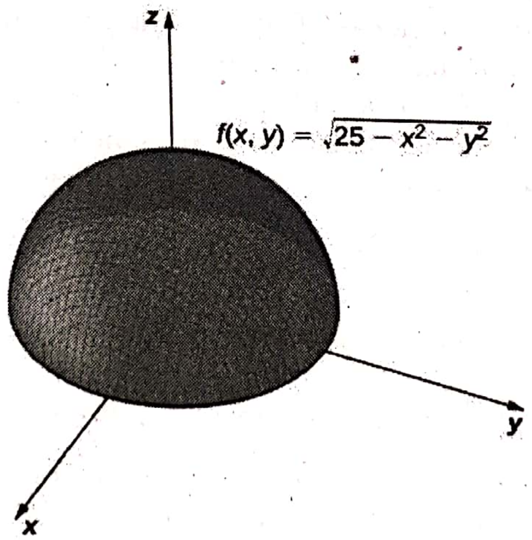
Chapter 2

Multivariable Calculus: Differentiation

2.1 Partial Derivative

The physical property of a system may depend on more than one variable. For example when we analyse the relation among volume, pressure and temperature, we can see that each quantity is depending on the other two quantities. Suppose we want to study the variation of one characteristic, say, pressure when volume is kept constant and temperature is allowed to change. This kind of situation lead us to the idea of partial differentiation of function of more than one variable. A simple example of graph of a function of two variables is given below.



2.1.1 Limit and Continuity

Let $z = f(x, y)$ be a function of two independent characteristics (independent variables) x and y. The domain of variation of the variables x and y is a subset of two dimensional plane. The point (x, y) can approach a specified point (a, b) along an infinite number of paths. In the case of functions $f(x)$ of one variable, the variable x can change along two direction. When we want to compute the limit or check the continuity or compute the derivative of a function $f(x)$ at a point a, the variable x can approach the point a from either the left or from right. Accordingly we compute the left hand limit and right hand limit. Then we will say the function $f(x)$ has a limit l at a if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = l$$

and it is said to be continuous at a if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

If we want to compute the derivative of the function $y = f(x)$ at the point $x = a$, compute the limit of the ratio

$$\frac{f(x) - f(a)}{x - a}$$

as x varies from either side of the point a.

When both the limits are finite and equal

$$\lim_{x\to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a^-} \frac{f(x) - f(a)}{x - a}$$

we will say the function has a derivative at that point a.

So when we measure some characters of a physical quantity at some point (a,b) should be independent of the path along which the point (x,y) approach the specific point (a,b). If the function $f(x,y)$ has different values at the point (a,b) when (x,y) approach the point (a,b) along different paths, uniqueness of the limit is lost. With this background we can develop ideas of limit, continuity, partial derivative and total derivative of function of more than one variables.

Let $f(x,y)$ be a function of two variables x and y defined in some region D in the two dimensional plane. Let (a,b) be a point in the two dimensional plane (not necessarily point in D). A real number L is said to be the **limit** of the function $f(x,y)$ at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

where the limit is independent of the path taken by the point (x,y) to approach the point (a,b).

Using Mathematical symbols this is expressed as follows:

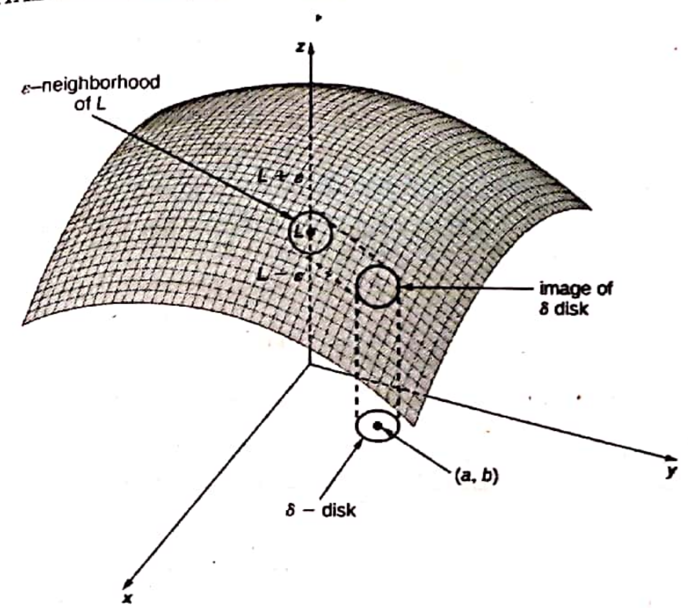
Definition 2.1.1. *A function $f(x,y)$ is said to have a* **limit** *at the point (a,b) if for every given $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $|(x,y) - (a,b)| < \delta$, we have*

$$|f(x,y) - L| < \epsilon$$

That is, when the distance between the points (x,y) and (a,b) is less than δ, the distance between $f(x,y)$ and L is no more than ϵ.

Symbolically this is expressed as

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$



Definition 2.1.2. *A function $f(x,y)$ is said to be* **continuous** *at the point (a,b) if for every given $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $|(x,y) - (a,b)| < \delta$, we have*

$$|f(x,y) - f(a,b)| < \epsilon$$

Symbolically this is expressed as

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Remark:1 Note that when we find the limit of a function at a point, the function need not be defined at that point and even point need not be in the domain at all. But when we define continuity of a function at a point, the function should be defined at that point.

Remark: 2 The distance between the points (x,y) and (a,b) is the usual Euclidean distance defined by

$$|(x,y) - (a,b)| = \sqrt{(x-a)^2 + (y-b)^2}$$

Remark: 3 Limit of linear combination of functions is equal to linear combination of limit of the functions. That is,

$$\lim\,(\alpha\, f(x,y) + \beta\, g(x,y)) = \alpha\, \lim f(x,y) + \beta \lim g(x,y)$$

Remark: 4 Limit of product (and ratio) of functions is equal to the product (ratio with non-zero denominator) of limit of the functions. The above results also true for continuous functions: linear combination of continuous functions is continuous, product and quotient of continuous functions is continuous where ever they are defined.

Example 2.1.1. *Show that the functions (a)* $\frac{2xy}{3x^2+y^2}$ *and (b)* $\frac{4xy^2}{x^2+3y^4}$ *do not have limit at the origin.*

Solution: (a) Let $f(x,y) = \frac{2xy}{3x^2+y^2}$. We shall compute the limit of this function at the origin along two different paths. First we approach the origin along the line $x = 0$, that is, along the y-axis.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{2xy}{3x^2+y^2} = \lim_{y\to 0} \frac{2.0.y}{3.0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} =$$

Secondly, we take a path along the line $y = x$, which passes through the origin.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{2x^2}{3x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{4x^2} = \frac{1}{2}$$

In both cases, we are approaching the origin, but the function takes different values and so the function does not have limit at the origin.

(b) For the function $\frac{4xy^2}{x^2+3y^4}$, we choose two paths $y = 0$ and $x = y^2$ through the origin. We can see that the two limits are different and so the function does not possess limit at the origin.

Remark:1 Note that these functions are discontinuous at the origin. So these are example of discontinuous functions of more than one variables. Any polynomial function of more than one variables are always continuous functions.

Remark:2 Ratio of two polynomials functions in one or more than one variables is called **rational functions**. Rational functions are always continuous at those points where the denominator is non-zero. At points where the denominator vanishes, the rational function may or may not be continuous.

2.1.2 Partial Derivative

Let $z = f(x,y)$ be a function giving pressure as a function of temperature and volume. Suppose we want to analyse the instantaneous rate of change of pressure at a point (a,b) when volume $y = b$ is kept constant and temperature x is allowed to vary. When y is kept constant $z = f(x,y)$ becomes a function of x alone.

The derivative $z = f(x,y)$ when y is kept constant and x is allowed to change is called **first order partial derivative of $z = f(x,y)$ with respect to x** and is given

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}\right) \quad (2.1)$$

provided the limit is finite and unique. We use the following notations $\frac{\partial z}{\partial x}$ or $f_x(x,y)$ or $\frac{\partial f}{\partial x}$ to denote first order partial derivative of $z = f(x,y)$ with respect to x.

The derivative of $z = f(x,y)$ when x is kept constant and y is allowed to change is called **first order partial derivative of $z = f(x,y)$ with respect to y** and is given by

$$\frac{\partial z}{\partial y} = \lim_{\Delta y\to 0}\left(\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}\right) \quad (2.2)$$

provided the limit is finite and unique. We use the following notations $\frac{\partial z}{\partial y}$ or $f_y(x,y)$ or $\frac{\partial f}{\partial y}$ to denote first order partial derivative of $z = f(x,y)$ with respect to y. The first order partial derivative at a specific point (x_0,y_0) with respect to x or y can be computed by taking $x = x_0$ and $y = y_0$ in the above formulae.

Example 2.1.2. *Let $f(x,y) = 5x^3y^2+2x^2-3y^2+7x+3y$. Find (i) $f_x(x,y)$ (ii) $f_y(x,y)$ (iii) $f_x(1,y)$ (iv) $f_x(x,1)$ (v) $f_y(1,y)$ (vi) $f_y(x,1)$ (vii) $f_x(1,2)$ (viii) $f_y(1,2)$*

$$f(x,y) = 5x^3y^2+2x^2-3y^2+7x+3y$$

$$(i)\quad f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = 15x^2y^2+4x+7$$

$$(ii)\quad f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = 10x^3y-6y+3$$

$$\begin{aligned}(iii)\quad f_x(1,y) &= \frac{\partial f}{\partial x}(1,y)\\ &= 15x^2y^2+4x+7\Big]_{(x=1,y=y)}\\ &= 15y^2+4+7\\ &= 15y^2+11\end{aligned}$$

$$\begin{aligned}(iv)\quad f_x(x,1) &= \frac{\partial f}{\partial x}(x,1)\\ &= 15x^2y^2+4x+7\Big]_{(x=x,y=1)}\\ &= 15x^2+4x+7\end{aligned}$$

$$\begin{aligned}(v)\quad f_y(1,y) &= \frac{\partial f}{\partial y}(1,y)\\ &= 10x^3y-6y+3\Big]_{(x=1,y=y)}\\ &= 10y-6y+3\\ &= 4y+3\end{aligned}$$

$$\begin{aligned}(vi)\quad f_y(x,1) &= \frac{\partial f}{\partial y}(x,1)\\ &= 10x^3y-6y+3\Big]_{(x=x,y=1)}\\ &= 10x^3-6+3\\ &= 10x^3-3\end{aligned}$$

(vii)
$$f_x(1,2) = \frac{\partial f}{\partial x}(1,2)$$
$$= 15x^2y^2 + 4x + 7\Big]_{(x=1,y=2)}$$
$$= 15(2)^2 + 4 + 7$$
$$= 71$$

(viii)
$$f_y(1,2) = \frac{\partial f}{\partial y}(1,2)$$
$$= 10x^3y - 6y + 3\Big]_{(x=1,y=2)}$$
$$= 10(2) - 6(2) + 3$$
$$= 11$$

Example 2.1.3. *Let $f(x,y) = \sin(5x + 3xy + 2y^2)$. Find $f_x(x,y)$ and $f_y(x,y)$*

$$f(x,y) = \sin(5x + 3xy + 2y^2)$$
$$f_x(x,y) = \cos(5x + 3xy + 2y^2) \times (5 + 3y)$$
$$= (5 + 3y)\cos(5x + 3xy + 2y^2)$$
$$f_y(x,y) = \cos(5x + 3xy + 2y^2) \times (3x + 4y)$$
$$= (3x + 4y)\cos(5x + 3xy + 2y^2)$$

Example 2.1.4. *Let $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x,y) \neq 0 \\ 0, & (x,y) = (0,0) \end{cases}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at origin*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0}$$
$$= \lim_{x \to 0} \frac{\left(\frac{x^3}{x^2}\right) - 0}{x}$$
$$= 1$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0}$$
$$= \lim_{y \to 0} \frac{\left(-\frac{y^3}{y^2}\right) - 0}{y}$$
$$= -1$$

Example 2.1.5. *Let $f(x,y) = e^{x^2y^3}\sin(x^2 - y^2)$. Find $f_x(x,y)$ and $f_y(x,y)$.*

$$f(x,y) = e^{x^2y^3}\sin(x^2 - y^2)$$
$$f_x(x,y) = e^{x^2y^3}\Big(\cos(x^2 - y^2) \times (2x)\Big) + \sin(x^2 - y^2)\Big(e^{x^2y^3} \times y^3(2x)\Big)$$
$$= 2xe^{x^2y^3}\cos(x^2 - y^2) + 2xy^3e^{x^2y^3}\sin(x^2 - y^2)$$
$$f_y(x,y) = e^{x^2y^3}\Big(\cos(x^2 - y^2) \times (-2y)\Big) + \sin(x^2 - y^2)\Big(e^{x^2y^3} \times x^2(3y^2)\Big)$$
$$= -2ye^{x^2y^3}\cos(x^2 - y^2) + 3x^2y^2e^{x^2y^3}\sin(x^2 - y^2)$$

Example 2.1.6. *Let $f(x,y) = \dfrac{x - y}{x^2 + y^2}$. Find $f_x(x,y)$ and $f_y(x,y)$.*

$$f(x,y) = \frac{x - y}{x^2 + y^2}$$
$$f_x(x,y) = \frac{(x^2 + y^2)(1) - (x - y)(2x)}{(x^2 + y^2)^2}$$
$$= \frac{y^2 + 2xy - x^2}{(x^2 + y^2)^2}$$
$$f_y(x,y) = \frac{(x^2 + y^2)(-1) - (x - y)(2y)}{(x^2 + y^2)^2}$$
$$= \frac{y^2 - 2xy - x^2}{(x^2 + y^2)^2}$$

2.2 Implicit partial differentiation

We can write a given function either in explicit form or in implicit form. A function expressed as $z = f(x,y)$ is said to be in the explicit form and if it is in the form $\phi(x,y,z) = 0$, it is said to be in its implicit form. For example $y^2 + xz - yz = 0$ is a function in implicit form and this function can expressed in the explicit form as $z = \dfrac{y^2}{y - x}$.

We need not express the function in explicit form to find the partial derivative. We can simply differentiate the implicit function with respect to x or y. While taking partial derivative, we treat z as a function of x and y. One advantage of implicit differentiation is we can find both slopes $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in an implicit differentiation.

Example 2.2.1. *Consider the function $y^2 + xz - yz = 0$.*

Differentiating partially with respect to x, we get

$$0 + z + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} = 0$$

Solving for $\frac{\partial z}{\partial x}$, we get

$$\frac{\partial z}{\partial x} = \frac{z}{y-x}$$

Differentiating the given function partially with respect to y, we get,

$$2y + +x\frac{\partial z}{\partial y} - z - y\frac{\partial z}{\partial y} = 0$$

Solving for $\frac{\partial z}{\partial y}$, we get

$$\frac{\partial z}{\partial y} = \frac{z-2y}{x-y}$$

Remark1: Using these relations,we can compute both slopes at the given point.

Remark2: Partial derivatives of functions of more than two variables can be computed analogously.

Let $w = f(x, y, z)$ be a function of three variables x, y and z. Keeping two of the variables constant, we can compute the different partial derivatives such as $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$.

Example 2.2.2. *If $f(x, y, z) = 5x^2y^3z^2$, find $f_x(x, y, z)$, $f_y(x, y, z)$ and $f_z(x, y, z)$.*

$$f(x, y, z) = 5x^2y^3z^2$$
$$f_x(x, y, z) = 10xy^3z^2$$
$$f_y(x, y, z) = 15x^2y^2z^2$$
$$f_z(x, y, z) = 10x^2y^3z$$

Example 2.2.3. *If $xy + yz + zx = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where x and y are independent variables.*

Given that

$$xy + yz + zx = 0$$

Differentiating partially with respect to x.

$$y + y\frac{\partial z}{\partial x} + z(1) + x\frac{\partial z}{\partial x} = 0$$
$$(y+x)\frac{\partial z}{\partial x} = -(y+z)$$
$$\frac{\partial z}{\partial x} = -\frac{(y+z)}{(x+y)}$$

Similarly, $$\frac{\partial z}{\partial y} = -\frac{(x+z)}{(x+y)}$$

Example 2.2.4. *Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\log(2x^2+y-z^3) = 2x$, where x and y are independent variables.*

Given that

$$\log(2x^2 + y - z^3) = 2x \qquad (1)$$

Differentiating partially with respect to x.

$$\frac{1}{2x^2+y-z^3}\left(4x - 3z^2\frac{\partial z}{\partial x}\right) = 2$$
$$4x - 3z^2\frac{\partial z}{\partial x} = 2(2x^2 + y - z^3)$$
$$-3z^2\frac{\partial z}{\partial x} = 4x^2 + 2y - 2z^3 - 4x$$
$$\frac{\partial z}{\partial x} = -\frac{(4x^2 + 2y - 2z^3 - 4x)}{3z^2}$$

Differentiating (2) partially with respect to y.

$$\frac{1}{2x^2+y-z^3}\left(1 - 3z^2\frac{\partial z}{\partial y}\right) = 0$$
$$1 - 3z^2\frac{\partial z}{\partial y} = 0$$
$$\frac{\partial z}{\partial y} = \frac{1}{3z^2}$$

Example 2.2.5. *Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2 + z\sin(xyz) = 0$, where x and y are independent variables.*

Given that

$$x^2 + z\sin(xyz) = 0 \qquad (1)$$

Differentiating partially with respect to x.

$$2x + z\left\{\cos(xyz)\left[y\left(x\frac{\partial z}{\partial x} + z\right)\right]\right\} + \sin(xyz)\frac{\partial z}{\partial x} = 0$$
$$2x + z^2y\cos(xyz) + \frac{\partial z}{\partial x}[xyz\cos(xyz) + \sin(xyz)] = 0$$
$$\frac{\partial z}{\partial x} = -\frac{[2x + z^2y\cos(xyz)]}{[xyz\cos(xyz) + \sin(xyz)]}$$

Differentiating (2) partially with respect to y.

$$z\left\{\cos(xyz)\left[x\left(y\frac{\partial z}{\partial y} + z\right)\right]\right\} + \sin(xyz)\frac{\partial z}{\partial y} = 0$$
$$z^2x\cos(xyz) + \frac{\partial z}{\partial y}[xyz\cos(xyz) + \sin(xyz)] = 0$$
$$\frac{\partial z}{\partial y} = -\frac{z^2x\cos(xyz)}{[xyz\cos(xyz) + \sin(xyz)]}$$

Example 2.2.6. *If $f(x,y,z) = \sin\left(\frac{xy}{z}\right)$, find $f_x(x,y,z), f_y(x,y,z)$ and $f_z(x,y,z)$.*

Given

$$f(x,y,z) = \sin\left(\frac{xy}{z}\right)$$
$$f_x(x,y,z) = \frac{y}{z}\cos\left(\frac{xy}{z}\right)$$
$$f_y(x,y,z) = \frac{x}{z}\cos\left(\frac{xy}{z}\right)$$
$$f_z(x,y,z) = -\frac{xy}{z^2}\cos\left(\frac{xy}{z}\right)$$

Example 2.2.7. *If $u = \log(\tan x + \tan y + \tan z)$, prove that*

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$$

$$u = \log(\tan x + \tan y + \tan z)$$
$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}\sec^2 x$$
$$\sin 2x\frac{\partial u}{\partial x} = 2\sin x\cos x \times \frac{1}{\tan x + \tan y + \tan z} \times \frac{1}{\cos^2 x}$$
$$= \frac{2\tan x}{\tan x + \tan y + \tan z}$$

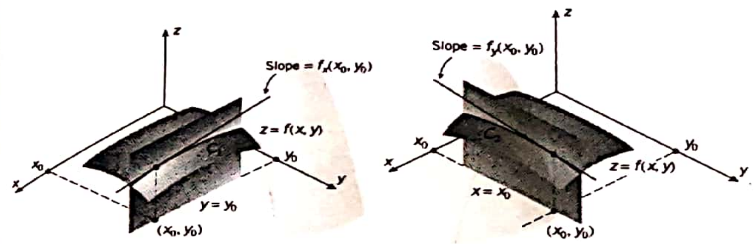
Similarly, $\sin 2y\frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z}$

$$\sin 2z\frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = \frac{2\tan x}{\tan x + \tan y + \tan z} + \frac{2\tan y}{\tan x + \tan y + \tan z} + \frac{2\tan z}{\tan x + \tan y + \tan z}$$
$$= \frac{2}{\tan x + \tan y + \tan z}(\tan x + \tan y + \tan z)$$
$$= 2$$

2.3 Interpretations of Partial Derivative

Geometrically $z = f(x,y)$ is a surface in three dimensional space. When we take $y =$ a constant, then we are taking the intersection of the surface $z = f(x,y)$ and the pla



$y = y_0$, which is parallel to the XOZ plane, the figure we get is a curve in space whose tangent is parallel to XOZ plane.

Then the partial derivative $\frac{\partial z}{\partial x}$ measure the instantaneous rate of change z tangential to the curve. The value of the partial derivative gives slope of the tangent to the curve. When we compute $\frac{\partial z}{\partial y}$, it measure the rate of change of z with respect to y along a curve which lie in a plane parallel to the plane YOZ.

Example 2.3.1. *Find the slopes of the surface $z = 10 - 4x^2 - y^2$ at the point $(1,2,2)$ in the $x-$direction and $y-$direction(slopes of the traces to the surface at $(1,2)$.*

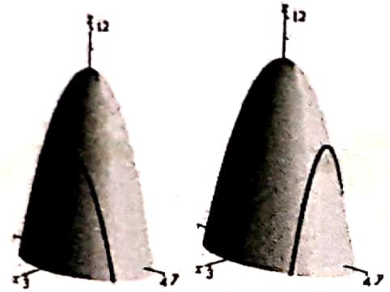
The slopes of the surface $z = 10 - 4x^2 - y^2$ at the point $(1,2,2)$ in the $x-$direction= $\left.\frac{\partial z}{\partial x}\right]_{(1,2,2)}$

Now, $$z = 10 - 4x^2 - y^2 \implies \frac{\partial z}{\partial x} = -8x$$
$$\left.\frac{\partial z}{\partial x}\right]_{(1,2,2)} = -8$$

The slopes of the surface $z = 10 - 4x^2 - y^2$ at the point $(1,2,2)$ in the $y-$direction= $\left.\frac{\partial z}{\partial y}\right]_{(1,2,2)}$

Now, $$z = 10 - 4x^2 - y^2 \implies \frac{\partial z}{\partial y} = -2y$$
$$\left.\frac{\partial z}{\partial y}\right]_{(1,2,2)} = -4$$

(So, the tangent line at $(1,2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $y = 2$ has a slope of -8. Also the tangent line at $(1,2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $x = 1$ has a slope of -4. Here the partial derivative with respect to x and y are negative and so the function is decreasing at $(1,2)$ as we vary x and y fixed, and as we vary y and hold x fixed.)



Trace for $x = 1$ Trace for $y = 2$

Example 2.3.2. *Find the slopes of the sphere $x^2+y^2+z^2 = 14$ in the $y-$direction a points $(1,2,3)$ and $(1,2,-3)$.*

The slopes of the surface $x^2+y^2+z^2 = 14$ at the point $(1,2,3)$ in the $y-$direc $\left.\frac{\partial z}{\partial y}\right]_{(1,2,3)}$

Given, $x^2+y^2+z^2 = 14$

Differentiating (1) with respect to y.

$$2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\left.\frac{\partial z}{\partial y}\right]_{(1,2,3)} = -\frac{2}{3}$$

The slopes of the surface $x^2+y^2+z^2 = 14$ at the point $(1,2,-3)$ in the $y-$directio

$$= \left.\frac{\partial z}{\partial y}\right]_{(1,2,-3)} = -\left.\frac{y}{z}\right]_{(1,2,-3)} = \frac{2}{3}$$

(The point $(1,2,3)$ lies on the upper hemisphere $z = \sqrt{14-x^2-y^2}$ and the poi $(1,2,-3)$ lies on the lower hemisphere $z = -\sqrt{14-x^2-y^2}$. Since the partial deriv tive with respect to y at $(1,2,3)$ is negative, the function is decreasing at $(1,2,3)$ as vary y and hold x fixed. Also the partial derivative with respect to y at $(1,2,-3)$ i positive, the function is increasing at $(1,2,-3)$ as we vary y and hold x fixed.)

Example 2.3.3. *Find the rate of change of $z = \frac{1}{x+y}$ with respect to x at the poi $(-1,4)$ with y held fixed.*

The rate of change of $z = \frac{1}{x+y}$ with respect to x at the point $(-1,4)$ with y hel fixed $= \left.\frac{\partial z}{\partial x}\right]_{(-1,4)}$

Given,

$$z = \frac{1}{x+y}$$

$$\frac{\partial z}{\partial x} = (-1)(x+y)^{-2} = -\frac{1}{(x+y)^2}$$

$$\left.\frac{\partial z}{\partial x}\right]_{(-1,4)} = -\frac{1}{9}$$

Example 2.3.4. *Find the rate of change of $z = \sin(y^2-4x)$ with respect to y at the point $(3,1)$ with x held fixed.*

The rate of change of $z = \sin(y^2-4x)$ with respect to y at the point $(3,1)$ with x held fixed $= \left.\frac{\partial z}{\partial y}\right]_{(3,1)}$

Given,

$$z = \sin(y^2-4x)$$

$$\frac{\partial z}{\partial y} = 2y\cos(y^2-4x)$$

$$\left.\frac{\partial z}{\partial y}\right]_{(3,1)} = 2\cos(-11) = 2\cos(11)$$

Example 2.3.5. *A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the the plane $y = 1$. At what rate z changes with respect to x when the point is at $(3,1,12)$.*

Given,

$$z = x^2+3y^2$$

$$\frac{\partial z}{\partial x} = 2x$$

$$\left.\frac{\partial z}{\partial x}\right]_{(3,1,12)} = 6$$

Therefore rate of change of z with respect to x when the point is at $(3,1,12)$ with y held fixed at 2 is 6.

2.4 Higher order Partial Derivatives

Consider the first order partial derivative of $z = f(x.y)$ with respect to x given by

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}\right) \tag{2.3}$$

Treating the partial derivative as an operator, we may note that $\frac{\partial z}{\partial x}$ is equivalent to operating $\frac{\partial}{\partial x}$ over $z = f(x,y)$ and $\frac{\partial z}{\partial y}$ is equivalent to operating $\frac{\partial}{\partial y}$ over $z = f(x,y)$.

So if we operate $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$ over $\frac{\partial z}{\partial x}$, we get second order partial derivatives. That i

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \lim_{\Delta x \to 0}\left(\frac{\frac{\partial}{\partial x}f(x+\Delta x, y) - \frac{\partial}{\partial x}f(x,y)}{\Delta x}\right) \quad (2$$

provided the limit is unique and finite (That is, finite and independent of path.) Also, can define the second order mixed partial derivative $\frac{\partial^2 z}{\partial y \partial x}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \lim_{\Delta y \to 0}\left(\frac{\frac{\partial}{\partial x}f(x, y+\Delta y) - \frac{\partial}{\partial x}f(x,y)}{\Delta y}\right) \quad (2$$

provided the limit is finite and unique. In a similar way we can define higher order partial derivative of function of two variables and function of three variables.

Remark: let $f(x,y)$ be a function of two variables x and y such that $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are continuous on some open disk, then

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad (2$$

at all points on that disk.

Example 2.4.1. *If $f(x,y) = 2x^5y^3 + 5x + 7y$, find $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$*

$$f(x,y) = 2x^5y^3 + 5x + 7y$$
$$\frac{\partial f}{\partial y} = 6x^5y^2 + 7$$
$$\frac{\partial^2 f}{\partial x \partial y} = 30x^4y^2$$
$$\frac{\partial f}{\partial x} = 10x^4y^3 + 5$$
$$\frac{\partial^2 f}{\partial y \partial x} = 30x^4y^2$$

Remark: Note that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Example 2.4.2. *If $f(x,y) = x^3y + e^{xy^2}$, show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.*

$$f(x,y) = x^3y + e^{xy^2}$$
$$\frac{\partial f}{\partial x} = 3x^2y + y^2e^{xy^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 2xy^3e^{xy^2} + 2ye^{xy^2}$$
$$\frac{\partial f}{\partial y} = x^3 + 2xye^{xy^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 2ye^{xy^2} + +2xy^3e^{xy^2}$$
$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Example 2.4.3. *Let* $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$. *Prove that, at origin* $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$.

Note that the mixed second order partial derivative at the point $(0,0)$ is defined by

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0}\frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x - 0}$$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y \to 0}\frac{f(x,y) - f(x,0)}{y - 0}$$
$$= \lim_{y \to 0}\frac{\frac{xy(x^2-y^2)}{x^2+y^2} - 0}{y}$$
$$= \lim_{y \to 0}\frac{x(x^2-y^2)}{x^2+y^2}$$
$$= \frac{x(x^2)}{x^2} = x$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0}\frac{f(0,y) - f(0,0)}{y - 0}$$
$$= \lim_{y \to 0}\frac{0 - 0}{y}$$
$$= 0$$

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$$\therefore \frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y-0}$$

$$\frac{\partial f}{\partial x}(0,y) = \lim_{x\to 0} \frac{f(x,y)-f(0,y)}{x-0}$$

$$= \lim_{x\to 0} \frac{\frac{xy(x^2-y^2)}{x^2+y^2} - 0}{x}$$

$$= \lim_{x\to 0} \frac{y(x^2-y^2)}{x^2+y^2}$$

$$= \frac{y(-y^2)}{y^2} = -y$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0}$$

$$= \lim_{y\to 0} \frac{0-0}{x}$$

$$= 0$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{-y-0}{y} = -1$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

Example 2.4.4. *If* $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, *show that*

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2} = \frac{\partial^2 u}{\partial y \partial x}$$

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = (x^2)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2} \times y(-1)(x^{-2})\right] + \tan^{-1}\left(\frac{y}{x}\right)(2x) - y^2\left[\frac{1}{1+\left(\frac{x}{y}\right)^2} \times \frac{1}{y}\right]$$

$$= -\frac{x^2 y}{x^2+y^2} + 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{y^3}{x^2+y^2}$$

$$= -\frac{y(x^2+y^2)}{x^2+y^2} + 2x\tan^{-1}\left(\frac{y}{x}\right)$$

$$= -y + 2x\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 u}{\partial y \partial x} = -1 + 2x\left[\frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}\right]$$

$$= -1 + \frac{2x^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = (x^2)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}\right] - \left\{(y^2)\left[\frac{1}{1+\left(\frac{x}{y}\right)^2} \times x(-1)y^{-2}\right] + \tan^{-1}\left(\frac{x}{y}\right)(2y)\right\}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - 2y\left[\frac{1}{1+\left(\frac{x}{y}\right)^2} \times \frac{1}{y}\right]$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2} = \frac{\partial^2 u}{\partial y \partial x}$$

Example 2.4.5. *If* $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, *prove that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[x\left(-\frac{3}{2}(x^2+y^2+z^2)^{-5/2}(2x)\right) + (x^2+y^2+z^2)^{-3/2}(1)\right]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\text{Similarly,}\quad \frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} &= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}\\ &= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}\\ &= 0\end{aligned}$$

Example 2.4.6. *If* $u = \frac{1}{r}$ *where* $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, *prove that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$$

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

Differentiating with respect to x,

$$2r\frac{\partial r}{\partial x} = 2(x-a)$$

$$\frac{\partial r}{\partial x} = \frac{x-a}{r}$$

$$\text{Similarly,}\quad \frac{\partial r}{\partial y} = \frac{y-b}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z-c}{r}$$

$$\text{Now}\quad \frac{1}{r} = r^{-1}$$

$$\frac{\partial u}{\partial x} = (-1)r^{-2}\frac{\partial r}{\partial x}$$

$$= -\frac{1}{r^2}\times\frac{x-a}{r}$$

$$= -\frac{(x-a)}{r^3}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= -\left[\frac{r^3(1) - (x-a)(3r^2)\frac{\partial r}{\partial x}}{r^6}\right]\\ &= -\left[\frac{r^3 - 3r^2(x-a)\frac{x-a}{r}}{r^6}\right]\\ &= -\left[\frac{r^2 - 3(x-a)^2}{r^5}\right]\end{aligned}$$

$$\text{Similarly,}\quad \frac{\partial^2 u}{\partial y^2} = -\left[\frac{r^2 - 3(y-b)^2}{r^5}\right]$$

$$\frac{\partial^2 u}{\partial z^2} = -\left[\frac{r^2 - 3(z-c)^2}{r^5}\right]$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} &= -\left[\frac{3r^2 - 3[(x-a)^2(y-b)^2 + (z-c)^2]}{r^5}\right]\\ &= -\left[\frac{3r^2 - 3r^2}{r^5}\right]\\ &= 0\end{aligned}$$

Example 2.4.7. *If* $u = f(r)$ *where* $r^2 = x^2 + y^2 + z^2$, *prove that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$$

$$r^2 = x^2 + y^2 + z^2$$

Differentiating with respect to x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\text{Similarly,}\quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{Now,}\quad u = f(r)$$

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

$$= \frac{x}{r}f'(r)$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{x}{r}\left[f''(r)\frac{\partial r}{\partial x}\right] + f'(r)\left[\frac{r(1) - x\frac{\partial r}{\partial x}}{r^2}\right]\\ &= \frac{x}{r}\left[\frac{x}{r}f''(r)\right] + f'(r)\left[\frac{r - x\times\frac{x}{r}}{r^2}\right]\\ &= \frac{x^2}{r^2}f''(r) + f'(r)\left[\frac{r^2 - x^2}{r^3}\right]\end{aligned}$$

$$\text{Similarly,}\quad \frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}f''(r) + f'(r)\left[\frac{r^2 - y^2}{r^3}\right]$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{z^2}{r^2}f''(r) + f'(r)\left[\frac{r^2 - z^2}{r^3}\right]$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{f''(r)}{r^2}(x^2+y^2+z^2)+\frac{f'(r)}{r^3}[3r^2-(x^2+y^2+z^2)]$$
$$=\frac{f''(r)}{r^2}(r^2)+\frac{f'(r)}{r^3}[3r^2-r^2)]$$
$$=f''(r)+\frac{2}{r}f'(r)$$

Example 2.4.8. *If $x^x y^y z^z = c$, show that at $x=y=z$, $\frac{\partial^2 z}{\partial x\partial y}=-(x\log ex)^{-1}$*

Given

$$x^x y^y z^z = c$$

Taking natural logarithm on both sides,

$$x\log x + y\log y + z\log z = \log c$$

Differentiating with respect to y

$$y\frac{1}{y}+\log y(1)+z\frac{1}{z}\frac{\partial z}{\partial y}+\log z\frac{\partial z}{\partial y}=0$$
$$\frac{\partial z}{\partial y}=-\frac{(1+\log y)}{(1+\log z)}$$

Similarly, $\frac{\partial z}{\partial x}=-\frac{(1+\log x)}{(1+\log z)}$

$$\frac{\partial^2 z}{\partial x\partial y}=-(1+\log y)(-1)(1+\log z)^{-2}\frac{1}{z}\frac{\partial z}{\partial x}$$
$$=\frac{1}{z}\frac{(1+\log y)}{(1+\log z)^2}\left(-\frac{(1+\log x)}{(1+\log z)}\right)$$
$$=-\frac{1}{z}\frac{(1+\log x)(1+\log y)}{(1+\log z)^3}$$

When $x=y=z$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{x}\frac{1}{(1+\log x)}$$
$$=-\frac{1}{x\log ex}$$
$$=-(x\log ex)^{-1}$$

Example 2.4.9. *If $u=\log(x^3+y^3+z^3-3xyz)$, prove that*

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$
$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

Given, $u=\log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$
$$=\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

Similarly, $\frac{\partial u}{\partial y}=\frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$

$$\frac{\partial u}{\partial z}=\frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}$$
$$=\frac{3(x^2+y^2+z^2-yz-xz-xy)}{(x+y+z)(x^2+y^2+z^2-yz-xz-xy)}$$
$$=\frac{3}{x+y+z}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=3(-1)(x+y+z)^{-2}=-\frac{3}{(x+y+z)^2}$$

Similarly, $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=-\frac{3}{(x+y+z)^2}$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=-\frac{3}{(x+y+z)^2}$$
$$\therefore\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}$$

Example 2.4.10. *If $f(x,y,z)=e^{xyz}$, find $\frac{\partial^3 f}{\partial x\partial y\partial z}$*

$$f(x,y,z)=e^{xyz}$$
$$\frac{\partial f}{\partial z}=e^{xyz}(xy)$$
$$=xye^{xyz}$$

$$\frac{\partial^2 f}{\partial y \partial z} = x[ye^{xyz}(xz) + e^{xyz}(1)]$$
$$= e^{xyz}(x + x^2yz)$$
$$\frac{\partial^3 f}{\partial x \partial y \partial z} = e^{xyz}(1 + 2xyz) + (x + x^2yz)e^{xyz}(yz)$$
$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

Example 2.4.11. *If $u = x^y$, show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

$$u = x^y$$
$$\frac{\partial u}{\partial y} = x^y \log x \qquad \left[\frac{d}{dx}(a^x) = a^x \log a\right]$$
$$\frac{\partial^2 u}{\partial x \partial y} = x^y \times \frac{1}{x} + \log x \times yx^{y-1}$$
$$= x^{y-1}(1 + y\log x)$$
$$\frac{\partial u}{\partial x} = yx^{y-1}$$
$$= yx^y\frac{1}{x}$$
$$\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{x}\left[y(x^y \log x) + x^y(1)\right]$$
$$= x^{y-1}(y\log x + 1)$$
$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$
$$\Longrightarrow \frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$$

2.5 Differentiability, differentials and local linearity

One implication of differentiability at $x = c$ of a function $y = f(x)$ of one variable is that the function can be approximated by a linear function (a straight line) in the near vicinity of the point c. This property of differentiable function is most desired in most of the engineering applications. A generalization of this property is approximating the function in the vicinity of a point c by a polynomial of suitable degree or in general by a power series. Examples are Taylor series and Maclaurin series etc. The derivative of a function $y = f(x)$ of one variable x at c is given by

$$f'(c) = \left.\frac{df(x)}{dx}\right]_{x=c} = \lim_{x\to c}\frac{f(x) - f(c)}{x - c} \approx \frac{f(x) - f(c)}{x - c} \tag{2.7}$$

so that

$$f(x) \approx f(c) + (x - c)f'(c) \tag{2.8}$$

The above approximation shows that every differentiable function of one variable can be suitably approximated by a linear function (a straight line segment) in the vicinity of the point of differentiability and the function will have a unique tangent and it is continuous at the point of differentiablility. We would like to define differentiability of functions of more than one variables in such a way that analogous characteristics of differentiability of function of one variable are retained for functions of more than one variables.

We may presume that existence of first order partial derivatives may guarantee differentiability of functions of more than one variable. But, we can have functions $f(x, y)$ whose first order partial derivatives exist, but the function $f(x, y)$ is not even continuous at those points where the function possess first order partial derivatives. So the existence of partial derivatives are not sufficient to assure differentiability. We need more stringent conditions on functions of more than one variables so that the desired characteristics are retained for the differentiable functions.

Let $\Delta f = f(x + \Delta x) - f(x)$, then we can observe that

$$\lim_{\Delta x\to 0}\frac{\Delta f - \Delta x f'(x)}{\Delta x} = 0 \tag{2.9}$$

This observation gives a hint to the additional condition needed to define the derivative of functions of more than one variable.

2.5.1 Derivative of a function at a point

A function $z = f(x, y)$ of two variables x and y is said to be differentiable at a point (a, b) if the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ exist at the point (a, b) and

$$\lim_{(\Delta x, \Delta y)\to(0,0)}\frac{\Delta f - \Delta x\frac{\partial f(a,b)}{\partial x} - \Delta y\frac{\partial f(a,b)}{\partial y}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \tag{2.10}$$

where $\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$. The above definition can be generalized to functions of three variables.

A function $w = f(x, y, z)$ of three variables x, y and z is said to be differentiable at a point (a, b, c) if the first order partial derivatives $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ exist at the point (a, b, c) and

$$\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)}\frac{\Delta f - \Delta x\frac{\partial f(a,b,c)}{\partial x} - \Delta y\frac{\partial f(a,b,c)}{\partial y} - \Delta z\frac{\partial f(a,b,c)}{\partial z}}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0 \tag{2.11}$$

where $\Delta f = f(a + \Delta x, b + \Delta y, c + \Delta z) - f(a, b, c)$

Remark: If a function $f(x, y, z)$ is differentiable at (a, b, c), then it will be continuous at (a, b, c).

Since the function is differentiable at (a, b, c), we have

$$\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)}\frac{\Delta f - \Delta x\frac{\partial f(a,b,c)}{\partial x} - \Delta y\frac{\partial f(a,b,c)}{\partial y} - \Delta z\frac{\partial f(a,b,c)}{\partial z}}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0 \tag{2.12}$$

This implies that the numerator goes to zero much faster than the denominator as $(\Delta x, \Delta y, \Delta z) \to (0,0,0)$. That is,

$$\lim_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)}\left(\Delta f - \Delta x\frac{\partial f(a,b,c)}{\partial x} - \Delta y\frac{\partial f(a,b,c)}{\partial y} - \Delta z\frac{\partial f(a,b,c)}{\partial z}\right) = 0 \quad (2.1\ldots$$

So that taking limit of individual terms as $(\Delta x, \Delta y, \Delta z) \to (0,0,0)$, we get

$$\lim_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \Delta f = 0$$

and hence the function is continuous at the point of differentiability.

Example 2.5.1. *Using definition, prove that $f(x,y) = x^2 + y^2$ is differentiable at* $(0,0)$.

$$\begin{aligned} f(x,y) &= x^2 + y^2 \\ \frac{\partial f}{\partial x} &= 2x, \frac{\partial f}{\partial y} = 2y \\ \therefore \frac{\partial f}{\partial x}(0,0) &= 0, \frac{\partial f}{\partial y}(0,0) = 0 \\ \Delta f &= f(0+\Delta x, 0+\Delta y) - f(0,0) = (\Delta x)^2 + (\Delta y)^2 \end{aligned}$$

$$\begin{aligned} \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - \Delta x\frac{\partial f(0,0)}{\partial x} - \Delta y\frac{\partial f(0,0)}{\partial y}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} &= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= 0 \end{aligned}$$

Therefore, f is differentiable at $(0,0)$.

Example 2.5.2. *Let* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. *Find $f_x(x,y)$ and $f_y(x,y)$ if it exists. Also prove that f is not continuous at $(0,0)$.*

Being a rational expression, clearly $f_x(x,y)$ and $f_y(x,y)$ exists at all points $(x,y) \neq (0,0)$.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{(x^2+y^2)(y) - (xy)(2x)}{(x^2+y^2)^2} \\ &= \frac{y^3 - x^2y}{(x^2+y^2)^2}, \ (x,y) \neq (0,0) \\ \frac{\partial f}{\partial y} &= \frac{(x^2+y^2)(x) - (xy)(2y)}{(x^2+y^2)^2} \\ &= \frac{x^3 - xy^2}{(x^2+y^2)^2}, \ (x,y) \neq (0,0) \\ \frac{\partial f}{\partial x}(0,0) &= \lim_{x\to 0}\frac{f(x,0) - f(0,0)}{x - 0} \\ &= \lim_{x\to 0}\frac{\left(\frac{0}{x^2}\right) - 0}{x} \\ &= 0 \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{y\to 0}\frac{f(0,y) - f(0,0)}{y - 0} \\ &= \lim_{y\to 0}\frac{\left(\frac{0}{y^2}\right) - 0}{y} \\ &= 0 \end{aligned}$$

Let $(x,y) \to (0,0)$ along $y = mx$. Then

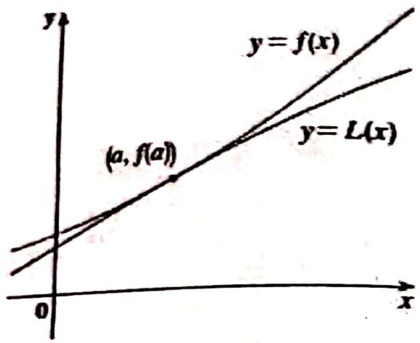
$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0}\frac{x(mx)}{x^2 + (mx)^2} \\ &= \lim_{x\to 0}\frac{mx^2}{x^2(1+m^2)} \\ &= \frac{m}{1+m^2} \end{aligned}$$

which depends on the value of m. Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence f is not continuous at $(0,0)$.

Example 2.5.3. *Let* $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. *Prove that f is not continuous at $(0,0)$.*

Let $(x,y) \to (0,0)$ along $y^3 = mx$. Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0}\frac{x(mx)}{x^2 + (mx)^2} \\ &= \lim_{x\to 0}\frac{mx^2}{x^2(1+m^2)} \\ &= \frac{m}{1+m^2} \end{aligned}$$



which depends on the value of m. Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence f is not continuous at $(0,0)$.

Applying the definition to check differentiability of a function is not desirable as it is very difficult. Next we give criterion for differentiability of function of more than one variables $f(x,y,z)$ without proof.

2.5.2 Criterion for checking differentiability of a function

Theorem 2.5.4. *A function $f(x,y,z)$ is differentiable at all points where its first order partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ exists and are continuous.*

Now we discuss analogous notion of differentials of function of one variable which are useful tools for finding the errors or increments in the dependent variable when there is an error in the independent variable.

2.5.3 Tangent planes and Linear approximations

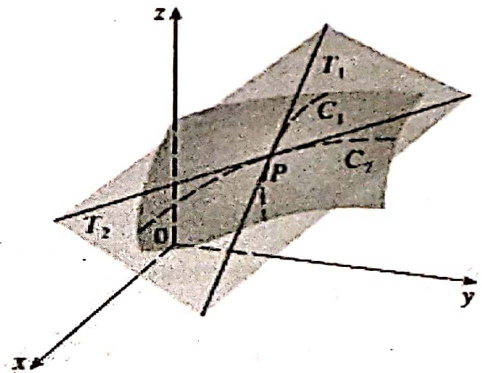
For a differentiable function of one variable, $y = f(x)$, we can see that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we can see that the graph looks more and more like its tangent line. We use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x-a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the linear approximation or tangent line approximation of $f(x)$ at a. The linear function whose graph is this tangent line, that is, is called the linearization of $f(x)$ at a.



Tangent planes

One of the most important ideas in single-variable calculus is that as we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function. Here we develop similar ideas in three dimensions. As we zoom in toward a point on a surface, that is, the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane) and we can approximate the function by a linear function of two variables. We also extend the idea of a differential to functions of two or more variables.

Suppose a surface S has equation $z = f(x,y)$, where $f(x,y)$ has continuous first order partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $x = x_0$ and $y = y_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

The tangent plane at a point on the surface is the plane that most closely approximates the surface near the point .

Equation of any plane passing through the point $P(x_0, y_0, z_0)$ has the form

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

By dividing this equation by C and letting $a = -\frac{A}{C}$ and $b = -\frac{B}{C}$, we can write it in the form

$$z - z_0 = a(x-x_0) + b(y-y_0)$$

The intersection of the tangent plane at P and the plane $x = x_0$ is the tangent line T_1. Hence the equation of T_1 is

$$z - z_0 = b(y - y_0)$$

where b is the slope of T_1.

Therefore $b = f_y(x_0, y_0)$. Similarly we get $a = f_x(x_0, y_0)$. Hence the equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0) the point is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Local linear approximation

We use the tangent plane at $(x_0, y_0, f(x_0, y_0))$ as an approximation to the surface $z = f(x, y)$ when (x, y) is near (x_0, y_0). An equation of the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linear function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

whose graph is the above tangent plane is called the linearization of $f(x, y)$ at (x_0, y_0) and the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the linear approximation or tangent plane approximation of $f(x, y)$ at (x_0, y_0).

Similarly

$$f(x, y, z) \approx f(x_0, y_0, z_0) + (x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0)$$

which is the linear approximation of $f(x, y, z)$ at (x_0, y_0, z_0).

Example 2.5.5. *Find the local linear approximation $L(x, y)$ to the function $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$ at a point (x, y) given that the function is differentiable at $(3, 4)$. Compare the error in approximating the value $f(3.04, 3.98)$ by $L(3.04, 3.98)$ with the distance between the points $(3, 4)$ and $(3.04, 3.98)$.*

The local linear approximation of f at (x, y) using the partial derivatives f at (x_0, y_0) is given by

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Given that $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$.

$$f_x(x, y) = -\frac{1}{2}(x^2 + y^2)^{-3/2}(2x) = -\frac{x}{(\sqrt{x^2 + y^2})^3}$$

$$f_y(x, y) = -\frac{y}{(\sqrt{x^2 + y^2})^3}$$

$$f(3, 4) = \frac{1}{\sqrt{3^2 + 4^2}} = \frac{1}{5}$$

$$f_x(3, 4) = -\frac{3}{(\sqrt{3^2 + 4^2})^3} = -\frac{3}{125}$$

$$f_y(3, 4) = -\frac{4}{(\sqrt{3^2 + 4^2})^3} = -\frac{4}{125}$$

$$\therefore L(x, y) = \frac{1}{5} - \frac{3}{125}(x - 3) - \frac{4}{125}(y - 4)$$

Hence $f(3.04, 3.98) \approx L(3.04, 3.98)$

$$= \frac{1}{5} - \frac{3}{125}(3.04 - 3) - \frac{4}{125}(3.98 - 4) = 0.19968$$

But $f(3.04, 3.98) = \dfrac{1}{\sqrt{(3.04)^2 + (3.98)^2}} = 0.1996728047$

Error in approximation $= 0.1996728047 - 0.19968 = -0.0000071967$

Distance between the points $(3, 4)$ and $(3.04, 3.98)$

$$= \sqrt{(3.04 - 3)^2 + (3.98 - 4)^2}$$
$$= 0.04472135955$$

$$\frac{\text{Error in approximation}}{\text{Distance between the points}} = \frac{-0.000007196}{0.044721359} = -0.000160923104$$

Example 2.5.6. *Find the local linear approximation $L(x, y, z)$ to the function $f(x, y, z) = \log(x + yz)$ at a point (x, y, z) given that the function is differentiable at $(2, 1, -1)$. Compare the error in approximating the value $f(2.02, 0.97, -1.01)$ by $L(2.02, 0.97, -1.01)$ with the distance between the points $(2, 1, -1)$ and $(2.02, 0.97, -1.01)$.*

The local linear approximation of f at (x, y, z) using the partial derivatives f at (x_0, y_0, z_0) is given by

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0)$$
$$+ f_z(x_0, y_0, z_0)(z - z_0)$$

Given that $f(x,y,z) = \log(x + yz)$.

$$f_x(x,y,z) = \frac{1}{(x+yz)}$$
$$f_y(x,y,z) = \frac{z}{(x+yz)}$$
$$f_z(x,y,z) = \frac{y}{(x+yz)}$$
$$f(2,1,-1) = \log(2 + (1)(-1)) = 0$$
$$f_x(2,1,-1) = \frac{1}{(2-1)} = 1$$
$$f_y(2,1,-1) = \frac{-1}{(2-1)} = -1$$
$$f_z(2,1,-1) = \frac{1}{(2-1)} = 1$$
$$\therefore L(x,y,z) = 0 + (1)(x-2) - (y-1) + (1)(z+1) = x - y + z$$

Hence $f(2.02, 0.97, -1.01) \approx L(2.02, 0.97, -1.01)$
$$= 2.02 - 0.97 - 1.01 = 0.04$$

But $f(2.02, 0.97, -1.01) = \log(2.02 + (0.97)(-1.01)) = 0.039509133$

Error in approximation $= 0.039509133 - 0.04 = -0.00049086691$

Distance between the points $(2,1,-1)$ and $(2.02, 0.97, -1.01)$
$$= \sqrt{(2.02-2)^2 + (0.97-1)^2 + (-1-(-1.01))^2}$$
$$= 0.03741657387$$

$$\frac{\text{Error in approximation}}{\text{Distance between the points}} = \frac{-0.00049086709}{0.03416573} = -0.013118972$$

Example 2.5.7. *Suppose that a function $f(x,y)$ is differentiable at the point $(2,5)$ with $f_x(2,5) = 5$ and $f(2,5) = 9$. Let $L(x,y)$ denote the local linear approximation of f at $(2,5)$. If $L(2.1, 4.8) = 9.15$, find the value of $f_y(2,5)$.*

The local linear approximation of f at (x,y) using the partial derivatives f at (x_0, y_0) is given by

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Therefore the local linear approximation of f at $(2,5)$ is

$$L(x,y) = f(2,5) + f_x(2,5)(x-2) + f_y(2,5)(y-5)$$
$$= 9 + 5(x-2) + f_y(2,5)(y-5)$$

Given that $L(2.1, 4.8) = 9.15$

$$\therefore \quad 9 + 5(2.1-2) + f_y(2,5)(4.8-5) = 9.15$$
$$-0.2 f_y(2,5) = -0.35$$
$$\implies f_y(2,5) = 1.75$$

Example 2.5.8. *Let $f(x,y,z)$ be differentiable at $(1,2,2)$ and Let $L(x,y,z) = 3x - y + 2z + 7$ be the local linear approximation to f at (x,y,z). Find $f(1,2,2)$, $f_x(1,2,2)$, $f_y(1,2,2)$, and $f_z(1,2,2)$.*

The local linear approximation of f at (x,y) using the partial derivatives f at (x_0, y_0) is given by

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The local linear approximation of f at (x,y,z) using partial derivatives at $(1,2,2)$ is given by

$$L(x,y,z) = f(1,2,2) + f_x(1,2,2)(x-1) + f_y(1,2,2)(y-2) + f_z(1,2,2)(z-2)$$

Given that

$$L(x,y,z) = 3x - y + 2z + 7$$
$$= 3(x-1) + 3 - (y-2) - 2 + 2(z-2) + 4 + 7$$
$$= 3(x-1) - (y-2) + 2(z-2) + 12$$

On comparison, we get
$f(1,2,2) = 12$, $f_x(1,2,2) = 3$, $f_y(1,2,2) - 1$, and $f_z(1,2,2) = 2$

Example 2.5.9. *Suppose that a function $f(x,y) = x^2y$ is differentiable at a point $P(a,b)$. If $L(x,y) = -4x + 4y + 8$ be the local linear approximation to f at (x,y), determine P.*

Solution: The local linear approximation of f at (x,y) using the partial derivatives f at (x_0, y_0) is given by

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Given that

$$L(x,y) = -4x + 4y + 8$$
$$= -4(x-x_0) + 4(y-y_0) - 4x_0 + 4y_0 + 8$$
$$\implies f(x_0,y_0) = -4x_0 + 4y_0 + 8,\ f_x(x_0,y_0) = -4,\ f_y(x_0,y_0) = 4$$
$$f(x,y) = x^2y \implies f_x(x,y) = 2xy,\ f_y(x,y) = x^2$$
$$f_x(x_0,y_0) = -4 \implies 2x_0y_0 = -4 \implies x_0y_0 = -2$$
$$f_y(x_0,y_0) = 4 \implies x_0^2 = 4 \implies x_0 = \pm 2$$
$$x_0 = 2 \implies y_0 = -1$$
$$x_0 = -2 \implies y_0 = 1$$
$$f(2,-1) = (2)^2(-1) = -4 = -4(2) + 4(-1) + 8$$
$$f(2,1) = (2)^2(1) = 4 = -4(2) + 4(1) + 8$$

Therefore P is $(2,-1)$ or $(-2,1)$.

Example 2.5.10. *Suppose that a function $f(x,y) = xy + z^2$ is differentiable at a po $P(a,b,c)$. If $L(x,y,z) = y + 2z + 5$ be the local linear approximation to f at (x,y,z) determine P.*

The local linear approximation of f at (x,y) using the partial derivatives f at (x_0,y_0) is given by

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0)$$

Given that

$$\begin{aligned} L(x,y,z) &= y + 2z + 5 \\ &= (y - y_0) + 2(z - z_0) + y_0 + 2z_0 + 5 \end{aligned}$$

$$\Longrightarrow f(x_0,y_0,z_0) = y_0 + 2z_0 + 5, f_x(x_0,y_0,z_0) = 0, f_y(x_0,y_0,z_0) = 1, f_z(x_0,y_0,z_0) = 2$$

$$f(x,y,z) = xy + z^2 \Longrightarrow f_x(x,y,z) = y, f_y(x,y,z) = x, f_z(x,y,z) = 2z$$

$$f_x(x_0,y_0,z_0) = 0 \Longrightarrow y_0 = 0$$

$$f_y(x_0,y_0,z_0) = 1 \Longrightarrow x_0 = 1$$

$$f_z(x_0,y_0,z_0) = 2 \Longrightarrow 2z_0 = 2 \Longrightarrow z_0 = 1$$

$$\therefore P = (1,0,1)$$

2.5.4 Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable: that is, dx can be given the value of any real number. The differential of y is then defined as

$$dy = f'(x)\,dx$$

If dx is given a specific value and x is taken to be some specific number in the domain of $f(x)$, then the numerical values of dy is determined.

The geometrical meaning of differentials is shown in the figure. Let $P(a, f(a))$ and $Q(a + \Delta x, f(a + \Delta x))$ be points on the graph of $f(x)$ and let $dx = \Delta x$

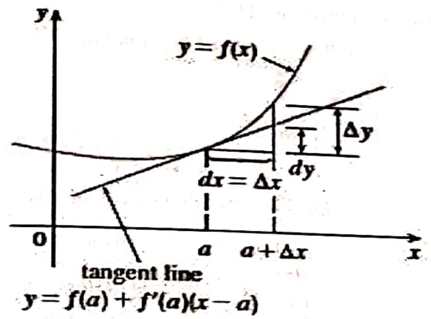
The corresponding change in y is

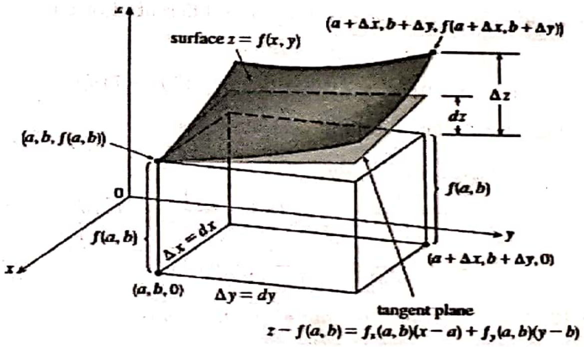
$$\Delta y = f(a + \Delta x) - f(a)$$

The slope of the tangent line is the derivative $f'(a)$. Therefore $dy = f'(a)\,dx$ represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve rises or falls when changes by an amount dx.

For a differentiable function of two variables $z = f(x,y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then the differential dz , also called the total differential, is defined by

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$





If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in the above equation, then the differential of z is

$$dz = f_x(a,b)\,(x - a) + f_y(a,b)\,(y - b)$$

The geometrical meaning of differentials is shown in the figure. dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x,y)$ when (x,y) changes from (a,b) to $(a + \Delta x, b + \Delta y)$.

Differential of function $f(x,y,z)$ of three variables is defined as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{2.14}$$

Note that differentials are functions with dx, dy, dz as independent variables and df as dependent variable. We can see that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \approx \Delta f \tag{2.15}$$

When there are errors of $\Delta x, \Delta y, \Delta z$ in the independent variables x, y, z respectively, the differential df gives the total error in $f(x, y, z)$.

Problem 2.5.11. *Find the differential dz of the functions*

(i) $z = 5x - 3y$

(ii) $z = 2x^3y^2 - 5y^3 - 2x + 7y - 5$

(iii) $z = \tan^{-1}(x^2y)$

Solution: (i) Given $z = 5x - 3y$. The total differential of z is

$$\begin{aligned} dz &= z_x(x,y)\,dx + z_y(x,y)\,dy \\ &= \frac{\partial z}{\partial x}(x(y)\,dx + \frac{\partial z}{\partial i}(x,q)\,dy \\ &= 1\,dx + (-3)\,dy = 5dx - 3dy \end{aligned}$$

(ii) Given $z = 2x^3y^2 - 5y^3 - x + 7y - 5$. Phe total differential of z is

$$\begin{aligned} dz &= z_x\,dx + z_y\,dy \\ &= (6x^2y^2 - 2)\,dx + (4x^3y - 15y^2 + 7)\,dy \end{aligned}$$

(iii) Given $z = \tan^{-1}(x^2y)$. The total differential of z is

$$\begin{aligned} dz &= z_x\,dx + z_y\,dy \\ &= \frac{1}{1+(x^2y)^2}(2xy)\,dx + \frac{1}{1+(x^2y)^2}(x^2)\,dy \\ &= \frac{2xy}{1+x^4y^2}\,dx + \frac{x^2}{1+x^4y^2}\,dy \end{aligned}$$

Problem 2.5.12. *Find the differential dw of the functions*

(i) $w = 6x + 2y - 5z + 3$

(ii) $w = e^{xyz}$

(ii) $w = \sqrt{x} + \sqrt{y} + \sqrt{z}$

Solution: (i) Given that $w = 6x + 2y - 5z + 3$.The total differential of w is

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz \\ &= 6\,dx + 2\,dy - 5\,dz \end{aligned}$$

(ii) Given that $w = e^{xyz}$. The total differential of w is

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz \\ &= yze^{xyz}\,dx + xze^{xyz}\,dy + xye^{xyz}\,dz \end{aligned}$$

(iii) Given that $w = \sqrt{x} + \sqrt{y} + \sqrt{z}$. The total differential of w is

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz \\ &= \frac{1}{2\sqrt{x}}\,dx + \frac{1}{2\sqrt{y}}\,dy + \frac{1}{2\sqrt{z}}\,dz \end{aligned}$$

Problem 2.5.13. *Use the total differential of $z = \log\sqrt{1+xy}$ at $(0,2)$ to approximate the change in the value of z from $(0,2)$ to its value at $(-0.01, 2.02)$. Compare your estimate with the change in z.*

Solution: Given that $z = \log\sqrt{1+xy}$. The total differential of z is

$$\begin{aligned} dz &= z_x\,dx + z_y\,dy \\ &= \frac{y}{2(1+xy)}\,dx + \frac{x}{2(1+xy)}\,dy \end{aligned}$$

Evaluating this differential at $(x,y) = (0,2)$, $dx = \Delta x = -0.01 - 0 = -0.01$, and $dy = \Delta y = 2.02 - 2 = 0.02$.

$$dz = \frac{2}{2}(-0.01) + \frac{0}{2}(0.02) = -0.01$$

Now, $\Delta z \approx dz = -0.01$

$$\begin{aligned} z(0,2) &= \log\sqrt{1+(0)(2)} = 0 \\ z(-0.01, 2.02) &= \log\sqrt{1+(-0.01)(2.02)} = -0.010203404 \\ \Delta z &= z(-0.01, 2.02) - z(0,2) \\ &= -0.010203404 - 0 = -0.010203404 \end{aligned}$$

$$\begin{aligned} \text{Error in approximating } \Delta z \text{ by } dz &= |dz - \Delta z| \\ &= |-0.01 - (-0.010203404)| = 0.000203404 \end{aligned}$$

Problem 2.5.14. *Use a total differential to approximate the change in the values of $f(x,y,z) = \dfrac{xyz}{x+y+z}$ from $(1,-2,5)$ to $(1.01, -2.003, 4.98)$. Compare your estimate with the change in f.*

Solution: Given that $f(x,y,z) = \dfrac{xyz}{x+y+z}$. The total differential of f is

$$\begin{aligned} df &= f_x\,dx + f_y\,dy + f_z\,dz \\ &= yz\left[\frac{(x+y+z)(1) - x(1)}{(x+y+z)^2}\right]dx + xz\left[\frac{(x+y+z)(1) - y(1)}{(x+y+z)^2}\right]dy \\ &\quad + xy\left[\frac{(x+y+z)(1) - z(1)}{(x+y+z)^2}\right]dz \\ &= \frac{yz(y+z)}{(x+y+z)^2}\,dx + \frac{xz(x+z)}{(x+y+z)^2}\,dy + \frac{xy(x+y)}{(x+y+z)^2}\,dz \end{aligned}$$

Evaluating this differential at $(x, y, z) = (1, -2, 5)$, $dx = \Delta x = 1.01 - 1 = 0.01$, $dy = \Delta y = -2.003 - (-2) = -0.003$ and $dz = \Delta z = 4.98 - 5 = -0.02$.

$$\begin{aligned} df &= \frac{(-2)(5)(-2+5)}{(1-2+5)^2}(0.01) + \frac{(1)(5)(1+5)}{(1-2+5)^2}(-0.003) \\ &\quad + \frac{(1)(-2)(1-2)}{(1-2+5)^2}(-0.02) \\ &= -0.026875 \end{aligned}$$

$$\begin{aligned} \text{Now,}\quad \Delta f \approx df &= -0.026875 \\ f(1,-2,5) &= \frac{(1)(-2)(5)}{1-2+5} = -2.5 \\ f(1.01,-2.003,4.98) &= \frac{(1.01)(-2.003)(4.98)}{1.01-2.003+4.98} = -2.526884725 \\ \Delta f &= f(1.01,-2.003,4.98) - f(1,-2,5) \\ &= -2.526884725 - (-2.5) = -0.026884725 \end{aligned}$$

$$\begin{aligned} \text{Error in approximating } \Delta f \text{ by } df &= |df - \Delta f| \\ &= |-0.026875 - (-0.026884725)| \\ &= 0.000009725 \end{aligned}$$

Problem 2.5.15. *Use a total differential to approximate the change in $z = xy^2$ from its value at $(0.5, 1)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in the approximation with the distance between the points $(0.5, 1)$ and $(0.503, 1.004)$.*

Solution: Given that $z = xy^2$. The total differential of z is

$$\begin{aligned} dz &= z_x\,dx + z_y\,dy \\ &= y^2\,dx + 2xy\,dy \end{aligned}$$

Evaluating this differential at $(x, y) = (0.5, 1)$, $dx = \Delta x = 0.503 - 0.5 = 0.003$, and $dy = \Delta y = 1.004 - 1 = 0.004$.

$$dz = (1)^2(0.003) + 2(0.5)(1)(0.004) = 0.007$$

$$\begin{aligned} \text{Now,}\quad \Delta z \approx dz &= 0.007 \\ z(0.5,1) &= (0.5)(1)^2 = 0.5 \\ z(0.503,1.004) &= (0.503)(1.004)^2 = 0.507032048 \\ \Delta z &= z(0.503,1.004) - z(0.5,1) \\ &= 0.507032048 - 0.5 = 0.007032048 \end{aligned}$$

$$\begin{aligned} \text{Error in approximating } \Delta z \text{ by } dz &= |dz - \Delta z| \\ &= |0.007 - 0.007032048| = 0.000032048 \end{aligned}$$

$$\begin{aligned} \text{The distance between}(0.5,1.0) \text{ and } (0.503,1.004) &= \sqrt{(0.503-0.5)^2 + (1.004-1)^2} \\ &= 0.005 \end{aligned}$$

$$\begin{aligned} \frac{\text{Error in approximating } \Delta z \text{ by } dz}{\text{The distance between}(0.5,1.0) \text{ and } (0.503,1.004)} &= \frac{0.000032048}{0.005} \\ &= 0.0064096 < \frac{1}{150} \end{aligned}$$

Problem 2.5.16. *Suppose that a function $f(x, y)$ is differentiable at the point $(1, 3)$ with $f_x(1, 3) = 1$ and $f_y(1, 3) = 3$. If $f(1, 3) = 2$, estimate the value of $f(0.98, 3.03)$.*

Solution:

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta f$$

We approximate Δf with the total differential df of f. The total differential of f is

$$df = f_x\,dx + f_y\,dy$$

Evaluating this differential at $(x, y) = (1, 3)$, $dx = \Delta x = 0.98 - 1 = -0.02$, and $dy = \Delta y = 3.03 - 3 = 0.03$.

$$\begin{aligned} df &= f_x(1,3)\,\Delta x + f_y(1,3)\,\Delta y \\ &= (1)(-0.02) + (3)(0.03) = 0.07 \end{aligned}$$

$$\begin{aligned} \text{Now,}\quad \Delta f \approx df &= 0.07 \\ f(0.98,3.03) &= f(1,3) + \Delta f = 2 + 0.07 = 2.07 \end{aligned}$$

Problem 2.5.17. *Suppose that a function $f(x, y, z)$ is differentiable at the point $(1, 2, -3)$ with $f_x(1, 2, -3) = 1$, $f_y(1, 2, -3) = 2$, and $f_z(1, 2, -3) = 3$. If $f(1, 2, -3) = 7$, estimate the value of $f(1.02, 1.99, -2.97)$.*

Solution:

$$f(x + \Delta x, y + \Delta y, z + \Delta z) = f(x, y, z) + \Delta f$$

We approximate Δf with the total differential df of f. The total differential of f is

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

Evaluating this differential at $(x, y, z) = (1, 2, -3)$, $dx = \Delta x = 1.02 - 1 = 0.02$, $dy = \Delta y = 1.99 - 2 = -0.01$ and $dz = \Delta z = -2.97 - (-3) = 0.03$.

$$\begin{aligned} df &= f_x(1,2,-3)\,\Delta x + f_y(1,2,-3)\,\Delta y + f_z(1,2,-3)\,\Delta z \\ &= (1)(0.02) + 2(-0.01) + 3(0.03) = 0.09 \end{aligned}$$

$$\begin{aligned} \text{Now,}\quad \Delta f \approx df &= 0.09 \\ f(1.02,1.99,-2.97) &= f(1,2,-3) + \Delta f = 7 + 0.09 = 7.09 \end{aligned}$$

Problem 2.5.18. *The length and width of a rectangle are measured with errors of at most 3% and 5%, respectively. Use differentials to approximate the maximum percentage error in the calculated area.*

Solution: Let x and y be the length and width of a rectangle with area A. Then

$$A = xy$$

Let x_0, y_0 and $A_0 = x_0 y_0$ denote the actual value of length, width and area of the rectangle. The total differential dA of A at (x_0, y_0) is given by

$$\begin{aligned} dA &= A_x(x_0, y_0)\,dx + A_y(x_0, y_0)\,dy \\ &= y_0\,dx + x_0\,dy \end{aligned}$$

Let x, y and $A = xy$ are the measured and computed values of length, width and area respectively of the rectangle. Then $\Delta x = x - x_0, \Delta y = y - y_0$. Given that $\left|\frac{\Delta x}{x}\right| \leq 0.03$ and $\left|\frac{\Delta y}{y}\right| \leq 0.05$ and we have to estimate the maximum size of $\left|\frac{\Delta A}{A_0}\right|$. Now,

$$\begin{aligned}
\frac{\Delta A}{A_0} \approx \frac{dA}{A_0} &= \frac{y_0\,\Delta x + x_0\,\Delta y}{x_0 y_0} \\
&= \frac{\Delta x}{x_0} + \frac{\Delta y}{y_0} \\
\left|\frac{\Delta A}{A_0}\right| &= \left|\frac{\Delta x}{x_0} + \frac{\Delta y}{y_0}\right| \\
&\leq \left|\frac{\Delta x}{x_0}\right| + \left|\frac{\Delta y}{y_0}\right| \\
&\leq 0.03 + 0.05 = 0.08
\end{aligned}$$

Thus maximum percentage error in the estimated value of area is 8%.

Problem 2.5.19. *The radius and height of a right circular cone are measured with errors of at most 1% and 4%, respectively. Use differentials to approximate the maximum percentage error in the calculated volume.*

Solution: Let x and y be the radius and height of a right circular cone with volume V. Then

$$V = \frac{1}{3}\pi x^2 y$$

Let x_0, y_0 and $V_0 = \frac{1}{3}\pi x_0^2 y_0$ denote the actual value of radius, height and volume of the right circular cone. The total differential dV of V at (x_0, y_0) is given by

$$\begin{aligned}
dV &= V_x(x_0, y_0)\,dx + V_y(x_0, y_0)\,dy \\
&= \frac{2}{3}\pi x_0 y_0\,dx + \frac{1}{3}\pi x_0^2\,dy
\end{aligned}$$

Let x, y and $V = \frac{1}{3}\pi x^2 y$ are the measured and computed values of radius, height and volume respectively of the cone. Then $\Delta x = x - x_0, \Delta y = y - y_0$. Given that $\left|\frac{\Delta x}{x}\right| \leq 0.01$ and $\left|\frac{\Delta y}{y}\right| \leq 0.04$ and we have to estimate the maximum size of $\left|\frac{\Delta V}{V_0}\right|$. Now,

$$\begin{aligned}
\frac{\Delta V}{V_0} \approx \frac{dV}{V_0} &= \frac{\frac{2}{3}\pi x_0 y_0\,\Delta x + \frac{1}{3}\pi x_0^2\,\Delta y}{\frac{1}{3}\pi x_0^2 y_0} \\
&= 2\frac{\Delta x}{x_0} + \frac{\Delta y}{y_0} \\
\left|\frac{\Delta V}{V_0}\right| &= \left|2\frac{\Delta x}{x_0} + \frac{\Delta y}{y_0}\right| \\
&\leq 2\left|\frac{\Delta x}{x_0}\right| + \left|\frac{\Delta y}{y_0}\right| \\
&\leq 2(0.01) + 0.04 = 0.06
\end{aligned}$$

Thus maximum percentage error in the estimated value of volume is 6%.

Problem 2.5.20. *The period T of a simple pendulum with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum and g is the acceleration due to gravity. Suppose that measured values of l and g have errors of at most 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.*

Solution: Given that

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Let l_0, g_0 and $T_0 = 2\pi\sqrt{\frac{l_0}{g_0}}$ denote the actual value of length of the pendulum, acceleration due to gravity and period of the pendulum. The total differential dT of T at (l_0, g_0) is given by

$$\begin{aligned}
dT &= T_l(l_0, g_0)\,dl + T_g(l_0, g_0)\,dg \\
&= \frac{\pi}{\sqrt{g_0\,l_0}}\,dl - \frac{\pi\sqrt{l_0}}{g_0\sqrt{g_0}}dg
\end{aligned}$$

Let l, g and $T = 2\pi\sqrt{\frac{l}{g}}$ are the measured and computed values of length of the pendulum, acceleration due to gravity and period of the pendulum. Then $\Delta l = l - l_0, \Delta g = g - g_0$. Given that $\left|\frac{\Delta l}{l}\right| \leq 0.005$ and $\left|\frac{\Delta g}{g}\right| \leq 0.001$ and we have to estimate the maximum size

Thus maximum error in the estimated value of surface area is $0.42cm^2$.

$$\text{Also}\quad S_0 = 2[(30)(50) + (50)(25) + (25)(30)]$$
$$= 7000$$
$$\left|\frac{\Delta S}{S_0}\right| \le \frac{0.42}{7000}$$
$$\le 0.00006$$

Thus maximum percentage error in the area$= 0.006$

2.6 The Chain rule for function of more than one variable

Let $y = f(x)$ and $x = x(t)$ be differentiable functions with respect to x and t respectively. The chain rule for function of one variable states that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad (2.1$$

Now we discuss analogous chain rule for functions of more than one variable. We develop chain rule for computing total derivative of a function and also chain rule for computing partial derivatives.

2.6.1 Chain rule for total derivative

Let $f(x, y)$ be a differentiable function of x and y and $x = x(t)$, $y = y(t)$ be differentiable function of t. When we put x and y in $f(x, y)$, we get $f(x, y) = f(x(t), y(t))$, which function of a single variable t. Then the chain rule for finding the total derivative with respect to t is given as follows:

Let $f(x, y)$ be differentiable at (x, y) and $x(t)$, $y(t)$ be differentiable at t. Then the chain rule for a function states that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad (2.1$$

and for a function of three variables satisfying conditions of differentiability we get

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \qquad (2.1$$

where partial derivatives are computed at (x, y, z) and ordinary derivatives are computed at t.

Example 2.6.1. *Suppose that $z = 5x^3y^2, x = t^3, y = t^5$. Use the chain rule to find $\frac{dz}{dt}$, and check the result by expressing z as a function of t and differentiating directly.*

Given that $z = 5x^3y^2, x = t^3, y = t^5$. By chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (15x^2y^2)(3t^2) + (10x^3y)(5t^4)$$
$$= 45t^6t^{10}t^2 + 50t^9t^5t^4$$
$$= 95t^{18}$$

Substituting the the values of x, y in z, we get

$$z = 5x^3y^2 = 5t^9t^{10} = 5t^{19}$$
$$\frac{dz}{dt} = 95t^{18}$$

Example 2.6.2. *Suppose that $z = \log(3x^2 + y), x = \sqrt{t}, y = t^{1/3}$. Use the chain rule to find $\frac{dz}{dt}$.*

Given that $z = \log(3x^2 + y), x = \sqrt{t}, y = t^{1/3}$. By chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= \frac{6x}{(3x^2 + y)} \times \frac{1}{2\sqrt{t}} + \frac{1}{(3x^2 + y)} \times \frac{1}{3}t^{-2/3}$$
$$= \frac{3\sqrt{t}}{(3t + t^{1/3})} \times \frac{1}{\sqrt{t}} + \frac{1}{(3t + t^{1/3})} \times \frac{1}{3}t^{-2/3}$$
$$= \frac{3}{(3t + t^{1/3})} + \frac{1}{3t^{2/3}(3t + t^{1/3})}$$

Example 2.6.3. *Suppose that $z = f(x, y)$ is differentiable at the point $(5, 8)$ with $f_x(5, 8) = 5$ and $f_y(5, 8) = -1$. If $x = t^2 + 1$ and $y = t^3$, find $\frac{dz}{dt}$ when $t = 2$.*

Given that $z = f(x, y), x = t^2 + 1, y = t^3$. By chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= f_x(x, y)(2t) + f_y(x, y)(3t^2)$$

When $t = 2$,

$$\frac{dz}{dt} = f_x(5, 8)(4) + f_y(5, 8)(12)$$
$$= (5)(4) + (-1)(12)$$
$$= 8$$

Example 2.6.4. *The length and breadth of a rectangle are increasing at the rate of 1.5 cm/sec and 0.5 cm/sec respectively Find the rate at which the area is increasing at the instant when the length is 40 cm and breadth is 30 cm respectively.*

Let x, y and A be the length, breadth and area of a rectangle. Then

$$A = xy$$

Given that $\frac{dx}{dt} = 1.5$ cm/sec and $\frac{dy}{dt} = 0.5$ cm/sec. By chain rule,

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$

When $x = 40, y = 30$

$$\frac{dA}{dt} = (30)(1.5) + (40)(0.5) = 65$$

That is, area is increasing with 65 cm^2/sec.

Example 2.6.5. *The altitude of a right circular cone is increasing at the rate of 0.2 cm/sec. The radius of its base is decreasing at the rate of 0.3 cm/sec.. Find the rate at which its volume is changing at the instant when the altitude is 15 cm and radius is 10 cm.*

Let x, y and V be the radius, altitude and volume of a right circular cone. Then

$$V = \frac{1}{3}\pi x^2 y$$

Given that $\frac{dx}{dt} = -0.3$ cm/sec and $\frac{dy}{dt} = 0.2$ cm/sec. By chain rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{2}{3}\pi xy\frac{dx}{dt} + \frac{1}{3}\pi x^2\frac{dy}{dt}$$

When $x = 10, y = 15$

$$\frac{dV}{dt} = \frac{2}{3}\pi(10)(15)(-0.3) + \frac{1}{3}\pi(10)^2(0.2) = -\frac{70}{3}\pi$$

That is volume is decreasing with $\frac{70}{3}\pi$ cm^3/sec.

Example 2.6.6. *Given that $z = 2xy^2 - 3x^2y$, and x increases at the rate of 2 cm/sec. Find the rate at which y changes at the instant when $x = 3$ cm and $y = 1$ cm if z remains a constant.*

Given that $z = 2xy^2 - 3x^2y$. It is required to find $\frac{dy}{dt}$ when $x = 3$ cm and $y = 1$, given that z is a constant and $\frac{dx}{dt} = 2$ cm/sec. $z =$ a constant $\implies \frac{dz}{dt} = 0$. By chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2y^2 - 6xy)\frac{dx}{dt} + (4xy - 3x^2)\frac{dy}{dt}$$

When $x = 3, y = 1$

$$0 = (2 - 18)(2) + (12 - 27)\frac{dy}{dt}$$

$$\implies \frac{dy}{dt} = -\frac{32}{15}$$

That is y is decreasing at the rate of $\frac{32}{15}$ cm/sec.

Example 2.6.7. *If $u = x^2 + y^2 + z^2$ where $x = e^t\cos t$, $y = e^t\sin t$ and $z = e^t$, find $\frac{du}{dt}$ by using the chain rule for total derivative and verify the answer by obtaining $\frac{du}{dt}$ after substituting for x, y and z in terms of t in u.*

Given that $u = x^2 + y^2 + z^2, x = e^t\cos t,\ y = e^t\sin t,\ z = e^t$. By chain rule,

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}\\ &= (2x)[e^t(-\sin t) + (\cos t)e^t] + (2y)[e^t(\cos t) + (\sin t)e^t] + 2z(e^t)\\ &= (2e^t\cos t)[-e^t\sin t + e^t\cos t] + (2e^t\sin t)[e^t\cos t + e^t\sin t] + 2e^te^t\\ &= 2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t + 2e^{2t}\\ &= 2e^{2t}(\cos^2 + t\sin^2 t) + 2e^{2t}\\ &= 4e^{2t}\end{aligned}$$

Substituting the the values of x, y in u, we get

$$\begin{aligned}u &= (e^t\cos t)^2 + (e^t\sin t)^2 + (e^t)^2\\ &= e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}\\ &= 2e^{2t}\end{aligned}$$

$$\frac{dz}{dt} = 4e^{2t}$$

Example 2.6.8. *Suppose that* $w = \sqrt{1+x^2+3xy^2z^3}, x = \sqrt{t}, y = \log t, z = e^t$. *Use the chain rule to find* $\frac{dw}{dt}$.

Given that $w = \sqrt{1+x^2+3xy^2z^3}, x = \sqrt{t}, y = \log t, z = e^t$. By chain rule,

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \frac{(2x+3y^2z^3)}{2\sqrt{1+x^2+3xy^2z^3}} \times \frac{1}{2\sqrt{t}} + \frac{(6xyz^3)}{2\sqrt{1+x^2+3xy^2z^3}} \times \frac{1}{t} \\
&\quad + \frac{(9xy^2z^2)}{2\sqrt{1+x^2+3xy^2z^3}} \times e^t \\
&= \frac{(2x+3y^2z^3)}{4\sqrt{t(1+x^2+3xy^2z^3)}} + \frac{(3xyz^3)}{t\sqrt{1+x^2+3xy^2z^3}} + \frac{(9xy^2z^2)e^t}{2\sqrt{1+x^2+3xy^2z^3}}
\end{aligned}$$

Example 2.6.9. *Suppose that* $u = \sqrt{x^2+y^2+z^2}, x = \cos\theta, y = \sin\theta, z = \tan\theta$. *Use the chain rule to find* $\frac{du}{d\theta}$ *when* $\theta = \frac{\pi}{4}$ *and then check the result by expressing* u *as a function of* θ *and differentiating directly.*

Given that $u = \sqrt{x^2+y^2+z^2}, x = \cos\theta, y = \sin\theta, z = \tan\theta$. By chain rule,

$$\begin{aligned}
\frac{du}{d\theta} &= \frac{\partial u}{\partial x}\frac{dx}{d\theta} + \frac{\partial u}{\partial y}\frac{dy}{d\theta} + \frac{\partial u}{\partial z}\frac{dz}{d\theta} \\
&= \frac{2x}{2\sqrt{x^2+y^2+z^2}}(-\sin\theta) + \frac{2y}{2\sqrt{x^2+y^2+z^2}}(\cos\theta) \\
&\quad + \frac{2z}{2\sqrt{x^2+y^2+z^2}}(\sec^2\theta)
\end{aligned}$$

Now, $\sqrt{x^2+y^2+z^2} = \sqrt{\cos^2\theta + \sin^2\theta + \tan^2\theta}$

$$= \sqrt{1+\tan^2\theta} = \sec\theta$$

$$\therefore \quad \frac{du}{d\theta} = \frac{\cos\theta}{\sec\theta}(-\sin\theta) + \frac{\sin\theta}{\sec\theta}(\cos\theta) + \frac{\tan\theta}{\sec\theta}(\sec^2\theta)$$

When, $\theta = \frac{\pi}{4}, \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \tan\left(\frac{\pi}{4}\right) = 1$

$$\therefore \quad \left.\frac{du}{d\theta}\right]_{\theta=\frac{\pi}{4}} = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}(2)$$

$$= \sqrt{2}$$

Substituting the the values of x, y and z in u, we get

$$\begin{aligned}
z &= \sqrt{\cos^2\theta + \sin^2\theta + \tan^2\theta} \\
&= \sqrt{1+\tan^2\theta} = \sec\theta \\
\frac{du}{d\theta} &= \sec\theta\tan\theta \\
\left.\frac{du}{d\theta}\right]_{\theta=\frac{\pi}{4}} &= \sqrt{2}
\end{aligned}$$

Example 2.6.10. *Suppose that* $u = x\cos(yz^2), x = \cos t, y = t^2, z = e^t$. *Find the rate of change of* u *with respect to* t *at* $t = \pi$ *by using the chain rule, and then check the result by expressing* u *as a function of* t *and differentiating directly.*

Given that $u = x\cos(yz^2), x = \cos t, y = t^2, z = e^t$. By chain rule,

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \\
&= \cos(yz^2)(-\sin t) + x(-\sin(yz^2)(z^2))(2t) + x(-\sin(yz^2)(2yz))(e^t) \\
&= -\sin t\cos(t^2e^{2t}) - 2te^{2t}\cos t\sin(t^2e^{2t}) - 2t^2e^{2t}\cos t\sin(t^2e^{2t}) \\
\left.\frac{du}{dt}\right]_{t=\pi} &= 0 - 2\pi e^{2\pi}(-1)\sin(\pi^2e^{2\pi}) - 2\pi^2e^{2\pi}(-1)\sin(\pi^2e^{2\pi}) \\
&= 2\pi e^{2\pi}\sin(\pi^2e^{2\pi}) + 2\pi^2e^{2\pi}\sin(\pi^2e^{2\pi})
\end{aligned}$$

Substituting the the values of x, y, z in u, we get

$$\begin{aligned}
u &= \cos t\cos(t^2e^{2t}) \\
\frac{du}{dt} &= \cos t[(-\sin(t^2e^{2t}))(t^2 2e^{2t} + e^{2t}(2t))] + \cos(t^2e^{2t})(-\sin t) \\
\left.\frac{du}{dt}\right]_{t=\pi} &= (-1)[-\sin(\pi^2e^{2\pi})(\pi^2 2e^{2\pi} + e^{2\pi}(2\pi))] + 0 \\
&= 2\pi e^{2\pi}\sin(\pi^2e^{2\pi}) + 2\pi^2e^{2\pi}\sin(\pi^2e^{2\pi})
\end{aligned}$$

Example 2.6.11. *Suppose that* $w = f(x, y, z)$ *is differentiable at the point* $(1, -1, 2)$ *with* $f_x(1, -1, 2) = 3$, $f_y(1, -1, 2) = 5$, *and* $f_z(1, -1, 2) = 3$. *If* $x = t, y = \cos(\pi t)$, *and* $z = t^2 + 1$, *find* $\frac{dw}{dt}$ *when* $t = 1$.

By chain rule,

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= f_x(x, y, z)(1) + f_y(x, y, z)(-\pi\sin(\pi t)) + f_z(x, y, z)(2t)
\end{aligned}$$

When $t = 1, (x, y, z) = (1, -1, 2)$ so that

$$\begin{aligned}
\frac{dw}{dt} &= (3)(1) + (5)(0) + (3)(2) \\
&= 9
\end{aligned}$$

2.6.2 Chain rule for Partial Derivatives

Let $f(x, y)$ be differentiable at (x, y) and $x = x(u, v)$, $y = y(u, v)$ be differentiable functions at (u, v). Then, chain rule for partial derivatives $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ are given by

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad (2.1)$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \quad (2.2)$$

If $f(x, y, z)$ be differentiable at (x,y,z) and $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ be differentiable functions at (u, v, w). Then, chain rule for partial derivatives $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ are given by

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$
$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

The above formula can be extended analogously for functions having more number of variables.

Example 2.6.12. *Suppose that $z = x^2 - y\tan x$, $x = \frac{u}{v}$, $y = uv$. Use appropriate form of chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$*

By chain rule,

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= (2x - y\sec^2 x) \times \frac{1}{v} + (-\tan x)(v) \\
&= \frac{2x - y\sec^2 x - v^2\tan x}{v} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= (2x - y\sec^2 x) \times \left(-\frac{u}{v^2}\right) + (-\tan x)(u) \\
&= -\frac{u(2x - y\sec^2 x + v^2\tan x)}{v^2}
\end{aligned}$$

Example 2.6.13. *Suppose that $z = e^{x^2 y}$, $x = \sqrt{uv}$, $y = \frac{1}{v}$. Use appropriate form of chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$*

By chain rule,

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= e^{x^2 y}(2xy) \times \sqrt{v}\frac{1}{2\sqrt{u}} + e^{x^2 y}(x^2) \times 0 \\
&= xye^{x^2 y}\sqrt{\frac{v}{u}} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= e^{x^2 y}(2xy) \times \sqrt{u}\frac{1}{2\sqrt{v}} + e^{x^2 y}(x^2) \times \left(-\frac{1}{v^2}\right) \\
&= xye^{x^2 y}\sqrt{\frac{u}{v}} - \frac{x^2 e^{x^2 y}}{v^2}
\end{aligned}$$

Example 2.6.14. *If $z = x^2 + 2xy + 4y^2$ and $y = e^{ax}$, find $\frac{dz}{dx}$.*

By chain rule,

$$\begin{aligned}
\frac{dz}{dx} &= \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \\
&= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \\
&= (2x + 2y) + (2x + 8y)ae^{ax}
\end{aligned}$$

Example 2.6.15. *If $u = f(x - y, y - z, z - x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.*

Putting $r = x - y$, $s = y - z$, $t = z - x$, we get $u = f(r, s, t)$.By chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \\
&= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1) \\
&= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \\
&= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0) \\
&= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}
\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}\\ &= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)\\ &= -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\\ &= 0\end{aligned}$$

Example 2.6.16. *If $u = f(x/y, y/z, z/x)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.*

Putting $r = x/y, s = y/z, t = z/x$, we get $u = f(r, s, t)$.By chain rule,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\\ &= \frac{\partial u}{\partial r}\frac{1}{y} + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)\\ &= \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\\ &= \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\frac{1}{z} + \frac{\partial u}{\partial t}(0)\\ &= -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}\\ &= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\frac{1}{x}\\ &= -\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}\end{aligned}$$

$$\begin{aligned}x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} &= \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} - \frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} - \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}\\ &= 0\end{aligned}$$

Example 2.6.17. *Using partial derivative, find $\frac{dy}{dx}$ if $x^3+y^3 = 3axy$ and verify the result using implicit differentiation.*

If the equation $f(x,y) = c$ defines y implicitly as a differentiable function of x, and if $\frac{\partial f}{\partial y} \neq 0$ then we have,

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

Here
$$\begin{aligned}f(x,y) &= x^3 + y^3 - 3axy\\ \frac{\partial f}{\partial x} &= 3x^2 - 3ay\\ \frac{\partial f}{\partial y} &= 3y^2 - 3ax\\ \therefore\quad \frac{dy}{dx} &= -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}\end{aligned}$$

Now, we use implicit differentiation to find $\frac{dy}{dx}$.

$$x^3 + y^3 = 3axy$$

Differentiating with respect to x, we get

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 3a\left[x\frac{dy}{dx} + y(1)\right]\\ \implies 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} &= 3ay - 3x^2\\ \implies \frac{dy}{dx} &= \frac{ay - x^2}{y^2 - ax}\end{aligned}$$

Example 2.6.18. *Find $\frac{du}{dx}$ if $u = x^2 + y^2 - a^2$ where $x^3 + y^3 = a^3$.*

$$\begin{aligned}\frac{du}{dx} &= \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}\\ &= (2x) + (2y)\frac{dy}{dx}\end{aligned}$$

Now, $\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$ where $f(x,y) = x^3 + y^3 - a^3$

$$= -\frac{3x^2}{3y^2} = -\frac{x^2}{y^2}$$

$$\begin{aligned}\therefore\quad \frac{du}{dx} &= 2x + (2y)\left(-\frac{x^2}{y^2}\right)\\ &= 2x - 2\frac{x^2}{y}\end{aligned}$$

Example 2.6.19. *Find* $\frac{du}{dx}$ *if* $u = \sin(x^2 + y^2)$ *where* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$
$$= (2x)\cos(x^2+y^2) + (2y)\cos(x^2+y^2)\frac{dy}{dx}$$

Now, $$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} \quad \text{where } f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$
$$= -\frac{\left(\frac{2x}{a^2}\right)}{\left(\frac{2y}{b^2}\right)} = -\frac{b^2x}{a^2y}$$

$$\therefore \quad \frac{du}{dx} = 2x\cos(x^2+y^2) + (2y)\cos(x^2+y^2)\left(-\frac{b^2x}{a^2y}\right)$$
$$= 2x\cos(x^2+y^2)\left(1 - \frac{b^2}{a^2}\right)$$

Example 2.6.20. *Suppose that* $w = xy + yz$, $y = \sin x, z = e^x$. *Use an appropriate form of the chain rule to find* $\frac{dw}{dx}$.

By chain rule,

$$\frac{dw}{dx} = \frac{\partial w}{\partial x}\frac{dx}{dx} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$
$$= y + (x+z)\cos x + ye^x$$

Example 2.6.21. *If* $u = f(x - ct) + g(x + ct)$ *where* f *and* g *are arbitrary functions of* $x - ct$ *and* $x + ct$ *respectively and* c *is a constant, then show that* $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

$$u = f(x-ct) + g(x+ct)$$
$$= f(r) + g(s) \text{ where } r = x - ct, s = x + ct$$
$$\frac{\partial u}{\partial t} = \frac{df}{dr}\frac{\partial r}{\partial t} + \frac{dg}{ds}\frac{\partial s}{\partial t}$$
$$= \frac{df}{dr}(-c) + \frac{dg}{ds}(c)$$
$$= -c\frac{df}{dr} + c\frac{dg}{ds}$$

$$\frac{\partial^2 u}{\partial t^2} = -c\frac{d^2 f}{dr^2}\frac{\partial r}{\partial t} + c\frac{d^2 g}{ds^2}\frac{\partial s}{\partial t}$$
$$= c^2\frac{d^2 f}{dr^2} + c^2\frac{d^2 g}{ds^2}$$
$$\frac{\partial u}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} + \frac{dg}{ds}\frac{\partial s}{\partial x}$$
$$= \frac{df}{dr}(1) + \frac{dg}{ds}(1)$$
$$= \frac{df}{dr} + \frac{dg}{ds}$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 f}{dr^2}\frac{\partial r}{\partial x} + \frac{d^2 g}{ds^2}\frac{\partial s}{\partial x}$$
$$= \frac{d^2 f}{dr^2} + \frac{d^2 g}{ds^2}$$
$$\therefore \quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

2.6.3 Maxima and Minima of functions of two variables

When we consider functions $f(x)$ of one variable, the tangent through the points where it attains a maximum or minimum (extreme values) will be parallel to the x-axis. . In this case the function $f(x)$ attains a maximum at a point $x = a$ where the first order derivative $\frac{dy}{dx} = 0$ and its second order derivative $\frac{d^2y}{dx^2} < 0$. The function attains local minimum at a point $x = a$ where the first order derivative $\frac{dy}{dx} = 0$ and its second order derivative $\frac{d^2y}{dx^2} > 0$. We can derive an analogously generalized result for functions of two variables x and y.

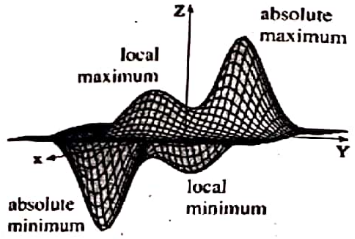
2.6.4 Local maximum and local minimum

A function $f(x.y)$ of two variables x and y is said to attain a **local maximum or relative maximum** at (a,b) if there is an open disk (open ball) $S_r(a,b)$ centred at (a,b) such that for all points (x,y) in the open ball $S_r(a,b)$, the value of the function satisfies $f(x,y) \leq f(a,b)$ and the function is said to have a **local minimum or relative minimum** at (a,b) if for all points (x,y) in the open ball $S_r(a,b)$, the value of the function satisfies $f(x,y) \geq f(a,b)$.

The function $f(x,y)$ is said to attain **global maximum or absolute maxima** at (a,b) if for all points (x,y) in the domain of the function we have $f(x,y) \leq f(a,b)$ and the function is said to have a **global minimum or absolute minima** if for all points (x,y) in the domain of the function we have $f(x,y) \geq f(a,b)$.

When the function $f(x,y)$ has a relative maximum or relative minimum at a point, the function $f(x,y)$ is said to have a **relative extremum (relative extrema)** and if the function $f(x, y)$ has a global maximum or a global minimum at a point, the function is said to have a **global extrema (absolute extrema)** at that point.

Now we discuss methods for finding the local extreme and global extreme values of functions. In the case of functions of a real variable, continuous functions defined on finite closed interval (bounded intervals) will attain its extreme values at points in the interval. If the interval is infinite (unbounded intervals), then there is no guarantee the function will attain its extremum value at points in the interval. For example, if take the function $f(x) = x^2$ defined in $[3, 7]$, it will attain its maximum and minimum points in the interval itself. Here the interval is closed and bounded. The same function defined in the open interval $(3, 7)$ will not attain its minimum and maximum at points in the interval. Similarly the same function treated as a function in $\mathbf{R}$, the set of real numbers, will not attain its maximum and minimum in $\mathbf{R}$ as it is unbounded. We can get analogous results for functions of more than one variable.



A subset R of two dimensional space (or three dimensional space) is said to be bounded if the set S can be enclosed by a rectangle or circle of finite radius (cuboid or sphere of finite radius). Otherwise the set S is said to be unbounded. The set $R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is a unit square and is bounded subset of two dimensional plane, whereas the set $G = \{(x,y) : 0 \leq x \leq 1, y \geq 0\}$ is an infinite strip parallel to the y-axis and is an unbounded subset of two dimensional plane.

Let F be a subset of $\mathbf{R}$, the real line (or of $\mathbf{R^2}$, the two dimensional plane or of $\mathbf{R}^3$ the three dimensional space or of an n-dimensional Euclidean space). A point $p \in R$ is said to be an **adherent point or closure point** of F if every neighbourhood $S_r(p)$ (open ball centred at p with non-zero radius r. In the case of real line it is an open interval and in R^2 it is a circular region without boundary and in R^3 it is a sphere without its outer surface) has points of F. A subset F of $\mathbf{R}$, the real line (or of $\mathbf{R^2}$, the two dimensional plane or of $\mathbf{R^3}$, the three dimensional space or of an n-dimensional Euclidean space) is said to be **closed set** if it contains all of its adherent points. For example closed intervals are closed sets and finite open intervals are not closed sets.

Extreme value theorem: This theorem asserts that if $f(x,y)$ is continuous on a closed and bounded subset R of two dimensional plane, then the function $f(x,y)$ attains both an absolute maximum and absolute minimum at a point on R.

There is no guarantee of absolute maximum or absolute minimum if any one condition such as continuity or closedness or boundedness, is not satisfied.

The first derivative of real function of one variable vanishes at points of local extremes values and global extreme values if it is attained at an interior point of the domain of definition. We have a similar result for functions of more than one variable.

If f(x,y) has a relative extremum (local maximum or local minimum) at a point (a,b) and if the first order partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at (a,b), then both $\dfrac{\partial f}{\partial x} = 0$ and

$\dfrac{\partial f}{\partial y} = 0$ at (a,b)

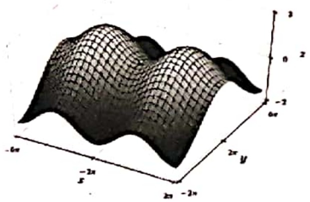
Points at which both the first order partial derivatives of $f(x,y)$ vanishes or one or both of them does not exist are called **critical points.**

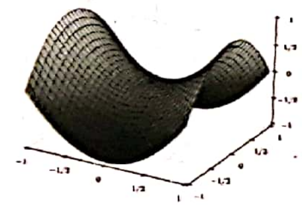
So points at which $f(x,y)$ attains its local extremes or global extremes are critical points.

A critical point at which a function $f(x,y)$ has neither relative minimum nor relative maximum is called a **saddle point** of the function.

The function $f(x,y) = x^2 - y^2$ has neither a relative maximum nor a relative minimum at $(0,0)$. So the point $(0,0)$ is a saddle point of the surface $f(x,y) = x^2 - y^2$.

The following result gives a practical procedure to find the local extremes of function $f(x,y)$.





2.6.5 The second order partial derivative test

Let $f(x,y)$ be a function of two variables x and y such that the second order partial derivatives $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$ are continuous at a critical point (a,b) of $f(x,y)$.

(i) If $rt - s^2 > 0$ and $r < 0$ at the critical point (a,b) then, the function $f(x,y)$ has a relative maximum at that point

(ii) If $rt - s^2 > 0$ and $r > 0$ at the critical point (a,b) then, the function $f(x,y)$ has a relative minimum at that point.

(iii) If $rt - s^2 < 0$ at the critical point (a,b) then, the function $f(x,y)$ has neither a relative maximum nor a relative minimum at that point and the point (a,b) is called a saddle point of $f(x,y)$.

(iv) If $rt - s^2 = 0$, the nature of the $f(x,y)$ at (a,b) cannot be determined using second order partial derivatives. Some other tools are needed to find the nature of the function.

An important special result of function $f(x,y)$ is the following

If a function $f(x, y)$ of two variables has an absolute extremum at an interior point of its domain, then this absolute extremum occurs at a critical point.

Method of finding absolute extreme value of a function

1. Find all critical points (a, b) by solving the system of equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$
2. Find all boundary points at which the absolute extremes can occur.
3. Compute the function value $f(x, y)$ at these points. The points at which the function value is the largest (smallest) are the point of extremas

Example 2.6.22. *Find the critical points of the function $f(x, y) = 2xy - x^3 - y^3$*

$$f(x,y) = 2xy - x^3 - y^3$$
$$\frac{\partial f}{\partial x} = 2y - 3x^2$$
$$\frac{\partial f}{\partial y} = 2x - 3y^2$$

At extreme points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 0 \implies 2y - 3x^2 = 0 \implies y = \frac{3}{2}x^2 \quad (1)$$
$$\frac{\partial f}{\partial y} = 0 \implies 2x - 3y^2 = 0 \quad (2)$$

Solving (1) and (2) we get

$$2x - 3\left(\frac{3}{2}x^2\right)^2 = 0 \implies 8x - 27x^4 = 0 \implies x(27x^3 - 8) = 0$$
$$\implies x = 0, \frac{2}{3}$$

Thus the critical points are $(0, 0)$ and $(2/3, 2/3)$

Example 2.6.23. *Find the relative minima of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$.*

$$f(x,y) = 3x^2 - 2xy + y^2 - 8y$$
$$\frac{\partial f}{\partial x} = 6x - 2y$$
$$\frac{\partial f}{\partial y} = -2x + 2y - 8$$

At extreme points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 0 \implies 6x - 2y = 0 \quad (1)$$
$$\frac{\partial f}{\partial y} = 0 \implies -2x + 2y - 8 = 0 \quad (2)$$

Solving (1) and (2) we get

$$x = 2, y = 6$$

Thus the critical points is $(2, 6)$.

$$r = \frac{\partial^2 f}{\partial x^2} = 6$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = -2$$
$$t = \frac{\partial^2 f}{\partial y^2} = 2$$
$$\text{At } (2,6), r = 6, s = -2, t = 2 \implies rt - s^2 = 8 > 0$$

Therefore $(2, 6)$ is a point of local minima and its relative minimum $= f(2, 6) = 3(2)^2 - 2(2)(6) + 6^2 - 8(6) = -24$.

Example 2.6.24. *Find the extreme values of*

$$u(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$u(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$
$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$
$$\frac{\partial u}{\partial y} = 6xy - 30y$$

At extreme points $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

$$\frac{\partial u}{\partial x} = 0 \implies 3x^2 + 3y^2 - 30x + 72 = 0 \implies x^2 + y^2 - 10x + 24 = 0 \quad (1)$$
$$\frac{\partial u}{\partial y} = 0 \implies 6xy - 30y = 0 \implies y(x-5) = 0 \implies y = 0, x = 5 \quad (2)$$
$$x = 5 \implies 5^2 + y^2 - 10(5) + 24 = 0 \implies y^2 - 1 = 0 \implies y = \pm 1$$

Therefore $(5, 1)$ and $(5, -1)$ are two stationary points.

$$y = 0 \implies x^2 - 10x + 24 = 0 \implies x = 4, 6$$

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Therefore (4, 0) and (6, 0) are also stationary points.

$$r = \frac{\partial^2 u}{\partial x^2} = 6x - 30$$
$$s = \frac{\partial^2 u}{\partial x \partial y} = 6y$$
$$t = \frac{\partial^2 u}{\partial y^2} = 6x - 30$$

At $(4,0), r = -6, s = 0, t = -6 \implies rt - s^2 = 36 > 0, r < 0$

Therefore (4, 0) is a maxima point and maxima= $f(4,0) = 112$.

At $(6,0), r = 6, s = 0, t = 6 \implies rt - s^2 = 36 > 0, r > 0$

Therefore (6, 0) is a minima point and minima $= f(6,0) = 108$.

At $(5,1), r = 0, s = 6, t = 0 \implies rt - s^2 = -36 < 0$

Therefore (5, 1) is a saddle point.

At $(5,-1), r = 0, s = -6, t = 0 \implies rt - s^2 = -36 < 0$

Therefore (5, −1) is a saddle point.

Example 2.6.25. *Locate all relative extrema and saddle points of*

$$f(x,y) = 4xy - x^4 - y^4$$

$$f(x,y) = 4xy - x^4 - y^4$$
$$\frac{\partial f}{\partial x} = 4y - 4x^3$$
$$\frac{\partial f}{\partial y} = 4x - 4y^3$$

At extreme points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 0 \implies 4y - 4x^3 = 0 \implies y = x^3 \qquad (1)$$
$$\frac{\partial f}{\partial y} = 0 \implies 4x - 4y^3 = 0 \implies x = y^3 \qquad (2)$$

Solving (1) and (2) we get

$$x = (x^3)^3 \implies x^9 - x = 0 \implies x(x^4 - 1)(x^4 + 1) = 0$$
$$\implies x(x^2 + 1)(x^- 1)(x^4 + 1) = 0$$
$$\implies x = 0, 1, -1$$

Thus the critical points are (0, 0), (1, 1) and (−1, −1)

$$r = \frac{\partial^2 f}{\partial x^2} = -12x^2$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = 4$$
$$t = \frac{\partial^2 f}{\partial y^2} = -12y^2$$

At $(0,0), r = 0, s = 4, t = 0 \implies rt - s^2 = -16 < 0$

Therefore (0, 0) is a saddle point.

At $(1,1), r = -12, s = 4, t = -12 \implies rt - s^2 = 128 > 0, r < 0$

Therefore (1, 1) is a relative maxima point and maxima $= f(1,1) = 2$.

At $(-1,-1), r = -12, s = 4, t = -12 \implies rt - s^2 = 128 > 0, r < 0$

Therefore (−1, −1) is a relative maxima point and maxima $= f(-1,-1) = 2$.

Example 2.6.26. *Discuss the maxima and minima of $xy(a - x - y)$*

$$f(x,y) = axy - x^2y - xy^2$$
$$\frac{\partial f}{\partial x} = ay - 2xy - y^2$$
$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy$$

At extreme points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 0 \implies ay - 2xy - y^2 = 0$$
$$\implies y(a - 2x - y) = 0 \implies y = 0 \text{ or } 2x + y = a \qquad (1)$$
$$\frac{\partial f}{\partial y} = 0 \implies ax - x^2 - 2xy = 0$$
$$\implies x(a - x - 2y) = 0 \implies x = 0 \text{ or } x + 2y = a \qquad (2)$$

Solving (1) and (2), we get the critical (0, 0), (a, 0), (0, a) and (a/3, a/3)

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y$$
$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

At $(0,0), r = 0, s = a, t = 0 \implies rt - s^2 = -a^2 < 0$

Therefore $(0,0)$ is a saddle point.

$$\text{At } (a,0), r=0, s=-a, t=-2a \implies rt-s^2=-a^2<0$$

Therefore $(a,0)$ is a saddle point.

$$\text{At } (0,a), r=-a, s=-a, t=0 \implies rt-s^2=-a^2<0$$

Therefore $(0,a)$ is a saddle point.

$$\text{At } (a/3,a/3), r=-\frac{2a}{3}, s=-\frac{a}{3}, t=-\frac{2a}{3} \implies rt-s^2=\frac{a^2}{3}>0$$

$r=-\frac{2a}{3}>0$ when $a<0$ and $r<0$ when $a>0$. Therefore $(a/3,a/3)$ is a relative mi point when $a<0$ and maxima point when $a>0$.

Example 2.6.27. *Find the maxima and minima of x^3+y^3-3axy*

$$f(x,y)=x^3+y^3-3axy$$
$$\frac{\partial f}{\partial x}=3x^2-3ay$$
$$\frac{\partial f}{\partial y}=3y^2-3ax$$

At extreme points $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

$$\frac{\partial f}{\partial x}=0 \implies 3x^2-3ay=0 \implies x^2=ay \quad (1)$$
$$\frac{\partial f}{\partial y}=0 \implies 3y^2-3ax=0 \implies y^2=ax \quad (2)$$

Solving (1) and (2) we get

$$\left(\frac{x^2}{a}\right)^2=ax \implies x^4-a^3x=0 \implies x(x^3-a^3)=0$$
$$\implies x=0,a$$

Thus the critical points are $(0,0)$ and (a,a)

$$r=\frac{\partial^2 f}{\partial x^2}=6x$$
$$s=\frac{\partial^2 f}{\partial x\partial y}=-3a$$
$$t=\frac{\partial^2 f}{\partial y^2}=6y$$

$$\text{At } (0,0), r=0, s=-3a, t=0 \implies rt-s^2=-9a^2<0$$

Therefore $(0,0)$ is a saddle point.

$$\text{At } (a,a), r=6a, s=-3a, t=6a \implies rt-s^2=27a^2>0$$

$r=6a<0$ when $a<0$ and $r>0$ when $a>0$. Therefore (a,a) is a minimum point of if $a>0$ and (a,a) is a maximum point of f if $a<0$.

Example 2.6.28. *A rectangular box open at the top is to have volume 32 cubic feet. Find its dimensions if the total surface area is minimum.*

Let x,y,z be the length, breadth and height of the box with surface area S and volume $V=32\ ft^3$. Then

$$S=xy+2xz+2yz$$

Given that

$$xyz=32 \implies z=\frac{32}{xy}$$
$$\therefore S=xy+\frac{64}{y}+\frac{64}{x}$$
$$\frac{\partial S}{\partial x}=y-\frac{64}{x^2}$$
$$\frac{\partial S}{\partial y}=x-\frac{64}{y^2}$$

At extreme points $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

$$\frac{\partial S}{\partial x}=0 \implies y-\frac{64}{x^2}=0 \implies x^2y=64$$
$$\frac{\partial S}{\partial y}=0 \implies x-\frac{64}{y^2}=0 \implies xy^2=64$$

Solving we get

$$x^2y=64=xy^2 \implies x^2y=xy^2 \implies xy(x-y)=0 \implies x=y$$
$$x=y \implies x^3=64 \implies x=4 \implies y=4 \text{ and } z=2$$
$$r=\frac{\partial^2 S}{\partial x^2}=\frac{128}{x^3}$$
$$s=\frac{\partial^2 S}{\partial x\partial y}=1$$
$$t=\frac{\partial^2 S}{\partial y^2}=\frac{128}{y^3}$$
$$\text{At } x=4, y=4, z=2$$
$$rt-s^2>0, r>0$$

Therefore S is minimum when $x=y=4, z=2$.

Example 2.6.29. *Find the absolute extrema of the function $f(x,y)=xy-4x$ on R where R is the rectangular region with vertices $(0,0),(0,4)$ and $(4,0)$.*

$$f(x,y)=xy-4x$$
$$\frac{\partial f}{\partial x}=y-4$$
$$\frac{\partial f}{\partial y}=x$$

At extreme points $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

$$\frac{\partial f}{\partial x}=0 \implies y-4=0 \implies y=4$$
$$\frac{\partial f}{\partial y}=0 \implies x=0$$

Thus $(0,4)$ is the only one critical point and it is a boundary point of R.

Next we will find all boundary points at which the absolute extrema can occur. On the line segment between $(0,0)$ and $(4,0)$, $y=0$ and so

$$f(x,y)=f(x,0)=-4x,\ 0\le x\le 4$$

Now, $f'(x,0)=-4\neq 0$ for $0\le x\le 4$, $f(x,0)$ has no critical points. Hence extreme values occur at $(0,0)$ and $(4,0)$.

On the line segment between $(4,0)$ and $(0,4)$, $\frac{x}{4}+\frac{y}{4}=1 \implies x+y=4$ and so

$$f(x,y)=x(4-x)-4x=-x^2,\ 0\le x\le 4$$

Now, $f'(x,x)=-2x$, $f'(x,x)=0 \implies -2x=0 \implies x=0$. Hence extreme values occur at $(4,0)$ and $(0,4)$.
Similarly, on the line segment between $(0,0)$ and $(0,x)$, $x=0$ and so

$$f(x,y)=f(y,0)=0,\ 0\le y\le 4$$

Hence extreme values occur at $(0,0)$ and $(0,4)$.
Now,

$$f(0,0)=0,\ f(4,0)=-16, f(0,4)=0$$

Hence absolute minimum $=-16$ and absolute maximum$=0$

2.7 Exercise

1. Let $f(x,y)=\begin{cases}\frac{x^2-xy}{x+y}, & (x,y)\neq 0\\ 0, & (x,y)=(0,0)\end{cases}$. Find $f_x(0,0)$ and $f_y(0,0)$ at origin.

2. Let $f(x,y)=\frac{x}{x^2+y^2}$. Find $f_x(x,y)$ and $f_y(x,y)$.

3. If $u=x^2\sin\left(\frac{y}{x}\right)$, find $\frac{\partial^2 u}{\partial x\partial y}$ and $\frac{\partial^2 u}{\partial y\partial x}$.

4. If $u=\log\left(\frac{xy}{x+y}\right)$, verify that $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$.

5. If $u=\log r$ where $r^2=(x-a)^2+(y-b)^2+(z-c)^2$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{r^2}$

6. The deflection at the center of an iron rod of length l and diameter d, supported at the ends and loaded at center with a weight w is known to be proportional to $\frac{wl^3}{d^4}$, what is the percentage increase in the deflection if the load w is increased by 4% , the length by 2% and the diameter by 1%

7. The tortional rigidity of a length of wire is obtained from the formula $N=\frac{8\pi Il}{t^2r^4}$. If l is decreased by 2% , r is increased by 2% and t is increased by 1%, show that the value of N is diminished by 12% approximately.

8. Find the percentage error in the area of an ellipse if 1% error made in measuring each of major and minor axis.

9. The range R of a projectile which starts with a velocity v at an elevation α is given by $R=\frac{v^2\sin 2\alpha}{g}$. Find the percentage of error in R due to an error of 1% in v and $\frac{1}{2}$% in α when $\alpha\frac{\pi}{3}$

10. The length,width, and height of a rectangular box are measured with an error of at most 5%. Use a total differential to estimate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

11. using the formula $R=\frac{E}{C}$, find the maximum error and the percentage error in R if $C=20$ with a possible error of 0.1 and $E=120$ with a possible error of 0.05

12. If z is a function of x and y and $x=u-v\,y=uv$, show that
(i)$(u+v)\frac{\partial z}{\partial x}=u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}$ and
(ii)$(u+v)\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$

13. If $u=\sin(xy^2)$ where $x=\log t\,y=e^t$, find $\frac{du}{dt}$ by using the chain rule for total derivative and verify the answer by obtaining $\frac{du}{dt}$ after substituting for x,y in terms of t in u.
Solution: $\frac{du}{dt}=\frac{e^{2t}}{t}\cos(e^{2t}\log t)(1+2t\log t)$

14. If $u=\sin^{-1}(x-y)$ where $x=3t, y=4t^3$, show that $\frac{du}{dt}=\frac{3}{\sqrt{1-t^2}}$

15. If $u=x^2+y^2+z^2$ where $x=e^{2t}\,y=e^{2t}\cos 3tz=e^{2t}\sin 3t$, find $\frac{du}{dt}$ by using the chain rule for total derivative and verify the answer by obtaining $\frac{du}{dt}$ after substituting for

x, y and z in terms of t in u.

Solution: $\frac{du}{dt} = 8e^{4t}$

16. If $u = xyz$ where $x = e^{-t}$ $y = e^{-t}\sin^2 3t$ $z = \sin t$, find $\frac{du}{dt}$ by using the ch for total derivative and verify the answer by obtaining $\frac{du}{dt}$ after substituting x, and z in terms of t in u.

Solution: $\frac{du}{dt} = e^{-2t}\sin^2 t(3\cos t - 2\sin t)$

17. Using partial derivative, find $\frac{dy}{dx}$ if $x^3 + y^2x - 3 = 0$ and verify the result implicit differentiation.

Solution: $\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$

18. Find $\frac{du}{dx}$ if $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$.

Solution: $\frac{du}{dx} = 1 + \log(xy) - \frac{x(x^2 + y)}{y(y^2 + x)}$

19. Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their na
Solution: The critical points are $(0,0), (\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. Nature of $(0,0)$ is undecided. At $(\pm\sqrt{2}, \pm\sqrt{2})$, f has relative minimum value 8.

20. Find the maxima and minima of $x^3y^2(12 - x - y)$.
Solution: At $(1/2, 1/3)$ the function has a relative maximum with value 1/4

21. Examine the extrema of $f(x, y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$

Solution: At $\left((1/3)^{1/3}, (1/3)^{1/3}\right)$ the function has relative minimum with v $3^{4/3}$

22. Show that of all rectangular parallelopiped with given volume , the cube has least surface area.