Rules of Inference

2. Law of Syllogism

In symbolic form, this rule is expressed by the logical implication

$[(p \to q) \land (q \to r)] \to (p \to r)$ where p, q, r are any statements.

In tabular form, it is written as

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Example1**:**

1. If the integer 1000 is divisible by 50, then the integer 1000 is divisible by 25.

2. If the integer 1000 is divisible by 25 , then the integer 1000 is divisible by 5 .

3. Therefore, if the integer 1000 is divisible by 50 , then the integer 1000 is divisible by 5 .

Here p: the integer 1000 is divisible by 50.

q: the integer 1000 is divisible by 25.

r: the integer 1000 is divisible by 5.

1. $p \to q$ 2. $q \to r$ 3. $\therefore p \to r$

Example2:

Consider the following argument.
1. Rita is baking a cake.
2. If Rita is baking a cake, then she is not practicing her flute.
3. If Rita is not practicing her flute, then her father will not buy her a car.
4. Therefore Rita's father will not buy her a car.

Here p: Rita is baking a cake.
q: she is practicing her flute.
r : her father will buy her a car.
We can write the argument as

$$\begin{array}{c} p \\ p \to \sim q \\ \sim q \to \sim r \\ \hline \therefore \sim r \end{array}$$

The three premises are $p,\ p \to \sim q,\ \sim q \to \sim r$.

To validate the argument , we can use the two different ways.

Steps	**Reasons**
1. p	Premise
2. $p \rightarrow \sim q$	Premise
3. $\sim q$	Steps (1) and (2) and Rule of Detachment
4. $\sim q \rightarrow \sim r$	Premise
5. $\therefore \sim r$	Steps (3) and (4) and Rule of Detachment

Or

Steps	**Reasons**
1. $p \rightarrow \sim q$	Premise
2. $\sim q \rightarrow \sim r$	Premise
3. $p \rightarrow \sim r$	This follows from steps (1) and (2) and the Law of Syllogism
4. p	Premise
5. $\therefore \sim r$	This follows from Steps (4) and (3) and Rule of Detachment

3. Modus Tollens

The rule of inference is given by

$$\begin{array}{c} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

In symbolic form, this rule is expressed by the logical implication

$$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$$

Example3:

For the following primitive statements p, q, r, s, t, u, show that the following argument is valid

$p \rightarrow r$

$r \rightarrow s$

$t \vee \sim s$

$\sim t \vee u$

$\sim u$

$\therefore \sim p$

Steps	Reasons
1. $p \to r\ ,\ r \to s$	Premises
2. $p \to s$	step(1) and the Law of the Syllogism
3. $t \vee \sim s$	Premise
4. $\sim s \vee t$	step(3) and the Commutative Law of $\vee$
5. $s \to t$	step(4) and use the result $\sim s \vee t \equiv s \to t$
6. $p \to t$	steps (2),(5), and the Law of the Syllogism
7. $\sim t \vee u$	Premise
8. $t \to u$	step (7) and use the result $\sim t \vee u \equiv t \to u$
9. $p \to u$	step (6),(8) and the Law of the Syllogism
10. $\sim u$	Premise
11. $\therefore \sim p$	Steps (9),(10) and Modus Tollens

4. Rule of Conjunction

In tabular form, the rule is given by

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

5. Rule of Disjunctive Syllogism

In symbolic form, this rule is expressed by the logical implication $[(p \vee q) \wedge \sim p] \rightarrow q$.

In tabular form, we write

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

6. Rule of Contradiction

The implication $(\sim p \rightarrow F) \rightarrow p$ is a tautology. In tabular form, this rule is written as

$$\begin{array}{l} \sim p \rightarrow F \\ \hline \therefore p \end{array}$$

Examples:

4. Check the validity of the argument

$$\begin{array}{l} p \rightarrow r \\ \sim p \rightarrow q \\ q \rightarrow s \\ \hline \therefore \sim r \rightarrow s \end{array}$$

Steps	**Reasons**
1. $p \rightarrow r$	Premise
2. $\sim r \rightarrow \sim p$	Step (1) and $p \rightarrow r \equiv \sim r \rightarrow \sim p$
3. $\sim p \rightarrow q$	Premise
4. $\sim r \rightarrow q$	Steps (2),(3) and the Law of the Syllogism

5. $q \to s$ Premise

6. $\therefore \sim r \to s$ Steps (4),(5) and the Law of the Syllogism

Rule of Inference	Logical Implication	Name of Rule
1. p $p \to q$ $\therefore q$	$p \wedge (p \to q) \to q$	Rule of Detachment (Modus Ponens)
2. $p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Law of the Syllogism
3. $p \to q$ $\sim q$ $\therefore \sim p$	$[(p \to q) \wedge \sim q] \to \sim p$	Modus Tollens
4. p q $\therefore p \wedge q$		Rule of Conjunction
5. $p \vee q$ $\sim p$ $\therefore q$	$[(p \vee q) \wedge \sim p] \to q$	Rule of Disjunctive Syllogism
6. $\sim p \to F$ $\therefore p$	$(\sim p \to F) \to p$	Rule of Contradiction
7. $p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Rule of Conjunctive Simplification

5. Establish the validity of the argument $p \to q$

$q \to r \wedge s$

$$\sim r \vee (\sim t \vee u)$$

$$p \wedge t$$

$$\therefore u$$

Steps	**Reasons**
1. $p \rightarrow q$	Premise
2. $q \rightarrow (r \wedge s)$	Premise
3. $p \rightarrow (r \wedge s)$	Steps (1),(2) and the Law of the Syllogism
4. $p \wedge t$	Premise
5. p	Step (4), Rule of Conjunctive Simplification
6. $r \wedge s$	Step (5),(3), Rule of Detachment
7. r	Step (6), Rule of Conjunctive Simplification
8. $\sim r \vee (\sim t \vee u)$	Premise
9. $\sim (r \wedge t) \vee u$	Step (8), Associative Law, De Morgan's Law
10. t	Step (4), Rule of Conjunctive Simplification
11. $r \wedge t$	Steps (7), (10), Rule of Conjunction
12. $\therefore u$	Steps (9),(11), Double negation, Rule of Disjunctive Syllogism

6. Establish the validity of the argument $p \rightarrow (q \rightarrow r)$

$$p \vee \sim s$$

$$q$$

$$\therefore s \rightarrow r$$

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Steps	Reasons
1. $p \to (q \to r)$	Premise
2. $p \vee \sim s$	Premise
3. q	Premise
4. $\sim s \vee p$	step (2), Commutative Law
5. $s \to p$	Implication
6. $s \to (q \to r)$	steps (5), (1), the Law of the Syllogism
7. $(s \wedge q) \to r$	step (6), Rule of Conditional Proof
8. $q \to [s \to (q \wedge s)]$	by $p \to [q \to (p \wedge q)]$
9. $s \to (q \wedge s)$	steps (3),(8), Modus Ponens
10. $s \to (s \wedge q)$	step (9), Commutative Law
11. $s \to r$	Steps (7),(10), the Law of the Syllogism

7. Establish the validity of the argument

$$\begin{array}{l} \sim p \leftrightarrow q \\ q \to r \\ \sim r \\ \hline \therefore p \end{array}$$

Steps	Reasons
1. $\sim p \leftrightarrow q$	Premise
2. $(\sim p \to q) \wedge (q \to \sim p)$	step (1), $\sim p \leftrightarrow q \equiv (\sim p \to q) \wedge (q \to \sim p)$
3. $\sim p \to q$	step (2), Rule of Conjunctive Simplification
4. $q \to r$	Premise
5. $\sim p \to r$	steps (3),(4), the Law of the Syllogism

6. $\sim p$ Premise

7. r steps (5),(6), Rule of Detachment

8. $\sim r$ Premise

9. $r \wedge \sim r \equiv F$ steps (7),(8), Rule of Conjunction

10. $\therefore p$ steps (6),(9), proof by Contradiction

The Use of Quantifiers

Sentences that involve a variable, such as x, need not be statements. For example, the sentence "The number $x + 2$ is an eve integer" is not necessarily true or false unless we know what value is substituted for x. If we restrict our choices to integers, then when x is replaced by $-5, -1, or\ 3$, for instance, the resulting statement is false. In fact, it is false whenever x is replaced by an odd integer. When an even integer is substituted for x, the resulting statement is true. So the sentence "The number $x + 2$ is an even integer" is an open statement.

Definition - Open statement

A declarative sentence is an open statement if
1. it contains one or more variables, and
2. it is not a statement, but
3. it becomes a statement when the variables in it are replaced by certain allowable choices.

These allowable choices constitute what is called the universe or universe of discourse for the open statement.

The open statement "The number $x + 2$ is an even integer" is denoted by $p(x), q(x), r(x), ...$Then $\sim p(x)$ is the open statement "The number $x + 2$ is not an even integer".

Quantifiers

Quantifiers are words that refer to quantities. They indicate how frequently a certain statement is true. There are two types of quantifiers.

(i) Universal quantifiers (ii) Existential quantifiers

The universal quantifier is denoted by $\forall x$ and is read as "For all x", "For any x", "For each x" or "For every x".

"For all x, y" , "For any x, y", "For every x, y" or "For all x and y" is denoted by $\forall x \forall y$ which can be abbreviated to $\forall x, y$.

The existential quantifier uses the words "For some x" , "for at least one x" or "there exists an x such that" . This quantifier is written in symbolic form as $\exists x$. Hence the statement "For some x, $p\ (\ x\ )$" becomes $\exists\ x\ p(x)$, in symbolic form.

In symbolic form, "For some $x, y\ , q(x, y\ )$ " is $\exists x, y\ \ q(x, \mathrm{y})$.

In the open statement $r(x)$: "$2x$ is an even integer", the variable x is called free variable of the open statement. In the symbolic representation $\exists x\ \ p(x)$ the variable x is said to be a bound variable — it is bound by the existential quantifier $\exists$.

For the statement $\forall x\ \ r(x)$, the variable x is bound by the universal quantifier $\forall$.

The open statement $q(x, y)$ has two free variables x, y. In the open statements $\exists x\ \exists y\ q(x, y)$ and $\exists x, y\ q(x, y)$, the variables x and y are the bound variables and they are bound by the existential quantifier $\exists$.

<u>Example1:</u>

Identify the bound variables and the free variables in each of the following statements. In both cases the universe comprises all real numbers.
(i) $\forall y \exists z\ \{\cos(x + y) = \sin\ (z - x)\}$ (ii) $\exists x \exists y\ \{x^2 - y^2 = z\}$

Ans:
(i) x is the free variable and y, z are the bound variables.
(ii) z is the free variable and x, y are the bound variables.

Statement	When is it true ?	When is it false ?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \ {\sim}p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation ${\sim}p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ {\sim}p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation ${\sim}p(a)$ is true.	There is at least one replacement a from the universe for which ${\sim}p(a)$ is false and $p(a)$ is true.

2. Let $p(x), q(x)$ denote the following open statements.
$p(\ x): x\ \leq\ 3$ $\quad q(x): x\ +\ 1\ is\ odd$. The universe consists of all integers.

Determine the truth values of the following statements

(i) $q(1)$ (ii) $\sim p(3)$ (iii) $p(7) \vee q(7)$ (iv) $p(3) \wedge q(4)$
(v) $\sim [p(-4) \vee q(-3)]$ (vi) $\sim p(-4) \wedge \sim q(-3)$

Ans: (i) $q(1): 2\ is\ odd$ is false.

(ii) $\sim p(3)$ is false.

(iii) $p(7) \vee q(7)$ is false.

(iv) $p(3) \wedge q(4)$ is true.

(v) $\sim [p(-4) \vee q(-3)]$ is false.

(vi) $\sim p(-4) \wedge \sim q(-3)$ is false.

3. Determine the truth value of each statement if the universe consists of all nonzero integers.

a. $\exists x \exists y\ (xy = 2)$

b. $\exists x \forall y\ (xy = 2)$

c. $\forall x \exists y\ (xy = 2)$

d. $\exists x \exists y\ [(3x + y = 8) \wedge (2x - y = 7)]$

e. $\exists x \exists y\ [(4x + 2y = 3) \wedge (x - y = 1)]$

<u>Answers:</u>

a. For some $x = 1$ and some $y = 2$, we have $1.2 = xy = 2$ is true. So it is true.

b. For some $x = 2, y = 3, xy = 2.3 = 6 \neq 2$. So for every y, (b) is not true.

c. For some $y = 1, x = 4, xy = 4.1 = 4 \neq 2$. So (c)is not true.

d. Solving the equations $3x + y = 8$ and $2x - y = 7$, we get $x = 3, y = -1$. So (d) is true for some $x = 3$ and some $y = -1$. Hence (d) is true.

e. Solving the equations $4x + 2y = 3$ and $x - y = 1$, we get $x = \frac{5}{6}$, $y = -\frac{1}{6}$ which are not integers. So (e) is false.

4. Let the universe consists of all real numbers. The open statements $p(x), q(x), r(x)$ are

given by $p(x): x \geq 0$ $q(x): x^2 \geq 0$

$r(x): x^2 - 3 > 0$

Then (i) the statement $\exists x\, [p(x) \wedge r(x)]$ is true since for example, if $x = 4$, then 4 is a member of the universe and both the statements $p(4)$ and $r(4)$ are both true.

(ii) the statement $\forall x\, [p(x) \rightarrow q(x)]$ is true since if we replace x in $p(x)$ and $q(x)$ by a negative real number a, then both $p(a), q(a)$ are false. So $p(a) \rightarrow q(a)$ is true. If we replace x in $p(x)\ and\ q(x)$ by a nonnegative real number b, then both $p(b)\ and\ q(b)$ are true. So $p(b) \rightarrow q(b)$ is true. The statement $\forall x\, [p(x) \rightarrow q(x)]$ can be read as " For every real number x, if $x \geq 0$, then $x^2 \geq 0$.

(iii) the statement $\forall x\, [q(x) \rightarrow r(x)]$ is false. To show that the statement is false, prove that there is one value of x for which the given statement is false. If we replace x by 1 in $q(x)$ and $r(x)$, $q(1)$ is true and $r(1)$ is false. So it is false.

Logically equivalent statements:

Let $p(x), q(x)$ be open statements defined for a given universe. The open statements $p(x)$ and $q(x)$ are called (logically) equivalent, and we write $\forall x\, [p(x) \equiv q(x)]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement a from the universe (ie, $p(a) \equiv q(a)$ for each a in the universe. If the implication $p(a) \rightarrow q(a)$ is true for each a in the universe (that is, $p(a) \Longrightarrow q(a)$ for each a in the universe), then we write $[\forall p(x) \Longrightarrow q(x)]$ and read as $p(x)$ logically implies $q(x)$.

Converse, Contrapositive and Inverse Statements

For open statements $p(x), q(x)$ defined for a prescribed universe and the universally quantified statement $\forall x\ [p(x) \rightarrow q(x)]$ we define the following:

1. The contrapositive of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [\sim q(x) \rightarrow \sim p(x)]$.

2. The converse of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [q(x) \rightarrow p(x)]$.

3. The inverse of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [\sim p(x) \rightarrow \sim q(x)]$.

Example1 :

For the universe of all quadrilaterals in the plane let $p(x)$ and $q(x)$ denote the open statements $p(x)$: x is a square $q(x)$: x is equilateral.

i) The statement $\forall x\, [p(x) \rightarrow q(x)]$ is a true statement and is logically equivalent to its contrapositive $\forall x\ [\sim q(x) \rightarrow \sim p(x)]$ because $[p(a) \rightarrow q(a)] \equiv [\sim q(a) \rightarrow \sim p(a)]$ for each replacement a.

ii) The statement $\forall x\ [q(x) \rightarrow p(x)]$ is a false statement and is the converse of the true statement $\forall x\ [p(x) \rightarrow q(x)]$.

iii) The statement $\forall x\ [\sim p(x) \rightarrow \sim q(x)]$ is a false statement and is the inverse of the statement

$\forall x\ [p(x) \rightarrow q(x)]$.

Example2:

Let $p(x)$ and $q(x)$ are the open statements $p(x): |x| > 3$ $q(x): x > 3$ and the universe consists of all real numbers.

i) The statement $\forall x\ [p(x) \rightarrow q(x)]$ is a false statement. Since if $x = -5$, then $p(-5)$ is true,

but $q(-5)$ is false. So $p(-5) \rightarrow q(-5)$ is false and hence $\forall x\ [p(x) \rightarrow q(x)]$ is false.

ii) The converse of the above statement is $\forall x\ [q(x) \rightarrow p(x)]$ is true. ie, Every real number

greater than 3 has absolute value greater than 3.

iii) The inverse of the above statement is $\forall x\ [\sim p(x) \rightarrow \sim q(x)]$ is also true. ie, If the magnitude of a real number is less than or equal to 3, then the number itself is less than or equal to 3.

iv) The contrapositive of the above statement is $\forall x\ [\sim q(x) \rightarrow \sim p(x)]$ is false. ie, If a real number is less than or equal to 3, then its magnitude is also less than or equal to 3.

Logical Equivalences and Logical Implications for Quantified Statements in One Variable

For a prescribed universe and any open statements $p(x), q(x)$ in the variable x:

1. $\exists x\ [p(x) \wedge q(x)] \Rightarrow [\exists x\ p(x) \wedge \exists x\ q(x)]$
2. $\exists x\ [p(x) \vee q(x)] \equiv [\exists x\ p(x) \vee \exists x\ q(x)]$
3. $\forall x\ [p(x) \wedge q(x)] \equiv [\forall x\ p(x) \wedge \forall x\ q(x)]$
4. $\forall x\ p(x) \vee \forall x\ q(x) \Longrightarrow \forall x[p(x) \vee q(x)]$
5. $\exists x\ [p(x) \rightarrow q(x)] \equiv \exists x\ [\sim p(x) \vee q(x)]$
6. $\forall x\ \sim [p(x) \wedge q(x)] \equiv \forall x\ [\sim p(x) \vee \sim q(x)]$

7. $\forall x \sim [p(x) \vee q(x)] \equiv \forall x\, [\sim p(x) \wedge \sim q(x)]$

Rules for Negating Statements with One Quantifier

1. $\sim[\forall x\ \ p(x)] \equiv \exists x\ \ \sim p(x)$

2. $\sim[\exists x\ \ p(x)] \equiv \forall x\ \ \sim p(x)$

3. $\sim[\forall x\ \ \sim p(x)] \equiv \exists x\ \ \sim\sim p(x) \equiv \exists x\ \ p(x)$

4. $\sim[\exists x\ \ \sim p(x)] \equiv \forall x\ \ \sim\sim p(x) \equiv \forall x\ \ p(x)$

5. $\forall x\ \ \sim\sim p(x) \equiv \forall x\, p(x)$

Example1: Let the universe consists of all the integers, and the open statements $p(x), q(x)$ be given by $p(x)$: x is odd $q(x)$: $x^2 - 1$ is even.

The statement "If x is odd, then $x^2 - 1$ is even" can be symbolized as

$\forall x\ \ [p(x) \to q(x)]$ which is a true statement. The negation is determined as follows.

$$\sim\{\forall x\ \ [p(x) \to q(x)]\} \equiv \exists x\, [\sim(p(x) \to q(x)]$$

$$\equiv \exists x\, [\sim(\sim p(x) \vee q(x))] \equiv \exists x\, [\sim\sim p(x) \wedge \sim q(x)]$$

$$\equiv \exists x\ \ [p(x) \wedge \sim q(x)]$$

The negation is that "There exists an integer x such that x is odd and $x^2 - 1$ is odd." This statement is false.

Results:
(i) If $p(x, y)$ is an open statement in two variables x, y, then
$\forall x \forall y\ \ p(x, y) \equiv \forall y \forall x\ \ p(x, y)$ and $\exists x \exists y\, p(x, y) \equiv \exists y \exists x\ \ p(x, y)$

(ii) If $p(x, y, z)$ is an open statement in three variables, then
$\forall x, y, z\ \ p(x, y, z) \equiv \forall y, x, z\ \ p(x, y) \equiv \forall x, z, y\ \ p(x, y, z)$

(iii) If $p(x, y)$ is an open statement in two variables x, y, then
$\forall x \exists y\ \ p(x, y)$ and $\exists y \forall x\ \ p(x, y)$ are generally not logically equivalent.

2. Find the negation of the statement $\forall x \exists y\, [(p(x, y) \wedge q(x, y)) \to r(x, y)]$ where $p(x, y)$, $q(x, y), r(x, y)$ represent three open statements, with replacements for the variables x, y chosen from some given universe.

$$\sim\{\, \forall x\, \exists y\, [(p(x, y) \wedge q(x, y)) \to r(x, y)]\, \} \equiv \exists x\, \{\sim\exists y\, [(p(x, y) \wedge q(x, y)) \to r(x, y)]\, \}$$

$$\equiv \exists x\, \forall y\, [\sim[(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]\,]$$

$$\equiv \exists x\, \forall y \sim\{\sim[p(x,y) \wedge q(x,y)] \vee r(x,y)\}$$

$$\equiv \exists x\, \forall y\{\sim\sim[p(x,y) \wedge q(x,y)] \wedge\sim r(x,y)\}$$

$$\equiv \exists x\, \forall y\, [(p(x,y) \wedge q(x,y)) \wedge\sim r(x,y)]$$

3. The universe for the variables in the following statements consist of all real numbers. In each case negate and simplify the given statement.

(i) $\forall x \forall y\, [(x > y) \rightarrow (x - y > 0)]$

(ii) $\forall x \forall y\, [(x < y) \rightarrow \exists z\, (x < z < y)]$

Answers:

(i) $\sim\{\forall x \forall y\, [(x > y) \rightarrow (x - y > 0)]\} \equiv \exists x \exists y \sim[(x > y) \rightarrow (x - y > 0)]$

$$\equiv \exists x \exists y \sim[\sim(x > y) \vee (x - y > 0)]$$

$$\equiv \exists x \exists y\, [\sim\sim(x > y) \wedge \sim (x - y > 0)]$$

$$\equiv \exists x \exists y\, [(x > y) \wedge (x - y \leq 0)]$$

(ii) $\sim \{\forall x \forall y\, [(x < y) \rightarrow \exists z\, (x < z < y)]\}$

$$\equiv \exists x \exists y \sim[(x < y) \rightarrow \exists z (x < z < y)]$$

$$\equiv \exists x \exists y \sim\{\sim(x < y) \vee \exists z (x < z < y)\}$$

$$\equiv \exists x \exists y\, \{\sim\sim(x < y) \wedge \forall z \sim(x < z < y)\}$$

$$\equiv \exists x \exists y\, \{\sim\sim(x < y) \wedge \forall z \sim\{(x < z) \wedge (z < y)\}$$

$$\equiv \exists x \exists y\, [(x < y) \wedge \forall z\{(x \geq z) \vee (z \geq y)\}]$$

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Module I - FUNDAMENTALS OF LOGIC

Logic

Logic is the discipline that deals with the methods of reasoning. But logic is not an empirical (ie, experimental or observational) science like Physics or Biology. Logic is a non-empirical science like Mathematics. Logic provides rules and techniques for determining whether a given argument is valid or not. Logical reasoning is used in Mathematics to prove theorems, in Computer Science to verify the correctness of programs and to prove theorems, in Physical Science to draw conclusions from experiments, and in Social Science and in our daily lives to solve a multitude of problems. One component of Logic is Propositional Calculus.

Propositions

A **statement** or **proposition** is a declarative sentence that is either true or false, but not both. A proposition which cannot be further broken into simpler propositions is called simple proposition. Two or more simple propositions can be combined to form a new proposition called compound proposition.

A proposition has two possible values called truth values - true and false denoted by T and F or 1 and 0.

Simple propositions are represented by letters p, q, r ,... which are known as **propositional variables** or statement variables.

Examples:

1. The earth is round.
2. 2+3=5
3. Do you speak English ?
4. 3-x = 5
5. Take two tablets.
6. The integer 5 is a prime number.

Here (1),(2),(6) are statements that happen to be true.
(3) is a question, so it is not a statement.

(4) is a declarative sentence, but not a statement, since it is true or false depending on the value of x.

(5) is not a statement, it is a command.

Logical Connectives

Statements or propositional variables can be combined by logical connectives to obtain compound statements. Important connectives are and, negation, or, implication and biconditional.

- **Conjunction**

  If p and q are statements, the conjunction of p and q is the compound statement "p and q" denoted by $p \wedge q$. The compound statement $p \wedge q$ is true when both p and q are true; otherwise it is false. The truth table for $p \wedge q$ is given below.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples:

1. p: It is snowing. q: I am cold. Then
   $p \wedge q$: It is snowing and I am cold.
2. $p: 2 < 3$ $q: -5 > -8$. Then
   $p \wedge q: 2 < 3 \; and - 5 > -8$.

- **Disjunction**

  If p and q are statements, the disjunction of p and q is the compound statement "p or q" denoted by $p \vee q$. The compound statement $p \vee q$ is true if at least one of p or q is true; it is false when both p and q are false. The truth table for $p \vee q$ is given below.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Examples:

1. p: 2 is a positive integer. q: $\sqrt{2}$ is a rational number. Then
   $p \vee q$: 2 is a positive integer or $\sqrt{2}$ is a rational number.

2. p: $2 + 3 \neq 5$. q: London is the capital of France. Then
   $p \vee q$: $2 + 3 \neq 5$ or London is the capital of France.

- **Negation**

If p is a statement, the negation of p is the statement not p, denoted by $\sim p$. The truth table for $\sim p$ is given below.

p	$\sim p$
T	F
F	T

Examples:

1. $p: 2 + 3 > 1$. Then $\sim p: 2 + 3 \leq 1$.
2. q: It is cold. Then $\sim q$: It is not cold.

- **Implication (Conditional statement)**

If p and q are statements, the compound statement if p then q is called a conditional statement or implication denoted as $p \rightarrow q$. The statement p is called the hypothesis or antecedent and the statement q is called conclusion or consequent. The connective if ... then is denoted by the symbol $\rightarrow$. The truth table for $p \rightarrow q$ is given below.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

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Examples:

1. Write the implication $p \rightarrow q$ for each of the following.

   (i) p: I am hungry. q: I will eat. Then the implication is

   If I am hungry, then I will eat.

   (ii) p: It is snowing. q: 3+5=8. Then the implication is

   If it is snowing, then 3+5=8.

2. Which of the following propositions are true.

   (a) If the earth is round then the earth travels around the sun.
   (b) If Graham Bell invented telephone, then tigers have wings.

   Solution:

   (a) Let p: The earth is round.
   q: The earth travels around the sun.
   Here p is true and q is also true. So the conditional proposition is true.

   (b) Let p: Graham Bell invented telephone.
   q: Tigers have wings.
   Here p is true and q is false. So the conditional proposition is false.

3. Let p: It is cold and q: It is raining. Give a statement which describes each of the following.

   (i) $\sim p$ (ii) $p \wedge q$ (iii) $p \vee q$ (iv) $q \vee \sim p$

   Answers:

   (i) It is not cold. (ii) It is cold and it is raining.
   (iii) It is either cold or raining.
   (iv) It is raining or it is not cold.

4. Write down the symbols for the connectives.

   (i) either p nor p (ii) p and not q
   (iii) not p and not q (iv) not p and q

Answers:

(i) $p \vee \sim p$ (ii) $p \wedge \sim q$

(iii) $\sim p \wedge \sim q$ (iv) $\sim p \wedge q$

- **Biconditional (Equivalence)**

  If p and q are statements, the compound statement p if and only if q is called a biconditional statement or an equivalence denoted as $p \leftrightarrow q$. The connective if and only if is denoted by the symbol $\leftrightarrow$. The equivalence $p \leftrightarrow q$ is true only when both p and q are true or when both p and q are false. The truth table for $p \leftrightarrow q$ is given below.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Eg1. Is the following equivalence a true statement ? 3>2 if and only if 0<3-2.

Solution: Let p be the statement 3>2 and q be the statement 0<3-2.

Since both p and q are true, the equivalence $p \leftrightarrow q$ is true.

Eg2. Compute the truth table of the statement $(p \to q) \leftrightarrow (\sim q \to \sim p)$.

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Note: If a statement involves three propositions p, q, r then truth table contains 2^3=8 rows.

Tautology:

A statement that is true for all possible values of its propositional variables is called a tautology. A statement that is always false is called a contradiction or an absurdity. A statement that can be either true or false, depending on the truth values of its propositional variables is called a contingency.

Examples:

1. Show that the statement $p \vee \sim p$ is a tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Since the entries in the last column are all T's, it is a tautology.

2. Show that the statement $(p \vee q) \vee \sim p$ is a tautology.

p	q	$(p \vee q)$	$\sim p$	$(p \vee q) \vee \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

Since the entries in the last column are all T's, it is a tautology.

3. Show that the statement $(p \vee q) \leftrightarrow (q \vee p)$ is a tautology.

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

4. Show that the statement $(\sim q \wedge p) \wedge q$ is a contradiction.

p	q	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

Since the entries in the last column are all F's, it is a contradition.

5. Construct truth table for the statement $(\sim(p \wedge q) \vee r) \rightarrow \sim p$

p	q	r	$p \wedge q$	$\sim (p \wedge q)$	$(\sim (p \wedge q) \vee r$	$\sim p$	$(\sim (p \wedge q) \vee r) \rightarrow \sim p$
T	T	T	T	F	T	F	F
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	F
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

<u>Logical Equivalence:</u>

Two statements s_1 and s_2 are said to be logically equivalent when the statement s_1 is true (or false) if and only if the statement s_2 is true (or false) and is denoted by the symbol $\equiv$ or $\Leftrightarrow$. To test whether two propositions s_1 and s_2 are logically equivalent, construct truth tables for s_1 and s_2. If the truth values of the two statements are identical, then they are equivalent.

<u>Examples:</u>

1. Using truth table, prove that $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$.

p	q	$p \wedge q$	$\sim$ (p $\wedge$ q)	$\sim p$	$\sim$ q	$(\sim p) \vee (\sim q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since 4th and 7th columns are identical, both the statements are equivalent.

2. Using truth table, prove that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since 5^{th} and 8^{th} columns are equal, both the statements are equivalent.

The Laws of Logic

For any primitive statements p, q, r

1. Idempotent Laws

   $p \vee p \equiv p$

   $p \wedge p \equiv p$

2. Associative Laws
   $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

3. Commutative Laws
   $p \vee q \equiv q \vee p$
   $p \wedge q \equiv q \wedge p$

4. Distributive Laws
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5. Absorption Laws
   $p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

6. De Morgan's Laws

$\sim (p \vee q) \equiv \sim p \wedge \sim q$

$\sim (p \wedge q) \equiv \sim p \vee \sim q$

7. Double Negation

$\sim (\sim p) \equiv p$

8. Inverse Laws

$p \vee \sim p \equiv T$

$p \wedge \sim p \equiv F$

9. Domination Laws

$p \vee T \equiv T$

$p \wedge F \equiv F$

10. Identity Laws

$p \vee F \equiv p$

$p \wedge T \equiv p$

3. Is there any simpler statement which is logically equivalent to the compound statement $(p \vee q) \wedge \sim (\sim p \wedge q)$?

Ans. $(p \vee q) \wedge \sim (\sim p \wedge q)$	Reasons
$\equiv (p \vee q) \wedge (\sim\sim p \vee \sim q)$	DeMorgan's Law
$\equiv (p \vee q) \wedge (p \vee \sim q)$	Double negation
$\equiv p \vee (q \wedge \sim q)$	Distributive Law
$\equiv p \vee F$	Inverse Law
$\equiv p$	Identity Law

So the compound statement $(p \vee q) \wedge \sim (\sim p \wedge q)$ is logically equivalent to the statement p.

4. For any three primitive statements p, q, r, is there any simpler statement which is logically equivalent to the compound statement

$$\sim \{\sim [(p \vee q) \wedge r] \vee \sim q\}$$

Ans. $\sim \{\sim [(p \vee q) \wedge r] \vee \sim q\}$ Reasons

$\equiv \sim\sim [(p \vee q) \wedge r] \wedge \sim\sim q$ DeMorgan's Law

$\equiv [(p \vee q) \wedge r] \wedge q$ Double negation

$\equiv (p \vee q) \wedge (r \wedge q)$ Associative Law

$\equiv (p \vee q) \wedge (q \wedge r)$ Commutative Law

$\equiv [(p \vee q) \wedge q] \wedge r$ Associative Law

$\equiv q \wedge r$ Absorption and Commutative Laws

So the original statement is logically equivalent to the statement $q \wedge r$.

<u>Dual of a statement</u>

Let s be a statement containing no logical connectives other than $\vee$ and $\wedge$, then the dual of s is the statement obtained from s by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$ denoted by s^d.

Eg: Find the duals of the following statements

(i) $p \vee \sim p$ (ii) $p \vee T$ (iii) $(p \vee q) \wedge r$

(iv) $(p \wedge q) \vee T$ (v) $(p \wedge \sim q) \vee (r \wedge T)$

Ans:

(i) $p \wedge \sim p$ (ii) $p \wedge F$ (iii) $(p \wedge q) \vee r$

(iv) $(p \vee q) \wedge F$ (v) $(p \vee \sim q) \wedge (r \vee F)$

Principle of Duality

Let s and t be statements that contain no logical connectives other than $\vee$ and $\wedge$. If s is logically equivalent to t $(s \equiv t)$, then $s^d \equiv t^d$.

Substitution Rules:

1. Suppose that the compound statement P is a tautology. If p is a primitive statement that appears in P and we replace p by the same statement q , then the resulting compound statement P_1 is also a tautology.

2. Let P be a compound statement where p is an arbitrary statement that appears in P , and let q be a statement such that $q \equiv p$. Suppose that in P we replace one or more occurrences of p by q. Then this replacement gives the compound statement P_1. Under these circumstances $P_1 \equiv P$.

Examples:

1. Using truth tables, prove that $p \rightarrow q \equiv {\sim}p \vee q$

p	q	$p \rightarrow q$	${\sim}p$	${\sim}p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since 3rd and 5th columns are same, the two statements are equivalent.

2. Without using truth tables, show that

$$p \rightarrow (q \rightarrow r) \equiv p \rightarrow ({\sim}q \vee r) \equiv (p \wedge q) \rightarrow r$$

Ans. $p \rightarrow (q \rightarrow r) \equiv p \rightarrow ({\sim}\, q \vee r)$, by above eg.

$\equiv ({\sim}\, p) \vee ({\sim}\, q \vee r)$

$\equiv ({\sim}\, p \vee {\sim}\, q) \vee r$, Associative law

$\equiv {\sim}\, (p \wedge q) \vee r$, De Morgan's law

$\equiv (p \wedge q) \rightarrow r$

3. Without using truth tables, show that $p \rightarrow (q \rightarrow p) \equiv ({\sim}\, p) \rightarrow (p \rightarrow q)$

Ans. $LHS = p \rightarrow (q \rightarrow p) \equiv ({\sim}p) \vee (q \rightarrow p)$, by above eg.

$\equiv ({\sim}p) \vee \big(({\sim}\, q) \vee p\big)$

$\equiv ({\sim}\, p \vee p) \vee ({\sim}\, q)$, by Associative law

$$\equiv T \vee (\sim q) \equiv T \text{, by Domination law}$$

$$RHS = (\sim p) \to (p \to q) \equiv \sim(\sim p) \vee (p \to q)$$

$$\equiv p \vee ((\sim p) \vee q \text{ , by Double negation, eg1}$$

$$\equiv p \vee (\sim p) \vee q \text{ , by Associative law}$$

$$\equiv \big(p \vee (\sim p)\big) \vee q$$

$$\equiv T \vee q \equiv T \text{ , by Domination law}$$

Since $LHS = RHS$, $p \to (q \to p) \equiv (\sim p) \to (p \to q)$

4. Without using truth tables, show that $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

Ans. $LHS = p \to (q \vee r) \equiv (\sim p) \vee (q \vee r)$, by eg1

$\equiv (\sim p \vee q) \vee r$, by Associative

$RHS = (p \to q) \vee (p \to r) \equiv \big((\sim p) \vee q\big) \vee ((\sim p) \vee r)$

$\equiv (\sim p) \vee (q \vee r)$

5. Without using truth tables, show that $\sim (p \to q) \equiv p \wedge \sim q$

Ans. $\sim (p \to q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$

Converse, Contrapositive and Inverse Implications

There are some related implications that can be formed from $p \to q$. Let p and q be two propositions and $p \to q$ is an implication, then

(i) the converse of $p \to q$ is the implication $q \to p$

(ii) the inverse of $p \to q$ is the implication $\sim p \to \sim q$ and

(iii) the contrapositive of $p \to q$ is the implication $\sim q \to \sim p$.

p	q	$\sim p$	$\sim q$	Conditional $p \to q$	Converse $q \to p$	Inverse $\sim p \to \sim q$	Contrapositive $\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Example:

Consider the statements

p: It rains, q: The crops will grow.

Then the implication, $p \to q$ is the statement r: If it rains then the crops will grow.

(i) The converse of the implication is $q \to p$, s: If the crops grow, then there has been rain.

(ii) The inverse of the of the implication is $\sim p \to \sim q$, t: If it does not rain, then the crops will not grow.

(iii) The contrapositive of of the implication is $\sim q \rightarrow \sim p$, u: If the crops do not grow, then there has been no rain.

Logical Implication

If p, q are statements such that $p \rightarrow q$ is a tautology, then we say that p logically implies q and the implication $p \rightarrow q$ is called logical implication denoted by $p \Longrightarrow q$.

Note:

Let p, q be two statements

(i) If $p \Leftrightarrow q$, then the statement $p \leftrightarrow q$ is a tautology. So the statements p, q have the same truth values. Then the statements $p \rightarrow q$ and $q \rightarrow p$ are tautologies and we have $p \Rightarrow q$ and $q \Rightarrow p$.

(ii) The notation $p \nRightarrow q$ is used to indicate that $p \rightarrow q$ is not a tautology. So the given implication $p \rightarrow q$ is not a logical implication.

Rules of Inference:

If an implication $p \rightarrow q$ is a tautology, where p and q may be compound statements involving any number of propositional variables, then we can say that q logically follows from p.

Suppose an implication of the form $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology. Then this implication is true regardless of the truth values of any of its components. If any one of the components $p_1, p_2, \dots, p_n$ is false, then the hypothesis $p_1 \wedge p_2 \wedge \dots \wedge p_n$ is false and the implication$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is automatically true. In this case, we say that q logically follows from $p_1, p_2, \dots, p_n$. When q logically follows from $p_1, p_2, \dots, p_n$, we write

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

where the symbol $\therefore$ means therefore. This means that if p_1 is true, p_2 is true, ... , p_n is true, then q is true.

In the implication of the form $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$, the propositions $p_1, p_2, \dots, p_n$ are called the hypotheses or premises and q is called the conclusion. The collection of propositions $p_1, p_2, \dots, p_n$ is called an argument.

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An argument is logically valid if and only if the conjunction of the premises tautologically implies conclusion.

If the argument consists of statements $p_1, p_2, \ldots, p_n$ and q is the conclusion, then the **argument is valid** if and only if $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is a tautology and we have a valid argument.

Examples:

1. Let p, q, r be the statements given as

p: Ajith studies. q: He plays football. r: He passes Discrete Maths.

Let p_1, p_2, p_3 denote the premises

p_1: If Ajith studies, then he will pass Discrete Maths.

p_2: If he doesn't play a football, then he will study.

p_3: He failed Discrete Maths.

Determine whether the argument $(p_1 \wedge p_2 \wedge p_3) \to q$ is valid.

Here the statements are $p_1: p \to r$

$p_2: \sim q \to p$ and $p_3: \sim r$

Construct truth table for the implication $[(p \to r) \wedge (\sim q \to p) \wedge \sim r] \to q$

p	q	r	$p \to r$	$\sim q$	$\sim q \to p$	$\sim r$	$p_1 \wedge p_2 \wedge p_3$	$(p_1 \wedge p_2 \wedge p_3) \to q$
T	T	T	T	F	T	F	F	T
T	T	F	F	F	T	T	F	T
T	F	T	T	T	T	F	F	T
T	F	F	F	T	T	T	F	T
F	T	T	T	F	T	F	F	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	F	F	T
F	F	F	T	T	F	T	F	T

Final column contains all T values. So the implication is a tautology. Hence the implication $(p_1 \wedge p_2 \wedge p_3) \to q$ is a valid argument.

1. **<u>Rule of Detachment or Modus Ponens</u>**

In symbolic form this rule of inference is expressed by the logical implication $[p \wedge (p \to q)] \to q$. Construct the truth table of the implication.

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The actual rule will be written in the tabular form
$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

This means that q is the conclusion of the premises p and $p \rightarrow q$ which above the horizontal line. Since the implication is a tautology, the argument is valid.

2. Is the following argument valid ?
If smoking is healthy, then cigarettes are prescribed by physicians.

$\therefore$ Cigarettes are prescribed by physicians.

Ans.

Let the first premise be p: Smoking is healthy is false.
The second premise be $p \rightarrow q$ is true.
Construct truth table for the implication $p \wedge (p \rightarrow q) \rightarrow q$.

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The argument is valid since it is of the form Modus Ponens. However the conclusion is false. The conjunction of the two premises $p \wedge (p \rightarrow q)$ is false.

5. Establish the validity of the argument $p \to q$

$$q \to r \wedge s$$

$$\sim r \vee (\sim t \vee u)$$

$$p \wedge t$$

$$\therefore u$$

Steps	Reasons
1. $p \to q$	Premise
2. $q \to (r \wedge s)$	Premise
3. $p \to (r \wedge s)$	Steps (1),(2) and the Law of the Syllogism
4. $p \wedge t$	Premise
5. p	Step (4), Rule of Conjunctive Simplification
6. $r \wedge s$	Step (5),(3), Rule of Detachment
7. r	Step (6), Rule of Conjunctive Simplification
8. $\sim r \vee (\sim t \vee u)$	Premise
9. $\sim (r \wedge t) \vee u$	Step (8), Associative Law, De Morgan's Law
10. t	Step (4), Rule of Conjunctive Simplification
11. $r \wedge t$	Steps (7), (10), Rule of Conjunction
12. $\sim\sim(r \wedge t)$	step (11), Double negation
13. $\therefore u$	Steps (9),(12), Rule of Disjunctive Syllogism

Result: **8. Rule of Conditional Proof:**

$$p \wedge q$$

$$p \to (q \to r)$$

$$\therefore r$$

In symbolic form, $[(p \land q) \land \{p \to (q \to r)\}] \to r$

7. Establish the validity of the argument

$$\begin{array}{l} \sim p \leftrightarrow q \\ q \to r \\ \sim r \\ \hline \therefore p \end{array}$$

Steps	Reasons
1. $\sim p \leftrightarrow q$	Premise
2. $(\sim p \to q) \land (q \to \sim p)$	step (1), $\sim p \leftrightarrow q \equiv (\sim p \to q) \land (q \to \sim p)$
3. $\sim p \to q$	step (2), Rule of Conjunctive Simplification
4. $q \to r$	Premise
5. $\sim p \to r$	steps (3),(4), the Law of the Syllogism
6. $\sim p$	Premise (the one assumed)
7. r	steps (5),(6), Rule of Detachment
8. $\sim r$	Premise
9. $r \land \sim r \equiv F$	steps (7),(8), Rule of Conjunction
10. $\therefore p$	steps (6),(9), proof by Contradiction