Module 2

Fundamentals of Counting Theory

Dr.Binoy Balan K

AHALIA SCHOOL OF ENGINEERING & TECHNOLOGY

2.1 The Pigeon-hole Principle

The Pigeonhole principle-If n pigeons are assigned to m pigeonholes, and m<n, then atleast one pigeonhole contains two or more pigeons.

Ex. An office employs 13 file clerks, so atleast two of them must have birthdays during the same month.

Sol. Here we have 13 pigeons and 12 pigeonholes (the month of the year). By pigeonhole principle atleaast two of them must have birthdays during the same month

Ex. Show that if any five numbers from 1 to 8 are chosen, then two of them will add up to 9.

Sol. Construct four different sets, A1= {1, 8}, A2= {2, 7}, A3= {3, 6}, A4= {4, 5}. Each of the five numbers chosen must belong to one of these sets. Since there are only four sets, by pigeonhole principle, two of the chosen numbers belong to the same set. These numbers add up to 9.

2.1 The Pigeon-hole Principle

Ex. Show that if any 11 numbers are chosen from the set {1, 2,….., 20}, then one of them will be multiple of another.

Sol. Every positive integer n can be written as $n = 2^k m$, where m is odd and k>=0. If 11 numbers are chosen from the set {1, 2, 3, ……20}, then two of them must have the same odd part. This follows from pigeonhole principle since there are 11 pigeons and 10 pigeonholes (10 odd numbers between 1 to 20)

2.1 The Pigeon-hole Principle

The Extended Pigeonhole principle

If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain atleast $\lfloor (n-1)/m \rfloor$ +1 pigeons.

Ex. Show that if any 30 peoples are selected, then we may choose a subset of 5 so that all 5 were born on the same day of the week.

Sol. Assign each person to the day of the week on which she or he was born. Then 30 pigeons are being assigned to 7 pigeonholes. By the extended pigeonhole principle with n= 30 and m =7, atleast $\lfloor (30-1)/7 \rfloor$ +1 or 5 of the people must have been born on the same day of the week.

2.1 The Pigeon-hole Principle

Ex. A school has 550 students show that atleast two of them were born on the same day of the year.

Sol. Pegionholes-365 days

pegions- 550 students

By pegionhole principle, atleast two students have the birthday on the same date

2.2 The Rule of Sum

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of m+n ways.

Ex. A college library has 40 textbooks on Sociology and 50 textbooks dealing with Anthropology. By the rule of sum, a student at this college can select among 40+50 = 90 textbooks in order to learn more about one or the other of these two subjects.

2.3 Extension of Sum Rule

Extension of Sum Rule: If tasks $T_1, T_2, \ldots\ldots, T_m$ can be done in $n_1, n_2, n_3, \ldots\ldots, n_m$ ways respectively and no two of these tasks can be performed at the same time, then the number of ways to do one of these tasks is $n_1 + n_2 + \ldots + n_m$.

Ex. If a student can choose a project either 20 from Mathematics or 35 from Computer Science or 15 from Engineering, then the student can choose a project in 20 + 35+15= 75 ways.

2.4 The Rule of Product

If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

Ex. The drama club of central University isis holding tryouts for a spring play. With 6 men and 8 women auditioning for the leading male and female roles. By the rule of product, the director can cast his leading couple in 6* 8= 48 ways

2.5Extension of Product Rule , Permutations

Extension of Product Rule: Suppose a procedure consists of performing tasks T_1, T_2,……, T_m in that order. Suppose task T_i can be performed in n_i ways after the tasks T_1, T_2,……, T_{i-1} are performed, then the number of ways the procedure can be executed in the designated order is n_1, n_2,……, n_m.

Ex.Charmas brand shirt available in 12 colors, has a male and female version. It comes in four sizes for each sex, comes in three makes of economy, standard and luxury. Then the number of different types of shirts produced are 12*2*4*3=288 ways.

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2.5 Permutations

Def: Given a collection of n distinct objects, any (linear) arrangement of these objects is called a permutation of the collection.

If there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects are

$$P(n, r) = n \times (n-1) \times (n-2) \times \ldots \ldots \times (n-r+1) = n! / (n-r)!$$

2.5 Permutations

Ex. In how many ways can eight men and eight women be one seated in a row if (a) any person may sit next to any other (b) men and women must occupy alternate seats (c) generalize this result for n men and n women.

Sol. (a) $P(16,16)=16!$

(b)

M	W	M	W	M	W	M	W	M	W	M	W	M	W	M	W

men sitting first: the number of ways *is 8! 8!*

women sitting first: the number of ways is 8! 8!

Thus the number of ways men and women occupy alternatively is $8!8!+ 8!\,8!= 2(8!\,8!)$

(c) Any person may sit = $(2n)!$

Men and women sit alternatively: 2(n!)^2

2.5 Permutations

Ex. How many arrangements are there of all the letters in COMPUTER be arranged? Sol: 8!

Ex. How many arrangements are there of all the letters in DATABASES be arranged?

Sol: 9!/ (2! 3!) = 30, 240.

Ex. In how many ways can the letters of the word ALLAHABAD be arranged ?

Ex.(a) In how many ways can seven people be arranged about a circular table? (7-1)!

(b) If two of the people insist on sitting next to each other, how many arrangements are possible? H &W+5=6 members= (6-1)!*2!=5! 2!

Ex. How many distinct four digit integers can one make from the digits 1, 3, 3, 7, 7 and 8?

2.6 Combinations

If we start with n distinct objects, each selection, or combination, of r of these objects, with no reference of order, corresponds to r! permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is $C(n, r) = P(n, r)/ r! = n!/ [r! (n-r)!]$; $0 \le r \le n$.

Ex. A student taking a history examination is directed to answer any seven of 10 essay questions.
Student can answer the examination in $C(10, 7) = 120$ ways

2.6 Combinations

Ex.A student taking a history examination is directed to answer three questions from the first five and four questions from the last five. Student can complete the examination in C(5, 3) * C(5, 4)= 50 ways.
Ex. A student taking a history examination is directed to answer 7 of the ten questions where atleast three are selected from the first five.
Then there are three cases
(i) The student answers three of the first five questions and four of the last five questions, C(5,3) C(5,4)= 50 ways.
(ii) C(5, 4) C(5,3)= 50 ways
(iii) C(5,5) C(5, 2)= 10 ways
Hence student can complete the examinations in 50+50+10 ways= 110 ways.

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2.6 Combinations

Ex. How many arrangements of the letters in MISSISSIPPI have no consecutive S's

Sol. Number of arrangements of the letters is 11!/(4!4!2! 1!)

When we disregard the S's, there are 7!/(4!2! 1!)=105 ways to arrange the remaining letters.

One of the arrangements as follows

I I M I P I P

and eight possible locations for the four S's. Four of these locations can be selected in C(8, 4)= 70 ways and because this is also possible for all other 104 arrangements. Hence by rule of product there are 105×70=7,350 arrangements of the letters in MISSISSIPPI with no consecutive S's.

2.6 Combination with repetition

The formula for distributing n objects into r categories with repetitions is C(n+r-1, r) = C(n+r-1, n-1)

Ex. A donut shop offers 20 kinds of donuts. Assuming that there are atleast a dozen of each kind when we enter the shop.

Sol. We can select a dozen donuts in C(20+12-1, 12)= C(31, 12)= 141120525 ways.

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2.6 Combination with repetition

The solution of the following example will turn out to be equivalent to counting combinations with repetitions.

Ex. Determine all integer solution to the equation $x1+x2+x3+x4=7$, where $xi>=0$ for all $1\leq i\leq 4$.

One solution of the equation is $x1=3$, $x2=3$, $x3=0$, $x4=1$.

A possible interpretation for the solution is that we are distributing 7 donuts among four children , and here we have given 3 donuts to each of the first two children, nothing to third child, and the last donut to the fourth child. That is, each non negative integer solution of the equation corresponds to a selection with repetition, of size 7 (the identical donuts) from a collection of size 4 (the distinct children), so there are $C(4+7-1, 7)= 120$ solutions.

2.6 Combination with repetition

Ex. In how many ways can one distribute 10 (identical) white marbles among six distinct containers?

Sol. C(6+10-1,10)=3003

Ex. In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among three student helpers if each helper wants atleast one donut of each kind.

Sol. There are 8 chocolates and 7 jelly donuts. Suppose x1, x2, x3 are the number of chocolate and jelly which 3 students will get.

From question (atleast one item) x1, x2, x3 $\geq$1 ie. xi$\geq$1

ie.xi-1$\geq$0

ie. yi$\geq$0; where xi-1= yi

2.6 Combination with repetition

Given that 8 chocolates need to give to 3 children

$x1+x2+x3 =8$

$x1-1+x2-1+x3-1=8-3$ (subtract 3 on both sides)

(ie.) $y1+y2+y3=5$. If sum of the non negative integer is given then number of different combinations for integer is $C(n+r-1, r)$ where n= number of non negative integers, r= sum of integers.

ie. $C(3+5-1, 5)= C(7, 5)$

Now the condition given, 7 jelly donuts need to give to 3 children, $x1+x2+x3=7$, $x1-1+x2-1+x3-1=7-3$

ie. $y1+y2+y3=4$

Ie. $C(3+4-1, 4)= C(6, 4)$

By the rule of product, total solution are

$C(7,5)\ C(6,4)=315$ ways

2.7 The Binomial Theorem

If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \ldots + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n} x^n y^0$$

Ex. Coefficient of $x^5 y^2$ in the expansion of $(x + y)^7$ is $\binom{7}{5} = 21$

Ex. Coefficient of $a^5 b^2$ in the expansion of $(2a - 3b)^7$, replace 2a by x and -3b by y. From the binomial theorem the coefficient of $x^5 y^2$ in $(x + y)^7$ is $\binom{7}{5} = 21$ and $21 x^5 y^2 = 21 (2a)^5 (-3b)^2 = 6048 a^5 b^2$

2.7 The Binomial Theorem

For positive inetgers n, t the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots\ldots x_t^{n_t}$ in the expansion of $(x_1+x_2+\ldots\ldots+x_t)^n$ is $\dfrac{n!}{n_1!\,n_2!\ldots\ldots n_t!}$

Where n1+n2+…..+nt=n

Ex. Coefficient of $x^2y^2 z^3$ in $(x+y+z)^7$ is 210.

Ex. Coefficient of $a^2b^3 c^2d^5$ in $(a+2b-3c+2d+5)^{16}$ is 435,891,456,000,000 $a^2b^3 c^2d^5$

Ex. Determine the sum of all the coefficients in the expansion of (a) $(x+y)^3$ (b) $(x+y+z)^{10}$ (c) $(v+w-2x+y+5z+3)^{12}$

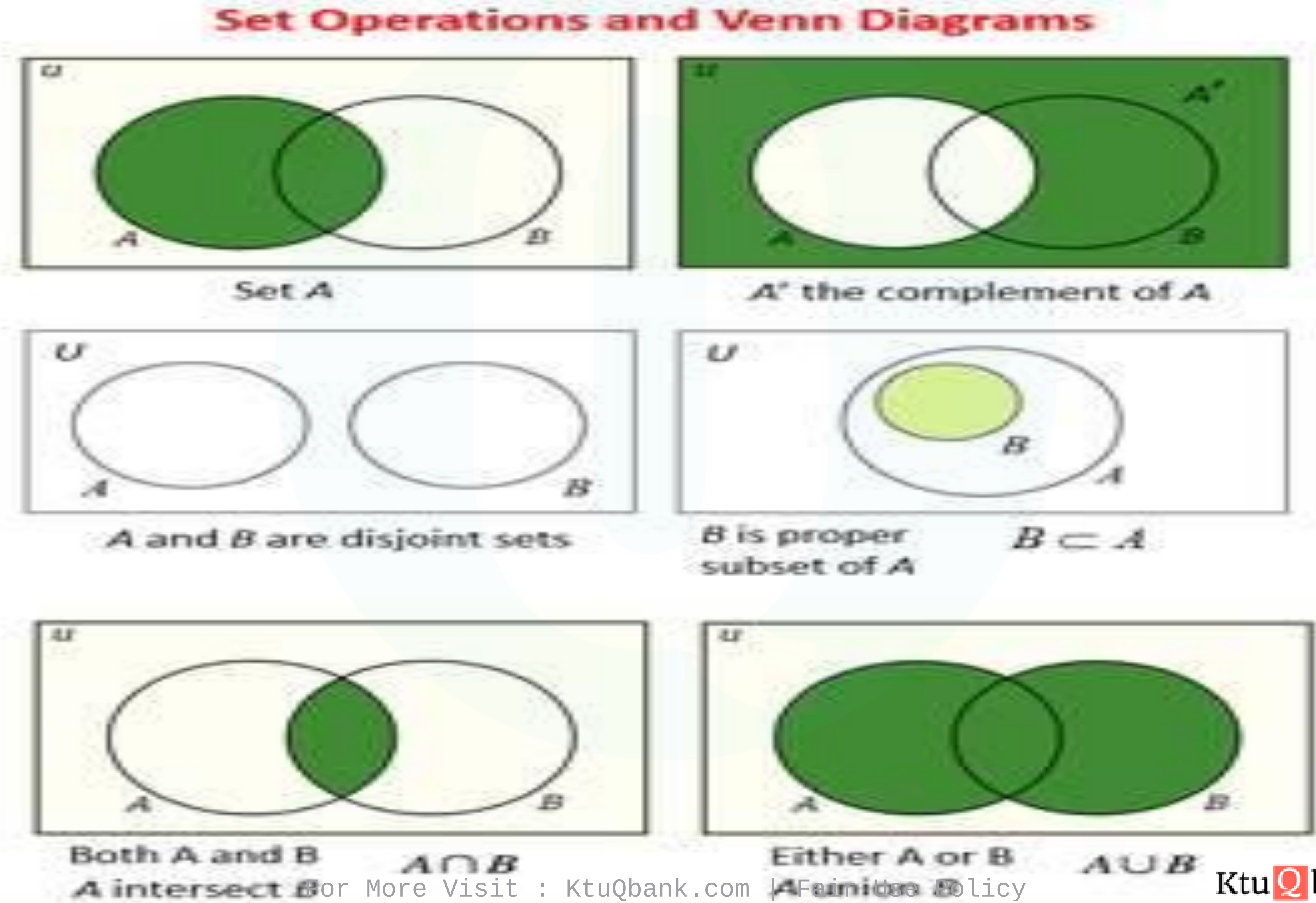
Ans. 2^n= 2^3=8

2.8 The Principle of Inclusion and Exclusion Theorem

The **principle of inclusion and exclusion (PIE)** is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

An underlying idea behind PIE is that summing the number of elements that satisfy at least one of two categories and subtracting the overlap prevents double counting. For instance, the number of people that have at least one cat or at least one dog can be found by taking the number of people who own a cat, adding the number of people that have a dog, then subtracting the number of people who have both.

2.8 The Principle of Inclusion and Exclusion Theorem



2.8 The Principle of Inclusion and Exclusion Theorem

For two sets,

$N(A \cup B) = N(A) + N(B) - N(A \cap B)$

For three sets,

$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$

2.8 The Principle of Inclusion and Exclusion Theorem

Ex. Let S represent the set of 100 students enrolled in the freshman engineering program at Central College. Then |S| =100. Now let $c_{1,}$ c_2 denote the following conditions satisfied by some of the elements of S:

c_1 : A student at Central college is among the 100 students in the freshman engineering program and is enrolled in freshman competition.

c_2 : A student at Central college is among the 100 students in the freshman engineering program and is enrolled in Introduction to Economics. Suppose that 35 of these 100 students are enrolled in Freshman Competition and that 30 of them are enrolled in Introduction to economics. If nine of these 100 students are enrolled in both Freshman Composition and Introduction to Economics. Find the number of students not taking both the course. Ans.44

2.8 The Principle of Inclusion and Exclusion Theorem (Without Proof) Generalization of the Principle

Consider a set S, with $|S| = N$, and conditions $c_i, 1 \leq i \leq t$, each of which may be satisfied by some of the elements of S. The number of elements of S that satisfy none of the conditions c_i , $1 \leq i \leq t$, is denoted by $\overline{N} = N(\overline{c_1}\,\overline{c_2} \ldots\ldots \overline{c_t})$ where

$\overline{N} = N - [N(c_1) + N(c_2) + \ldots\ldots + N(c_t)] + [N(c_1c_2) + N(c_1c_3) + \ldots\ldots + N(c_1c_t) + N(c_2c_3) + \ldots\ldots\ldots + N(c_{t-1}c_t)] - [N(c_1c_2c_3) + N(c_1c_2c_4) + \ldots\ldots\ldots + N(c_1c_2c_t) + N(c_1c_3c_4) + \text{------} + N(c_1c_3c_t) + \ldots\ldots\ldots + N(c_{t-2}c_{t-1}c_t)] + \ldots\ldots + (-1)^t N(c_1c_2c_3\ldots\ldots c_t)$

2.9 Derangements

derangement is a permutation of the elements of a set, such that no element appears in its original position.

Ex. The number of derangements of a set with n elements is $D_n = n![1- \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots\ldots\ldots + (-1)^n \frac{1}{n!}$

Ex. The number of derangements of 1, 2, 3, 4 is D4 = 9

These 9 derangements are

2143	3142	4123
2341	3412	4312
2413	3421	4321

Among the 24-9= 15 permutations of 1, 2, 3, 4 that are not derangements.

2.9 Derangements

Ex. Determine the number of derangements of (2, 4, 6, 1, 3, 5) that end with integer 2, 4 and 6 in some order?
Sol. The place of 2, 4, 6 is specified i.e. each of them will get their place out of the last 3 places only. So 1, 3, 5 will automatically get one of the places in the first 3 places. This must ensure that 2, 4 and 6 occupies one of the last 3 places each and 1, 3 and 5 one of 1st 3 places each. Hence, 1, 3 and 5 can be arranged in 3! ways and 2, 4 and 6 also in 3! Ways. So, no of such derangements = 3! * 3! = 6 * 6 = 36.

2.9 Derangements

A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters (a) no letters is put into the correct envelope (b) exactly 1 letter is put into the correct envelope (c) exactly 98 letters are put into the correct envelope (d) exactly 99 letters are put into the correct envelope (e) all letters are put into the correct envelopes?

Sol: (a) Number of favorable cases is Dn.

The probability of no letter being put in the correct envelope is Dn/100!

(b)When exactly 1 letter is put correctly, the number of derangements for the remaining 99 letters is D99. This can happen in C(100,99) ways. So the probability is C(100, 99) D99/100!

(c) C(100, 2) D2/100! (d) zero (e)1/100!

2.9 Derangements

Ex. A nursery teacher has 5 pencil boxes to give out to her five students. Determine the probability that at least one student gets their name tag?

Sol.There are 5!= 120 ways to give out the pencil boxes. By using complementary probability, the number of ways where nobody gets their pencil boxes is

5!(10!–11!+…–15!)

= 44. Hence, the required probability

is (120–44)/120=19/30.

2.9 Derangements

Ex. A postman can put 12 letters into their respective envelopes such that exactly 5 will go into the right envelope. Find the number of ways of doing this work

Sol. The number of ways in which the 5 correct envelopes can be selected = ${}^{12}C_5$ = 864

Derangement of the remaining 7 envelopes & letters = 1864 (derangement value for 7 is 1864)

Total No of ways of arrangement = 1864 * 864 = 1610496.