CS201: DISCRETE COMPUTATIONAL STRUCTURES	**Semester III**
Module V	
Lecturer: Jestin Joy	*Class: CSE-B*

Syllabus: ***Propositional Logic***:*- Propositions – Logical connectives – Truth tables Tautologies and contradictions - Contra positive - Logical equivalences and implications*

Rules of inference: *Validity of arguments.*

Disclaimer: *These may be distributed outside this class only with the permission of the Instructor.*

Federal Institute of Science And Technology (FISAT)

Contents

5.1 Propositional Logic **1**

5.2 Logical connectives **1**

5.2.1 Negation 2

5.2.2 Conjunction 2

5.2.3 Disjunction 2

5.2.4 Statement formulas and truth tables 2

5.3 Tautologies and Contradiction **3**

5.4 Contrapositive **3**

5.5 Equivalence of formulas **3**

5.6 Rules of inference **4**

5.6.1 Validity of arguments 4

5.1 Propositional Logic

Definition 5.1 *A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.*

5.2 Logical connectives

Connectives are used to construct complicated statements from simple statements. Statements along with the connectives define an algebra that satisfies a set of properties.

5.2.1 Negation

Definition 5.2 *Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement*

"It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

P	$\neg P$
True	False
False	True

Table 5.1: Truth table

5.2.2 Conjunction

Definition 5.3 *The conjunction of two statements P and Q is the statement $P \wedge Q$ which is read as "P and Q". The statement $P \wedge Q$ has the truth value T whenever both P and Q have the truth value T; otherwise it has the truth value F.*

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

5.2.3 Disjunction

Definition 5.4 *The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P or Q". The statement $P \vee Q$ has the truth value F only when both P and Q have the truth value F; otherwise it has the truth value T.*

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

5.2.4 Statement formulas and truth tables

Those statements which contain one or more primary statements and some connectives are called molecular or composite or compound statements. Compound statements are statement formulas derived from statement variables. For example if P and Q are any two statements, then

$$(P \vee Q) \wedge Q$$

is a statement formula derived from the statement variables P and Q. A statement formula has no truth value. It is only when the statement variables are replaced by definite statements that we get a statement. To find truth value of a statement formula we need to construct truth table including values of all possible truth values of its constituent variables.

5.3 Tautologies and Contradiction

Definition 5.5 *A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a **universally valid formula** or a **tautology** or a **logical truth**.*

*A statement formula which is false regardless of the truth values of the statements which replace the variables in it is called a **contradiction**.*

*We may say that a statement formula which is a tautology is **identically true** and a formula which is a contradiction is **identically false**.*

5.4 Contrapositive

Definition 5.6 *For any statement formula $P \to Q$, the statement formula $Q \to P$ is called its **converse**, $\neg P \to \neg Q$ is called its **inverse**, and $\neg Q \to \neg P$ is called its contrapositive.*

Example: $P \to Q$: “If it rains, then the match is cancelled”

- Converse ($Q \to P$): “If the match is cancelled, then it rains”
- Inverse ($\neg P \to \neg Q$): “If it doesnt rain, then the match is not cancelled”
- Contrapositive ($\neg Q \to \neg P$): “If the match is not cancelled, then it doesnt rain”

5.5 Equivalence of formulas

Definition 5.7 *Let A and B be two statement formulas. If the truth values of A is equal to the truth values of B for truth values assigned to the variables occurring in A and B, then A and B are said to be equivalent.*

For example the statement formulas $\neg\neg P$ is equivalent to P

It is not necessary to assume that the two statement formulas contain same variables.

The biconditional, $P \rightleftarrows Q$ is true whenever both P and Q have the same truth values. That is if $P \rightleftarrows Q$ is a tautology, then P and Q are equivalent. It is written as $P \Leftrightarrow Q$ and read as “P is equivalent to Q”.

$P \vee P \Leftrightarrow P$	$P \wedge P \Leftrightarrow P$	(Idempotent laws)	(1)
$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	(Associative laws)	(2)
$P \vee Q \Leftrightarrow Q \vee P$	$P \wedge Q \Leftrightarrow Q \wedge P$	(Commutative laws)	(3)
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	(Distributive laws)	(4)
$P \vee \mathbf{F} \Leftrightarrow P$	$P \wedge \mathbf{T} \Leftrightarrow P$		(5)
$P \vee \mathbf{T} \Leftrightarrow \mathbf{T}$	$P \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$		(6)
$P \vee \neg P \Leftrightarrow \mathbf{T}$	$P \wedge \neg P \Leftrightarrow \mathbf{F}$		(7)
$P \vee (P \wedge Q) \Leftrightarrow P$	$P \wedge (P \vee Q) \Leftrightarrow P$	(Absorption laws)	(8)
$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	(De Morgan's laws)	(9)

Figure 5.1: Equivalent Formulas

A statement A is said to tautologically imply a statement B if and only if $A \to B$ is a tautology. This is denoted by $A \Rightarrow B$ and is read as “A implies B”.

$$P \wedge Q \Rightarrow P \tag{1}$$
$$P \wedge Q \Rightarrow Q \tag{2}$$
$$P \Rightarrow P \vee Q \tag{3}$$
$$\neg P \Rightarrow P \rightarrow Q \tag{4}$$
$$Q \Rightarrow P \rightarrow Q \tag{5}$$
$$\neg (P \rightarrow Q) \Rightarrow P \tag{6}$$
$$\neg (P \rightarrow Q) \Rightarrow \neg Q \tag{7}$$
$$P \wedge (P \rightarrow Q) \Rightarrow Q \tag{8}$$
$$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P \tag{9}$$
$$\neg P \wedge (P \vee Q) \Rightarrow Q \tag{10}$$
$$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow P \rightarrow R \tag{11}$$
$$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Rightarrow R \tag{12}$$

Figure 5.2: Implications

5.6 Rules of inference

Rules are stated in terms of the statements (premises and conclusions) involved rather than in terms of the actual statements or their truth values.

5.6.1 Validity of arguments

In logic we concentrate our attention on the study of the rules of inference by which conclusions are derived from premises. Any conclusion which is arrived at by following these rules is called a valid conclusion and the argument is called a **valid argument**.

An invalid argument example

“All actors are robots
Tom Cruise is a robot
So Tom Cruise is an actor”

Here the conclusion can be false since there are robots in some other profession, which can make Tom Cruise not an actor. So here the conclusion can be false.

The truth of the premises **does not guarantee** the truth of the conclusion.

Validity and invalidity apply only to arguments, not statements.

An valid argument example

“All actors are robots
Tom Cruise is an actor
So Tom Cruise is a robot”

Valid argument says that if all the premsies were true then the conclusion must be true. Here the truth of premises **gurantee** truth of conclusion.

I_1	$P \wedge Q \Rightarrow P$	(simplification)
I_2	$P \wedge Q \Rightarrow Q$	(simplification)
I_3	$P \Rightarrow P \vee Q$	(addition)
I_4	$Q \Rightarrow P \vee Q$	(addition)
I_5	$\neg P \Rightarrow P \rightarrow Q$	
I_6	$Q \Rightarrow P \rightarrow Q$	
I_7	$\neg(P \rightarrow Q) \Rightarrow P$	
I_8	$\neg(P \rightarrow Q) \Rightarrow \neg Q$	
I_9	$P, Q \Rightarrow P \wedge Q$	
I_{10}	$\neg P, P \vee Q \Rightarrow Q$	(disjunctive syllogism)
I_{11}	$P, P \rightarrow Q \Rightarrow Q$	(modus ponens)
I_{12}	$\neg Q, P \rightarrow Q \Rightarrow \neg P$	(modus tollens)
I_{13}	$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$	(hypothetical syllogism
I_{14}	$P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$	(dilemma)

Figure 5.3: Implications

E_1	$\neg\neg P \Leftrightarrow P$	(double negation)
E_2	$P \wedge Q \Leftrightarrow Q \wedge P$	(commutative laws)
E_3	$P \vee Q \Leftrightarrow Q \vee P$	(commutative laws)
E_4	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	(associative laws)
E_5	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	(associative laws)
E_6	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	(distributive laws)
E_7	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	(distributive laws)
E_8	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	(De Morgan's laws)
E_9	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	(De Morgan's laws)
E_{10}	$P \vee P \Leftrightarrow P$	
E_{11}	$P \wedge P \Leftrightarrow P$	
E_{12}	$R \vee (P \wedge \neg P) \Leftrightarrow R$	
E_{13}	$R \wedge (P \vee \neg P) \Leftrightarrow R$	
E_{14}	$R \vee (P \vee \neg P) \Leftrightarrow \mathbf{T}$	
E_{15}	$R \wedge (P \wedge \neg P) \Leftrightarrow \mathbf{F}$	
E_{16}	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$	
E_{17}	$\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$	
E_{18}	$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$	
E_{19}	$P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$	
E_{20}	$\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$	
E_{21}	$P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$	
E_{22}	$(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$	

Figure 5.4: Equivalences