Module 1

Fundamentals of Logic

Dr.Binoy Balan K

AHALIA SCHOOL OF ENGINEERING & TECHNOLOGY

1.1 Mathematical logic

1.1 Basic connectives and truth tables

Statements (***propositions***): declarative sentences that are either ***true or false***--but not both.

Ex. Margaret Mitchell wrote *Gone with the Wind*.
2+3=5.

not statements:

What a beautiful morning!
Get up and do your exercises.

1.1 Basic connectives and truth tables

primitive and compound statements

combined from primitive statements by **logical connectives** or by **negation**

logical connectives:

(a) conjunction (AND): $p \wedge q$
(b) disjunction(inclusive OR): $p \vee q$
(c) exclusive or: $p \oplus q$
(d) implication: $p \rightarrow q$ (if p then q)
(e) biconditional: $p \leftrightarrow q$ (p if and only if q, or p iff q)

1.1 Basic connectives and truth tables

Truth Tables

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
F	F	F	F	F	T	T
F	T	F	T	T	T	F
T	F	F	T	T	F	F
T	T	T	T	F	T	T

1.1 Basic connectives and truth tables

Ex. 2.1 s: Phyllis goes out for a walk.
t: The moon is out.
u: It is snowing.

$(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.

If it is snowing and the moon is not out, then Phyllis will not go out for a walk. $(u \wedge \neg t) \rightarrow \neg s$

1.2 Statements, Logical Connectives, Tautology, Contradiction

Def. . A compound statement is called a ***tautology(T_0)*** if it is true for all truth value assignments for its component statements. If a compound statement is false for all such assignments, then it is called a ***contradiction(F_0)***.

$p \rightarrow (p \vee q)$: tautology

$p \wedge (\neg p \wedge q)$: contradiction

1.2 Logical Connectives:

Ex.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Def . logically equivalent

$s_1 \Leftrightarrow s_2$

1.2 Logical Connectives

Def. . If p, q are any arbitrary statements such that $p \rightarrow q$ is a tautology, then we say that *p logically implies q* and we write $p \Rightarrow q$ to denote this situation.

$p \Leftrightarrow q$ *means* $p \leftrightarrow q$ *is a tautology.*

$p \Rightarrow q$ *means* $p \rightarrow q$ *is a tautology.*

1.3 Logical Equivalence: The Laws of Logic

logically equivalent $(p \to q) \Leftrightarrow \neg p \vee q$

$$(p \leftrightarrow q) \Leftrightarrow (p \to q) \wedge (q \to p)$$

$$(p \leftrightarrow q) \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$$

$$(p \oplus q) \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$$

$$\Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$$

We can eliminate the connectives $\to$ and $\leftrightarrow$ from compound statements.

(and,or,not) is a complete set.

1.3 Logical Equivalence: The Laws of Logic

Ex *DeMorgan's Laws*

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

p and q can be any compound statements.

1.3 Logical Equivalence: The Laws of Logic

$(1) \neg\neg p \Leftrightarrow p$ — Law of *Double Negation*

$(2) \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ — *Demorgan's* Laws

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$(3) p \vee q \Leftrightarrow q \vee p$ — *Commutative* Laws

$p \wedge q \Leftrightarrow q \wedge p$

$(4) p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ — *Associative* Laws

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

1.3 Logical Equivalence: The Laws of Logic

$(5) p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ *Distributive* Law

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$(6) p \vee p \Leftrightarrow p, p \wedge p \Leftrightarrow p$ *Idempotent* Law

$(7) p \vee F_0 \Leftrightarrow p, p \wedge T_0 \Leftrightarrow p$ *Identity* Law

$(8) p \vee \neg p \Leftrightarrow T_0, p \wedge \neg p \Leftrightarrow F_0$ *Inverse* Law

$(9) p \vee T_0 \Leftrightarrow T_0, P \wedge F_0 \Leftrightarrow F_0$ *Domination* Law

$(10) p \vee (p \wedge q) \Leftrightarrow p, p \wedge (p \vee q) \Leftrightarrow p$ *Absorption* Law

1.4 The principle of duality, substitution rules

Def. Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the dual of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

Eg.

$$s: (p \wedge \neg q) \vee (r \wedge T_0), s^d : (p \vee \neg q) \wedge (r \vee F_0)$$

The dual of $p \to q$ is $(\neg p \vee q)^d \Leftrightarrow \neg p \wedge q$

1.4 The principle of duality, substitution rules

Theorem (*The Principle of Duality*) Let s and t be statements. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

First Substitution Rule (replace each *p* by another statement *q*)

Ex. 2.10 $P: \neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology. Replace each occurrence of p by $r \wedge s$

$P_1: \neg[(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$ is also a tautology.

1.4 The principle of duality, substitution rules

Second Substitution Rule

Ex. $P: (p \to q) \to r$ Then, $P \Leftrightarrow P_1$

$P_1: (\neg p \vee q) \to r$ because $(p \to q) \Leftrightarrow (\neg p \vee q)$

Ex. Negate and simplify the compound statement $(p \vee q) \to r$

$\neg[(p \vee q) \to r] \Leftrightarrow \neg[\neg(p \vee q) \vee r] \Leftrightarrow$

$\neg[(\neg p \wedge \neg q) \vee r] \Leftrightarrow \neg(\neg p \wedge \neg q) \wedge \neg r \Leftrightarrow$

$(p \vee q) \wedge \neg r$

1.5 The implication, The Contrapositive, the Converse , the Inverse

Ex. *contrapositive of* $p \to q$

p	q	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

converse

inverse

1.5 The implication, The Contrapositive, the Converse , the Inverse

p: Jeff is concerned about his cholesterol levels

q: Jeff walks at least two miles three times week

Ex. *The implication:* $p \to q$

$$\neg q \to \neg p$$

$$q \to p$$

$$\neg p \to \neg q$$

1.6 Logical Implication: Rules of Inference

an argument: $p_1, p_2, \cdots, p_n \Rightarrow q$ or

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Rightarrow q$$

premises

conclusion

is a valid argument iff

$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is a tautology

1.6 Logical Implication: Rules of Inference

Ex. **statements**:

p: Roger studies. q: Roger plays tennis.

r: Roger passes discrete mathematics.

premises: p_1: If Roger studies, then he will pass discrete math.

p_2: If Roger doesn't play tennis, then he'll study.

p_3: Roger failed discrete mathematics.

Determine whether the argument $p_1, p_2, p_3 \Rightarrow q$ is valid.

$$p_1 : p \to r, p_2 : \neg q \to p, p_3 : \neg r$$

$$\therefore (p_1 \wedge p_2 \wedge p_3) \to q \Leftrightarrow$$

$$\boxed{[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q}$$

which is a tautology, the original argument is true

1.6 Logical Implication: Rules of Inference

$$p, (p \wedge r) \to s) \Rightarrow (r \to s)$$

Ex.

p	r	s	$p \wedge r$	$(p \wedge r) \to s$	$r \to s$	$[p \wedge ((p \wedge r) \to s)] \to (r \to s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

a tautology ***deduced or inferred from*** *the two premises*

1.6 Logical Implication: Rules of Inference

rule of inference: use to *validate or invalidate* a logical implication without resorting to truth table (which will be prohibitively large if the number of variables are large)

1. Modus Ponens (the method of affirming) or the *Rule of Detachment*

$$p, (p \to q) \Rightarrow q$$

2. Law of the Syllogism

$$(p \to q), (q \to r) \Rightarrow (p \to r)$$

1.6 Logical Implication: Rules of Inference

3. *Modus Tollens (method of denying)*

$$p \rightarrow q, \neg q \Rightarrow \neg p$$

4.` *Rule of Conjunction*

$$p, q \Rightarrow p \wedge q$$

5.Rule of conjunctive simplification

$$p \wedge q \Longrightarrow p$$

$$p \wedge q \Longrightarrow q$$

1.6 Logical Implication: Rules of Inference

6. *Rule of disjunctive Amplification*

$$p \Longrightarrow p \vee q$$

$$q \Longrightarrow p \vee q$$

1.6 Logical Implication: Rules of Inference

7.Method of Proof by contradiction

Ex. Establish the validity of the arguments,~p<--> q, q→r, ~r ==> p

Sol: Assume the negation of the conclusion as another premise.

(1)	~p<-->q	Rule P
(2)	~p	Rule P, additional premise
(3)	q	Rule T, using (1) , (2), p, p→q ==>q
(4)	q→r	Rule P
(5)	r	Rule T, using (3), (4), p, p→q ==>q
(6)	~r	Rule P
(7)	$r \wedge \sim r$	

-contradiction (using p,q==> p^q)s

Hence by method of proof by contradiction, the argument is valid.

Rules of inference:

Rule P: A premises can be introduced at any step of derivation.

Rule T: A formula can be introduced provided it is Tautologically implied by previously introduced formulas in the derivation.

Rule CP: If the conclusion is of the form $r \to s$ then we include r as an additional premises and derive s.

Indirect method: We use negation of the conclusion as an additional premise and try to arrive a contradiction.

Inconsistent: A set of premises are inconsistent provided their conjunction implies a contradiction.

Example: Prove that $p \to q$, $p \to r$, $q \to \neg r$ and p are inconsistent.
Solution: The desired result is false.

Step	Derivation	Rule
1	p	P
2	$p \to q$	P
3	q	$\{1,2\}$, $\mathcal{I}_4$
4	$q \to \neg r$	P
5	$\neg r$	$\{3,4\}$, $\mathcal{I}_4$
6	$p \to r$	P
7	$\neg p$	$\{5,6\}$, $\mathcal{I}_6$
8	F	$\{1,7\}$,$\mathcal{I}_3$

III. Problems:

(i) Show that $r \vee s$ is tautologically implied by $c \vee d$, $(c \vee d) \rightarrow \neg h$, $\neg h \rightarrow (a \wedge \neg b)$ and $(a \wedge \neg b) \rightarrow (r \vee s)$.

(ii) Show that $r \wedge (p \vee q)$ is tautologically implied by $p \vee q$, $q \rightarrow r$, $p \rightarrow m$ and $\neg m$.

(iii) Show that $r \rightarrow s$ is tautologically implied by $\neg r \vee p$, $p \rightarrow (q \rightarrow s)$ and q.

(iv) Show that $p \rightarrow s$ is tautologically implied by $\neg p \vee q$, $\neg q \vee r$ and $r \rightarrow s$.

(v) Show that $p \to (q \to s)$ is tautologically implied by $p \to (q \to r)$ and $q \to (r \to s)$ using CP rule.

(vi) Show that the following premises are inconsistent.
$v \to l$, $l \to b$, $m \to \neg b$ and $v \wedge m$.

(vii) Show that $p \to \neg s$ logically follows from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$ and p by indirect method.

(viii) Show that r logically follows from the premises $p \to q$, $\neg q$ and $p \vee r$ by indirect method.

(i) Show that the following set of premises is inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books.

Let us consider,

E : Jack misses many classes through illness

S : Jack fails high school

A : Jack reads a lot of books

H : Jack is uneducated.

The premises are,

- $E \to S$
- $S \to H$
- $A \to \neg H$ and
- $E \wedge A$

1.7The use of Quantifiers, Open Statement, Quantifier, Negation

Def. A declarative sentence is an *open statement* if

(1) it contains one or more **variables**, and

(2) it is not a statement, but

(3) it becomes a statement when the variables in it are replaced by certain **allowable choices (Universe)**

1.7The Use of Quantifiers

existential quantifier: For some x: $\exists x$

universal quantifier: For all x: $\forall x$

x in $p(x)$: *free* variable

x in $(\exists x)p(x)$: ***bound*** variable $\longrightarrow$ $\exists x, p(x)$ is either true or false.

1.8 Logically Equivalent, Logical Implication

Def. Let p(x), q(x) be open statements defined for a given Universe. The open statements p(x) and q(x) are called *logically equivalent*, *and we write*

$\forall x[p(x) \Leftrightarrow q(x)]$ When the biconditional $p(a) \leftrightarrow q(a)$

is true for each replacement a from the universe. If the implication p(a) → *q(a) is true for each a in the universe, we write* $\forall x[p(x) \Rightarrow q(x)]$ *and say that p(x) logically implies q(x).*

1.8 Logically Equivalent, Contra positive, The Converse, The Inverse

Def. For open statements p(x), q(x) defined for a prescribed universe and the universally quantified statement

$$\forall x[p(x) \to q(x)]$$

We define: (1) The contrapositive of . $\forall x[p(x) \to q(x)]$

to be $\forall x[\neg q(x) \to \neg p(x)]$

(2) The converse of $\forall x[p(x) \to q(x)]$ to be

$$\forall x[q(x) \to p(x)]$$

1.8 Logically Equivalent, Contra positive, The Converse, The Inverse

(3) The inverse of . $\forall x[p(x) \rightarrow q(x)]$ to be

$$\forall x[\neg p(x) \rightarrow \neg q(x)]$$

1.8 The Use of Quantifiers

How do we negate quantified statements that involve a single variable?

$$\neg[\forall x p(x)] \Leftrightarrow \exists x \neg p(x)$$

$$\neg[\exists x p(x)] \Leftrightarrow \forall x \neg p(x)$$

$$\neg[\forall x \neg p(x)] \Leftrightarrow \exists x p(x)$$

$$\neg[\exists x \neg p(x)] \Leftrightarrow \forall x p(x)$$

1.8 The Use of Quantifiers

Symbolize the statement

(i) All the world loves a lover
(ii) All men are giant
(iii) Every apple is red
(iv) Everybody loves a lover
(v) Any integer is either positive or negative
(vi) x is the father of the mother of y.
(vii) For any x and y, if x is taller than y, then y is not taller than x

1.8 The Use of Quantifiers

Sol:

(i) P(x): x is a person, L(x); x is a lover, R(x, y): x loves y. The required expression is $(x)[P(x) \rightarrow (y)\{P(y) \wedge L(y) \rightarrow Q(x,y)\}]$

(ii) For all x, if x is a man then x is giant

(iii) For all x, x is an apple then x is red

(iv) For all x, if x is a person and x is a lover than x loves y

(v) For all x , If x is an integer then x is either positive or negtive

(vi) P(x): x is a person, M(x, y); x is the mother of y, F(x, y): x is the father of y

$(\exists z)\ (P(z) \wedge F(x, z) \wedge M(z,y))$

(i) G(x, y): x is taller than y

$(x)(y)[G(x, y) \rightarrow \sim G(y, x)]$

1.8 The Use of Quantifiers

(i) It is not true that all roads lead to Rome
(ii) For any given positive integer, there is a greater positive integer.
(iii) Some people are not admired by everyone.
(iv) Every book with a blue cover is a mathematics book
(v) Some people who trusts others are rewarded

1.8 The Use of Quantifiers

Rule US;(x) A(x)==>A(y)

Rule ES: $\exists x$ A(x) ==> A(y), provided y is not free in any given premise and also not free in any prior step of the derivations.

Rule EG:A(x) ==> $\exists y$ A(y)

Rule UG: A(x) ==> (y) A(y), provided that x is not free in any of the given premises and provided if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in A(x).

1.8 The Use of Quantifiers

1. Show that $\exists x$ M(x) follows logically from the premises (x) (H(x)→M(x)) and $\exists x$H(x).
2. Show that $\exists x$(P(x) ∧Q(x))==> $\exists x$ P(x) ∧ $\exists x$ Q(x)
3. Indicate the variables that are free and bound

(a) (x)(P(x) ∧ R(x))

(b) (x) P(x,y)

(c) $\exists x$ $\exists y$ Q(x,y)