MODULE 1

HARMONIC OSCILLATIONS

INTRODUCTION

A motion that is repeated at regular intervals of time is calledperiodic motion. Examples of periodic motion are the oscillation of simple pendulum,the motion of earth around the sun,the motion of satellite around a planet,oscillations of a loaded spring etc.

If the particle moves back and forth along the same path and repeats itself at regular intervals of time is called oscillatory or vibratory motion. The motion of the prongs of a tuning fork, motion of simple pendulum, the vertical oscillations of a loaded spring, the motion of pendulum of clock are the examples of oscillatory motion.

Here we shall define few terms regarding the oscillatory motion.

1. **Frequency (f) is the number of oscillations per second.**
2. **Time period (T) is the time required for one oscillation**

$$T = \frac{1}{f}$$

3. **Displacement (x) is the distance of the particle from the equilibrium position.**
4. **Amplitude (A) is the maximum displacement**

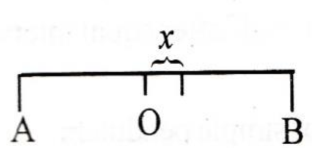
The solution of the equations of periodic motion of systems can always be expressed as functions of sines and cosines. The motion described by functions of Sines and cosines are called harmonic motion.

SIMPLE HARMONIC MOTION (SHM)

A particle is said to execute SHM if it moves to and fro periodically along a path such that the restoring force acting on it is proportional to its displacement from a fixed point and is always directed towards that point.A body moving simple harmonically is called a harmonic oscillator.

Example of simple harmonic motion is the oscillations of a simple pendulum.

Equation of SHM



Consider a particle of mass 'm' executing SHM alonga straight line. 'O' is the equilibrium position. Let 'x' be the displacement at an instant.

$$Acceleration\ of\ the\ particle = \frac{d^2x}{dt^2}$$

$$\text{Re}\ storing\ \ force\ on\ the\ particle = ma = m\frac{d^2x}{dt^2}$$

$$By\ definition\ of\ SHM,\ \ m\frac{d^2x}{dt^2} \propto -x$$

$$m\frac{d^2x}{dt^2} = -kx$$

Where 'k' is the proportionality constant. Negative sign indicates that restoring force acts against displacement.

$$m\frac{d^2x}{dt^2} + \ \ kx = 0$$

$$\frac{d^2x}{dt^2} + \ \ \frac{k}{m}x = 0$$

$$put\ \frac{k}{m} = \omega^2$$

$$\frac{d^2x}{dt^2} + \omega^2\ \ x = 0 ------(1)$$

This is the differential equation for SHM.

Solution

$$We\ have\ \ \frac{d^2x}{dt^2} + \omega^2\ \ x = 0$$

$$Multiplying\ \ the\ equation\ by\ \ 2\frac{dx}{dt},$$

$$2\frac{dx}{dt}.\frac{d^2x}{dt^2} + 2\frac{dx}{dt}\omega^2\ \ x ------(2)$$

$$Eq(2)\ can\ be\ written\ as$$

$$\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \omega^2\ \ x^2\right] = 0$$

$$Integrating,\ \left(\frac{dx}{dt}\right)^2 + \omega^2 x^2\ = C ------(3)$$

Where C is the constant of integration.

To find C:

The velocity of the particle at the extreme position is zero. If 'a' is the amplitude, Then

$$\frac{dx}{dt} = 0, \; at \; x = a$$

$putting \; in \, equation \, (3) \, we \, get \; C = \omega^2 a^2$

$putting \; this \, in \, eq \, (3)$

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = \omega^2 a^2$$

$$\left(\frac{dx}{dt}\right)^2 = \omega^2 (a^2 - x^2)$$

$$\frac{dx}{dt} = \omega\sqrt{(a^2 - x^2)} ---------(4)$$

$$ie \quad velocity = \omega\sqrt{(a^2 - x^2)} -------(5)$$

$$From \, eq \, (4), \quad \frac{dx}{\sqrt{(a^2 - x^2)}} = \omega dt$$

$$Integrating, \quad Sin^{-1}\frac{x}{a} = \omega t + \phi$$

$where \, \phi \; is \, cons\tan t \, of \; \int egration.$

$$ie \qquad \frac{x}{a} = Sin(\omega t + \phi)$$

$$x = aSin(\omega t + \phi) -------(6)$$

$(\omega t + \phi)$ $is \, the \; phase \, of \; oscillation \, at \, any \, ins\tan t.$

$$let \; initial \; phase \, \phi = \delta + \frac{\pi}{2}$$

$$Then \qquad x = aSin(\omega t + \delta + \frac{\pi}{2})$$

$$x = aCos(\omega t + \delta) -----------(7)$$

$this \; also \; represents \; SHM.$

$If \;\; time \; t \;\; in \, eq \, (6) \, is \, increased \; by \; \frac{2\pi}{\omega},$

$$x = aSin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right] =$$

$$x = aSin(\omega t + 2\pi + \phi)$$

$$x = aSin(\omega t + \phi)$$

$The \; function \, repeats \, itself \; after \, a \, time \;\; \frac{2\pi}{\omega}, \frac{4\pi}{\omega} \, etc.$

$$Hence \, it \, is \, called \;\; Period \;\; T \qquad \therefore T = \frac{2\pi}{\omega}$$

$$But \;\; \omega^2 = \frac{k}{m}; \quad \omega = \sqrt{\frac{k}{m}} \qquad \therefore \; T = 2\pi\sqrt{\frac{m}{k}}$$

FREE OSCILLATIONS

Consider a particle executing oscillations. If no forces are acting on it, the oscillations will continue for an indefinite period without change in amplitude. In this oscillation total energy of the system remains the same always. This type of oscillations is called free oscillations

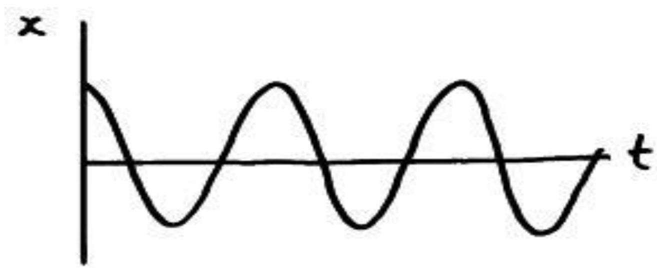


Fig: Free Oscillations

DAMPED HARMONIC OSCILLATION

A harmonic oscillator in which the motion is damped by the action of an additional force is called a damped harmonic oscillator. Damped oscillations are oscillations under the action of resistive forces.

Damping force is α to its velocity.

ie Damping force = b $\frac{dx}{dt}$

'b' is called the damping constant.Including the damping force, the differential eq. for a damped harmonic oscillator is

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = 0$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 ---------(1)$$

$$Let\ \frac{b}{m} = 2\gamma \quad where\ \gamma\ is\ the\ damping\ coefft.$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = 0 ---------(2)$$

This is the differential eqn for damped harmonic oscillator

Solutions to the Equation

Assume the solution is of the form

$$x = Ae^{\alpha t}$$

$$\frac{dx}{dt} = \alpha Ae^{\alpha t}$$

$$\frac{d^2x}{dt^2} = \alpha^2 Ae^{\alpha t} = \alpha^2 x$$

$$\text{Putting these values in (2), we get}$$

$$\alpha^2 x + 2\gamma\alpha x + \omega^2 x = 0$$

$$ie \quad \alpha^2 + 2\gamma\alpha + \omega^2 = 0$$

$$\text{The solution} \quad \alpha = \frac{-2\gamma \pm \sqrt{\left(4\gamma^2 - 4\omega^2\right)}}{2}$$

$$x = Ae^{\left(-\gamma \pm \sqrt{\left(\gamma^2 - \omega^2\right)}\right)t}$$

$$\text{The solutions can be written as}$$

$$x_1 = A_1 e^{\left(-\gamma + \sqrt{\left(\gamma^2 - \omega^2\right)}\right)t} \quad and \quad x_2 = A_2 e^{\left(-\gamma - \sqrt{\left(\gamma^2 - \omega^2\right)}\right)t}$$

$$\therefore \text{The general solution is } x = x_1 + x_2$$

$$x = A_1 e^{\left(-\gamma + \sqrt{\left(\gamma^2 - \omega^2\right)}\right)t} + A_2 e^{\left(-\gamma - \sqrt{\left(\gamma^2 - \omega^2\right)}\right)t} ----(3)$$

$$\text{Where } A_1 \text{ and } A_2 \text{ are constants which depends}$$

$$\text{on the initial values of position and velocity.}$$

$$'\gamma' \text{ determines the behaviour of the system.}$$

<u>Case 1</u>: γ>ω

The roots of the auxiliary equation are real

$$\alpha_1 = -\gamma + \sqrt{\left(\gamma^2 - \omega^2\right)}; \qquad \alpha_2 = -\gamma - \sqrt{\left(\gamma^2 - \omega^2\right)} ---(4)$$

$$\text{The solution of equation (3) is}$$

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} -----------(5)$$

Where A_1 and A_2 are constants.

Let the mass be given a displacement x_0 and then released ,So that

$$x = x_0, \quad \frac{dx}{dt} = 0 \ at \ t = 0$$

$$From\, eq(5)\, we\, get \quad x_0 = A_1 + A_2$$

$$and \quad A_1\alpha_1 + A_2\alpha_2 = 0$$

$$Simplifying, \ A_1 = -\frac{x_0\alpha_2}{\alpha_1 - \alpha_2} \quad and \quad A_2 = \frac{x_0\alpha_1}{\alpha_1 - \alpha_2} ---(6)$$

$$\therefore Eq(5) \ reduces \ to \quad x = \frac{x_0}{\alpha_1 - \alpha_2}\left(\alpha_1 e^{\alpha_2 t} - \alpha_2 e^{\alpha_1 t}\right)$$

This shows that x is always positive and decreases exponentially to zero without any oscillations.The motion is called over damped or dead beat motion

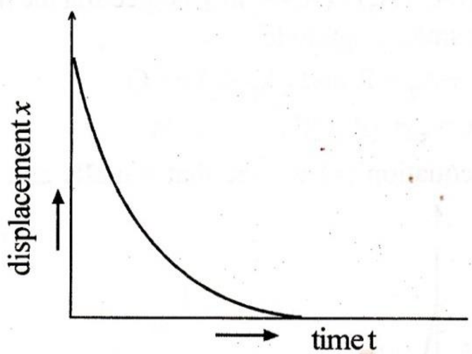


fig: Over damped

Case 2: γ=ω

The roots of the auxiliary equation are real and equal each being equal to –γ.

The general solution of eq (2) is then

$$x = (A_1 + A_2 t)e^{-\gamma t}$$

Here also if $x = x_0, \quad \frac{dx}{dt} = 0 \ at \ t = 0$

The constants $A_1 = x_0$ and $A_2 = \gamma x_0$

Hence the solution to the equation (2) is

$$x = x_0(1 + \gamma t)e^{-\gamma t}$$

Again x is always positive and decreases to zero.The nature of the motion is similar to that of case 1.But it is different from the non-oscillatory motion of case 1 .Hence this motion is termed as critically damped. Here the particle acquires the position of equilibrium very rapidly.

Application

It is applied in automobile shock absorbers door closer mechanism and recoil mechanism in gun.If the door closes without performing any oscillation, it is an example of critically damped system.

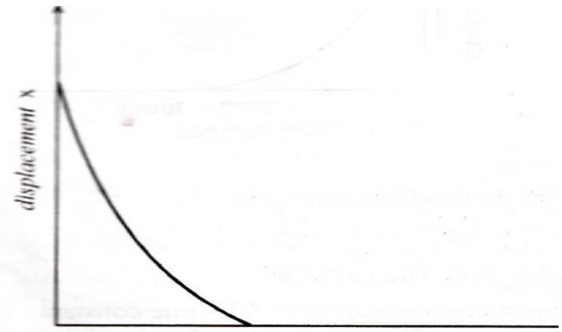


Fig. Critically damped

Case3: $\gamma < \omega$ $The\ damping\ is\ low$

Here $\sqrt{\gamma^2 - \omega^2}$ $is\ imaginary$

Let $\sqrt{\gamma^2 - \omega^2} = i\omega' = i\sqrt{\omega^2 - \gamma^2}$

Where $\omega' = \sqrt{\omega^2 - \gamma^2}$ -----------(8)

From eq (3),

X=$A_1 e^{(-\gamma + i\omega')t} + A_2\, e^{(-\gamma - i\omega')t}$ ------(9)

=$e^{-\gamma t}$ ($A_1 e^{i\omega' t} + A_1 e^{-i\omega' t}$)

=$e^{-\gamma t}$(A_1($Cos\ \omega' t + i\ Sin\ \omega' t$)) + A_2($Cos\ \omega' t -$

$i\ Sin\ \omega' t$)

X=$e^{-\gamma t}$((A_1+A_2)$Cos\ \omega' t + i$(A_1-A_2) $Sin\ \omega' t$)

Put (A_1+A_2) = $x_0 \sin\varphi$

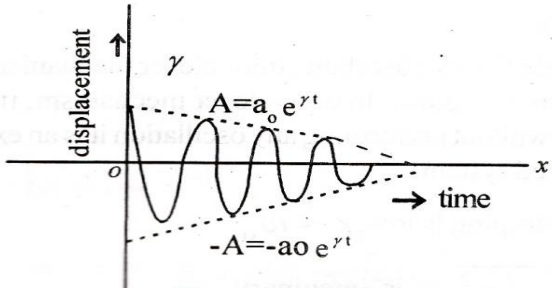
And i(A_1-A_2) = $x_0 \cos\varphi$

X = $x_0\, e^{-\gamma t}$ Sin($\omega' t + \varphi$) $- - - - -$ (10)

Equation (10) shows that the motion is oscillatory.The amplitude $x_0 e^{-\gamma t}$ Is not a constant but decreases with time. The motion is not periodic.

Period of oscillation $T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega^2-\gamma^2}}$

The oscillations are as shown in the figure.



damped oscillations

Effect of damping

1. **Amplitude of oscillations decreases exponentially with time.**
2. **The time period of oscillation of a damped oscillator is greater than the time period of un damped(free) oscillation**
3. **The frequency of oscillation of a damped oscillator is less than the frequency of undamped oscillation.**

FORCED HARMONIC OSCILLATOR

If an external force acts on a damped oscillatory system it is called a Forced Harmonic Oscillator. The oscillations produced, under the action of external periodic force on the body is called forced oscillations.

The energy of a damped oscillator decreases with the passage of time. It is possible to compensate for the energy loss by applying a suitable external periodic force.Foreg: a swing.

The frequency of the forced oscillation will be different from the natural frequency of the body.Let ω be the frequency of natural and ω_f the frequency of the forced vibrations when the body is subjected to an oscillatory external force $F_0 \sin \omega_f t$.

The Equation of motion is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = \mathrm{F_0} \sin \omega_f \mathrm{t}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{\mathrm{F_0}}{m}\sin \omega_f \mathrm{t} --------(1)$$

$$Let\ \frac{b}{m} = 2\gamma \quad where\ \gamma\ is\ the\ damping\ coefft\ and\ \omega^2 = \frac{k}{m}$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = \mathrm{f_0} \sin \omega_f \mathrm{t} --------(2)$$

$$where\ \mathrm{f_0} = \frac{\mathrm{F_0}}{m}$$

This is the differential eqn for damped harmonic oscillator

Here a tussle occurs between the external periodic force and the damping force. After some time the system reaches a steady state. The resulting oscillations are forced oscillations. Its frequency is that of the external periodic force and not that of the natural frequency of the body.When steady state is reached, the solution of the above equation is given by

$$x = A\sin(\omega_f t - \theta) ------(3)$$

$$\frac{dx}{dt} = A\omega_f \cos(\omega_f t - \theta)$$

$$\frac{d^2x}{dt^2} = -A\omega_f^{\,2} \sin(\omega_f t - \theta)$$

Putting these values in (2)

$$-A\omega_f^{\,2}\sin(\omega_f t-\theta) + 2\gamma A\omega_f\cos(\omega_f t-\theta) + \omega^2 A\sin(\omega_f t-\theta)$$
$$= f_0 \sin\left[(\omega_f t-\theta)+\theta\right] -------(4)$$

Expanding RHS

$$= f_0\sin(\omega_f t-\theta)\cos\theta + f_0\cos(\omega_f t-\theta)\sin\theta$$

$$\left[-A\omega_f^{\,2} - f_0\cos\theta + \omega^2 A\right]\sin(\omega_f t-\theta) +$$
$$\left[2\gamma A\omega_f - f_0\sin\theta\right]\cos(\omega_f t-\theta) = 0 ---(5)$$

To find A:

For this eq. to hold good for all values of t, *the coeffts of the term* $\sin(\omega_f t-\theta)$ *and* $\cos(\omega_f t-\theta)$ *must vanish separately.*

ie $\left[-A\omega_f^{\,2} - f_0\cos\theta + \omega^2 A\right] = 0$

$$\therefore\ -A\omega_f^{\,2} + \omega^2 A = f_0\cos\theta ---------(6)$$

$$\left[2\gamma A\omega_f - f_0\sin\theta\right] = 0$$

$$\therefore\ \left[2\gamma A\omega_f = f_0\sin\theta\right] ----------(7)$$

Squaring and eq. (6) *and* (7),

$$\left(-A\omega_f^{\,2} + \omega^2 A\right) + 4\gamma^2 A^2\omega_f^{\,2} = f_0^{\,2}$$

$$A = \frac{f_0}{\sqrt{\left(\omega^2 - \omega_f^{\,2}\right)^2 + 4\gamma^2\omega_f^{\,2}}} ----(8)$$

This is the amplitude of the Forced vibration.

Dividing eq. (7) *by* (6)

$$Tan\,\theta = \frac{2\gamma\omega_f}{\omega^2 - \omega_f^{\,2}} -----------(9)$$

This gives the phase difference between the forced oscillations and the applied force.

Substituting for A in eq.(3)

$$x = \frac{f_0 \quad .Sin(\omega_f t - \theta)}{\sqrt{(\omega^2 - \omega_f^{\,2}) + 4\gamma^2 \omega_f^{\,2}}}$$

The above equation shows that the system vibrates with the frequency of the applied periodic force and is having a phase lag of θ.

Different cases :

Case(1) $\omega_f < \omega$ *(low driving frequency)*

We have $$A = \frac{f_0}{\sqrt{(\omega^2 - \omega_f^{\,2}) + 4\gamma^2 \omega_f^{\,2}}}$$

Neglecting $\omega_f^{\,2}$, $$A = \frac{f_0}{\omega^2} = \frac{F_0}{m} \Big/ \frac{k}{m} = \frac{F_0}{k}$$

Amplitude does not depend on the mass of the oscillating body, but depends on the force constant.

case 2 : $\omega_f = \omega$ (Resonance condition)

The particular frequency of the applied periodic force at which the amplitude becomes maximum is called rsonant frequency. when the frequency of the applied force is equal to the natural frequency of the oscillator the smplitude will be maximum. This condition is known as resonance.

Amplitude at Resonance $A_{max} = \dfrac{f_0}{2\gamma\omega_f}$

The amplitude of the forced oscillations at resonance is called resonant amplitude

$$Tan\,\theta = \frac{2\gamma\omega_f}{0} = \infty \;; \qquad \theta = \frac{\pi}{2}$$

The displacement lags the applied voltage by $\frac{\pi}{2}$.

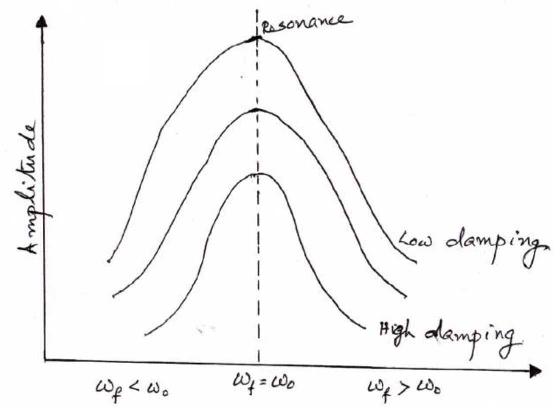
The value of amplitude depends upon the damping coefft, γ.

$$Case\ 3:\quad \omega_f > \omega\ (high\ driving\ \ frequency)$$

$$A = \frac{f_0}{\sqrt{\left(\omega^2 - \omega_f{}^2\right)^2 + 4\gamma^2\omega_f{}^2}}$$

$$When\ \ \omega_f > \omega\ and\ for\ low\ damping\quad A = \frac{f_0}{\omega_f{}^2}$$

Graph: Variation of Amplitude A with ω



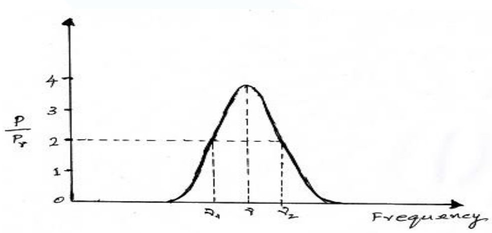
Sharpness of Resonance

When ω_f is increased or decreased from resonance frequency,the amplitude falls from the maximum value.

The term sharpness of resonance refers to the rate of decrease of amplitude with the change in frequency of the applied periodic forceon either side of the resonant frequency.

Let P_r be the power absorbed at resonance and P , the power absorbed at any other frequency γ.

The graph between P/ P_r and frequency is as follows



At the frequency values the power absorbed is half of the maximum and these γ_1 and γ_2 values are called half power points.The frequency difference between these two half power points is called the bandwidth of the oscillator.

ie Bandwidth Δ γ= γ2 - γ1

The average energy over a cycle is given by

$$E = E_0 e^{-2\gamma t}$$

$$E = E_0 e^{-\frac{t}{\tau}}$$

$$where\ \tau = \frac{1}{2\gamma}\ is\ called\ the\ relaxation\ time(\tau)$$

$$E_0\ is\ the\ initial\ energy\ and\ E\ is\ the\ average\ energy\ per\ cycle.$$

$$When\ t = \tau, E = \frac{E_0}{e}$$

$$(where\ e\ is\ the\ base\ of\ natural\ \log arithm.)$$

Hence the relaxation time of an oscillating system may be defined as the time taken by the system to reduce its average total energy to 1/e (1/2.72 =37%) of its initial energy.

Quality Factor (Q - factor)

Quality factor represents the efficiency of the oscillator.

Q - factor is defined as 2π times the energy stored in the oscillator to the energy dissipated per cycle.

ie

$$Q = \frac{2\pi \times \text{Energy stored in the oscillator}}{\text{Energy lost per cycle}}$$

$$= \frac{2\pi \times \text{Energy stored in the oscillator}}{\text{Energy lost per sec} \times \text{Time period}}$$

$$= \frac{2\pi E}{-\frac{dE}{dt}T} \quad \text{where E is the energy. } -\frac{dE}{dt} \text{ is the power dissipated per sec, P.} \quad \text{But } P = 2\gamma E$$

$$Q = \frac{2\pi E}{PT} = \frac{2\pi E}{2\gamma E T} = \frac{\pi}{\gamma T} = \frac{\pi}{\gamma \frac{2\pi}{\omega}} = \frac{\omega}{2\gamma}$$

$$\therefore\ Q = \frac{\omega}{2\gamma} = \omega\tau,$$

$$\frac{1}{2\gamma} = \tau, relaxation\ time$$

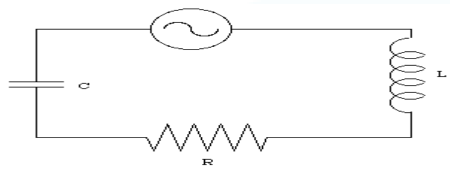
LCR circuit as an Electrtical analogue of mechanical oscillator

A pure LC circuit is an electrical analogue of simple pendulum. In the case of simple pendulum, the energy alternates betweenpotential and kinetic energy.

In the case of LC circuit when the electrical oscillation takes place, the energy is alternately shared in the capacitor as electric field and in the inductor as magnetic field.

The frequency oscillation in LC circuit, $n = \frac{1}{2\pi\sqrt{LC}}$

Forced oscillation in a series LCR circuit



$$\text{Applying Kirchoff's law to the circuit}$$

$$V_L + iR + V_C = V_0 \sin \omega t$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin \omega t$$

$$ie \quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = V_0 \sin \omega t$$

This is the differential equation in the case of forced oscillations in LCR circuit

Mechanical Oscillator → $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega_f t$

Electrical Oscillator → $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{V_0}{L}\sin \omega t$

Comparison between Mechanical and Electrical Oscillator

Mechanical Oscillator	Electrical Oscillator
1. Displacement x	Charge q
2. Velocity dx/dt	Current dq/dt
3. Mass m	Inductance L
4. Damping coefficient γ	Resistance R
5. Force amplitude f_0	Voltage amplitude v_0
6. Driving frequency ω_f	Oscillator frequency ω

WAVES

The phenomenon of wave motion is prevalent in almost all branches of physics. The concept of wave is very fundamental in nature. In this chapter we shall discuss the ideas related to travelling waves, wave equation and solution of one dimensional and three dimensional problems.

WAVE MOTION

In daily life we come across different types of waves.Water waves, light waves, sound waves and e.m.waves. When sound is produced the sound waves travel through air and reach our ear.The medium(air) does not move. Sound energy propagates without actual movement of matter. This is called wave motion.

Wave motion is one of the most important means of transferring of energy. When we throw a stone in a pool of water, waves form and travel outward. When we shout, sound waves are generated and travel in all directions. Light waves are generated and travel outward when an electric bulb is switched on. In all these cases, there is a transfer of energy in the form of waves.

Wave motion is a form of disturbance which travels through medium due to the repeated periodic motion of the particles of the medium . Only the wave travels forward while the particles of the medium vibrate about their mean positions. It transfer energy from one region of space to another region without transferring matter along with it. Wave motion is produced due to the harmonic vibration of the particles of the medium.

TWO CATEGORIES OF WAVES

1. **Mechanical Waves**

The waves which require a medium for travelling are called mechanical waves.Waves are produced due to the vibration of particles in this medium

egs: Sound waves and seismic waves

2. **Electromagnetic waves**

The waves which require no medium for propagation are called electromagnetic waves. They travel through vacuum. Electromagnetic waves are produced by oscillating electric charges. E.F and M.F acts perpendicular to the direction of propagation.

egs: Radio waves, microwaves, light waves, X-rays etc

TYPES OF WAVE MOTION

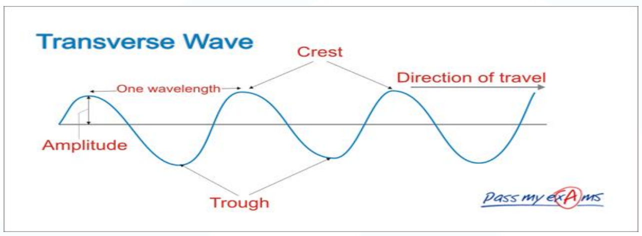
There are two distinct types of wave motion.

1.Transverse wave motion

When the particle of the medium vibrate about their mean position in a direction perpendicular to the direction of propagation of waves is called transverse wave.

eg: Light wave, Ripples on the surface of water, waves produced in a stretched string under tension.

Wavelength λ -distance between two consecutive crest or two consecutive trough

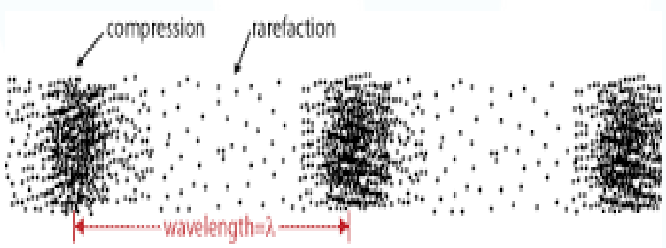


2. Longitudinal wave motion

When the particle of the medium vibrate about their mean position parallel to the direction of propagation of waves is called longitudinal wave.

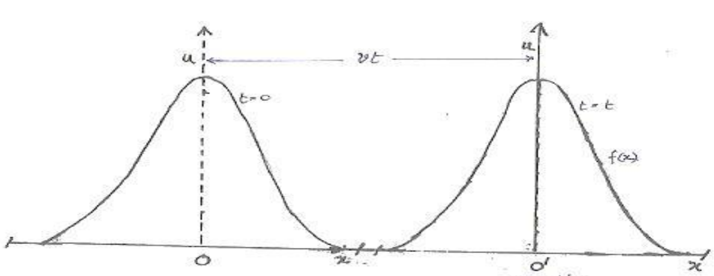
eg: sound waves.

Wavelength λ -distance between two consecutive compressions or rarefactions



GENERAL EQUATION OF WAVE MOTION

Consider a transverse pulse move in the positive direction of X-axis with a velocity v. After a time t , the pulse has travelled through a distance vt



Let u(x,t) be the transverse displacement at x which is a function of x and t. We can represent the wave form as

u(x, t)= f(x, t)

u(x, 0)= f(x, 0)(At t=0)

Where f describes the shape of the wave function.After a time t the pulse has travelled through a distance vt. Since the shape of the wave does not change as it travels, the wave form must be represented by the same wave function f.

Let o' be the new origin.

x'= x - vt

therefore u(x,t)= f(x - vt)

If the pulse is moving in the opposite direction (-ve direction of X-axis), then

u(x, t)= f(x + vt)

SINUSOIDAL WAVES

Consider a transverse wave having a sinusoidal Shape at t=0

ie u(x, 0)= f(x, 0) = a sin ωt = $a \sin \frac{2\pi}{\lambda} x$

If the wave travels with a velocity v in the positive direction of X- axis,

u(x, t) = $a \sin \frac{2\pi}{\lambda}(x - vt)$

$\frac{2\pi}{\lambda}(x - vt)$ is called the phase of the wave at a time

$$u = a \sin \frac{2\pi}{\lambda}(x - vt)$$

$$u = a \sin (\frac{2\pi}{\lambda} x - \frac{2\pi}{\lambda} vt)$$

$$u = a \sin (kx - \omega t)$$

The general expression for a sinusoidal wave travelling in positive x- direction

$$u(x,t) = a \sin (kx - \omega t + \phi)$$

ϕ is initial phase of wave

PARTICLE VELOCITY AND WAVE VELOCITY

Particle velocity (v_p) is the velocity of the mediumundergoing SHM .

$$v_p = \frac{du}{dt}$$

Wave velocity (v)is the velocity of the wave moving in x direction.

$$v = \frac{dx}{dt}$$

THE GENERAL WAVE EQUATION

ONE DIMENSIONAL WAVE EQUATION

The equation of wave motion is given by

U=f(x-vt) ....(1)

Differentiating eqn (1) w.r.t. x twice

$$\frac{\partial u}{\partial x} = f^{1}(x - vt).......(2)$$

$$\frac{\partial^2 u}{\partial x^2} = f^{11}(x - vt)......(3)$$

Differentiating eqn (1) w.r.t t twice

$$\frac{\partial u}{\partial t} = -f^{1}(x - vt)v.........(4)$$

$$\frac{\partial^2 u}{\partial t^2} = v^2 f^{11}(x - vt)........(5)$$

Substituting for f^{11} (x-vt) from eqn (3) in eqn (5)

We get

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} --------(7)$$

This is calledone dimensionaldifferential equation of wave motion.

This eqn is the same for a wave moving in the –ve direction of x-axis also.

$$Combining\ (2)\ and\ (4),$$

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x}$$

$\frac{\partial u}{\partial t}$ **represents particle velocity, v is the wave velocity and** $\frac{\partial u}{\partial x}$ **is the slope of the xy curve**

$$particle\ velocity = wave\ velocity \times slope\ of\ XY\ curve.$$

SOLUTION TO ONE DIMENSIONAL WAVE EQUATION

Consider one dimensional differential wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} ------(1)$$

We assume a solution of the form

$$u(x,t) = X(x)\,T(t) ----(2)$$

where $X(x)$ is a function of x and $T(t)$ is a function of t.

Differentiating eq.(2) twice w.r.t x and w.r.t t and substituting in eq.(1)

$$T\frac{\partial^2 X}{\partial x^2} = \frac{X}{v^2}\frac{\partial^2 T}{\partial t^2} ------(3)$$

Deviding (3) by XT, $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{v^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} --------(4)$$

The LHS and RHS of the eqn(4) contains only one variable. A change in x will not change the right side of the eqn and a change in t will not change the left side. Hence each side must be equal to a constant $-k^2$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2 \quad or \quad \frac{\partial^2 X}{\partial x^2} = -k^2 X -----(5)$$

$$\frac{1}{v^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -k^2 \quad or \quad \frac{\partial^2 T}{\partial t^2} = -k^2 v^2 T ----(6)$$

Writing $k^2 v^2 = \omega^2$ *eqn* (5) *becomes*

$$\frac{\partial^2 T}{\partial t^2} = -\omega^2 T -------------(7)$$

Eqn (5) and (7) are second degree differential equations. Their solution can be written in terms of exponential functions.

X(x) = constant .$e^{\pm ikx}$---------------(8)

T(t) = constant . $e^{\pm i\omega t}$--------------(9)

Putting these values in eqn(2) we have

U(x,t)= A e $^{i(kx\pm\omega t)}$--------------------(10)

This is the solution for one dimensional wave equation. Where A is a constant to be found by the initial conditions

THREE DIMENSIONAL WAVE EQUATION

In three dimension equation becomes

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$$\nabla^2 u=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}---------(8)$$

$$where\ \nabla^2\ is\ the\ Laplacian\ operator\ \ defined\ by$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

Eq(8) represents the differential eqn for a wave propagating in a 3D space.

SOLUTION OF 3 DIMENSIONAL WAVE EQUATION

In 3D we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{------------(1)}$$

The solution of the above eqn is assumed as

U(x,y,z,t)= X(x) Y(y) Z(z) T(t)

Proceeding as in the case of 1D we arrive at

X(x) Y(y) Z(z) = const.$e^{i(kxx+kyy+kz\,z)}$ ----(2)

The solution to the eqn is

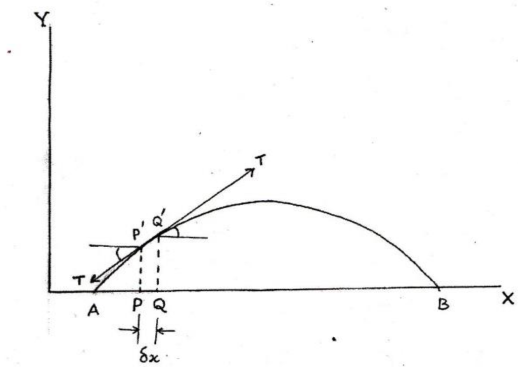
u(x,y,z,t)= A $e^{\,i(k.r\pm\omega t+\phi)}$ -----------------(3)

Where A is amplitude and ϕ initial Phase.

TRANSVERSE WAVES IN A STRETCHED STRING

Consider an infinitely long thin and uniform string stretched between two points by a constant tension T along the X-axis. Let the string be slightly displaced aside along the Y-axis and released. Transverse vibrations are set up in the string.

Consider a small element PQ of the string of length δx. The magnitude of tension in the string will be same everywhere since the string is perfectly flexible.The tension T acts tangentially at every point on the string. P'Q' is the displaced position of the string.



The tension at P' and Q' acts along the tangent making angles θ_1 and θ_2 with the horizontal. Resolving the tension along x- axis and y- axis,

Thenet force on PQ acting X and Y direction are

$$F_x = T\cos\theta_2 - T\cos\theta_1$$

$$F_y = T\sin\theta_2 - T\sin\theta_1$$

For small oscillations θ_1 and θ_2 are very small.

$$\cos\theta_1 = 1 \quad ; \quad \cos\theta_2 = 1$$

Also $\sin\theta_1 = \tan\theta_1$; $\sin\theta_2 = \tan\theta_2$ $\quad$ ($\tan\theta_1 = \sin\theta_1 / \cos\theta_1 = \sin\theta_1$)

Then $F_x = 0$

$$F_y = T\tan\theta_2 - T\tan\theta_1$$

So the net force acting on the element δx in the displaced position is only along Y-axis.

$$F_y = T(\tan\theta_2 - \tan\theta_1)$$

If μ is the mass per unit length of the string (linear density) ,

mass of the element δx, $m = \mu\,\delta x$

$$\text{Acceleration} = \frac{\partial^2 y}{\partial t^2}$$

According to the Newton's second law of motion

$$F = ma$$

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$$\mathbf{F} = \mu\, \delta x \frac{\partial^2 y}{\partial t^2}$$

$$\mathrm{T}\left(\tan\theta_2 - \tan\theta_1\right) = \mu\, \delta x . \frac{\partial^2 y}{\partial t^2}$$

$$But \quad \tan\theta_1 = \left(\frac{\partial y}{\partial x}\right)_x \quad ; \quad \tan\theta_2 = \left(\frac{\partial y}{\partial x}\right)_{x+\partial x}$$

$$or \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\left[\frac{\tan\theta_2 - \tan\theta_1}{\delta x}\right]$$

$$ie \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\partial x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right]$$

$$when \ \delta x --> 0$$

$$Lim\, \delta x --> 0 \left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\partial x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right] = \frac{\partial^2 y}{\partial x^2} ---(5)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} ----------(6)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

This is the differential equation of a vibrating string.

Comparing this equation with the standard wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{1}{v^2} = \frac{m}{T}$$

$$v^2 = \frac{T}{m}$$

$$v = \sqrt{\frac{T}{m}}$$

This is the velocity of transverse wave on a stretched string.

$$v = \upsilon\lambda$$

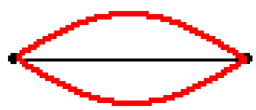
$$\upsilon = \frac{v}{\lambda}$$

$$v = \frac{1}{\lambda}\sqrt{\frac{T}{m}}$$

This is the frequency of transverse waves developed on a stretched string.

NOTE :

If the string is ' l ' meters long and vibrating in one segment only ,then l = $\frac{\lambda}{2}$



λ =2l

The frequency of transverse wave is given by

$$v = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

Frequency of the standing wave with *n* antinodes is given by

$$v = \frac{n}{2l}\sqrt{\frac{T}{m}}$$

Where ,
n is the no. of antinode

n = 1	1^{st} Harmonic
n = 2	2^{nd} Harmonic
n = 3	3^{rd} Harmonic

Law of transverse vibrations of a stretched string

Frequency of transverse vibrations of stretched string

$$v = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

Frequency $\alpha\sqrt{T}$
Frequency $\alpha 1/\sqrt{m}$
Frequency α 1/l