# Module 3

# Relations and Functions

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# Cartesian Product, Binary Relation

For sets A, B the **Cartesian product, or cross product**, of A and B is denoted by $A \times B$ and equals $\{(a, b) / a \in A, b \in B\}$
Ex. Let A= {2, 3, 4} B={4, 5}.
Find (i) $A \times B$ (ii) $B \times B$ (iii) $B^3$

For sets A, B, any subset of $A \times B$ is called a (binary) relation from A to B. Any subset of $A \times A$ is called a (binary) relation on A.
For finite sets A, B with $|A| = m$ and $|B| = n$, there are $2^{mn}$ relations from A to B

# Cartesian Product, Binary Relation

Ex. Let A= {a, b, c, d} and B={w, x, y}(a) Give examples of three nonempty relations from A to B (b) and three examples of nonempty relations on A.

Sol:R1= {(a, w), (d, x)}

R2= {(c, x)}

R3= {(d,w), (d, x), (d,y)}

R4={(w, c), (y, w)}- not a relation from A to B because R4 is not subset of AXB 2^12 relations=

(b) R5={(b,b), (c, b), (d, a)} A= {a, b, c, d}

A= {a, b, c, d}

Ex. Let A , B be sets with |B|=3. If there are 4096 relations from A to B, what is |A|?

N(AXB)=mn=3m

Number of reltions= 2^3m

4096= 2^3m ==> 2^3

# Reflexive Relations, Symmetric Relations, Transitive relations, Antisymmetric Relations

1. Let R and S be two relations on a set A . If R and S are symmetric, Prove that $(R \cap S)$ is also symmetric.

# Equivalence Relation, Irreflexive Relations.

# Equivalence Relations and Partitions ,Equivalence Class

# Equivalence Relations and Partitions ,Equivalence Class

1. Let X = {1, 2 ......7 }and R = {< X,Y> / X-Y is divisible by 3}. Show that R is an equivalence relation. Draw the graph R.
2. Let Z be the set of integers and R be the relation called congruence modulo 3 defined by R = {<x,y>| x and y are elements in Z and (x-y) is divisible by 3}.Determine the equivalence classes generated by the elements of Z.

# Partial Order relations

# Partially ordered Set, Hasse Diagram

Def : If (A,R) is a poset, we say that A is totally ordered if for all x, y€ A either xRy or yRx. In this case R is called a total order.

1. Draw the Hasse diagram for the following sets under the partial ordering relation “Divides”, and indicates those which are totally ordered.{2,6,24}, {1,2,3,6,12}, {2,4,8,16}, {3,9,27,54}
2. Show that the divisibility relation ' / ' is a partial ordering on the set Z+.
3. Draw the Hasse diagram for the set A= {2, 3, 4,6,12,18,24,36} with partial order of divisibility.
4. Let A be the set of factors of a particular positive integer m and let <= be the relation divides, ie relation <= be such that x<=y if x divides y. Draw the Hasse diagrams for m= 30 and m= 45.

# Partially ordered Set, Hasse Diagram

1. Draw the Hasse diagram for the following sets under the partial ordering relation, with A= {1, 2, 3} and R is the subset relation on P(A), power set of A.

# Maximal-Minimal Element, Least Upper bound, Greatest Lower Bound

Def: If (A, R) is a poset, then an element x€A is called a maximal element of A if for all a€A, a≠x==>x$\cancel{R}$a. An element y€A is called a minimal element of A if whenever b€A and b≠y then b$\cancel{R}$a.

Def: If (A, R) is a poset, then an element x€A is called a least element if xRa for all a€A. Element y€A is called a greatest element if aRy for all a€A.

Def: If (A, R) be a poset with B subset of A . An elemet x€A is called a lower bound of B if xRb for all b €B. Likewise, an element y€A is called an upperbound of B if bRy for all b €B.

# Maximal-Minimal Element, Least Upper bound, Greatest Lower Bound

Ex. Let U= {1,2,3} with A = P(U), and let R be the subset relation on A. For each of the following subset B of A, determine the lub and glb of B

(a) B= {{1}, {2}, {3}, {1,2}}

(b) B={{1}, [1,2}, {1,3}, {1, 2,3}}

# Lattice

Def. A lattice is a partially ordered set (A, R) in which for every pair of element a, b €A, the least upper bound LUB{a, b}and the greatest lower bound GLB{a,b} both exists in A. We denote LUB{a, b} by $a \vee b$ or $a \oplus b$ and call it the join or sum of a and b. Similarly we denote GLB {a, b} by $a \wedge b$ or $a * b$ and call it the meet or product of a and b

# Lattice- Dual Lattice, sub lattice

Ex. Show that $(D_{35},|)$ and $(D_{42},|)$ are lattices, where $D_n$ is the set of all positive divisors of n.

Ex. A= {2, 3, 4,6,12,18,24,36} with partial order of divisibility. Determine the POSET is a lattice.

Def: If $[A, \vee,\wedge]$ is a lattice then $[A, \wedge,\vee]$ is known as dual or reversed lattice, which is obtained by interchanging glb and lub.

Def. Let (A, R) be a lattice. A nonempty subset B of A is called a sublattice of A if for any a, b€B, $a\vee b$ and $a\wedge b$ €B

Ex. Find all sublattices of $D_{24}$ that contains atleast five elements.

Sol. $B_1$ = {1, 2, 3,6, 12}, $B_2$ = {1, 2, 3, 6, 12, 24}, $B_3$ = {1, 2, 6, 12, 24}, $B_4$ = {1, 3, 6, 12, 24}are sublattices.

$B_5$ = {1, 2, 3, 4, 6, 8, 12} is not sublattice.

# Lattic-Properties of glb and lub

Ex. Show that $(D_{35},|)$ and $(D_{42},|)$ are lattices, where $D_n$ is the set of all positive divisors of n.

Def: If $[A, \vee, \wedge]$ is a lattice then $[A, \wedge, \vee]$ is known as dual or reversed lattice, which is obtained by interchanging glb and lub.

# Properties of Lattice

1. $GLB(x, y) \leq x$ and $GLB(x, y) \leq y$
2. $a \leq x$ and $a \leq y \Longrightarrow a \leq GLB(x, y)$
3. $x \leq LUB(x, y)$ and $y \leq LUB(x, y)$
4. $x \leq b$ and $y \leq b \Longrightarrow LUB(x, y) \leq b$

Theorem: Let $(A, *, +)$ be a Lattice. For any $x, y \in A$, then

1.Idempotent law   $x+x=x$, $x*x=x$

2. Commutative law, $x+y=y+x$, $x*y=y*x$

3. Associative law, $x+(y+z)= (x+y)+z$, $x*(y*z)= (x*y)*z$

4. Absorption law, $x+(x*y)= x$, $x*(x+y)=x$

Proof: refer text book

# Special Lattice

Def: A lattice is called **complete** if each of its nonempty subsets have a lub and glb.

Ex. $(D_{20}, |)$ is a complete lattice.

Ex. Infinite lattice $(Z^+ , \leq)$ is not complete, because infinite subset consisting even positive integers has no lub.

Def: A lattice is said to be **bounded** if it has a greatest element (denoted as 1) and least element (denoted as 0).

Ex. The lattice $(P(A), \leq)$ is bounded.

Ex. The Infinite lattice $(Z^+ , \leq)$ is not bounded.

Def: For a bounded lattice an element b is said to be complement of a if $a*b=0$ and $a+b=1$. Complement of a is denoted by a'.

# Special Lattice

Def: A lattice is said to be **complemented lattice** if every element has atleast one complement.

Ex. $(D_{20}, |)$ is not a complemented lattice, because 2 and 10 not have complements.

Ex. Find the complement of each element in $D_{42}$

Def: A lattice (A, *, +) is said to be a **distributive lattice** if for any a, b, c € A

a+(b*c)= (a+b)*(a+c), + is distributive over *

a*(b+c)= (a*b)+(a*c), * is distributive over +

Ex. Let A= {1, 2, 3, 5, 30}. (a) Show that (A, |} is a lattice. (b) prove that * is not distributive over + (c) prove that + is not distributive over *.

Sol: a=2, b=3, c=5

# Special Lattice

Ex. Show that the lattice (A, |), where A={1, 2,3, 4, 12} is not distributive.

Ex. Find the complements of lattice A= {1,2,3,4, 12}

# Reachability Relations

Ex. Let A = {a, b, c, d, e}and R= {(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)}. Compute (a) $R^2$ (b) $R^\infty$

Sol: $R^2 = R \circ R$={(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)}

$R^\infty$ = {(a, a), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c,d), (c, e ), (d, e)}.We need all ordered pairs of vertices for which there is a path of any length from the first vertex to the second.

# Reachability Relations

Def: If n is a fixed positive integer, we define a relation $R^n$ on A , by letting $xR^n y$ mean that there is a path of length n from x to y. We may also define a relation $R^\infty$ on A, by letting $x R^\infty y$ mean that there is some path in R from x to y. The relation $R^\infty$ is sometimes called the **connectivity relation** for R.

Def: The **reachability relation** R* of a relation R on a set A that has n elements is defined as follows: xR*y means that $x=y$ or $x R^\infty y$. The idea is that y is reachable from x if either y is x or there is some path from x to y.

# Function, Domain, Range , One to One Function Image - Restriction

Def: For nonempty sets A, B, **a function, or mapping**, f from A to B, denoted f:A→B, is a relation from A to B in which every element of A has (appears exactly once as the first component of an ordered pair in the relation) unique image in B.

Def: For the function f:A→B, A is called the **domain** of f and B the **codomain** of f. The subset of B consisting of those elements that appears second components in the ordered pairs of f is called the **range** of f and is also denoted by f(A) because it is the set of images under f.

# Function, Domain, Range , One to One Function Image - Restriction

Def: A function $f:A \rightarrow B$ is called **one-to-one, or injective**, if each element of B appears at most once as the image of an element of A.

(ie) for all $a_1, a_2 \in A$, $f(a_1)= f(a_2) \implies a_1=a_2$

Def: If $f:A \rightarrow B$ and $A_1 \subseteq A$, then $f(A_1)= \{b \in B \mid b = f(a)$, for some $a \in A_1\}$ and $f(A_1)$ is called the **image** of $A_1$ under f.

Def: If $f:A \rightarrow B$ and $A_1 \subseteq A$, then $f|A_1: A_1 \rightarrow B$ is called the restriction of f to $A_1$ if $f|A_1(a) = f(a)$ for all $a \in A_1$.

Ex. For $A= \{1, 2, 3, 4, 5\}$.Let $f:A \rightarrow R$ be defined by $f = \{(1, 10), (2, 13), (3, 16), (4, 19), (5, 22)\}$. Let $g: Q \rightarrow R$, where $g(q)= 3q+7$ for all $q \in Q$. Here f is the restriction of g (from Q) toA.

# Function, Domain, Range , One to One Function Image - Restriction

Ex. Determine which of the following functions are one-to-one and find its range.

(i) f:Z→Z, f(x)=2x (ii) f:Q→ Q, f(x)= 2x (iii) f:R→R, f(x)= $e^{x^2}$

(iv) f:R→ R, f(x)= cos x.

Sol: (i) yes, one-to-one, Range is set of integers.

(ii) yes, one-to-one, Range is Q.

(iii) Not one-to-one, range= [0, ∞]

(iv) Not one-to-one, range=[-1, 1]

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