MODULE 4

COMPLEX VARIABLE INTEGRATION

Q: Evaluate $\int_{-\pi i}^{\pi i} \cos z \, dz$

$$\int_{-\pi i}^{\pi i} \cos z \, dz = \left[\sin z\right]_{-\pi i}^{\pi i}$$

$$= \sin \pi i - \sin(-\pi i)$$

$$= \sin \pi i + \sin \pi i$$

$$= 2 \sin \pi i$$

$$= \underline{\underline{2 i \sinh \pi}}$$

Q: Evaluate $\int_{0}^{1+i} z^2 \, dz$

$$\int_{0}^{1+i} z^2 dz = \left[\frac{z^3}{3}\right]_{0}^{1+i}$$

$$= \underline{\underline{\frac{(1+i)^3}{3}}}$$

a: $\displaystyle\int_{8+\pi i}^{8-3\pi i} e^{z/2}\, dz$

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2}\, dz = \left[\frac{e^{z/2}}{1/2}\right]_{8+\pi i}^{8-3\pi i}$$

$$= 2\left[e^{\frac{8-3\pi i}{2}} - e^{\frac{8+\pi i}{2}}\right]$$

$$= 2\left[e^4 . e^{\frac{-3\pi i}{2}} - e^4 . e^{\frac{\pi i}{2}}\right]$$

$$= 2e^4\left[e^{\frac{-3\pi i}{2}} - e^{\frac{\pi i}{2}}\right]$$

$$\left\{ e^{i\theta} = \cos\theta + i\sin\theta \right\}$$

$$= 2e^4\left[\overbrace{\cos\left(-\tfrac{3\pi}{2}\right)}^{0} + i\overbrace{\sin\left(-\tfrac{3\pi}{2}\right)}^{+1} - \left(\underbrace{\cos\tfrac{\pi}{2}}_{0} + i\underbrace{\sin\tfrac{\pi}{2}}_{1}\right)\right]$$

$$= 2e^4(i - i)$$

$$= \underline{\underline{0}}$$

Q: $\int_{-i}^{i} \frac{dz}{z}$

$$\int_{-i}^{i} \frac{dz}{z} = [\log z]_{-i}^{i}$$

$$= \log i - \log(-i)$$

$$\left\{ \log z = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left[\frac{y}{x}\right] \right\}$$

$$= \frac{1}{2}\log 1 + i\tan^{-1}(\infty) - \left[\frac{1}{2}\log 1 - i\tan^{-1}(\infty)\right]$$

$$= 2 \times i\tan^{-1}(\infty) \qquad \left\{ \tan^{-1}\infty = \frac{\pi}{2} \right.$$

$$= 2 \times i\,\frac{\pi}{2} \qquad \left\{ \log 1 = 0 \right.$$

$$= \underline{\underline{\pi i}}$$

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Q: $\displaystyle\int_{1-\pi i}^{1+\pi i} e^{z/3}\,dz$

$$\int_{1-\pi i}^{1+\pi i} e^{z/3}\,dz = 3\left[e^{z/3}\right]_{1-\pi i}^{1+\pi i}$$

$$= 3\left[e^{\frac{1+\pi i}{3}} - e^{\frac{1-\pi i}{3}}\right].$$

$$= 3e^{1/3}\left[e^{\frac{\pi}{3}i} - e^{-\pi/3\, i}\right]$$

$$\left\{ e^{i\theta} = \cos\theta + i\sin\theta \right\}$$

$$= 3e^{1/3}\left[\underbrace{\cos\frac{\pi}{3}}_{1/2} + i\sin\frac{\pi}{3} - \left(\underbrace{\cos(-\pi/3)}_{1/2} + i\sin(-\pi/3)\right)\right]$$

$$= 3e^{1/3}\; 2\times i\sin\frac{\pi}{3}$$

$$= 3e^{1/3}\cdot\cancel{2}\times i\times\frac{\sqrt{3}}{\cancel{2}}$$

$$= \underline{\underline{3\sqrt{3}\, i\, e^{1/3}}}$$

Q: $\displaystyle\int_0^{\pi i} \sin 3z \cos z \, dz$

$$= \frac{1}{2}\int_0^{\pi i} \left[\sin(3z+z) + \sin(3z-z)\right] dz$$

$$= \frac{1}{2}\left[-\frac{\cos(4z)}{4} - \frac{\cos 2z}{2}\right]_0^{\pi i}$$

$$= \frac{1}{2}\left[-\frac{\overbrace{\cos 4\pi}^{1}}{4} - \frac{\overbrace{\cos 2\pi}^{1}}{2} - \left(-\frac{\cos 0}{4} - \frac{\cos 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4} - \frac{1}{2} - \left(-\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \underline{\underline{0}}$$

Q: Evaluate $\displaystyle\int_0^{1+i} (x^2 + iy)\, dz$ along the line $y = x$

$y = x$

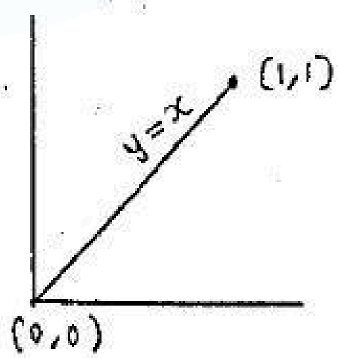
diff,

$dy = dx$

and $z = x + iy$

diff.

$dz = dx + i\,dy$



$\left\{\begin{array}{l} 1+i \\ (1,1) \end{array}\right.$

Now,
$$\int_0^{1+i} (x^2+iy)\,dz = \int_0^{1+i} (x^2+iy)(dx+idy)$$

$$= \int_0^{1} (x^2+ix)(dx+idx)$$

$$= (1+i)\int_0^{1} (x^2+ix)\,dx$$

$$= 1+i\left[\frac{x^3}{3}+\frac{ix^2}{2}\right]_0^{1}$$

$$= 1+i\left[\frac{1}{3}+\frac{i}{2}-0\right]$$

$$= 1+i\left[\frac{2+3i}{6}\right]$$

$$= \frac{2+2i+3i-3}{6}$$

$$= \frac{-1+5i}{6}$$

Q: Evaluate $\int_C (x^2 - iy^2)\, dz$ along $y = 2x^2$ from $(1,2)$ to $(2,8)$

given $y = 2x^2$

diff,

$dy = 4x\,dx$

and $z = x + iy$

$dz = dx + i\,dy$

$$\int_C (x^2 - iy^2)\, dz = \int (x^2 - i(2x^2)^2)\,(dx + i\,dy)$$

$$= \int_1^2 (x^2 - i4x^4)\,(dx + i4x\,dx)$$

$$= \int_1^2 (x^2 - i4x^4)(1 + i4x)\, dx \quad \begin{array}{c} x \;\; y \\ (1,2) \\ \downarrow \\ (2,8) \\ x \;\; y \end{array}$$

$$= \int_1^2 (x^2 + i4x^3 - i4x^4 + 16x^5)\, dx$$

$$= \left[\frac{x^3}{3} + ix^4 - i\frac{4}{5}x^5 + 16\frac{x^6}{6}\right]_1^2$$

$$= \frac{8}{3} + i16 - i\frac{128}{5} + \frac{1024}{6} - \left(\frac{1}{3} + i - i\frac{4}{5} + \frac{16}{6}\right)$$

$$= \frac{511}{3} - \frac{49}{5}i$$

Q: Evaluate $\int_{1-i}^{2+i} (2x+1+iy)\,dz$ from $(1-i)$ to $(2+i)$

$1-i \Rightarrow (1,-1)$

$2+i \Rightarrow (2,1)$

line $\Rightarrow$ $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$$\frac{y+1}{1+1} = \frac{x-1}{2-1}$$

$$y+1 = 2x-2$$

$$y = 2x-3$$

diff,

$$dy = 2\,dx$$

also $z = x+iy$

$dz = dx + idy$

$$\int_{1-i}^{2+i} (2x+1+iy)\, dz$$

$$= \int \left[2x+1+i(2x-3)\right] (dx+idy)$$

$$= \int (2x+1+i2x-3i)\,(dx+i\,2dx)$$

$$= 1+2i \int_1^2 (2x+1+2ix-3i)\, dx$$

$$= 1+2i \left[x^2+x+ix^2-3ix\right]_1^2$$

$$= 1+2i\left[4+2+4i-6i-(1+1+i-3i)\right]$$

$$= 1+2i\left[4+0\right]$$

$$= \underline{\underline{4+8i}}$$

Q: Evaluate $\int_C \mathrm{Re}(z)\,dz$ where C is the shortest path from $1+i$ to $3+i$

$1+i \Rightarrow (1,1)$

$3+i \Rightarrow (3,1)$ $\Rightarrow y=1$

$dy=0$

$z = x+iy$

$dz = dx + i\,dy$

$$\int_C \mathrm{Re}(z)\,dz = \int x\,(dx+i\,dy)$$

$$= \int_1^3 x\,(dx+0)$$

$$= \left[\frac{x^2}{2}\right]_1^3$$

$$= \frac{9}{2} - \frac{1}{2}$$

$$= \underline{\underline{4}}$$

Q: Evaluate $\int_C \text{Im}(z)\,dz$ where C is the straight line from 0 to $1+2i$

$0 \Rightarrow (0,0)$

$1+2i \Rightarrow (1,2)$

line $\Rightarrow$ $\dfrac{y-0}{2-0} = \dfrac{x-0}{1-0}$

$y = 2x$

$dy = 2dx$

also $z = x+iy$

$dz = dx+idy$

$$\int_C \text{Im}(z)\,dz = \int y\,(dx+idy)$$

$$= \int_0^1 2x\,(dx+i2dx)$$

$$= 1+2i \int 2x\,dx$$

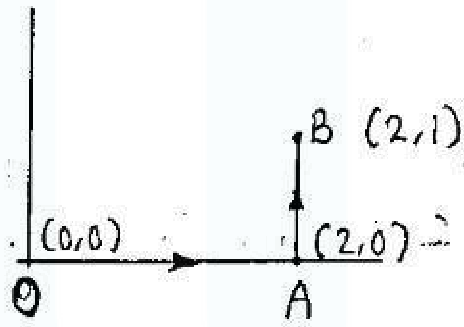
$$= (1+2i)\cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^1$$

$$= \underline{\underline{1+2i}}$$

Q: Evaluate $\int_C z^2 dz$ where C is a line along the real axis from (0,0) to (2,0) and then vertically to (2,1)

or

Evaluate $\int_C z^2 dz$ where C is a line along $z=0$ horizontally to $z=2$ and then vertically to $2+i$



$$\int_C z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz$$

<u>along OA</u>

$y = 0$

$dy = 0$

$z = x + iy$

$dz = dx + i dy$

$$\int_{OA} z^2 dz = \int_0^2 (x+0i)^2 (dx + i \times 0)$$

$$= \int_0^2 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{8}{3}$$

along AB

$$\int_{AB} z^2 dz = \int_0^1 (2+iy)^2 (0+idy) \qquad \left\{ \begin{array}{l} x = 2 \\ dx = 0 \end{array} \right.$$

$$= i\int_0^1 (2+iy)^2 dy$$

$$= \cancel{i}\left[\frac{(2+iy)^3}{3\times \cancel{i}}\right]_0^1$$

$$= \frac{(2+i)^3}{3} - \frac{2^3}{3}$$

$$= \frac{(2+i)^3 - 8}{3}$$

Now $\int_C z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz$

$$= \frac{8}{3} + \frac{(2+i)^3 - 8}{3}$$

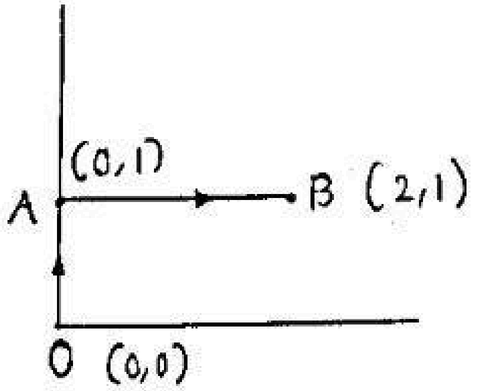
$$= \frac{\cancel{8} + (2+i)^3 - \cancel{8}}{3}$$

$$= \frac{2^3 + 3 \times 2 \times i^2 + 3 \times i \times 2^2 + i^3}{3}$$

$$= \frac{8 - 6 + 12i - i}{3}$$

$$= \frac{2 + 11i}{3}$$

Q: Evaluate $\int_C z^2 dz$ where, C is a line from $z=0$ vertically to $z=i$ and then horizontally to $z=2+i$.



$$\int_C z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz$$

<u>along OA</u>

$x = 0$

$dx = 0$

$z = x + iy$

$dz = dx + idy$

$$\int_{OA} z^2 dz = \int_0^1 (0+iy)^2 (0+idy)$$

$$= -1 \times i \int_0^1 y^2 dy$$

$$= -i \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{-i}{3}$$

along AB

$y = 1$

$dy = 0$

$$\int_{AB} z^2 dz = \int_0^2 (x+1)^2 (dx + i \times 0)$$

$$= \int_0^2 (x+1)^2 dx$$

$$= \int_0^2 (x^2 + 2x + 1) dx$$

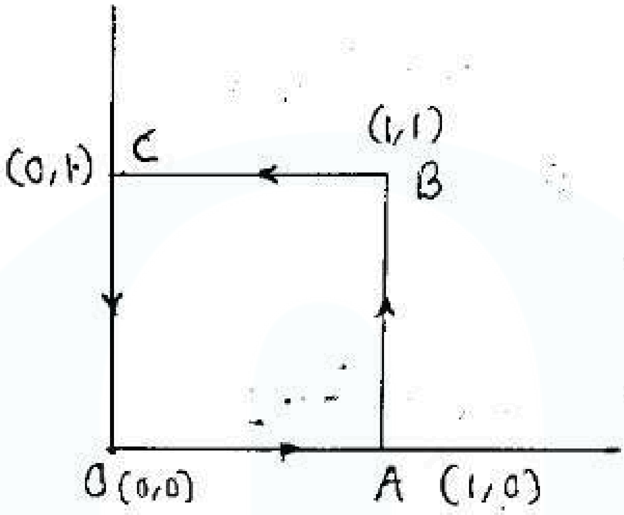
$$= \left[\frac{x^3}{3} + x^2 + x\right]_0^2$$

$$= \frac{8}{3} + 4 + 2 - 0$$

$$= \frac{26}{3}$$

Now, $\displaystyle\int_C z^2 dz = \frac{-i}{3} + \frac{26}{3} = \frac{26 - i}{3}$

Q : Evaluate $\int_C |z|^2 dz$ where C is the square having vertices (0,0), (1,0), (1,1), and (0,1) in anticlockwise direction.



$$\int_C |z|^2 dz = \int_{OA} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{CO} |z|^2 dz$$

$$|z|^2 = x^2 + y^2$$

$$z = x + iy$$

$$dz = dx + i\,dy$$

along OA

$y=0$

$dy=0$

$$\int_{OA} |z|^2 dz = \int_0^1 (x^2+0)\,(dx+i\times 0)$$

$$= \int_0^1 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

along AB

$x=1$

$dx=0$

$$\int_{AB} |z|^2 dz = \int_0^1 (1+y^2)(0+i\,dy)$$

$$= i\int_0^1 (1+y^2)\,dy$$

$$= i\left[y+\frac{y^3}{3}\right]_0^1$$

$$= i\left[1+\frac{1}{3}-0\right]$$

$$= \frac{4}{3}i$$

<u>along BC</u>

$y = 1$

$dy = 0$

$$\int_{BC} |z|^2 dz = \int_1^0 (x^2+1)(dx+0)$$

$$= \left[\frac{x^3}{3}+x\right]_1^0$$

$$= 0-\left(\frac{1}{3}+1\right)$$

$$= -\frac{4}{3}$$

along CO

$x = 0$

$dx = 0$

$$\int_{CO} |z|^2 dz = \int_1^0 (0+y^2)(0+i\,dy)$$

$$= i\left[\frac{y^3}{3}\right]_1^0$$

$$= i\left[0 - \frac{1}{3}\right]$$

$$= -\frac{i}{3}$$

Now,

$$\int_C |z|^2 dz = \int_{OA} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{CO} |z|^2 dz$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= -\frac{3}{3} + \frac{3i}{3}$$

$$= \underline{\underline{-1+i}}$$

Q: Evaluate $\int_C (x^2+ixy)\,dz$ from $(1,1)$ to $(2,4)$ along the curve $x=t$, $y=t^2$

$z = x+iy$

$dz = dx + i\,dy$

given, $x = t$, $y = t^2$

$dx = dt$ $\quad$ $dy = 2t\,dt$

$$\int_C (x^2+ixy)\,dz = \int (t^2+it^3)\,(dx+i\,dy)$$

$$= \int (t^2+it^3)\,(dt+i2t\,dt)$$

$$= \int (t^2+it^3)\,(1+i2t)\,dt$$

$$= \int (t^2+i2t^3+it^3-2t^4)\,dt$$

$\{$ from $(1,1)$ to $(2,4)$ $\}$

$(1,1) \Rightarrow x=1 \Rightarrow t=1$

$$\left.\begin{array}{l} x=1 \Rightarrow t=1 \\ y=1 \Rightarrow t^2=1 \Rightarrow t=\pm 1 \end{array}\right\} t=1$$

$(2,4) \Rightarrow$

$$\left.\begin{array}{l} x=2 \Rightarrow t=2 \\ y=4 \Rightarrow t^2=4 \Rightarrow t=\pm 2 \end{array}\right\} t=2$$

i.e, $t=1$ to $t=2$

Now,

$$\int_c (x^2+ixy)\,dz = \int_1^2 (t^2+\underbrace{i2t^3+it^3}_{i3t^3}-2t^4)\,dt$$

$$= \left[\frac{t^3}{3}+i3\frac{t^4}{4}-2\frac{t^5}{5}\right]_1^2$$

$$= \frac{8}{3}+i12-\frac{64}{5}-\left(\frac{1}{3}+\frac{3i}{4}-\frac{2}{5}\right)$$

$$= \frac{-151}{15}+\frac{45i}{4}$$

Q: Evaluate $\int_0^{4+2i} \bar{z}\,dz$ along the curve given by $z = t^2 + it$

$z = x + iy$

$dz = dx + i\,dy$

$\therefore \bar{z} = x - iy$

$$\int_0^{4+2i} \bar{z}\,dz = \int_0^{4+2i} (x - iy)(dx + i\,dy)$$

$$\left\{\begin{array}{ll} \text{given} \quad z = t^2 + it & \\ x = t^2 & y = t \\ dx = 2t\,dt & dy = dt \end{array}\right\}$$

$$= \int_C (t^2 - it)(2t\,dt + i\,dt)$$

$$= \int_C (t^2 - it)(2t + i)\,dt$$

$$= \int_C (2t^3 + it^2 - 2it^2 + t)\,dt$$

$$= \int_c (2t^3 - it^2 + t)\,dt$$

0 to $4+2i$

$\Rightarrow (0,0)$ to $(4,2)$

$(0,0) \Rightarrow x = 0 \Rightarrow t^2 = 0 \Rightarrow t = 0$
$y = 0 \Rightarrow t = 0$ $\Big\}$ $t=0$

$(4,2) \Rightarrow x = 4 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$
$y = 2 \Rightarrow t = 2$ $\Big\}$ $t = 2$

i.e, $t = 0$ to $t = 2$

Now,

$$= \int_0^2 (2t^3 - it^2 + t)\,dt$$

$$= \left[\frac{t^4}{2} - \frac{it^3}{3} + \frac{t^2}{2}\right]_0^2$$

$$= \frac{16}{2} - \frac{i8}{3} + \frac{4}{2} - 0$$

$$= \underline{\underline{10 - \frac{8i}{3}}}$$

Q: Evaluate $\int_c \bar{z}\,dz$ where c is $x = 3t$ and ~~and~~ $y = t^2$, $-1 \leq t \leq 4$

$$z = x + iy$$

$$dz = dx + i\,dy$$

$$\bar{z} = x - iy$$

$x = 3t$ $\quad\quad$ $y = t^2$

$dx = 3\,dt$ $\quad\quad$ $dy = 2t\,dt$

$$\int_c \bar{z}\,dz = \int_c (x - iy)(dx + i\,dy)$$

$$= \int_c (3t - it^2)(3dt + i2t\,dt)$$

$$= \int_C (3t - it^2)(3 + i2t)\,dt$$

$$= \int_C (9t + i6t^2 - 3it^2 + 2t^3)\,dt$$

$$= \int_C (9t + i3t^2 + 2t^3)\,dt$$

$\{\text{given } -1 \leq t \leq 4\}$

$$= \int_{-1}^{4} (9t + i3t^2 + 2t^3)\,dt$$

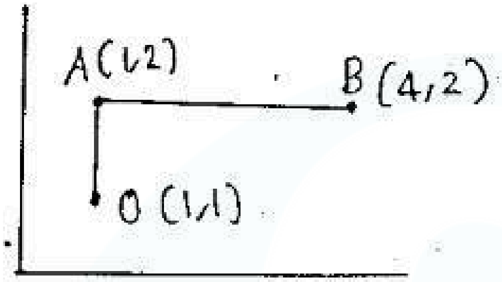
$$= \left[\frac{9t^2}{2} + it^3 + \frac{t^4}{2}\right]_{-1}^{4}$$

$$= \frac{144}{2} + i64 + \frac{256}{2} - \left(\frac{9}{2} - i + \frac{1}{2}\right)$$

$$= \underline{\underline{195 + 65i}}$$

Q: Evaluate $\int_{(1,1)}^{(4,2)} (x+y)dx + (y-x)dy$ along

a straight line from (1,1) to (1,2) and then to (4,2)



$$\int_C = \int_{OA} + \int_{AB}$$

along OA

(1,1) to (1,2)

$x = 1$

$dx = 0$

$$\int_{OA} (x+y)dx + (y-x)dy = \int (1+y)\times 0 + (y-1)dy$$

$$= \int_1^2 (y-1)\,dy$$

$$= \left[\frac{y^2}{2} - y\right]_1^2$$

$$= 2-2 - \left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2}$$

<u>along AB</u>

(1,2) to (4,2)

$y = 2,$

$dy = 0$

$$\int_{AB} (x+y)\,dx + (y-x)\,dy = \int (x+2)\,dx + 0$$

$$= \int_1^4 (x+2)\,dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_1^4$$

$$= 8+8 - \left(\frac{1}{2}+2\right)$$

$$= \frac{27}{2}$$

Now,

$$\int_{(1,1)}^{(4,2)} (x+y)dx + (y-x)dy = \frac{1}{2} + \frac{27}{2}$$

$$= \underline{\underline{14}}$$

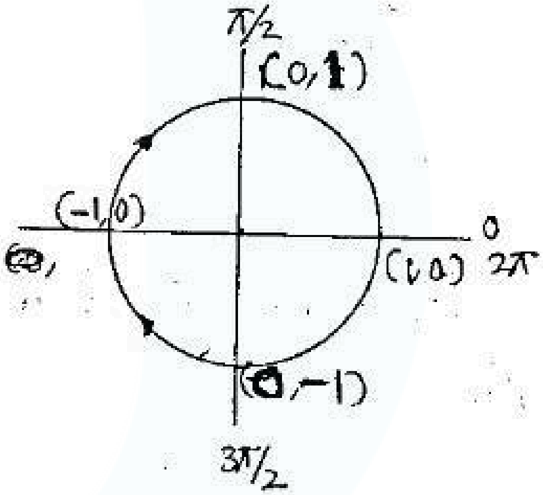
Q: Evaluate $\int_C |z|\,dz$ where C is the left half of the circle $|z|=1$ from $z=-i$ to $z=i$

$z = re^{i\theta}$

$\{|z|=1 \Rightarrow r=1\}$

$z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$



$$\int_C |z|\,dz = \int_{\frac{3\pi}{2}}^{\pi/2} 1 \times i e^{i\theta} d\theta \qquad \{|z|=1$$

$$= \cancel{i}\left[\frac{e^{i\theta}}{\cancel{i}}\right]_{\frac{3\pi}{2}}^{\pi/2}$$

$$= \left[\cos\theta + i\sin\theta\right]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

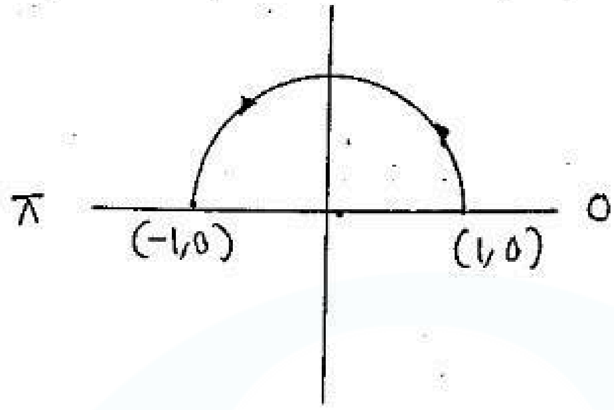
$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= 0 + i\times 1 - (0 + i\times -1)$$

$$= \underline{\underline{2i}}$$

Q: Show that $\int_C \frac{1}{z}\,dz = \pi i$ or $-\pi i$ according as C is the semicircle $|z|=1$ above or below the real axis from $(1,0)$ to $(-1,0)$

C is the semi-circle above the real axis



$$z = re^{i\theta}$$

$$\{ |z| = 1 \Rightarrow r = 1 \}$$

$$z = e^{i\theta} \ldots$$

$$dz = i e^{i\theta} d\theta$$

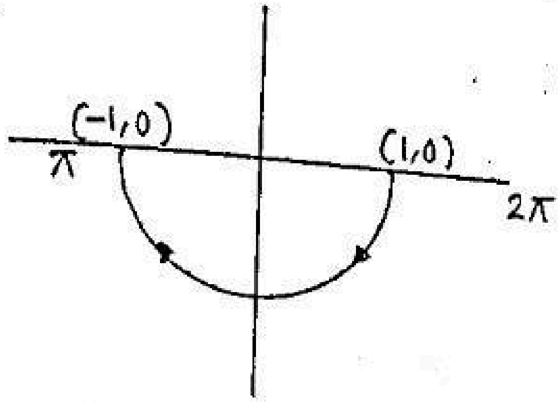
$$\int_C \frac{1}{z} dz = \int_0^{\pi} \frac{1}{e^{i\theta}} \, i e^{i\theta} d\theta$$

$$= i \int_0^{\pi} d\theta$$

$$= i [\theta]_0^{\pi}$$

$$= \underline{\underline{\pi i}}$$

C is the semi-circle below the real axis



$$\int_C \frac{1}{z} dz = \int_{2\pi}^{\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta$$

$$= i \int_{2\pi}^{\pi} d\theta$$

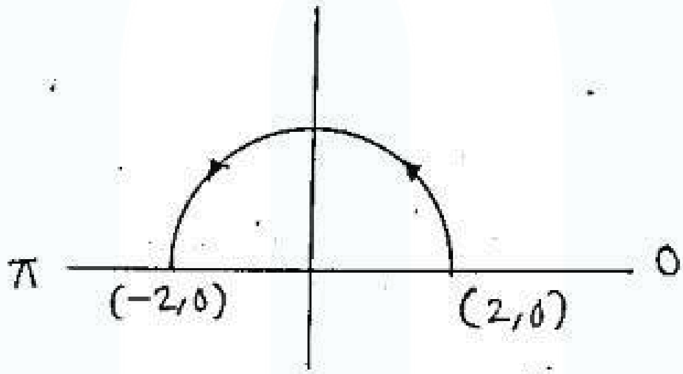
$$= i \left[\theta \right]_{2\pi}^{\pi}$$

$$= i \left[\pi - 2\pi \right]$$

$$= -\pi i$$

hence proved $\int_C \frac{1}{z} dz = \pi i$ or $-\pi i$

Q: Evaluate $\int_c \frac{z+2}{z}\,dz$ where c is the semi circle $|z|=2$ above the real axis from $(2,0)$ to $(-2,0)$



$z = re^{i\theta}$

$\{ |z| = 2 \Rightarrow r = 2 \}$

$z = 2e^{i\theta}$

$dz = 2ie^{i\theta}\,d\theta$

$$\int_C \frac{z+2}{z}\,dz = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} \cdot 2ie^{i\theta}\,d\theta$$

$$= \int_0^{\pi} (2e^{i\theta}+2)\; i\,d\theta$$

$$= 2i\int_0^{\pi} (e^{i\theta}+1)\,d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i} + \theta\right]_0^{\pi}$$

$$= 2i\left[\frac{e^{i\pi}}{i} + \pi - \left(\frac{e^{0}}{i} + 0\right)\right]$$

$$= 2i\left[\frac{e^{i\pi}}{i} + \pi - \frac{1}{i}\right]$$

$$= 2e^{i\pi} + 2\pi i - 2$$

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$$= 2(\cos\pi + i\sin\pi) + 2\pi i - 2$$

$$= 2(-1+0) + 2\pi i - 2$$

$$= \underline{\underline{-4 + 2\pi i}}$$

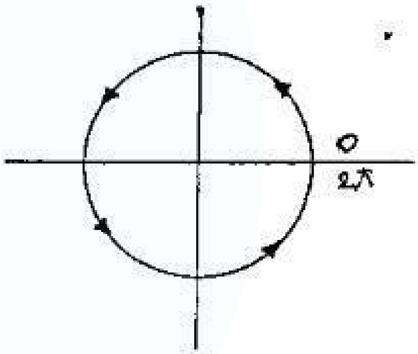
Q: Evaluate $\int_C \log z\, dz$. where C is the circle $|z|=1$:

$z = re^{i\theta}$

$\{\, |z|=1 \Rightarrow r=1 \,\}$

$z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$



$$\int_C \log z\, dz = \int_0^{2\pi} \log e^{i\theta}\; i e^{i\theta} d\theta$$

$$= \int_0^{2\pi} i\theta \cdot i e^{i\theta} d\theta$$

$$= -\int_0^{2\pi} \theta\, e^{i\theta} d\theta$$

$$= -\left[(\theta)\frac{e^{i\theta}}{i} - (1)\frac{e^{i\theta}}{i^2} \right]_0^{2\pi}$$

$$= -\left[\frac{\theta e^{i\theta}}{i} + e^{i\theta} \right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{i2\pi}}{i} + e^{i2\pi} - (0 + e^0) \right]$$

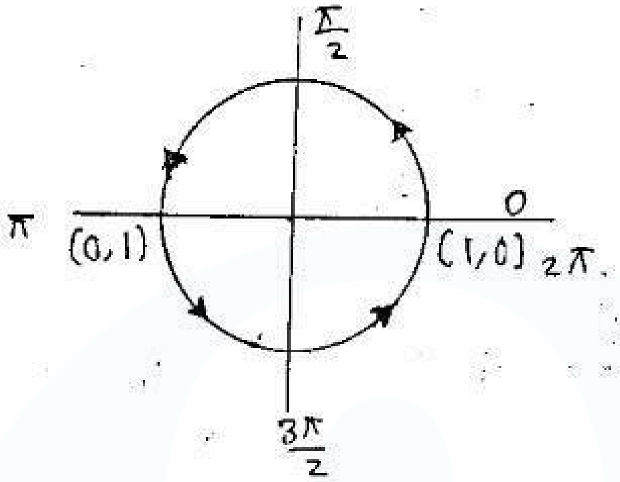
$$= -\left[\frac{2\pi}{i}\left(\cos 2\pi + i\sin 2\pi\right) + \cos 2\pi + i\sin 2\pi - 1 \right]$$

$$= -\left[\frac{2\pi}{i}\left[1 + 0\right] + 1 + 0 - 1 \right]$$

$$= -\frac{2\pi}{i} \qquad \left\{ \frac{1}{i} = -i \right.$$

$$= \underline{\underline{2\pi i}}$$

Q: Evaluate the integral $\int_C Re(z^2)\,dz$ where C is the unit circle in anticlockwise direction.



$z = re^{i\theta}$

{unit circle ⇒ $r=1$}

$z = e^{i\theta}$

$dz = i\,e^{i\theta}\,d\theta$

$z^2 = (e^{i\theta})^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

$Re(z^2) = \cos 2\theta$

Now,

$$\int_C Re(z^2)\,dz = \int_0^{2\pi} \cos 2\theta \; i\,e^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} \cos 2\theta\,(\cos\theta + i\sin\theta)\,d\theta$$

$$= i\int_0^{2\pi} \cos 2\theta \cos\theta\,d\theta + i\int_0^{2\pi} \cos 2\theta . i\sin\theta\,d\theta$$

$$= i\int_0^{2\pi} \cos 2\theta \cos\theta\,d\theta + i^2\int_0^{2\pi} \cos 2\theta \sin\theta\,d\theta$$

$$\boxed{\begin{array}{ll} \displaystyle\int_0^{2\pi} \cos m\theta \cos n\theta\,d\theta = 0 & \{\ m \neq n \\ \displaystyle\int_0^{2\pi} \cos m\theta \sin n\theta\,d\theta = 0 & \{\ \text{for all } m, n \end{array}}$$

$$= i \times 0 + i^2 \times 0$$

$$= \underline{\underline{0}}$$

Q: Show that $\int_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & , \text{if } n=1 \\ 0 & , \text{if } n \neq 1 \end{cases}$

where c is a circle $|z-a| = r$

here centre $\Rightarrow$ $(a,0)$

radius $\Rightarrow$ r

$$z-a = re^{i\theta}$$

$$z = re^{i\theta} + a$$

$$dz = ire^{i\theta} d\theta$$

$\theta \Rightarrow$ from 0 to 2π

<u>if $n=1$</u>

$$\int_C \frac{dz}{(z-a)^n} = \int_C \frac{dz}{z-a}$$

$$= \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}}$$

$$= i \int_0^{2\pi} d\theta$$

$$= i\left[\theta\right]_0^{2\pi}$$

$$= \underline{\underline{2\pi i}}$$

<u>if $n \neq 1$</u>

$$\int_C \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{i\theta}\, d\theta}{(re^{i\theta})^n}$$

$$= i\int_0^{2\pi} \frac{r\, e^{i\theta}\quad d\theta}{r^n\ e^{in\theta}}$$

$$= i\int_0^{2\pi} r^{(1-n)} \cdot e^{i\theta(1-n)}\, d\theta$$

$$= i\, r^{(1-n)} \int_0^{2\pi} e^{i\theta(1-n)}\, d\theta$$

$$= \cancel{i}\, r^{(1-n)} \left[\frac{e^{i\theta(1-n)}}{\cancel{i}(1-n)}\right]_0^{2\pi}$$

$$= \frac{r^{(1-n)}}{(1-n)} \left[e^{i2\pi(1-n)} - e^{0} \right]$$

$$= \frac{r^{(1-n)}}{1-n} \left[\cos 2\pi(1-n) + i \sin 2\pi(1-n) - 1 \right]$$

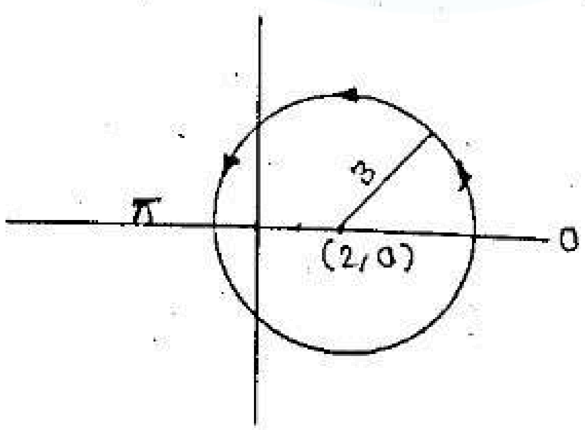
$$= \frac{r^{(1-n)}}{1-n} \left[1 + 0 - 1 \right]$$

$$= \underline{\underline{0}}$$

Q: Evaluate $\int_C (z - z^2)\, dz$ where C is upper half of circle $|z-2| = 3$.

here, centre $\Rightarrow$ $(2, 0)$

radius $\Rightarrow$ $r = 3$



$\theta \Rightarrow$ from 0 to π

$$z = re^{i\theta}$$

$$\{ r = 3 \}$$

$$z = 3e^{i\theta}$$

$$z^2 = 9e^{i2\theta}$$

$$dz = 3ie^{i\theta}d\theta$$

$$\int_C (z - z^2)dz = \int_0^{\pi} (3e^{i\theta} - 9e^{i2\theta}) \, 3ie^{i\theta}d\theta$$

$$= 3i\int_0^{\pi} (3e^{i\theta} - 9e^{i2\theta}) e^{i\theta}d\theta$$

$$= 3i\int_0^{\pi} (3e^{i2\theta} - 9e^{i3\theta}) d\theta$$

$$= 3\cancel{i}\left[\frac{3e^{i2\theta}}{2\cancel{i}} - \frac{\overset{3}{\cancel{9}}e^{i3\theta}}{\cancel{3}\cancel{i}}\right]_0^{\pi}$$

$$= 3\left[\frac{3e^{i2\theta}}{2} - 3e^{i3\theta}\right]_0^{\pi}$$

$$= 3\left[\frac{3}{2}e^{i2\pi} - 3e^{i3\pi} - \left(\frac{3}{2}e^{0} - 3e^{0}\right)\right]$$

$$= 3\left[\frac{3}{2}\left[\cos 2\pi + i\sin 2\pi\right] - 3\left[\cos 3\pi + i\sin 3\pi\right) - \frac{3}{2} + 3\right]$$

$$= 3\left[\frac{3}{2}(1+0) - 3(-1+0) - \frac{3}{2} + 3\right]$$

$$= \underline{\underline{18}}$$

Cauchy Integral formula

If a function $f(z)$ is analytic at all points inside and on a simple curve C and a is any point inside C, then,

$$\boxed{\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)}$$

- $\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$, if a lies inside C
- $\int_c \frac{f(z)}{z-a}\,dz = 0$, if a lies outside C

Note

* $$\int_c \frac{f(z)}{(z-a)^n}\,dz = \begin{cases} \dfrac{2\pi i}{(n-1)!} f^{(n-1)}(a) & \text{, if a lies inside c} \\ 0 & \text{, if a lies outside c} \end{cases}$$

Q: Evaluate $\int_C \frac{\cos\pi z}{z-2}\, dz$ where C is $|z|=3$

$|z|=3 \Rightarrow r=3$

let, $z-2=0$

$z=2$

{ the integrand $\frac{\cos\pi z}{z-2}$ is analytic at all points except at $z=2$

Now, $|z| = |2|$

$= \sqrt{2^2}$

$= 2$

$2 < 3$

$\therefore$ $z=2$ ~~inside~~ lies inside C

by Cauchy integral formula,

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

$$\int_C \frac{\cos\pi z}{z-2}\, dz = 2\pi i\, f(2)$$

$$= \underline{\underline{2\pi i}}$$

{ $f(z) = \cos\pi z$

$f(2) = \cos\pi\times 2$

$= 1$

Q: Evaluate $\int_C \frac{e^z}{z+1} dz$, C is $|z+1| = \frac{1}{2}$

$z+1 = 0$

$z = -1$

at $z=-1 \Rightarrow |z+1| = |-1+1| = |0| = 0$

$0 < \frac{1}{2}$

$\therefore$ $z=-1$ lies inside C

by Cauchy Integral formula

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_C \frac{e^z}{z+1} dz = \underline{\underline{2\pi i e^{-1}}}$$

$\left\{ \begin{array}{l} f(z) = e^z \\ f(-1) = e^{-1} \end{array} \right.$

Q: Evaluate $\int_C \frac{e^z}{z} dz$ where C is $|z| = 1$

$z = 0$

$|z| = |0| = 0 < 1$

by Cauchy Integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_C \frac{e^z}{z} dz = 2\pi i \times 1 = \underline{\underline{2\pi i}}$$

$$\left\{ \begin{array}{l} f(z) = e^z \\ f(0) = e^0 = 1 \end{array} \right.$$

Q: Evaluate $\int_C \frac{e^{-2z}}{z+i} dz$ where C is $|z+2| = 10$

$z + i = 0$

$z = -i$

at $z = -i \Rightarrow |z+2| = |-i+2|$

$= \sqrt{1+4}$

$= \sqrt{5}$

$\sqrt{5} < 10$.

$\therefore$ $z = -i$ lies inside C

by Cauchy Integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\left\{ \begin{array}{l} f(z) = e^{-2z} \\ f(-i) = e^{-2 \times -i} \\ \quad = e^{2i} \end{array} \right.$$

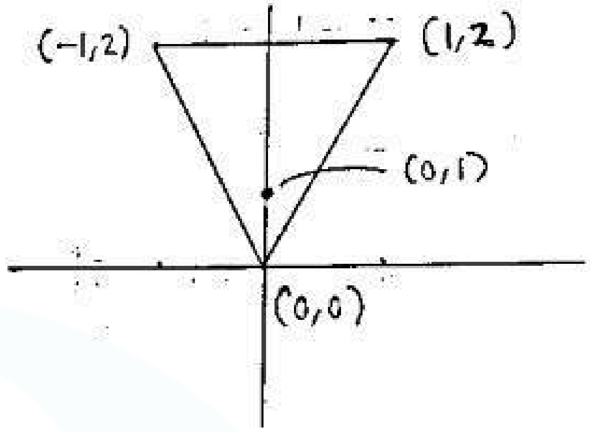
$$\int_C \frac{e^{-2z}}{z+i} dz = \underline{\underline{2\pi i\, e^{2i}}}$$

Q: Evaluate $\int_C \frac{\tan z}{z-i}\, dz$ where C is a triangle with vertices $0, \pm 1 + 2i$

(0,0)

(1+2i) ⇨

(-1+2i)



$z - i = 0$

$z = i \Rightarrow (0,1)$

$\therefore$ $z = i$ lies inside C

by Cauchy Integral formula,

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

$$\begin{cases} f(z) = \tan z \\ f(i) = \tan i \end{cases}$$

$$\int_C \frac{\tan z}{z-i}\, dz = \underline{\underline{2\pi i \tan i}}$$

Q: Evaluate $\int_C \frac{\sin \pi z}{z-1} dz$ where C is $|z| = \frac{1}{2}$

$z - 1 = 0$

$z = 1$

at $z = 1 \Rightarrow |z| = |1| = 1$

$1 > \frac{1}{2}$

$\therefore$ $z = 1$ lies outside C

by Cauchy Integral formula,

$\int_C \frac{f(z)}{z-a} dz = 0$ $\{$ a lies outside C

$\int_C \frac{\sin \pi z}{z-1} dz = 0$

Q: Evaluate $\int_C \frac{\cot z}{z+i}\,dz$ where: C is $|z+2|=1$

$z+i=0$

$z=-i$

at $z=-i \Rightarrow |z+2| = |-i+2| = \sqrt{1+4}$

$= \sqrt{5}$

$\sqrt{5} > 1$

$\therefore$ by Cauchy Integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 0$$

$$\int_C \frac{\cot z}{z+i}\,dz = \underline{\underline{0}}$$

Q: Evaluate $\int_C \frac{e^{2z}}{(z-2)^2}\,dz$ where C is $|z|=3$

$(z-2)^2=0$

$z-2=0$

$z=2$

at $z=2$, $|z|=|2|=2<3$

$z=2$ lies inside C

by Cauchy-Integral formula,

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$\int_C \frac{e^{2z}}{(z-2)^2}\,dz = \frac{2\pi i}{1!} f'(2)$$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f'(2) = 2e^{2\times 2} = 2e^4$

$$= 2\pi i \times 2e^4$$

$$= \underline{\underline{4\pi i e^4}}$$

Q: Evaluate $\int_C \frac{\sin \pi z^2}{(z-1)^3} dz$ where C is $|z|=2$

$(z-1)^3 = 0$

$z-1 = 0$

$z = 1$

at $z=1$, $|z| = |1| = 1 < 2$

$\therefore$ $z=1$ lies inside C

by Cauchy integral formula,

$$\int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$\int_C \frac{\sin \pi z^2}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}\times 1} \times -2\pi$$

$$= -2\pi^2 i$$

$f(z) = \sin \pi z^2$

$f'(z) = \cos \pi z^2 \times \pi \times 2z$

$f''(z) = \cos \pi z^2 \times 2\pi$

$+ 2\pi z \bar{x} \sin \pi z^2$

$\times 2\pi z$

$f''(1) = \cos \pi \times 2\pi$

$+ 2\pi \times^- \sin \pi$

$\times 2\pi$

$= -2\pi$

Q: Evaluate $\int_c \frac{e^{-z}}{z^3} dz$ where C is $|z|=1$

$z^3 = 0$

$z = 0$

at $z=0$, $|z| = |0| = 0 < 1$

$\therefore$ $z=0$ lies inside c

by Cauchy integral formula,

$$\int_c \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)'}(a)$$

$$\int_c \frac{e^{-z}}{z^3} dz = \frac{2\pi i}{2!} f''(0)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}\times 1} \times 1$$

$$= \underline{\underline{\pi i}}$$

$f(z) = e^{-z}$

$f'(z) = -e^{-z}$

$f''(z) = e^{-z}$

$f''(0) = e^0 = 1$

Q: Evaluate $\int_C \frac{\cos^2 \pi z}{(z-2)^3} dz$ where C is $|z|=1$

$(z-2)^3 = 0$

$z-2 = 0$

$z = 2$

at $z=2$, $|z| = |2| = 2 > 1$

$\therefore$ $z=2$ lies outside C

by Cauchy-integral formula,

$$\int_C \frac{f(z)}{(z-a)^n} dz = 0$$

$$\int_C \frac{\cos^2 \pi z}{(z-2)^3} dz = \underline{\underline{0}}$$

Q: Evaluate $\int_C \frac{\cos^2 \pi z}{(z-1)^3} dz$ where C is $|z|=1$

$(z-1)^3 = 0$

$z-1 = 0$

$z = 1$

at $z=1$, $|z| = |1| = 1 \quad \{1 = 1$

$\therefore$ $z=1$ lies inside C.

by Cauchy integral formula,

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$\int_C \frac{\cos^2 \pi z}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f''(1)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}\times 1} \times -2\pi^2$$

$$= -2\pi^3 i$$

$f(z) = \cos^2 \pi z$

$f'(z) = 2\cos \pi z \times -\sin \pi z \times \pi$

$f'(z) = -\pi \sin 2\pi z$

$f''(z) = -\pi \times \cos 2\pi z \times 2\pi$

$f''(1) = -2\pi^2 \times \cos 2\pi$

$= -2\pi^2$

Q: Evaluate $\int_C \frac{e^z}{(z+1)^3} dz$ where C is $|z+1| = 1$

$(z+1)^3 = 0$

$z+1 = 0$

$z = -1$

at $z = -1$, $|z+1| = |-1+1| = |0| = 0 < 1$

$\therefore$ $z = -1$ lies inside C

by Cauchy integral formula,

$$\int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)'}(a)$$

$$\int_C \frac{e^z}{(z+1)^3} dz = \frac{2\pi i}{2!} f''(-1)$$

$f(z) = e^z$

$f'(z) = e^z$

$f''(z) = e^z$

$f''(-1) = e^{-1}$

$$= \frac{\cancel{2}\pi i}{\cancel{2} \times 1} \times e^{-1}$$

$$= \frac{\pi i}{e}$$

Q: Evaluate $\int_C \frac{dz}{z^2 e^{2z}}$ where C is $|z|=1$

$$\int_C \frac{dz}{z^2 e^{2z}} = \int_C \frac{e^{-2z}\,dz}{z^2}$$

$z^2 = 0$

$z = 0$

at $z=0$, $|z| = |0| = 0 < 1$

$\therefore$ $z=0$ lies inside C

by Cauchy-integral formula

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$\int_C \frac{e^{-2z}}{(z^2)}\,dz = \frac{2\pi i}{1!} f'(0)$$

$$= \frac{2\pi i}{1} \times -2$$

$$= \underline{\underline{-4\pi i}}$$

$f(z) = e^{-2z}$

$f'(z) = e^{-2z} \times -2$

$f'(0) = e^0 \times -2$

$= -2$

Q: Evaluate $\int_C \frac{\cosh 4z}{(z-4)^3}\,dz$ where C is: $|z|=6$.

$(z-4)^3 = 0$

$z-4 = 0$

$z = 4$

at $z=4$, $|z| = |4| = 4 < 6$

$\therefore z=4$ lies inside C.

by Cauchy integral formula,

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)'}(a)$$

$$\int_C \frac{\cosh 4z}{(z-4)^3}\,dz = \frac{2\pi i}{2!} f''(4)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}\times 1} \times 16\cosh 16$$

$$= 16\pi i \cosh 16$$

$f(z) = \cosh 4z$

$f'(z) = \sinh 4z \times 4$

$f''(z) = \cosh 4z \times 16$

$f''(4) = \cosh(4\times 4) \times 16$

$= 16\cosh 16$

Q: Evaluate $\int_C \frac{\tan z}{z^2} dz$ where C is the ellipse

$16x^2+y^2=1$

$z^2=0$

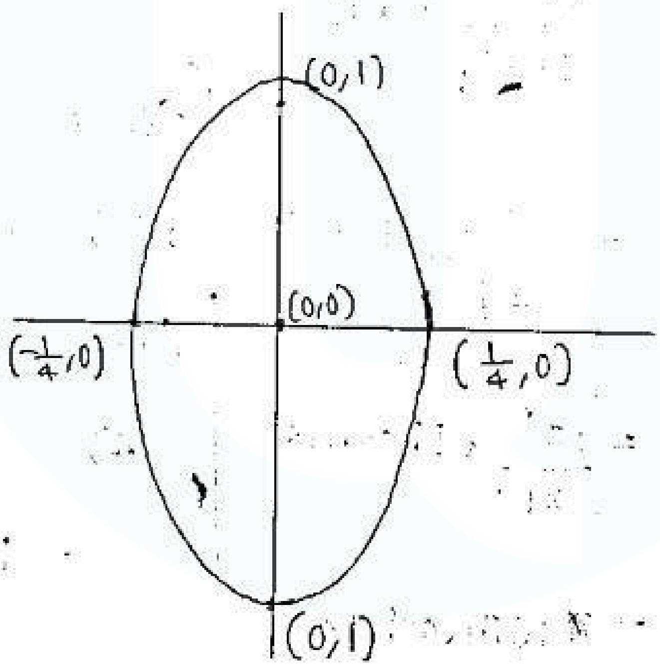
$z=0$

at $z=0$, $16x^2+y^2=0+0=0$

at $z=0 \Rightarrow (x,y)=(0,0)$

$16x^2+y^2=1$

$\frac{x^2}{(\frac{1}{4})^2}+\frac{y^2}{(1)^2}=1$



$\therefore$ $z=0$ lies inside C

by Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$f(z) = \tan z$

$f'(z) = \sec^2 z$

$f'(0) = \sec^2 0$

$= 1$

$$\int_C \frac{\tan z}{z^2}\,dz = \frac{2\pi i}{1!} f'(0)$$

$$= \frac{2\pi i \times 1}{1}$$

$$= \underline{\underline{2\pi i}}$$

Q: If $f(a) = \int_C \frac{3z^2+7z+1}{z-a}\,dz$ where C is the circle $x^2+y^2=4$, find $f(3), f(1), f'(1-i)$ and $f''(1+i)$

$x^2+y^2=4 \Rightarrow r = 2$

$$f(3) = \int_C \frac{3z^2+7z+1}{z-3}\,dz$$

$z - 3 = 0$

$z = 3 \Rightarrow (x,y) = (3,0)$. $\{3 > 2$

$\therefore$ ~~at~~ $z = 3$ lies outside the circle

by Cauchy integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 0$$

$$f(3) = \int_c \frac{3z^2+7z+1}{z-3}\,dz = \underline{\underline{0}}$$

$$f(1) = \int_c \frac{3z^2+7z+1}{z-1}\,dz.$$

$z-1=0$

$z=1 \Rightarrow (x,y) = (1,0)$ $\{1<2$

$z=1$ lies inside c

by Cauchy integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

here,

$f(1) = ?$

$\therefore$ let $f(1) = \int_c \frac{g(z)}{z-a}\, dz = 2\pi i\, g(a)$

$f(1) = \int_c \frac{3z^2+7z+1}{z-1}\, dz = 2\pi i\, g(1)$

$= 2\pi i \times 11$

$= \underline{\underline{22\pi i}}$

$$\left\{ \begin{aligned} g(z) &= 3z^2+7z+1 \\ g(1) &= 3+7+1 \\ &= 11 \end{aligned} \right.$$

$f'(1-i) = ?$

We've $f(a) = \int_c \frac{g(z)}{z-a}\, dz = 2\pi i\, g(a)$ {if inside point

since, $1-i \Rightarrow (x,y) = (1,-1)$

it is an inside point of circle

$f(1-i) = 2\pi i\, g(1-i)$

$f'(1-i) = 2\pi i\, g'(1-i)$

$= 2\pi i \times (13-6i)$

$= \underline{\underline{2\pi(13i+6)}}$

$$\left\{ \begin{aligned} g(z) &= 3z^2+7z+1 \\ g'(z) &= 6z+7 \\ g'(1-i) &= 6(1-i)+7 \\ &= 6-6i+7 \\ &= 13-6i \end{aligned} \right.$$

$f''(1+i) = ?$

$1+i \Rightarrow (1,1)$

it is an inside point of circle.

for inside points,
We've $f(a) = 2\pi i\, g(a)$

$f'(a) = 2\pi i\, g'(a)$

$f''(a) = 2\pi i\, g''(a)$

$f''(1+i) = 2\pi i \times g''(1+i)$

$= 2\pi i \times 6$

$= \underline{\underline{12\pi i}}$

$$\left\{\begin{array}{l} g(z) = 3z^2+7z+1 \\ g'(z) = 6z+7 \\ g''(z) = 6 \\ g''(1+i) = 6 \end{array}\right.$$

⊙ Let $\int_c \frac{f(z)}{(z-a)(z-b)}\, dz$

① If a and b ~~are~~ lies inside c,

then use partial fraction

i.e,

- $$\frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$$

also,

- $$\frac{1}{(z-a)^3(z-b)} = \frac{A}{z-a} + \frac{B}{(z-a)^2} + \frac{C}{(z-a)^3} + \frac{D}{z-b}$$

- $$\frac{1}{(z^2+a)(z+b)} = \frac{Az+B}{z^2+a} + \frac{C}{z+b}$$

{partial fraction can be used only if degree of numerator is less than the degree of denominator.}

② If 'a' lies inside and 'b' lies outside of C,

$$\boxed{\int_C \frac{f(z)}{(z-a)(z-b)}\,dz = \int_C \frac{\frac{f(z)}{(z-b)}}{z-a}\,dz}$$

Q: Evaluate $\int_C \frac{z}{(z-1)(z+2)} dz$ where C is $|z| = 3$

$(z-1)(z+2) = 0$

$z-1 = 0$ $\quad\quad$ $z+2 = 0$

$z = 1$ $\quad\quad$ $z = -2$

at $z=1$, $|z| = |1| = 1 < 3$

$\therefore$ $z=1$ lies inside C

at $z=-2$, $|z| = |-2| = 2 < 3$

$\therefore$ $z=-2$ lies inside C

$$\frac{1}{(z-1)(z+2)} = \frac{A}{(z-1)} + \frac{B}{(z+2)}$$

$$1 = A(z+2) + B(z-1)$$

put $z = -2$

$1 = 0 + B \times -3$

$B = \frac{-1}{3}$

put $z = 1$

$1 = A \times 3 + 0$

$A = \frac{1}{3}$

$$\frac{1}{(z-1)(z+2)} = \frac{1/3}{z-1} + \frac{-1/3}{z+2}$$

$$\int_c \frac{z}{(z-1)(z+2)}\,dz = \frac{1}{3}\int_c \frac{z}{z-1}\,dz - \frac{1}{3}\int_c \frac{z}{z+2}\,dz$$

$$\left\{ \begin{array}{l} \text{by Cauchy integral formula} \\ \displaystyle\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \end{array} \right\}$$

$$= \frac{1}{3} \times 2\pi i\, f(1) - \frac{1}{3} 2\pi i\, f(-2)$$

$$\left\{ \begin{array}{l} f(z) = z \\ f(1) = 1 \\ f(-2) = -2 \end{array} \right.$$

$$= \frac{2}{3}\pi i \times 1 - \frac{2}{3}\pi i \times -2$$

$$= \frac{6}{3}\pi i$$

$$= \underline{\underline{2\pi i}}$$

Q: Evaluate $\displaystyle\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$ where C is $|z|=3$

$(z-1)(z-2) = 0$

$z-1 = 0 \qquad z-2 = 0$

$z = 1 \qquad z = 2$

at $z=1$, $|z|=|1| = 1 < 3$

$\therefore$ $z=1$ lies inside C

at $z=2$, $|z|=|2| = 2 < 3$

$\therefore$ $z=2$ lies inside C

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

put $z=2$

$1 = 0 + B$

$B = 1$

put $z=1$

$1 = A \times -1 + 0$

$A = -1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} = -\int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-1}\,dz + \int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\,dz$$

$\left\{ \text{by Cauchy integral formula,} \quad \int_c \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a) \right\}$

$$= -2\pi i\, f(1) + 2\pi i\, f(2)$$

$$= -2\pi i \times -1 + 2\pi i \times 1$$

$$= \underline{\underline{4\pi i}}$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$

$f(1) = \sin \pi + \cos \pi$

$= -1$

$f(2) = \sin 4\pi + \cos 4\pi$

$= 1$

Q: Evaluate $\int_c \frac{e^z}{z^2+4}\, dz$ where c is $|z|=3$

$z^2+4 = 0$

$z^2 = -4$

$z = \pm 2i$ $\quad \{ z^2+4 = (z+2i)(z-2i)$

at $z = 2i$, $|z| = |2i| = \sqrt{2^2} = 2 < 3$

$\therefore$ $z = 2i$ lies inside c

at $z = -2i$, $|z| = |-2i| = \sqrt{(-2)^2} = 2 < 3$

$\therefore$ $z = -2i$ lies inside c

$$\frac{1}{(z+2i)(z-2i)} = \frac{A}{z+2i} + \frac{B}{z-2i}$$

$$1 = A(z-2i) + B(z+2i)$$

put $z = 2i$

$1 = 0 + B \times 4i$

$B = \frac{1}{4i}$

put $z = -2i$

$1 = A \times {}^{-}4i$

$A = \frac{-1}{4i}$

$$\frac{1}{(z+2i)(z-2i)} = \frac{-1/4i}{z+2i} + \frac{1/4i}{z-2i}$$

$$\int_C \frac{e^z}{(z+2i)(z-2i)} dz = -\frac{1}{4i} \int_C \frac{e^z}{z+2i} dz + \frac{1}{4i} \int_C \frac{e^z}{z-2i} dz$$

$$\left\{ \text{by Cauchy Integral formula, } \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \right\}$$

$$= \frac{-1}{4i} \times 2\pi i f(-2i) + \frac{1}{4i} \times 2\pi i f(2i)$$

$$= -\frac{\pi}{2} f(-2i) + \frac{\pi}{2} f(2i)$$

$$f(z) = e^{z}$$

$$f(-2i) = e^{-2i}$$

$$f(2i) = e^{2i}$$

$$= -\frac{\pi}{2} e^{-2i} + \frac{\pi}{2} e^{2i}$$

$$= \pi \frac{e^{2i} - e^{-2i}}{2}$$

$$= \pi \sinh 2i$$

Q: Evaluate $\int_C \frac{3z-1}{z^3-z} dz$ where C is $|z|=2$

$z^3 - z = 0$

$z(z^2-1) = 0$

$z(z+1)(z-1) = 0$

$z=0$ $\quad$ $z+1=0$ $\quad$ $z-1=0$

$\quad\quad z=-1$ $\quad$ $z=1$

at $z=0$, $|z| = |0| = 0 < 2$

$z=0$ lies inside C

at $z=-1$, $|z| = |-1| = 1 < 2$

$z=-1$ lies inside C

at $z=1$, $|z| = |1| = 1 < 2$

$z=1$ lies inside C.

$$\frac{1}{z^3-z} = \frac{1}{z(z+1)(z-1)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-1}$$

$$1 = A(z+1)(z-1) + Bz(z-1) + Cz(z+1)$$

put $z=-1$

$1 = 0 + 2B + 0$

$\underline{B = \frac{1}{2}}$

put $z = 1$

$1 = 0 + C + 2C$

$\underline{C = \frac{1}{2}}$

put $z = 0$

$1 = -A + 0 + 0$

$\underline{A = -1}$

$$\frac{1}{z(z+1)(z-1)} = \frac{-1}{z} + \frac{1/2}{z+1} + \frac{1/2}{z-1}$$

$$\int_c \frac{3z-1}{z(z+1)(z-1)}\,dz = -\int_c \frac{3z-1}{z}\,dz + \frac{1}{2}\int_c \frac{3z-1}{z+1}\,dz + \frac{1}{2}\int_c \frac{3z-1}{z-1}\,dz$$

By Cauchy integral formula,

$$\int \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$= -2\pi i f(0) + \frac{1}{\cancel{2}}\cancel{2}\pi i f(-1) + \frac{1}{\cancel{2}}\cancel{2}\pi i f(1)$$

$$\begin{cases} f(z) = 3z-1 \\ f(0) = -1 \\ f(-1) = -4 \\ f(1) = 2 \end{cases}$$

$$= -2\pi i \times -1 + \pi i \times -4 + \pi i \times 2$$

$$= 2\pi i - 4\pi i + 2\pi i$$

$$= \underline{\underline{0}}$$

Q: Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz$ where C is $|z| = 3$

$(z+2)(z+1)^2 = 0$

$z+2 = 0$ $\qquad$ $(z+1)^2$

$z = -2$ $\qquad$ $z+1 = 0$

$\qquad$ $z = -1$

at $z = -2$, $|z| = |-2| = 2 < 3$

$z = -2$ lies inside C.

at $z=-1$, $|z| = |-1| = 1 < 3$

$z=-1$ lies inside C

$$\frac{1}{(z+2)(z+1)^2} = \frac{A}{z+2} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$

$$1 = A(z+1)^2 + B(z+2)(z+1) + C(z+2)$$

put $z=-1$

$1 = 0 + 0 + C$

$\underline{C = 1}$

put $z=-2$

$1 = A + 0 + 0$

$\underline{A = 1}$

put $z=0$

$1 = A + 2B + 2C$

$2B = 1 - A - 2C$

$= 1 - 1 - 2$

$= -2$

$\underline{B = -1}$

$$\frac{1}{(z+2)(z+1)^2} = \frac{1}{z+2} + \frac{-1}{z+1} + \frac{1}{(z+1)^2}.$$

$$\int_c \frac{e^z}{(z+2)(z+1)^2}\,dz = \int_c \frac{e^z}{z+2}\,dz - \int_c \frac{e^z}{z+1}\,dz + \int_c \frac{e^z}{(z+1)^2}\,dz$$

by Cauchy integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

and

$$\int_c \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)'}(a)$$

$$= 2\pi i f(-2) - 2\pi i f(-1) + \frac{2\pi i}{1!} f'(-1)$$

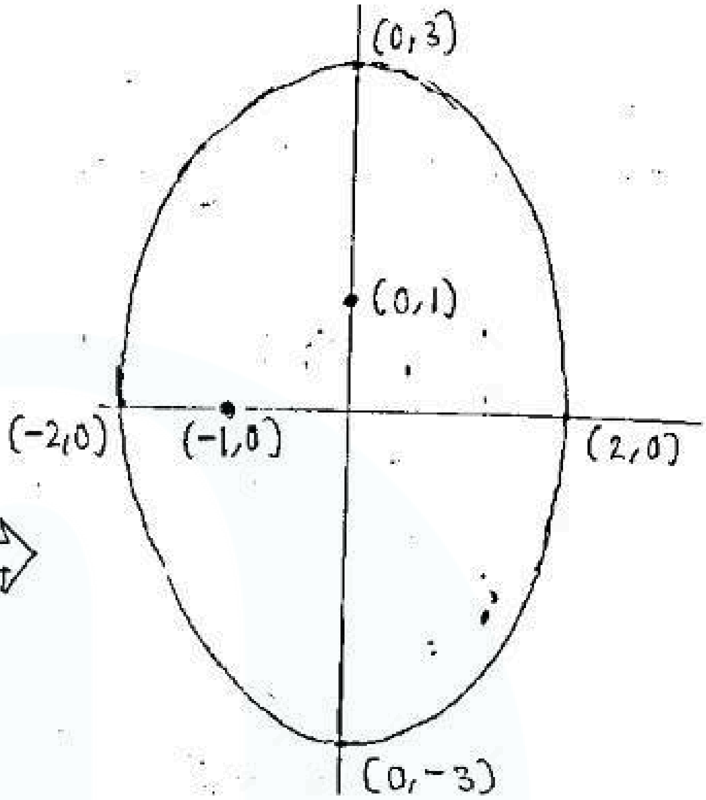
$$\begin{cases} f(z) = e^z & f'(z) = e^z \\ f(-2) = e^{-2} & f'(-1) = e^{-1} \\ f(-1) = e^{-1} & \end{cases}$$

$$= 2\pi i \times e^{-2} - \cancel{2\pi i \times e^{-1}} + \cancel{\frac{2\pi i}{1} \times e^{-1}}$$

$$= \frac{2\pi i}{e^2}$$

Q: Evaluate $\int_C \frac{z^4}{(z+1)(z-i)^2} dz$ where C is a) ellipse.

$9x^2+4y^2=36$.



$9x^2+4y^2=36$

$\frac{\cancel{9}x^2}{\cancel{36}_4}+\frac{\cancel{4}y^2}{\cancel{36}_9}=1$

$\frac{x^2}{2^2}+\frac{y^2}{3^2}=1 \Rightarrow$

$(z+1)(z-i)^2=0$

$z+1=0$ $\quad (z-i)^2=0$

$z=-1$ $\quad z-i=0$

$z=i$

$z=-1$ and $z=i$ lie inside C

$$\frac{1}{(z+1)(z-i)^2}=\frac{A}{z+1}+\frac{B}{(z-i)}+\frac{C}{(z-i)^2}$$

$$1=A(z-i)^2+B(z+1)(z-i)+C(z+1)$$

put $z = i$

$$1 = 0 + 0 + C(i+1)$$

$$C = \frac{1}{1+i}$$

put $z = -1$

$$1 = A(-1-i)^2 + 0 + 0$$

$$1 = A(1 + 2i - 1)$$

$$A = \frac{1}{2i}$$

put $z = 0$

$$1 = A(-i)^2 + B \times -i + C$$

$$1 + Bi = -A + C$$

$$= -\frac{1}{2i} + \frac{1}{1+i}$$

$$= \frac{-1-i+2i}{2i-2} = \frac{(i-1)}{2(i-1)} = \frac{1}{2}$$

$$Bi = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$B = -\frac{1}{2i}$$

$$\frac{1}{(z+1)(z-i)^2} = \frac{\frac{1}{2i}}{z+1} + \frac{-\frac{1}{2i}}{z-i} + \frac{\frac{1}{1+i}}{(z-i)^2}.$$

$$\int_C \frac{z^4}{(z+1)(z-i)^2}\,dz = \frac{1}{2i}\int_C \frac{z^4}{z+1}\,dz - \frac{1}{2i}\int_C \frac{z^4}{z-i} + \frac{1}{1+i}\int_C \frac{z^4}{(z-i)^2}\,dz$$

by Cauchy integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

and

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i\, f^{(n-1)}(a)}{(n-1)!}$$

$$= \frac{1}{2i} \times 2\pi i\, f(-1) - \frac{1}{2i} \times 2\pi i\, f(i) + \frac{1}{1+i}\,\frac{2\pi i}{1!}\, f'(i)$$

$$= \pi f(-1) - \pi f(i) + \frac{2\pi i}{1+i}\, f'(i)$$

$f(z) = z^4$

$f(-1) = (-1)^4 = 1$

$f(i) = (i)^4 = 1$

$f'(z) = 4z^3$

$f'(i) = 4 \times i^3 = -4i$

$$= \pi \times 1 - \pi \times 1 + \frac{2\pi i}{1+i} \times -4i$$

$$= \frac{8\pi}{1+i}$$

~~Q: Evaluate~~ $\int_C \cancel{2z-1}$

Q: Evaluate $\int_C \frac{e^z}{z(1-z)^3} dz$ where C is $|z| = \frac{1}{2}$

$z(1-z)^3 = 0$

$z = 0$ $\quad$ $(1-z)^3 = 0$

$1 - z = 0$

$z = 1$

at $z = 0$, $|z| = 0 < \frac{1}{2}$

$z = 0$ lies inside C

at $z = 1$, $|z| = |1| = 1 > \frac{1}{2}$

$z = 1$ lies outside C

$$\int_C \frac{e^z}{z(1-z)^3} dz = \int_C \frac{\frac{e^z}{(1-z)^3}}{z} dz$$

$$= 2\pi i f(0) \qquad \left\{ f(z) = \frac{e^z}{(1-z)^3} \right.$$

$$= \underline{\underline{2\pi i}} \qquad \left. f(0) = \frac{e^0}{1^3} = 1 \right.$$

Q: Evaluate $\int_C \frac{z}{z^2+1}\,dz$ where C is $|z+i| = 1$

$z^2+1 = 0$

$z^2 = -1$

$z = \pm i \Rightarrow (z+i)(z-i) = z^2+1$

at $z = i$, $|z+i| = |i+i| = |2i| = \sqrt{2^2} = 2 > 1$

$\therefore z = i$ lies outside C

at $z = -i$, $|z+i| = |-i+i| = |0| = 0 < 1$

$\therefore z = -i$ lies ~~outside~~ inside C

$$\int_C \frac{z}{z^2+1}\,dz = \int_C \frac{z}{(z+i)(z-i)}\,dz = \int_C \frac{\frac{z}{z-i}}{z+i}\,dz$$

$$= 2\pi i f(-i)$$

$$= 2\pi i \times \frac{1}{2}$$

$$= \pi i$$

$$\left\{ \begin{aligned} f(z) &= \frac{z}{z-i} \\ f(-i) &= \frac{-i}{-i-i} \\ &= \frac{-i}{-2i} = \frac{1}{2} \end{aligned} \right.$$

Q : Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)} dz$ where C is $|z-2| = 2$

$(z-1)^2 (z+2) = 0$

$(z-1)^2 = 0$ $\qquad$ $z+2 = 0$

$z-1 = 0$ $\qquad$ $z = -2$

$z = 1$

at $z=1$, $|z-2| = |1-2| = |-1| = 1 < 2$

$z=1$ lies inside C

at $z=-2$, $|z-2| = |-2-2| = |-4| = 4 > 2$

$z=-2$ lies outside C

$$\int_C \frac{z^2}{(z-1)^2(z+2)} dz = \int_C \frac{\frac{z^2}{z+2}}{(z-1)^2} dz$$

$$= \frac{2\pi i}{1!} \times f'(1)$$

$$= \frac{2\pi i}{1} \times \frac{5}{9}$$

$$= \frac{10\pi i}{9}$$

$$f(z) = \frac{z^2}{z+2}$$

$$\cancel{f(1) = \frac{1^2}{1+2} = 1}$$

$$f'(z) = \frac{(z+2)\times 2z - z^2 \times 1}{(z+2)^2}$$

$$f'(1) = \frac{3\times 2 - 1}{9}$$

$$f'(1) = \frac{5}{9}$$

Q: Evaluate $\int \frac{3z-1}{z^3-z}\,dz$ where C is $|z| = \frac{1}{2}$

$$z^3 - z = 0$$

$$z(z^2-1) = 0$$

$z = 0$ $\qquad$ $z^2 - 1 = 0$ $\qquad$ $\Rightarrow z^3 - z = z(z+1)(z-1)$

$$z^2 = 1$$

$$z = \pm 1$$

at $z=0$, $|z| = |0| = 0 < \frac{1}{2}$

$z=0$ lies inside C

at $z=1$, $|z| = |1| = 1 > \frac{1}{2}$

$\therefore$ $z=1$ lies outside c

at $z=-1$, $|z| = |-1| = 1 > \frac{1}{2}$

$\therefore$ $z=-1$ lies outside c

$$\int_c \frac{3z-1}{z^3-z}\,dz = \int_c \frac{3z-1}{z(z^2-1)}\,dz$$

$$= \int_c \frac{\frac{3z-1}{(z^2-1)}}{z}\,dz$$

$$= 2\pi i\, f(0) \qquad \left\{ \begin{array}{l} f(z) = \dfrac{3z-1}{z^2-1} \\ f(0) = \dfrac{0-1}{0-1} = 1 \end{array} \right.$$

$$= 2\pi i \times 1$$

$$= \underline{\underline{2\pi i}}$$

Q: Evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz$ where C is $|z| = \frac{3}{2}$

$z(z-1)(z-2) = 0$

$z = 0$ $\quad$ $z-1=0$ $\quad$ $z-2=0$

$z=1$ $\quad$ $z=2$

at $z=0$, $|z| = |0| = 0 < \frac{3}{2}$

$z=0$ lies inside C

at $z=1$, $|z| = |1| = 1 < \frac{3}{2}$

$z=1$ lies inside C

at $z=2$, $|z| = |2| = 2 > \frac{3}{2}$

$z=2$ lies outside C

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_C \frac{\frac{4-3z}{z-2}}{z(z-1)}\,dz$$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$1 = A(z-1) + Bz$$

put $z=1$

$1 = 0 + B \Rightarrow B = 1$

put $z=0$

$1 = -A + 0 \Rightarrow A = -1$

$$\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{\frac{4-3z}{z-2}}{z(z-1)}\,dz = -\int_C \frac{\frac{4-3z}{z-2}}{z}\,dz + \int_C \frac{\frac{4-3z}{z-2}}{z-1}\,dz$$

$$= -2\pi i f(0) + 2\pi i f(1)$$

$$f(z) = \frac{4-3z}{z-2}$$

$$f(0) = \frac{4-0}{0-2} = -2$$

$$f(1) = \frac{4-3}{1-2} = -1$$

$$= -2\pi i \times -2 + 2\pi i \times -1$$

$$= 4\pi i - 2\pi i$$

$$= \underline{\underline{2\pi i}}$$

Q: Evaluate $\int_C \frac{e^z}{(z^2+4)(z-1)^2}\,dz$ where C is the circle $|z-1| = 2$

$$(z^2+4)(z-1)^2 = 0$$

$z^2+4 = 0$

$z^2 = -4$

$z = \pm 2i$

$(z-1)^2 = 0$

$z-1 = 0$

$z = 1$

at $z = 2i$, $|z-1| = |2i-1| = \sqrt{2^2+(-1)^2} = \sqrt{5} > 2$;

$z = 2i$ lies outside C

at $z = -2i$, $|z-1| = |-2i-1| = \sqrt{(-2)^2+(-1)^2} = \sqrt{5} > 2$

$z = -2i$ lies outside C

at $z = 1$, $|z-1| = |1-1| = |0| = 0 < 2$

$z = 1$ lies inside C

$$\int_C \frac{e^z}{(z^2+4)(z-1)^2}\,dz = \int_C \frac{\frac{e^z}{z^2+4}}{(z-1)^2}\,dz$$

$$= \frac{2\pi i}{1!} f'(1)$$

$$= \frac{2\pi i}{1} \times \frac{3e}{25}$$

$$= \frac{6}{25}\pi i e$$

$$f(z) = \frac{e^z}{z^2+4}$$

$$f'(z) = \frac{(z^2+4)\times e^z - e^z(2z)}{(z^2+4)^2}$$

$$f'(1) = \frac{(1^2+4)e^1 - e^1 \times 2\times 1}{(1^2+4)^2}$$

$$= \frac{3e}{25}$$

Q: Evaluate $\int_C \frac{5z+7}{z^2+2z-3}\,dz$ where C is $|z-2|=2$

$$z^2+2z-3=0$$

$$(z+3)(z-1)=0$$

$z+3=0 \qquad z-1=0$

$z=-3 \qquad z=1$

at $z=-3$, $|z-2|=|-3-2|=|-5|=5>2$

$z=-3$ lies outside C

at $z=1$, $|z-2|=|1-2|=|-1|=1<2$

$z=1$ lies inside C

$$\int_C \frac{5z+7}{(z+3)(z-1)}\,dz = \int_C \frac{\frac{5z+7}{z+3}}{z-1}\,dz$$

$= 2\pi i f(1)$

$= 2\pi i \times 3$

$= \underline{\underline{6\pi i}}$

$f(z)=\frac{5z+7}{z+3}$

$f(1)=\frac{5+7}{1+3}=\frac{12}{4}$

$=3$

Tylor's series

Let $f(z)$ be an analytic function within a circle with centre at z_0 and radius r.
Then, for any z within the circle $|z-z_0|<r$, we can represent the function $f(z)$ in the form,

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \dots$$

Maclaurin series

Maclaurin series is a Tylor series with centre $z_0 = 0$

$$f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \dots$$

Standard expansions

- $(1-x)^{-1} = 1+x+x^2+x^3+\dots$
- $(1+x)^{-1} = 1-x+x^2-x^3+\dots$
- $(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$
- $(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$
- $(1-x)^{-3} = 1+3x+6x^2+10x^3+\dots$
- $(1+x)^{-3} = 1-3x+6x^2-10x^3+\dots$

- $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
- $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Q: Find the ~~Macc~~ Maclaurin series of $f(z) = \cos z$

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \frac{z^4}{4!} f''''(0) \dots$$

$f(z) = \cos z$ $\qquad$ $f(0) = \cos 0 = 1$

$f'(z) = -\sin z$ $\qquad$ $f'(0) = -\sin 0 = 0$

$f''(z) = -\cos z$ $\qquad$ $f''(0) = -\cos 0 = -1$

$f'''(z) = \sin z$ $\qquad$ $f'''(0) = \sin 0 = 0$

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$f''''(z) = \cos z \qquad f''''(0) = \cos 0 = 1$

$$f(z) = \cos z = 1 + 0 + \frac{z^2}{1!} \times -1 + 0 + \frac{z^4}{4!} \times 1 + \dots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

{ or simply use standard expansion

Q: find Maclaurin series of $f(z) = \cos(2z^2)$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$$\cos(2z^2) = 1 - \frac{(2z^2)^2}{2!} + \frac{(2z^2)^4}{4!} - \frac{(2z^2)^6}{6!} + \dots$$

$$\cos(2z^2) = 1 - 2z^4 + \frac{2}{3}z^8 - \frac{4}{45}z^{12} + \dots$$

Q: find Maclaurin series of $f(z) = e^{-\frac{z^2}{2}}$

$$e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{-\frac{z^2}{2}} = 1 - \frac{\left(\frac{z^2}{2}\right)}{1!} + \frac{\left(\frac{z^2}{2}\right)^2}{2!} - \frac{\left(\frac{z^2}{2}\right)^3}{3!} + \frac{\left(\frac{z^2}{2}\right)^4}{4!} + \dots$$

$$= 1 - \frac{z^2}{2} + \frac{z^4}{8} - \frac{z^6}{48} + \frac{z^8}{384} - \dots$$

Q: find the Maclaurin series of $\frac{1}{z+3i}$

$$\frac{1}{z+3i} = \frac{1}{3i\left(\frac{z}{3i}+1\right)} = \frac{1}{3i}\left(1+\frac{z}{3i}\right)^{-1}$$

$$\frac{1}{3i}\left(1+\frac{z}{3i}\right)^{-1} = \frac{1}{3i}\left[1-\frac{z}{3i}+\left(\frac{z}{3i}\right)^2-\left(\frac{z}{3i}\right)^3+\dots\right]$$

Q: find the Tylor series expansion of $f(z)=\sin z$ about $\frac{\pi}{4}$

here $z_0 = \frac{\pi}{4}$

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \frac{(z-z_0)^3}{3!}f'''(z_0) + \dots$$

$$f(z) = f(\pi/4) + (z-\pi/4)f'(\pi/4) + \frac{(z-\pi/4)^2}{2!}f''(\pi/4) + \frac{(z-\pi/4)^3}{3!}f'''(\pi/4) + \dots$$

$f(z) = \sin z \qquad f(\pi/4) = \sin \pi/4 = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z \qquad f'(\pi/4) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z \qquad f''(\pi/4) = -\sin \pi/4 = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z \qquad f'''(\pi/4) = -\cos \pi/4 = -\frac{1}{\sqrt{2}}$

$$f(z) = \frac{1}{\sqrt{2}} + (z-\pi/4)\frac{1}{\sqrt{2}} - \frac{(z-\pi/4)^{2}}{2!}\frac{1}{\sqrt{2}} - \frac{(z-\pi/4)^{3}}{3!}\frac{1}{\sqrt{2}} + \ldots$$

OR

$f(z) = \sin z$

$f(z) = \sin\left((z-\pi/4) + \pi/4\right)$

$$f(z) = \sin(z-\pi/4)\underbrace{\cos \pi/4}_{\frac{1}{\sqrt{2}}} + \cos(z-\pi/4)\underbrace{\sin \pi/4}_{\frac{1}{\sqrt{2}}}$$

$$f(z) = \frac{1}{\sqrt{2}}\left[\sin(z-\pi/4) + \cos(z-\pi/4)\right]$$

$$f(z) = \frac{1}{\sqrt{2}}\left[\left((z-\pi/4) - \frac{(z-\pi/4)^3}{3!} + \frac{(z-\pi/4)^5}{5!} + \dots\right) + \left(1 - \frac{(z-\pi/4)^2}{2!} + \frac{(z-\pi/4)^4}{4!} - \frac{(z-\pi/4)^6}{6!} + \dots\right)\right]$$

* find region

region $\Rightarrow$ $|z - \pi/4| < 1$

Q: Find Tylor series expansion of $f(z) = \cos z$ at $z_0 = \frac{\pi}{2}$

$$f(z) = \cos z$$

$$= \cos\left(\left(z - \frac{\pi}{2}\right) + \frac{\pi}{2}\right)$$

$$= -\sin\left(z - \frac{\pi}{2}\right)$$

$$= -\left[(z-\pi/2) - \frac{(z-\pi/2)^3}{3!} + \frac{(z-\pi/2)^5}{5!} + \dots\right]$$

region $\Rightarrow$ $|z - \pi/2| < 1$

Q: find T.S expansion of $f(z) = \frac{\sin z}{z-\pi}$ at $z=\pi$

$$f(z) = \frac{\sin z}{z-\pi}$$

$$= \frac{1}{z-\pi}\left[\sin\left((z-\pi)+\pi\right)\right]$$

$$= \frac{1}{z-\pi}\left[-\sin(z-\pi)\right]$$

$$= \frac{-1}{z-\pi}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} + \dots\right]$$

region $\Rightarrow$ $|z-\pi| < 1$

Q: find T.S expansion of $f(z) = \frac{1}{z}$ at $z=1$

$$f(z) = \frac{1}{z}$$

$$= \frac{1}{((z-1)+1)}$$

$$= \left[1+(z-1)\right]^{-1}$$

$$= 1-(z-1)+(z-1)^2-(z-1)^3+\dots$$

region $\Rightarrow$ $|z-1| < 1$

Q: Find T.S expansion of $f(z) = \frac{1}{z+2}$ at $z=1$

$$f(z) = \frac{1}{z+2} = \frac{1}{(z-1)+1+2}$$

$$= \frac{1}{3+(z-1)}$$

$$= \frac{1}{3\left(1+\frac{(z-1)}{3}\right)}$$

$$= \frac{1}{3}\left(1+\frac{(z-1)}{3}\right)^{-1}$$

$$f(z) = \frac{1}{3}\left(1+\frac{(z-1)}{3}\right)^{-1}$$

$$= \frac{1}{3}\left[1 - \frac{(z-1)}{3} + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \dots\right]$$

region ⇨ $\left|\frac{z-1}{3}\right| < 1$

$$|z-1| < 3$$

Q: Find T.S expansion of $f(z) = \frac{1}{(1+z)^2}$ at $z=1$

$$f(z) = \frac{1}{(1+z)^2} = \frac{1}{[1+1+(z-1)]^2} = \frac{1}{[2+(z-1)]^2}$$

$$= \frac{1}{2^2\left[1+\frac{(z-1)}{2}\right]^2}$$

$$= \frac{1}{4}\left[1+\frac{(z-1)}{2}\right]^{-2}$$

$$= \frac{1}{4}\left[1 - 2\left[\frac{z-1}{2}\right] + 3\left[\frac{z-1}{2}\right]^2 - 4\left(\frac{z-1}{2}\right)^3 + \dots\right]$$

region $\Rightarrow$ $\left|\frac{z-1}{2}\right| < 1$

$$|z-1| < 2$$

Q: Find T.S expansion of $f(z) = \frac{1}{z^2}$ at $z=3$

$$f(z) = \frac{1}{z^2} = \frac{1}{[(z-3)+3]^2}$$

$$= \frac{1}{3^2\left[1+\frac{(z-3)}{3}\right]^2}$$

$$f(z) = \frac{1}{9}\left(1+(z-3)\right)^{-2}$$

$$= \frac{1}{9}\left[1 - 2\left(\frac{z-3}{3}\right) + 3\left(\frac{z-3}{3}\right)^2 - 4\left(\frac{z-3}{3}\right)^3 + \ldots\right]$$

region $\Rightarrow$ $\left|\frac{z-3}{3}\right| < 1$

$$|z-3| < 3$$

Q: find T.S expansion of $f(z) = \frac{z-1}{z+1}$ at $z=1$

$$f(z) = \frac{z-1}{(z-1)+1+1} = \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$$

$$= \frac{z-1}{2}\left(1+\frac{z-1}{2}\right)^{-1}$$

$$= \frac{z-1}{2}\left[1 - \left(\frac{z-1}{2}\right) + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \ldots\right]$$

region $\Rightarrow$ $\left|\frac{z-1}{2}\right| < 1$

$$|z-1| < 2$$