FUNDAMENTALS OF LOGIC

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1 INTRODUCTION

1.1 Propositions

A proposition is a declarative sentence that is either true or false (but not both). The truth value of proposition is true or false.

Example : The following are propositions:

(1) "Paris is in France" (true)

(2) "London is in Denmark" (false)

(3) " $2 < 4$ " (true)

(4) " $4 = 7$ " (false)

However the following are not propositions:

(1) "What is your name?" (This is a question)

(2) "Do your homework" (This is a command)

(3) "This sentence is false" (neither true nor false)

(4) " x is an even number" (it depends on what x)

(5) "Socrates" (it is not even a sentence)

Note: The truth or falsehood of a proposition is called its truth value.

Truth table: It displays the relationship between the truth values of proposition.

1.2 Negation of a proposition

If p is a proposition, then its negation is denoted by $\neg p$ or $\sim p$ and is defined by the following truth table.

p	$\neg p$
T	F
F	T

Example: P: Ram is intelligent. $\Rightarrow$ $\neg p$: Ram is not intelligent.

2 COMPOUND PROPOSITION

It is a proposition consisting of two or more simple proposition using logical operators.

2.1 Logical Connectives

2.1.1 Conjunction (AND)

If p and q are two propositions, then the conjunction of p and q is denoted by $p \wedge q$ (read as p and q) and is defined by following truth table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that $p \wedge q$ is true only when both p and q are true.

2.1.2 Disjunction (OR)

The disjunction of two proposition p and q is the proposition $p \vee q$ [read as p or q] and is defined by the following truth table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that $\vee$ represents a non-exclusive or. $ie, p \vee q$ is true when any of p, q is true and also when both are true.

2.1.3 Exclusive Disjunction (XOR)

The exclusive or of two proposition p and q is the proposition $p \oplus q$ or $p \veebar q$ and is defined by the following truth table.

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

Note that $\oplus$ or $\veebar$ represents an exclusive or. $p \veebar q$ is true only when exactly one of p and q is true.

2.2 Conditional and Bi-conditional Propositions

2.2.1 Conditional proposition

A proposition of the form "if p then q " or " p implies q ", represented " $p \rightarrow q$ " is called a conditional proposition. The proposition p is called **hypothesis** or **antecedent**, and the proposition q is the **conclusion** or **consequent** .

Truth Table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that $p \rightarrow q$ is true always except when p is true and q is false.

So, the following sentences are true:

(1) "if $2 < 4$ then Paris is in France" (true $\rightarrow$ true)

(2) "if London is in Denmark then $2 < 4$ "(false $\rightarrow$ true)

(3) "if $4 = 7$ then London is in Denmark" (false $\rightarrow$ false)

However the following one is false:
"if $2 < 4$ then London is in Denmark" (true $\rightarrow$ false).

It might seem strange that " $p \rightarrow q$ " is considered true when p is false, regardless of the truth value of q. This will become clearer when we study predicates such as "if x is a multiple of 4 then x is a multiple of 2." That implication is obviously true. Although for the particular case $x = 3$ it becomes "if 3 is a multiple of 4 then 3 is a multiple of 2".

Note: $p \rightarrow q$ is false when p is true and q is false. Otherwise it is true. The different situations where the conditional statements applied are listed below:

(1) If p then q

(2) p only if q

(3) q whenever p

(4) q is necessary for p

(5) q follows from p

(6) q when p

(7) p is sufficient for q

(8) p implies q

Example: p : Ram is a computer science student , q : Ram study DBMS

$p \rightarrow q$: If Ram is a computer science student, then he will study DBMS.

2.2.2 Bi-conditional Proposition

If p and q are proposition, then the proposition p if and only if q, denoted by $p \leftrightarrow q$ is called the bi-conditional statement and is defined by the following truth table.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: $p \leftrightarrow q$ is true if both p and q have same truth values, *ie*, they are both true or both false. Otherwise $p \leftrightarrow q$ is false.

Example: p: You can take the flight , q : You buy a ticket $p \leftrightarrow q$: You can take the flight if and only if buy a ticket.

2.3 Problems

1. Symbolize the statements using Logical Connectives:

   The automated reply cannot be sent when the file system is full.

   Solution: P: The automated reply can be sent , q: The file system is full

   Symbolic form: $q \rightarrow \neg p$

2. Write the symbolized form of the statement. If either Ram takes C++ or Kumar takes Pascal, then Latha will take Lotus.

   Solution: R: Ram takes $C++$; K: Kumar takes Pascal ; L: Latha takes Lotus

   Symbolic form: $(R \vee K) \rightarrow L$

3. Let p, q, r represent the following propositions,

   P: It is raining

   q: The sun is shining

   r: There are clouds in the sky Symbolize the following statements.

   Symbolize the following statements:

   (a) If it is raining, then there are clouds in the sky.

   (b) If it is not raining, then the sun is not shining and there are clouds in the sky.

   (c) The sun is shining if and only if it is not raining.

   Solution: Symbolic form:

(a) $p \to r$

(b) $\neg p \to (\neg q \wedge r)$

(c) $q \leftrightarrow \neg p$

4. Symbolize the following Let the proposition statements:

(a) If the moon is out and it is not snowing, then Ram goes out for a walk.

(b) If the moon is out, then if it is not snowing, Ram goes out for a walk.

(c) It is not the case that Ram goes out for a walk if and only if it is not snowing or the moon is out.

Solution: Let the propositions be,

p: The moon is out

q: It is snowing

r: Ram goes out for a walk.

Symbolic form:

(a) $(p \wedge \neg q) \to r$

(b) $p \to (\neg q \to r)$

(c) $\neg(r \leftrightarrow (\neg q \vee p))$

5. Symbolize the following using the propositions.

p: I finish writing my computer program before lunch.

q: I shall play Tennis in afternoon.

r: The sun is shining.

s: The boundary is low.

(a) If the sun is shining, I shall play tennis in the afternoon.

(b) Finishing the writing of my computer program before lunch is necessary for playing tennis in this afternoon.

(c) Low boundary and sunshine are sufficient to play Tennis in this afternoon.

Solution: Symbolic form:

(a) $r \to q$ (b) $q \to p$ (c) $(s \wedge r) \to q$

6. Show that the truth values of the formula $p \wedge (p \to q) \to q$ are independent of their components.

Solution: The truth table for the formula is

p	q	$p \to q$	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The truth values of the given formula are all true for every possible truth values of p and q. Therefore, the truth value of the given formula is independent of their components.

7. Construct the compound statement and develop the truth table for the following:

   "Margaret Mitchell wrote Gone with the wind, and if $2+3 \neq 5$, then Combinatorics is a course for sophomores".

   Solution: Let p : Margaret Mitchell wrote Gone with the wind ;

   q: $2+3=5$;

   r: Combinatorics is a course for sophomores

   Thus compound statement is $p \wedge (\neg q \to r)$ and its Truth Table is as follows:

p	q	r	$\neg q$	$\neg q \to r$	$p \wedge (\neg q \to r)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	T	T	F
F	F	F	T	F	F

3 LOGICAL EQUIVALENCE

Two compound propositions 'u' and 'v' are said to be logically equivalent whenever u and v have the same truth value, or equivalently. Then we write $u \equiv v$ (or) $u \Leftrightarrow v$. Here the symbol $\equiv$ (or) $\Leftrightarrow$ stands for "logically equivalent to". When the propositions u and v are not logically equivalent we write $u \not\Leftrightarrow v$ (or) $u \not\equiv v$

Result: $p \to q \equiv \neg p \vee q$

Result: De Morgan's Laws for Logic

(a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	F	T	T	F	T	T

3.1 Laws of Logic

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
$\neg(p \to q) \equiv p \wedge \neg q$	Negation of a Conditional

Example: The compound propositions $p \to q$ and $\neg p \vee q$ have the same truth values.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Thus, $p \to q \equiv \neg p \vee q$

3.2 Problems

1. Show that the following propositions are equivalent:

   $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

   Solution: This can be checked with the truth tables:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

2. Prove that $p \to q$ is logically equivalent to $(\neg p \vee q)$

   Solution:

p	q	$p \to q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

3. Check the following logical equivalences:

   (a) $\neg(p \to q) \equiv p \wedge \neg q$

   (b) $p \to q \equiv \neg q \to \neg p$

   (c) $\neg(p \leftrightarrow q) \equiv p \oplus q$

3.3 Converse, Inverse & Contradiction

Converse: The converse of a conditional proposition $p \to q$ is the proposition $q \to p$.

Inverse: The inverse of a conditional proposition $p \to q$ is the proposition $\neg p \to \neg q$.

Contrapositive: The contrapositive of a conditional proposition $p \to q$ is the proposition $\neg q \to \neg p$.

Example: If $x^2 = 4$, then $x = 2$. ($p \to q$ form)

Converse: If $x = 2$, then $x^2 = 4$.

Inverse: If $x^2 \neq 4$, then $x \neq 2$. (or, it is not the case that If $x^2 = 4$, then $x = 2$.)

Contrapositive: If $x \neq 2$, then $x^2 \neq 4$.

4 TAUTOLOGIES & CONTRADICTIONS

Tautology: A compound proposition which is always true regardless of the truth values of its components is called a Tautology.

Contradiction or Absurdity: A compound proposition which is always false regardless of the truth values of its components is called a Contradiction or an Absurdity.

Contingency: A compound proposition that can be true or false (depending upon the truth values of its components) is called a Contingency.

ie, Contingency is a compound proposition which is neither a Tautology nor a Contradiction.

4.1 Principle of Duality

Dual of a Compound Proposition: The dual of a compound proposition that contains only the logical operators $\vee, \wedge$ and $\neg$ is the proposition that obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T, where T and F are representing Tautologies and Contradictions respectively. The dual of a proposition A is denoted by A^*.

4.2 Problems

1. Write down the dual of the following statements:

   (a) $\neg(p \vee q) \vee [(\neg p) \wedge q] \wedge p$

   (b) $\neg p \rightarrow (p \rightarrow q)$

   (c) $(p \wedge q) \rightarrow (p \rightarrow q)$

   (d) $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$

   Solution:

   (a) $\neg(p \wedge q) \wedge [(\neg p) \vee q] \vee p$

   (b) $$\begin{aligned} \neg p \rightarrow (p \rightarrow q) &\equiv \neg p \rightarrow (\neg p \vee q) \\ &\equiv \neg\neg p \vee (\neg p \vee q) \\ &\equiv p \vee (\neg p \vee q) \\ &\equiv p \vee \neg p \vee q \end{aligned}$$
   Thus, the dual is $p \wedge \neg p \wedge q$.

   (c) $$\begin{aligned} (p \wedge q) \rightarrow (p \rightarrow q) &\equiv (p \wedge q) \rightarrow (\neg p \vee q) \\ &\equiv \neg(p \wedge q) \vee (\neg p \vee q) \end{aligned}$$
   Thus, the dual is $\neg(p \vee q) \wedge (\neg p \wedge q)$.

(d) $(p \to q) \to (\neg q \to \neg p) \equiv (\neg p \vee q) \to (\neg\neg q \vee \neg p)$

$$\equiv (\neg p \vee q) \to (q \vee \neg p)$$

$$\equiv \neg(\neg p \vee q) \vee (q \vee \neg p)$$

Thus, the dual is $\neg(\neg p \wedge q) \wedge (q \wedge \neg p)$

2. Prove that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a Tautology.

Solution: Let $S : (p \to q) \wedge (q \to r) \to (p \to r)$

To prove: S is a Tautology

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The last column shows that S is a Tautology.

3. Check whether $\neg(q \to r) \wedge r \wedge (p \to q)$ is a Tautology or Contradiction or Contingency.

Solution: Let $S : \neg(q \to r) \wedge r \wedge (p \to q)$

p	q	r	$q \to r$	$\neg(q \to r)$	$\neg(q \to r) \wedge r$	$p \to q$	S
T	T	T	T	F	F	T	F
T	T	F	F	T	F	T	F
T	F	T	T	F	F	F	F
T	F	F	T	F	F	F	F
F	T	T	T	F	F	T	F
F	T	F	F	T	F	T	F
F	F	T	T	F	F	T	F
F	F	F	T	F	F	T	F

The last column shows that the given statement is a Contradiction.

4. Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Solution: $\neg(p \to q) \equiv \neg(\neg p \vee q)$

$$\equiv \neg(\neg p) \wedge \neg q, \quad \textit{by the second De Morgan's law}$$

$$\equiv p \wedge \neg q, \quad \textit{by the double negation law}$$

5. Without using truth table, show that $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \equiv r$.

   Solution: $LHS \equiv [\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)$

   $$\equiv [(\neg p \wedge \neg q) \wedge r] \vee (q \vee p) \wedge r, \quad \textit{Associative law \& Distributive property}$$

   $$\equiv [\neg(p \vee q) \wedge r] \vee (p \vee q) \wedge r, \quad \textit{De-Morgan's law \& Commutative law}$$

   $$\equiv [\neg(p \vee q) \vee (p \vee q)] \wedge r, \quad \textit{Distributive Law}$$

   $$\equiv T \wedge r, \quad \textit{Since, } p \vee \neg p \equiv T$$

   $$\equiv r = RHS$$

6. Without using truth table, show that $(p \wedge q) \to p$ is a Tautology.

   Solution: $(p \wedge q) \to p \equiv \neg(p \wedge q) \vee p$

   $$\equiv (\neg p \vee \neg q) \vee p$$

   $$\equiv \neg p \vee (\neg q \vee p)$$

   $$\equiv \neg p \vee (p \vee \neg q)$$

   $$\equiv (\neg p \vee p) \vee \neg q$$

   $$\equiv T \vee \neg q$$

   $$\equiv T, \quad \textit{Tautology}$$

7. Use substitution rule to show that $(p \vee q) \wedge \neg(\neg p \wedge q) \equiv p$.

   Solution: $(p \vee q) \wedge \neg(\neg p \wedge q) \equiv (p \vee q) \wedge (\neg\neg p \vee \neg q), \quad \textit{De-Morgan's Law}$

   $$\equiv (p \vee q) \wedge (p \vee \neg q)$$

   $$\equiv p \vee (q \wedge \neg q), \quad \textit{Distributive Law}$$

   $$\equiv p \vee F_o, \quad \textit{Inverse Law}$$

   $$\equiv p, \quad \textit{Identity Law}$$

8. Use substitution rule to show that $\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow q \wedge r$

   Solution: $\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \equiv \neg\neg[(p \vee q) \wedge r] \wedge \neg\neg q], \quad \textit{De-Morgan's Law}$

   $$\equiv [(p \vee q) \wedge r] \wedge q]$$

   $$\equiv (p \vee q) \wedge (r \wedge q)]$$

   $$\equiv (p \vee q) \wedge (q \wedge r)]$$

   $$\equiv [(p \vee q) \wedge q] \wedge r$$

   $$\equiv q \wedge r$$

9. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

   Solution: $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q), \quad by\ the\ second\ De-Morgan's\ law$

   $$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q], \quad by\ the\ first\ De-Morgan's\ law$$

   $$\equiv \neg p \wedge (p \vee \neg q), \quad by\ the\ double\ negation\ law$$

   $$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q), \quad by\ distributive\ law$$

   $$\equiv F \vee (\neg p \wedge \neg q), \quad because \neg p \wedge p \equiv F$$

   $$\equiv (\neg p \wedge \neg q) \vee F, \quad by\ the\ commutative\ disjunction$$

   $$\equiv \neg p \wedge \neg q, \quad by\ the\ identity\ law\ for\ F$$

   Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

10. Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

    Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

    (**Note:** This could also be done using a truth table.)

    $(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$

    $$\equiv (\neg p \vee \neg q) \vee (p \vee q), \quad by\ the\ first\ De-Morgan's\ law$$

    $$\equiv (\neg p \vee p) \vee (\neg q \vee q), \quad by\ the\ associative\ and\ commutative\ laws\ for\ disjunction$$

    $$\equiv T \vee T, \quad by\ the\ commutative\ law\ for\ disjunction$$

    $$\equiv T, \quad by\ the\ domination\ law$$

5 LOGICAL IMPLICATION - RULE OF INFERENCE

5.1 Argument

The rules of inference are used to draw a conclusion from a set of premises (assumptions or hypothesis) in a finite sequence of steps called argument. An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion. An argument is valid if the truth of all its premises implies that the conclusion is true.

The general form of an argument is $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \to C$. Here $p_1, p_2, p_3, \ldots, p_n$ are called the premises of the argument and the statement C is called the conclusion for the argument. An argument is valid if $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \to C$ is a Tautology.

5.2 Rules of Inference

(1) **Modus Ponens**

p

$p \to q$

$\therefore q$

(2) **Modus Tollens**

$\neg q$

$p \to q$

$\therefore \neg p$

(3) **Hypothetical Syllogism**

$p \to q$

$q \to r$

$\therefore p \to r$

(4) **Disjunctive Syllogism**

$p \vee q$

$\neg p$

$\therefore q$

(5) **Addition**

p

$\therefore p \vee q$

(6) **Simplification**

$p \wedge q$

$\therefore p$

(7) **Conjunction**

p

q

$\therefore p \wedge q$

(8) **Resolution**

$p \vee q$

$\neg p \vee r$

$\therefore q \wedge r$

5.3 Problems

1. Let p, q, r denote the primitive statements given as

   - p: Roger studies.
   - q: Roger plays racket ball.
   - r: Roger passes discrete mathematics.

   Let the premises be

   - p_1 : If Roger studies, then he will pass discrete mathematics.
   - p_2 : If Roger doesn't play racket ball, then he will study.
   - p_3 : Roger failed discrete mathematics.

   Determine whether the argument $(p_1 \wedge p_2 \wedge p_3) \to q$ is valid.

   Solution: Given premises can be written as:

   $p_1 : p \to r$, $p_2 : \neg q \to p$, $p_3 : \neg r$

Truth Table:

p	q	r	$\neg q$	p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
T	T	T	F	T	T	F	T
T	T	F	F	F	T	T	T
T	F	T	T	T	T	F	T
T	F	F	T	F	T	T	T
F	T	T	F	T	T	F	T
F	T	F	F	T	T	T	T
F	F	T	T	T	F	F	T
F	F	F	T	T	F	T	T

Last column is a Tautology. Thus, the given argument is valid.

2. Use truth table to verify that the following is a logical implication.

   $[p \wedge ((p \wedge r) \rightarrow s)] \Rightarrow (r \rightarrow s)$

   Solution: Let $p_1 : p,$ $p_2 : (p \wedge r) \rightarrow s,$ $C : (r \rightarrow s)$

$p_1 : p$	r	s	$p \wedge r$	p_2	$C : (r \rightarrow s)$	$(p_1 \wedge p_2) \rightarrow C$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	T	F	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Last column is a Tautology. Thus , given argument is valid.

3. Establish the validity of the argument:

$$p \rightarrow q$$

$$q \rightarrow (r \wedge s)$$

$$\neg r \vee (\neg t \vee u)$$

$$p \wedge t$$

$$\overline{\qquad\qquad}$$

$$\therefore u$$

Solution:

Steps	Reasons
(1). $p \to q$	Premise
(2). $q \to (r \wedge s)$	Premise
(3). $p \to (r \wedge s)$	(1) & (2) and Hypothetical syllogism
(4). $p \wedge t$	Premise
(5). p	(4) and Simplification
(6). $r \wedge s$	(5) & (3) and Modus Ponens
(7). r	(6) and Simplification
(8). $\neg r \vee (\neg t \vee u)$	Premise
(9). $\neg(r \wedge t) \vee u$	(8), Associative law & De Morgan's law
(10). t	(4) and Simplification
(11). $r \wedge t$	(7) & (10) and Rule of Conjunction
(12). $\therefore u$	(9) & (11), double negation & Disjunctive syllogism

4. If the band could not play rock music or the refreshments were not delivered on time, them the New Years' party would have been cancelled and Alicia would have been angry. If the party were cancelled, then refunds would had to be made. No refunds were made. Therefore the band could play rock music. Test the validity of the conclusion.

Solution: Let p: The band could play rock music.

q: The refreshments were delivered on time.

r: The New Years' party was cancelled.

s: Alicia was angry.

t: refunds had to be made.

Thus the given argument become:

$$(\neg p \vee \neg q) \to (r \wedge s)$$
$$r \to t$$
$$\neg t$$
$$\overline{\hphantom{xxxxxx}}$$
$$\therefore p$$

Steps	Reasons
(1). $r \to t$	Premise
(2). $\neg t$	Premise
(3). $\neg r$	(1) & (2) and Modus Tollens
(4). $\neg r \vee \neg s$	(3) and Disjunctive Amplification
(5). $\neg(r \wedge s)$	(4) and De Morgan's laws
(6). $(\neg p \vee \neg q) \to (r \wedge s)$	Premise
(7). $\neg(\neg p \vee \neg q)$	(6) and (5) and Modus Tollens
(8). $p \wedge q$	(7), De Morgan's laws & Law of double negation
(9). $\therefore p$	(8) and Rule of Conjunctive Simplification

5. Show that $\neg p$ is a valid inference from the premises $p \to q$, $r \to \neg q$, r

Solution:

Steps	Reasons
(1). $p \to q$	Premise
(2). $r \to \neg q$	Premise
(3). r	Premise
(4). $\neg q \to \neg p$	(1) & Contrapositive
(5). $r \to \neg p$	(2) & (4) and Hypothetical syllogism
(6). $\neg p$	(3) & (5) and Modus Ponens

6. Establish the validity of the argument:

$$\neg p \leftrightarrow q$$
$$q \to r$$
$$\neg r$$
$$\overline{\qquad\qquad}$$
$$\therefore p$$

Solution:

Steps	Reasons
(1). $\neg p \leftrightarrow q$	Premise
(2). $(\neg p \to q) \wedge (q \to \neg p)$	(1) & $(\neg p \leftrightarrow q) \Leftrightarrow [(\neg p \to q) \wedge (q \to \neg p)]$
(3). $\neg p \to q$	(2) and Rule of Conjunctive Simplification
(4). $q \to r$	Premise
(5). $\neg p \to r$	(3) & (4) and the law of Syllogism
(6). $\neg p$	Premise (the one assumed)
(7). r	(5) & (6) and the Rule of Detachment
(8). $\neg r$	Premise
(9). $r \wedge \neg r (\equiv F)$	(7) & (8) and the Rule of Conjunction
(10). $\therefore p$	(6) & (9) and by contradiction.

7. Show that $r \wedge (p \vee q)$ is a valid conclusion from $p \vee q,\ q \to r,\ p \to m,\ \neg m$

Solution:

Steps	Reasons
(1). $p \vee q$	Premise
(2). $q \to r$	Premise
(3). $p \to m$	Premise
(4). $\neg m$	Premise
(5). $\neg p$	(3) & (4) and Modus Tollens
(6). q	(1) & (5) and Disjunctive Syllogism
(7). r	(6) & (2) and Modus Ponens
(8). $r \wedge (p \vee q)$	(1) & (7) and Conjunction

8. If today is Tuesday, then I have a test in CS or in Economics. If my Economics Professor is sick, then I will not have a test in Economics. Today is Tuesday and my Economics Professor is sick. Therefore I have a test in CS. Test the validity of the conclusion.

Solution: Let

p: Today is Tuesday

q: Test in CS

r: Test in Economics

s: My Economics Professor is sick

Thus the given argument is:

$$p \to (q \vee r)$$
$$s \to \neg r$$
$$p \wedge s$$
$$\overline{\qquad\qquad}$$
$$\therefore q$$

Steps	Reasons
(1). $p \to (q \vee r)$	Premise
(2). $s \to \neg r$	Premise
(3). $p \wedge s$	Premise
(4). p	(3) & Conjunctive Simplification
(5). s	(3) & Conjunctive Simplification
(6). $q \vee r$	(4) & (1) and Modus Ponens
(7). $\neg r$	(5) & (2) and Modus Ponens
(8). q	(6) & (7) and Disjunctive Syllogism

9. Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”

   Solution: Let p:It is sunny this afternoon.

   q: It is colder than yesterday

   r: We will go swimming

   s: We will take a canoe trip

   t: We will be home by sunset

   Then the premises become

$$\neg p \wedge q$$
$$r \to p$$
$$\neg r \to s$$
$$s \to t$$

   The conclusion is simply t.

We construct an argument to show that our premises lead to the desired conclusion as follows.

Steps	Reasons
(1). $\neg p \wedge q$	Premise
(2). $\neg p$	Simplification using (1)
(3). $r \rightarrow p$	Premise
(4). $\neg r$	Modus Tollens using (2) and (3)
(5). $\neg r \rightarrow s$	Premise
(6). s	(4) & (5) and Modus Ponens
(7). $s \rightarrow t$	Premise
(8). t	(6) & (7) and Modus Ponens

10. Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p : You send me an e-mail message $\quad q$: I will finish writing the program

r : I will go to sleep early $\quad s$: I will wake up feeling refreshed

Then the premises become

$$p \rightarrow q$$
$$\neg p \rightarrow r$$
$$r \rightarrow s$$

Conclusion is $\neg q \rightarrow s$.

This argument form shows that the premises lead to the desired conclusion.

Steps	Reasons
(1). $p \rightarrow q$	Premise
(2). $\neg q \rightarrow \neg p$	Contrapositive of (1)
(3). $\neg p \rightarrow r$	Premise
(4). $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
(5). $r \rightarrow s$	Premise
(6). $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

5.4 Consistency of Premises:

Premises $p_1, p_2, p_3, \ldots, p_n$ are consistent if $p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n \equiv T$, Tautology.

Example: Show that the following premises are inconsistent.

(1) If Jack misses many classes through illness, then he fails high school.

(2) If Jack fails high school, then he is uneducated.

(3) If Jack reads a lot of books, then he is not uneducated.

(4) Jack misses many classes through illness and he reads a lot of books.

Solution: Let p: Jack misses many classes through illness.

q : Jack fails high school.

r : Jack is un educated.

s : Jack reads a lot of books.

Thus the given argument is

$$\begin{array}{c} p \to q \\ q \to r \\ s \to \neg r \\ p \wedge s \\ \hline \therefore F \end{array}$$

Steps	Reasons
(1). $p \to q$	Premise
(2). $q \to r$	Premise
(3). $s \to \neg r$	Premise
(4). $p \wedge s$	Premise
(5). p	(4) & Conjunctive Simplification
(6). s	(4) & Conjunctive Simplification
(7). $\neg r$	(3) & (6) and Modus Ponens
(8). $p \to r$	(1) & (2) and Hypothetical syllogism
(9). r	(8) & (5) and Modus Ponens
(10). F	(7) & (9) and Conjunction

6 PREDICATE LOGIC

6.1 Predicate

A predicate is a statement that contains variables (predicate variables), and they may be true or false depending on the values of these variables. *ie*, A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

Example: $P(x) =$ "x^2 is greater than x " is a predicate.

If we choose $x = 1$, $P(1)$ is " 1 is greater than 1 ",

which is a proposition (always false).

Domain of Discourse or the Universe of Discourse: The domain of a predicate variable is the collection of all possible values that the variable may take.

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), often just referred to as the domain. Such a statement is expressed using **universal quantification**.

6.2 Quantifiers

We call the symbols ∀ and ∃, the quantifiers. Formulae involving them are called **quantified formulae**.

A **universal quantification** is a quantifier meaning "given any" or "for all". We use the following symbol:

$$\boxed{\forall \text{ (universal quantifier)}}$$

An **existential quantification** is a quantifier meaning "there exists", "there is at least one" or "for some". We use the following symbol:

$$\boxed{\exists \text{ (existential quantifier)}}$$

The **existential quantification** of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$." We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here ∃ is called the **existential quantifier**.

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

Example 1: Let $A = 1, 2, 3, 4, 5$ and $P(x) : x$ is a prime number.

Now, $\forall x \in A, P(x) : F$ and $\exists x \in A, P(x) : T$ are propositions.

Example 2: Let $E = 2,4,6,...$ and $Q(x) : x$ is an even number.

Now, $(x)Q(x) : T$ and $\exists x Q(x) : T$ are propositions.

Example 3: Let $O = 1,3,5,...$ and $Q(x) : x$ is an even number.

Now, $(x)Q(x) : F$ and $\exists x Q(x) : F$ are propositions.

Free and bound variables A particular occurrence of x in A is bound in A if it immediately follows an occurrence of the symbol $\forall$ or $\exists$ lies within the scope of an occurrence of $\forall$ or $\exists$. If an occurrence of x in A is not bound, it is free in A.

Example:

(1) In the statement, $(\exists x P(x)) \vee Q(y)$; x is the bound variable and y is the free variable

(2) In the statement, $\forall x, \exists y[P(x) \to Q(y)]$; x, y are the bound variable

Properties of Predicates:

(1) $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

(2) $\exists x \exists y P(x,y) \equiv \exists y \exists x P(x,y)$

(3) $\forall x \exists y P(x,y) \not\Leftrightarrow \exists y \forall x P(x,y)$

(4) $\neg[\forall x P(x)] \equiv \exists x \neg P(x)$

(5) $\neg[\exists x P(x)] \equiv \forall x \neg P(x)$

(6) $\neg[\forall x \neg P(x)] \equiv \exists x \neg\neg P(x) \equiv \exists x P(x)$

(7) $\neg[\exists x \neg P(x)] \equiv \forall x \neg\neg P(x) \equiv \forall x P(x)$

Example:

(1) $\exists x(\neg P(x)) \equiv \neg(\forall x P(x))$

(2) $\neg \forall x[P(x) \wedge Q(x)] \equiv \exists x \neg(P(x) \wedge Q(x)) \equiv \exists x(\neg P(x) \vee \neg Q(x))$

6.3 Rules of Inference for Quantified Statements

Rule of Inference	Name
$\forall x P(x) \equiv P(c)$	Universal instantiation
$P(c)$ for an arbitrary $c \equiv \forall x P(x)$	Universal generalization
$\exists x P(x) \equiv P(c)$ for some element c	Existential instantiation
$P(c)$ for some element $c \equiv \exists x P(x)$	Existential generalization

6.4 Problems

1. Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

   Solution: Let $D(x)$: " x is in this discrete mathematics class,"

   $C(x)$: " x has taken a course in computer science."

   Then the premises: $\forall x(D(x) \rightarrow C(x))$

   $D(Marla)$

   The conclusion: $C(Marla)$

   The following steps can be used to establish the conclusion from the premises.

Steps	Reasons
(1). $\forall x(D(x) \rightarrow C(x))$	Premise
(2). $D(Marla) \rightarrow C(Marla)$	Universal instantiation from (1)
(3). $D(Marla)$	Premise
(4). $C(Marla)$	Modus ponens from (2) and (3)

2. "All men are mortal", "Socrates is a man". Prove that "Socrates is mortal" using rule of inference.

   Solution: Let $M(x)$: x is a man

   $R(x)$: x is mortal

   The premises: $\forall x(M(x) \rightarrow R(x))$

   $M(Socrates)$

   The conclusion: $R(Socrates)$

   The following steps can be used to establish the conclusion from the premises.

Steps	Reasons
(1). $\forall x(M(x) \rightarrow R(x))$	Premise
(2). $M(Socrates) \rightarrow R(Socrates)$	Universal instantiation from (1)
(3). $M(Socrates)$	Premise
(4). $R(Socrates)$	Modus ponens from (2) & (3)

3. Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

   Solution: Let $C(x)$: " x is in this class,"

   $B(x)$: " x has read the book,"

$P(x)$: " x passed the first exam."

The premises: $\exists x(C(x) \wedge \neg B(x))$

$\forall x(C(x) \rightarrow P(x))$

The conclusion: $\exists x(P(x) \wedge \neg B(x))$

The following steps can be used to establish the conclusion from the premises.

Steps	Reasons
(1). $\exists x(C(x) \wedge \neg B(x))$	Premise
(2). $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3). $C(a)$	Simplification from (2)
(4). $\forall x(C(x) \rightarrow P(x))$	Premise
(5). $C(a) \rightarrow P(a)$	Universal instantiation from (4)
(6). $P(a)$	Modus ponens from (3) and (5)
(7). $\neg B(a)$	Simplification from (2)
(8). $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
(9). $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

4. Prove the implication using inference:

$\forall x(P(x) \rightarrow Q(x)) \wedge \forall x(R(x) \rightarrow \neg Q(x)) \Rightarrow \forall x(R(x) \rightarrow \neg P(x))$

Solution: Let the Premises: $\forall x(P(x) \rightarrow Q(x))$

$\forall x(R(x) \rightarrow \neg Q(x))$

The conclusion: $\forall x(R(x) \rightarrow \neg P(x))$

The following steps can be used to establish the conclusion from the premises.

Steps	Reasons
(1). $\forall x(P(x) \rightarrow Q(x))$	Premise
(2). $\forall x(R(x) \rightarrow \neg Q(x))$	Premise
(3). $P(a) \rightarrow Q(a)$	Universal instantiation from (1)
(4). $R(a) \rightarrow \neg Q(a)$	Universal instantiation from (2)
(5). $\neg Q(a) \rightarrow \neg P(a)$	Contrapositive from (3)
(6). $R(a) \rightarrow \neg P(a)$	Hypothetical syllogism using (4) & (5)
(7). $\forall x(R(x) \rightarrow \neg P(x))$	Universal generalization from (6)

5. Show that the premises, "one student in the class knows how to write programme in JAVA", "Everyone who knows how to write programme in JAVA can get a high paying job" imply the conclusion that "someone in the class can get a high paying job".

Solution: Let $P(x)$: x is a student in the class

$Q(x)$: x knows how to write programme in JAVA

$R(x)$: x can get a high paying job

The Premises: $\exists x(P(x) \wedge Q(x))$

$\forall x(Q(x) \rightarrow R(x))$

The conclusion: $\exists x(P(x) \wedge R(x))$

The following steps can be used to establish the conclusion from the premises.

Steps	Reasons
(1). $\exists x(P(x) \wedge Q(x))$	Premise
(2). $\forall x(Q(x) \rightarrow R(x))$	Premise
(3). $P(a) \wedge Q(a)$	Existential instantiation from (1)
(4). $Q(a) \rightarrow R(a)$	Universal instantiation from (2)
(5). $P(a)$	Simplification from (3)
(6). $Q(a)$	Simplification from (3)
(7). $R(a)$	Modus ponens from (4) and (6)
(8). $P(a) \wedge R(a)$	Conjunction from (5) and (7)
(9). $\exists x(P(x) \wedge R(x))$	Existential generalization from (8)

6. Determine the validity of the arguments:

   Every living thing is a plant or an animal John's gold fish is alive and it's not a plant. All animals have hearts. Therefore John's Gold fish have a heart.

   Solution: Let $L(x)$: x is a living thing

   $P(x)$: x is a plant

   $A(x)$: x is an animal

   $H(x)$: x have a heart

   The Premises: $\forall x(L(x) \rightarrow (P(x) \vee A(x)))$

   $L(J) \wedge \neg P(J)$

   $\forall x(A(x) \rightarrow H(x))$

   The conclusion: $H(J)$

The following steps can be used to establish the conclusion from the premises.

Steps	Reasons
(1). $\forall x(L(x) \to (P(x) \lor A(x)))$	Premise
(2). $L(J) \land \neg P(J)$	Premise
(3). $\forall x(A(x) \to H(x))$	Premise
(4). $L(J) \to (P(J) \lor A(J))$	Universal instantiation from (1)
(5). $A(J) \to H(J)$	Universal instantiation from (3)
(6). $L(J)$	Simplification from (2)
(7). $P(J) \lor A(J)$	Modus ponens from (4) & (6)
(8). $\neg P(J)$	Simplification from (2)
(9). $A(J)$	Disjunctive Syllogism from (7) & (8)
(10). $H(J)$	Modus ponens from (5) & (9)