MAT-201
Partial Differential Equations and Complex Analysis

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Contents

MODULE 1

Chapter 1

Partial Differential Equations

1.1 Introduction

A differential equation which involves partial derivatives with respect to two or more independent variables is called a partial differential equation(PDE). Such equations appear in physical processes in applied sciences and engineering. If the number of independent variable is two, then independent variables are denoted by x and y and dependent variable by z and the partial derivatives are denoted as follows:

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

If we have three independent variables we will denote it by x,y, z and the dependent variable by w.

For example,

1. The two dimensional Lapalce equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$
2. The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
3. The heat equation $\frac{\partial^2 u}{\partial x^2} = k \frac{\partial u}{\partial t}$

The order of the highest partial derivative occuring in a PDE is called the order of the PDE. The degree of a PDE is the degree of the highest partial derivative appearing in the PDE free from radicals and fractions.

1.2 Formation of Partial Differential Equations

A Partial Differential Equation can be formulated in two ways

1. By eliminating arbitrary constants
2. By eliminating arbitrary function

Note: If the number of arbitrary constants to be eliminated equals the number of independent variables, then the PDE formed is of **first order**. If the number of arbitrary constants is more than the number of independent variables, then the PDE formed will be of **second or higher order**. While eliminating arbitrary functions, the order of the differential equation is same as the number of arbitrary functions involved in the equation.

1.2.1 Formation of PDE by eliminating arbitrary constants

Problems:

1. Form a P.D.E. by eliminating constants from the equation

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Ans.
Let $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ − − − − − − − − − − − − − −(1)
Differentiate (1) partially w.r.t x and y. Then

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2}, or \ \ \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2}, or \ \ \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substitute values of $\frac{1}{a^2}$ and $\frac{1}{b^2}$ in equation (1),

$$2z = x^2\frac{p}{x} + y^2\frac{q}{y}$$

or $2z = px + qy$, which is the required P.D.E.

2. Form a P.D.E. from $x^2 + y^2 + (z - c)^2 = a^2$
   Ans.
   Given $x^2 + y^2 + (z - c)^2 = a^2$ − − − − − − − − − − − (1).
   Equation (1) contains two arbitrary constants a, c. Differentiate (1) partially w.r.t x and y. Then

$$x + (z - c)\frac{\partial z}{\partial x} = 0, \text{ i.e; } x + (z - c)p = 0$$
$$\text{or } (z - c) = \frac{-x}{p} - - - - - - (2)$$

   and

$$y + (z - c)\frac{\partial z}{\partial y} = 0, \text{ i.e; } y + (z - c)q = 0$$
$$\text{or } (z - c) = \frac{-y}{q} - - - - - - (3)$$

   Equating (2) and (3) we get $\frac{x}{p} = \frac{y}{q}$ or $py - qx = 0$, which is the required P.D.E.

3. Form a P.D.E. from $z = (x + a)(y + b)$
   Ans.
   Given $z = (x + a)(y + b)$ − − − − − − −(1)
   Differentiate (1) partially w.r.t x and y. Then

$$\frac{\partial z}{\partial x} = y + b, p = y + b \text{ and}$$
$$\frac{\partial z}{\partial y} = x + a, q = x + a$$

   Substituting for $x + a$ and $y + b$ in equation (1) we get $z = pq$, which is the required P.D.E.

4. Form a P.D.E. from $z = ax + by + ab$
   Ans.
   Given $z = ax + by + ab - - - - - (1)$
   Differentiate (1) partially w.r.t x and y. Then
$$\frac{\partial z}{\partial x} = a, \frac{\partial z}{\partial y} = b. \text{ i.e; } p = a, q = b.$$
   Then equation (1) becomes $z = px + qy + pq$, which is the required P.D.E.

5. Form a P.D.E from $z = (x-a)^2 + (y-b)^2$.
   Ans.
   Given $z = (x-a)^2 + (y-b)^2 - - - - - - (1)$. Differentiate (1) partially w.r.t $x \& y$. Then
$$p = 2(x-a), q = 2(y=b) \text{ or } (x-a) = \frac{p}{2}, (y-b) = \frac{q}{2}.$$
   Substituting in (1), $z = \frac{p^2}{4} + \frac{q^2}{4}$ or $p^2 + q^2 = 4z$, which is the required P.D.E.

6. Form a P.D.E from $z = a \log \frac{b(y-1)}{(1-x)}$
   Ans.
   Given $z = a \log \frac{b(y-1)}{(1-x)} = a(\log b(y-1) - \log(1-x))$. Then
$$p = \frac{a}{1-x} \text{ and } q = \frac{ab}{b(y-1)} = \frac{a}{y-1}.$$
   or
$$a = p(1-x) \text{ and } a = q(y-1).$$
   Equating these, we get $p(1-x) = q(y-1)$ or $p + q = px + qy$, which is the required P.D.E.

7. Form a P.D.E. from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where a, b, c are constants
   Ans.
   Given $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$. Differentiating this equation w.r.t. x we get,
   $\frac{2x}{a^2} + \frac{2zp}{c^2} = 0$ or, $\frac{x}{a^2} + \frac{zp}{c^2} = 0 - - - (1)$
   Differentiating (1) w.r.t. x,
   $\frac{1}{a^2} + \frac{p^2 + zr}{c^2} = 0$ or $\frac{1}{a^2} = -\frac{zr + p^2}{c^2}$
   Substituting this value of $\frac{1}{a^2}$ in (1)
   $xzr + xp^2 - zp = 0$, which is the required P.D.E.

Homework

Form P.D.E. by eliminating the arbitrary constants

1. $z = ax + a^2y^2 + b$. [Ans. $q = 2p^2y$]
2. $z = (x^2 + a)(y^2 + b)$. [Ans.$pq = 4xyz$]
3. $z = ax + by + a^2 + b^2$. [Ans.$z = px + qy + p^2 + q^2$]
4. Find the differential equation of all spheres of fixed radius having their centers in the XY-plane. [HINT: $(x-h)^2 + (y-k)^2 + z^2 = a^2$ is the equation of the sphere where h and k are arbitrary constants] [Ans: $z^2(p^2 + q^2 + 1) = a^2$]
5. $ax^2 + by^2 + z^2 = 1$. [Ans. $z(px + qy) = z^2 - 1$]
6. $x^2 + y^2 = (z-c)^2 \tan^2 \alpha$. [Ans. $py - qx = 0$]

1.2.2 Formation of PDE by eliminating arbitrary function

Problems:

1. Form the P.D.E. from $z = f(x^2 - y^2)$

   Ans.

   Given $z = f(x^2 - y^2) - - - - - - (1)$.

   Differentiate (1) partially w.r.t x and y. Then

   $$p = f'(x^2 - y^2)2x,\ f'(x^2 - y^2) = \frac{p}{2x}$$
   $$q = f'(x^2 - y^2)(-2y),\ f'(x^2 - y^2) = \frac{-q}{2y}$$

   So $\dfrac{p}{2x} = \dfrac{-q}{2y}$ or $py + qx = 0$, which is the required P.D.E.

2. Form the P.D.E. by eliminating the arbitrary function ϕ from $z = e^{ny}\phi(x - y)$.

   Ans.

   Given $z = e^{ny}\phi(x - y) - - - - - - (1)$.

   Differentiate (1) partially w.r.t x and y, we get

   $$p = e^{ny}\phi'(x - y) \text{ and } q = ne^{ny}\phi(x - y) - e^{ny}\phi'(x - y)$$

   i.e; $q = nz - p$ or $p + q = nz$, which is the required P.D.E.

3. Form the P.D.E from $z = f\left(\dfrac{y}{x}\right)$.

   Ans.

   Given $z = f\left(\dfrac{y}{x}\right) - - - - - - (1)$. Differentiate (1) partially w.r.t $x\&y$, we get

$$p = f'\left(\frac{y}{x}\right) \times \left(\frac{-y}{x^2}\right) \text{ or } f'\left(\frac{y}{x}\right) = \frac{-px^2}{y} \text{ and}$$
$$q = f'\left(\frac{y}{x}\right) \times \frac{1}{x} \text{ or } f'\left(\frac{y}{x}\right) = qx$$

Then $\dfrac{-px^2}{y} = qx$. i.e, $px + qy = 0$, which is the required P.D.E.

4. Form the P.D.E from $z = f\left(\frac{xy}{z}\right)$.

   Ans.

   Given $z = f\left(\frac{xy}{z}\right) - - - - - - (1)$. Differentiate (1) partially w.r.t x & y, we get

$$p = f'\left(\frac{xy}{z}\right) \times \left(\frac{zy - xyp}{z^2}\right) \text{ or } f'\left(\frac{xy}{z}\right) = \frac{pz^2}{zy - xyp} \text{ and}$$
$$q = f'\left(\frac{xy}{z}\right) \times \left(\frac{zx - xyq}{z^2}\right) \text{ or } f'\left(\frac{xy}{z}\right) = \frac{qz^2}{zx - xyq}$$

   Then we get $px - qy = 0$, which is the required P.D.E.

5. Form the P.D.E from $xyz = \phi(x + y + z)$.

   Ans.

   Given $xyz = \phi(x + y + z) - - - - - - (1)$. Differentiate (1) partially w.r.t $x \& y$, we get

$$yz + xyp = \phi'(x + y + z)(1 + p) \text{ and}$$
$$xz + xyq = \phi'(x + y + z)(1 + q)$$

$$\phi'(x + y + z) = \frac{yz + xyp}{(1 + p)} \text{ and } \phi'(x + y + z) = \frac{xz + xyq}{(1 + q)}$$

   So we get

$$\frac{yz + xyp}{(1 + p)} = \frac{xz + xyq}{(1 + q)}$$
$$\text{i.e; } x(y - z)p + y(z - x)q = z(x - y)$$

6. Eliminate the arbitrary functions f and ϕ from the function $z = f(x+ay)+\phi(x-ay)$

   Ans.

   Given $z = f(x + ay) + \phi(x - ay) - - - - - - (1)$. Differentiate (1) partially w.r.t $x \& y$, we get

$$\frac{\partial z}{\partial x} = f'(x + ay) + \phi'(x - ay) \ , \ \frac{\partial z}{\partial y} = af'(x + ay) - a\phi'(x - ay)$$
$$\frac{\partial^2 z}{\partial x^2} = f''(x + ay) + \phi''(x - ay)$$
$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2\phi''(x - ay) = a^2\frac{\partial^2 z}{\partial x^2}$$

   i.e $t = a^2 r$.

Homework

Form P.D.E by eliminating the arbitrary functions

1. $z = f(x^2 + y^2)$. [Ans. $py - qx = 0$]
2. $z = y^2 + 2f(\frac{1}{x} + \log y)$ [Ans. $px^2 + qy = 2y^2$]
3. $lx + my + nz = \phi(x^2 + y^2 + z^2)$.
   [Ans. $y(l + np) + z(lq - mp) = x(m + np)$]

1.3 Solutions of a PDE

Solution of a PDE is a relation between the independent and the dependent variables which satisfies the PDE. Solution of a PDE is also known as integral of the PDE.
A solution of a PDE which contains as many arbitrary constants as the number of independent variables is called a complete solution or a complete integral.
A solution of a PDE which contains as many arbitrary functions as the order of the equation is called the general solution or general integral. A solution obtained by giving particular values to the arbitrary constants in the complete solution or to the arbitrary functions in the general solution is known as the particular solution or particular integral.

1.3.1 Equations Solvable by Direct Integration

A PDE which contains only one partial derivative, which is of z can be solved by direct integration.

Problems:

1. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when $x = 0, z = e^y$ and $\frac{\partial z}{\partial x} = 1$.
   Ans. If z is a function of x alone then

$$(D^2 + 1)z = 0, D^2 + 1 = 0 \text{ or } D = \pm i$$

so that $z = A\cos x + B\sin x$. In this case z is a function of x and y, then the solution is $z = f(y)\cos x + g(y)\sin x$. So

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x.$$

Given that

$$\begin{aligned} &\text{when } x = 0, z = e^y, & \therefore \quad e^y = f(y) \\ &\text{when } x = 0, \frac{\partial z}{\partial x} = 1 & \therefore \quad 1 = g(y) \end{aligned}$$

Therefore the solution is $z = e^y \cos x + \sin x$

2. Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$
Ans. Integrating given equation w.r.t x, keeping y fixed

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

Integrating w.r.t y, we get $z = \dfrac{x^2}{2} \log y + axy + \int f(y)dy + v(x)$. Therefore the solution is $z = \dfrac{x^2}{2} \log y + axy + u(y) + v(x)$ where $u(y) = \int f(y)dy$.

3. Solve $\dfrac{\partial^2 z}{\partial x^2} = xy$
Ans. Integrating given equation w.r.t x,

$$\frac{\partial z}{\partial x} = \frac{x^2}{2}y + f_1(y)$$

Integrating this equation w.r.t x, we get

$$z = \frac{x^3}{6}y + xf_1(y) + f_2(y).$$

4. Solve $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$.
Ans. Integrating given equation w.r.t x,

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f_1(t)$$

Integrating this equation w.r.t t, we get

$$u = -e^{-t} \sin x + \int f_1(t)dt + f_2(x)$$

Therefore the solution is $u = -e^{-t} \sin x + \phi(t) + f_2(x)$, $\phi(t) = \int f_1(t)dt$

5. Solve $\dfrac{\partial^2 u}{\partial x^2} = a^2 z$, given that when $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0$.
Ans. When z is a function of x alone, $(D^2 - a^2)z = 0$. Then $D = \pm a$ so that $z = c_1 e^{ax} + c_2 e^{-ax}$.
If z is a function of x and y, then

$$z = f(y)e^{ax} + g(y)e^{-ax}$$
$$\frac{\partial z}{\partial x} = af(y)e^{ax} - ag(y)e^{-ax}$$
$$\frac{\partial z}{\partial y} = f'(y)e^{ax} + g'(y)e^{-ax}$$

It is given that when $x=0, \frac{\partial z}{\partial x} = a\sin y$, so $a\sin y = af(y) - ag(y)$.
$f(y) - g(y) = \sin y$ —— (1)
When $x = 0, \frac{\partial z}{\partial y} = 0$, and so $f'(y) + g'(y) = 0$ ——(2)
Differentiate (1), we get $f'(y) - g'(y) = \cos y$ ——(3)

$$(2)+(3) \implies 2f'(y) = \cos y \text{ or } f(y) = \frac{1}{2}\sin y \text{ and}$$
$$(2)\text{-}(3) \implies g(y) = \frac{-1}{2}\sin y$$

$$\therefore \quad z = \frac{1}{2}\sin y e^{ax} - \frac{1}{2}\sin y e^{-ax} = \sin y \frac{e^{ax} - e^{-ax}}{2} = \sin y \sinh ax$$

Homework

1. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$
   Ans: $z = \frac{-1}{12}\sin(2x+3y) + xf_1(y) + \phi_1(y) + \psi(x)$
2. Solve $\frac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0, z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.
   Ans: $z = e^y \cosh x + e^{-y}\sinh x$.
3. Solve $\frac{\partial^2 z}{\partial y^2} = -z$, given that when $y = 0, z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.
   Ans: $z = e^x \cos y + e^{-x}\sin y$.

1.3.2 Types of PDE

A PDE is said to be linear if the dependent variable z and its derivative are of degree one and product of z and its derivative do not appear in the equation. Otherwise, the PDE is called non-linear.

1.4 Linear Equations of First Order

1.4.1 Lagrange's Linear Equation

General form of a quasi-linear PDE of first order is $Pp + Qq = R$ where P, Q, R are functions of x, y, z and this equation is known as Lagrange's Linear equation.
If P and Q are independent of z, then the above equation become a linear PDE.

Method : To solve the equation $Pp + Qq = R$,

1. Form the Auxiliary Equation (A.E.), $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

2. Solve the A.E. by the **method of grouping** or by **the method of multipliers** or both to get two independent solutions $u(x,y,z)=a$ and $v(x,y,z)=b$, where a and b are arbitrary constants.

3. Then $\phi(u,v)=0$ or $u=\phi(v)$ or $v=\phi(u)$ is the general solution of the equation $Pp+Qq=R$

Method of Grouping :

By grouping any two of the three ratios, we may get an ordinary differential equation containing atmost two variables which can be solved easily.

1. Solve $\dfrac{y^2z}{x}p+xzq=y^2$.
   Ans. Given equation can be written as $y^2zp+x^2zq=y^2x$. Comparing with $Pp+Qq=R$, we get $P=y^2z, Q=x^2z, R=y^2x$. Then A.E is
   $$\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$$
   Taking the first two, $\dfrac{dx}{y^2z}=\dfrac{dy}{x^2z}$, by the method of grouping we get $x^2dx=y^2dy$.
   Integrating, we get $\dfrac{x^3}{3}-\dfrac{y^3}{3}=c_1$ or $x^3-y^3=3c_1=a$ —–(1)
   Again by taking the first & third by the method of grouping, we get $\dfrac{dx}{y^2z}=\dfrac{dz}{y^2x}\implies$ $xdx=zdz\implies x^2-z^2=b$—-(2)
   From (1) & (2), the general solution is $\phi(x^3-y^3,x^2-z^2)=0$ or $x^3-y^3=\phi(x^3-y^3)$.

2. Solve $pz-qz=z^2+(x+y)^2$
   Ans. Here $P=z, Q=-z, R=z^2+(x+y)^2$. A.E is
   $$\frac{dx}{z}=\frac{dy}{-z}=\frac{dz}{z^2+(x+y)^2}$$
   Taking the first two by the method of grouping, we get
   $$\frac{dx}{z}=\frac{dy}{-z}\implies dx=-dy\implies x+y=a \text{ —-(1)}$$
   which is of the form $u=a$.
   Taking first and third,
   $$\begin{aligned}\frac{dx}{z}=\frac{dz}{z^2+(x+y)^2}&\implies\frac{dx}{z}=\frac{dz}{z^2+a^2}\\&\implies\frac{z}{z^2+a^2}dz=dx\\&\implies\frac{1}{2}\log(z^2+a^2)=x+c\end{aligned}$$
   $\therefore\ \log(z^2+a^2)=2x+2c$ or $\log(z^2+(x+y)^2)-2x=b$ —(2), where $b=2c$.
   From (1) & (2), the solution is $\phi(x+y,\log(z^2+(x+y)^2)-2x)=0$.

3. Solve $xp + yq = 3z$.
   Ans. A.E is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

Taking first two by the method of grouping, $\frac{dx}{x} = \frac{dy}{y}$
Integrating, we get $\log x - \log y = \log c \implies \frac{x}{y} = a - - - (1)$
Taking last two, $\frac{dy}{y} = \frac{dz}{3z}$, integrating we get
$\log y = \frac{1}{3}\log z + \log c \implies y^3 = c^3 z$ or $\frac{y^3}{z} = b - - - (2)$ where $c^3 = b$. Therefore the solution is $\phi(\frac{x}{y}, \frac{y^3}{z}) = 0$

4. Solve $yzp + zxq = xy$
   Ans. A.E is

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Taking first two, $\frac{dx}{yz} = \frac{dy}{zx} \implies xdx = ydy$.
Integrating, we get $x^2 - y^2 = a$— (1)
Taking first and last, $\frac{dx}{yz} = \frac{dz}{xy} \implies xdx = zdz$
Integrating, we get $x^2 - z^2 = b - - - (2)$
Therefore the solution is $\phi(x^2 - y^2, x^2 - z^2) = 0$.

The method of multipliers:
If l, m, n are multipliers, then by the principle of algebra each ratio is $\frac{ldx + mdy + ndz}{lP + mQ + nR}$. If it is possible to choose l, m, n such that $lP + mQ + nR = 0$, then $ldx + mdy + ndz = 0$ which can be integrated to give $u(x, y, z) = a$. This method may be repeated to get another independent solution $v(x, y, x) = b$. The multipliers l, m, n are called *Lagrange multipliers*. These multipliers may be functions of x, y, z or constants.

5. Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$
   Ans. A.E is

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking second and last by the method of grouping,

$$\frac{dy}{2xy} = \frac{dz}{2xz} \implies \frac{dy}{y} = \frac{dz}{z}$$

Integrating, $\log y - \log z = \log a \implies \frac{y}{z} = a - - - (1)$
Using x, y, z as multipliers, each ratio is

$$\frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

Then $\frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz} \implies \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)} = \frac{dz}{z}$
Integrating we get

$$\log(x^2 + y^2 + z^2) = \log z + \log b \implies \frac{x^2 + y^2 + z^2}{z} = b - - - (2)$$

Therefore the solution is $\phi(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}) = 0.$

6. Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx.$
Ans. A.E is

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

Taking the last two, $\frac{dy}{xy + zx} = \frac{dz}{xy - zx}$ we get

$$\begin{aligned} (y - z)dy &= (y + z)dz \\ \implies ydy - (zdy + ydz) - zdz &= 0 \\ \implies ydy - d(yz) - zdz &= 0 \end{aligned}$$

Integrating, we get $\frac{y^2}{2} - yz - \frac{z^2}{2} = c_1$
i.e; $y^2 - 2yz - z^2 = a$ ——(1)
Again choosing x, y, z as multipliers,
each ratio $= \frac{xdx + ydy + zdz}{xz^2 - 2xyz - xy^2 - +xy^2 + zxy + xyz - xz^2} = \frac{xdx + ydy + zdz}{0}$
Then $xdx + ydy + zdz = 0.$
Integrating, we get $x^2 + y^2 + z^2 = 2c_1 = b$ ——(2)
Therefore the solution is $\phi(y^2 - 2yz - z^2, x^2 + y^2 + z^2).$

7. Solve $(y - z)p + (x - y)q = z - x.$
Ans. A.E is

$$\frac{dx}{y - z} = \frac{dy}{x - y} = \frac{dz}{z - x}$$

Choosing $1, 1, 1$ as multipliers, each ratio $= \frac{dx + dy + dz}{y - z + x - y + z - x}$
Then $dx + dy + dz = 0 \implies x + y + z = a$ ——(1)
Choosing x, z, y as multipliers, each ratio $= \frac{xdx + zdy + ydz}{xy - xz + xz - yz + zy - xy}$

Then $xdx + zdy + ydz = 0 \implies xdx + d(yz) = 0$
Integrating, $\frac{x^2}{2} + yz = b$ —–(2)
Therefore the solution is $\phi(x + y + z, \frac{x^2}{2} + yz) = 0$.

8. Solve $x(y - z)p + y(z - x)q = z(x - y)$
Ans. A.E is

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \text{ —–(1)}$$

Choosing 1,1,1 as multipliers,

$$\text{each ratio} = \frac{dx + dy + dz}{0} \implies dx + dy + dz = 0$$

Integrating, we get $x + y + z = a$—–(2)
Equation (1) can be rewritten as

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y}$$

Choosing $1, 1, 1$ as multipliers,

$$\text{each ratio}=\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

Then $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$
Integrating, we get $\log x + \log y + \log z = \log b \implies xyz = b$—–(3)
Therefore the solution is $\phi(x + y + z, xyz) = 0$.

9. Solve $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$
Ans. A.E is

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \text{ —–(1)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\text{each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} \implies \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get $\log x + \log y + \log z = \log a \implies xyz = a$ —– (2)
Taking $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers,

$$\text{each ratio} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0} \implies \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating, we get $-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_1 \implies \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b$ —-(3)

Therefore the solution is $\phi(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}) = 0$.

Homework:

1. Solve $(mz - ny)p + (nx - lz)q = ly - mx$
   Ans:(Hint: Choose x, y, z as multipliers and l, m, n as multipliers)
   $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$

2. Solve $p \tan x + q \tan y = \tan z$
   Ans: $\phi(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}) = 0$

3. Solve $(z - y)p + (x - z)q = y - x$
   Ans: $\phi(x + y + z, x^2 + y^2 + z^2) = 0$, [Hint: Take $1, 1, 1$ and x, y, z as multipliers]

4. Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.
   Ans: $\phi(x^2 + y^2 + z^2, xyz) = 0$ [Hint: Take x, y, z and $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers]

5. Solve $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x + y)z$.
   Ans: $\phi(\frac{1}{x} - \frac{1}{y}, \frac{x - y}{z})$ [Hint: Take $1, -1, 0$ as multipliers, $\frac{dx - dy}{x^2 - y^2} = \frac{dz}{z(x + y)}$]

6. Solve $(y + zx)p - (x + yz)q = x^2 - y^2$.
   Ans: $\phi(x^2 + y^2 - z^2, x + y + z) = 0$ [Hint: Take $x, y, 0$ and $y, x, 0$ as multipliers]

1.5 Non-linear first order P.D.E.

1.5.1 Charpit's Method

Charpit's method is a general method for solving first order non-linear equations.

Let the given equation be

$$f(x, y, z, p, q) = 0 \text{ —- } (1)$$

If we can find another relation

$$F(x, y, z, p, q) = 0 \text{ ——}(2)$$

involving x, y, z, p and q, then we can solve equations (1) and (2) for p, q and substitute in

$$dz = pdx + qdy \text{ —-(3)}$$

Solution of equation (3), if it exists is the complete solution of equation (1).

NOTE:

1. Since z is a function of two independent variables x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \implies dz = pdx + qdy$$

2. Proof of finding $F(x, y, z, p, q)$ is not necessary. To find the solution write Charpit's auxiliary equation as

$$A.E. = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} \text{ —-(4)}$$

Any integral of (4) which involves p or q or both can be taken as the assumed relation. Usually we choose the simplest of the integral of (4).

Problems:

1. Solve $(p^2 + q^2)y = qz$.

Ans: Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ —-(1)

Charpit's A.E. is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$
$$\frac{dx}{-2py} = \frac{dy}{-2qy + z} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

Consider last two, $\dfrac{dp}{-pq} = \dfrac{dq}{p^2} \implies pdp + qdq = 0$

Integrating, we get $p^2 + q^2 = c^2$—–(2).

To solve (1) and (2), put $p^2 + q^2 = c^2$ in (1).

Then $c^2y - qz = 0$ or $q = \dfrac{c^2y}{z}$. Substituting this in (1), we get

$$p^2 + \frac{c^4y^2}{z^2} = c^2 \implies p^2 = \frac{c^2z^2 - c^4y^2}{z^2} \text{ or } p = \frac{c\sqrt{z^2 - c^2y^2}}{z}$$

Hence

$$dz = pdx + qdy = \frac{c\sqrt{z^2 - c^2y^2}}{z}dx + \frac{c^2y}{z}dy$$
$$\implies zdz - c^2ydy = c\sqrt{z^2 - c^2y^2}dx$$
$$\implies \frac{1}{2}\frac{d(z^2 - c^2y^2)}{\sqrt{z^2 - c^2y^2}} = cdx$$

Integrating, we get $\sqrt{z^2 - c^2y^2} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$.

2. Solve $2xz - px^2 - 2qxy + pq = 0$.

   Ans: Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ —-(1). A.E. are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2q} = \frac{dq}{0}$$

   Therefore $dq = 0$ or $q = a$. Putting $q = a$ in (1) , we get

$$2xz - px^2 - 2axy + ap = 0 \implies 2x(z - ay) + p(a - x^2) = 0$$
$$p = \frac{2x(z - ay)}{x^2 - a}$$

   Therefore

$$dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}dx + ady \implies \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$$

   Integrating, we get

$$\log(z - ay) = \log(x^2 - a) + \log b \implies z - ay = b(x^2 - a)$$

   Therefore $z = ay + b(x^2 - a)$ is the required complete solution.

3. Solve $2z + p^2 + qy + 2y^2 = 0$.

   Ans: Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2 = 0$ —-(1). A.E. are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

   Taking first and fourth ratios, $\frac{dx}{-2p} = \frac{dp}{2p}$, we get $dp = -dx$ or $p = -x+a$. Substituting $p = a - x$ in (1), we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a - x)^2]$$

   $dz = pdx + qdy = (a - x)dx + \frac{1}{y}[-2z - 2y^2 - (a - x)^2]dy$

   Multiplying both sides by $2y^2$,

$$2y^2dz + 4yzdy = 2y^2(a - x)dx - 4y^3dy - 2y(a - x)^2dy$$

   Integrating, e get

$$2zy^2 = -[y^2(a - x)^2 + y^4] + b$$

   Therefor $y^2[(x - a)^2 + 2z + y^2] = b$ is the solution.

4. Solve $px + qy = pq$.

Ans: Let $f(x, y, z, p, q) = px + qy - pq = 0$ —-(1).

Then $f_p = x - q, f_q = y - q, f_z = 0, f_x = p, f_y = q$. A.E. are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$
$$\implies \frac{dx}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p} = \frac{dq}{q}$$

Consider the last two ratios $\frac{dp}{p} = \frac{dq}{q}$.

Integrating, we get $\log p = \log q + \log a \implies p = aq$.

Substituting in (1), $aqx + qy = aq^2 \implies ax + y = aq$.

Therefore $q = \frac{ax + y}{a}$ and $p = ax + y$. Then $dz = pdx + qdy$ becomes

$$dz = (ax + y)dx + \frac{ax + y}{a}dy \implies adz = (ax + y)(adx + dy)$$

Integrating, we get $az = \frac{1}{2}(ax + y)^2 + b$, which is the required solution.

5. Solve $px + q^2y = z$.

Ans: Let $f(x, y, z, p, q) = px + q^2y - z = 0$ —-(1). A.E. are

$$\frac{dx}{-x} = \frac{dy}{-2qy} = \frac{dz}{-px - 2q^2y} = \frac{dp}{0} = \frac{dq}{q^2 - q}$$

The fourth ratio gives $dp = 0$ so that $p = a$.

Putting $p = a$ in (1), $ax + q^2y - z = 0$ or $q = \frac{z - ax}{y}$.

Therefore from $dz = pdx + qdy$ we get

$$dz = adx + \frac{z - ax}{y}dy$$
$$\frac{dz - adx}{\sqrt{z - ax}} = \frac{1}{\sqrt{y}}dy$$

Integrating,we get $\sqrt{z - ax} = \sqrt{y} + b$ where a, b are constants.

6. Solve $pxy + pq + qy - yz = 0$.

Ans: Let $f(x, y, z, p, q) = pxy + pq + qy - yz = 0$ —-(1). Then

$$f_x = py, f_y = px + +q - z, f_z = -y, f_p = xy + q, f_q = p + y$$

The A.E. are

$$\frac{dx}{-(xy + q)} = \frac{dy}{-(p + y)} = \frac{dz}{-pxy - qy - 2pq} = \frac{dp}{0} = \frac{dq}{px + q - z - qy}$$

The fourth ratio gives $dp = 0$ so that $p = a$.
Putting $p = a$ in (1),

$$axy + aq + qy - yz = 0 \implies q = \frac{y(z - ax)}{a + y}$$

Now $dz = pdx + qdy$ becomes

$$\begin{aligned} dz &= adx + \frac{y(z-ax)}{a+y}dy \\ \implies \frac{dz - adx}{z - ax} &= \frac{y}{a+y}dy \\ \implies \frac{dz - adx}{z - ax} &= \left(1 - \frac{a}{a+y}\right) dy \end{aligned}$$

Integrating, we get

$$\log(z - ax) = y - a\log(y + a) + \log b \text{ or}$$
$$(z - ax)(a + y)^a = be^y$$

Homework:

1. Solve $p = (qy + z)^2$. [Ans: $yz = ax + 2\sqrt{a}y^{\frac{1}{2}} + b$]
2. Solve $p^2x + q^2y = z$ [Ans: $(1 + a)^{\frac{1}{2}}z^{\frac{1}{2}} = a^{\frac{1}{2}}x^{\frac{1}{2}} + y^{\frac{1}{2}} + b$]
3. Solve $(1 + q^2)z = px$. [Ans: $x^2 - az^2 = (b - \sqrt{a}y)^2$]
4. Solve $(x^2 - y^2)pq - xy(p^2 - q^2) - 1 = 0$.
   Ans: $z = \frac{a}{2}\log(x^2 + y^2) + \frac{1}{a}\tan^{-1}\left(\frac{y}{x}\right) + b$

1.6 Method of Seperation of Variables

The method of seperating variables or product method involves finding solutions of PDE which are of product form. In this method we assume that a solution to the given PDE has the form $u(x, y) = X(x)Y(y)$, where $X(x)$ is a function of x only and $Y(y)$ is a function of y only. We substitute in the given equation and obtain the functions X and Y.

is used to determine the solution of a P.D.E.

Problems:

1. Solve the equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ by the method of seperation of variables.
   Ans: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ —- (1)
   Let $u = XY$, where X is a function of x alone and Y is a function of y alone be a solution of the given equation. Then equation(1) becomes $X'Y + XY' = 0$ or

$\frac{X'}{X} = -\frac{Y'}{Y}$.
In the above equation, L.H.S. is a function of x alone and R.H.S. is a function of y alone. So each side must reduce to a constant k. (A pure function of x cannot be equal to a pure function of y alone unless the functions are both constant and of the same value). Therefore

$$\frac{X'}{X} = -\frac{Y'}{Y} = k$$

Then $\frac{X'}{X} = k$—- (2) and $-\frac{Y'}{Y} = k$—– (3)
These two P.D.E. are such that each contain only one variable and its derivative and hence behave like ordinary differential equation. So integrating both sides of $\frac{X'}{X} = k$ w.r.t. x, we get

$$\log X + \log a = kx \implies \log aX = kx$$
$$\implies aX = e^{kx} \text{ or } X = c_1 e^{kx} \text{ —- (4) where } c_1 = \frac{1}{a}$$

Similarly integrating both sides of $-\frac{Y'}{Y} = k$ w.r.t. y, we get

$$\log Y + \log b = -ky \implies \log bY = -ky$$
$$\implies bY = e^{-ky} \text{ or } Y = c_2 e^{-ky} \text{ —- (5) where } c_2 = \frac{1}{b}$$

Therefore the solution is $u = XY = c_1 e^{kx} c_2 e^{-ky}$ from (3) and (4).
$\therefore u = ce^{k(x-y)}$ where $c = c_1c_2$ is the required solution.

2. Solve $y^2 u_x - x^2 u_y = 0$.
Ans: $y^2 \frac{\partial u}{\partial x} - x^2 \frac{\partial u}{\partial y} = 0$ —-(1)
Let $u = XY$ be the required solution where X is a function of x alone and Y is a function of y alone. Then $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$. Substituting these values in (1), we get

$$y^2 X'Y - x^2 XY' = 0$$
$$\text{i.e; } \frac{X'}{x^2 X} = \frac{Y'}{y^2 Y} = k$$
$$\frac{X'}{x^2 X} = k \text{ —-(2), } \frac{Y'}{y^2 Y} = k \text{ —-(3)}$$

Integrating (2) w.r.t x, $\log X + \log a = k\frac{x^3}{3}$ or $X = c_1 e^{\frac{kx^3}{3}}$.

Similarly,integrating (3) w.r.t y, $\log Y + \log b = k\frac{y^3}{3}$ or $Y = c_2 e^{\frac{ky^3}{3}}$.
Therefore

$$u = XY = c_1 e^{\frac{kx^3}{3}} c_2 e^{\frac{ky^3}{3}} = ce^{\frac{k(x^3+y^3)}{3}}$$

is the required solution.

3. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

   Ans: Let $u = XY$ be the required solution. Substitute $\dfrac{\partial^2 u}{\partial x^2} = X''Y$ and $\dfrac{\partial^2 u}{\partial y^2} = XY''$ in the given equation, we get

$$X''Y + XY'' = 0 \text{ or } \frac{X''}{X} = -\frac{Y''}{Y} = k$$

   i.e; $X'' - kX = 0$ or $\dfrac{d^2X}{dx^2} - kX = 0 \implies (D^2 - k)X = 0$.
   A.E. is $D^2 - k = 0$ and so $D = \pm\sqrt{k}$.
   Therefore $X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$.
   Similarly $Y'' + kY = 0$ or $\dfrac{d^2Y}{dy^2} + kY = 0 \implies (D^2 + k)Y = 0$.
   A.E. is $D^2 + k = 0$ and so $D = \pm i\sqrt{k}$.
   Therefore $Y = c_3 \cos(\sqrt{k}y) + c_4 \sin(\sqrt{k}y)$.
   The solution is $u = (c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x})(c_3 \cos(\sqrt{k}y) + c_4 \sin(\sqrt{k}y))$.

4. Solve $u_{xy} - u = 0$.

   Ans: $\dfrac{\partial^2 u}{\partial x \partial y} - u = 0$ ——(1)

   Let $u = XY$ be the solution. Then $\dfrac{\partial u}{\partial x} = X'Y, \dfrac{\partial^2 u}{\partial x \partial y} = X'Y'$. Now equation (1) becomes

$$X'Y' - XY = 0 \text{ or } \frac{X'}{X} = \frac{Y}{Y'} = k$$

   i.e; $\dfrac{X'}{X} = k$. Integrating it, we get

$$\log X + \log a = kx \implies \log aX = kx = \text{or } X = c_1 e^{kx}$$

   Similarly $\dfrac{Y}{Y'} = k$ or $\dfrac{Y'}{Y} = \dfrac{1}{k} = k'$. Integrating it, we get

$$\log Y + \log b = k'y \implies \log bY = k'y = \text{or } Y = c_2 e^{k'y}$$

   The solution is $u = c_1 e^{kx} c_2 e^{k'y} = ce^{kx+k'y}$.

5. Solve $x\dfrac{\partial^2 u}{\partial x \partial y} + 2yu = 0$.

   Ans: $x\dfrac{\partial^2 u}{\partial x \partial y} + 2yu = 0$ ——(1)

   $u = XY$ be the solution. Then $\dfrac{\partial u}{\partial x} = X'Y, \dfrac{\partial^2 u}{\partial x \partial y} = X'Y'$. Now equation (1) becomes

$$xX'Y' + 2yXY = 0 \text{ or } x\frac{X'}{X} = -2y\frac{Y}{Y'} = k$$

i.e; $\frac{X'}{X} = \frac{k}{x}$. Integrating,

$$\log X + \log a = k \log x \implies \log aX = \log x^k \text{ or } X = c_1 x^k$$

Similarly, $\frac{Y'}{Y} = \frac{-2y}{k}$. Integrating,

$$\log Y + \log b = \frac{-y^2}{k} \text{ or } Y = c_2 e^{\frac{-y^2}{k}}$$

The solution is $u = cx^k e^{\frac{-y^2}{k}}$.

6. Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$.
   Ans: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ —-(1)
   Let $u = XY$ be the solution. Then $\frac{\partial u}{\partial x} = X'Y, \frac{\partial^2 u}{\partial x \partial y} = X'Y'$. Now equation (1) becomes

$$X'Y + XY' = 2(x+y)XY \implies (X' - 2xX)Y + (Y' - 2yY)X = 0$$

i.e $\frac{X' - 2xX}{X} = -\frac{Y' - 2yY}{Y} = k$.
Now $\frac{X'}{X} = k + 2x$. Integrating,

$$\log X + \log a = kx + x^2 \implies \log aX = kx + x^2 \text{ or } X = c_1 e^{kx+x^2}$$

Similarly $\frac{Y' - 2yY}{Y} + k = 0 \implies \frac{Y'}{Y} = 2y - k$. Integrating,

$$logY + logb = y^2 - ky \implies \log bY = y^2 - ky \text{ or } Y = c_2 e^{y^2 - ky}$$

Therefore the solution is $u = ce^{k(x-y)+x^2+y^2}$.

7. Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, given $u(x,0) = 6e^{-3x}$.
   Ans: Let $u = XT$ where X is a function of x alone and T is a function of t alone. Then $\frac{\partial u}{\partial x} = X'T$ and $\frac{\partial u}{\partial t} = XT'$. Substituting in given equation, we get

$$X'T = 2XT' + XT = (2T' + T)X \implies \frac{X'}{X} = \frac{2T' + T}{T} = 1 + \frac{2T'}{T} = k$$

Now

$$\frac{X'}{X} = k \implies \log X + \log a = kx \text{ or } X = c_1 e^{kx}$$

Similarly, $1 + \dfrac{2T'}{T} = k \implies \dfrac{T'}{T} = \dfrac{k-1}{2}$
Integrating,

$$\log T + \log b = \frac{k-1}{2}t \implies T = c_2 e^{\dfrac{k-1}{2}t}$$

Therefore $u = c_1 e^{kx} c_2 e^{\frac{k-1}{2}t} = c_1 c_2 e^{kx + \frac{k-1}{2}t}$. It is given that

$$u(x,0) = 6e^{-3x}$$
$$\therefore \quad 6e^{-3x} = c_1 c_2 e^{kx} \implies c_1 c_2 = 6, k = -3$$

The solution is $u = XT = 6e^{-(3x+2t)}$.

Homework:

1. Solve $\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} = 0$. [Ans: $u = ce^{k(x+y)}$]
2. Solve $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$. [Ans: $u = (c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x})(c_3 e^{\sqrt{k}y} + c_4 e^{-\sqrt{k}y})$]
3. Solve $2x\dfrac{\partial z}{\partial x} - 3y\dfrac{\partial z}{\partial y} = 0$. [Ans: $z = cx^{\frac{k}{2}} y^{\frac{k}{3}}$]
4. Solve $4\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 3z$ subject to $z = e^{-5y}$ when $x = 0$. [Ans: $z = e^{2x-5y}$]
5. Solve $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$.
   [Ans: $u = (c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x})c_3 e^{-ky}$]