# Module 4

## Generating Functions and Recurrence Relations

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# Generating Function

A generating function is a way of encoding an infinite sequence of numbers ($a_n$) by treating them as the coefficients of a formal power series. This series is called the generating function of the sequence.

Def: Let $a_0$ , $a_1$ , $a_2$ , …… be a sequence of real numbers. The function $f(x)= a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots$ is called the generating function for the give sequence.

Ex. For n€ $Z^+$ , $(1- x^{n+1}) = (1-x) (1+x+ x^2 + x^3 +\text{-----}+ x^n)$

So $(1+x+ x^2 + x^3 +\text{-----}+ x^n)= (1- x^{n+1})/(1-x)$ and $(1- x^{n+1})/(1-x)$ is the generating function for the sequence 1, 1,1,…., 1,0,0,…… where the first n+1 terms are 1.

Ex. We have $1= (1-x)(1+x+ x^2 + x^3 +\text{-----}+ x^n +\ldots\ldots.)$

So $1/(1-x)$ is the generating function for the sequence 1, 1, 1,….

# Generating Function

Ex. $d/dx[1/(1-x)] = 1/(1-x)^2$ is the generating function for the sequence 1, 2, 3, 4,…..

$x/(1-x)^2$ is the generating function for the sequence 0, 1, 2, 3, 4,…..

$(x+1)/(1-x)^3$ generates $1^2, 2^2, 3^2$ ,……..

$x(x+1)/(1-x)^3$ generates $0^2, 1^2, 2^2, 3^2$ ,……..

$$d/dx[1/(1-x)] = d/dx[1+x+x^2+x^3+\text{-----}+x^n+\ldots\ldots]$$

$$1/(1-x)^2 = 1+2x+3x^2+4x^3+\text{-----}+nx^{n-1}+\ldots\ldots$$

$$2/(1-x)^3 = 2[1+3x+6x^2+\text{-----}+\ldots\ldots]$$

# Generating Function

Ex. Find the generating function for the sequence 0, 2, 6, 12, 20, 30, 42,……

Sol: $0= 0^2 +0$

$2= 1^2 +1$

$6= 2^2 + 2$

$12= 3^2 +3$,…….

In general $\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{2x}{(1-x)^3}$ is the generating function of the given sequence

# Exponential Generating Function.

Def: Let $a_0$ , $a_1$ , $a_2$ , …… be a sequence of real numbers. The function $f(x)= a_0 + a_1 x+ a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!}$ ……… is called the Exponential generating function for the give sequence.

Ex. $e^x$ is the exponential generating function for the sequence 1,1,1,…….

$e^{-x}$ is the exponential generating function for the sequence 1,-1, 1,-1, …….

Ex. Find the generating function for the sequence 0, 1, 3, 6, 10, 15, …….

Sol: $x/(1-x)^3$ is the generating function

# First Order Linear Recurrence Relations with Constant Coefficients

If $\mathbf{a_n = A\, a_{n-1} + C}$, where A and C are constants, we call this a first order recurrence relation. By first order, we mean that we are looking back only one unit in time to $a_{n-1}$.

The unique solution of the recurrence relation $a_{n+1} = d\, a_n$ , where $n \geq 0$, d is a constant and $a_0 = A$, is given by $\mathbf{a_n = A\, d^n}$ , $n \geq 0$.

**Ex. Solve the recurrence relation $a_n = 7\, a_{n-1}$ , where $n \geq 1$ and $a_2 = 98$**

Sol: Alternative form of given equation is $a_{n+1} = 7\, a_n$ , $n \geq 0$ and $a_2 = 98$. Hence the solution has the form $a_n = a_0 (7^n)$. since $a_2 = 98 \Rightarrow a_2 = a_0 (7^2) \Rightarrow 98 = a_0 (7^2)$. Thus $a_0 = 2$ and $a_n = 2 (7^n)$, $n \geq 0$, is the unique solution.

# First Order Linear Recurrence Relations with Constant Coefficients

Ex. Find a recurrence relation with initial condition that uniquely determines each of the following sequences that begin with the given terms.

(a)3, 7, 11, 15, 19, …….

(b)8, 24/7, 72/49, 216/343,…… …

(c)6, -18, 54, -162,-----

Sol:(a) Here first term is $a_0 = 3$, and increases by 4. So the recurrence relation is $a_n = a_{n-1} + 4$ for $n \geq 1$

(b) Here first term is $a_0 = 8$, and each terms gets multiplied by 3/7. So the recurrence relation is $a_n = (3/7)\, a_{n-1}$ for $n \geq 1$

(c) Here first term is $a_0 = 6$, and $a_n = (-3)^n\, a_{n-1}$ for $n \geq 1$.

# Homogeneous Solution

Ex. Find the unique solution for each of the following recurrence relation.

(a)$a_{n+1} – 1.5a_n = 0$ , $n \geq 0$

(b) $2a_n – 3a_{n-1} = 0$ , $n \geq 1$, $a_4 = 81$

Sol: (b) Solving $a_n = (3/2)a_{n-1}$

# Non homogeneous Solution

Ex. Solve the following recurrence relation.

(a)$a_n - a_{n-1} = 3n^2$ , where $n \geq 1$ and $a_0 = 7$.

(b)$a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$

(c)$a_n - 3a_{n-1} = 5(3^n)$, where $n \geq 1$ and $a_0 = 2$

Sol: (a) Here $f(n) = 3n^2$ , so the unique solution is

$$a_n = a_0 + f(1) + f(2) + \ldots\ldots + f(n)$$

$$= 7 + 3(1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2) = 7 + 3[n(n+1)(2n+1)]/6$$

$$= 7 + [n(n+1)(2n+1)]/2$$

(b) $a_n = -(1/4)(3^{n+1}) + (5/4)(7^{n+1})$

(c) $a_n = (2+5n)(3^n)$

# Second order linear recurrence relations with constant coefficients

Ex. Solve the recurrence relation $2a_{n+3} = a_{n+2} + 2\, a_{n+1} – a_n$ , $n \geq 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$

Sol: $a_n = (5/2) + (1/6)(-1)^n + (-8/3)(1/2)^n$ , $n \geq 0$

Ex. Solve the recurrence relation $a_n = 2( a_{n-1} – a_{n-2} )$ , $n \geq 2$, $a_0 = 1$, $a_1 = 2$

Sol. $a_n = (\sqrt{2})^n [\cos(n\pi/4) + \sin(n\pi/4)]$, $n \geq 0$

Ex. Solve the recurrence relation $a_{n+2} = 4a_{n+1} – 4a_n$ , $n \geq 0$, and $a_0 = 1$, $a_1 = 3$.

Sol: $a_n = 2^n + (1/2)\, n\, (2)^n$ , $n \geq 0$

Ex. Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$ ; $n \geq 2$, $a_0 = 5$, $a_1 = 12$

# Second order linear recurrence relations with constant coefficients

Ex. Solve the recurrence relation $a_n + a_{n-1} - 6\, a_{n-2} = 0$, $n \geq 2$, $a_0 = -1$, $a_1 = 8$

Ex. Solve the recurrence relation $a_n - 3a_{n-1} = 5\,(7)^n$, $n \geq 1$, and $a_0 = 2$

Ex. Solve $a_n - 3a_{n-1} = 5\,(3)^n$; $n \geq 1$, $a_0 = 2$.

Ex. Solve $a_{n+2} - 8a_{n+1} + 16\, a_n = 8(5)^n + 6\,(4)^n$; $n \geq 0$, $a_0 = 12$, $a_1 = 5$

Ex. Solve $a_{n+2} - 4a_{n+1} + 3\, a_n = -200$; $n \geq 0$, $a_0 = 3000$, $a_1 = 3300$ .

Ex.Solve $a_{n+2} - 10a_{n+1} + 21\, a_n = 5$

Ex. Solve $a_{n+2} - 10a_{n+1} + 21\, a_n = 2(3)^n - 8\,(9)^n$

# Second order linear recurrence relations with constant coefficients

Ex. Solve the recurrence relation

$a_n + 4 a_{n-1} + 4 a_{n-2} = 7n (-2)^n$

Ex. Solve the recurrence relation

$a_n + 4a_{n-1} + 4 a_{n-2} = -11n^2(-2)^n$

Ex. Solve $a_n = 4a_{n-1} - 4 a_{n-2} + (n+1)(2)^n$

# Homogeneous Solution

The roots $r_1$ , $r_2$ of characteristic equations determine the following three cases:

(a) $r_1$ , $r_2$ are distinct real numbers

Solution is $a_n = A(r_1)^n+B(r_2)^n$

(b) $r_1$, $r_2$ are complex roots

where $r_1 = a+ib$, $r_2 = a-ib$

Solution is $a_n = (r)^n[A \cos n\theta+B \sin n\theta]$ where $a+ib=r(\cos\theta+\sin \theta)$

(c) $r_1$ , $r_2$ are repeated real numbers, ie. $r_1 = r_2$

Solution is $a_n = [A+ Bn](r_1)^n$

# Second order linear recurrence relations with constant coefficients

| | |
|---|---|
| $F(n) = \text{Constant}$ | $a_n^{(p)} = A$ |
| $F(n) = n$ | $a_n^{(p)} = An + B$ |
| $F(n) = n^2$ | $a_n^{(p)} = An^2 + Bn + C$ |
| $F(n) = \sin \theta n$ or $\mathbf{cos}\ \theta n$ | $a_n^{(p)} = A \sin \theta n + \mathbf{B\ cos}\ \theta n$ |
| $F(n) = k\, r^n$ ,where k is const | $a_n^{(p)} = A\, r^n$ , where A is const |
| $F(n) = k\, r^n$ ,where k is const | $a_n^{(p)} = A\, nr^n$ , when r is the root of homogenous equation |
| $F(n) = k\, r^n$ ,where k is const | $a_n^{(p)} = A\, n^2\, r^n$ , when r is the double repeated root of homogenous equation |
| $F(n) = n^3\, r^n$ | $a_n^{(p)} = (An^3 + Bn^2 + Cn + D)\, r^n$ |
| $F(n) = n^3\, r^n$ | $a_n^{(p)} = (An^3 + Bn^2 + Cn + D)n\, r^n$ , when r is the rootof homogenous equation |

# Second order linear recurrence relations with constant coefficients

Solve the recurrence relation using the method of generating function.

$a_n - 3 a_{n-1} = n$ ; $n \geq 1$, $a_0 = 1$

Sol: we first multiply this given relation by $x^n$ because n is the largest subscript that appears.

$a_n x^n - 3 a_{n-1} x^n = n x^n$ ………(i)

Then sum all of the equations

$$\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n \quad \text{……….(ii)}$$

# Second order linear recurrence relations with constant coefficients

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the solution. The equation (ii) takes the form,

$$(f(x) - a_0) - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^n$$

$$f(x) - a_0 - 3x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} nx^n$$

$$f(x) - a_0 - 3x\, f(x) = x + x^2 + x^3 + \ldots\ldots\ldots$$

$$f(x) - 1 - 3x\, f(x) = x/(1-x)^2$$

$$f(x)[1-3x] = 1 + x/(1-x)^2$$

$$f(x) = (1/1-3x) + x/(1-3x)\,(1-x)^2$$

# Second order linear recurrence relations with constant coefficients

Using partial fraction,

We get $f(x) = \frac{1}{(1-3x)} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2} + \frac{(3/4)}{(1-3x)}$

$== \frac{(7/4)}{(1-3x)} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2}$

We find $a_n$ by determining the coefficient of $x^n$ in each of the three summands.

Therefore $a_n = (7/4)\, 3^n - (1/2)\, n - (3/4), n \geq 0$

# Second order linear recurrence relations with constant coefficients

Solve the following recurrence relation using the method of generating function.

(i) $a_{n+2} -3 a_{n+1} +6 a_n=2$ ; $n\geq 0$, $a_0 = 3$, $a_1 = 7$

Sol: $a_n = 2(3^n) +1$, $n\geq 0$

(ii) $a_{n+1} -a_n =3^n$ ; $n\geq 0$, $a_0 = 1$

(iii) $a_{n+2} -3 a_{n+1} +2 a_n=0$ ; $n\geq 0$, $a_0 = 1$, $a_1 = 6$