MODULE-5

COMPLEX VARIABLE RESIDUE INTEGRATION

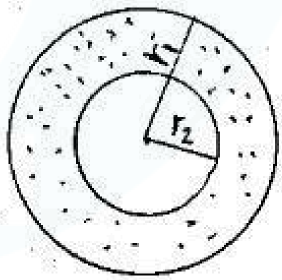
Laurent's series

Let C_1 and C_2 be two concentric circles with centre at z_0 and radius r_1 and r_2 with $r_2 < r_1$. If $f(z)$ is analytic inside and on a ring shaped region bound by C_1 and C_2, then at each point z in the region can be represented by,

$$f(z) = \sum_{n=0}^{\infty} \underbrace{a_n (z-z_0)^n}_{\text{Analytic part}} + \sum_{n=1}^{\infty} \underbrace{b_n (z-z_0)^{-n}}_{\text{Principal part}}$$

where, $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \, dz$$



Q: Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in

(i) $|z| < 1$ (ii) $1 < |z| < 2$

(iii) $|z| > 2$ (iv) $0 < |z-1| < 1$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

put $z = 2$ put $z = 1$

$1 = 0 + B$ $1 = -A + 0$

$B = 1$ $A = -1$

(i) $|z| < 1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{-(1-z)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \left[1+z+z^2+z^3+\ldots\right] - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\right]$$

(ii) $1 < |z| < 2$

$1 < |z|$ $\qquad$ $|z| < 2$

$\frac{1}{|z|} < 1$ $\qquad$ $\frac{|z|}{2} < 1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z(1-\frac{1}{z})} + \frac{1}{-2(1-\frac{z}{2})}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right] - \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\right]$$

(iii) $|z| > 2$

$\frac{2}{|z|} < 1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z(1-\frac{1}{z})} + \frac{1}{z(1-\frac{2}{z})}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right)+\frac{1}{z}\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\dots\right)$$

(iv) $0 < |z-1| < 1$

$|z-1| < 1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z-1} + \frac{1}{(z-1)-1}$$

$$= \frac{-1}{z-1} + \frac{1}{-1(1-(z-1))}$$

$$= \frac{-1}{z-1} - (1-(z-1))^{-1}$$

$$= \frac{-1}{z-1} - \left[1+(z-1)+(z-1)^2+(z-1)^3+\dots\right]$$

Q: find the Laurent's series expansion of

$f(z) = \frac{1}{(z+1)(z+3)}$ in (i) $|z| > 3$

(ii) $1 < |z| < 3$

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$1 = A(z+3) + B(z+1)$$

put $z=-3$

$$1 = 0 - 2B$$

$$B = -\frac{1}{2}$$

put $z=-1$

$$1 = 2A + 0$$

$$A = \frac{1}{2}$$

$$\frac{1}{(z+1)(z+3)} = \frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{z+3}$$

(i) $|z| > 3$

$$\frac{3}{|z|} < 1$$

$$\frac{1}{(z+1)(z+3)} = \frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{z+3}$$

$$= \frac{1}{2} \times \left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{z\left(1+\frac{3}{z}\right)} \right]$$

$$= \frac{1}{2z} \left[\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{3}{z}\right)^{-1} \right]$$

$$= \frac{1}{2z}\left[\left(1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\dots\right)-\left(1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\dots\right)\right]$$

(ii) $1 < |z| < 3$

$1 < |z|$ $\qquad$ $|z| < 3$

$\frac{1}{|z|} < 1$ $\qquad$ $\frac{|z|}{3} < 1$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z(1+\frac{1}{z})} - \frac{1}{3(\frac{z}{3}+1)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\dots\right] - \frac{1}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\dots\right]\right]$$

Q: Expand $f(z) = \frac{1}{z(1-z)}$ in the region $|z+1| > 2$

$$\frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z}$$

$$1 = A(1-z) + Bz$$

put $z=1$ | put $z=0$

$1 = 0 + B$ | $1 = A + 0$

$B = 1$ | $A = 1$

$$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$

now, $|z+1| > 2$

$$\frac{2}{|z+1|} < 1$$

$\therefore$

$$\frac{1}{z(1-z)} = \frac{1}{(z+1)-1} + \frac{1}{1-(z+1)+1}$$

$$= \frac{1}{(z+1)-1} + \frac{1}{2-(z+1)}$$

$$= \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} + \frac{1}{-(z+1)\left(1-\frac{2}{z+1}\right)}$$

$$= \frac{1}{z+1}\left[\left(1-\frac{1}{z+1}\right)^{-1} - \left(1-\frac{2}{z+1}\right)^{-1}\right]$$

$$= \frac{1}{z+1}\left[\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\left(\frac{1}{z+1}\right)^3+\dots\right]-\left[1+\left(\frac{2}{z+1}\right)+\left(\frac{2}{z+1}\right)^2+\left(\frac{2}{z+1}\right)^3+\dots\right]\right]$$

Q: find the Laurent's series expansion of $f(z) = \frac{1}{z-z^3}$ in $1 < |z+1| < 2$

$$\frac{1}{z-z^3} = \frac{1}{z(1-z^2)} = \frac{1}{z(1+z)(1-z)}$$

$$\frac{1}{z(1+z)(1-z)} = \frac{A}{z} + \frac{B}{1+z} + \frac{C}{1-z}$$

$$1 = A(1+z)(1-z) + Bz(1-z) + Cz(1+z)$$

put $z = 1$

$$1 = 0 + 0 + 2C$$

$$C = \frac{1}{2}$$

put $z = -1$

$$1 = 0 + B \times -2$$

$$B = \frac{-1}{2}$$

equating coefficients of z^2,

$$0 = -A + \frac{1}{2} + \frac{1}{2}$$

$$A = 1$$

$$\frac{1}{z(1+z)(1-z)} = \frac{1}{z} + \frac{-1/2}{1+z} + \frac{1/2}{1-z}$$

now, $1 < |z+1| < 2$

$1 < |z+1|$ $\qquad$ $|z+1| < 2$

$\frac{1}{|z+1|} < 1$ $\qquad$ $\frac{|z+1|}{2} < 1$

$\therefore$

$$\frac{1}{z(1+z)(1-z)} = \frac{1}{(z+1)-1} - \frac{1/2}{1+z} + \frac{1/2}{1-(z+1)+1}$$

$$= \frac{1}{(z+1)-1} - \frac{1/2}{1+z} + \frac{1/2}{2-(z+1)}$$

$$= \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} - \frac{1/2}{z+1} + \frac{1/2}{2\left(1-\frac{z+1}{2}\right)}$$

$$= \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{1}{2}\frac{1}{(z+1)} + \frac{1}{4}\left(1-\frac{z+1}{2}\right)^{-1}$$

$$= \frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\left(\frac{1}{z+1}\right)^3+\dots\right] - \frac{1}{2(z+1)} + \frac{1}{4}\left[1+\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2+\left(\frac{z+1}{2}\right)^3+\dots\right]$$

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Q: Find Laurent's series expansion of

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} \quad \text{in the region}$$

(i) $|z| < 3$

(ii) $2 < |z| < 3$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6}$$

$$z^2+5z+6 \overset{1}{\overline{\big)\, z^2-1}}$$
$$\underline{z^2+5z+6}$$
$$0-5z-7$$

Q = quotient
R = Reminder
D = Divisor

$$\frac{z^2-1}{(z+2)(z+3)} = Q + \frac{R}{D}$$

$$= 1 + \frac{(-5z-7)}{z^2+5z+6}$$

$$= 1 - \frac{5z+7}{\underbrace{z^2+5z+6}_{(z+2)(z+3)}} \quad \text{------- (1)}$$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$5z+7 = A(z+3) + B(z+2)$$

put $z = -3$

$$-15+7 = 0 - B$$

$$B = 8$$

put $z = -2$

$$-10+7 = A + 0$$

$$A = -3$$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3}$$

put this in ①

$$\frac{z^2-1}{(z+1)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) $|z| < 3$

$\left|\frac{z}{3}\right| < 1$

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{2\left(\frac{z}{2}+1\right)} - \frac{8}{3\left(\frac{z}{3}+1\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\dots\right] - \frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\dots\right]$$

(ii) $2 < |z| < 3$

$2 < |z|$ $\qquad$ $|z| < 3$

$\frac{2}{|z|} < 1$ $\qquad$ $\frac{|z|}{3} < 1$

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(\frac{z}{3}+1\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\dots\right] - \frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\dots\right]$$

Q: Find the Laurent's series expansion of $f(z) = \frac{z}{(z+1)(z+2)}$ about $z=-2$ in the region $0 < |z+2| < 2$

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+1)$$

put $z=-2$

$$-2 = 0 - B$$

$$B = 2$$

put $z=-1$

$$-1 = A + 0$$

$$A = -1$$

$$\frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$0 < |z+2| < 2$$

$$|z+2| < 2$$

$$\frac{|z+2|}{2} < 1$$

$$\frac{z}{(z+1)(z+2)} = \frac{-1}{(z+2)-2+1} + \frac{2}{z+2}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= \frac{-\cancel{1}}{-\cancel{1}(1-(z+2))} + \frac{2}{z+2}$$

$$= \left(1-(z+2)\right)^{-1} + \frac{2}{z+2}$$

$$= \left[1+(z+2)+(z+2)^2+(z+2)^3+\dots\right] + \frac{2}{z+2}$$

Q: find Laurent's series expansion of $f(z) = \dfrac{7z-2}{z(z+1)(z-2)}$ in $1 < |z+1| < 3$

$$\frac{7z-2}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$$

$$7z-2 = A(z+1)(z-2) + B(z)(z-2) + Cz(z+1)$$

put $z=-1$

$$-7-2 = 0 + 3B$$

$$B = -\frac{9}{3} = -3$$

put $z=2$

$$14-2 = 0+0+6C$$

$$C = \frac{12}{6} = 2$$

put $z=0$

$$-2 = -2A + 0 + 0$$

$$A = 1$$

$$\frac{7z-2}{z(z+1)(z-2)} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

$1 < |z+1| < 3$

$1 < |z+1|$ $\qquad$ $|z+1| < 3$

$\frac{1}{|z+1|} < 1$ $\qquad$ $\frac{|z+1|}{3} < 1$

$$\frac{7z-2}{z(z+1)(z-2)} = \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

$$= \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} - \frac{3}{z+1} + \frac{2}{-3\left(1-\frac{z+1}{3}\right)}$$

$$= \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} - \frac{2}{3}\left(1-\frac{z+1}{3}\right)^{-1}$$

$$= \frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\left(\frac{1}{z+1}\right)^3+\dots\right] - \frac{3}{z+1}$$

$$- \frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\dots\right]$$

SINGULAR POINT

A point $z = z_0$ at which function $f(z)$ fails to be analytic is called singular point or singularity of $f(z)$.

eg: $f(z) = \frac{1}{z-3}$

$z - 3 = 0$

$z = 3$ is a singular point.

Isolated singularity

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if,

(1) $f(z)$ is not analytic at $z = z_0$

(2) there exist a neighbourhood of $z = z_0$ containing no other singularities.

eg: $f(z) = \frac{1}{z}$

this function is analytic everywhere except at $z = 0$.

3-types of isolated singularity

⇨ ① Removable singularity

② Pole

③ Essential singularity

We've Laurent's series expansion of f(z) as,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

① **Removable singularity**

If a Laurent's series about $z=z_0$ of f(z) has no negative powers, then it is known as removable singularity.

② **Pole**

If a Laurent's series about $z=z_0$ of f(z) has finite number of negative powers, then it is known as pole.

③ **Essential singularity**

If a Laurent's series about $z=z_0$ of f(z) has infinite number of negative powers, then it is known as essential singularity.

Q: Find the singularities of following functions.

(1) $f(z) = e^{1/z}$

$$f(z) = e^{1/z}$$

$$= 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \dots$$

$\therefore$ $z = 0$ is an essential singularity.

(2) $f(z) = e^{\frac{1}{z-4}}$

$$f(z) = e^{1/z-4}$$

$$= 1 + \frac{\frac{1}{z-4}}{1!} + \frac{\left(\frac{1}{z-4}\right)^2}{2!} + \frac{\left(\frac{1}{z-4}\right)^3}{3!} + \dots$$

here $z-4 \Rightarrow z=4$

$\therefore$ $z=4$ is an essential singularity.

(3) $f(z) = \frac{\sin z}{z}$

$$f(z) = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right]$$

$z=0$ is a removable singularity.

(4) $f(z) = \frac{e^z}{z}$

$$f(z) = \frac{e^z}{z}$$

$$= \frac{1}{z}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right]$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$$

$z = 0$ is a pole.

(5) $f(z) = \frac{1}{z - \sin z}$

$$f(z) = \frac{1}{z - \sin z} = \frac{1}{\cancel{z} - \left[\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right]}$$

$$= \frac{1}{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots}$$

$$= \frac{1}{\frac{z^3}{3!}\left[1 - \frac{z^2}{5 \times 4} + \frac{z^4}{7 \times 6 \times 5 \times 4} - \dots\right]}$$

$$= \frac{1}{\frac{z^3}{3!}\left[1 - \left(\frac{z^2}{5 \times 4} - \frac{z^4}{7 \times 6 \times 5 \times 4} + \dots\right)\right]}$$

$$= \frac{3!}{z^3}\left[1-\left(\frac{z^2}{5\times4}-\frac{z^4}{7\times6\times5\times4}+\dots\right)\right]^{-1}$$

$$= \frac{3!}{z^3}\left[1+\left[\frac{z^2}{5\times4}-\frac{z^4}{7\times6\times5\times4}+\dots\right]+\left[\frac{z^2}{5\times4}-\frac{z^4}{7\times6\times5\times4}+\dots\right]^2+\dots\right]$$

$z=0$ is a pole

RESIDUES

Let $f(z) = \dfrac{1}{(z-a)(z-b)^2(z-c)^5}$

$(z-a)(z-b)^2(z-c)^5 = 0$

$z-a=0$ $\quad (z-b)^2=0$ $\quad (z-c)^5=0$

$z=a$ $\quad z=b,b$ $\quad z = c,c,c,c,c$

$z=a$ is simple pole

$z=b$ is pole of order 2

$z=c$ is pole of order 5

① If $z=a$ is a simple pole,

$$\underset{z=a}{\text{Res}}[f(z)] = \lim_{z\to a} (z-a)\, f(z)$$

② If $z=a$ is pole of order 'n'

$$\underset{z=a}{\text{Res}}[f(z)] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

③ If $f(z) = \dfrac{h(z)}{g(z)}$

$$\underset{z=a}{\text{Res}}[f(z)] = \frac{h(a)}{g'(a)}$$

④ $\text{Res}[f(z)]$ = coefficient of $\frac{1}{z}$

Q: find the residues of $f(z) = \dfrac{2z-3}{(z-3)(z+4)}$

$(z-3)(z+4)=0$

$z-3=0$ $\qquad$ $z+4=0$

$z=3$ $\qquad$ $z=-4$

here, $z=3,-4$ are simple poles.

∴ here

$$\underset{z=a}{\operatorname{Res}}[f(z)] = \lim_{z \to a} (z-a)\, f(z)$$

at $z = 3$

$$\underset{z=3}{\operatorname{Res}}[f(z)] = \lim_{z \to 3} \cancel{(z-3)} \times \frac{2z-3}{\cancel{(z-3)}(z+4)}$$

$$= \frac{2 \times 3 - 3}{3+4}$$

$$= \frac{3}{7}$$

at $z = -4$

$$\underset{z=-4}{\operatorname{Res}}[f(z)] = \lim_{z \to -4} \cancel{(z+4)} \times \frac{2z-3}{(z-3)\cancel{(z+4)}}$$

$$= \frac{2 \times -4 - 3}{-4-3}$$

$$= \frac{-11}{-7}$$

$$= \frac{11}{7}$$

Q: Find poles and residues of $f(z) = \frac{z e^{2iz}}{z^2+9}$

$$z^2+9=0$$

$$(z+3i)(z-3i)=0$$

$$z+3i=0 \qquad z-3i=0$$

$$z=-3i \qquad z=3i$$

$z=-3i$ and $z=3i$ are simple poles.

$\therefore$ here

$$\underset{z=a}{\text{Res}}[f(z)] = \lim_{z\to a} (z-a)f(z)$$

at $z=-3i$

$$\underset{z=-3i}{\text{Res}}[f(z)] = \lim_{z\to -3i} (z+\cancel{3i}) \frac{z e^{2iz}}{(\cancel{z+3i})(z-3i)}$$

$$= \frac{-3i\, e^{2\times i\times -3i}}{-3i-3i}$$

$$= \frac{-\cancel{3i}\, e^{6}}{2\times -\cancel{3i}}$$

$$= \frac{1}{2} e^{6}$$

at $z=3i$

$$\operatorname*{Res}_{z=3i}[f(z)] = \lim_{z\to 3i} \cancel{(z-3i)} \frac{z e^{2iz}}{(z+3i)\cancel{(z-3i)}}$$

$$= \frac{3i \times e^{2i\times 3i}}{3i+3i}$$

$$= \frac{3i\, e^{-6}}{2\,3i}$$

$$= \frac{e^{-6}}{2}$$

$$= \frac{1}{2e^{6}}$$

Q: Find poles and residues of $f(z) = \frac{ze^{z}}{(z+1)^3}$

$(z+1)^3 = 0$

$z+1 = 0$

$z = -1$ $\{z=-1,-1,-1$

$z=-1$ is pole of order 3. $\{n=3$

∴ here

$$\operatorname*{Res}_{z=a}[f(z)] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

$$\underset{z=-1}{\text{Res}}\,[f(z)] = \frac{1}{2!} \lim_{z \to -1} \frac{d^2}{dz^2} \cancel{(z+1)^3} \times \frac{ze^z}{\cancel{(z+1)^3}}$$

$$= \frac{1}{2!} \lim_{z \to -1} \frac{d}{dz} \left[ze^z + e^z \right]$$

$$= \frac{1}{2} \lim_{z \to -1} ze^z + e^z + e^z$$

$$= \frac{1}{2} \times \left[-1 \times e^{-1} + e^{-1} + e^{-1} \right]$$

$$= \frac{e^{-1}}{2}$$

$$= \frac{1}{2e}$$

Q: Find poles and residues of $f(z) = \frac{z+1}{z^2(z-2)}$

$z^2(z-2) = 0$ $\quad$ $z-2=0$

$z^2 = 0$ $\quad$ $z = 2$

$z = 0$

$z=0$ is pole of order 2.

$z=2$ is a simple pole.

at $z=0$ $\quad \{n=2$

$$\underset{z=a}{\text{Res}}[f(z)] = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

$$\underset{z=0}{\text{Res}}[f(z)] = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} z^2 f(z)$$

$$= \lim_{z \to 0} \frac{d}{dz} \cancel{z^2} \times \frac{z+1}{\cancel{z^2}(z-2)}$$

$$= \lim_{z \to 0} \frac{(z-2)\times 1 - (z+1)\times 1}{(z-2)^2}$$

$$= \frac{0-2-(0+1)}{(0-2)^2}$$

$$= \frac{-3}{4}$$

at $z=2$

$$\underset{z=a}{\operatorname{Res}}\,[f(z)] = \lim_{z\to a} (z-a)\,f(z)$$

$$\underset{z=2}{\operatorname{Res}}\,[f(z)] = \lim_{z\to 2} \cancel{(z-2)} \times \frac{z+1}{z^2\cancel{(z-2)}}$$

$$= \frac{2+1}{2^2}$$

$$= \frac{3}{4}$$

Q: find the residue of $\frac{1+e^z}{\sin z + z\cos z}$ at the point $z=0$

$$f(z) = \frac{1+e^z}{\sin z + z\cos z} = \frac{h(z)}{g(z)}$$

$g(z) = \sin z + z\cos z$

$g'(z) = \cos z + -z\sin z + \cos z$

Here $\underset{z=a}{\operatorname{Res}}\,[f(z)] = \frac{h(a)}{g'(a)}$

$$\underset{z=0}{\operatorname{Res}}\,[f(z)] = \frac{1+e^0}{\cos 0 - 0 + \cos 0}$$

$$= \frac{1+1}{2}$$

$$= 1$$

Q: Find poles and residues of $f(z) = \tan z$

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

$$\cos z = 0$$

$$z = (2n+1)\frac{\pi}{2}$$ { simple pole.

here, $$\underset{z=a}{\text{Res}}[f(z)] = \frac{h(a)}{g'(a)}$$

$h(z) = \sin z$
$g(z) = \cos z$
$g'(z) = -\sin z$

$$\underset{z=(2n+1)\frac{\pi}{2}}{\text{Res}}[f(z)] = \frac{\sin\left[(2n+1)\frac{\pi}{2}\right]}{-\sin\left[(2n+1)\frac{\pi}{2}\right]}$$

$$= -1$$

Q: Find poles and residues of $f(z) = \cot z$

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

$$\sin z = 0$$

$$z = n\pi$$ { simple pole

here, $$\underset{z=a}{\text{Res}}[f(z)] = \frac{h(a)}{g'(a)}$$

$h(z) = \cos z$
$g(z) = \sin z$
$g'(z) = \cos z$

$$\underset{z=n\pi}{\text{Res}}\,[f(z)] = \frac{\cos n\pi}{\cos n\pi}$$

$$= \underline{\underline{1}}$$

Q: Find poles and residues of $f(z) = \frac{1-e^{2z}}{z^4}$

$z^4 = 0$

$z = 0$

$z=0$ is pole of order 4

here, $\underset{z=a}{\text{Res}}\,[f(z)] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$

$$\underset{z=0}{\text{Res}}\,[f(z)] = \frac{1}{3!} \lim_{z\to 0} \frac{d^3}{dz^3} \cancel{z^4} \times \frac{1-e^{2z}}{\cancel{z^4}}$$

$$= \frac{1}{6} \lim_{z\to 0} \frac{d^2}{dz^2} (0-2e^{2z})$$

$$= \frac{1}{6} \lim_{z\to 0} \frac{d}{dz} -2\times e^{2z}\times 2$$

$$= \frac{1}{6} \lim_{z\to 0} -4\, e^{2z}\times 2$$

$$= \frac{1}{6} \times -4\times e^{0}\times 2$$

$$= \frac{-8}{6}$$

$$= -\underline{\underline{\frac{4}{3}}}$$

OR

$$f(z) = \frac{1-e^{2z}}{z^4}$$

$z^4 = 0$

$z = 0$ is a pole of order 4

$z=0$ ~~is pole of order or simple pole~~

$$f(z) = \frac{1-e^{2z}}{z^4}$$

$$= \frac{1}{z^4}\left(1-e^{2z}\right)$$

$$= \frac{1}{z^4}\left[\cancel{1} - \left(\cancel{1} + \frac{2z}{1!} + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \dots\right)\right]$$

$$= \frac{1}{z^4}\left[-2z - \frac{\cancel{4}^{2} z^2}{\cancel{2}} - \frac{\cancel{8}^{4} z^3}{\cancel{6}_{3}} - \dots\right]$$

$$= -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3z} - \dots$$

$\underset{z=a}{\text{Res}}[f(z)]$ = coefficient of $\frac{1}{z}$

$$\underset{z=0}{\text{Res}}[f(z)] = -\frac{4}{3}$$

Q: Find poles and residues of $f(z) = \frac{\sinh z}{z^4}$

$$f(z) = \frac{\sinh z}{z^4}$$

$$z^4 = 0$$

$$z = 0$$

$z=0$ is pole of order 4

$$f(z) = \frac{1}{z^4}\left[\sinh z\right]$$

$$= \frac{1}{z^4}\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots\right]$$

$$= \frac{1}{z^3} + \frac{1}{3!\,z} + \frac{z}{5!} + \frac{z^3}{7!} + \dots$$

$$\operatorname{Res}_{z=0}[f(z)] = \frac{1}{3!}$$

$$= \frac{1}{6}$$

Q: find residue of $f(z) = ze^{1/z}$ at $z=0$

$$f(z) = ze^{1/z}$$

$$= z\left[1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \dots\right]$$

$$= z + 1 + \frac{1}{2!\,z} + \frac{1}{3!\,z^2} + \dots$$

$$\operatorname{Res}_{z=0}[f(z)] = \frac{1}{2!} = \frac{1}{2}$$

CAUCHY RESIDUE THEORY

Let $f(z)$ be a function which is analytic inside and on a simple curve C, except for a finite number of singular points $z_1, z_2, \ldots, z_n$ inside C, then

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

Q: Evaluate $\int_C \frac{z\,dz}{(z-4)(z-2)}$ where C is the circle

(i) $|z|=1$ ~~(ii) $|z|=3$~~ (ii) $|z|=3$

$(z-4)(z-2)=0$

$z-4=0$ $\quad$ $z-2=0$

$z=4$ $\quad$ $z=2$

$z=2,4$ are simple poles.

(i) $|z|=1$

at $z=2$

$|z|=|2|=2>1$

$\therefore$ $z=2$ lies outside C

at $z=4$

$|z| = |4| = 4 > 1$

$\therefore z=4$ lies outside c

$$\therefore \int_c \frac{z\,dz}{(z-4)(z-2)} = 2\pi i \times 0$$

$$= \underline{\underline{0}}$$

(ii) $|z|=3$

at $z=2$

$|z| = |2| = 2 < 3$

$\therefore$ $\underline{z=2 \text{ lies inside } c}$

at $z=4$

$|z| = |4| = 4 > 3$

$\therefore z=4$ lies outside c

here, $\underset{z=a}{\text{Res}}[f(z)] = \lim_{z \to a} (z-a) f(z)$

$$\underset{z=2}{\text{Res}}[f(z)] = \lim_{z \to 2} \cancel{(z-2)} \times \frac{z}{(z-4)\cancel{(z-2)}}$$

$$= \frac{2}{2-4}$$

$$= \underline{\underline{-1}}$$

by Cauchy residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residue inside } c$$

$$\int_C \frac{z}{(z-4)(z-2)}\,dz = 2\pi i \times -1$$

$$= -2\pi i$$

Q: Evaluate $\int_C \frac{dz}{z^3(z-1)}$ where c is $|z|=2$

$z^3(z-1)=0$

$z^3=0$ $\quad$ $z-1=0$

$z=0$ $\quad$ $z=1$

$z=0$ is pole of order 3

$z=1$ is a simple pole

$|z|=2$

at $z=0$

$|z|=|0|=0<2$

$\therefore z=0$ lies inside c

at $z=1$

$|z| = |1| = 1 < 2$

$\therefore z=1$ lies inside c

at $z=0$ $\{n=3$

$$\operatorname*{Res}_{z=a}[f(z)] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

$$\operatorname*{Res}_{z=0}[f(z)] = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} (\cancel{z})^3 \times \frac{1}{\cancel{z^3}(z-1)}$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz} \frac{-1}{(z-1)^2}$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz} -(z-1)^{-2}$$

$$= \frac{1}{2} \lim_{z\to 0} - -2 \times (z-1)^{-3}$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \cdot (0-1)^{-3}$$

$$= \frac{1}{(-1)^{-3}}$$

$$= \underline{\underline{-1}}$$

at $z = 1$

$$\underset{z=a}{\text{Res}}[f(z)] = \lim_{z \to a} (z-a) f(z)$$

$$\underset{z=1}{\text{Res}}[f(z)] = \lim_{z \to 1} \cancel{(z-1)} \times \frac{1}{z^3\cancel{(z-1)}}$$

$$= \frac{1}{1^3}$$

$$= \underline{\underline{1}}$$

by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$\int_C \frac{dz}{z^3(z-1)} = 2\pi i \cdot (-1+1)$$

$$= \underline{\underline{0}}$$

Q: Evaluate $\displaystyle\int_C \frac{\sin z}{(z-1)^2\,(z^2+9)}\,dz$ where C is the circle $|z-3i| = 1$.

$$(z-1)^2\,(z^2+9) = 0$$

$(z-1)^2 = 0$ $\qquad\qquad$ $z^2+9 = 0$

$z = 1$ $\qquad\qquad$ $z^2 = -9$

$\qquad\qquad$ $z = \pm 3i$

$z=1$ is pole of order 2

$z=\pm 3i$ are simple poles.

at $z=1$ $\{|z-3i|=1$

$|z-3i| = |1-3i|$

$= \sqrt{1^2+9}$

$= \sqrt{10} > 1$

$\therefore$ $z=1$ lies outside C

at $z=3i$

$|z-3i| = |3i-3i|$

$= 0 < 1$

$z=3i$ lies inside C

at $z=-3i$

$|z-3i| = |-3i-3i|$

$= |-6i|$

$= \sqrt{36}$

$= 6 > 1$

$z=-3i$ lies outside C

at $z=3i$

$$\underset{z=a}{\mathrm{Res}}[f(z)] = \lim_{z\to a} (z-a)f(z)$$

$$\underset{z=3i}{\mathrm{Res}}[f(z)] = \lim_{z\to 3i} \cancel{(z-3i)} \times \frac{\sin z}{(z-1)^2(z+3i)\cancel{(z-3i)}}$$

$$= \frac{\sin 3i}{(3i-1)^2(3i+3i)}$$

$$= \frac{\sin 3i}{6i\,(3i-1)^2}$$

by Cauchy's residue theorem,

$$\int_c f(z)dz = 2\pi i \times \text{sum of residues inside } c$$

$$\int_c \frac{\sin z}{(z-1)^2(z^2+9)}dz = \cancel{2}\pi \cancel{i} \times \frac{\sin 3i}{\cancel{6i}_{3}(3i-1)^2}$$

$$= \frac{\pi \sin 3i}{3(3i-1)^2}$$

Q: Evaluate $\int_C \frac{z-1}{(z+1)^2 (z-2)} dz$ where C is $|z+i|=2$

$(z+1)^2 (z-2) = 0$

$(z+1)^2 = 0$ $\quad$ $z-2=0$

$z=-1$ $\quad$ $z=2$

$z=-1$ is pole of order 2

$z=2$ is a simple pole

$|z+i|=2$

at $z=-1$

$|z+i| = |-1+i|$

$= \sqrt{1+1}$

$= \sqrt{2} < 2$

$z=-1$ lies inside c

at $z=2$

$|z+i| = |2+i|$

$= \sqrt{4+1}$

$= \sqrt{5} > 2$

$z=2$ lies outside c

at $z=-1$ $\{n=2$

$$\underset{z=a}{\text{Res}}[f(z)] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

$$\underset{z=-1}{\text{Res}}[f(z)] = \frac{1}{1!} \lim_{z\to -1} \frac{d}{dz} \cancel{(z+1)^2} \times \frac{(z-1)}{\cancel{(z+1)^2}(z-2)}$$

$$= \lim_{z\to -1} \frac{(z-2)\times 1 - (z-1)\times 1}{(z-2)^2}$$

$$= \frac{-1-2-(-1-1)}{(-1-2)^2}$$

$$= \frac{-1}{9}$$

by Cauchy residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } c$$

$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = 2\pi i \times \frac{-1}{9}$$

$$= -\frac{2\pi i}{9}$$

Q: Evaluate $\int_C \frac{\cos\pi z^2 + \sin\pi z^2}{(z+1)(z+2)}\, dz$ where C is

$|z| = 3$

$(z+1)(z+2) = 0$

$z+1=0 \qquad z+2=0$

$z=-1 \qquad z=-2$

$z=-1, -2$ are simple poles.

$|z| = |-1| = 1 < 3$ {at $z=-1$

$\therefore$ $z=-1$ lies inside C

at $z=-2$

$|z| = |-2| = 2 < 3$

$\therefore$ $z=-2$ lies inside C

here, $\underset{z=a}{\text{Res}}\,[f(z)] = \lim_{z\to a} (z-a) f(z)$

at $z=-1$

$$\underset{z=-1}{\text{Res}}\,[f(z)] = \lim_{z\to -1} \cancel{(z+1)} \times \frac{\cos\pi z^2 + \sin\pi z^2}{\cancel{(z+1)}(z+2)}$$

$$= \frac{\cos\pi + \sin\pi}{-1+2} = \frac{-1+0}{1} = -1$$

at $z=-2$

$$\underset{z=-2}{\text{Res}}[f(z)] = \lim_{z\to -2} \cancel{(z+2)} \times \frac{\cos\pi z^2+\sin\pi z^2}{(z+1)\cancel{(z+2)}}$$

$$= \frac{\cos 4\pi + \sin 4\pi}{-2+1}$$

$$= \frac{1+0}{-1}$$

$$= \underline{\underline{-1}}$$

by Cauchy's Residue theorem

$$\int_c f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$\int_c \frac{\cos\pi z^2+\sin\pi z^2}{(z+1)(z+2)}\,dz = 2\pi i \times (-1-1)$$

$$= \underline{\underline{-4\pi i}}$$

Q: Evaluate $\int_c \tan z\,dz$ where C is $|z|=\pi$

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

$$\cos z = 0$$

$$z = (2n+1)\frac{\pi}{2}$$

here $|z|=\pi \Leftarrow C$

when $n=0$, $z = \frac{\pi}{2} \Rightarrow |z| = |\frac{\pi}{2}| = \frac{\pi}{2} < \pi$

$\therefore z = \frac{\pi}{2}$ lies inside C ✓

when $n=1$, $z = \frac{3\pi}{2} \Rightarrow |z| = |\frac{3\pi}{2}| = \frac{3\pi}{2} > \pi$

$\therefore z = \frac{3\pi}{2}$ lies outside C

when $n=-1$, $z = -\frac{\pi}{2} \Rightarrow |z| = |-\frac{\pi}{2}| = \frac{\pi}{2} < \pi$

$\therefore z = -\frac{\pi}{2}$ lies inside C ✓

when $n=-2$, $z = -\frac{3\pi}{2} \Rightarrow |z| = |-3\pi/2| = \frac{3\pi}{2} > \pi$

$\therefore z = -\frac{3\pi}{2}$ lies outside C

here

$$\operatorname*{Res}_{z=a}[f(z)] = \frac{h(a)}{g'(a)}$$

$$\left\{\begin{array}{l} h(z) = \sin z \\ g(z) = \cos z \\ g'(z) = -\sin z \end{array}\right.$$

at $z = \frac{\pi}{2}$

$$\operatorname*{Res}_{z=\frac{\pi}{2}}[f(z)] = \frac{\sin \pi/2}{-\sin \pi/2}$$

$$= -1$$

at $z = -\pi/2$

$$\underset{z=-\pi/2}{\text{Res}}\,[f(z)] = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)}$$

$$= -1$$

by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$\int_C \tan z\,dz = 2\pi i \times (-1-1)$$

$$= -\underline{\underline{4\pi i}}$$

Q: Evaluate $\int_C \cot z\,dz$ where C is $|z| = \pi$

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

$\sin z = 0$

$z = \pm n\pi$

here $C \rightarrow |z| = \pi$

when $n = 0$, $z = 0 \Rightarrow |z| = 0 < \pi$

$\therefore z = 0$ lies inside C ✓

when $n=1$, $z=\pi \Rightarrow |z|=|\pi|=\pi$

$\therefore$ $z=\pi$ lies on C ✓

when, $n=2$, $z=2\pi \Rightarrow |z|=|2\pi|=2\pi > \pi$

$\therefore$ $z=2\pi$ lies outside C

when, $n=-1$, $z=-\pi \Rightarrow |z|=|-\pi|=\pi$

$\therefore$ $z=-\pi$ lies on C ✓

here $\underset{z=a}{\text{Res}}[f(z)] = \dfrac{h(a)}{g'(a)}$

$$\left\{\begin{array}{l} h(z)=\cos z \\ g(z)=\sin z \\ g'(z)=\cos z \end{array}\right.$$

at $z=0$

$$\underset{z=0}{\text{Res}}[f(z)] = \frac{\cos 0}{\cos 0}$$

$$= 1$$

at $z=\pi$

$$\underset{z=\pi}{\text{Res}}[f(z)] = \frac{\cos\pi}{\cos\pi} = 1$$

at $z=-\pi$

$$\underset{z=-\pi}{\text{Res}}[f(z)] = \frac{\cos(-\pi)}{\cos(-\pi)} = 1$$

by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \times \text{ sum of residues inside } C$$

$$\int_C f(z)dz = 2\pi i \times (1+1+1)$$

$$= \underline{\underline{6\pi i}}$$

Q : Evaluate $\int_C \tan\pi z \, dz$ where C is $|z|=2$.

$$f(z) = \tan\pi z = \frac{\sin\pi z}{\cos\pi z}$$

$$\cos\pi z = 0$$

$$\pi z = (2n+1)\frac{\pi}{2}$$

$$z = \frac{2n+1}{2}$$

when $n=0$, $z = \frac{1}{2}$

$$|z| = |\tfrac{1}{2}| = \tfrac{1}{2} < 2$$

$\therefore z = \frac{1}{2}$ lies inside C ✓

when $n=1$, $z = \frac{3}{2}$

$$|z| = |\tfrac{3}{2}| = \tfrac{3}{2} < 2$$

$\therefore z = \frac{3}{2}$ lies inside C ✓

when $n = -1$, $z = -\frac{1}{2}$

$$|z| = |-1/2| = \frac{1}{2} < 2$$

$\therefore$ $z = -\frac{1}{2}$ lies inside c ✓

when $n = -2$, $z = -\frac{3}{2}$

$$|z| = |-3/2| = \frac{3}{2} < 2$$

$\therefore$ $z = -\frac{3}{2}$ lies inside c ✓

here, $\operatorname{Res}_{z=a} [f(z)] = \frac{h(a)}{g'(a)}$

$h(z) = \sin \pi z$

$g(z) = \cos \pi z$

$g'(z) = -\pi \sin z$

at $z = \frac{1}{2}$

$$\operatorname{Res}_{z=\frac{1}{2}} [f(z)] = \frac{\sin \pi/2}{-\pi \sin \pi/2}$$

$$= -\frac{1}{\pi}$$

at $z = \frac{3}{2}$

$$\operatorname{Res}_{z=\frac{3}{2}} [f(z)] = \frac{\sin \frac{3\pi}{2}}{-\pi \sin \frac{3\pi}{2}} = -\frac{1}{\pi}$$

at $z = -\frac{1}{2}$

$$\operatorname{Res}_{z=-\frac{1}{2}} [f(z)] = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\pi \sin\left(-\frac{\pi}{2}\right)} = -\frac{1}{\pi}$$

at $z=-\frac{3}{2}$

$$\underset{z=-\frac{3}{2}}{\text{Res}}[f(z)] = \frac{\sin(-\frac{3\pi}{2})}{-\pi\sin(-\frac{3\pi}{2})} = -\frac{1}{\pi}$$

by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$\int_C \tan\pi z\,dz = 2\pi i \times \left[-\frac{1}{\pi}-\frac{1}{\pi}-\frac{1}{\pi}-\frac{1}{\pi}\right]$$

$$= 2\pi i \times -\frac{4}{\pi}$$

$$= \underline{\underline{-8i}}$$

Q: Evaluate $\int_C z\cos\frac{1}{z}\,dz$ where C is $|z|=1$

$$f(z) = z\cos\frac{1}{z}$$

$$= z\left[1-\frac{(1/z)^2}{2!}+\frac{(1/z)^4}{4!}-\frac{(1/z)^6}{6!}+\dots\right]$$

$$= z-\frac{1}{2!z}+\frac{1}{4!z^3}-\frac{1}{6!z^5}+\dots$$

here $z=0$ $\{z-0=0$

$$\underset{z=0}{\text{Res}}[f(z)] = \text{coefficient of } \frac{1}{z} = -\frac{1}{2!} = -\frac{1}{2}$$

by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$\int_C z\cos\frac{1}{z}\,dz = 2\pi i \times -\frac{1}{2}$$

$$= -\pi i$$

Q: Evaluate $\int_C z^4 e^{1/z}\,dz$ where C is $|z|=1$

$$f(z) = z^4 e^{1/z}$$

$$= z^4\left[1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \frac{(1/z)^4}{4!} + \frac{(1/z)^5}{5!} + \dots\right]$$

$$= z^4 + \frac{z^3}{1!} + \frac{z^2}{2!} + \frac{z}{3!} + \frac{1}{4!} + \frac{1}{5!z} + \dots$$

$$\underset{z=0}{\text{Res}}\,[f(z)] = \frac{1}{5!} = \frac{1}{120}$$

by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$\int_C z^4 e^{1/z}\,dz = 2\pi i \times \frac{1}{120}$$

$$= \frac{\pi i}{60}$$

Real integrals

Type - I

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

OR

$$\int_{-\pi}^{\pi} f(\cos\theta, \sin\theta)\, d\theta$$

let C be $|z|=1$

$$z = re^{i\theta}$$

$$\{r=1\}$$

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta$$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

and, • $\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$

• $\sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right]$

- $\cos 2\theta = \frac{1}{2}\left[z^2 + \frac{1}{z^2}\right]$

- $\sin 2\theta = \frac{1}{2i}\left[z^2 - \frac{1}{z^2}\right]$

- $\cos 3\theta = \frac{1}{2}\left[z^3 + \frac{1}{z^3}\right]$

- $\sin 3\theta = \frac{1}{2i}\left[z^3 - \frac{1}{z^3}\right]$

Q: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2}$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

$$= \frac{1}{2}\,\frac{(z^2+1)}{z}$$

$$\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = \int_C \frac{\frac{dz}{iz}}{\left[5 - 3\times\frac{1}{2}\frac{(z^2+1)}{z}\right]^2}$$

$$= \int_C \frac{\frac{dz}{iz}}{\frac{(10z - 3z^2 - 3)^2}{(2z)^2}}$$

$$= \int_C \frac{\frac{dz}{iz}}{\frac{(-3z^2+10z-3)^2}{4z^2}}$$

$$= \frac{4}{i}\int_C \frac{z}{(3z^2-10z+3)^2}\,dz \quad \text{———} (1)$$

solve (1) using Cauchy's Residue theorem,

$$(3z^2-10z+3)^2 = 0$$

$$\left(z^2-\frac{10z}{3}+1\right)^2 = 0$$

$$\left[(z-3)\left(z-\frac{1}{3}\right)\right]^2 = 0$$

$$(z-3)^2\left(z-\frac{1}{3}\right)^2 = 0$$

$$(z-3)^2\,\frac{(3z-1)^2}{3^2} = 0$$

$$(z-3)^2(3z-1)^2 = 0$$

$z-3=0$ $\quad\quad$ $3z-1=0$

$z=3$ $\quad\quad$ $z=\frac{1}{3}$

$z=3$, $z=\frac{1}{3}$ are poles of order 2

at $z=3$

$|z|=|3|=3>1$

$z=3$ lies outside c

at $z=\frac{1}{3}$

$|z|=|\frac{1}{3}|=\frac{1}{3}<1$

$z=\frac{1}{3}$ lies inside c

here,

$$\operatorname*{Res}_{z=a}[f(z)] = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)$$

at $z=\frac{1}{3}$

$$\operatorname*{Res}_{z=\frac{1}{3}}[f(z)] = \frac{1}{1!}\lim_{z\to\frac{1}{3}}\frac{d}{dz}\left(z-\frac{1}{3}\right)^2 \times \frac{z}{(z-3)^2(3z-1)^2}$$

$$= \lim_{z\to\frac{1}{3}}\frac{d}{dz}\frac{\cancel{(3z-1)^2}}{3^2} \times \frac{z}{(z-3)^2\cancel{(3z-1)^2}}$$

$$= \frac{1}{9}\lim_{z\to\frac{1}{3}}\frac{d}{dz}\frac{z}{(z-3)^2}$$

$$= \frac{1}{9}\lim_{z\to\frac{1}{3}}\frac{(z-3)^2(1) - z\times 2(z-3)}{(z-3)^4}$$

$$= \frac{1}{9} \times \frac{\left(\frac{1}{3}-3\right)^2 - \frac{1}{3} \times 2 \times \left(\frac{1}{3}-3\right)}{\left(\frac{1}{3}-3\right)^4}$$

$$= \frac{5}{256}$$

by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

compare ①

$$\Rightarrow \frac{4}{i}\int_C f(z)\,dz = \frac{4}{\cancel{i}} \times 2\pi \cancel{i} \times \frac{5}{256}$$

$$= \frac{5\pi}{32}$$

Q: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

$$d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right]$$

$$= \frac{1}{2i}\frac{(z^2-1)}{z}$$

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{\frac{dz}{iz}}{5+\cancel{4}^{2}\times\frac{1}{\cancel{2}i}\frac{(z^2-1)}{z}}$$

$$= \int_C \frac{\frac{dz}{\cancel{i}z}}{\frac{5iz+2z^2-2}{\cancel{iz}}}$$

$$= \int_C \frac{dz}{2z^2+5iz-2} \quad \text{———} \; ①$$

Use Cauchy's Residue theorem for ①

$$2z^2+5iz-2=0 \qquad z = \frac{-5i \pm \sqrt{-25-4\times2\times-2}}{2\times2}$$

$$\left(z+\frac{i}{2}\right)(z+2i)=0 \qquad z = \frac{-5i \pm 3i}{4}$$

$$\frac{(2z+i)}{2}(z+2i)=0 \qquad z=\frac{-2i}{4} \quad z=\frac{-8i}{4}$$

$$(2z+i)(z+2i)=0 \qquad z=-\frac{i}{2} \quad z=-2i$$

here

$$2z+i=0 \qquad z+2i=0$$

$$2z=-i \qquad z=-2i$$

$$z=\frac{-i}{2}$$

$z=\frac{-i}{2}, -2i$ are simple poles

at $z = \frac{-i}{2}$

$|z| = \left|-\frac{i}{2}\right| = \sqrt{\left(-\frac{1}{2}\right)^2}$

$= \frac{1}{2} < 1$

$\therefore$ $z = \frac{-i}{2}$ lies inside c

at $z = -2i$

$|z| = |-2i| = \sqrt{(-2)^2}$

$= 2 > 1$

$z = -2i$ lies outside c

at $z = \frac{-i}{2}$

$$\underset{z=a}{\text{Res}}\,[f(z)] = \lim_{z \to a} (z-a)\, f(z)$$

$$\underset{z=-\frac{i}{2}}{\text{Res}}\,[f(z)] = \lim_{z \to -\frac{i}{2}} \left(z+\frac{i}{2}\right) \times \frac{1}{(2z+i)(z+2i)}$$

$$= \lim_{z \to -\frac{i}{2}} \frac{\cancel{2z+i}}{2} \times \frac{1}{(\cancel{2z+i})(z+2i)}$$

$$= \frac{1}{2} \times \frac{1}{-\frac{i}{2}+2i}$$

$$= \frac{1}{-i+4i} = \frac{1}{3i}$$

by Cauchy's residue theorem,

$$\int_c f(z)dz = 2\pi i \times \text{sum of residues inside } c$$

now, in ①

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c f(z)\,dz$$

$$= 2\pi i \times \frac{1}{3i}$$

$$= \frac{2\pi}{3}$$

Q: Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

let C be $|z|=1$

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

$$\cos 2\theta = \frac{1}{2}\left[z^2 + \frac{1}{z^2}\right] = \frac{(z^4+1)}{2z^2}$$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] = \frac{(z^2+1)}{2z}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \int_C \frac{\frac{z^4+1}{2z^2}}{5+\cancel{4}^{2}\frac{(z^2+1)}{\cancel{2}z}} \frac{dz}{iz}$$

$$= \frac{1}{2i} \int_C \frac{z^4+1}{z^2\frac{(5z+2z^2+2)}{\cancel{z}}} \frac{dz}{\cancel{z}}$$

$$= \frac{1}{2i} \int_C \frac{z^4+1}{z^2(2z^2+5z+2)}\, dz$$

$$= \frac{1}{2i} \int_C f(z)\, dz \quad \text{--- (1)}$$

Solve (1) using Cauchy's Residue theorem

$$z^2(2z^2+5z+2)=0$$

$$z^2=0$$

$z=0$ { pole of order 2

$$2z^2+5z+2=0$$

$$(z+2)\left(z+\tfrac{1}{2}\right)=0$$

$$(z+2)\left(\frac{2z+1}{2}\right)=0$$

$$(z+2)(2z+1)=0$$

$$z=\frac{-5\pm\sqrt{25-16}}{4}$$

$$=\frac{-5\pm 3}{4}$$

$$z=-2\ ,\ z=-\tfrac{1}{2}$$

$z=-2,\ -\frac{1}{2}$ { simple poles

at $z=0$

$|z|=|0|=0<1$

$z=0$ lies inside C

at $z=-2$

$|z|=|-2|=2>1$

$z=-2$ lies outside C

at $z=-\frac{1}{2}$

$|z|=\left|-\frac{1}{2}\right|=\frac{1}{2}<1$

$z=-\frac{1}{2}$ lies inside C

at $z=0$, $n=2$

$$\operatorname*{Res}_{z=a}[f(z)] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

$$\operatorname*{Res}_{z=0}[f(z)] = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz} \cancel{(z)^2} \times \frac{z^4+1}{\cancel{z^2}(2z^2+5z+2)}$$

$$= \lim_{z\to 0} \frac{(2z^2+5z+2)\,4z^3 - (z^4+1)(4z+5)}{(2z^2+5z+2)^2}$$

$$= \frac{0-5}{4}$$

$$= -\frac{5}{4}$$

at $z=-\frac{1}{2}$

$$\operatorname*{Res}_{z=a}(f(z)) = \lim_{z\to a} (z-a) f(z)$$

$$\operatorname*{Res}_{z=-\frac{1}{2}}[f(z)] = \lim_{z\to -\frac{1}{2}} (z+\tfrac{1}{2}) \times \frac{z^4+1}{z^2(z+1)(2z+1)}$$

$$= \lim_{z\to -\frac{1}{2}} \frac{\cancel{(2z+1)}}{2} \times \frac{z^4+1}{z^2(z+2)\cancel{(2z+1)}}$$

$$= \frac{1}{2} \times \frac{(-\frac{1}{2})^4+1}{(-\frac{1}{2})^2(-\frac{1}{2}+2)}$$

$$= \frac{1}{2} \times \frac{\frac{17}{16}}{\frac{3}{8}}$$

$$= \frac{17}{12}$$

by CRT,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$= 2\pi i \times \left(-\frac{5}{4} + \frac{17}{12}\right)$$

$$= \frac{\pi i}{3}$$

put in ①

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2\pi} \times \frac{i\pi}{3} = \frac{\pi}{6}$$

Q: Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

let $|z| = 1$

$z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] = \frac{(z^2+1)}{2z}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{\frac{dz}{iz}}{2+\frac{(z^2+1)}{2z}}$$

$$= \frac{1}{i}\int_C \frac{1}{\frac{4z+z^2+1}{2\cancel{z}}} \frac{dz}{\cancel{z}}$$

$$= \frac{2}{i}\int_C \frac{1}{z^2+4z+1}\, dz$$

$$= \frac{2}{i}\int_C f(z)dz \quad \text{—(1)}$$

Solve (1) using CRT

$$z^2+4z+1 = 0 \qquad z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$[z-(-2+\sqrt{3})][z-(-2+\sqrt{3})] = 0 \qquad z = \frac{-4 \pm 2\sqrt{3}}{2}$$

$$z = -2+\sqrt{3}, \quad z = -2-\sqrt{3}$$

{ Simple poles.

at $z = -2+\sqrt{3}$

$$|z| = |-2+\sqrt{3}| = |-0.268| = 0.268 < 1$$

$z = -2+\sqrt{3}$ lies inside c

at $z = -2-\sqrt{3}$

$$|z| = |-2-\sqrt{3}| = |-3.732| = 3.732 > 1$$

lies outside c

at $z = -2+\sqrt{3}$

$$\operatorname{Res}_{z=a}[f(z)] = \lim_{z \to a} (z-a) f(z)$$

$$\operatorname{Res}_{z=-2+\sqrt{3}}[f(z)] = \lim_{z \to -2+\sqrt{3}} \cancel{(z-(-2+\sqrt{3}))} \times \frac{1}{\cancel{(z-(-2+\sqrt{3}))}(z-(-2-\sqrt{3}))}$$

$$= \frac{1}{-2+\sqrt{3}-(-2-\sqrt{3})}$$

$$= \frac{1}{2\sqrt{3}}$$

by CRT

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues inside } C$$

$$= 2\pi i \times \frac{1}{2\sqrt{3}} = \frac{\pi i}{\sqrt{3}}$$

put in ①

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i} \times \frac{\pi i}{\sqrt{3}}$$

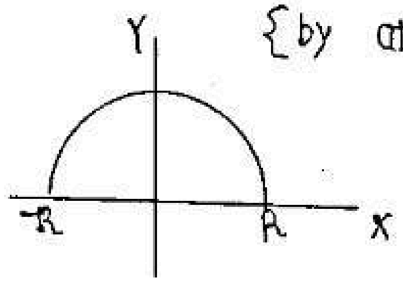
$$= \frac{2\pi}{\sqrt{3}}$$

Type - II

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\,dx$$

$f(x)$, $g(x)$ ⇨ polynomials

degree of numerator $<$ degree of denominator {by atleast 2}

Γ ⇨ 

Semi circle

Choose Γ consisting of real axis from $-R$ to R and upper semicircle c such that $|z|=R$

as $R \to \infty$, $\displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\, dx = \int_{\Gamma} \frac{f(z)}{g(z)}\, dz$

Q: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\, dx$

$6-2=4$.

Choose Γ consisting of real axis from $-R$ to R and upper semi circle such that $|z|=R$

as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\, dx = \int_{\Gamma} \frac{z^2}{(z^2+a^2)^3}\, dz$$

$$= \int_{\Gamma} f(z)\, dz$$

using Cauchy's residue theorem

$(z^2+a^2)^3 = 0$

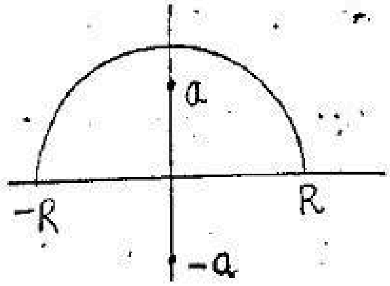
$z^2+a^2 = 0$

$z^2 = -a^2$

$z = \pm ia$ { pole of order 3

at $z = ia \Rightarrow$ inside point

$z = -ia \Rightarrow$ outside point



$\therefore$ at $z = ia$

$$\operatorname*{Res}_{z=a}[f(z)] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

$$\operatorname*{Res}_{z=ia}[f(z)] = \frac{1}{2!} \lim_{z\to ia} \frac{d^2}{dz^2} (z-ia)^3 \times \frac{z^2}{(z^2+a^2)^3}$$

$$= \frac{1}{2} \lim_{z\to ia} \frac{d^2}{dz^2} \cancel{(z-ia)^3} \times \frac{z^2}{(z+ia)^3 \cancel{(z-ia)^3}}$$

$$= \frac{1}{2} \lim_{z\to ia} \frac{d}{dz} \frac{(z+ia)^3 \times 2z - z^2 . 3(z+ia)^2}{(z+ia)^6}$$

$$= \frac{1}{2} \lim_{z\to ia} \frac{d}{dz} \frac{\cancel{(z+ia)^2}\left[(z+ia)2z - 3z^2\right]}{(z+ia)^{\cancel{6}\,4}}$$

$$= \frac{1}{2} \lim_{z\to ia} \frac{d}{dz} \frac{2iaz - z^2}{(z+ia)^4}$$

$$= \frac{1}{2} \lim_{z\to ia} \frac{(z+ia)^4 (2ia - 2z) - (2iaz - z^2) 4 (z+ia)^3}{(z+ia)^8}$$

$$= \frac{1}{2} \times \frac{0 - (-2a^2+a^2)\, 4\, (2ia)^3}{(2ia)^8}$$

$$= \frac{1}{2} \times \frac{a^2 \times 4 \times 2^3 \times i^3 \times a^3}{2^8 . i^8 . a^8}$$

$$= \frac{1}{16a^3 i}$$

by CRT

$$\int_\Gamma f(z)\,dz = 2\pi i \times \text{sum of residues inside } \Gamma$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\,dx = 2\pi i \times \frac{1}{16a^3 i}$$

$$= \underline{\underline{\frac{\pi}{8a^3}}}$$

- $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

- $\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$

Q: Evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$

{ 4-2=2

$$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$$

Choose Γ consisting of real axis from $-R$ to R and upper semicircle C such that $|z|=R$

as $R\to\infty$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \int_\Gamma \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz$$

$$= \int_\Gamma f(z)dz \quad \text{—①}$$

using CRT, for ①

$(z^2+a^2)(z^2+b^2)=0$

$z^2+a^2=0$ $\qquad$ $z^2+b^2=0$

$z^2 = -a^2$ $\qquad$ $z^2=-b^2$

$z = \pm ia$ $\qquad$ $z = \pm ib$

$z = \pm ia, \ \pm ib$ are simple poles.

here $\left.\begin{array}{l} z = ia \\ z = ib \end{array}\right\}$ lies inside Γ

$$\operatorname*{Res}_{z=a} [f(z)] = \lim_{z \to a} (z-a) f(z)$$

at $z = ia$

$$\operatorname*{Res}_{z=ia} [f(z)] = \lim_{z \to ia} (z - ia) \times \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \lim_{z \to ia} \cancel{(z-ia)} \times \frac{z^2}{(z+ia)\cancel{(z-ia)}(z^2+b^2)}$$

$$= \frac{(ia)^2}{2ia\,((ia)^2 + b^2)}$$

$$= \frac{-a^2}{2i\cancel{a}(b^2 - a^2)}$$

$$= \frac{-a}{2i(b^2 - a^2)}$$

at $z = ib$

$$\operatorname*{Res}_{z=ib}[f(z)] = \lim_{z \to ib} \cancel{(z-ib)} \frac{z^2}{(z^2+a^2)(z+ib)\cancel{(z-ib)}}$$

$$= \frac{(ib)^2}{((ib)^2+a^2)\, 2ib}$$

$$= \frac{-b^{\cancel{2}}}{2i\cancel{b}(a^2-b^2)}$$

$$= \frac{-b}{2i(a^2-b^2)}$$

$$= \frac{b}{2i(b^2-a^2)}$$

by CRT

$$\int_\Gamma f(z)\,dz = 2\pi i \times \text{sum of residues inside } \Gamma$$

$$= 2\pi i \times \left(\frac{-a}{2i(b^2-a^2)} + \frac{b}{2i(b^2-a^2)}\right)$$

$$= 2\pi i \times \left(\frac{b-a}{2i(b^2-a^2)}\right)$$

$$= 2\pi i \times \left(\frac{\cancel{b-a}}{2i(b+a)\cancel{(b-a)}}\right)$$

$$= \frac{\pi}{a+b}$$

now,

$$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{1}{2} \times \frac{\pi}{a+b} = \frac{\pi}{2(a+b)}$$

Q: Using contour integration, evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx$.

$\{ 4-2=2$

Choose Γ consisting of real axis from $-R$ to R and upper semicircle C such that $|z|=R$

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \int_\Gamma \frac{z^2-z+2}{z^4+10z^2+9}\,dz$$

Solve using CRT $\Rightarrow$ $= \int_\Gamma f(z)\,dz$

$$z^4+10z^2+9 = 0$$

$$(z^2+9)(z^2+1) = 0$$

$z^2+9=0$ $\qquad$ $z^2+1=0$

$z=\pm 3i$ $\qquad$ $z=\pm i$

$z=\pm 3i, \pm i$ { simple poles

$z = 3i$, i lies inside Γ

$$\underset{z=a}{\text{Res}}[f(z)] = \lim_{z \to a} (z-a) f(z)$$

at $z = 3i$

$$\underset{z=3i}{\text{Res}}[f(z)] = \lim_{z \to 3i} (z-3i) \times \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$$

$$= \lim_{z \to 3i} (z-3i) \times \frac{z^2 - z + 2}{(z^2+9)(z^2+1)}$$

$$= \lim_{z \to 3i} \cancel{(z-3i)} \times \frac{z^2 - z + 2}{(z+3i)\cancel{(z-3i)}(z^2+1)}$$

$$= \frac{-9 - 3i + 2}{6i(-9+1)}$$

$$= \frac{-7-3i}{-48i}$$

$$= \frac{7+3i}{48i}$$

at $z = i$

$$\underset{z=i}{\text{Res}}\,[f(z)] = \lim_{z \to i} (\cancel{z-i}) \times \frac{z^2 - z + 2}{(z^2+9)(z+i)(\cancel{z-i})}$$

$$= \frac{-1 - i + 2}{(-1+9)\,2i}$$

$$= \frac{1-i}{16i}$$

by CRT

$$\int_\Gamma f(z)\,dz = 2\pi i \times \text{sum of residues inside } \Gamma$$

$$= 2\pi i \times \left(\frac{7+3i}{48i} + \frac{1-i}{16i} \right)$$

$$= 2\pi i \left(\frac{7+\cancel{3i}+3-\cancel{3i}}{48i} \right)$$

$$= \cancel{2}\pi \cancel{i} \times \frac{\cancel{10}\,5}{\cancel{48}\cancel{i}\;\cancel{24}\,12}$$

$$= \frac{5\pi}{12}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx = \frac{5\pi}{12}$$

Q: Using contour integration, evaluate $\int_0^\infty \frac{dx}{1+x^4}$

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

$\{ 4-0=4$

Choose Γ consisting of real axis from $-R$ to R and upper semicircle C such that $|z|=R$

as $R\to\infty$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \int_\Gamma \frac{dz}{1+z^4}$$

$$= \int_\Gamma f(z)dz$$

using CRT

$$1+z^4 = 0$$

$$z^4 = -1$$

$$z = (-1)^{1/4}$$

$\{ \cos\theta + i\sin\theta = e^{i\theta}$

$\{ \underbrace{\cos(2n+1)\pi}_{-1} + \underbrace{i\sin(2n+1)\pi}_{0} = e^{i(2n+1)\pi}$

$-1 = e^{i(2n+1)\pi}$

$$z = \left(e^{i(2n+1)\pi}\right)^{\frac{1}{4}}$$

$$Z = e^{\frac{i(2n+1)\pi}{4}}$$

$n = 0, 1, 2, 3$

$$Z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$$

$$Z = e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}} \text{ lies inside } \Gamma.$$

$$\underset{z=a}{\text{Res}}[f(z)] = \frac{h(a)}{g'(a)}$$

$$\left\{ \begin{array}{l} h(z) = 1 \\ g(z) = 1 + z^4 \\ g'(z) = 4z^3 \end{array} \right.$$

at $z = e^{\frac{i\pi}{4}}$

$$\underset{z=e^{\frac{i\pi}{4}}}{\text{Res}}[f(z)] = \frac{1}{4\left(e^{\frac{i\pi}{4}}\right)^3}$$

$$= \frac{1}{4} \cdot e^{-\frac{i3\pi}{4}}$$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]$$

$$= \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

at $z = e^{i\frac{3\pi}{4}}$

$$\operatorname*{Res}_{z=e^{i\frac{3\pi}{4}}}[f(z)] = \frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3}$$

$$= \frac{1}{4} \cdot e^{-i\frac{9\pi}{4}}$$

$$= \frac{1}{4}\left[\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

by CRT,

$$\int_\Gamma f(z)\,dz = 2\pi i \times \text{sum of residues inside } \Gamma$$

$$= 2\pi i \times \frac{1}{4}\left[-\cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}}\right]$$

$$= 2\pi i \times \frac{1}{4} \times -\frac{2i}{\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2} \times \frac{\pi}{\sqrt{2}}$$

$$= \frac{\pi}{2\sqrt{2}}$$

Q: Using contour integrals, evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

$\{ 6-0=6$

choose Γ consisting of real axis from $-R$ to R and upper semi circle C such that $|z|=R$

as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \int_{\Gamma} \frac{dz}{1+z^6}$$

$$= \int_{\Gamma} f(z)dz$$

using CRT

$$1+z^6=0$$

$$z^6 = (-1)$$

$$z = (-1)^{\frac{1}{6}}$$

$\{ -1 = e^{i(2n+1)\pi}$

$$z = \left[e^{i(2n+1)\pi}\right]^{\frac{1}{6}}$$

$$z = e^{i\frac{(2n+1)\pi}{6}}$$

$n = 0,1,2,3,4,5$

$$z = e^{\frac{i\pi}{6}}, e^{\frac{i3\pi}{6}}, e^{\frac{i5\pi}{6}}, e^{\frac{i7\pi}{6}}, e^{\frac{i9\pi}{6}}, e^{\frac{i11\pi}{6}}$$

$$z = e^{i\frac{\pi}{6}},\ e^{i\frac{3\pi}{6}},\ e^{i\frac{5\pi}{6}} \quad \text{lies inside } \Gamma$$

$$\operatorname*{Res}_{z=a}[f(z)] = \frac{h(a)}{g'(a)} \qquad h(z) = 1$$

$$g(z) = 1+z^6$$

at $z = e^{i\frac{\pi}{6}}$ $\qquad g'(z) = 6z^5$

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{6}}}[f(z)] = \frac{1}{6\times\left(e^{i\frac{\pi}{6}}\right)^5}$$

$$= \frac{1}{6} \cdot e^{-i\frac{5\pi}{6}}$$

$$= \frac{1}{6}\left[\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right]$$

$$= \frac{1}{6}\left[-\frac{\sqrt{3}}{2} - \frac{i}{2}\right]$$

at $z = e^{i\frac{3\pi}{6}}$

$$\operatorname*{Res}_{z=e^{i\frac{3\pi}{6}}}[f(z)] = \frac{1}{6\times\left(e^{i\cancel{3}\pi/\cancel{6}_2}\right)^5}$$

$$= \frac{1}{6} \cdot e^{-i\frac{5\pi}{2}}$$

$$= \frac{1}{6}\left[\cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}\right]$$

$$= \frac{1}{6}\left[0 - i\right] = -\frac{i}{6}$$

at $z = e^{i\frac{5\pi}{6}}$

$$\underset{z=e^{i\frac{5\pi}{6}}}{\text{Res}}[f(z)] = \frac{1}{6\left(e^{i\frac{5\pi}{6}}\right)^5}$$

$$= \frac{1}{6} \cdot e^{-i\frac{25\pi}{6}}$$

$$= \frac{1}{6}\left[\cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6}\right]$$

$$= \frac{1}{6}\left[\frac{\sqrt{3}}{2} - \frac{i}{2}\right]$$

by C.R.T

$$\int_\Gamma f(z)dz = 2\pi i \times \text{sum of residues inside } \Gamma$$

$$= 2\pi i \times \frac{1}{6}\left[\cancel{\frac{-\sqrt{3}}{2}} - \frac{i}{2} - i + \cancel{\frac{\sqrt{3}}{2}} - \frac{i}{2}\right]$$

$$= 2\pi i \times \frac{1}{6} \times -2i$$

$$= \frac{2\pi}{3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3}$$

Type - III

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \sin mx \, dx \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \cos mx \, dx$$

$f(x), g(x) \Rightarrow$ polynomials

degree of numerator $<$ degree of denominator {by atleast 1}

$$\left\{ \begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ \cos\theta = \text{Re}\left[e^{i\theta}\right] \\ \sin\theta = \text{Im}\left[e^{i\theta}\right] \end{array} \right\}$$

Choose Γ consisting of real axis from $-R$ to R and upper semicircle C such that $|z| = R$

as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \sin mx \, dx = \int_{\Gamma} \frac{f(z)}{g(z)} \text{Im}\left[e^{imz}\right] dz$$

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \cos mx \, dx = \int_{\Gamma} \frac{f(z)}{g(z)} \text{Re}\left[e^{imz}\right] dz$$

Q: Show that $\int_0^\infty \frac{\cos ax}{1+x^2}\,dx = \frac{\pi}{2}e^{-a}$

$$\int_0^\infty \frac{\cos ax}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx$$

$\{2-0=2$

Choose Γ consisting of real axis from $-R$ to R and upper semi circle C such that $|z|=R$

as $R\to\infty$

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx = \int_\Gamma \frac{1}{1+z^2} Re[e^{iaz}]\,dz$$

$$= Re\int_\Gamma \frac{e^{iaz}}{1+z^2}\,dz$$

$$= Re\int_\Gamma f(z)\,dz$$

using CRT

$1+z^2=0$

$z^2=-1$

$z=\pm i$ {simple pole

$z=i$ lies inside Γ

$z=-i$ lies outside Γ

$$\operatorname{Res}_{z=a} [f(z)] = \lim_{z \to a} (z-a) f(z)$$

at $z = i$

$$\operatorname{Res}_{z=i} [f(z)] = \lim_{z \to i} (z-i) \frac{e^{iaz}}{1+z^2}$$

$$= \lim_{z \to i} \cancel{(z-i)} \frac{e^{iaz}}{(z+i)\cancel{(z-i)}}$$

$$= \frac{e^{iai}}{2i}$$

$$= \frac{e^{-a}}{2i}$$

by CRT

$$\int_\Gamma f(z) dz = 2\pi i \times \text{sum of residues inside } \Gamma$$

$$= 2\pi i \times \frac{e^{-a}}{2i}$$

$$= \pi e^{-a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2} dx = \operatorname{Re} [\pi e^{-a}]$$

$$= \pi e^{-a}$$

$\therefore$

$$\int_0^\infty \frac{\cos ax}{1+x^2}\,dx = \frac{1}{2} \times \pi e^{-a} = \frac{\pi}{2} e^{-a}$$

Q: Using contour integration, evaluate $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx$, $a>0$

Choose Γ consisting of real axis from $-R$ to R and upper semi circle C such that $|z|=R$

as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \int_\Gamma \frac{z}{z^2+a^2} \operatorname{Im}[e^{iz}]\,dz$$

$$= \operatorname{Im} \int_\Gamma \frac{z\, e^{iz}}{z^2+a^2}\,dz$$

$$= \operatorname{Im} \int_\Gamma f(z)\,dz$$

Using CRT

$z^2+a^2 = 0$

$z^2 = -a^2$

$z = \pm ia$

$z = ia$ lies inside Γ

$z = -ia$ lies outside Γ

$$\underset{z=a}{\text{Res}}[f(z)] = \lim_{z \to a} (z-a) f(z)$$

at $z = ia$

$$\underset{z=ia}{\text{Res}}[f(z)] = \lim_{z \to ia} (z-ia) \frac{z \cdot e^{iz}}{z^2+a^2}$$

$$= \lim_{z \to ia} \cancel{(z-ia)} \times \frac{z \cdot e^{iz}}{(z+ia)\cancel{(z-ia)}}$$

$$= \frac{\cancel{ia} \cdot e^{i\,ia}}{2\cancel{ia}}$$

$$= \frac{e^{-a}}{2}$$

by CRT.

$$\int_\Gamma f(z)\,dz = 2\pi i \times \text{sum of residues inside } \Gamma$$

$$= 2\pi i \times \frac{e^{-a}}{2}$$

$$= i\pi e^{-a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\, dx = \operatorname{Im}\left[i\pi e^{-a} \right]$$

$$= \pi e^{-a}$$

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