A| = -44$. Therefore the system has only trivial solution.

Example 1.2.34. *Show that the equations*

$$x + 2y - z = 0$$
$$3x + y - z = 0 \; \textit{and}$$
$$2x - y = 0$$

have non-trivial solutions and find them.

Its augmented matrix is $[A] = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

$$[A] \equiv \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & -5 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & -5 & 2 \end{bmatrix} \quad R_2 \to -\frac{1}{5}R_2$$

$$\equiv \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

Rank of the matrix $[A] = 2$ <Number of unknowns= 3. Therefore the system has nontrivial solution. The given system of equations equivalent to

$$x + 2y - z = 0$$
$$y - \frac{2}{5}z = 0$$

Choose $n - r = 3 - 2 = 1$ variable arbitrarily. Let $z = t$. Then $y = \frac{2}{5}t$ and $x = \frac{1}{5}t$.

1.2.8 *Gauss - Jordan method to find inverse of a matrix

Using this method we can compute the inverse of a non-singular matrix using elementary row transformations. Let A be a non-singular square matrix of order n. Let B be the inverse of A, where $|A| \neq 0$. Then $AB = I$, identity matrix. Then using a finite sequence of elementary row operations, we can transform the A into it equivalent normal form. Since A is a non-singular matrix, its normal form will be the identity matrix of order n.

Let $R_1, R_2, \ldots R_m$ be the elementary row operations which when applied on A transforms it into the identity matrix.

That is, $R_m \cdots R_3 R_2 R_1 A = I$. Applying these operations, we have

$$\begin{aligned} AB &= I \\ \Longrightarrow R_m \cdots R_3 R_2 R_1(AB) &= R_m \cdots R_3 R_2 R_1(I) \\ \Longrightarrow (R_m \cdots R_3 R_2 R_1 A)B &= R_m \cdots R_3 R_2 R_1 \\ \Longrightarrow B &= R_m \cdots R_3 R_2 R_1 \end{aligned}$$

which is the required inverse of the matrix A

1.2.9 *Rank of a matrix and linearly (in)dependent vectors

The above problems reveal that determining whether a set of vectors be linearly dependent or independent reduces to solving a system of homogeneous equations.

Suppose we want to check whether the vectors $u_1 = (a_1, b_1, c_1)$, $u_2 = (a_2, b_2, c_2)$ and $u_3 = (a_3, b_3, c_3)$ be linearly dependent or not. Then we will consider the equation

$$p\,u_1 + q\,u_2 + r\,u_3 = (0, 0, 0)$$

for finding the possible values of the scalars p, q and r which will satisfy the equation. Since each vector has three components, equating the components on both sides of the equation, we get three equations:

$$p\,a_1 + q\,a_2 + r\,a_3 = 0$$
$$p\,b_1 + q\,b_2 + r\,b_3 = 0$$
$$p\,c_1 + q\,c_2 + r\,c_3 = 0$$

This system of homogeneous equations can be expressed in the matrix form as

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is of the form

$$AX = 0$$

representing a system of homogeneous equations, where the coefficient matrix A is obtained by taking each vector as a column. In the present context existence of non-trivial solutions for $AX = 0$ will imply that the given set of vectors are linearly dependent and existence of no solution other than trivial solution will imply that the set of vectors are linearly independent. We know that the system

$$AX = 0$$

will have trivial solution if and only if the rank of the matrix is same as the number of rows or columns since it is a square matrix. Since rank of a matrix and its transpose are same, we have the following theorem.

Theorem 1.2.35. A set of m vectors $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_m}\}$ are linearly independent if the rank of the matrix A is equal to the number of rows of the matrix, where A is a matrix obtained by taking each vector as a row. The vectors are linearly dependent if the rank of A is less than the number of rows of A.

It the vectors are linearly dependent, then one of the vectors can be expressed as a linear combination of the remaining vectors with atleast one non-zero scalar.

Example 1.2.36. *Show that the vectors* $(1,1,1), (1,2,3)$ *and* $(2,-1,1)$ *are linearly independent.*

Let

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \\ &\equiv \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \\ &\equiv \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad R_3 \to R_3 + 3R_2 \\ &\equiv \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to \frac{1}{5}R_3 \end{aligned}$$

Since there are three non-zero rows in the echelon form and is equal to the number of vectors, the given vectors are linearly independent.

Example 1.2.37. *Prove that the vectors* $(1,2,1), (2,1,4)$ *and* $(4,5,6)$ *are linearly dependent in* $\mathbb{R}^3$ *and also find a relation connecting them.*

Let $X_1 = (1,2,1), X_2 = (2,1,4)$ and $X_3 = (4,5,6)$, and

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A \equiv \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \end{bmatrix} \quad \begin{matrix} X_2' = X_2 - 2X_1 \\ X_3' = X_3 - 4X_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & -3 & 2 \end{bmatrix} \quad X_2'' = -\frac{1}{3}X_2'$$

$$\equiv \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \quad X_3'' = X_3' + 3X_2''$$

This shows that the third row of the matrix can expressed as a linear combination of the first two rows (or since the rank is less than the number of rows) and so the given vectors are linearly dependent. Observe that

$$\begin{aligned} \implies X_3'' &= 0 \\ X_3' + 3X_2'' &= 0 \\ X_3 - 4X_1 + 3\left(-\frac{1}{3}X_2'\right) &= 0 \\ X_3 - 4X_1 - (X_2 - 2X_1) &= 0 \\ X_3 - 2X_1 - X_2 &= 0 \end{aligned}$$

This is the required relationship connecting the vectors.

Example 1.2.38. *Prove that the vectors* $(2,-1,3,2), (1,3,4,2)$ *and* $(3,-5,2,2)$ *are linearly dependent in* $\mathbb{R}^4$ *and also find a relation connecting them.*

Let $X_1 = (2,-1,3,2), X_2 = (1,3,4,2)$ and $X_3 = (3,-5,2,2)$, and

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 3 & 4 & 2 \\ 3 & -5 & 2 & 2 \end{bmatrix}$$

$$A \equiv \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \end{bmatrix} \quad \begin{matrix} X_1' = X_2 \\ X_2' = X_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \end{bmatrix} \quad \begin{matrix} X_2'' = X_2' - 2X_1' \\ X_3' = X_3 - 3X_1' \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & \frac{5}{7} & \frac{2}{7} \\ 0 & -14 & -10 & -4 \end{bmatrix} \quad X_2''' = -\frac{1}{7}X_2''$$

$$\equiv \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & \frac{5}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_3'' = X_3' + 14X_2'''$$

This shows that the third row of the matrix can expressed as a linear combination of the first two rows (or since the rank is less than the number of rows) and so the given vectors are linearly dependent. Note that

$$\begin{aligned} \implies X_3'' &= 0 \\ X_3' + 14X_2''' &= 0 \\ X_3 - 3X_1' + 14\left(-\frac{1}{7}X_2''\right) &= 0 \\ X_3 - 3X_2 - 2(X_2' - 2X_1') &= 0 \\ X_3 - 3X_2 - 2(X_1 - 2X_2) &= 0 \\ X_3 + X_2 - 2X_1 &= 0 \end{aligned}$$

which is the required relationship connecting the vectors.

1.2.10 *Row space, column space and null space of a matrix

Consider an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Each row of this matrix can be thought of as a vector and each column can be thought as a vector. So when we are given an $m \times n$ matrix, we can think of m row vectors or n column vectors. Each of these m row vectors are elements of the vector space $\mathbb{R}^n$ and each these n column vectors are elements of the vector space $\mathbb{R}^m$.

We know that if $S = \{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_m}\}$ is a subset of a vector space V, the linear spa of S will be subspace of V (proof is given in the next section). So the linear span of ro vectors of the matrix A is a subspace of $\mathbb{R}^n$ and the linear span of the column vecto of the matrix A is a subspace of $\mathbb{R}^m$. These two subspaces are respectively called ro **space and column space** of the matrix A.

Definition 1.2.9. *Let A be an $m \times n$ order matrix. The linear span of row vectors of th matrix A is a subspace of $\mathbb{R}^n$ and is called* **Row space** *of the matrix. The linear span the column vectors of the matrix A is a subspace of $\mathbb{R}^m$ and it is called* **Column spac** *of the matrix A.*

Note that if all the rows of A are linearly independent, the row space will be same a $\mathbb{R}^n$ and if all the column vectors are linearly independent, the column space will be sam as $\mathbb{R}^m$.

Number of linearly independent rows is same as the rank of the matri Since a matrix and its transpose have the same rank, we get that row spac and column space will have the same dimension which is equal to the rank the matrix.

Another important subspace related to a matrix is the **null space** of a matrix. W give the formal definition below:

Definition 1.2.10. *Let A be an $m \times n$ matrix of real numbers. The* **null space** *of th matrix A denoted by $Null(A)$ is the set of all vectors $X = (x_1, x_2, \ldots, x_n)^T$ such that*

$$AX = 0$$

That is, the null space of an $m \times n$ matrix A is set of solutions of the homogeneo equation $AX = 0$. So we have

$$Null(A) = \{X \in \mathbb{R}^n : AX = 0\}$$

Method for finding basis for Null space, row space and column space: Sinc row space is spanned by the row vectors of a matrix, we will choose maximum numbe of linearly independent rows from the matrix and basis for column space is obtained b choosing maximum number of columns from the matrix. This can be done after convertin the matrix into its equivalent row reduced echelon form. The procedure for finding basi of null space is explained in the following examples.

Example 1.2.39. **Find a basis for row space, column space and null space of the matri*

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

$$A \equiv \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\equiv \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} R_2 \to (-1)R_2$$

$$\equiv \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} R_3 \to R_3 + 3R_2$$

$$\equiv \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} R_3 \to -\frac{1}{5}R_3$$

$$\equiv \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 3 \\ R_4 \to R_4 + 13R_3 \end{matrix}$$

The row reduced matrix has three non-zero rows and so the maximal number of linearly independent rows is three. Hence basis for row space is

$$\{(1, 2, 0, 2, 5), (0, 1, -1, -3, -2), (0, 0, 0, 1, 1)\}$$

Since rank of a matrix and its transpose are same, the maximum number of linearly independent columns is also three. Choosing three linearly independent columns, the basis for column space is

$$\{(1, -2, 0, 3), (2, -5, -3, 6), (2, -1, 4, -7)\}$$

To find a basis for the null space, note that Null space is the solution space of $AX = 0$. The system $AX = 0$ is equivalent to

$$x_1 + 2x_2 + 2x_4 + 5x_5 = 0$$
$$x_2 - x_3 - 3x_4 - 2x_5 = 0$$
$$x_4 + x_5 = 0$$

Here we can choose $n - r = 5 - 3 = 2$ variables arbitrarily. Let $x_3 = t$ and $x_5 = s$. Then $x_4 = -s, x_2 = t - s, x_1 = -2t - s$.

$$\begin{aligned}
X &= (-2t-s, t-s, t, -s, s)\\
&= (-2t, t, t, 0, 0) + (-s, -s, 0, -s, s)\\
&= t(-2,1,1,0,0) + s(-1,-1,0,-1,1)
\end{aligned}$$

$$\begin{aligned}
\text{Let}\quad B &= \begin{bmatrix} -2 & 1 & 1 & 0 & 0\\ -1 & -1 & 0 & -1 & 1 \end{bmatrix}\\
&\equiv \begin{bmatrix} -2 & 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 1 & -1 \end{bmatrix} R_2 \to (-1)R_2\\
&\equiv \begin{bmatrix} 1 & 1 & 0 & 1 & -1\\ -2 & 1 & 1 & 0 & 0 \end{bmatrix} R_1 \leftrightarrow R_2\\
&\equiv \begin{bmatrix} 1 & 1 & 0 & 1 & -1\\ 0 & 3 & 1 & 2 & -2 \end{bmatrix} R_2 \to R_2 + 2R_1\\
&\equiv \begin{bmatrix} 1 & 1 & 0 & 1 & -1\\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} R_2 \to \frac{1}{3}R_2
\end{aligned}$$

Since rank is same as the number linearly independent rows of row reduced echelon form of the given matrix, the vectors $(-2,1,1,0,0), (-1,-1,0,-1,1)$ are linearly independent. Therefore basis for null space $= \{(-2,1,1,0,0), (-1,-1,0,-1,1)\}$.

Example 1.2.40. **Find a basis for row space, column space and null space of the matr*

$$A = \begin{bmatrix} -1 & 2 & -1 & 5 & 6\\ 4 & -4 & -4 & -12 & -8\\ 2 & 0 & -6 & -2 & 4\\ -3 & 1 & 7 & -2 & 12 \end{bmatrix}$$

$$\begin{aligned}
A &\equiv \begin{bmatrix} 1 & -2 & 1 & -5 & -6\\ 4 & -4 & -4 & -12 & -8\\ 2 & 0 & -6 & -2 & 4\\ -3 & 1 & 7 & -2 & 12 \end{bmatrix} R_1 \to (-1)R_1\\
&\equiv \begin{bmatrix} 1 & -2 & 1 & -5 & -6\\ 0 & 4 & -8 & 8 & 16\\ 0 & 4 & -8 & 8 & 16\\ 0 & -5 & 10 & -17 & -6 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 4R_1\\ R_3 \to R_3 - 2R_1\\ R_4 \to R_4 + 3R_1 \end{matrix}\\
&\equiv \begin{bmatrix} 1 & -2 & 1 & -5 & -6\\ 0 & 1 & -2 & 2 & 4\\ 0 & 4 & -8 & 8 & 16\\ 0 & -5 & 10 & -17 & -6 \end{bmatrix} R_2 \to \frac{1}{4}R_2
\end{aligned}$$

$$\begin{aligned}
&\equiv \begin{bmatrix} 1 & -2 & 1 & -5 & -6\\ 0 & 1 & -2 & 2 & 4\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -7 & 14 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 4R_2\\ R_4 \to R_4 + 5R_2 \end{matrix}\\
&\equiv \begin{bmatrix} 1 & -2 & 1 & -5 & -6\\ 0 & 1 & -2 & 2 & 4\\ 0 & 0 & 0 & -7 & 14\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \leftrightarrow R_4\\
&\equiv \begin{bmatrix} 1 & -2 & 1 & -5 & -6\\ 0 & 1 & -2 & 2 & 4\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to -\frac{1}{7}R_3
\end{aligned}$$

The row reduced matrix has three non-zero rows and so the maximal number of linearly independent rows is three. Hence basis for row space is

$$\{(1,-2,1,-5,-6), (0,1,-2,2,4), (0,0,0,1,-2)\}$$

Since rank of a matrix and its transpose are same, the maximum number of linearly independent columns is also three. Choosing three linearly independent columns, the basis for column space is

$$\{(-1,4,2,-3), (2,-4,0,1), (5,-12,-2,-2)\}$$

To find a basis for the null space, note that Null space is the solution space of $AX = 0$. The system $AX = 0$ is equivalent to

$$\begin{aligned}
x_1 - 2x_2 + x_3 - 5x_4 - 6x_5 &= 0\\
x_2 - 2x_3 + 2x_4 + 4x_5 &= 0\\
x_4 - 2x_5 &= 0
\end{aligned}$$

Here we can choose $n - r = 5 - 3 = 2$ variables arbitrarily. Let $x_3 = t$ and $x_5 = s$. Then $x_4 = 2s, x_2 = 2t - 8s, x_1 = 3t$.

$$\begin{aligned}
X &= (3t, 2t - 8s, t, 2s, s)\\
&= (3t, 2t, t, 0, 0) + (0, 0, -8s, 0, 2s, s)\\
&= t(3,2,1,0,0) + s(0,-8,0,2,1)
\end{aligned}$$

$$\text{Let}\quad B = \begin{bmatrix} 3 & 2 & 1 & 0 & 0\\ 0 & -8 & 0 & 2 & 1 \end{bmatrix}$$

Clearly the vectors $(3,2,1,0,0), (0,-8,0,2,1)$ are linearly independent. Therefore basis for null space $= \{(3,2,1,0,0), (0,-8,0,2,1)\}$.

Example 1.2.41. *Find a basis for the null space of*

$$A = \begin{bmatrix} 2 & -2 & 0\\ 0 & 4 & 8\\ 2 & 0 & 4 \end{bmatrix}$$

[KTU. March 2017]

Null space is the solution space of $AX = 0$.

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 8 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 8 \\ 0 & 2 & 4 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\equiv \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 1/2R_2$$

The system $AX = 0$ is equivalent to

$$2x - 2y = 0$$
$$4y + 8z = 0$$

Here we can choose $n - r = 3 - 3 = 1$ variables arbitrarily. Let $z = t$. Then $y = -2t$ $x = -2t$. Therefore $X = \begin{bmatrix} -2t \\ -2t \\ t \end{bmatrix}$. Therefore basis for null space= $\{(2, 2, -1)\}$