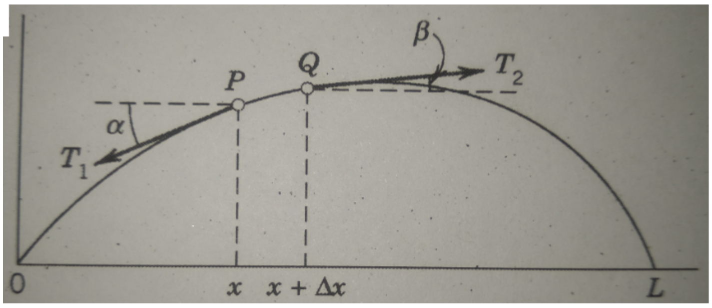
MODULE 2

Chapter 2

Applications of Partial Differential Equations

2.1 Derivation of Wave Equation



Consider a tightly stretched elastic string of length l kept along the $x-$ axis with one end at the origin $O(0,0)$ and the other end at $A(l,0)$. Let the string be allowed to vibrate in a direction normal to its length. Such vibrations are called **transverse vibrations**. Our aim is to analyze the position of a general point P at a a distance x from its origin after time t. We shall find the displacement y as a function of the distance x and the time t.

For the derivation of the P.D.E. describing the transverse vibration of a string, we make the following assumptions:

1. The string is made up of homogeneous material so that mass per unit length is constant.
2. The motion takes place entirely in the xy- plane and each particle of the string moves perpendicular to the equilibrium position OA of the string.
3. The string is perfectly flexible and does not offer resistance to bending.
4. The tension in the string is so large that the force due to the weight of the string can be neglected.

Let m be the mass per unit length of the string. Consider a small segment PQ of the string with end points $P(x,y)$ and $Q(x+\Delta x, y+\Delta y)$. Consider the motion of element PQ of length Δx. Since the string do not offer resistance to bending, the tensions T_1 and T_2 at P and Q are tangential to the curve.

Since there is no motion in the horizontal direction,

$$T_2 \cos\beta = T_1 \cos\alpha = T \qquad ------(1)$$

Mass of the element $PQ = m\Delta x$. By Newton's second law of motion, the equation of motion in the horizontal direction is

$$m\Delta x\frac{\partial^2 y}{\partial t^2} = T_2\sin\beta - T_1\sin\alpha$$
$$\frac{m\Delta x}{T}\frac{\partial^2 y}{\partial t^2} = \frac{T_2\sin\beta}{T_2\cos\beta} - \frac{T_1\sin\alpha}{T_1\cos\alpha} \text{ by using (1)}$$
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\Delta x}(\tan\beta - \tan\alpha)$$
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\Delta x}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x}$$

Taking the limit as $\Delta x \to 0$ on both sides,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\lim_{\Delta x\to 0}\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

or

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

where $c^2 = \frac{T}{m}$.

This is the P.D.E. giving the transverse vibration of the string. It is called one dimensional wave equation.

Initial Conditions:

If the string is initially held in the form of of a curve $y = f(x)$ and then released from this position, the initial conditions of vibrations will be

1. $y = f(x)$ at $t = 0$.
2. $\frac{\partial y}{\partial t} = 0$ at $t = 0$

Boundary Conditions:

At the end points O and L there is no motion. Therefore the boundary conditions are $y = 0$ at $x = 0$ and $y = 0$ at $x = l$. These conditions must hold for all values of t. [i.e; $t \geq 0$]

2.1.1 Solution of Wave Equation

Wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - - - - - (1)$$

Let

$$y = XT - - - - - (2)$$

where X is a function of x alone and T is a function of t alone be the solution of (1). Then $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$. Substitute these values in (1),

$$XT'' = c^2 X''T$$

Separating the variables,

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \text{ ——(3)}$$

L.H.S. of (3) is a function of x alone and R.H.S. is a function of t alone. Since x and t are independent variables, this equation can hold only when both reduces to a constant k. Then equation (3) leads to the ordinary differential equations

$$X'' - kX = 0, T'' - kc^2T = 0$$

Three possibilities arises, namely when k is positive, k is negative and $k = 0$.

Case I:

When k is positive, say $k = p^2$. Then $X'' - p^2X = 0$.

$$\frac{d^2X}{dx^2} - p^2X = 0$$
$$(D^2 - p^2)X = 0$$

A.E. is

$$D^2 - p^2 = 0 \implies D = \pm p$$

Therefore $X = c_1e^{px} + c_2e^{-px}$.

Now $T'' - p^2c^2T = 0$.

$$\frac{d^2T}{dt^2} - p^2c^2T = 0$$
$$(D^2 - p^2c^2)T = 0$$

A.E. is

$$D^2 - p^2c^2 = 0 \implies D = \pm pc$$

Therefore $T = c_3e^{pct}t + c_4e^{-pct}$.

Case II:

When $k = -p^2$, then $X'' + p^2X = 0 \implies (D^2 + p^2)X = 0$.

A.E. is

$$D^2 + p^2 = 0 \implies D = \pm ip$$

Therefore $X = c_1 \cos px + c_2 \sin px$

Now $T'' + p^2c^2T = 0$.

$$\frac{d^2T}{dt^2} + p^2c^2T = 0$$
$$(D^2 + p^2c^2)T = 0$$

A.E. is

$$D^2 + p^2c^2 = 0 \implies D = \pm ipc$$

Therefore $T = c_3 \cos pct + c_2 \sin pct$.

Case III:

When $k = 0$, $X'' = 0 \implies \dfrac{d^2X}{dx^2} = 0$.

Integrating w.r.t. x, we get $\dfrac{dX}{dx} = c_1$. Again integrating w.r.t. x, we get $X = c_1x + c_2$.

Similarly $T'' = 0 \implies \dfrac{d^2T}{dt^2} = 0$.

Integrating w.r.t. t, we get $\dfrac{dT}{dt} = c_3$. Again integrating w.r.t. t, we get $T = c_3t + c_4$.

Thus the various possible solutions of wave equations are

1. $y = (c_1e^{px} + c_2e^{-px})(c_3e^{pct} + c_4e^{-pct})$
2. $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$
3. $y = (c_1x + c_2)(c_3t + c_4)$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problems of vibrations, y must be a periodic function of x w.r.t. x and t. Therefore the solution must involve trigonometric series.

Accordingly, $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$ is the only suitable solution of the wave equation and it corresponds to $k = -p^2$.

Now applying the boundary condition that $y = 0$ when $x = 0$ and $y = 0$ when $x = l$,

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct) \text{ ——(1)}$$
$$0 = c_1(c_3 \cos pct + c_2 \sin pct) \implies c_1 = 0$$

Therefore equation (1) reduces to

$$y = c_2 \sin px(c_3 \cos pct + c_4 \sin pct) \text{ ——(2)}$$

Applying $y = 0, x = l$, second boundary condition in (2),

$$0 = c_2 \sin pl(c_3 \cos pct + c_4 \sin pct) \implies \sin pl = 0 \implies pl = n\pi, p = \frac{n\pi}{l}, n = 1, 2, ...$$

Therefore solution of wave equation satisfying the boundary condition is

$$y = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right)$$
$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \text{ ——(3)}$$

on replacing c_2c_3 by a_n and c_2c_4 by b_n. This equation satisfies the given condition for all integral values of n. Adding up the solution for different values of n, we get

$$y = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \text{ ——(4)}$$

Applying the initial condition $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \text{ —— } (5)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-a_n \frac{n\pi c}{l} \sin \frac{n\pi ct}{l} + b_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

Since $\frac{\partial y}{\partial t} = 0$ at $t = 0$,

$$0 = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l} \implies b_n = 0 \text{ for all } n.$$

Therefore equation (5) represents the Fourier series for $f(x)$ where

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \text{ ——} (7)$$

Then equation (4) reduces to $y = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$ where a_n is given by equation (7) when $f(x)$ is known.

NOTE:

$\int_0^l \sin mx \sin nx dx = 0, m \neq n$

$\int_0^l \sin^2 nx dx = \frac{l}{2}, m = n$

PROBLEMS:

1. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from the end at time t is given by $y(x,t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$.

   Ans: As the end points of the string are fixed for all time, $y(0,t) = 0, y(l,t) = 0$. These are the boundary conditions. Since the initial transverse velocity of any point of the string is zero, $\frac{\partial y}{\partial t} = 0$ at $t = 0$ and also $y(x,0) = a \sin \frac{\pi x}{l}$. Then the equation of wave motion is

   $$y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ ——-} (1)$$

   where

   $$\begin{aligned} a_n &= \frac{2}{l} \int_0^l y(x,0) \sin \frac{n\pi x}{l} dx \\ &= \frac{2}{l} \int_0^l a \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx \\ &= \frac{2a}{l} \int_0^l \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx \end{aligned}$$

   which vanishes for all values of n except when $n = 1$. So when $n = 1$, $a_1 = \frac{2a}{l} \frac{l}{2} = a$. Substituting in equation (1), the required solution is

   $$y(x,t) = a_1 \cos \frac{\pi ct}{l} \sin \frac{\pi x}{l} = a \cos \frac{\pi ct}{l} \sin \frac{\pi x}{l}.$$

2. Solve the equation $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$ under the conditions $y = 0$ when $x = 0$, $y = 0$ when $x = \pi$, $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$ and $y(x, 0) = x, 0 < x < \pi$.
Ans: The boundary conditions and initial conditions are given. Therefore the equation of motion is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} \text{ ——-(1)}$$
$$= \sum_{n=1}^{\infty} a_n \cos\frac{n\pi at}{\pi} \sin\frac{n\pi x}{\pi} \text{ since } c^2 = a^2, l = \pi$$

Therefore

$$y = \sum_{n=1}^{\infty} a_n \cos nat \sin nx \text{ ——(2)}$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\sin\frac{n\pi x}{\pi}dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx dx$$
$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n\right] = \frac{2}{n}(-1)^{n+1}$$

Therefore equation (2) becomes

$$y = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos nat \sin nx$$

3. Find the deflection $y(x, t)$ of the vibrating string of length π and ends fixed, corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$, given $c^2 = 1$.
Ans: The boundary conditions are $y = 0$ when $x = 0$, $y = 0$ when $x = \pi$. The initial conditions are $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$ and $f(x) = k(\sin x - \sin 2x)$ when $t = 0$.
Therefore equation of motion is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$
$$= \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{\pi} \sin\frac{n\pi x}{\pi}, \text{ since } c^2 = 1, l = \pi$$

So

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos nt \sin nx \text{ ——(1)}$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin\frac{n\pi x}{\pi}dx$$
$$= \frac{2k}{\pi}\int_0^\pi(\sin x - \sin 2x)\sin nx$$

which vanishes for all values of n except for $n = 1$ and $n = 2$.

$a_1 = \frac{2k}{\pi}\left[\int_0^\pi \sin^2 x dx - \int_0^\pi \sin 2x \sin x dx\right] = \frac{2k}{\pi}\frac{\pi}{2} - 0 = k$

$a_2 = \frac{2k}{\pi}\left[\int_0^\pi \sin 2x \sin x dx - \int_0^\pi \sin^2 x dx\right] = \frac{-2k}{\pi}\frac{\pi}{2} = -k$

Therefore equation (1) becomes

$$y = a_1\cos t\sin x + a_2\cos 2t\sin 2x$$
$$= k[\cos t\sin x - \cos 2t\sin 2x]$$

4. A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point x of the string at any time t.

Ans: The boundary conditions are $y = 0$ when $x = 0$, $y = 0$ when $x = l$. The initial conditions are $y = k(lx - x^2)$ when $t = 0$ and $\frac{\partial y}{\partial t} = 0$ when $t = 0$.

The wave equation is

$$y = \sum_{n=1}^\infty a_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \text{ ---(1)}$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$
$$= \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$
$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + \left((-2)\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]$$
$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}((-1)^n - 1)\right]$$
$$= \frac{4kl^2}{n^3\pi^3}(1 - (-1)^n)$$
$$= 0, \text{ when } n \text{ is even}$$
$$= \frac{8kl^2}{n^3\pi^3}, \text{ when } n \text{ is odd}$$

Therefore equation (1) becomes

$$y = \frac{8kl^2}{\pi^3}\sum_{n=1,3,5,...}\frac{1}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$
$$= \frac{8kl^2}{\pi^3}\sum_{m=1}^\infty\frac{1}{(2m-1)^3}\cos\frac{(2m-1)\pi ct}{l}\sin\frac{(2m-1)\pi x}{l}$$

where $n = (2m - 1)$.

NOTE: When $n = 1$, $2m - 1 = 1 \implies m = 1$; $n = 3 \implies 2m - 1 = 3 \implies m = 2$ and so on. So when $n = 2m - 1$, m takes values from 1 to ∞.

5. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in equilibrium position. If it is set vibrating by giving to each of its points with a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end point at any time t.

Ans: The boundary conditions are $y = 0$ at $x = 0$, $y = 0$ when $x = l$. After applying boundary conditions, the equation of motion is

$$y(x,t) = \sum \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \text{ ——(1)}$$

Since the string was at rest initially, $y = 0$ at $t = 0$. So equation (1) becomes

$$0 = \sum a_n \sin \frac{n\pi x}{l} \implies a_n = 0$$

Therefore

$$y(x,t) = \sum b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ ——(2)}$$
$$\frac{\partial y}{\partial t} = \sum b_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

Given $\frac{\partial y}{\partial t} = \lambda x(l - x)$ at $t = 0$. (This value is true for all values of t). Then

$$\lambda x(l - x) = \sum b_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$
$$b_n \frac{n\pi c}{l} = \frac{2\lambda}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx$$
$$= \frac{2\lambda}{l} \left[(lx - x^2) \left(\frac{-cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (l - 2x) \left(\frac{-sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + (-2) \left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) \right]_0^l$$
$$= \frac{2\lambda}{l} \left[\frac{-2l^3}{n^3\pi^3} ((-1)^n - 1) \right]$$
$$= \begin{cases} \frac{8\lambda l^2}{n^3\pi^3} &, \ when \quad n \quad is \quad odd \\ 0 &, \ when \quad n \quad is \quad even \end{cases}$$

Therefore

$$b_n = \frac{8\lambda l^2}{n^3\pi^3} \frac{l}{n\pi c} = \frac{8\lambda l^3}{n^4\pi^4 c}, n \text{ odd}$$

Then equation (2) becomes

$$y = \frac{8\lambda l^3}{\pi^4 c} \sum_{n=1,3,5,\ldots} \frac{1}{n^4} \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$y = \frac{8\lambda l^3}{\pi^4 c} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin \frac{(2m-1)\pi ct}{l} \sin \frac{(2m-1)\pi x}{l}, n = 2m - 1$$

6. A string of length l is initially at rest in equilibrium position and each of its points is given by the velocity $\frac{\partial y}{\partial t}_{t=0} = V_0 \sin^3 \frac{\pi x}{l}$. Find the displacement $y(x,t)$.
Ans: The boundary conditions are $y = 0$ at $x = 0$, $y = 0$ when $x = l$. Equation of motion is

$$y(x,t) = \sum \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \text{ ——(1)}$$

Since the string was at rest initially, $y = 0$ at $t = 0$. So equation (1) becomes

$$0 = \sum a_n \sin \frac{n\pi x}{l} \implies a_n = 0$$

Therefore

$$y(x,t) = \sum b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ ——(2)}$$
$$\frac{\partial y}{\partial t} = \sum b_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

Given $\frac{\partial y}{\partial t} = V_0 \sin^3 \frac{\pi x}{l}$ at $t = 0$. Then

$$V_0 \sin^3 \frac{\pi x}{l} = \sum b_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$
$$b_n \frac{n\pi c}{l} = \frac{2V_0}{l} \int_0^l \sin^3 \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx$$
$$= \frac{2V_0}{l} \frac{1}{4} \int_0^l \left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right) \sin \frac{n\pi x}{l} dx$$
$$= \frac{V_0}{2l} \left[\int_0^l 3 \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx - \int_0^l \sin \frac{3\pi x}{l} \sin \frac{n\pi x}{l} dx \right]$$

which vanishes for all values of n except for $n = 1$ and $n = 3$. When $n = 1$,

$$b_1 \frac{\pi c}{l} = \frac{V_0}{2l} \int_0^l 3 \sin^2 \frac{\pi x}{l} dx = \frac{3V_0}{2l} \frac{l}{2} = \frac{3V_0}{4}$$

Therefor $b_1 = \frac{3V_0 l}{4\pi c}$.
When $n = 3$,

$$b_3 \frac{3\pi c}{l} = \frac{-V_0}{2l} \int_0^l 3 \sin^2 \frac{3\pi x}{l} dx = \frac{-V_0}{2l} \times \frac{l}{2} = \frac{-V_0}{4}$$

Therefore $b_3 = \frac{-V_0 l}{12\pi c}$.
Then

$$y(x,t) = b_1 \sin \frac{\pi ct}{l} \sin \frac{\pi x}{l} + b_3 \sin \frac{3\pi ct}{l} \sin \frac{3\pi x}{l}$$
$$= \frac{3V_0 l}{4\pi c} \sin \frac{\pi ct}{l} \sin \frac{\pi x}{l} - \frac{V_0 l}{12\pi c} \sin \frac{3\pi ct}{l} \sin \frac{3\pi x}{l}$$
$$= \frac{V_0 l}{12\pi c} \left[9 \sin \frac{\pi ct}{l} \sin \frac{\pi x}{l} - \sin \frac{3\pi ct}{l} \sin \frac{3\pi x}{l} \right]$$

[**NOTE:** $\sin 3x = 3\sin x - 4\sin^3 x$]

7. A string is stretched between the fixed points $(0,0)$ and $(l,0)$ and released at rest from the initial deflection given by $f(x) = \begin{cases} \dfrac{2k}{l}x & , \quad 0 < x < \dfrac{l}{2} \\ \dfrac{2k}{l}(l-x) & , \quad \dfrac{l}{2} < x < l \end{cases}$
Find the deflection of the string at any time t.
Ans: The equation of motion is

$$y(x,t) = \sum a_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \text{ ——(1)}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\times\frac{2k}{l}\left[\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}dx + \int_{\frac{l}{2}}^l (l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{4k}{l^2}\left[\left(x\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)_0^{\frac{l}{2}} + \left((l-x)\times\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1)\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)_{\frac{l}{2}}^l\right]$$

$$= \frac{4k}{l^2}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2 jkl}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4k}{l^2}\times\frac{2l^2}{n^2\pi^2}\times\sin\frac{n\pi}{2}$$

$$= \begin{cases} \dfrac{8k}{n^2\pi^2}(-1)^{\frac{n-1}{2}} & , \quad n \ is \ odd \\ 0 & , \quad n \ is \ even \end{cases}$$

Therefore $y = \sum_{n=1,3,5,\ldots}\dfrac{(-1)^{\frac{(n-1)}{2}}}{n^2}\cos\dfrac{n\pi ct}{l}\sin\dfrac{n\pi x}{l}$

i.e; $y = \dfrac{8k}{\pi^2}\sum_{m=1}^{\infty}\dfrac{(-1)^{m-1}}{(2m-1)^2}\cos\dfrac{(2m-1)\pi ct}{l}\sin\dfrac{(2m-1)\pi x}{l}, n = 2m-1$

NOTE: $\sin\dfrac{n\pi}{2} = \begin{cases} (-1)^{\frac{n-1}{2}} & , \quad n \ is \ odd \\ 0 & , \quad n \ is \ even \end{cases}$

HOME WORK:

1. A tightly stretched string with fixed end points $x = 0, x = l$ is initially in a position given by $y = y_0\sin^3\dfrac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x,t)$.
Ans: $y = \dfrac{y_0}{4}\left[3\cos\dfrac{\pi ct}{l}\sin\dfrac{\pi x}{l} - \cos\dfrac{3\pi ct}{l}\sin\dfrac{3\pi x}{l}\right]$

2. The vibration of an electric string is governed by P.D.E. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$. The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $u(x,0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x,t)$ of the vibrating string for $t > 0$.
Ans: $u(x,t) = 2(\cos t\sin x + \cos 3t\sin 3x)$

3. Find the solution of the vibrating string of unit length having wave velocity $c = 1$. The end points of the string are fixed. The initial velocity is zero and the initial deflection is given by $u(x,0) = \begin{cases} 1 & , \quad 0 \le x \le \frac{1}{2} \\ -1 & , \quad \frac{1}{2} \le x \le l \end{cases}$

   Ans: The equation of motion is $u(x,t) = \sum a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$. Since $l = 1$,

$$u(x,t) = \sum a_n \cos n\pi t \sin n\pi x$$

   where

$$\begin{aligned} a_n &= \frac{2}{1} \int_0^1 f(x) \sin n\pi x dx \\ &= 2\left[\int_0^{\frac{1}{2}} \sin n\pi x dx - \int_{\frac{1}{2}}^1 \sin n\pi x dx\right] \\ &= 2\left[\left(\frac{-\cos n\pi x}{n\pi}\right)_0^{\frac{1}{2}} + \left(\frac{\cos n\pi x}{n\pi}\right)_{\frac{1}{2}}^1\right] \\ &= 2\left[\frac{-1}{n\pi}\cos\frac{n\pi}{2} + \frac{1}{n\pi} + \frac{\cos n\pi}{n\pi} - \frac{1}{n\pi}\cos\frac{n\pi}{2}\right] \\ & \frac{2}{n\pi}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\right] \end{aligned}$$

   Therefore $u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left(1 + (-1)^n - 2\cos\frac{n\pi}{2}\right) \cos n\pi t \sin n\pi x$

4. A tightly stretched homogeneous string of length l, with its fixed ends at $x = 0$ and $x = l$ executes transverse vibrations. Motion start with zero initial velocity by displacing the string into the form $f(x) = k(x^2 - x^3)$. Find the deflection $y(x,t)$ at any time t.

   Ans: $y = 2k\left[(-1)^{n+1}\frac{l^2 - l^3}{n\pi} + \frac{l^2}{n^3\pi}((2-6l)(-1)^n - 2)\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}\right]$

5. Solve the boundary value problem $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}, y(0,t) = y(5,t) = 0, y(x,0) = 0, \left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x)$ where $f(x) = 5\sin \pi x$.

   Ans: $y = \frac{5}{2\pi}\sin 2\pi t \sin \pi x$

2.1.2 D'Alembert's Solution of Wave Equation

Wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \text{ ——— } (1)$$

Let $y = f(u,v)$ where $u = x + ct, v = x - ct$. Then

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \end{aligned}$$

Therefore $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}$. Now

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\\ &= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial x}\right)\end{aligned}$$

Therefore

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \text{ —— } (2)$$

Similarly

$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}\\ &= \frac{\partial y}{\partial u} \times c + \frac{\partial y}{\partial v} \times (-c)\\ &= c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\end{aligned}$$

So $\dfrac{\partial}{\partial t} = c\left(\dfrac{\partial}{\partial u} - \dfrac{\partial}{\partial v}\right)$. Then

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)\\ &= c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial t}\right)\\ &= c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\\ &= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \text{ ——} (3)\end{aligned}$$

Substitute equation (2), (3) in equation (1),

$$\begin{aligned}c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) &= c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)\\ \implies -4\frac{\partial^2 y}{\partial u \partial v} &= 0\\ \implies \frac{\partial^2 y}{\partial u \partial v} &= 0\end{aligned}$$

Integrating w.r.t. v, $\dfrac{\partial y}{\partial u} = f(u)$ where $f(u)$ is constant in respect of v. Again integrating w.r.t. u,

$$\begin{aligned}y &= \int f(u)du + \phi_2(v)\\ &= \phi_1(u) + \phi_2(v)\\ &= \phi_1(x+ct) + \phi_2(x-ct)\end{aligned}$$

$$y = \phi_1(x+ct) + \phi_2(x-ct) \text{ ——} (4)$$

is the D' Alembert's solution of the wave equation (1).

To determine ϕ_1 and ϕ_2, let us apply initial conditions $y(x,0) = f(x)$, $\dfrac{\partial y}{\partial t} = 0$ at $t = 0$,

$$f(x) = \phi_1(x) + \phi_2(x) \text{ ———-(5)}$$
$$\frac{\partial y}{\partial t} = c\phi_1'(x+ct) - c\phi_2'(x-ct)$$
$$\implies 0 = c(\phi_1'(x) - \phi_2'(x))$$
$$\implies \phi_1'(x) = \phi_2'(x)$$
$$\implies \phi_1(x) = \phi_2(x) + b$$

Substitute $\phi_1(x)$ in equation (5),

$$f(x) = \phi_2(x) + b + \phi_2(x) = 2\phi_2(x) + b$$
$$\implies \phi_2(x) = \frac{1}{2}[f(x) - b]$$
$$\implies \phi_1(x) = \phi_2(x) + b = \frac{1}{2}[f(x) - b] + b$$
$$\implies \phi_1(x) = \frac{1}{2}[f(x) + b]$$

Then equation (4) becomes,

$$y = \phi_1(x+ct) + \phi_2(x-ct)$$
$$= \frac{1}{2}\left[f(x+ct) + b + f(x-ct) - b\right]$$
$$= \frac{1}{2}[f(x+ct) + f(x-ct)]$$

which is D' Alembert's solution of wave equation.

PROBLEMS:

1. Using D' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.
   Ans: By D' Alembert's method,

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$
$$= \frac{k}{2}\left[\sin(x+ct) - \sin 2(x+ct) + \sin(x-ct) - \sin 2(x-ct)\right]$$
$$= \frac{k}{2}\left[(\sin(x+ct) + \sin(x-ct)) - (\sin 2(x+ct) + \sin 2(x-ct))\right]$$
$$= \frac{k}{2}\left[2\sin x \cos ct - 2\sin 2x \cos 2ct\right]$$
$$= k\left[\sin x \cos ct - \sin 2x \cos 2ct\right]$$

   [NOTE: $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$]

2. Find the deflection $y(x,t)$ of the vibrating string of length T and ends fixed, corresponding to zero initial velocity and initial deflection given by $y(x,0) = kx(\pi - x)$.
   Ans: D' Alembert's solution of wave equation is,

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$
$$= \frac{k}{2}\left[\pi(x+ct) - (x+ct)^2 + \pi(x-ct) - (x-ct)^2\right]$$

$$= \frac{k}{2}\left[2\pi x - x^2 - 2xct - c^2t^2 - x^2 + 2xct - c^2t^2\right]$$
$$= \frac{k}{2}\left[2\pi x - 2x^2 - 2c^2t^2\right]$$
$$= k(\pi x - x^2 - c^2t^2)$$

3. Using D' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection (1) $f(x) = a(x - x^3)$ (2) $f(x) = a\sin^2 \pi x$

   Ans: D' Alembert's solution is,

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

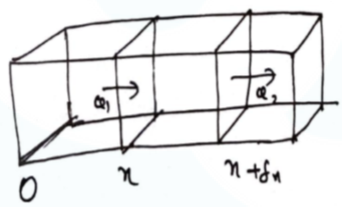
(1) $f(x) = a(x - x^3)$

$$y = \frac{a}{2}\left[(x+ct) - (x+ct)^3 + (x-ct) - (x-ct)^3\right]$$
$$= \frac{a}{2}\left[2x - x^3 - 3x^2ct - 3xc^2t^2 - c^3t^3 - x^3 + 3x^2ct - 3xc^2t^2 + c^3t^3\right]$$
$$= \frac{a}{2}[2x - 2x^3 - 6c^2t^2x]$$
$$= a(x - x^3 - 3c^2t^2x)$$

(2) $f(x) = a\sin^2 \pi x$

Ans: $\frac{a}{2}(1 - \cos 2\pi x \cos 2\pi ct)$

2.2 One Dimensional Heat Equation



Consider the flow of heat by conduction in a uniform bar. It is assumed that sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible. Take one end of the bar as the origin and the direction of flow as the +ve $x-$ axis. The temperature u at any point of the bar depends on the distance x of the point from the end of the bar and the time t. Also the temperature of all points of any cross section is the same. The amount of heat crossing any section of the bar per second depends on the area A of the cross section, the conductivity k of the material of the bar and the temperature gradient $\frac{\partial u}{\partial x}$(i.e; the rate of change of temperature w.r.t. distance.)

Therefore Q_1, the quantity of heat flowing into the section at a distance $x = -kA\left(\frac{\partial u}{\partial x}\right)_x$ (negative sign on the right is attached because as x increases, u decreases)

Q_2, the quantity of heat flowing out of the section at a distance $x+\delta x = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ per second. Hence the amount of heat retained by the slab with thickness δx is

$$Q_1 - Q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{—— (1)}$$

per second. (This follows from the definition of conduction)

But the rate of increase of heat in the slab $= S\rho A\delta x\frac{\partial u}{\partial t}$ ——(2)
where S is the specific heat , ρ the density of the material (This follows from the definition of specific heat). Equations (1) and (2) are the same. Therefore

$$\begin{aligned} S\rho A\delta x\frac{\partial u}{\partial t} &= kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \\ \implies \frac{\partial u}{\partial t} &= \frac{k}{S\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right] \end{aligned}$$

Taking the limit as $\delta x \to 0$,

$$\frac{\partial u}{\partial t} = \frac{k}{S\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where $c^2 = \frac{k}{S\rho}$ is known as the difficulty of the material of the bar. Therefore $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ is the heat equation. Since the temperature u depends on x coordinate only, it is called one dimensional heat equation.

2.2.1 Solution of Heat Equation

Heat equation is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \text{ —-(1)}$$

Let $u = XT$ where X is a function of x only and T is a function of t only be a solution of equation (1). Then $\frac{\partial^2 u}{\partial x^2} = X''T, \frac{\partial u}{\partial t} = XT'$. Substitute these values in equation (1),

$$\begin{aligned} XT' &= c^2X''T \\ \implies \frac{X''}{X} &= \frac{1}{c^2}\frac{T'}{T} \end{aligned}$$

L.H.S. is a function of x only and R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both reduces to a constant say k. Therefore

$$\begin{aligned} \frac{X''}{X} &= \frac{1}{c^2}\frac{T'}{T} = k \\ \implies X'' - kX &= 0 \text{——-(2) and} \\ T' - kc^2T &= 0 \text{ ——(3)} \end{aligned}$$

Three cases arise. i.e; k is +ve, -ve and $k = 0$.

Case I-k is positive

When $k = p^2$, equation (2) becomes

$$X'' - p^2X = 0$$
$$\implies \frac{d^2X}{dx^2} - p^2X = 0$$
$$\implies (D^2 - p^2)X = 0, D = \frac{d}{dx}$$

A.E. is $D^2 - p^2 = 0 \implies D = \pm p$.

Therefore $X = c_1e^{px} + c_2e^{-px}$.

Equation (3) becomes

$$\frac{dT}{dt} - p^2c^2T = 0$$
$$\implies \frac{dT}{T} = p^2c^2dt$$

Integrating w.r.t t,

$$\log T + \log a = p^2c^2t$$
$$\implies \log aT = p^2c^2t$$
$$\implies T = c_3e^{p^2c^2t}$$

Case II: k is negative

When $k = -p^2$, equation (2) becomes

$$X'' + p^2X = 0$$
$$\implies \frac{d^2X}{dx^2} + p^2X = 0$$
$$\implies (D^2 + p^2)X = 0, D = \frac{d}{dx}$$

A.E. is $D^2 + p^2 = 0 \implies D = \pm ip$.

Therefore $X = c_1 \cos px + c_2 \sin px$.

Equation (3) becomes

$$\frac{dT}{dt} + p^2c^2T = 0$$
$$\implies (D + p^2c^2)T = 0$$

A.E. is $D + p^2c^2 = 0 \implies D = -p^2c^2$. Therefore $T = c_3e^{-p^2c^2t}$.

Case III:

When $k = 0$, equation (2) becomes $\frac{d^2X}{dx^2} = 0$. Integrating w.r.t. x, $\frac{dX}{dx} = c_1$. Again integrating w.r.t. x, we get

$$X = c_1x + c_2$$

Equation (3) becomes $\frac{dT}{dt} = 0$. integrating w.r.t. t, we get $T = c_3$.

Thus the various possible solutions are

(i) $u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{p^2 c^2 t}$

(ii) $u = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2 c^2 t}$

(iii) $u = (c_1 x + c_2) c_3$

Of these three solutions we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of heat equation is

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2 c^2 t}$$

Problems:

1. Find the solution of $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$ for which $u(0,t) = 0 = u(l,t), u(x,0) = \sin \frac{\pi x}{l}$ by the method of seperation of variables.
   Ans: Heat equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ and given equation is $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$. Then $c^2 = \frac{1}{h^2}$. Therefore the solution is

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{\frac{-p^2 t}{h^2}} \text{——}(1)$$

Given $u = 0$ when $x = 0$. So equation (1) becomes,

$$0 = c_1 c_3 e^{\frac{-p^2 t}{h^2}}$$
$$\implies c_1 = 0$$
$$u = c_2 \sin px c_3 e^{\frac{-p^2 t}{h^2}} \text{——}(2)$$

Apply the second boundary condition $u = 0$ when $x = l$ in (2),

$$0 = c_2 \sin pl c_3 e^{\frac{-p^2 t}{h^2}}$$
$$\implies \sin pl = 0$$
$$\implies pl = n\pi \implies p = \frac{n\pi}{l}$$

Then equation (2) becomes

$$u = c_2 \sin \frac{n\pi x}{l} c_3 e^{\frac{-n^2\pi^2}{l^2 h^2} t}$$

i.e; $u = b \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2}{l^2 h^2} t}$. The most general solution is obtained by adding all such solutions for $n = 1, 2, 3, \ldots$. Therefore

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2}{l^2 h^2} t} \text{——}(3)$$

Applying the initial condition $u(x,0) = \sin \frac{\pi x}{l}$,

$$\sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$
$$= \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}dx$$

which vanishes for all values of n except when $n = 1$. Therefore when $n = 1$,

$$b_1 = \frac{2}{l}\int_0^l \sin^2\frac{\pi x}{l}dx$$
$$= \frac{2}{l}\times\frac{l}{2} = 1$$

Therefore equation (3) becomes, $u = b_1 \sin\frac{\pi x}{l}e^{\frac{-\pi^2 t}{l^2 h^2}} = \sin\frac{\pi x}{l}e^{\frac{-\pi^2 t}{l^2 h^2}}$, which is the required solution.

2. Let $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, given $u(0,t) = u(10,t) = 0$. Solve the above equation. Also $u(x,0) = x^2 - 10x, 0 < x < 10$.
Ans: Solution of heat equation is

$$u = (c_1\cos px + c_2\sin px)c_3 e^{-p^2 t} \text{ ——(1) } (\because c^2 = 1)$$

Applying boundary condition $u(0,t) = u(10,t) = 0$,

$$0 = c_1 c_3 e^{-\rho^2 t} \implies c_1 = 0$$
$$\implies u = c_2 \sin px c_3 e^{-p^2 t} \text{ ——(2)}$$

And

$$0 = c_2 \sin p10 c_3 e^{-p^2 t}$$
$$\implies \sin p10 = 0 \implies p10 = n\pi$$
$$\implies p = \frac{n\pi}{10}$$

Then equation (2) becomes $u = b_n \sin\frac{n\pi x}{10}e^{\frac{-n^2\pi^2 t}{10^2}}$. The most general solution is obtained by adding up all such solutions for $n = 1, 2, 3, \ldots$. Therefore

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}e^{\frac{-n^2\pi^2 t}{10^2}} \text{ —— (3)}$$

Applying the initial condition $u(x,0) = x^2 - 10x$,

$$x^2 - 10x = \sum b_n \sin \frac{n\pi x}{10} \text{ where}$$
$$b_n = \frac{2}{10} \int_0^{10} (x^2 - 10x) \sin \frac{n\pi x}{10} dx$$
$$= \frac{2}{10} \left[(x^2 - 10x) \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - (2x - 10) \left(\frac{-\sin \frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}} \right) + 2 \left(\frac{\cos \frac{n\pi x}{10}}{\frac{n^3\pi^3}{10^3}} \right) \right]_0^{10}$$
$$= \frac{2}{10} \left[\frac{2 \times 10^3}{n^3\pi^3} ((-1)^n - 1) \right]$$
$$= \frac{400}{n^3\pi^3} ((-1)^n - 1)$$
$$= \begin{cases} \frac{-800}{n^3\pi^3} & , \quad n \ \ is \ \ odd \\ 0 & , \quad n \ \ is \ \ even \end{cases}$$

Then equation (3) becomes, $u = \sum_{n=1,3,5,...} \frac{1}{n^3} \sin \frac{n\pi x}{10} e^{\frac{-n^2\pi^2 t}{10^2}}$. That is

$$u = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{10} e^{\frac{-(2m-1)^2\pi^2 t}{10^2}}$$

3. A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0^oC and are kept at that temperature. Find $u(x,t)$.
Ans: Solution of heat equation is

$$u = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2c^2t} \text{ ——(1)}$$

Since the ends $x = 0, x = l$ are cooled to 0^oC and kept at that temperature throughout, $u(0,t) = 0 = u(l,t)$ for all t. Also $u(x,0) = u_0$ is the initial condition.

$$0 = c_1 c_3 e^{-p^2c^2t} \implies c_1 = 0$$

Therefore

$$u = c_2 \sin px c_3 e^{-p^2c^2t} \text{ —-(2)}$$

Now

$$0 = c_2 \sin pl c_3 e^{-p^2c^2t}$$
$$\implies \sin pl = 0 \implies pl = n\pi$$
$$p = \frac{n\pi}{l}$$

Therefore $u = b_n \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2c^2t}{l^2}}$. The most general solution is obtained by adding all such solutions for $n = 1, 2, 3...$ Therefore

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2c^2t}{l^2}} \text{ ——(3)}$$

Applying initial condition $u(x,0) = u_0$,

$$\begin{aligned} u_0 &= \textstyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where} \\ b_n &= \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}dx \\ &= \frac{2u_0}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l \\ &= \frac{-2u_0}{l}\frac{l}{n\pi}((-1)^n - 1) \\ &= \begin{cases} \dfrac{4u_0}{n\pi} & , \quad n \;\; is \;\; odd \\ 0 & , \quad n \;\; is \;\; even \end{cases} \end{aligned}$$

Substituting in equation (3), $u = \frac{4u_0}{\pi}\sum_{n=1,3,5,...}\frac{1}{n}\sin\frac{n\pi x}{l}e^{\frac{-n^2\pi^2c^2t}{l^2}}$. Therefore

$$u = \frac{4u_0}{\pi}\textstyle\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin\frac{(2m-1)\pi x}{l}e^{\frac{-(2m-1)^2\pi^2c^2t}{l^2}}$$

4. A homogeneous rod of conducting material of length 100cm has its ends kept at zero temperature and the temperature initially is $u(x,0) = \begin{cases} x & , \quad 0 \le x \le 50 \\ 100 - x & , \quad 50 \le x \le 100 \end{cases}$
Find the temperature $u(x,t)$.
Ans: Solution of heat equation is

$$u = (c_1 \cos px + c_2 \sin px)c_3 e^{-p^2c^2t} \text{——}(1)$$

Applying boundary condition $u = 0$ when $x = 0$ and $u = 0$ when $x = 100$,

$$\begin{aligned} 0 = c_1c_3e^{-p^2c^2t} &\implies c_1 = 0 \\ \implies u = c_2 \sin pxc_3e^{-p^2c^2t} &\text{——}(2) \end{aligned}$$

Also

$$\begin{aligned} 0 = c_2 \sin p100c_3e^{-p^2c^2t} &\implies \sin p100 = 0 \\ \implies p100 = n\pi &\implies p = \frac{n\pi}{100} \end{aligned}$$

Then equation (2) becomes $u = b_n \sin\frac{n\pi x}{100}e^{\frac{-n^2\pi^2c^2t}{100^2}}$. he most general solution is obtained by adding all such solutions for $n = 1, 2, 3...$ Therefore

$$u = \textstyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{100}e^{\frac{-n^2\pi^2c^2t}{100^2}} \text{——}(3)$$

When $t = 0$, $u(x,0) = \begin{cases} x & , \quad 0 \le x \le 50 \\ 100 - x & , \quad 50 \le x \le 100 \end{cases}$ and $u = \sum b_n \sin\frac{n\pi x}{100}$ where

$$b_n = \frac{2}{100}\int_0^{100} u(x,0)\sin\frac{n\pi x}{100}dx$$
$$= \frac{2}{100}\left[\int_0^{50} x\sin\frac{n\pi x}{100}dx + \int_{50}^{100}(100-x)\sin\frac{n\pi x}{100}dx\right]$$
$$= \frac{2}{100}\left[x\left(\frac{-cos\frac{n\pi x}{100}}{\frac{n\pi}{100}}\right) - \left(\frac{-sin\frac{n\pi x}{100}}{\frac{n^2\pi^2}{100^2}}\right)\right]_0^{50} +$$
$$\frac{2}{100}\left[(100-x)\left(\frac{-cos\frac{n\pi x}{100}}{\frac{n\pi}{100}}\right) - (-1)\left(\frac{-sin\frac{n\pi x}{100}}{\frac{n^2\pi^2}{100^2}}\right)\right]_{50}^{100}$$
$$= \frac{2}{100}\left[\frac{-5000}{n\pi}\cos\frac{n\pi}{2} + \frac{100^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{5000}{n\pi}\cos\frac{n\pi}{2} + \frac{100^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$
$$= \frac{2}{100}\times 2\times\frac{100^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$
$$= \begin{cases} \frac{400}{n^2\pi^2}(-1)^{\frac{n-1}{2}} & , \quad n \;\; is \;\; odd \\ 0 & , \quad n \;\; is \;\; even \end{cases}$$

Therefore

$$u(x,t) = \frac{400}{\pi^2}\sum\nolimits_{n=1,3,5,\ldots}\frac{1}{n^2}(-1)^{\frac{n-1}{2}}\sin\frac{n\pi x}{100}e^{\frac{-n^2\pi^2c^2t}{100^2}}$$
$$= \frac{400}{\pi^2}\sum\nolimits_{m=1}^{\infty}\frac{1}{(2m-1)^2}(-1)^{m-1}\sin\frac{(2m-1)\pi x}{100}e^{\frac{-(2m-1)^2\pi^2c^2t}{100^2}}$$

5. Find the temperature distribution in a rod of length l whose end points are fixed at temperatures zero and the initial temperature distribution is $f(x)$.
   Ans: If we take the end of the rod as origin, the boundary conditions are $u(0,t) = 0 = u(l,t)$. The initial temeperature is $u(x,0) = f(x)$. We know that

$$u(x,t) = (c_1\cos px + c_2\sin px)c_3e^{-p^2c^2t} \text{ ——(1)}$$

Then

$$u(0,t) = 0 \implies 0 = 0 = c_1c_3e^{-p^2c^2t} \implies c_1 = 0$$
$$\implies u(x,t) = c_2\sin pxc_3e^{-p^2t}$$

And

$$u(l,t) = 0 \implies 0 = c_2\sin plc_3e^{-p^2c^2t}$$
$$\implies \sin pl = 0 \implies pl = n\pi$$
$$\rho = \frac{n\pi}{l}$$

Therefore $u(x,t) = b_n\sin\frac{n\pi x}{l}e^{\frac{-n^2\pi^2c^2t}{l^2}}$. Adding up the solutions for different values of n

$$u = \sum\nolimits_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}e^{\frac{-n^2\pi^2c^2t}{l^2}} \text{ ——(2)}$$

Given $u(x,0) = f(x)$. So

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{\frac{-n^2\pi^2c^2t}{l^2}} \text{ where}$$
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \text{ ——-(3)}$$

Hence the required temperature distribution in the rod is obtained from (2) where b_n is given by equation (3).

HOMEWORK:

1. Find the temperature distribution in a rod of length 2m whose end points are maintained at temperature zero and the initial temperature is $f(x) = 100(2x - x^2)$.
   Ans: $u(x,t) = \frac{3200}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{2}e^{\frac{-(2m-1)^2\pi^2c^2t}{4}}, n = 2m-1$

2. If the initial temperature is $f(x) = \begin{cases} x & , \quad 0 < x < \frac{l}{2} \\ l - x & , \quad \frac{l}{2} < x < l \end{cases}$ when a homogeneous rod of length l cm has its end points kept at zero degree temperature, find the temperature $u(x,t)$ at any time t.
   Ans: $u = \frac{4l}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}(-1)^{m-1}\sin\frac{(2m-1)\pi x}{l}e^{\frac{-(2m-1)^2\pi^2c^2t}{l^2}}, n = 2m-1$

3. Determine the solution of heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial t^2}$ where the boundary conditions are $u(0,t) = 0, u(l,t) = 0, t > 0$ and the initial condition is $u(x,0) = x, l$ being the length of the bar.
   Ans: $u = \frac{2l}{\pi}\sum_{n=1}^{infty}\frac{1}{n}(-1)^{n+1}\sin\frac{n\pi x}{l}e^{\frac{-n^2\pi^2c^2t}{l^2}}$

4. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}$ with boundary conditions $u(x,0) = 3\sin n\pi x, u(0,t) = 0, u(1,t) = 0, 0 < t < 1$.
   Ans: $u = 3\sum_{n=1}^{infty}\sin n\pi x e^{-n^2\pi^2 t}$

5. Solve $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial t^2}$, $u = 0$ when $x = 0, u = 0$ when $x = l$ and $u = 3\sin\frac{\pi x}{l}$ at $t = 0$.
   Ans: $u = 3\sin\frac{\pi x}{l}e^{\frac{\pi^2 a^2 t}{l^2}}$

6. Find the temperature distribution in a bar of length π whose surface is thermally insulated with end points maintained at 0^0C. The initial temperature distribution in the rod is $u(x,0) = f(x) = \begin{cases} x & , \quad 0 \leq xleq\frac{\pi}{2} \\ \pi - x & , \quad \frac{\pi}{2} \leq x \leq \pi \end{cases}$
   Ans: $u(x,t) = 4\frac{(-1)^{m-1}}{(2m-1)^2}\sin(2m-1)xe^{-(2m-1)^2c^2t}$