Chapter 5

Taylor Series and Fourier Series

5.1 Series with variable terms

In the previous chapter we considered series in which the terms are constants. Now we consider series in which each term is a function of x in the form $\sum a_n u_n(x)$ where a_n's are constants and $u_n(x)$ is a function of x. We restrict our attention to functions with $u_n(x) = x^n$ or $u_n(x) = (x-a)^n$. Then the above series is called **Power series.** Our aim in this section is to express a function f(x) as a power series of the form $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$. Now the question is: Under what conditions on the function $f(x)$ such a power series representation is possible or valid?

What is the meaning of validity of power series representation? We know from the generalized binomial theorem (or using geometric series) that $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$. This equation is not an identity. This equation is valid only for a certain range of values of x. In this case the range of values of x is $-1 < x < 1$. The range of values of x for which a power series representation is valid is called **interval of convergence (region of convergence or domain of convergence)** of the power series. We will discuss such notions in the following sections. To discuss two important representations of function of real variable, we need the notion of neighbourhood of a point and open set.

Neighbourhood of a point and open sets

A neighbourhood $H_r(p)$ of a point p in the real line is the set of all points x for which $|x-p| < r$, where $r > 0$ is a real number. That is,

$$H_r(p) = \{x : |x-p| < r\} = \{x : p-r < x < p+r\}$$

So neighbourhood of a point consists of all points in a line segment excluding the end points. For example, for each positive integer n, $|x| < \dfrac{1}{n}$ is a neighbourhood of the origin. This in fact shows that if we have one neighbourhood of a point, then we can have infinite number of neighbourhoods for that point.

Let S be a region in the real line. A point $p \in S$ is said to be an interior point of S if there is a neighbourhood (a line segment) of p fully contained in S. If all the points of S are interior points of S, then S is called an open region (open set). The region defined by $|x| < 1$ is an open region(open set), but the region $|x| \le 1$ is not an open region (open set). For, what ever neighbourhood we construct for points on $x = \pm 1$, there will be some points outside $|x| \le 1$.

Taylor's series and Maclaurin's series

Let S be an open set in the real line and let f(x) be a function having derivatives of all orders at $p \in S$; then $f(x)$ can be expressed as a power series in the form

$$\sum_{n=0}^{\infty} \frac{(x-p)^n}{n!} f^{(n)}(p)$$

This is called the **Taylor series expansion** of $f(x)$ about the point p. We write this statement as $f(x) = \sum_{n=0}^{\infty} \frac{(x-p)^n}{n!} f^{(n)}(p)$ and we say that the series converges to f(x), where $f^{(n)}(p)$ is the nth derivative of the function $f(x)$ at the point p.

In Taylor's series, put $p = 0$, we get the Maclaurin's series for the function f(x). That is,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Since each term of the power series is a function of a variable x, there may be a range of values of x for which the series is convergent. That is, there is a set of values M such that for each $x \in M$, the series is convergent (has a finite and unique value).

The set of values of x for which the series $f(x) = \sum_{n=0}^{\infty} \frac{(x-p)^n}{n!} f^{(n)}(p)$ is convergent is called **interval of convergence (or region of convergence)** of the series. Note that the series is convergent at p.

Results: In general, a series expansion of a function $f(x)$ given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (x-p)^n$$

will converge for all x satisfying $-\rho < x - p < \rho$, where ρ is a non-negative extended real number $(0 \le \rho < \infty)$. Then ρ is called the radius of convergence of the series and $-\rho < x-p < \rho$ is called **region of convergence** or **interval of convergence** or **domain of convergence** of the series.

For any power series $\sum_{n=0}^{\infty} \frac{(x-p)^n}{n!} f^{(n)}(p)$, one of the following is true:

1. The series converges only at p and the radius of convergence is $\rho = 0$.
2. The series converges absolutely for all real values of x and the radius of convergence is $\rho = \infty$ and the region of convergence is the whole real line.
3. The series converges absolutely for all x in some finite open internal $-\rho < x - p < \rho$ and the radius of convergence is ρ. The series diverges for all $|x-p| > \rho$ and at $x = \pm\rho$, the series may converge absolutely or conditionally or diverge.

Example 5.1.1. *Find the Taylor series of $f(x) = x \sin x$ about the point $x = \frac{\pi}{2}$.* [KTU FEB 2016]

$$\begin{aligned} f(x) &= x\sin x \implies f(\pi/2) = \frac{\pi}{2} \\ f'(x) &= x\cos x + \sin x \implies f'(\pi/2) = 1 \\ f''(x) &= -x\sin x + 2\cos x \implies f''(\pi/2) = -\frac{\pi}{2} \\ f'''(x) &= -x\cos x - 3\sin x \implies f'''(\pi/2) = -3 \end{aligned}$$

The Taylor series expansion of the function about $x = \frac{\pi}{2}$ is given by

$$f(x) = \frac{\pi}{2} + (x - \frac{\pi}{2}) - \frac{\pi}{2}\frac{\left(x-\frac{\pi}{2}\right)^2}{2!} - 3\frac{\left(x-\frac{\pi}{2}\right)^3}{3!} + \cdots$$

Example 5.1.2. *Find the Maclaurin's series for the function xe^x* [KTU DEC 2016]

$$\begin{aligned} f(x) &= xe^x \implies f(0) = 0 \\ f'(x) &= xe^x + e^x = (1+x)e^x \implies f'(0) = 1 \\ f''(x) &= (x+2)e^x \implies f''(0) = 2 \\ f'''(x) &= (x+3)e^x \implies f'''(0) = 3 \end{aligned}$$

The Maclaurin's series expansion of the function is given by

$$f(x) = 0 + (1)\frac{x}{1!} + (2)\frac{x^2}{2!} + (3)\frac{x^3}{3!} + \cdots = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$$

Example 5.1.3. *Find the Maclaurin series expansion of the following functions.*
(*i*) e^x (*ii*) e^{-x} (*iii*) $\sin x$ (*iv*) $\cos x$ (*v*) $\sinh x$
(*vi*) $\cosh x$ (*vii*) $\frac{1}{1-x}$ (*viii*) $\frac{1}{1+x}$ (*ix*) $\frac{1}{(1-x)^2}$ (*x*) $\frac{1}{(1+x)^2}$

If $f(x)$ is a function having derivatives of all orders at the point $x = 0$, its Maclaurin series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(0)$ denotes the nth derivative of $f(x)$ at $x = 0$.

(i) Here $f(x) = e^x$ and its derivatives of all orders at $x = 0$ are given by the formula $f^{(n)}(0) = 1$. Hence the Maclaurins series expansion of

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(ii) Here $f(x) = e^{-x}$. The derivatives of $f(x) = e^{-x}$ at $x = 0$ are given by $f^n(0) =$ 1 or -1 according as n is even or odd. So the Maclaurin series expansion e^{-x} is given by

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

(iii) Here $f(x) = \sin x$
$\implies f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = -\sin x, \ldots$
$\implies f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{(4)}(0) = 0, \ldots$ So the Maclaurin series expansion e^{-x} is given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

(iv) Here $f(x) = \cos x$
$\implies f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, f^{(4)}(x) = \cos x, \ldots$
$\implies f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0, f^{(4)}(0) = 1, \ldots$ So the Maclaurin series expansion e^{-x} is given by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(v) Here $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$. The derivatives of $f(x) = \sinh x$ at $x = 0$ are given by $f^n(0) = 1$ or 0 according as n is odd or even. So the Maclaurin series expansion $\sinh x$ is given by

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

(vi) Here $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$. The derivatives of $f(x) = \cosh x$ at $x = 0$ are given by $f^n(0) = 1$ or 0 according as n is even or odd. So the Maclaurin series expansion of $\cosh x$ is given by

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

(vii) Here $f(x) = \frac{1}{1-x}$ and so $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$. Then $f^{(n)}(0) = n!$. So the Maclaurin series expansion of $f(x) = \frac{1}{1-x}$ is given by

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

(viii) Here $f(x) = \frac{1}{1+x}$ and so $f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$. Then $f^{(n)}(0) = (-1)^n n!$. So the Maclaurin series expansion is given by

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

(ix) Here $f(x) = \frac{1}{(1-x)^2}$ and so $f^{(n)}(x) = \frac{(n+1)!}{(1-x)^{n+2}}$. Then $f^{(n)}(0) = (n+1)!$. So the Maclaurin series expansion of $f(x) = \frac{1}{(1-x)^2}$ is given by

$$f(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

(x) Here $f(x) = \frac{1}{(1+x)^2}$ and so $f^{(n)}(x) = \frac{(-1)^n(n+1)!}{(1+x)^{n+2}}$. Then $f^{(n)}(0) = (-1)^n(n+1)!$. So the Maclaurin series expansion of $f(x) = \frac{1}{(1+x)^2}$ is given by

$$f(x) = \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

Problem 5.1.4. *Find the Taylor's series representation of the following functions about the specified points. (i)$f(x) = \frac{1}{x}$ about $x = -1$ (ii)$f(x) = \sin \pi x$, about $x = 1$, (iii)$f(x) = \ln x$, about $x = e$ and (iv) $f(x) = \cos x$ about $x = \frac{\pi}{2}$*

Solution: If $f(x)$ is a function having derivatives of all orders at a point $x = a$ its Taylor series expansion about the point $x = a$ is given by $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(a)$ is the nth order derivative of $f(x)$ at $x = a$.

(i) Here $f(x) = \frac{1}{x}$ about the point $x = -1$. Its nth order derivative is given by $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$ so that $f^{(n)}(-1) = -(n!)$. Hence the Taylor's series expansion is given by $f(x) = \frac{1}{x} = -[1 + (x+1) + (x+1)^2 + (x+1)^3 + \cdots]$

(ii) Here $f(x) = \sin \pi x$, about $x = 1$. $f^{(1)}(1) = -\pi$, $f^{(2)}(1) = 0$, $f^{(3)}(1) = \pi^3$, $f^{(4)}(1) = 0$, $f^{(5)}(1) = -\pi^5$ etc. The Taylor's series representation of

$$f(x) = -\pi(x-1) + \frac{\pi^3(x-1)^3}{3!} - \frac{\pi^5(x-1)^5}{5!} + \cdots$$

(iii) Here the function is $f(x) = \ln x$ about the point $x = e$. The derivatives of the function at $x = e$ are given by $f^{(1)}(e) = \frac{1}{e}$, $f^{(2)}(e) = \frac{-1}{e^2}$, $f^{(3)}(e) = \frac{2}{e^3}$, $f^{(4)}(e) = \frac{-6}{e^4}$ etc. So the Taylor's series expansion is given by

$$f(x) = \ln x = 1 + \frac{x-e}{e} - \frac{(x-e)^2}{2e^2} + \frac{(x-e)^3}{3e^3} - \frac{(x-e)^4}{4e^4} + \cdots$$

(iv) here the function is $f(x) = \cos x$ about the point $x = \frac{\pi}{2}$. The derivatives of the function at $x = \frac{\pi}{2}$ is given by $f^{(1)}(\frac{\pi}{2}) = -1$, $f^{(1)}(\frac{\pi}{2}) = 0$, $f^{(1)}(\frac{\pi}{2}) = 1$, $f^{(1)}(\frac{\pi}{2}) =$

0, $f^{(7)}(\frac{\pi}{2}) = -1$ and so on. The Taylor series expansion of the function about $x = \frac{\pi}{2}$ is given by

$$f(x) = -(x - \tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} + \frac{(x-\frac{\pi}{2})^7}{7!} - \dots$$

Problem 5.1.5. *Find the radius of convergence (interval of convergence) and domain of convergence of the following functions (series with variable terms).*

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n(x-4)^n}{3^n}$, (ii) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$, (iii) $\sum_{n=0}^{\infty} \frac{x^n}{1+n^3}$, (iv) $\sum_{n=0}^{\infty} \frac{(2x-1)^n}{3^{2n}}$

Solution: We shall use the ratio test to compute the region of convergence of the series.

(i) Here the series is

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{(-1)^n(x-4)^n}{3^n}$$

so that the nth term is $u_n = \frac{(-1)^n(x-4)^n}{3^n}$. Consider the ratio $\left|\frac{u_{n+1}}{u_n}\right| = \frac{|x-4|}{3}$, (which is independent of n so need not apply limit $n \to \infty$). So by ratio test the series converges when $|x-4| < 3$. That is, $1 < x < 7$ and it diverges if $|x-4| > 3$. We can see that the series diverges when $x = 1$ and $x = 7$. So the domain of convergence of the function is $1 < x < 7$ and its radius of convergence is 3..

(ii) Here the series is

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$$

and the nth term is $u_n = \frac{(-1)^n x^{3n}}{(2n)!}$. Note that $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^3}{(2n+2)(2n+1)}\right| \to 0$ as $n \to \infty$. So by ratio test the series converges for all values of x. That is, the domain of the function is the whole real line and the radius of convergence is ∞.

(iii) Here the series is

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{x^n}{1+n^3}$$

and the nth term of the series is $u_n = \frac{x^n}{1+n^3}$. Note that

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^3}{(2n+2)(2n+1)}\right| = \left|\frac{x(1+n^3)}{(1+(1+n)^3)}\right| \to |x|$$

as $n \to \infty$. So by ratio test the series converges if $|x| < 1$ and diverges if $|x| > 1$. When $x = \pm 1$, we can see that the resulting series is convergent in both cases. (use comparison test with series $\sum \frac{1}{n^2}$). So the series converges if $|x| \leq 1$ and it diverges if $|x| > 1$. That is, the domain of the series is $-1 \leq x \leq 1$ or the interval $[-1, 1]$ and the radius of convergence is 1.

(iv) Here the series is

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{(2x-1)^n}{3^{2n}}$$

and so the nth term is $u_n = \frac{(2x-1)^n}{3^{2n}}$. Note that $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{2x-1}{9}\right|$. So by ratio test the series converges if $\left|\frac{2x-1}{9}\right| < 1$ and it diverges if $\left|\frac{2x-1}{9}\right| > 1$. That is the series converges if $-4 < x < 5$ and it diverges if $|2x-1| > 9$. When $x = -4$ and when $x = 5$, we can see that the resulting series is divergent. Hence the domain of the series is $-4 < x < 5$ and the series diverges if $|2x-1| \geq 9$. The radius of convergence of the series is $\frac{9}{2}$.

Example 5.1.6. *Determine the Maclaurin series representation of the function $f(x) = \tan^{-1} x$. What are its region of convergence and radius of convergence? Using this representation evaluate the value of the series $1 - \frac{1}{3} + \frac{1}{5} - \cdots$, if possible.*

The given function is $f(x) = \tan^{-1} x$ and the values of its derivatives at $x = 0$ are given by $f^{(0)}(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -2$, $f^{(4)}(0) = 0$, $f^{(5)}(0) = 24$. The Maclaurin series representation is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

That is,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$$

Now we shall determine the region of convergence of the this series. For this we consider the modulus terms series

$$\sum_{n=1}^{\infty} |U_n| = \sum_{n=1}^{\infty} \left|(-1)^{2n-1} \frac{x^{2n-1}}{2n-1}\right| = \sum_{n=1}^{\infty} \frac{|x|^{2n-1}}{2n-1}$$

Taking $V_n = \frac{|x|^{2n-1}}{2n-1}$ and using the ratio test we get

$$\lim_{n\to\infty} \frac{V_{n+1}}{V_n} = |x|^2$$

So that the given series is absolutely convergent if $|x| < 1$ and is divergent if $|x| > 1$. We can also see that the series is conditionally convergent when $x = \pm 1$. Hence the given series is convergent if $|x| \leq 1$ and is divergent if $|x| > 1$. Its radius of convergence is 1.

Putting $x = 1$, we get $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \tan^{-1}(1) = \frac{\pi}{4}$

Example 5.1.7. *Find the radius of convergence and interval of convergence (domain of convergence) of the series*

(a) $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(b) $1 + x + x^2 + x^3 + \cdots$

Solution: (a) Here the series is $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so that the nth term is $u_n = \frac{x^n}{n!}$. We compute the ratio $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x}{n+1}\right| \to 0$ as $n \to \infty$. So by ratio test the series converges for all x. That is, the domain of convergence is whole real line and its radius of convergence is ∞.

(b) The series is $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} x^n$ so that the nth term is $u_n = x^n$. Note that $\left|\frac{u_{n+1}}{u_n}\right| = |x|$. The series converges if $|x| < 1$ and it is diverges if $|x| > 1$. When $|x| = 1$, that is, when $x = \pm 1$, we observe the following: When $x = 1$, the series is $1 + 1 + 1 + 1 + \cdots$ and it is divergent. When $x = -1$, the series is $1 - 1 + 1 - 1 + 1 - 1 + \cdots$, again the series not convergent. Hence the series is convergent if $|x| < 1$ and it is divergent if $|x| \geq 1$. So the domain of convergence of the function is the open interval $(-1, 1)$ and the radius of convergence is 1.

5.1.1 Series representation of Trigonometric and exponential functions

Using Mclaurin's series expansion we can see that

$$Sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$Cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

If we assume that the series expansion for exponential function is valid for the function $e^{i\theta}$, where $i^2 = -1$, we can see that $e^{i\theta} = Cos\theta + i\, Sin\theta$. This is the famous **Euler identity. Note that while defining trigonometric functions we assume that the argument is measured in radians.**

Example 5.1.8. *Determine an approximate value of* $\sin(85°)$ *correct to four decimal places using Maclaurin series expansion.*

Solution: Maclaurin series expansion of Sine function is

$$Sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Here we assume that the argument x is measured in radians. $180° \equiv \pi^{rad}$ and so

$$85° = \frac{\pi}{180} \times 85 = \frac{17\pi}{36} = 1.483529$$

$$\therefore Sin 85° = Sin(1.483529) = 1.483529 - \frac{1.483529^3}{6} + \frac{1.483529^5}{120} - \frac{1.483529^7}{5040} + \cdots$$
$$= 1.483529 - 0.544173 + 0.059882 - 0.003137 + 0.000095 - 0.000001$$
$$= 0.996196$$

5.1.2 Generalized Binomial Theorem

The binomial theorem for positive integer index is given by

$$(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + \cdots + nC_r a^{n-r} b^r + \cdots + b^n$$

This is true for any choice of real numbers a and b provided n is positive integer. Generalization of binomial theorem for any real number n is called the generalized binomial theorem. Generalized Binomial theorem states that

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots$$

provided $|x| < 1$. That is, the above expansion is valid only if $-1 < x < 1$.

Remark: Note that the above series will have infinite number of terms when n is not a positive integer and in that case the series will converge to a unique finite number only if $|x| < 1$. But when n is a non-negative integer, the series reduces to a polynomial having $n+1$ number of terms, which is same as the case of binomial theorem for positive index.

Some useful series expansions

The following series expansions are useful while solving problems:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$$
$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

Example 5.1.9. *Determine the binomial series representation of the following functions*

$$(i)\frac{1}{\sqrt{(2+x)^3}}, \quad (ii)\frac{x}{1+x}, \quad (iii)\frac{3}{\sqrt{1+x^2}}$$

Also, find the region of validity of each of these expansions.

Solution: (i) $\frac{1}{\sqrt{(2+x)^3}}$

$$\frac{1}{\sqrt{(2+x)^3}} = \frac{1}{2^{\frac{3}{2}}}\left(1+\frac{x}{2}\right)^{\frac{3}{2}}$$
$$= \frac{1}{2^{\frac{3}{2}}}\left(1 + \frac{-\frac{3}{2}}{1!}\left(\frac{x}{2}\right) + \frac{-\frac{3}{2}\cdot -\frac{5}{2}}{2!}\left(\frac{x}{2}\right)^2 + \frac{-\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right)$$
$$= \frac{1}{2^{\frac{3}{2}}}\left(1 - \frac{3}{4}x + \frac{15}{32}x^2 - \frac{35}{128}x^3 + \cdots\right)$$

This expansion is valid only if $|\frac{x}{2}| < 1$, that is, $|x| < 2$

(ii) $\frac{x}{1+x}$

$$\frac{x}{1+x} = x\,(1+x)^{-1} = x\,(1 - x + x^2 - x^3 + x^4 - \cdots) = x - x^2 + x^3 - x^4 + x^5 - \cdots$$

This expansion is valid only if $|x| < 1$

(iii) $\frac{3}{\sqrt{1+x^2}}$

$$\frac{3}{\sqrt{1+x^2}} = 3\,(1+x^2)^{-\frac{1}{2}} = 3\left(1 + \frac{-\frac{1}{2}}{1!}x^2 + \frac{-\frac{1}{2}\cdot -\frac{3}{2}}{2!}x^4 + \frac{-\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}}{3!}x^6 + \cdots\right)$$
$$= 3\left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots\right)$$

This expansion is valid only if $|x^2| < 1$, that is, when $|x| < 1$

Example 5.1.10. *Using binomial theorem, show that*

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{2n-1}$$

Note that
$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2}$$

Using binomial theorem, we get

$$\frac{1}{1+t^2} = (1+t^2)^{-1} = 1 + \frac{(-1)}{1!}t^2 + \frac{(-1)(-2)}{2!}t^4 + \frac{(-1)(-2)(-3)}{3!}t^6 + \cdots = 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \cdots$$

so that

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \cdots)\,dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

5.1.3 Series representation of logarithmic function

The Maclaurin series representation of the function $\ln(1+x)$ and $\ln(1-x)$ (region of convergence is given in bracket) are given by

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \quad (-1 < x \le 1)$$

and

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right), \quad (-1 \le x < 1)$$

Example 5.1.11. *Using Binomial theorem, find the series representation of the function* (*a*) $\ln(1+x)$ (*b*) $\ln(1-x)$

Solution: (a) Note that

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$$

Using Binomial theorem,

$$\frac{1}{1+t} = (1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 - t^5 + \cdots \quad \text{with } |t| < 1$$

Putting this in the first expression, we get

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - t^5 + \cdots)\,dt$$
$$= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6} + \cdots\right]_0^x$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

(b) Similarly, we can see that

$$\ln(1-x) = -\int_0^x \frac{1}{1-t}\,dt$$

Using Binomial theorem,

$$\frac{1}{1-t} = (1-t)^{-1} = 1 + t + t^2 + t^3 + t^4 + t^5 + \cdots \quad \text{with } |t| < 1$$

Putting this in the previous expression, we get

$$\ln(1-x) = -\int_0^x \frac{1}{1-t}\,dt = -\int_0^x (1 + t + t^2 + t^3 + t^4 + t^5 + \cdots)\,dt$$
$$= -\left[t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \frac{t^5}{5} + \frac{t^6}{6} + \cdots\right]_0^x$$
$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots\right)$$

Example 5.1.12. *Using Binomial theorem, find the series representation of the function* (*a*) $\ln(1+x^2)$ (*b*) $\ln(1-x^2)$

Solution: (a) Note that

$$\ln(1+x^2) = \int_0^x \frac{2t}{1+t^2}\,dt$$

Using Binomial theorem,

$$\frac{1}{1+t^2} = (1+t^2)^{-1} = 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \cdots \quad \text{with } |t| < 1$$

5.2 Exercise

Find the radius of convergence and interval of convergence (domain of convergence) of the series

1. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$

2. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

3. $1-x+x^2-x^3+\cdots$

4. $1+2x+3x^2+4x^3+\cdots$

5. $1-2x+3x^2-4x^3+\cdots$

6. $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\cdots$

7. $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\cdots$

8. $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}\,x^n$

9. $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n^2+1}}\,x^n$

10. $\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$

11. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$

12. $u_n=\frac{x^n}{n^2+1}\quad 1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\cdots$

13. $u_n=\frac{2^n-2}{2^n+1}x^n\quad \sum_{n=1}^{\infty}\frac{x^n}{n^n}$

14. $\sum_{n=1}^{\infty}\frac{[(n+1)x]^n}{n^{n+1}}$

15. $\frac{x}{2.3}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\cdots$

16. $\frac{x^n n!}{(n+1)^n}$

17. Find an approximate value of $\tan^{-1}(0.2)$ correct to three places of accuracy using an appropriate Taylor series expansion of $tan^{-1}x$

18. Find an approximate value of $\tan^{-1}(\frac{1}{3})$ correct to three decimal places of accuracy using suitable series expansion.

19. Determine an approximate value of $Cos(-17$ correct to three places of accuracy.

20. Show that $(1+4)^3=125$ using generalized binomial theorem.

21. Show that $(1-x)^{-1}=1+x+x^2+x^3+x^4+\cdots$

22. Show that $(1+x)^{-1}=1-x+x^2-x^3+x^4-\cdots$

23. Show that $(1-x)^{-2}=1+2x+3x^2+4x^3+5x^4+\cdots$.

24. Show that $(1+x)^{-2}=1-2x+3x^2-4x^3+5x^4-\cdots$

25. Determine series representation of $(2-3x)^{\frac{2}{3}}$ using the generalized binomial theorem. Find the region of convergence.

26. Determine a series representation of $\ln 3$

27. Determine a series representation of $\ln 5$

28. Find first five non-zero terms in the expansion of the following functions (a) $\frac{\cos x}{e^x}$ (b) $e^{-x}Sin^2x$ (c) $e^{2x}Cos^3(2x)$

5.3 Fourier Series Representation

Most of the mathematical ideas, concepts and tools are developed to represent complex or difficult processes in terms of simple or elementary or known functions. Mostly we divide the process into simple components and analyze the major components to get a full picture of the process under investigation. In the case of a three dimensional vector, we resolve it into three components in three mutually perpendicular directions and we call each component the orthogonal projections. Weighted sum of these three components will give a complete understanding of the actual vector.

In Taylor (Maclaurin) series representation of a function $f(x)$ defined in $[a,b]$, we express the function in the form

$$f(x)=f^{(0)}(0)+x\frac{f^{(1)}(0)}{1!}+x^2\frac{f^{(2)}(0)}{2!}+\cdots+x^n\frac{f^{(n)}(0)}{n!}+\cdots$$

in terms of the simple (elementary) functions $1, x, x^2, \ldots$ and the coefficients $f^{(0)}(0)$, $\frac{f^{(1)}(0)}{1!}, \frac{f^{(2)}(0)}{2!}, \ldots$ are a type of "projections" (components) of $f(x)$ in the directions of $1, x, x^2, \ldots$. In Taylor's series, we use the sequence of functions $1, x, x^2, \ldots, x^n, \ldots$ as the basis functions for the representation.

For example using Maclaurin series expansion, we can represent e^x in the form

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

which is an infinite series representation. One serious limitation of Taylor's (Maclaurin) series representation is that function which has higher order derivatives can be represented 'faithfully' in this form. Then only Taylor's series representation becomes faithful to the characteristics of the function $f(x)$. This is a very stringent restriction which drastically reduces the field of application of Taylor's series representation though Taylor series representation can be applied to periodic and non-periodic functions.

Now we would like to develop a representation of periodic process using known basis functions which put fairly lesser restrictions on the basis functions and as a consequence of this we can include a fairly large family of functions for systematic analysis. Our aim is to represent a periodic function $f(x)$ defined in an interval $[c, c+2l]$ in terms of the basis functions $\sin\left(\frac{n\pi x}{l}\right)$ and $\cos\left(\frac{n\pi x}{l}\right)$, where n is an integer. Specifically we would like to have an infinite series representation of $f(x)$ in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

where $l>0$ is a real number, a_0, a_n, b_n are suitable constants (which will be clear as we move forward). **This is called Fourier series representation and these constants are called the Fourier coefficients.**

What is the advantage of representing functions using infinite series? In engineering problems, we are not too much worried about the exact analytical solutions rather we are bothered about getting a 'suitable' approximate solution. The system under consideration may be very complex and in most of the problems what we wanted is a 'sufficiently accurate' value of the characteristic under investigation. So when we express the function in the form of an infinite series, we are in a better position to find a good approximation by computing only those terms of the series which contribute the required accurate value rather than computing all the information about the process. When we include more number of terms, we get a more accurate value. So we have the dynamic freedom to fix the number of terms to be included depending on the accuracy of the approximation needed.

Another advantage is that by adding additional components to an already obtianed approximation, we will be able to construct reasonable complex processes from simple easy to understand components.

5.3.1 Time domain and frequency domain analogy

Fourier series methods describe a fundamental procedure by which complex physical waveform (signals) can be decomposed into simpler ones and conversely using simpler building blocks a very complex signal may be created. Fourier analysis is able to execute this process by establishing a simultaneous dual view of phenomena called frequency domain and time domain representations. Now we shall illustrate through some simple examples, how the Fourier series approximation capture the dynamics of a process.

The time and frequency domain relation can be related to the way in which human ear and eye interpret stimulus. The ear, for example, responds to minute variation in the atmospheric pressure. This causes the ear drum to vibrate and the various nerves in the inner ear then converts these vibrations into what the brain interprets as sounds. In the eye, in contrast, electromagnetic waves fall on the rods and cones in the back of the eye ball and are converted into what the brain interprets as colours.

For example, consider some programs of T.V. The speakers in the T.V vibrate producing minute compressions and rarefactions (increases and decreases in air pressure) which propagate across the room to the viewers' ear. These vibrations impact on the ear drum as a single continuously varying pressure. By the time the result is interpreted by the brain it has been separated into different actors' voices, the background music, background sounds etc. That is the nature of human ear to take a single complex signal, decompose it into simpler components, and recognize the simultaneous existence of those different components. This is the essence of frequency domain analysis in Fourier analysis.

In the case of T.V, it uses electron gun to illuminate groups of three different coloured - red, blue, green- phosphor dots on the screen. These different coloured light beams then propagate across the room and fall on the eyes. The eyes interpret the simultaneous reception of these different colours in exactly the reverse manner of the ear. That is, the nature of the human eye is to take a multiple simple component signals and combine or synthesize them into a simple complex signal. This is the time domain interpretation of a signal in Fourier series analysis.

Before plunging into the rigorous details of the Fourier series representation, we would like to recall some of the notions which are needed for the development of the Fourier series.

5.3.2 Waveform, Periodic function

A function of time $f(t)$ is called a **waveform**. A function $f(x)$ is said to be **periodic with period** $T > 0$ if $f(x+T) = f(x)$ for all x in the domain of $f(x)$. The smallest of those periods $T > 0$ is called the **fundamental period** of the function $f(x)$.

For example, the function $f(x) = \cos x$ is periodic with periods $T = 2n\pi$, where n is a positive integer. The fundamental period of $\cos x$ is 2π. Similarly $\sin x$ is also a function having periods $T = 2n\pi$, where n is a positive integer and its fundamental period is 2π. The function $\tan x$ is periodic with fundamental period π. Note that the fundamental period of $\sin mx$ and $\cos nx$ are $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$ respectively. In general, if $f(x)$ is a function having period T, then the period of $f(ax)$ is $\frac{T}{a}$. If T is the fundamental period of f(x), the number $\mathbf{f} = \frac{1}{\mathbf{T}}$ is called the **fundamental frequency** of $f(x)$ and the number $\omega = 2\pi\mathbf{f}$ is called the **angular fundamental frequency** of the function $f(x)$.

5.3.3 Continuous and Bounded functions

A function $f(x)$ defined on a set R is said to be bounded if for all x in the domain, $|f(x)| \leq M$ for some real number $M > 0$. The functions $\sin x$, $\cos x$ etc are bounded functions. But the function $\frac{1}{x}$ is unbounded in any interval containing zero and the function $\tan x$ is also unbounded in an interval containing any of the points $(2n+1)\frac{\pi}{2}$, where n is an integer.

A function $f(x)$ is said to have a **limit** at a point $x = a$ if the left hand limit at $x = a$ and the right hand limit at $x = a$ are finite and equal. That is,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L, \quad \text{where } |L| < \infty$$

The function $f(x)$ is said to be **continuous** at $x = a$ if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

Otherwise the function is said to be **discontinuous**.

A function $f(x)$ is said to have a **removable discontinuity** if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$$

That is, limit exists but this finite limit is not equal to the function value at that point. An important type of discontinuity which is often connected with Fourier series analysis is the **jump discontinuity**.

A function $f(x)$ is said to have a **jump discontinuity** at $x = a$ if the left hand limit and right hand limit at that point are finite but different. That is,

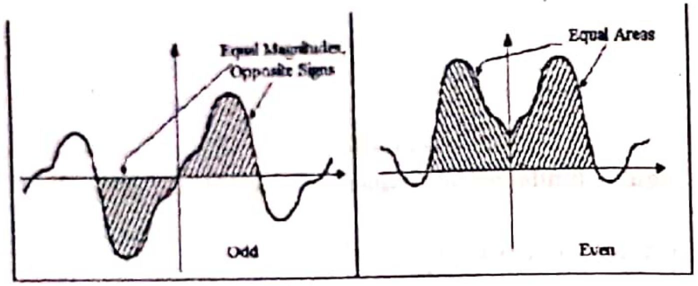
$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

and both limits are finite.

Even function and odd function

A function $f(x)$ is said to be an even function if $f(x) = f(-x)$ for all x in the domain of the function. A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$ for all x in its domain

Geometrically the graph of an even function is symmetric about the y- axis and the graph of an odd function is symmetric about the origin. An important property of odd and even function which is useful in the study of



Fourier series is the following. Let $f(x)$ be a function defined in $[-a, a]$. Then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even}$$

$$\int_{-a}^{a} f(x)\,dx = 0, \text{ if } f(x) \text{ is odd}$$

Local Maximum and Local Minimum

A function $f(x)$ is said to attain a local maximum at a point $x = a$ if $f(x) \le f(a)$ for all x in some neighborhood of the point a. The function $f(x)$ is said to attain a local minimum at $x = a$ if $f(x) \ge f(a)$ for all x in some neighborhood of the point a.

Integration by parts

Another important result which we frequently use while computing the Fourier coefficients is the product rule of integration.

Let $f(x)$ and $g(x)$ be two continuous functions defined in some interval in the real line line. Then product rule of integration states

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{df}{dx}\int g(x)dx\right)dx$$

When the first function $f(x)$ is a polynomial and the second function $g(x)$ is function which can be integrated sufficiently large number times, then following form of product rule of integration is more convenient.

$$\int f(x)g(x)dx = f(x)\int g(x)dx - df(x)\int\int g(x)dxdx + d^2f(x)\int\int\int g(x)dxdx\,dx - \cdots$$

where $df(x)$, $d^2f(x)$ etc. stands for first derivative, second derivative of $f(x)$ etc.

4 Fourier Series Representation of a function

we shall elaborate on the fundamentals of Fourier analysis. We already have indicated our aim is to represent a function $f(x)$ defined in some interval $[c, c+2l]$ in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

at is the value of the above series should converge to the function value $f(x)$ for each in $[c, c+2l]$. This is called Fourier series representation of the function $f(x)$. **Now the question is :** Can we represent any function $f(x)$ in the above form? obably the answer may be clear to you since $\sin x$ and $\cos x$ are periodic functions, expect $f(x)$ to be periodic. That is, non periodic function cannot be expressed as a urier series. The following are some of the important characteristics of the trigonometric nctions $\sin x$ and $\cos x$.

1. The functions $\sin x$ and $\cos x$ are finite in their domain of definition.
2. Both are bounded functions.
3. They are periodic functions with period 2π.
4. Both functions possess limit at each point in their domain.
5. Both are continuous functions.
6. Both are continuously differentiable functions.
7. Both $\sin x$ and $\cos x$ possess only one point of maximum and only one point of minimum in any interval of length 2π (Period).

We observed that both functions $\sin x$ and $\cos x$ possess a lot of properties. If we can present only those functions which have all the above properties, the set of functions ich we can analyze in terms of Fourier series will be drastically reduced to a very all family of functions. We know that the condition of differentiability will restrict the plication of Fourier series to the family of function which are differentiable as in the se of Taylor series expansion. The following theorem shows that the conditions that e function $f(x)$ should satisfy in order to represent it as a Fourier series can be limited a few of the above conditions so that we can include a large family of functions within e field of analysis of Fourier series expansion.

urier series representation Theorem

eorem 5.4.1. *Let $f(x)$ be a function defined in the interval $[c, c+2l]$ having the owing characteristics:*

1. *$f(x)$ is finite and bounded in $[c, c+2l]$*
2. *$f(x)$ is periodic with period $2l$.*

3. *$f(x)$ can have atmost a finite number of jump discontinuities in any interval of length $2l$ (period)*

4. *$f(x)$ can have atmost a finite number of points of maxima and minima in any interval of length $2l$.*

Then the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

with suitable choice of constants a_0, a_n and b_n will converge to the function value $f(x)$, if $f(x)$ is continuous at x and the above series will converge to the average of the left hand limit and the right hand limit at x if $f(x)$ is having jump discontinuity at x.

The above four conditions in the theorem are called **Dirichlet conditions**. That is, any function satisfying the Dirichlet conditions in $[c, c+2l]$ can be expressed as Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)=\frac{f(x^-)+f(x^+)}{2},\quad \forall x\in[c,c+2l]$$

Note that when $f(x)$ is continuous at x, the right hand side reduces to the function value $f(x)$. Here the constants a_0, a_n, and b_n are called the **Fourier coefficients** and a_0 is specifically called **DC component**. The term $a_1\cos\left(\frac{\pi x}{l}\right)+b_1\sin\left(\frac{\pi x}{l}\right)$ is called **first harmonics**, the term $a_2\cos\left(\frac{2\pi x}{l}\right)+b_2\sin\left(\frac{2\pi x}{l}\right)$ is called **second harmonics**, the term $a_3\cos\left(3\frac{\pi x}{l}\right)+b_3\sin\left(3\frac{\pi x}{l}\right)$ is called **third harmonics** and so on.

Another terminology connected with the Fourier series is the Fourier Polynomials. The **Fourier Polynomials** F_n are defined by $F_0=\frac{1}{2}\left(a_0\cos\left(\frac{0\pi x}{l}\right)+b_0\sin\left(\frac{0\pi x}{l}\right)\right)=\frac{a_0}{2}$, $F_1=F_0+a_1\cos\left(\frac{\pi x}{l}\right)+b_1\sin\left(\frac{\pi x}{l}\right)$, $F_2=F_1+a_1\cos\left(\frac{2\pi x}{l}\right)+b_1\sin\left(\frac{2\pi x}{l}\right)$ and in general

$$F_n=F_{n-1}+a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)$$

Problem 5.4.2. *Discuss the Fourier series representation of $f(x)=\sin\left(\frac{1}{x}\right)$, $0<x<1$ treating $f(x)$ as a periodic function with period 1.*

Solution: By assumption $f(x)=\sin\left(\frac{1}{x}\right)$, $0<x<1$ is periodic with period of length 1. Now we shall verify whether $f(x)$ satisfies the remaining Dirichlet conditions. The points of maxima and minima of the function are given by solving $f'(x)=\cos\left(\frac{1}{x}\right)\frac{-1}{x^2}=0$. We

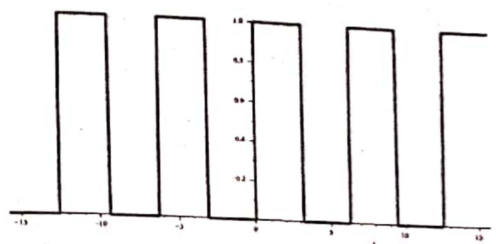
... $\frac{1}{x}=(2n-1)\frac{\pi}{2}$. That is, $x=\frac{2}{(2n-1)\pi}$, where n is an integer. This shows that there ... infinite number of points of maxima and minima within the interval $(0, 1)$ of length ... This violates the Dirichlet conditions and hence f(x) cannot be expressed as a Fourier ...ries in the above interval.

5.4.1 Nature of Convergence of Fourier series

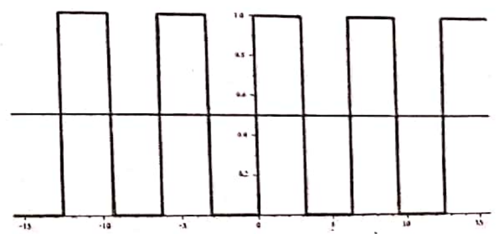
In the next section we shall discuss ideas needed to develop mathematical tools to compute the Fourier coefficients for the smooth convergence of Fourier series. Before that we shall give some simple examples which demonstrate how the Fourier series effectively capture the dynamics of a given process (signal) as the number of terms (harmonics) increases. Consider the square wave function defined by

$$f(x)=\begin{cases}0, & -\pi<x<0\\ 1, & 0<x<\pi\end{cases}$$

Now we would like to find the Fourier series representation of this function. The graph of this waveform is given.

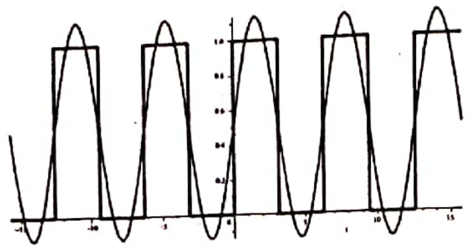


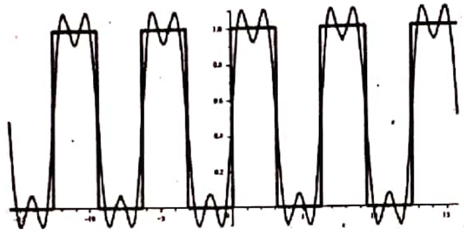
Here we assume that we know how to compute the Fourier coefficients (explained below). We compute a finite number of Fourier Polynomials of the square wave function and plot them along with the graph of the square wave function. We get zeroth Fourier Polynomials $F_0=\frac{1}{2}$ and the graph is



We get the first Fourier Polynomials as $F_1=\frac{1}{2}+\frac{2}{\pi}\sin x$ and the corresponding graph is

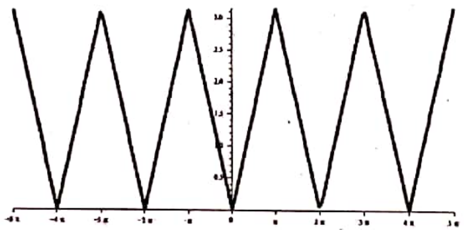
We get the second Fourier Polynomials as $F_2=\frac{1}{2}+\frac{2}{\pi}\sin x+\frac{2}{3\pi}\sin 3x$ and the corresponding graph is



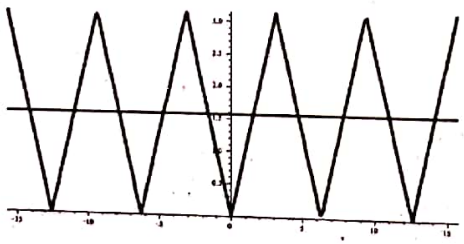


We can see that as the number of terms (harmonics) increases, we get more and more accurate value. But the behaviour of the approximation shows that in the vicinity of points where the function is not differentiable, we need more number of terms to get a good approximation. The following example illustrate this point more clearly.

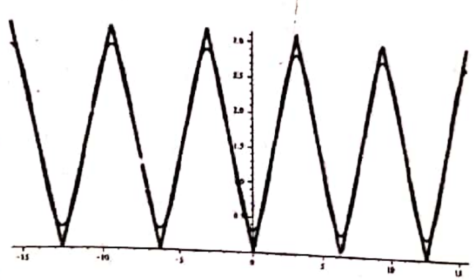
The next example we consider is the triangular wave function with period 2π defined by $f(x) = |x|, -\pi \leq x \leq \pi$. The graph of this function is given below.



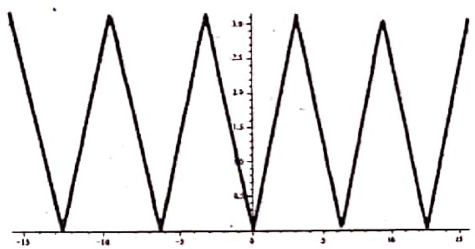
Here also we embed the graph of the various Fourier Polynomials with the graph of the triangular wave function. The zeroth Fourier Polynomial is $F_0 = \frac{\pi}{2}$ and the embedded graph is given below.



We compute the first Fourier Polynomial $F_1 = \frac{\pi}{2} - \frac{4}{\pi}\cos x$ and embed the graph of this Polynomial with the graph of the triangular wave function. Then we get the following graph.



We compute the second Fourier Polynomial $F_2 = \frac{\pi}{2} - \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x$ and embed the graph of this Polynomial with the graph of the triangular wave function. Then we get the following graph.



Now it is clear that as the number of harmonics increases, the approximation of the signals by the the Fourier Polynomials becomes more and more accurate.

5.4.2 Orthogonality of basis functions

For a reasonable representation of $f(x)$ using a sequence of basis functions $(\phi_n(x))_{n=1}^{\infty}$, the basis functions must have the following characteristics.

1. The basis functions must be independent. That is, $\phi_m(x)$ part of $\phi_n(x)$ is zero if $m \neq n$.
2. The basis functions should have uniformity. That is, $\phi_n(x)$ part of $\phi_n(x)$ should be 1.
3. The basis functions should be complete in the sense that $f(x)$ can be written in terms of $\phi_n(x)$'s alone.

A sequence of functions having these characteristics is said to form an orthonormal basis for the representation. So for a comprehensive and thorough explanation of the underlying ideas needed for deriving the formula for computing the Fourier coefficients, we need a generalized notion of orthogonality of vectors. In fact we want to introduce the notion of **orthogonality of basis functions**. Roughly speaking a_n and b_n are the "orthogonal components" of the waveform $f(x)$ in the "direction" of $\cos\left(\frac{n\pi x}{l}\right)$ and $\sin\left(\frac{n\pi x}{l}\right)$ respectively. In three dimensional space, we express a vector $\vec{u} = a_1 i + a_2 j + a_3 k = (a_1, a_2, a_3)$, where a_1, a_2 and a_3 are the components of the vector along the directions of three mutually perpendicular unit vectors i, j and k. (note that (a_1, a_2, a_3) is a finite sequence). So we express the vector in terms of its components in three mutually perpendicular directions. Vectors are tools needed to express physical processes or phenomena which are governed by two or more number of linearly independent factors (parameters). That is, if we have a process or phenomena $\vec{u}$ which is governed by infinite number of linearly independent parameters, we will need vectors having infinite number of components. So we can express vectors having infinite dimension in the form

$$\vec{u} = a_1 e_1 + a_2 e_2 + a_2 e_2 + \cdots + a_n e_n + \cdots = \sum_{k=1}^{\infty} a_k e_k = (a_1, a_2, a_3, \cdots, a_n, \cdots)$$

where e_k, $k = 1, 2, \ldots$ are unit vectors analogous to the unit vectors i, j and k and they are orthonormal vectors (orthogonal and unit vectors). That is, $e_k \cdot e_l = 1$ or 0 according as $k = l$ or $k \neq l$. Also note that $e_k \cdot \vec{u} = a_k$ for each $k = 1, 2, \ldots$. This shows that the components of the vector $\vec{u}$ along the mutually perpendicular directions are uniquely determined and this infact shows that the expression

$$\vec{u} = a_1 e_1 + a_2 e_2 + a_2 e_2 + \cdots + a_n e_n + \cdots = \sum_{k=1}^{\infty} a_k e_k = (a_1, a_2, a_3, \cdots, a_n, \cdots)$$

is unique. This shows that each constant vectors can be uniquely represented by (infinite) sequence and sequence as we know are functions defined on the set of positive integers.

We will say that the two vectors $\vec{u} = \sum_{k=1}^{\infty} a_k e_k$ and $\vec{v} = \sum_{k=1}^{\infty} b_k e_k$ are orthogonal (perpendicular) to each other if their dot product

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{\infty} a_k b_k = 0$$

Also we note that each of the infinite dimensional vectors can be treated as an infinite sequence of real numbers. That is, each of these vectors are functions defined on the set of positive integers. Hence the above dot product defines orthogonality of two functions whose domain is the set of positive integers.

So we generalize the notion of orthogonality of vectors which are functions with positive integer as domain to an analogous notion of orthogonality of functions which are defined on subsets of real numbers by means of integration in place of summation.

Two functions $f(x)$ and $g(x)$ defined in $[a, b]$ are said to be orthogonal if

$$< f(x), g(x) >= \int_a^b f(x)g(x)dx = 0 .$$

...her Mathematics $< f(x), g(x) >$ is called inner product which is a generalization of ...t product defined for the usual vectors. **Now the question is:** Are the functions ...x and $\cos nx$ orthogonal as defined in the sense just above?.

...e can see that for any two integers m and n with $m \neq n$,

$$\int_c^{c+2\pi} \sin mx \sin nx dx = 0 \text{ and } \int_c^{c+2\pi} \cos mx \cos nx dx = 0$$

$$\int_c^{c+2\pi} \sin mx \cos nx dx = \frac{1}{2}\int_c^{c+2\pi} (\sin(m+n)x + \sin(m-n)x)\, dx = 0$$

...he family of functions $\sin mx$ and $\cos nx$ are orthogonal. Also,

$$\int_c^{c+2\pi} \sin^2 mx dx = \int_c^{c+2\pi} \cos^2 mx dx = \pi$$

$$\int_c^{c+2\pi} \sin mx dx = \int_c^{c+2\pi} \cos mx dx = 0$$

...ng the orthogonality properties of the functions $\sin x$ and $\cos x$, we can derive simple ...mula for computing the Fourier coefficients suitable for the convergence of the Fourier ...ries.

Consider the Fourier series of $f(x)$ in $[c, c+2l]$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = f(x) \qquad (A)$$

...tegrating both sides of the above series from c to c+2l and then using above properties, ...e get

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$$

...ultiplying both sides of (A) by $\cos\left(\frac{n\pi x}{l}\right)$ and then integrating from c to $c+2l$ and ...sing the orthogonality property of $\cos\left(\frac{n\pi x}{l}\right)$, we get

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

...ultiplying both sides of (A) by $\sin\left(\frac{n\pi x}{l}\right)$ and then integrating from c to $c+2l$ and ...sing the orthogonality property of $\sin\left(\frac{n\pi x}{l}\right)$, we get

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

...he above formulae for computing the Fourier coefficients a_0, a_n and b_n are called **Euler** ...rmulae.

Thus a function $f(x)$ defined in $[c, c+2l]$ satisfying the Dirichlet conditions can be represented as Fourier series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)=\frac{f(x^-)+f(x^+)}{2}, \quad \forall x \in [c, c+2l]$$

where the Fourier coefficients a_0, a_n and b_n are computed by the Euler's formula given above.

5.4.3 Fourier series of a function with period 2π in $[-\pi, \pi]$

Let $f(x)$ be a function defined in the interval $[-\pi, \pi]$ and satisfying the Dirichlet conditions. Comparing this with the general interval $[c, c+2l]$, we get $c=-\pi$, $c+2l=\pi$ so that $l=\pi$. Hence the Fourier series takes the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx=\frac{f(x^-)+f(x^+)}{2}$$

and the Euler formula for computing the Fourier coefficients are given by

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Problem 5.4.3. *Find the Fourier series representation of $f(x)=x$ in $-\pi<x<\pi$*

Solution: The Fourier series representation of $f(x)$ in $-\pi<x<\pi$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$\text{where } a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$
$$\text{and } b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

Here $f(x)=x$ satisfies the Dirichlet conditions in $(-\pi,\pi)$ and is an odd function. So the Fourier coefficients are given by

$$a_0=0$$
$$a_n=0$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\ dx$$
$$=\frac{2}{\pi}\int_{0}^{\pi} x\sin nx\ dx$$
$$=\frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$
$$=\frac{2}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi}$$
$$=\frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n}+0\right]-[0]$$
$$=\frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx\ ,-\pi<x<\pi$$

Problem 5.4.4. *Find the Fourier series representation of $f(x)=x^2$ in $[-\pi,\pi]$ and deduce that $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$ and $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$*

Solution: The Fourier series representation of $f(x)$ in $-\pi\le x\le\pi$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$\text{where } a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$
$$\text{and } b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

Here $f(x)=x^2$ satisfies the Dirichlet conditions in $[-\pi,\pi]$ and is an even function. So the Fourier coefficients are given by

$$b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\times\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right] - [0+0+0] = \frac{4(-1)^n}{n^2}$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx = \frac{\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \cdots \quad (1)$$

Putting $x = 0$, the series becomes $\frac{\pi^2}{3} - \frac{4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \cdots$

Since the function is continuous at $x = 0$, the series converges to $f(0) = 0$.

$$\therefore \frac{\pi^2}{3} - \frac{4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \cdots = 0$$

$$\implies 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Putting $x = \pi$ in (1), the series becomes

$$\frac{\pi^2}{3} - \frac{4}{1^2}(-1) + \frac{4}{2^2}(1) - \frac{4}{3^2}(1) + \cdots = \frac{\pi^2}{3} + \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + \frac{4}{4^2} + \cdots$$

... the series converges to $\frac{1}{2}[f(-\pi+) + f(\pi-)] = \frac{1}{2}[\pi^2 + \pi^2] = \pi^2$.

$$\therefore \frac{\pi^2}{3} + \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + \frac{4}{4^2} + \cdots = \pi^2$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

...cation of Fourier series to Riemann Zeta function

...ove series is a special case of the generalized harmonic series $\zeta(p) = \sum_{n=1}^{\infty}\frac{1}{n^p}$

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

...s can further be generalized to get Riemann zeta function which defined for any com-... number $z \neq 1$ given by $\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}$

Using a suitable polynomial function x^{2m} we can prove by means of Fourier series that ...$) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, $\zeta(8) = \frac{\pi^8}{9450}$. There is an elegant formula for $\zeta(2n)$ in terms for ... for any positive integer n.

Finding a formula for $\zeta(2n+1)$ is still an open problem.

...roblem 5.4.5. *Find a Fourier series to represent $f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$ ...d hence deduce that $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$ and $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$*

...olution: The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Here $f(x) = x - x^2$ satisfies the Dirichlet conditions in $[-\pi, \pi]$. So the Fourier coefficients are given by

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx \\
&= 0 - \frac{2}{\pi}\int_0^{\pi} x^2\,dx \\
&= -\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \\
&= -\frac{2}{\pi}\times\frac{\pi^3}{3} \\
&= -\frac{2\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx \\
&= 0 - \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx \\
&= -\frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \\
&= -\frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi} \\
&= -\frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right] - [0+0+0] \\
&= \frac{4(-1)^{n+1}}{n^2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx + 0 \\
&= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n} + 0\right] - [0] \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx \\
&= -\frac{\pi^2}{3} + \frac{4}{1^2}\cos x - \frac{4}{2^2}\cos 2x + \frac{4}{3^2}\cos 3x - \dots \\
&\quad + \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x \qquad (1)
\end{aligned}$$

Putting $x = 0$, the series becomes $-\frac{\pi^2}{3} + \frac{4}{1^2} - \frac{4}{2^2} + \frac{4}{3^2} - \dots$

Since the function is continuous at $x = 0$, the series converges to $f(0) = 0$.

$$\therefore -\frac{\pi^2}{3} + \frac{4}{1^2} - \frac{4}{2^2} + \frac{4}{3^2} - \dots = 0$$

$$\Longrightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

Putting $x = \pi$ in (1), the series becomes

$-\frac{\pi^2}{3} + \frac{4}{1^2}(-1) - \frac{4}{2^2}(1) + \frac{4}{3^2}(1) - \dots = -\frac{\pi^2}{3} - \frac{4}{1^2} - \frac{4}{2^2} - \frac{4}{3^2} - \frac{4}{4^2} - \dots$

At $x = \pi$, the series converges to $\frac{1}{2}[f(-\pi+) + f(\pi-)] = \frac{1}{2}[-\pi - \pi^2 + \pi - \pi^2] = -\pi^2$.

$$\therefore -\frac{\pi^2}{3} - \frac{4}{1^2} - \frac{4}{2^2} - \frac{4}{3^2} - \frac{4}{4^2} - \dots = -\pi^2$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

Problem 5.4.6. *Find the Fourier series for $f(x) = |x|$, $-\pi < x < \pi$ and hence deduce that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.*

Solution: The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Here $f(x) = |x|$ satisfies the Dirichlet conditions in $[-\pi, \pi]$ and is an even function. So the Fourier coefficients are given by

$$b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\, dx \qquad [\text{In } 0 < x < \pi, |x| = x]$$
$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$
$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx$$
$$= \frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$
$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^n - 1] \cos nx$$
$$= \frac{\pi}{2} - \frac{4}{\pi(1)^2}\cos x + 0 - \frac{4}{\pi(3)^2}\cos 3x + 0 - \frac{4}{\pi(5)^2}\cos 5x + \cdots$$
$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

Putting $x = 0$, the series becomes $\frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

Since the function is continuous at $x = 0$, the series converges to $f(0) = 0$.

$$\frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = 0$$
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Problem 5.4.7. *Find the Fourier series for* $f(x) = |\sin x|$, $-\pi < x < \pi$.

Solution: The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$
$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Here $f(x) = |\sin x|$ satisfies the Dirichlet conditions in $[-\pi, \pi]$ and is an even function. So the Fourier coefficients are given by

$$b_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\sin x|\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \sin x\, dx$$
$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$
$$= \frac{2}{\pi}[2]$$
$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx \tag{1}$$
$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \qquad n \neq 1$$
$$= \frac{1}{\pi}\left\{\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\}$$
$$= \frac{1}{\pi}\left\{\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{1-n}\right]\right\}$$
$$= \frac{1}{\pi}\left\{(-1)^{n+1}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\}$$
$$= \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right][(-1)^{n+1} - 1]$$
$$= \frac{2}{\pi(n^2-1)}[(-1)^{n+1} - 1], \quad n \neq 1$$

When $n = 1$, (1) becomes

$$a_1 = \frac{1}{\pi}\int_0^{\pi} [\sin 2x - \sin 0]\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$
$$= 0$$
$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi(n^2-1)}[(-1)^{n+1} - 1]\cos nx$$

Problem 5.4.8. *Find the Fourier series for* $f(x) = \begin{cases} 1-x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$

The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$
$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$
$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Here $f(x)$ satisfies the Dirichlet conditions in $(-\pi, \pi)$.

$$x \in (-\pi, 0) \implies -x \in (0, \pi)$$
$$\therefore f(x) = 1 - x \text{ and } f(-x) = 1 + (-x) = 1 - x$$
$$x \in (0, \pi) \implies -x \in (-\pi, 0)$$
$$\therefore f(x) = 1 + x \text{ and } f(-x) = 1 - (-x) = 1 + x$$
$$\text{Hence } f(-x) = f(x)$$

Therefore $f(x)$ is an even function. So the Fourier coefficients are given by

$$b_n = 0$$
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$
$$= \frac{2}{\pi}\int_0^{\pi}(1+x)dx$$
$$= \frac{2}{\pi}\left[x + \frac{x^2}{2}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\pi + \frac{\pi^2}{2}\right]$$
$$= \pi + 2$$
$$a_n = \frac{2}{\pi}\int_0^{\pi}(1+x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[(1+x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{(1+x)\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$
$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$
$$\therefore f(x) = \frac{2+\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}[(-1)^n - 1]\cos nx$$

Problem 5.4.9. *Find the Fourier series for* $f(x) = x\cos x,\ -\pi < x < \pi$.

Solution: The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$
$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$
$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Here $f(x) = x\cos x$ satisfies the Dirichlet conditions in $(-\pi, \pi)$ and is an odd function. So the Fourier coefficients are given by

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin nx\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]\, dx \quad (1)$$
$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right) - (1)\left(-\frac{\sin(n-1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi} \quad n \neq 1$$
$$= \frac{1}{\pi}\left[(\pi)\left(-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}\right)\right]$$

$$= -\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}$$
$$= (-1)^{n+2}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$
$$= (-1)^{n+2}\frac{2n}{n^2-1}, \quad n \neq 1$$

When $n = 1$, (1) becomes

$$b_1 = \frac{1}{\pi}\int_0^\pi x\,[\sin 2x + \sin 0]\,dx$$
$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$
$$= \frac{1}{\pi}\left[(\pi)\left(-\frac{\cos 2\pi}{2}\right)\right]$$
$$= -\frac{1}{2}$$
$$\therefore f(x) = -\frac{1}{2}\sin x + \sum_{n=2}^\infty (-1)^{n+2}\frac{2n}{n^2-1}\sin nx$$

Problem 5.4.10. *Find the Fourier series for $f(x) = x\sin x$, $-\pi < x < \pi$ and show that* $1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots = \frac{\pi}{2}$.

Solution: The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Here $f(x) = x\sin x$ satisfies the Dirichlet conditions in $(-\pi, \pi)$ and is an even function. So the Fourier coefficients are given by

$$b_n = 0$$
$$a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,dx$$
$$= \frac{2}{\pi}\Big[(x)(-\cos x) - (1)(-\sin x)\Big]_0^\pi$$
$$= \frac{2}{\pi}[-\pi(-1)]$$
$$= 2$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx$$
$$= \frac{1}{\pi}\int_0^\pi x\,[\sin(n+1)x - \sin(n-1)x]\,dx \qquad (1)$$
$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) - (1)\left(-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^\pi \qquad n \neq 1$$
$$= \frac{1}{\pi}\left[(\pi)\left(-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right)\right]$$
$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$
$$= -(-1)^{n+1}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right]$$
$$= (-1)^{n+1}\frac{2}{n^2-1}, \quad n \neq 1$$

When $n = 1$, (1) becomes

$$a_1 = \frac{1}{\pi}\int_0^\pi x\,[\sin 2x - \sin 0]\,dx$$
$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$
$$= \frac{1}{\pi}\left[(\pi)\left(-\frac{\cos 2\pi}{2}\right)\right]$$
$$= -\frac{1}{2}$$
$$\therefore f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^\infty (-1)^{n+1}\frac{2}{n^2-1}\cos nx$$
$$= 1 - \frac{1}{2}\cos x - \frac{2}{1\cdot 3}\cos 2x + \frac{2}{3\cdot 5}\cos 3x - \frac{2}{5\cdot 7}\cos 4x + \cdots$$

Putting $x = \frac{\pi}{2}$, the series becomes $1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots$

Since $f(x)$ is continuous at $x = \frac{\pi}{2}$, the series converges to $f\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$$\therefore 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots = \frac{\pi}{2}$$

Problem 5.4.11. *Find the Fourier series for* $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi, & 0 < x < \pi \end{cases}$ *and deduce that* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Solution: The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(-\pi, \pi)$. So the Fourier coefficients are given by

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \pi dx\right]\\
&= \frac{1}{\pi} \times \pi[x]_0^{\pi}\\
&= \pi\\
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \pi \cos nx dx\right]\\
&= \frac{1}{\pi} \times \pi\left[\frac{\sin nx}{n}\right]_0^{\pi}\\
&= 0\\
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \pi \sin nx dx\right]\\
&= \frac{1}{\pi} \times \pi\left[-\frac{\cos nx}{n}\right]_0^{\pi}\\
&= \left[\left(-\frac{(-1)^n}{n}\right) - \left(-\frac{1}{n}\right)\right]\\
&= \frac{1}{n}[1-(-1)^n]\\
\therefore f(x) &= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{1}{n}[1-(-1)^n]\sin nx\\
&= \frac{\pi}{2} + \frac{2\sin x}{1} + 0 + \frac{2\sin 3x}{3} + 0 + \frac{2\sin 5x}{5}\ldots\\
&= \frac{\pi}{2} + 2\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\ldots\right]
\end{aligned}$$

Putting $x = \pi/2$, the series becomes

$$\frac{\pi}{2} + 2\left[\frac{\sin(\pi/2)}{1} + \frac{\sin(3\pi/2)}{3} + \frac{\sin(5\pi/2)}{5}\ldots\right] = \frac{\pi}{2} + 2\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\right]$$

Since the function is continuous at $x = \pi/2$, the series converges to $f(\pi/2) = \pi$.

$$\therefore \frac{\pi}{2} + 2\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\right] = \pi$$

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$$

5.4.4 Fourier series of a function with period 2π in $[0, 2\pi]$

Let $f(x)$ be a function defined in the interval $[0, 2\pi]$ and satisfying the Dirichlet conditions. Comparing this interval with the general interval $[c, c+2l]$, we get $c = 0$, $c + 2l = 2\pi$, so that $l = \pi$. Then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \frac{f(x^-) + f(x^+)}{2}$$

takes the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \frac{f(x^-) + f(x^+)}{2}$$

and the formula for computing the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

Problem 5.4.12. *Find the Fourier series for* $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi < x \le 2\pi \end{cases}$ *and deduce that* $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution: The Fourier series representation of $f(x)$ in $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0, 2\pi)$. So the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} 2\pi - x\,dx\right]$$
$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$
$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2}\right) + \left((4\pi^2 - 2\pi^2) - (2\pi^2 - \pi^2/2)\right)\right]$$
$$= \pi$$
$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$
$$= \frac{1}{\pi}\left\{\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$
$$= \frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} + \left[\frac{(2\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$
$$= \frac{1}{\pi}\left\{\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] + \left[\left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{(-1)^n}{n^2}\right)\right]\right\}$$
$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$
$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$
$$= \frac{1}{\pi}\left\{\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$
$$= \frac{1}{\pi}\left\{\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} + \left[-\frac{(2\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$
$$= \frac{1}{\pi}\left\{\left[\left(-\frac{\pi(-1)^n}{n} + 0\right) - (0 + 0)\right] + \left[(0 + 0) - \left(-\frac{\pi(-1)^n}{n} + 0\right)\right]\right\}$$
$$= 0$$
$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}[(-1)^n - 1]\cos nx$$
$$= \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \frac{2}{5^2}\cos 5x + 0 - \cdots\right]$$
$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Putting $x = \pi$, the series becomes

$$\frac{\pi}{2} - \frac{4}{\pi}\left[-1 - \frac{1}{3^2} - \frac{1}{5^2} - \cdots\right] = \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Since $f(x)$ is continuous at $x = \pi$, the series converges to $f(\pi) = \pi$

$$\therefore \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \pi$$
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Problem 5.4.13. *Find the Fourier series for* $f(x) = \left(\frac{\pi - x}{2}\right)^2$ *in* $(0, 2\pi)$

Solution: The Fourier series representation of $f(x)$ with period 2π is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0, 2\pi)$. So the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}dx$$
$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{(-1)(3)}\right]_0^{2\pi}$$
$$= -\frac{1}{12\pi}[(-\pi^3) - (\pi^3)] = \frac{\pi^2}{6}$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\cos nx\,dx$$
$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - (2(\pi - x)(-1))\left(-\frac{\cos nx}{n^2}\right) + ((-2)(-1))\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^2\sin nx}{n} - \frac{2(\pi - x)\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$
$$= \frac{1}{4\pi}\left[\left(0 - \frac{2(-\pi)}{n^2} - 0\right) - \left(0 - \frac{2(\pi)}{n^2} - 0\right)\right]$$
$$= \frac{1}{n^2}$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - (2(\pi - x)(-1))\left(-\frac{\sin nx}{n^2}\right) + ((-2)(-1))\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{(\pi - x)^2\cos nx}{n} + \frac{2(\pi - x)\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

$$= 0$$

$$\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \ , 0 < x < 2\pi$$

5.4.5 Fourier series of a function with period 2π in $[c, c + 2\pi]$

Let $f(x)$ be a function defined in the interval $[c, c + 2\pi]$ and satisfying the Dirichlet conditions. Comparing this interval with the general interval $[c, c + 2l]$, we get $c = c$, $c + 2l = c + 2\pi$, so that $l = \pi$. Then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \frac{f(x^-) + f(x^+)}{2} \quad \forall x \in [c, c + 2l]$$

takes the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \frac{f(x^-) + f(x^+)}{2} \quad \forall x \in [c, c + 2\pi]$$

and the formula for computing the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx \, dx$$

Problem 5.4.14. *Find the Fourier series for* $f(x) = \begin{cases} x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$ *and deduce that* $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution: The Fourier series representation of $f(x)$ with period 2π is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx \, dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx \, dx$

Here the function $f(x)$ satisfies the Dirichlet conditions in $-\frac{\pi}{2} < x < \frac{3\pi}{2}$. So the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x \, dx + \int_{\pi/2}^{3\pi/2} (\pi - x)\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{3\pi/2}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x \cos nx dx + \int_{\pi/2}^{3\pi/2} (\pi - x)\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{0 + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{3\pi/2}\right\}$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\pi\sin(3n\pi/2)}{2n} - \frac{\cos(3n\pi/2)}{n^2}\right] - \left[\frac{\pi\sin(n\pi/2)}{2n} - \frac{\cos(n\pi/2)}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{2n}[\sin(3n\pi/2) + \sin(n\pi/2)] - \frac{1}{n^2}[\cos(3n\pi/2) - \cos(n\pi/2)]\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{2n}\sin(n\pi)\cos(n\pi/2)] - \frac{1}{n^2}[-2\sin(n\pi)\sin(n\pi/2)]\right\}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x \sin nx dx + \int_{\pi/2}^{3\pi/2} (\pi - x)\sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{2\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}\right.$$

$$\left. + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{3\pi/2}\right\}$$

$$= \frac{1}{\pi}\left\{2\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi/2}+\left[-\frac{(\pi-x)\cos nx}{n}-\frac{\sin nx}{n^2}\right]_{\pi/2}^{3\pi/2}\right\}$$

$$= \frac{1}{\pi}\left\{2\left[-\frac{\pi\cos(n\pi/2)}{2n}+\frac{\sin(n\pi/2)}{n^2}\right]+\left[\frac{\pi\cos(3n\pi/2)}{2n}-\frac{\sin(3n\pi/2)}{n^2}\right]\right.$$

$$\left.-\left[-\frac{\pi\cos(n\pi/2)}{2n}-\frac{\sin(n\pi/2)}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\pi\cos(n\pi/2)}{n}+\frac{2\sin(n\pi/2)}{n^2}\right]+\frac{\pi}{2n}[\cos(3n\pi/2)+\cos(n\pi/2)]\right.$$

$$\left.-\frac{1}{n^2}[\sin(3n\pi/2)-\sin(n\pi/2)]\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\pi\cos(n\pi/2)}{n}+\frac{2\sin(n\pi/2)}{n^2}\right]\right.$$

$$\left.+\frac{\pi}{2n}[2\cos(n\pi)\cos(n\pi/2)]-\frac{1}{n^2}[2\cos(n\pi)\sin(n\pi/2)]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}\cos(n\pi/2)[(-1)^n-1]+\frac{2}{n^2}\sin(n\pi/2)[1-(-1)^n]\right\}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{1}{\pi}\left\{\frac{\pi}{n}\cos(n\pi/2)[(-1)^n-1]+\frac{2}{n^2}\sin(n\pi/2)[1-(-1)^n]\right\}\sin nx$$

$$= \frac{4}{\pi}\sin x-\frac{4}{\pi}\frac{\sin 3x}{3^2}+\frac{4}{\pi}\frac{\sin 5x}{5^2}-\cdots$$

$$= \frac{4}{\pi}\left[\sin x-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\cdots\right]$$

Putting $x=\frac{\pi}{2}$, the series becomes $\frac{4}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$

Since $f(x)$ is continuous at $x=\frac{\pi}{2}$, the series converges to $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$

$$\therefore \frac{4}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]=\frac{\pi}{2}$$

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi}{8}$$

5.4.6 Fourier series of a function with period $2l$ in $[-l,\ l]$

Let $f(x)$ be a function defined in the interval $[-l,\ l]$ and satisfying the Dirichlet conditions. The Fourier series representation of $f(x)$ with period $2l$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

where $a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and $b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Remark: When the function $f(x)$ defined in the interval $[-l,\ l]$ is even, using the properties of even and odd functions, we can observe that the formula for Fourier coefficients becomes

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx=\frac{2}{l}\int_0^{l}f(x)dx$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx=\frac{2}{l}\int_0^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and the value of $b_n=0$. So the Fourier series reduces to a series having cosine terms alone.

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)=\frac{f(x^-)+f(x^+)}{2}$$

If the function $f(x)$ is an odd function, both $a_0=0$ and $a_n=0$ and formula for computing b_n is given by

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx=\frac{2}{l}\int_0^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

and the Fourier series reduces to a series containing sine terms alone.

$$\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)=\frac{f(x^-)+f(x^+)}{2}$$

This observation is useful when we deal with half-range sine series and cosine series.

Problem 5.4.15. *Find the Fourier series of the periodic function $f(x)$ of period 2, where* $f(x)=\begin{cases}-1, & -1<x\le 0\\ 2x, & 0<x<1\end{cases}$ *and deduce that* $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$ *and* $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$.

Solution: The Fourier series representation of $f(x)$ with period $2l = 2$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$\text{where } a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\, dx$$

$$\text{and } b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\, dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(-1,1)$. So the Fourier coefficients are given by

$$\begin{aligned}
a_0 &= \int_{-1}^{0}(-1)dx + \int_0^1 2x\,dx \\
&= -[x]_{-1}^{0} + 2\left[\frac{x^2}{2}\right]_0^1 \\
&= -[0-(-1)] + [1-0] = 0 \\
a_n &= \int_{-1}^{0}(-1)\cos n\pi x\,dx + \int_0^1 (2x)\cos n\pi x\,dx \\
&= -\left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{0} + 2\left[(x)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 \\
&= -[0-0] + 2\left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 \\
&= 2\left[\left(0 + \frac{\cos n\pi}{n^2\pi^2}\right) - \left(0 + \frac{1}{n^2\pi^2}\right)\right] \\
&= \frac{2}{n^2\pi^2}[(-1)^n - 1] \\
b_n &= \int_{-1}^{0}(-1)\sin n\pi x\,dx + \int_0^1 (2x)\sin n\pi x\,dx \\
&= -\left[-\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + 2\left[(x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 \\
&= \left[\frac{1}{n\pi} - \frac{\cos(-n\pi)}{n\pi}\right] + 2\left[-\frac{x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 \\
&= \left[\frac{1}{n\pi} - \frac{(-1)^n}{n\pi}\right] + 2\left[-\frac{(-1)^n}{n\pi}\right] \\
&= \frac{1}{n\pi}[1 - 3(-1)^n]
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^n - 1]\cos n\pi x + \frac{1}{n\pi}[1-3(-1)^n]\sin n\pi x \\
&= -\frac{4}{\pi^2}\left[\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots\right] \\
&\quad + \frac{2}{\pi}\left[2\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{2\sin 3\pi x}{3} - \frac{\sin 4\pi x}{4} + \dots\right]
\end{aligned}$$

Putting $x = 0$, the series becomes

$$-\frac{4}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

Since $f(x)$ is discontinuous at $x = 0$, the series converges to

$$\frac{1}{2}[f(0-) + f(0+)] = \frac{1}{2}[-1+0] = -\frac{1}{2}$$

$$\therefore -\frac{4}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] = -\frac{1}{2}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Putting $x = \frac{1}{2}$, the series becomes

$$\frac{2}{\pi}\left[2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \dots\right] = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$$

Since $f(x)$ is discontinuous at $x = \frac{1}{2}$, the series converges to

$$f\left(\frac{1}{2}\right) = 1$$

$$\therefore \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right] = 1$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

Problem 5.4.16. *Find the Fourier series of the periodic function $f(x)$ of period 4, where* $f(x) = \begin{cases} 2, & -2 < x \le 0 \\ x, & 0 < x < 2 \end{cases}$ *and deduce that* $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$ *and* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

Solution: The Fourier series representation of $f(x)$ with period $2l = 4$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$\text{where } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$$
$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right) dx$$
$$\text{and } b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(-2, 2)$. So the Fourier coefficients are given by

$$a_0 = \frac{1}{2}\left[\int_{-2}^{0}(2)dx + \int_{0}^{2} x\,dx\right]$$
$$= \frac{1}{2}\left[2[x]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^2\right]$$
$$= \frac{1}{2}\left[2[0-(-2)] + [2-0]\right]$$
$$= 3$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0}(2)\cos\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2}(x)\cos\left(\frac{n\pi x}{2}\right) dx\right]$$
$$= \frac{1}{2}\left\{2\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_{-2}^{0} + \left[(x)\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right) - (1)\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n^2\pi^2}{2^2}\right)}\right)\right]_0^2\right\}$$
$$= \frac{1}{2}\left\{2\left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2\right\}$$
$$= \frac{1}{2}\left\{2[0-0] + \left[\left(0 + \frac{4(-1)^n}{n^2\pi^2}\right) - \left(0 + \frac{4}{n^2\pi^2}\right)\right]\right\}$$
$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0}(2)\sin\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2}(x)\sin\left(\frac{n\pi x}{2}\right) dx\right]$$
$$= \frac{1}{2}\left\{2\left[-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_{-2}^{0} + \left[(x)\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right) - (1)\left(-\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n^2\pi^2}{2^2}\right)}\right)\right]_0^2\right\}$$
$$= \frac{1}{2}\left\{2\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2\right\}$$
$$= \frac{1}{2}\left\{2\left[\left(-\frac{2}{n\pi}\right) - \left(-\frac{2(-1)^n}{n\pi}\right)\right] + \left[\left(-\frac{4(-1)^n}{n\pi}\right) - 0\right]\right\}$$
$$= -\frac{2}{n\pi}$$

$$\therefore f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$
$$= \frac{3}{2} - \frac{4}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right]$$
$$- \frac{2}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \cdots\right]$$

Putting $x = 0$, the series becomes

$$\frac{3}{2} - \frac{4}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Since $f(x)$ is discontinuous at $x = 0$, the series converges to

$$\frac{1}{2}[f(0-) + f(0+)] = \frac{1}{2}[2 + 0] = 1$$
$$\therefore \frac{3}{2} - \frac{4}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = 1$$
$$\frac{4}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{1}{2}$$
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Putting $x = 1$, the series becomes

$$\frac{3}{2} - \frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \cdots\right] = \frac{3}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

Since $f(x)$ is discontinuous at $x = 1$, the series converges to

$$f(1) = 1$$
$$\therefore \frac{3}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] = 1$$
$$\frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] = \frac{1}{2}$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Problem 5.4.17. *Find the Fourier series of the periodic function $f(x)$ of period 4, where*

$$f(x) = \begin{cases} 0, & -2 < x \le -1 \\ k, & -1 < x < 1 \\ 0, & 1 \le x < 2 \end{cases} \quad \text{and deduce that } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Solution: The Fourier series representation of $f(x)$ with period $2l = 4$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{2}\right)$$

$$\text{where } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$$
$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right) dx$$
$$\text{and } b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(-2,2)$. So the Fourier coefficients are given by

$$a_0 = \frac{1}{2}\left[\int_{-2}^{-1}(0)dx + \int_{-1}^{1}(k)\,dx + \int_{1}^{2} 0\,dx\right]$$
$$= \frac{1}{2}k[x]_{-1}^{1}$$
$$= k$$
$$a_n = \frac{1}{2}\left[\int_{-1}^{1}(k)\cos\left(\frac{n\pi x}{2}\right) dx\right]$$
$$= \frac{k}{2}\left[2\int_{0}^{1}\cos\left(\frac{n\pi x}{2}\right) dx\right]$$
$$= k\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^1$$
$$= k\left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^1$$
$$= \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$
$$b_n = \frac{1}{2}\int_{-1}^{1}(k)\sin\left(\frac{n\pi x}{2}\right) dx$$
$$= 0$$
$$\therefore f(x) = \frac{k}{2} + \sum_{n=1}^{\infty}\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)$$
$$= \frac{k}{2} + \frac{2k}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + \cdots\right]$$

Putting $x = 0$, the series becomes

$$\frac{k}{2} + \frac{2k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

Since $f(x)$ is continuous at $x = 0$, the series converges to $f(0) = k$.

$$\therefore \frac{k}{2} + \frac{2k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] = k$$

$$\frac{2k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] = \frac{k}{2}$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

5.4.7 Fourier series of a function with period $2l$ in $[0, 2l]$

Let $f(x)$ be a function defined in the interval $[0, 2l]$ and satisfying the Dirichlet conditions. The Fourier series representation of $f(x)$ with period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)$$
$$\text{where } a_0 = \frac{1}{l}\int_{0}^{2l} f(x)dx$$
$$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$
$$\text{and } b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

Problem 5.4.18. *Find the Fourier series of the periodic function $f(x)$ of period 3, where $f(x) = 2x - x^2$ in $(0,3)$ and deduce that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.*

Solution: The Fourier series representation of $f(x)$ with period $2l = 3$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2n\pi x}{3}\right) + b_n\sin\left(\frac{2n\pi x}{3}\right)$$
$$\text{where } a_0 = \frac{2}{3}\int_{0}^{3} f(x)dx$$
$$a_n = \frac{2}{3}\int_{0}^{3} f(x)\cos\left(\frac{2n\pi x}{3}\right) dx$$
$$\text{and } b_n = \frac{2}{3}\int_{0}^{3} f(x)\sin\left(\frac{2n\pi x}{3}\right) dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0,3)$. So the Fourier coefficients are given by

$$a_0 = \frac{2}{3}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 = 0$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - (2-2x)\left(-\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{4n^2\pi^2}{9}\right)}\right) + (-2)\left(-\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{8n^3\pi^3}{27}\right)}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3(2x-x^2)}{2n\pi}\sin\left(\frac{2n\pi x}{3}\right) + \frac{9(2-2x)}{4n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{54}{8n^3\pi^3}\sin\left(\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left(0 - \frac{9}{n^2\pi^2} + 0\right) - \left(0 + \frac{9}{2n^2\pi^2} + 0\right)\right]$$

$$= \frac{2}{3}\left[-\frac{27}{2n^2\pi^2}\right]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(-\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - (2-2x)\left(-\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{4n^2\pi^2}{9}\right)}\right) + (-2)\left(\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{8n^3\pi^3}{27}\right)}\right)\right]_0^3$$

$$= \frac{2}{3}\left[-\frac{3(2x-x^2)}{2n\pi}\cos\left(\frac{2n\pi x}{3}\right) + \frac{9(2-2x)}{4n^2\pi^2}\sin\left(\frac{2n\pi x}{3}\right) - \frac{54}{8n^3\pi^3}\cos\left(\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left(\frac{9}{2n\pi} + 0 - \frac{54}{8n^3\pi^3}\right) - \left(0 + 0 - \frac{54}{8n^3\pi^3} + 0\right)\right]$$

$$= \frac{3}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left(-\frac{9}{n^2\pi^2}\right)\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)$$

$$= -\frac{9}{\pi^2}\left[\cos\left(\frac{2\pi x}{3}\right) + \frac{1}{2^2}\cos\left(\frac{4\pi x}{3}\right) + \frac{1}{3^2}\cos\left(\frac{6\pi x}{3}\right) + \cdots\right]$$
$$+ \frac{3}{\pi}\left[\sin\left(\frac{2\pi x}{3}\right) + \frac{1}{2}\sin\left(\frac{4\pi x}{3}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{3}\right) + \cdots\right]$$

Putting $x = 0$, the series becomes $-\frac{9}{\pi^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$

The series converges to $\frac{1}{2}[f(0+) + f(3-)] = \frac{1}{2}[0 - 3] = -\frac{3}{2}$.

$$\therefore -\frac{9}{\pi^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = -\frac{3}{2}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Problem 5.4.19. *Obtain the Fourier series of the periodic function $f(x)$ of period 2, where $f(x) = \pi x$, $0 \le x \le 2$ and deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.*

Solution: The Fourier series representation of $f(x)$ with period $2l = 2$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$$

$$\text{where } a_0 = \frac{1}{1}\int_0^2 f(x)dx$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\,dx$$

$$\text{and } b_n = \frac{1}{1}\int_0^2 f(x)\sin n\pi x\,dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0, 3)$. So the Fourier coefficients are given by

$$a_0 = \int_0^2 \pi x\,dx = \pi\left[\frac{x^2}{2}\right]_0^2 = 2\pi$$

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$$
\begin{aligned}
a_n &= \int_0^2 (\pi x) \cos n\pi x \, dx \\
&= \pi\left[(x)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^2 \\
&= \pi\left[\frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^2 \\
&= \pi\left[\left(0 + \frac{\cos 2n\pi}{n^2\pi^2}\right) - \left(0 + \frac{1}{n^2\pi^2}\right)\right] \\
&= \pi\left[\frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\right] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \int_0^2 (\pi x) \sin n\pi x \, dx \\
&= \pi\left[(x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^2 \\
&= \pi\left[-\frac{x \cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^2 \\
&= \pi\left[-\frac{2}{n\pi}\right] \\
&= -\frac{2}{n}
\end{aligned}
$$

$$
\begin{aligned}
\therefore f(x) &= \pi + \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)\sin n\pi x \\
&= \pi - 2\left[\sin \pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \dots\right]
\end{aligned}
$$

Putting $x = \frac{1}{2}$, the series becomes

$$\pi - 2\left[\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \dots\right] = \pi - 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$$

Since $f(x)$ is continuous at $x = \frac{1}{2}$, the series converges to $f\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$$
\begin{aligned}
\pi - 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right] &= \frac{\pi}{2} \\
2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right] &= \frac{\pi}{2} \\
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots &= \frac{\pi}{4}
\end{aligned}
$$

Problem 5.4.20. *Find the Fourier series of the periodic function $f(x)$ of period 2, where*

$$f(x) = \begin{cases} \pi x, & 0 < x \le 1 \\ \pi(2 - x) & 1 < x < 2 \end{cases}$$

Solution: The Fourier series representation of $f(x)$ with period $2l = 2$ is

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x) \\
\text{where } a_0 &= \frac{1}{1}\int_0^2 f(x)dx \\
a_n &= \frac{1}{1}\int_0^2 f(x)\cos n\pi x \, dx \\
\text{and } b_n &= \frac{1}{1}\int_0^2 f(x)\sin n\pi x \, dx
\end{aligned}
$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0, 2)$. So the Fourier coefficients are given by

$$
\begin{aligned}
a_0 &= \int_0^1 \pi x \, dx + \int_1^2 \pi(2 - x) \, dx \\
&= \pi\left[\left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2\right] \\
&= \pi\left[\left(\frac{1}{2}\right) - 0 + (4 - 2) - \left(2 - \frac{1}{2}\right)\right] \\
&= \pi
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \int_0^1 \pi x \cos n\pi x \, dx + \int_1^2 \pi(2 - x)\cos n\pi x \, dx \\
&= \pi\left[(x)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 \\
&\quad + \pi\left[(2 - x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2 \\
&= \pi\left[\frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 + \pi\left[\frac{(2 - x)\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2 \\
&= \pi\left[\left(0 + \frac{(-1)^n}{n^2\pi^2}\right) - \left(0 + \frac{1}{n^2\pi^2}\right)\right] + \pi\left[\left(0 - \frac{1}{n^2\pi^2}\right) - \left(0 - \frac{(-1)^n}{n^2\pi^2}\right)\right] \\
&= \frac{2}{n^2\pi}[(-1)^n - 1]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2 - x)\sin n\pi x \, dx \\
&= \pi\left[(x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 \\
&\quad + \pi\left[(2 - x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2
\end{aligned}
$$

$$= \pi\left[-\frac{x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 + \pi\left[-\frac{(2-x)\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\left(-\frac{(-1)^n}{n\pi}\right) + 0 - (-0+0)\right] + \pi\left[(-0-0) - \left(-\frac{(-1)^n}{n\pi} - 0\right)\right]$$

$$= 0$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos n\pi x$$

5.5 Half-range cosine series and sine series

In most practical situations we would like to get a Fourier series representation of the function $f(x)$ defined in the interval $[0, l]$ in terms of sine function alone or cosine function alone. If we are able to represent the function $f(x)$ as a Fourier series involving cosine or sine terms, we can drastically reduce the computational complexity and also it is useful in some theoretical interpretations and practical applications. One of the important practical applications of such expansion as a Fourier series involving sine functions or cosine functions alone is to analyze the temperature distribution along a rod or in a plane sheet or in a solid. Another example where we apply Fourier series of sine functions alone is to describe the transverse vibrations of a string with both ends fixed or both ends moving in periodic pattern.

Let $f(x)$ be a function defined in the interval $[-l, l]$ satisfying Dirichlet conditions. The general form of Fourier series of $f(x)$ in the interval $[-l, l]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \frac{f(x^-) + f(x^+)}{2}$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

5.5.1 Half-range cosine series of a function

Here we are given a function $f(x)$ defined in the interval $[0, l]$ satisfying the Dirichlet conditions. Now we would like to reduce the Fourier series to a series having only cosine terms and so we have to make $b_n = 0$ for all n. Observe that if $f(x)$ were an even function defined in $[-l, l]$, then $f(x)\cos\left(\frac{n\pi x}{l}\right)$ is an even function and $f(x)\sin\left(\frac{n\pi x}{l}\right)$ is an odd function so that

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = 0$$

So if we redefine the function $f(x)$ which is defined in $[0, l]$ to an **even function** $g(x)$ defined in $[-l, l]$ such that $g(x) = f(x), \forall x \in [0, l]$, then we get a Fourier series having cosine terms alone in the following form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) = \frac{f(x^-) + f(x^+)}{2}, \quad \forall x \in [0, l]$$

where the Fourier coefficients a_0 and a_n are given by the formulae

$$a_0 = \frac{2}{l}\int_{0}^{l} f(x)dx$$

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

This is called half-range cosine series of the function $f(x)$. Actually this is the Fourier series representation of the function $g(x)$, but we can replace $g(x)$ by $f(x)$ as they are identical in $[0, l]$.

Problem 5.5.1. *Find the half range Fourier cosine series representation of $f(x) = x^2$ in $(0, \pi)$ and hence deduce that*

$$(i)\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad \textit{and}$$

$$(ii)\ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Solution: The Fourier cosine series representation of $f(x)$ in $0 < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$\text{and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0,\pi)$. So the Fourier coefficients are given by

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi x^2\,dx\\
&= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi\\
&= \frac{2}{\pi}\times\frac{\pi^3}{3}\\
&= \frac{2\pi^2}{3}\\
a_n &= \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx\\
&= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right)-(2x)\left(-\frac{\cos nx}{n^2}\right)+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi\\
&= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right]_0^\pi\\
&= \frac{2}{\pi}\left[0+\frac{2\pi(-1)^n}{n^2}+0\right]-[0+0+0]\\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) &= \frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx\\
&= \frac{\pi^2}{3}-\frac{4}{1^2}\cos x+\frac{4}{2^2}\cos 2x-\frac{4}{3^2}\cos 3x+\cdots \qquad (1)
\end{aligned}$$

Putting $x=0$, the series becomes $\frac{\pi^2}{3}-\frac{4}{1^2}+\frac{4}{2^2}-\frac{4}{3^2}+\cdots$

The series converges to $f(0+)=0$.

$$\therefore \frac{\pi^2}{3}-\frac{4}{1^2}+\frac{4}{2^2}-\frac{4}{3^2}+\cdots=0$$

$$\Longrightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$$

Putting $x=\pi$ in (1), the series becomes

$\frac{\pi^2}{3}-\frac{4}{1^2}(-1)+\frac{4}{2^2}(1)-\frac{4}{3^2}(1)+\cdots=\frac{\pi^2}{3}+\frac{4}{1^2}+\frac{4}{2^2}+\frac{4}{3^2}+\frac{4}{4^2}+\cdots$

At $x=\pi$, the series converges to $f(\pi-)=\pi^2$.

$$\therefore \frac{\pi^2}{3}+\frac{4}{1^2}+\frac{4}{2^2}+\frac{4}{3^2}+\frac{4}{4^2}+\cdots=\pi^2$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

Problem 5.5.2. *Obtain the half range Fourier cosine series representation of*

$$f(x)=\begin{cases}\cos x, & 0<x<\pi/2\\ 0, & \pi/2<x<\pi\end{cases}$$

Solution: The Fourier cosine series representation of $f(x)$ in $0<x<\pi$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

where $a_0=\frac{2}{\pi}\int_0^\pi f(x)\,dx$

and $a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0,\pi)$. So the Fourier coefficients are given by

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx+0\\
&= \frac{2}{\pi}\left[\sin x\right]_0^{\pi/2}\\
&= \frac{2}{\pi}\\
a_n &= \frac{2}{\pi}\int_0^{\pi/2}\cos x\cos nx\,dx\\
&= \frac{1}{\pi}\int_0^{\pi/2}[\cos(n+1)x+\cos(n-1)x]\,dx \qquad (1)\\
&= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} \qquad n\neq 1\\
&= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1}+\frac{\sin(n-1)\pi/2}{n-1}\right]\\
&= \frac{1}{\pi}\left[\frac{\cos(n\pi/2)}{n+1}-\frac{\cos(n\pi/2)}{n-1}\right]\\
&= \frac{1}{\pi}\cos(n\pi/2)\left[\frac{(n-1)-(n-1)}{(n+1)(n-1)}\right]\\
&= -\frac{2}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right) \qquad n\neq 1
\end{aligned}$$

When $n=1$, (1) becomes

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^{\pi/2}[\cos 2x+1]\,dx\\
&= \frac{1}{\pi}\left[\frac{\sin 2x}{2}+x\right]_0^{\pi/2}\\
&= \frac{1}{\pi}\times\frac{\pi}{2}=\frac{1}{2}
\end{aligned}$$

$$\therefore f(x)=\frac{1}{\pi}+\frac{1}{2}\cos x+\sum_{n=2}^{\infty}-\frac{2}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right)\cos nx$$

Problem 5.5.3. *Obtain the half range Fourier cosine series expansion of* $f(x) = x\sin x$ *in* $(0, \pi)$ *and show that* $1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots = \frac{\pi}{2}$

Solution: The Fourier cosine series representation of $f(x)$ in $0 < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$\text{and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0, \pi)$. So the Fourier coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx \\ &= \frac{2}{\pi}\Big[(x)(-\cos x) - (1)(-\sin x)\Big]_0^{\pi} \\ &= \frac{2}{\pi}[-\pi(-1)] \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\,[\sin(n+1)x - \sin(n-1)x]\,dx \qquad (1) \\ &= \frac{1}{\pi}\Bigg[(x)\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) \\ &\qquad - (1)\left(-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\Bigg]_0^{\pi} \qquad n \neq 1 \\ &= \frac{1}{\pi}\left[(\pi)\left(-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right)\right] \\ &= -\frac{(-1)^{n+1}}{1+n} + \frac{(-1)^{n-1}}{n-1} \\ &= (-1)^{n+1}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] \\ &= (-1)^{n+1}\frac{2}{n^2-1}, \quad n \neq 1 \end{aligned}$$

When $n = 1$, (1) becomes

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_0^{\pi} x\,[\sin 2x - \sin 0]\,dx \\ &= \frac{1}{\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\pi}\left[(\pi)\left(-\frac{\cos 2\pi}{2}\right)\right] \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{2}{n^2-1}\cos nx \\ &= 1 - \frac{1}{2}\cos x - \frac{2}{1\cdot 3}\cos 2x + \frac{2}{3\cdot 5}\cos 3x - \frac{2}{5\cdot 7}\cos 4x + \cdots \end{aligned}$$

Putting $x = \frac{\pi}{2}$, the series becomes $1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots$

Since $f(x)$ is continuous at $x = \frac{\pi}{2}$, the series converges to $f\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$$\therefore 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \cdots = \frac{\pi}{2}.$$

Problem 5.5.4. *Find the half range cosine series for* $f(x) = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l-x), & l/2 \le x < l \end{cases}$ *and hence show that* $\frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solution: The Fourier cosine series representation of $f(x)$ in $0 \le x \le l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$\text{where } a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$$

$$\text{and } a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0, l)$. So the Fourier coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right] \\ &= \frac{2k}{l}\left[\left(\frac{x^2}{2}\right)_0^{l/2} + \left(lx - \frac{x^2}{2}\right)_{l/2}^{l}\right] \\ &= \frac{2k}{l}\left[\frac{l^2}{8} + \frac{l^2}{2} - \frac{3l^2}{8}\right] \\ &= \frac{2k}{l}\times\frac{l^2}{4} = \frac{kl}{2} \end{aligned}$$

$$a_n = \frac{2}{l}\left[\int_0^{l/2} kx\cos\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} k(l-x)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2k}{l}\left\{\left[(x)\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right]_0^{l/2}\right.$$
$$\left. + \left[(l-x)\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right) - (-1)\left(-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n^2\pi^2}{l^2}\right)}\right)\right]_{l/2}^{l}\right\}$$
$$= \frac{2k}{l}\left\{\left[\frac{lx}{n\pi}\sin\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_0^{l/2}\right.$$
$$\left. + \left[\frac{l(l-x)}{n\pi}\sin\left(\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_{l/2}^{l}\right\}$$
$$= \frac{2k}{l}\left\{\left[\frac{l^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2}\right]\right.$$
$$\left. + \left[-\frac{l^2}{n^2\pi^2}(-1)^n - \frac{l^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right]\right\}$$
$$= \frac{2k}{l} \times \frac{l^2}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right]$$
$$= \frac{2kl}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right]$$
$$\therefore f(x) = \frac{kl}{4} + \sum_{n=1}^{\infty}\frac{2kl}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right]\cos\left(\frac{n\pi x}{l}\right)\Bigg]$$
$$= \frac{kl}{4} + \frac{2kl}{\pi^2}\left[\frac{1}{2^2}(-2-1-1)\cos\left(\frac{2\pi x}{l}\right) + \frac{1}{6^2}(-2-1-1)\cos\left(\frac{6\pi x}{l}\right)\right.$$
$$\left. + \frac{1}{10^2}(-2-1-1)\cos\left(\frac{10\pi x}{l}\right)\cdots\right]$$
$$= \frac{kl}{4} - \frac{2kl}{\pi^2}\left[\cos\left(\frac{2\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{6\pi x}{l}\right) + \frac{1}{5^2}\cos\left(\frac{10\pi x}{l}\right) + \cdots\right]$$

Putting $x = \frac{l}{2}$, the series becomes $\frac{kl}{4} - \frac{2kl}{\pi^2}\left[-1 - \frac{1}{3^2} - \frac{1}{5^2} - \cdots\right]$

$$= \frac{kl}{4} + \frac{2kl}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Since $f(x)$ is continuous at $x = \frac{l}{2}$, the series converges to $f\left(\frac{l}{2}\right) = \frac{kl}{2}$

$$\therefore \frac{kl}{4} + \frac{2kl}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{kl}{2}$$
$$\frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

5.5.2 Half-range sine series of a function

Here also we are given a function $f(x)$ defined in the interval $[0, l]$ satisfying the Dirichlet conditions. Now we would like to reduce the Fourier series to a series having only sine terms and so we have to make $a_0 = 0$ and $a_n = 0$ for all n. Observe that if $f(x)$ were an odd function defined in $[-l, l]$, then $f(x)\cos\left(\frac{n\pi x}{l}\right)$ is an odd function and $f(x)\sin\left(\frac{n\pi x}{l}\right)$ is an even function so that

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = 0$$
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = 0$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$= \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

So if we redefine the function $f(x)$ which is defined in $[0, l]$ to an **odd function** $g(x)$ defined in $[-l, l]$ such that $g(x) = f(x), \forall x \in [0, l]$, then we get a Fourier series having sine terms alone in the following form

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \frac{f(x^-) + f(x^+)}{2}, \quad \forall x \in [0, l]$$

where the Fourier coefficient b_n is given by the formula

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

This is called half-range sine series of the function $f(x)$. Actually this is the Fourier series representation of the function $g(x)$, but we can replace $g(x)$ by $f(x)$ as they are identical in $[0, l]$.

Problem 5.5.5. *Find the half range Fourier sine series representation of $f(x) = k$ in $(0, \pi)$*

Solution: The Fourier sine series representation of $f(x)$ in $0 < x < \pi$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$
$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0,\pi)$. So the Fourier coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} k \sin nx \, dx \\ &= \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi} \\ &= -\frac{2}{n\pi}[(-1)^n - 1] = \frac{2}{n\pi}[1-(-1)^n] \end{aligned}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n] \sin nx$$

Problem 5.5.6. *Find the half range Fourier sine series representation of* $f(x) = x$ *in* $(0,2)$

Solution: The Fourier sine series representation of $f(x)$ in $0 < x < 2$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{2}{2}\int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0,2)$. So the Fourier coefficients are given by

$$\begin{aligned} &= \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx \\ &= \left[(x)\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right) - (1)\left(-\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n^2\pi^2}{4}\right)}\right)\right]_0^2 \\ &= \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 \\ &= -\frac{4}{n\pi}(-1)^n \end{aligned}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{2}\right)$$

Problem 5.5.7. *Find the half range Fourier sine series representation of* $f(x) = e^x$ *in* $(0,1)$

Solution: The Fourier sine series representation of $f(x)$ in $0 < x < 1$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x \, dx$$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0,1)$. So the Fourier coefficients are given by

$$\begin{aligned} &= 2\int_0^1 e^x \sin n\pi x \, dx \\ &= 2\left[\frac{e^x}{1+n^2\pi2}[(1)\sin n\pi x - (n\pi)\cos n\pi x]\right]_0^1 \\ &= \frac{2}{1+n^2\pi^2}\left\{e[0 - n\pi(-1)^n] - [(0 - n\pi]\right\} \\ &= \frac{2n\pi}{1+n^2\pi2}[1 - e(-1)^n] \end{aligned}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi2}[1 - e(-1)^n] \sin n\pi x$$

Problem 5.5.8. *If* $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$ *, show that*

$$\begin{aligned} f(x) &= \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \dots\right) \\ &= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \dots\right) \end{aligned}$$

Hence deduce that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi}{8}$.

Solution: The Fourier cosine series representation of $f(x)$ in $0 < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

Here the function $f(x)$ satisfies the Dirichlet conditions in $(0,\pi)$. So the Fourier coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{\pi}\left[\int_0^{\pi/2} x \, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx\right] \\ &= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right] \\ &= \frac{2}{\pi}\left[\left(\frac{\pi^2}{8} - 0\right) + \left(\frac{\pi^2}{2} - \frac{3\pi^2}{8}\right)\right] \\ &= \frac{2}{\pi} \times \frac{\pi^2}{4} \\ &= \frac{\pi}{2} \end{aligned}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$
$$= \frac{2}{\pi}\left\{\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi/2} + \left[\frac{(\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{\left[\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right] - \left[0 + \frac{1}{n^2}\right] + \left[0 - \frac{(-1)^n}{n^2}\right] - \left[\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right]\right\}$$
$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$
$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{2}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]\cos nx$$
$$= \frac{\pi^2}{4} - \frac{2}{\pi}\left(\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \cdots\right) \tag{1}$$

Putting $x = 0$, the series becomes $\frac{\pi^2}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

The series converges to $f(0+) = 0$.

$$\therefore \frac{\pi^2}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = 0$$
$$\Longrightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{8}$$

The Fourier sine series representation of $f(x)$ in $0 < x < \pi$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$
$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$
$$= \frac{2}{\pi}\left\{\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[-\frac{(\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{\left[-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right] - \left[0 + 0\right] + \left[0 - 0\right] - \left[-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]\right\}$$
$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$
$$= \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)$$
$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)\sin nx$$
$$= \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right)$$

5.6 Parseval's Identity

Consider the Fourier series representation of a function $f(x)$ defined in the interval $(-l, l)$ and satisfying the Dirichlet conditions. We have,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Note that

$$f(x)\times f(x) = f(x)\times\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)\right)$$
$$= \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty} a_n f(x)\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n f(x)\sin\left(\frac{n\pi x}{l}\right)$$

Now we integrate from $-l$ to l on both sides and assuming that the convergence is uniform so that summation and integration can be interchanged, we get

$$\int_{-l}^{l}|f(x)|^2dx = \frac{a_0}{2}\int_{-l}^{l} f(x)dx + \sum_{n=1}^{\infty} a_n\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \sum_{n=1}^{\infty} b_n\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

This is called **Parseval's identity**.

Remark:1 If the function $f(x)$ is defined in the interval $(0, 2l)$ the form of Parseval's identity is

$$\int_{-l}^{l} |f(x)|^2 dx = \frac{a_0}{2} \times a_0 l + \sum_{n=1}^{\infty} a_n \times a_n l + \sum_{n=1}^{\infty} b_n \times b_n l,$$

using Euler formula for Fourier coefficients.

$$\frac{1}{l}\int_{-l}^{l} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

This is called **Parseval's identity.**

Remark:1 If the function $f(x)$ is defined in the interval $(0, 2l)$, the form of Parseval's identity is

$$\frac{1}{l}\int_{0}^{2l} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Remark: 2 When the function is expanded as half-range cosine series in $(0, l)$, the form of the Parseval's identity is

$$\frac{2}{l}\int_{0}^{l} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

Remark: 3 When the function is expanded as half-range sine series in $(0, l)$, the form of the Parseval's identity is

$$\frac{2}{l}\int_{0}^{l} |f(x)|^2 dx = \sum_{n=1}^{\infty} b_n^2$$

Example 5.6.1. *Find the Fourier series representation of $f(x) = x^2$ in $[-\pi, \pi]$ and deduce that $1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$*

The Fourier series representation of $f(x)$ in $-\pi \le x \le \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Here $f(x) = x^2$ satisfies the Dirichlet conditions in $[-\pi, \pi]$ and is an even function. So the Fourier coefficients are given by

$$\begin{aligned} b_n &= 0 \\ a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2 \, dx \\ &= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \\ &= \frac{2}{\pi} \times \frac{\pi^3}{3} \\ &= \frac{2\pi^2}{3} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \, dx \\ &= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right] - [0 + 0 + 0] \\ &= \frac{4(-1)^n}{n^2} \end{aligned}$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Parsevals's Identity

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{4\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

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$$\frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^\pi = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4}$$
$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = \sum_{n=1}^{\infty}\frac{16}{n^4}$$
$$\frac{8\pi^4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$
$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

Example 5.6.2. *Find the Fourier series for* $f(x) = \begin{cases}\pi - x, & -\pi < x < 0\\ \pi + x, & 0 < x < \pi\end{cases}$ *and deduce that* $1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$

The Fourier series representation of $f(x)$ in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$

Here $f(x)$ satisfies the Dirichlet conditions in $(-\pi, \pi)$.

$$x \in (-\pi, 0) \implies -x \in (0, \pi)$$
$$\therefore f(x) = \pi - x \text{ and } f(-x) = \pi + (-x) = 1 - x$$
$$x \in (0, \pi) \implies -x \in (-\pi, 0)$$
$$\therefore f(x) = \pi + x \text{ and } f(-x) = \pi - (-x) = \pi + x$$

Hence $f(-x) = f(x)$

Therefore $f(x)$ is an even function. So the Fourier coefficients are given by

$$b_n = 0$$
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$
$$= \frac{2}{\pi}\int_0^{\pi}(\pi + x)dx$$
$$= \frac{2}{\pi}\left[\pi x + \frac{x^2}{2}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\pi^2 + \frac{\pi^2}{2}\right]$$
$$= 3\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi + x)\cos nx\ dx$$
$$= \frac{2}{\pi}\left[(\pi + x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{(\pi + x)\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$
$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$
$$\therefore f(x) = \frac{3\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}[(-1)^n - 1]\cos nx$$

Parsevals's Identity

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$
$$\frac{2}{\pi}\int_0^{\pi}(\pi + x)^2dx = \frac{9\pi^2}{2} + \sum_{n=1}^{\infty}\frac{4}{n^4\pi^2}[(-1)^n - 1]^2$$
$$\frac{14\pi^2}{3} - \frac{9\pi^2}{2} = \sum_{n=1}^{\infty}\frac{4}{n^4\pi^2}[(-1)^n - 1]^2$$
$$\frac{\pi^2}{6} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4}[(-1)^n - 1]^2$$
$$\frac{\pi^4}{24} = 4 + \frac{4}{3^4} + \frac{4}{5^4} + \cdots$$
$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

5.7 Exercise

1. Find the Fourier series representation of $f(x) = x^3$ in $-\pi < x < \pi$

   Solution: $f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^3}(3 - n^2\pi^2)\sin n\pi x$

2. Find the Fourier series representation of $f(x) = e^x$ in $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$

   Solution: $\frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2} - \frac{1}{1^2+1}\cos x + \frac{1}{2^2+1}\cos 2x - \frac{1}{3^2+1}\cos 3x + \cdots\right)\right.$ $\left.\left(\frac{1}{1^2+1}\sin x - \frac{2}{2^2+1}\sin 2x + \frac{3}{3^2+1}\sin 3x - \cdots\right)\right]$

3. Find the Fourier series expansion of $f(x) = \begin{cases}0, & -\pi < x < 0\\ \sin x, & 0 < x < \pi\end{cases}$ and show that

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots=\frac{\pi-2}{4}$$

4. Find the Fourier series representation of $f(x)=|\cos x|$ in $-\pi<x<\pi$

Solution: $f(x)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{1-n^2}\cos\left(\frac{n\pi}{2}\right)\cos n\pi x$

5. Find the Fourier series representation of $f(x)=x\sin x$ in $0<x<2\pi$

Solution: $f(x)=-1-\frac{1}{2}\cos x+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx+\pi\sin x$

6. Find the Fourier series for $f(x)=\begin{cases}1, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ -1, & \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$ and deduce that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}.$$

Solution: $\frac{\pi}{4}\left[\cos x-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}-\frac{\cos 7x}{7}+\cdots\right]$

7. Find the Fourier series for $f(x)=\begin{cases}0, & -2<x<-1\\ 1-x, & -1<x<0\\ 1+x, & 0<x<1\\ 0, & 1<x<2\end{cases}$

Solution: $\frac{1}{4}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\sin nx$

8. Find the Fourier series representation of $f(x)=e^{-x}$ in $-c<x<c$

9. Find the Fourier series series for

$$f(x)=\begin{cases}x, & 0\le x\le l\\ l-x, & l\le x<2l\end{cases}$$

10. Find the Fourier series series for

$$f(x)=\begin{cases}1, & 0\le x\le l/2\\ 0, & l/2\le x<l\end{cases}$$

Solution: $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n\pi}[1-(-1)^n]\sin nx$

11. Obtain the half range sine series for $f(x)=\sin x$ in $(0,\pi)$.

12. Obtain the half range Fourier cosine series representation of

$$f(x)=\begin{cases}1, & 0<x<\pi/2\\ 0, & \pi/2<x<\pi\end{cases}$$

Solution: $f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos nx$

13. Obtain the half range Fourier cosine series representation of

$$f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\end{cases}$$

14. Obtain the half range sine series for $f(x)=\pi x-x^2$ in $(0,\pi)$.

15. Find the Fourier series for $f(x)=\begin{cases}-k, & -\pi<x<0\\ k, & 0<x<\pi\end{cases}$ and deduce that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$