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Perspective creates the illusion that the sequence of railroad ties continues indefinitely but converges toward a single point infinitely far away.

9 INFINITE SERIES

In this chapter we will be concerned with infinite series, which are sums that involve infinitely many terms. Infinite series play a fundamental role in both mathematics and science—they are used, for example, to approximate trigonometric functions and logarithms, to solve differential equations, to evaluate difficult integrals, to create new functions, and to construct mathematical models of physical laws. Since it is impossible to add up infinitely many numbers directly, one goal will be to define exactly what we mean by the sum of an infinite series. However, unlike finite sums, it turns out that not all infinite series actually have a sum, so we will need to develop tools for determining which infinite series have sums and which do not. Once the basic ideas have been developed we will begin to apply our work; we will show how infinite series are used to evaluate such quantities as ln 2, e, sin 3°, and π, how they are used to create functions, and finally, how they are used to model physical laws.

9.1 SEQUENCES

In everyday language, the term "sequence" means a succession of things in a definite order—chronological order, size order, or logical order, for example. In mathematics, the term "sequence" is commonly used to denote a succession of numbers whose order is determined by a rule or a function. In this section, we will develop some of the basic ideas concerning sequences of numbers.

DEFINITION OF A SEQUENCE

Stated informally, an ***infinite sequence***, or more simply a ***sequence***, is an unending succession of numbers, called ***terms***. It is understood that the terms have a definite order; that is, there is a first term a_1, a second term a_2, a third term a_3, a fourth term a_4, and so forth. Such a sequence would typically be written as

$$a_1, a_2, a_3, a_4, \ldots$$

where the dots are used to indicate that the sequence continues indefinitely. Some specific examples are

$$1, 2, 3, 4, \ldots, \qquad 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots,$$

$$2, 4, 6, 8, \ldots, \qquad 1, -1, 1, -1, \ldots$$

Each of these sequences has a definite pattern that makes it easy to generate additional terms if we assume that those terms follow the same pattern as the displayed terms. However, such patterns can be deceiving, so it is better to have a rule or formula for generating the terms. One way of doing this is to look for a function that relates each term in the sequence to its term number. For example, in the sequence

$$2, 4, 6, 8, \ldots$$

each term is twice the term number; that is, the nth term in the sequence is given by the formula $2n$. We denote this by writing the sequence as

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

We call the function $f(n) = 2n$ the *general term* of this sequence. Now, if we want to know a specific term in the sequence, we need only substitute its term number in the formula for the general term. For example, the 37th term in the sequence is $2 \cdot 37 = 74$.

▶ **Example 1** In each part, find the general term of the sequence.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

(c) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ (d) $1, 3, 5, 7, \ldots$

Table 9.1.1

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	⋯	$\frac{n}{n+1}$	⋯

Solution (a). In Table 9.1.1, the four known terms have been placed below their term numbers, from which we see that the numerator is the same as the term number and the denominator is one greater than the term number. This suggests that the nth term has numerator n and denominator $n + 1$, as indicated in the table. Thus, the sequence can be expressed as

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

Table 9.1.2

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	⋯	$\frac{1}{2^n}$	⋯

Solution (b). In Table 9.1.2, the denominators of the four known terms have been expressed as powers of 2 and the first four terms have been placed below their term numbers, from which we see that the exponent in the denominator is the same as the term number. This suggests that the denominator of the nth term is 2^n, as indicated in the table. Thus, the sequence can be expressed as

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

Solution (c). This sequence is identical to that in part (a), except for the alternating signs. Thus, the nth term in the sequence can be obtained by multiplying the nth term in part (a) by $(-1)^{n+1}$. This factor produces the correct alternating signs, since its successive values, starting with $n = 1$, are $1, -1, 1, -1, \ldots$. Thus, the sequence can be written as

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$$

Table 9.1.3

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	1	3	5	7	⋯	$2n-1$	⋯

Solution (d). In Table 9.1.3, the four known terms have been placed below their term numbers, from which we see that each term is one less than twice its term number. This suggests that the nth term in the sequence is $2n - 1$, as indicated in the table. Thus, the sequence can be expressed as

$$1, 3, 5, 7, \ldots, 2n - 1, \ldots \blacktriangleleft$$

When the general term of a sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{1}$$

is known, there is no need to write out the initial terms, and it is common to write only the general term enclosed in braces. Thus, (1) might be written as

$$\{a_n\}_{n=1}^{+\infty} \quad \text{or as} \quad \{a_n\}_{n=1}^{\infty}$$

For example, here are the four sequences in Example 1 expressed in brace notation.

SEQUENCE	BRACE NOTATION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$	$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$	$\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$	$\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$1, 3, 5, 7, \ldots, 2n-1, \ldots$	$\{2n-1\}_{n=1}^{+\infty}$

A sequence cannot be uniquely determined from a few initial terms. For example, the sequence whose general term is

$$f(n) = \tfrac{1}{3}(3 - 5n + 6n^2 - n^3)$$

has 1, 3, and 5 as its first three terms, but its fourth term is also 5.

The letter n in (1) is called the ***index*** for the sequence. It is not essential to use n for the index; any letter not reserved for another purpose can be used. For example, we might view the general term of the sequence $a_1, a_2, a_3, \ldots$ to be the kth term, in which case we would denote this sequence as $\{a_k\}_{k=1}^{+\infty}$. Moreover, it is not essential to start the index at 1; sometimes it is more convenient to start it at 0 (or some other integer). For example, consider the sequence

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$$

One way to write this sequence is

$$\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{+\infty}$$

However, the general term will be simpler if we think of the initial term in the sequence as the zeroth term, in which case we can write the sequence as

$$\left\{\frac{1}{2^n}\right\}_{n=0}^{+\infty}$$

We began this section by describing a sequence as an unending succession of numbers. Although this conveys the general idea, it is not a satisfactory mathematical definition because it relies on the term "succession," which is itself an undefined term. To motivate a precise definition, consider the sequence

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

If we denote the general term by $f(n) = 2n$, then we can write this sequence as

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

which is a "list" of values of the function

$$f(n) = 2n, \quad n = 1, 2, 3, \ldots$$

whose domain is the set of positive integers. This suggests the following definition.

9.1.1 DEFINITION A ***sequence*** is a function whose domain is a set of integers.

Typically, the domain of a sequence is the set of positive integers or the set of nonnegative integers. We will regard the expression $\{a_n\}_{n=1}^{+\infty}$ to be an alternative notation for the function $f(n) = a_n$, $n = 1, 2, 3, \ldots$, and we will regard $\{a_n\}_{n=0}^{+\infty}$ to be an alternative notation for the function $f(n) = a_n$, $n = 0, 1, 2, 3, \ldots$.

GRAPHS OF SEQUENCES

Since sequences are functions, it makes sense to talk about the graph of a sequence. For example, the graph of the sequence $\{1/n\}_{n=1}^{+\infty}$ is the graph of the equation

$$y = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$

Because the right side of this equation is defined only for positive integer values of n, the graph consists of a succession of isolated points (Figure 9.1.1*a*). This is different from the graph of

$$y = \frac{1}{x}, \quad x \geq 1$$

which is a continuous curve (Figure 9.1.1*b*).

When the starting value for the index of a sequence is not relevant to the discussion, it is common to use a notation such as $\{a_n\}$ in which there is no reference to the starting value of n. We can distinguish between different sequences by using different letters for their general terms; thus, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ denote three different sequences.

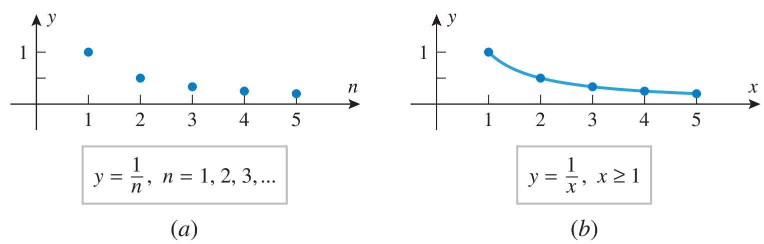


Figure 9.1.1

LIMIT OF A SEQUENCE

Since sequences are functions, we can inquire about their limits. However, because a sequence $\{a_n\}$ is only defined for integer values of n, the only limit that makes sense is the limit of a_n as $n \to +\infty$. In Figure 9.1.2 we have shown the graphs of four sequences, each of which behaves differently as $n \to +\infty$:

- The terms in the sequence $\{n + 1\}$ increase without bound.
- The terms in the sequence $\{(-1)^{n+1}\}$ oscillate between -1 and 1.
- The terms in the sequence $\{n/(n + 1)\}$ increase toward a "limiting value" of 1.
- The terms in the sequence $\left\{1 + \left(-\frac{1}{2}\right)^n\right\}$ also tend toward a "limiting value" of 1, but do so in an oscillatory fashion.

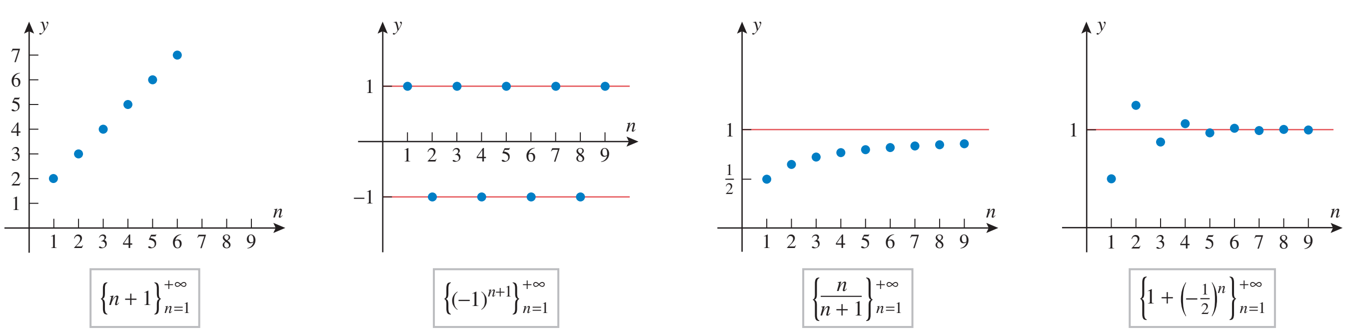


Figure 9.1.2

Informally speaking, the limit of a sequence $\{a_n\}$ is intended to describe how a_n behaves as $n \to +\infty$. To be more specific, we will say that *a sequence* $\{a_n\}$ *approaches a limit* L *if the terms in the sequence eventually become arbitrarily close to* L. Geometrically, this

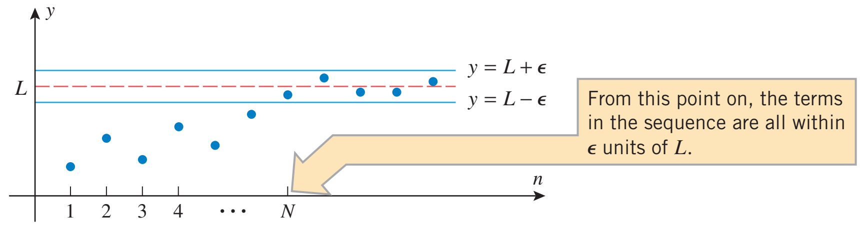


Figure 9.1.3

means that for any positive number ϵ there is a point in the sequence after which all terms lie between the lines $y = L - \epsilon$ and $y = L + \epsilon$ (Figure 9.1.3).

The following definition makes these ideas precise.

How would you define these limits?

$$\lim_{n \to +\infty} a_n = +\infty$$
$$\lim_{n \to +\infty} a_n = -\infty$$

9.1.2 DEFINITION A sequence $\{a_n\}$ is said to ***converge*** to the ***limit*** L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to ***diverge***.

Example 2 The first two sequences in Figure 9.1.2 diverge, and the second two converge to 1; that is,

$$\lim_{n \to +\infty} \frac{n}{n+1} = 1 \quad \text{and} \quad \lim_{n \to +\infty} \left[1 + \left(-\tfrac{1}{2}\right)^n\right] = 1 \blacktriangleleft$$

The following theorem, which we state without proof, shows that the familiar properties of limits apply to sequences. This theorem ensures that the algebraic techniques used to find limits of the form $\lim_{x \to +\infty}$ can also be used for limits of the form $\lim_{n \to +\infty}$.

9.1.3 THEOREM *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2, respectively, and c is a constant. Then:*

(*a*) $\lim_{n \to +\infty} c = c$

(*b*) $\lim_{n \to +\infty} ca_n = c \lim_{n \to +\infty} a_n = cL_1$

(*c*) $\lim_{n \to +\infty} (a_n + b_n) = \lim_{n \to +\infty} a_n + \lim_{n \to +\infty} b_n = L_1 + L_2$

(*d*) $\lim_{n \to +\infty} (a_n - b_n) = \lim_{n \to +\infty} a_n - \lim_{n \to +\infty} b_n = L_1 - L_2$

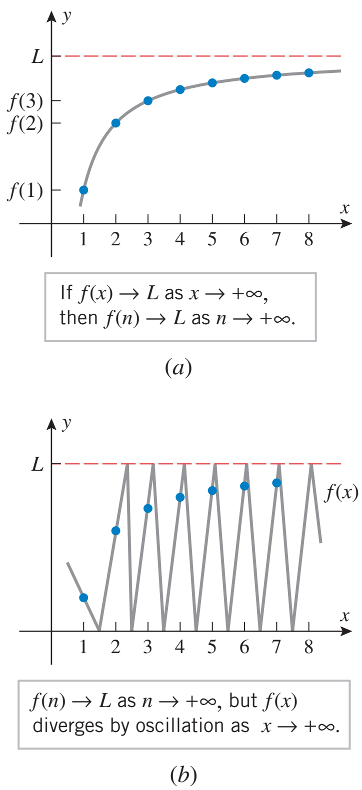
(*e*) $\lim_{n \to +\infty} (a_n b_n) = \lim_{n \to +\infty} a_n \cdot \lim_{n \to +\infty} b_n = L_1 L_2$

(*f*) $\lim_{n \to +\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim_{n \to +\infty} a_n}{\lim_{n \to +\infty} b_n} = \dfrac{L_1}{L_2}$ (*if* $L_2 \neq 0$)

Additional limit properties follow from those in Theorem 9.1.3. For example, use part (e) to show that if $a_n \to L$ and m is a positive integer, then

$$\lim_{n \to +\infty} (a_n)^m = L^m$$

If the general term of a sequence is $f(n)$, where $f(x)$ is a function defined on the entire interval $[1, +\infty)$, then the values of $f(n)$ can be viewed as "sample values" of $f(x)$ taken



▲ **Figure 9.1.4**

at the positive integers. Thus,

$$\text{if } f(x) \to L \text{ as } x \to +\infty, \quad \text{then} \quad f(n) \to L \text{ as } n \to +\infty$$

(Figure 9.1.4*a*). However, the converse is not true; that is, one cannot infer that $f(x) \to L$ as $x \to +\infty$ from the fact that $f(n) \to L$ as $n \to +\infty$ (Figure 9.1.4*b*).

▶ **Example 3** In each part, determine whether the sequence converges or diverges by examining the limit as $n \to +\infty$.

$$\text{(a)}\ \left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty} \qquad \text{(b)}\ \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$$

$$\text{(c)}\ \left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{+\infty} \qquad \text{(d)}\ \{8-2n\}_{n=1}^{+\infty}$$

Solution (a). Dividing numerator and denominator by n and using Theorem 9.1.3 yields

$$\lim_{n\to+\infty}\frac{n}{2n+1} = \lim_{n\to+\infty}\frac{1}{2+1/n} = \frac{\displaystyle\lim_{n\to+\infty}1}{\displaystyle\lim_{n\to+\infty}(2+1/n)} = \frac{\displaystyle\lim_{n\to+\infty}1}{\displaystyle\lim_{n\to+\infty}2+\lim_{n\to+\infty}1/n}$$

$$= \frac{1}{2+0} = \frac{1}{2}$$

Thus, the sequence converges to $\frac{1}{2}$.

Solution (b). This sequence is the same as that in part (a), except for the factor of $(-1)^{n+1}$, which oscillates between $+1$ and -1. Thus, the terms in this sequence oscillate between positive and negative values, with the odd-numbered terms being identical to those in part (a) and the even-numbered terms being the negatives of those in part (a). Since the sequence in part (a) has a limit of $\frac{1}{2}$, it follows that the odd-numbered terms in this sequence approach $\frac{1}{2}$, and the even-numbered terms approach $-\frac{1}{2}$. Therefore, this sequence has no limit—it diverges.

Solution (c). Since $1/n \to 0$, the product $(-1)^{n+1}(1/n)$ oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive values and the even-numbered terms approaching 0 through negative values. Thus,

$$\lim_{n\to+\infty}(-1)^{n+1}\frac{1}{n} = 0$$

so the sequence converges to 0.

Solution (d). $\lim_{n\to+\infty}(8-2n) = -\infty$, so the sequence $\{8-2n\}_{n=1}^{+\infty}$ diverges. ◀

▶ **Example 4** In each part, determine whether the sequence converges, and if so, find its limit.

$$\text{(a)}\ 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots \qquad \text{(b)}\ 1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$$

Solution. Replacing n by x in the first sequence produces the power function $(1/2)^x$, and replacing n by x in the second sequence produces the power function 2^x. Now recall that if $0 < b < 1$, then $b^x \to 0$ as $x \to +\infty$, and if $b > 1$, then $b^x \to +\infty$ as $x \to +\infty$ (Figure 6.1.1).

Thus,

$$\lim_{n\to+\infty} \frac{1}{2^n} = 0 \quad \text{and} \quad \lim_{n\to+\infty} 2^n = +\infty$$

So, the sequence $\{1/2^n\}$ converges to 0, but the sequence $\{2^n\}$ diverges. ◀

▶ Example 5 Find the limit of the sequence $\left\{\dfrac{n}{e^n}\right\}_{n=1}^{+\infty}$.

Solution. The expression

$$\lim_{n\to+\infty} \frac{n}{e^n}$$

is an indeterminate form of type ∞/∞, so L'Hôpital's rule is indicated. However, we cannot apply this rule directly to n/e^n because the functions n and e^n have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem we extend the domains of these functions to all real numbers, here implied by replacing n by x, and apply L'Hôpital's rule to the limit of the quotient x/e^x. This yields

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

from which we can conclude that

$$\lim_{n\to+\infty} \frac{n}{e^n} = 0 \quad ◀$$

▶ Example 6 Show that $\lim_{n\to+\infty} \sqrt[n]{n} = 1$.

Solution.

$$\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} n^{1/n} = \lim_{n\to+\infty} e^{(1/n)\ln n} = e^0 = 1$$

By L'Hôpital's rule applied to $(1/x)\ln x$ ◀

Sometimes the even-numbered and odd-numbered terms of a sequence behave sufficiently differently that it is desirable to investigate their convergence separately. The following theorem, whose proof is omitted, is helpful for that purpose.

9.1.4 THEOREM *A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.*

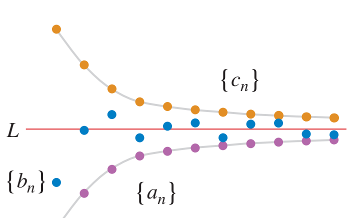
▶ Example 7 The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots$$

converges to 0, since the even-numbered terms and the odd-numbered terms both converge to 0, and the sequence

$$1, \tfrac{1}{2}, 1, \tfrac{1}{3}, 1, \tfrac{1}{4}, \ldots$$

diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0. ◀



If $a_n \to L$ and $c_n \to L$, then $b_n \to L$.

▲ **Figure 9.1.5**

THE SQUEEZING THEOREM FOR SEQUENCES

The following theorem, illustrated in Figure 9.1.5, is an adaptation of the Squeezing Theorem (1.6.2) to sequences. This theorem will be useful for finding limits of sequences that cannot be obtained directly. The proof is omitted.

9.1.5 THEOREM (*The Squeezing Theorem for Sequences*) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that*

$$a_n \le b_n \le c_n \quad (\text{for all values of } n \text{ beyond some index } N)$$

If the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.

Recall that if n is a positive integer, then $n!$ (read "n factorial") is the product of the first n positive integers. In addition, it is convenient to define $0! = 1$.

Example 8 Use numerical evidence to make a conjecture about the limit of the sequence

$$\left\{\frac{n!}{n^n}\right\}_{n=1}^{+\infty}$$

and then confirm that your conjecture is correct.

Table 9.1.4

n	$\dfrac{n!}{n^n}$
1	1.0000000000
2	0.5000000000
3	0.2222222222
4	0.0937500000
5	0.0384000000
6	0.0154320988
7	0.0061198990
8	0.0024032593
9	0.0009366567
10	0.0003628800
11	0.0001399059
12	0.0000537232

Solution. Table 9.1.4, which was obtained with a calculating utility, suggests that the limit of the sequence may be 0. To confirm this we need to examine the limit of

$$a_n = \frac{n!}{n^n}$$

as $n \to +\infty$. Although this is an indeterminate form of type ∞/∞, L'Hôpital's rule is not helpful because we have no definition of $x!$ for values of x that are not integers. However, let us write out some of the initial terms and the general term in the sequence:

$$a_1 = 1, \quad a_2 = \frac{1\cdot 2}{2\cdot 2} = \frac{1}{2}, \quad a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3} = \frac{2}{9} < \frac{1}{3}, \quad a_4 = \frac{1\cdot 2\cdot 3\cdot 4}{4\cdot 4\cdot 4\cdot 4} = \frac{3}{32} < \frac{1}{4}, \ldots$$

If $n > 1$, the general term of the sequence can be rewritten as

$$a_n = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} = \frac{1}{n}\left(\frac{2\cdot 3\cdots n}{n\cdot n\cdots n}\right)$$

from which it follows that $a_n \le 1/n$ (why?). It is now evident that

$$0 \le a_n \le \frac{1}{n}$$

However, the two outside expressions have a limit of 0 as $n \to +\infty$; thus, the Squeezing Theorem for Sequences implies that $a_n \to 0$ as $n \to +\infty$, which confirms our conjecture. ◀

The following theorem is often useful for finding the limit of a sequence with both positive and negative terms—it states that if the sequence $\{|a_n|\}$ that is obtained by taking the absolute value of each term in the sequence $\{a_n\}$ converges to 0, then $\{a_n\}$ also converges to 0.

9.1.6 THEOREM *If* $\lim_{n\to+\infty} |a_n| = 0$, *then* $\lim_{n\to+\infty} a_n = 0$.

PROOF Depending on the sign of a_n, either $a_n = |a_n|$ or $a_n = -|a_n|$. Thus, in all cases we have

$$-|a_n| \le a_n \le |a_n|$$

However, the limit of the two outside terms is 0, and hence the limit of a_n is 0 by the Squeezing Theorem for Sequences. ■

▶ **Example 9** Consider the sequence

$$1, -\frac{1}{2}, \frac{1}{2^2}, -\frac{1}{2^3}, \ldots, (-1)^n \frac{1}{2^n}, \ldots$$

If we take the absolute value of each term, we obtain the sequence

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$$

which, as shown in Example 4, converges to 0. Thus, from Theorem 9.1.6 we have

$$\lim_{n \to +\infty} \left[(-1)^n \frac{1}{2^n}\right] = 0 \blacktriangleleft$$

SEQUENCES DEFINED RECURSIVELY

Some sequences do not arise from a formula for the general term, but rather from a formula or set of formulas that specify how to generate each term in the sequence from terms that precede it; such sequences are said to be defined ***recursively***, and the defining formulas are called ***recursion formulas***. A good example is the mechanic's rule for approximating square roots. In Exercise 23 of Section 3.7 you were asked to show that

$$x_1 = 1, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{2}$$

describes the sequence produced by Newton's Method to approximate $\sqrt{a}$ as a zero of the function $f(x) = x^2 - a$. Table 9.1.5 shows the first five terms in an application of the mechanic's rule to approximate $\sqrt{2}$.

Table 9.1.5

n	$x_1 = 1, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$	DECIMAL APPROXIMATION
	$x_1 = 1$ (Starting value)	1.00000000000
1	$x_2 = \frac{1}{2}\left[1 + \frac{2}{1}\right] = \frac{3}{2}$	1.50000000000
2	$x_3 = \frac{1}{2}\left[\frac{3}{2} + \frac{2}{3/2}\right] = \frac{17}{12}$	1.41666666667
3	$x_4 = \frac{1}{2}\left[\frac{17}{12} + \frac{2}{17/12}\right] = \frac{577}{408}$	1.41421568627
4	$x_5 = \frac{1}{2}\left[\frac{577}{408} + \frac{2}{577/408}\right] = \frac{665{,}857}{470{,}832}$	1.41421356237
5	$x_6 = \frac{1}{2}\left[\frac{665{,}857}{470{,}832} + \frac{2}{665{,}857/470{,}832}\right] = \frac{886{,}731{,}088{,}897}{627{,}013{,}566{,}048}$	1.41421356237

It would take us too far afield to investigate the convergence of sequences defined recursively, but we will conclude this section with a useful technique that can sometimes be used to compute limits of such sequences.

▶ **Example 10** Assuming that the sequence in Table 9.1.5 converges, show that the limit is $\sqrt{2}$.

Solution. Assume that $x_n \to L$, where L is to be determined. Since $n+1 \to +\infty$ as $n \to +\infty$, it is also true that $x_{n+1} \to L$ as $n \to +\infty$. Thus, if we take the limit of the expression

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

as $n \to +\infty$, we obtain

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

which can be rewritten as $L^2 = 2$. The negative solution of this equation is extraneous because $x_n > 0$ for all n, so $L = \sqrt{2}$. ◄

✔QUICK CHECK EXERCISES 9.1 (See page 607 for answers.)

1. Consider the sequence 4, 6, 8, 10, 12, ...
(a) If $\{a_n\}_{n=1}^{+\infty}$ denotes this sequence, then $a_1 =$ ______, $a_4 =$ ______, and $a_7 =$ ______. The general term is $a_n =$ ______.
(b) If $\{b_n\}_{n=0}^{+\infty}$ denotes this sequence, then $b_0 =$ ______, $b_4 =$ ______, and $b_8 =$ ______. The general term is $b_n =$ ______.

2. What does it mean to say that a sequence $\{a_n\}$ converges?

3. Consider sequences $\{a_n\}$ and $\{b_n\}$, where $a_n \to 2$ as $n \to +\infty$ and $b_n = (-1)^n$. Determine which of the following sequences converge and which diverge. If a sequence converges, indicate its limit.
(a) $\{b_n\}$ (b) $\{3a_n - 1\}$ (c) $\{b_n^2\}$
(d) $\{a_n + b_n\}$ (e) $\left\{\dfrac{1}{a_n^2+3}\right\}$ (f) $\left\{\dfrac{b_n}{1000}\right\}$

4. Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences such that $a_n \le b_n \le c_n$ for all $n \ge 10$, and that $\{a_n\}$ and $\{c_n\}$ both converge to 12. Then the ______ Theorem for Sequences implies that $\{b_n\}$ converges to ______.

EXERCISE SET 9.1 Graphing Utility

1. In each part, find a formula for the general term of the sequence, starting with $n = 1$.
(a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (b) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$
(c) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ (d) $\frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \frac{16}{\sqrt[5]{\pi}}, \ldots$

2. In each part, find two formulas for the general term of the sequence, one starting with $n = 1$ and the other with $n = 0$.
(a) $1, -r, r^2, -r^3, \ldots$ (b) $r, -r^2, r^3, -r^4, \ldots$

3. (a) Write out the first four terms of the sequence $\{1 + (-1)^n\}$, starting with $n = 0$.
(b) Write out the first four terms of the sequence $\{\cos n\pi\}$, starting with $n = 0$.
(c) Use the results in parts (a) and (b) to express the general term of the sequence 4, 0, 4, 0, ... in two different ways, starting with $n = 0$.

4. In each part, find a formula for the general term using factorials and starting with $n = 1$.
(a) $1\cdot 2, 1\cdot 2\cdot 3\cdot 4, 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6, 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8, \ldots$
(b) $1, 1\cdot 2\cdot 3, 1\cdot 2\cdot 3\cdot 4\cdot 5, 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7, \ldots$

5–6 Let f be the function $f(x) = \cos\left(\frac{\pi}{2}x\right)$ and define sequences $\{a_n\}$ and $\{b_n\}$ by $a_n = f(2n)$ and $b_n = f(2n+1)$. ■

5. (a) Does $\lim_{x\to+\infty} f(x)$ exist? Explain.
(b) Evaluate a_1, a_2, a_3, a_4, and a_5.
(c) Does $\{a_n\}$ converge? If so, find its limit.

6. (a) Evaluate b_1, b_2, b_3, b_4, and b_5.
(b) Does $\{b_n\}$ converge? If so, find its limit.
(c) Does $\{f(n)\}$ converge? If so, find its limit.

7–22 Write out the first five terms of the sequence, determine whether the sequence converges, and if so find its limit. ■

7. $\left\{\dfrac{n}{n+2}\right\}_{n=1}^{+\infty}$ **8.** $\left\{\dfrac{n^2}{2n+1}\right\}_{n=1}^{+\infty}$ **9.** $\{2\}_{n=1}^{+\infty}$

10. $\left\{\ln\left(\dfrac{1}{n}\right)\right\}_{n=1}^{+\infty}$ **11.** $\left\{\dfrac{\ln n}{n}\right\}_{n=1}^{+\infty}$ **12.** $\left\{n\sin\dfrac{\pi}{n}\right\}_{n=1}^{+\infty}$

13. $\{1 + (-1)^n\}_{n=1}^{+\infty}$ **14.** $\left\{\dfrac{(-1)^{n+1}}{n^2}\right\}_{n=1}^{+\infty}$

15. $\left\{(-1)^n\dfrac{2n^3}{n^3+1}\right\}_{n=1}^{+\infty}$ **16.** $\left\{\dfrac{n}{2^n}\right\}_{n=1}^{+\infty}$

17. $\left\{\dfrac{(n+1)(n+2)}{2n^2}\right\}_{n=1}^{+\infty}$ **18.** $\left\{\dfrac{\pi^n}{4^n}\right\}_{n=1}^{+\infty}$

19. $\{n^2e^{-n}\}_{n=1}^{+\infty}$ **20.** $\{\sqrt{n^2+3n} - n\}_{n=1}^{+\infty}$

21. $\left\{\left(\dfrac{n+3}{n+1}\right)^n\right\}_{n=1}^{+\infty}$ **22.** $\left\{\left(1-\dfrac{2}{n}\right)^n\right\}_{n=1}^{+\infty}$

23–30 Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so find its limit.

23. $\dfrac{1}{2}, \dfrac{3}{4}, \dfrac{5}{6}, \dfrac{7}{8}, \ldots$ **24.** $0, \dfrac{1}{2^2}, \dfrac{2}{3^2}, \dfrac{3}{4^2}, \ldots$

25. $\dfrac{1}{3}, -\dfrac{1}{9}, \dfrac{1}{27}, -\dfrac{1}{81}, \ldots$ **26.** $-1, 2, -3, 4, -5, \ldots$

27. $\left(1-\dfrac{1}{2}\right), \left(\dfrac{1}{3}-\dfrac{1}{2}\right), \left(\dfrac{1}{3}-\dfrac{1}{4}\right), \left(\dfrac{1}{5}-\dfrac{1}{4}\right), \ldots$

28. $3, \dfrac{3}{2}, \dfrac{3}{2^2}, \dfrac{3}{2^3}, \ldots$

29. $(\sqrt{2}-\sqrt{3}), (\sqrt{3}-\sqrt{4}), (\sqrt{4}-\sqrt{5}), \ldots$

30. $\dfrac{1}{3^5}, -\dfrac{1}{3^6}, \dfrac{1}{3^7}, -\dfrac{1}{3^8}, \ldots$

31–34 True–False Determine whether the statement is true or false. Explain your answer.

31. Sequences are functions.

32. If $\{a_n\}$ and $\{b_n\}$ are sequences such that $\{a_n + b_n\}$ converges, then $\{a_n\}$ and $\{b_n\}$ converge.

33. If $\{a_n\}$ diverges, then $a_n \to +\infty$ or $a_n \to -\infty$.

34. If the graph of $y = f(x)$ has a horizontal asymptote as $x \to +\infty$, then the sequence $\{f(n)\}$ converges.

35–36 Use numerical evidence to make a conjecture about the limit of the sequence, and then use the Squeezing Theorem for Sequences (Theorem 9.1.5) to confirm that your conjecture is correct.

35. $\displaystyle\lim_{n\to+\infty} \frac{\sin^2 n}{n}$ **36.** $\displaystyle\lim_{n\to+\infty} \left(\frac{1+n}{2n}\right)^n$

FOCUS ON CONCEPTS

37. Give two examples of sequences, all of whose terms are between -10 and 10, that do not converge. Use graphs of your sequences to explain their properties.

38. (a) Suppose that f satisfies $\lim_{x\to 0^+} f(x) = +\infty$. Is it possible that the sequence $\{f(1/n)\}$ converges? Explain.
(b) Find a function f such that $\lim_{x\to 0^+} f(x)$ does not exist but the sequence $\{f(1/n)\}$ converges.

39. (a) Starting with $n = 1$, write out the first six terms of the sequence $\{a_n\}$, where

$$a_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even} \end{cases}$$

(b) Starting with $n = 1$, and considering the even and odd terms separately, find a formula for the general term of the sequence

$$1, \frac{1}{2^2}, 3, \frac{1}{2^4}, 5, \frac{1}{2^6}, \ldots$$

(c) Starting with $n = 1$, and considering the even and odd terms separately, find a formula for the general term of the sequence

$$1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{7}, \frac{1}{9}, \frac{1}{9}, \ldots$$

(d) Determine whether the sequences in parts (a), (b), and (c) converge. For those that do, find the limit.

40. For what positive values of b does the sequence $b, 0, b^2, 0, b^3, 0, b^4, \ldots$ converge? Justify your answer.

41. Assuming that the sequence given in Formula (2) of this section converges, use the method of Example 10 to show that the limit of this sequence is $\sqrt{a}$.

42. Consider the sequence

$$a_1 = \sqrt{6}$$
$$a_2 = \sqrt{6+\sqrt{6}}$$
$$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$$
$$a_4 = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}}$$
$$\vdots$$

(a) Find a recursion formula for a_{n+1}.
(b) Assuming that the sequence converges, use the method of Example 10 to find the limit.

43. (a) A bored student enters the number 0.5 in a calculator display and then repeatedly computes the square of the number in the display. Taking $a_0 = 0.5$, find a formula for the general term of the sequence $\{a_n\}$ of numbers that appear in the display.
(b) Try this with a calculator and make a conjecture about the limit of a_n.
(c) Confirm your conjecture by finding the limit of a_n.
(d) For what values of a_0 will this procedure produce a convergent sequence?

44. Let

$$f(x) = \begin{cases} 2x, & 0 \le x < 0.5 \\ 2x-1, & 0.5 \le x < 1 \end{cases}$$

Does the sequence $f(0.2), f(f(0.2)), f(f(f(0.2))), \ldots$ converge? Justify your reasoning.

45. (a) Use a graphing utility to generate the graph of the equation $y = (2^x + 3^x)^{1/x}$, and then use the graph to make a conjecture about the limit of the sequence

$$\{(2^n + 3^n)^{1/n}\}_{n=1}^{+\infty}$$

(b) Confirm your conjecture by calculating the limit.

46. Consider the sequence $\{a_n\}_{n=1}^{+\infty}$ whose nth term is

$$a_n = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(k/n)}$$

Show that $\lim_{n\to+\infty} a_n = \ln 2$ by interpreting a_n as the Riemann sum of a definite integral.

47. The sequence whose terms are 1, 1, 2, 3, 5, 8, 13, 21, ... is called the ***Fibonacci sequence*** in honor of the Italian mathematician Leonardo ("Fibonacci") da Pisa (c. 1170–1250). This sequence has the property that after starting with two 1's, each term is the sum of the preceding two.

(a) Denoting the sequence by $\{a_n\}$ and starting with $a_1 = 1$ and $a_2 = 1$, show that
$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} \quad \text{if } n \geq 1$$

(b) Give a reasonable informal argument to show that if the sequence $\{a_{n+1}/a_n\}$ converges to some limit L, then the sequence $\{a_{n+2}/a_{n+1}\}$ must also converge to L.

(c) Assuming that the sequence $\{a_{n+1}/a_n\}$ converges, show that its limit is $(1+\sqrt{5})/2$.

48. If we accept the fact that the sequence $\{1/n\}_{n=1}^{+\infty}$ converges to the limit $L = 0$, then according to Definition 9.1.2, for every $\epsilon > 0$ there exists a positive integer N such that $|a_n - L| = |(1/n) - 0| < \epsilon$ when $n \geq N$. In each part, find the smallest possible value of N for the given value of ϵ.

(a) $\epsilon = 0.5$ (b) $\epsilon = 0.1$ (c) $\epsilon = 0.001$

49. If we accept the fact that the sequence
$$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$$
converges to the limit $L = 1$, then according to Definition 9.1.2, for every $\epsilon > 0$ there exists an integer N such that
$$|a_n - L| = \left|\frac{n}{n+1} - 1\right| < \epsilon$$
when $n \geq N$. In each part, find the smallest value of N for the given value of ϵ.

(a) $\epsilon = 0.25$ (b) $\epsilon = 0.1$ (c) $\epsilon = 0.001$

50. Use Definition 9.1.2 to prove that

(a) the sequence $\{1/n\}_{n=1}^{+\infty}$ converges to 0

(b) the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{+\infty}$ converges to 1.

51. **Writing** Discuss, with examples, various ways that a sequence could diverge.

52. **Writing** Discuss the convergence of the sequence $\{r^n\}$ considering the cases $|r| < 1$, $|r| > 1$, $r = 1$, and $r = -1$ separately.

✔ QUICK CHECK ANSWERS 9.1

1. (a) 4; 10; 16; $2n+2$ (b) 4; 12; 20; $2n+4$ **2.** $\lim\limits_{n\to+\infty} a_n$ exists **3.** (a) diverges (b) converges to 5 (c) converges to 1 (d) diverges (e) converges to $\frac{1}{7}$ (f) diverges **4.** Squeezing; 12

9.2 MONOTONE SEQUENCES

There are many situations in which it is important to know whether a sequence converges, but the value of the limit is not relevant to the problem at hand. In this section we will study several techniques that can be used to determine whether a sequence converges.

TERMINOLOGY

We begin with some terminology.

9.2.1 DEFINITION A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$

increasing if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$

strictly decreasing if $a_1 > a_2 > a_3 > \cdots > a_n > \cdots$

decreasing if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$

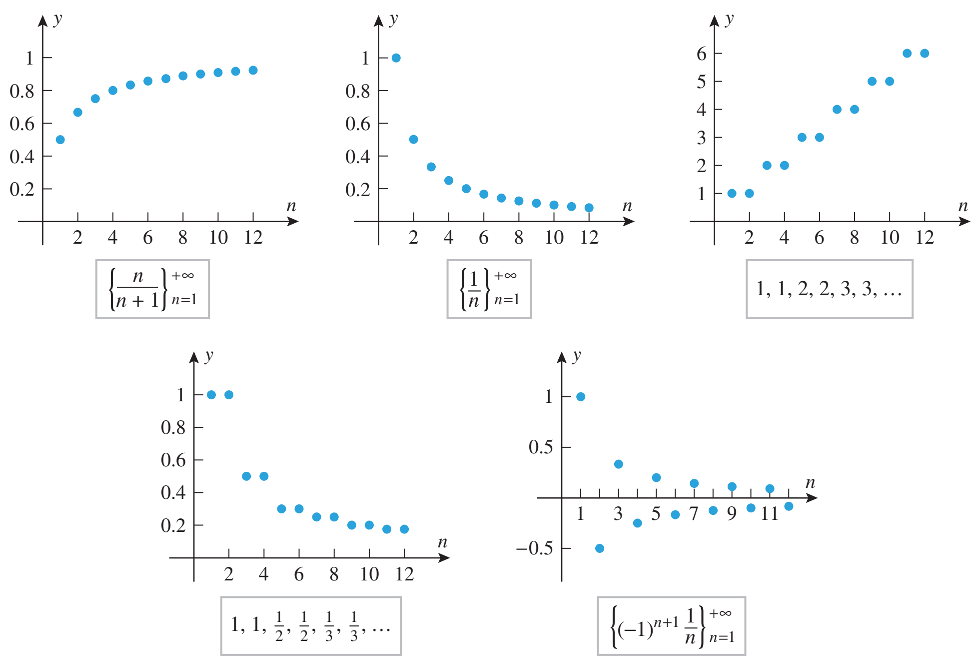
A sequence that is either increasing or decreasing is said to be ***monotone***, and a sequence that is either strictly increasing or strictly decreasing is said to be ***strictly monotone***.

Note that an increasing sequence need not be strictly increasing, and a decreasing sequence need not be strictly decreasing.

Some examples are given in Table 9.2.1 and their corresponding graphs are shown in Figure 9.2.1. The first and second sequences in Table 9.2.1 are strictly monotone; the third and fourth sequences are monotone but not strictly monotone; and the fifth sequence is neither strictly monotone nor monotone.

Table 9.2.1

SEQUENCE	DESCRIPTION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$	Strictly increasing
$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$	Strictly decreasing
$1, 1, 2, 2, 3, 3, \ldots$	Increasing; not strictly increasing
$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$	Decreasing; not strictly decreasing
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$	Neither increasing nor decreasing



Can a sequence be both increasing and decreasing? Explain.

▲ **Figure 9.2.1**

TESTING FOR MONOTONICITY

Frequently, one can *guess* whether a sequence is monotone or strictly monotone by writing out some of the initial terms. However, to be certain that the guess is correct, one must give a precise mathematical argument. Table 9.2.2 provides two ways of doing this, one based

Table 9.2.2

DIFFERENCE BETWEEN SUCCESSIVE TERMS	RATIO OF SUCCESSIVE TERMS	CONCLUSION
$a_{n+1} - a_n > 0$	$a_{n+1}/a_n > 1$	Strictly increasing
$a_{n+1} - a_n < 0$	$a_{n+1}/a_n < 1$	Strictly decreasing
$a_{n+1} - a_n \geq 0$	$a_{n+1}/a_n \geq 1$	Increasing
$a_{n+1} - a_n \leq 0$	$a_{n+1}/a_n \leq 1$	Decreasing

on differences of successive terms and the other on ratios of successive terms. It is assumed in the latter case that the terms are positive. One must show that the specified conditions hold for *all* pairs of successive terms.

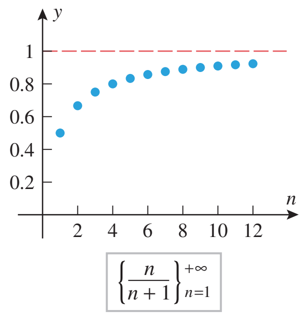


Figure 9.2.2

Example 1 Use differences of successive terms to show that

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

(Figure 9.2.2) is a strictly increasing sequence.

Solution. The pattern of the initial terms suggests that the sequence is strictly increasing. To prove that this is so, let

$$a_n = \frac{n}{n+1}$$

We can obtain a_{n+1} by replacing n by $n+1$ in this formula. This yields

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

Thus, for $n \geq 1$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$$

which proves that the sequence is strictly increasing. ◀

Example 2 Use ratios of successive terms to show that the sequence in Example 1 is strictly increasing.

Solution. As shown in the solution of Example 1,

$$a_n = \frac{n}{n+1} \quad \text{and} \quad a_{n+1} = \frac{n+1}{n+2}$$

Forming the ratio of successive terms we obtain

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/(n+2)}{n/(n+1)} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} \tag{1}$$

from which we see that $a_{n+1}/a_n > 1$ for $n \geq 1$. This proves that the sequence is strictly increasing. ◀

The following example illustrates still a third technique for determining whether a sequence is strictly monotone.

Example 3 In Examples 1 and 2 we proved that the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is strictly increasing by considering the difference and ratio of successive terms. Alternatively, we can proceed as follows. Let

$$f(x) = \frac{x}{x+1}$$

so that the nth term in the given sequence is $a_n = f(n)$. The function f is increasing for $x \geq 1$ since

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

29. Let $\{x_n\}$ be the sequence of population values defined recursively by $x_1 = 60$, and for $n \geq 1$, x_{n+1} is given by the Beverton–Holt model with $R = 10$ and $K = 300$.
(a) List the first four terms of the sequence $\{x_n\}$.
(b) If $0 < x_n < 300$, show that $0 < x_{n+1} < 300$. Conclude that $0 < x_n < 300$ for $n \geq 1$.
(c) Show that $\{x_n\}$ is increasing.
(d) Show that $\{x_n\}$ converges and find its limit L.

30. Let $\{x_n\}$ be a sequence of population values defined recursively by the Beverton–Holt model for which $x_1 > K$. Assume that the constants R and K satisfy $R > 1$ and $K > 0$.
(a) If $x_n > K$, show that $x_{n+1} > K$. Conclude that $x_n > K$ for all $n \geq 1$.
(b) Show that $\{x_n\}$ is decreasing.
(c) Show that $\{x_n\}$ converges and find its limit L.

31. The goal of this exercise is to establish Formula (5), namely,

$$\lim_{n\to+\infty} \frac{x^n}{n!} = 0$$

Let $a_n = |x|^n/n!$ and observe that the case where $x = 0$ is obvious, so we will focus on the case where $x \neq 0$.
(a) Show that

$$a_{n+1} = \frac{|x|}{n+1} a_n$$

(b) Show that the sequence $\{a_n\}$ is eventually strictly decreasing.
(c) Show that the sequence $\{a_n\}$ converges.

32. (a) Compare appropriate areas in the accompanying figure to deduce the following inequalities for $n \geq 2$:

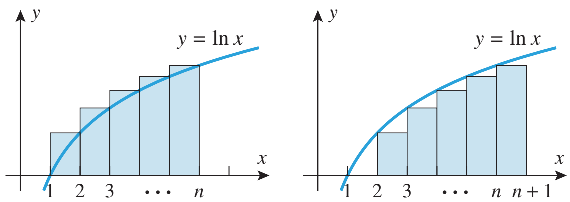
$$\int_1^n \ln x\, dx < \ln n! < \int_1^{n+1} \ln x\, dx$$

(b) Use the result in part (a) to show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \quad n > 1$$

(c) Use the Squeezing Theorem for Sequences (Theorem 9.1.5) and the result in part (b) to show that

$$\lim_{n\to+\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$



▲ **Figure Ex-32**

33. Use the left inequality in Exercise 32(b) to show that

$$\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty$$

34. Writing Give an example of an increasing sequence that is not eventually strictly increasing. What can you conclude about the terms of any such sequence? Explain.

35. Writing Discuss the appropriate use of "eventually" for various properties of sequences. For example, which is a useful expression: "eventually bounded" or "eventually monotone"?

✔ QUICK CHECK ANSWERS 9.2

1. I; D; N; I; N **2.** N; M; S **3.** 1; increasing **4.** 8; eventually; increasing

9.3 INFINITE SERIES

The purpose of this section is to discuss sums that contain infinitely many terms. The most familiar examples of such sums occur in the decimal representations of real numbers. For example, when we write $\frac{1}{3}$ in the decimal form $\frac{1}{3} = 0.3333\ldots$, we mean

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

which suggests that the decimal representation of $\frac{1}{3}$ can be viewed as a sum of infinitely many real numbers.

SUMS OF INFINITE SERIES

Our first objective is to define what is meant by the "sum" of infinitely many real numbers. We begin with some terminology.

9.3.1 DEFINITION An ***infinite series*** is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the ***terms*** of the series.

Since it is impossible to add infinitely many numbers together directly, sums of infinite series are defined and computed by an indirect limiting process. To motivate the basic idea, consider the decimal

$$0.3333\ldots \tag{1}$$

This can be viewed as the infinite series

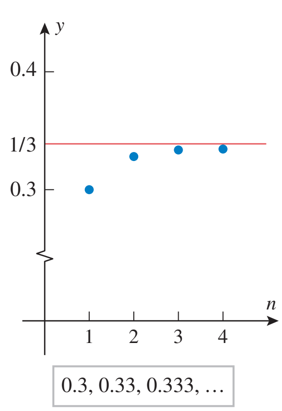
$$0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

or, equivalently,

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots \tag{2}$$

Since (1) is the decimal expansion of $\frac{1}{3}$, any reasonable definition for the sum of an infinite series should yield $\frac{1}{3}$ for the sum of (2). To obtain such a definition, consider the following sequence of (finite) sums:

$$\begin{aligned}
s_1 &= \frac{3}{10} = 0.3 \\
s_2 &= \frac{3}{10} + \frac{3}{10^2} = 0.33 \\
s_3 &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333 \\
s_4 &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} = 0.3333 \\
&\vdots
\end{aligned}$$



▲ **Figure 9.3.1**

The sequence of numbers $s_1, s_2, s_3, s_4, \ldots$ (Figure 9.3.1) can be viewed as a succession of approximations to the "sum" of the infinite series, which we want to be $\frac{1}{3}$. As we progress through the sequence, more and more terms of the infinite series are used, and the approximations get better and better, suggesting that the desired sum of $\frac{1}{3}$ might be the *limit* of this sequence of approximations. To see that this is so, we must calculate the limit of the general term in the sequence of approximations, namely,

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} \tag{3}$$

The problem of calculating

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \left(\frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} \right)$$

is complicated by the fact that both the last term and the number of terms in the sum change with n. It is best to rewrite such limits in a closed form in which the number of terms does not vary, if possible. (See the discussion of closed form and open form following Example 2 in Section 4.4.) To do this, we multiply both sides of (3) by $\frac{1}{10}$ to obtain

$$\frac{1}{10} s_n = \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \frac{3}{10^{n+1}} \tag{4}$$

and then subtract (4) from (3) to obtain

$$s_n - \frac{1}{10}s_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$
$$\frac{9}{10}s_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$
$$s_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

Since $1/10^n \to 0$ as $n \to +\infty$, it follows that

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}$$

which we denote by writing

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots$$

Motivated by the preceding example, we are now ready to define the general concept of the "sum" of an infinite series

$$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

We begin with some terminology: Let s_n denote the sum of the initial terms of the series, up to and including the term with index n. Thus,

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ s_3 &= u_1 + u_2 + u_3 \\ &\vdots \\ s_n &= u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^{n} u_k \end{aligned}$$

The number s_n is called the ***nth partial sum*** of the series and the sequence $\{s_n\}_{n=1}^{+\infty}$ is called the ***sequence of partial sums***.

As n increases, the partial sum $s_n = u_1 + u_2 + \cdots + u_n$ includes more and more terms of the series. Thus, if s_n tends toward a limit as $n \to +\infty$, it is reasonable to view this limit as the sum of *all* the terms in the series. This suggests the following definition.

WARNING

In everyday language the words "sequence" and "series" are often used interchangeably. However, in mathematics there is a distinction between these two words—a sequence is a *succession* whereas a series is a *sum*. It is essential that you keep this distinction in mind.

9.3.2 DEFINITION Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

If the sequence $\{s_n\}$ converges to a limit S, then the series is said to ***converge*** to S, and S is called the ***sum*** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to ***diverge***. A divergent series has no sum.

▶ **Example 1** Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

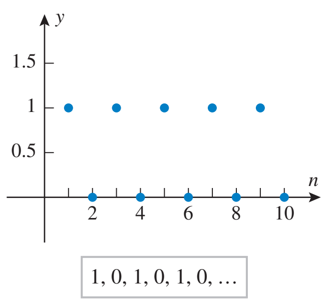
converges or diverges. If it converges, find the sum.

Solution. It is tempting to conclude that the sum of the series is zero by arguing that the positive and negative terms cancel one another. However, this is *not correct*; the problem is that algebraic operations that hold for finite sums do not carry over to infinite series in all cases. Later, we will discuss conditions under which familiar algebraic operations can be applied to infinite series, but for this example we turn directly to Definition 9.3.2. The partial sums are

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 - 1 = 0 \\ s_3 &= 1 - 1 + 1 = 1 \\ s_4 &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

and so forth. Thus, the sequence of partial sums is

$$1, 0, 1, 0, 1, 0, \ldots$$



▲ **Figure 9.3.2**

(Figure 9.3.2). Since this is a divergent sequence, the given series diverges and consequently has no sum. ◀

GEOMETRIC SERIES

In many important series, each term is obtained by multiplying the preceding term by some fixed constant. Thus, if the initial term of the series is a and each term is obtained by multiplying the preceding term by r, then the series has the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \quad (a \neq 0) \tag{5}$$

Such series are called ***geometric series***, and the number r is called the ***ratio*** for the series. Here are some examples:

$$1 + 2 + 4 + 8 + \cdots + 2^k + \cdots \qquad \boxed{a = 1, r = 2}$$

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots \qquad \boxed{a = \tfrac{3}{10}, r = \tfrac{1}{10}}$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{k+1}\frac{1}{2^k} + \cdots \qquad \boxed{a = \tfrac{1}{2}, r = -\tfrac{1}{2}}$$

$$1 + 1 + 1 + \cdots + 1 + \cdots \qquad \boxed{a = 1, r = 1}$$

$$1 - 1 + 1 - 1 + \cdots + (-1)^{k+1} + \cdots \qquad \boxed{a = 1, r = -1}$$

$$1 + x + x^2 + x^3 + \cdots + x^k + \cdots \qquad \boxed{a = 1, r = x}$$

Sometimes it is desirable to start the index of summation of an infinite series at $k = 0$ rather than $k = 1$, in which case we would call u_0 the *zeroth term* and $s_0 = u_0$ the *zeroth partial sum*. One can prove that changing the starting value for the index of summation of an infinite series has no effect on the convergence, the divergence, or the sum. If we had started the index at $k = 1$ in (5), then the series would be expressed as

$$\sum_{k=1}^{\infty} ar^{k-1}$$

Since this expression is more complicated than (5), we started the index at $k = 0$.

The following theorem is the fundamental result on convergence of geometric series.

9.3.3 THEOREM *A geometric series*

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \quad (a \neq 0)$$

converges if $|r| < 1$ *and diverges if* $|r| \geq 1$. *If the series converges, then the sum is*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

PROOF Let us treat the case $|r| = 1$ first. If $r = 1$, then the series is

$$a + a + a + a + \cdots$$

so the nth partial sum is $s_n = (n+1)a$ and

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} (n+1)a = \pm\infty$$

(the sign depending on whether a is positive or negative). This proves divergence. If $r = -1$, the series is

$$a - a + a - a + \cdots$$

so the sequence of partial sums is

$$a, 0, a, 0, a, 0, \ldots$$

which diverges.

Now let us consider the case where $|r| \neq 1$. The nth partial sum of the series is

$$s_n = a + ar + ar^2 + \cdots + ar^n \tag{6}$$

Multiplying both sides of (6) by r yields

$$rs_n = ar + ar^2 + \cdots + ar^n + ar^{n+1} \tag{7}$$

and subtracting (7) from (6) gives

$$s_n - rs_n = a - ar^{n+1}$$

or

$$(1-r)s_n = a - ar^{n+1} \tag{8}$$

Since $r \neq 1$ in the case we are considering, this can be rewritten as

$$s_n = \frac{a - ar^{n+1}}{1-r} = \frac{a}{1-r}(1 - r^{n+1}) \tag{9}$$

Note that (6) is an open form for s_n, while (9) is a closed form for s_n. In general, one needs a closed form to calculate the limit.

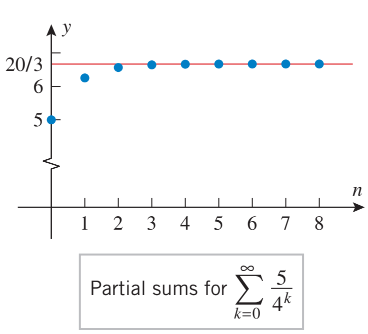
If $|r| < 1$, then r^{n+1} goes to 0 as $n \to +\infty$ (can you see why?), so $\{s_n\}$ converges. From (9)

$$\lim_{n\to+\infty} s_n = \frac{a}{1-r}$$

If $|r| > 1$, then either $r > 1$ or $r < -1$. In the case $r > 1$, r^{n+1} increases without bound as $n \to +\infty$, and in the case $r < -1$, r^{n+1} oscillates between positive and negative values that grow in magnitude, so $\{s_n\}$ diverges in both cases. ■

Example 2 In each part, determine whether the series converges, and if so find its sum.

$$\text{(a)} \sum_{k=0}^{\infty} \frac{5}{4^k} \qquad \text{(b)} \sum_{k=1}^{\infty} 3^{2k}5^{1-k}$$



▲ **Figure 9.3.3**

Solution (a). This is a geometric series with $a = 5$ and $r = \frac{1}{4}$. Since $|r| = \frac{1}{4} < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

(Figure 9.3.3).

Solution (b). This is a geometric series in concealed form, since we can rewrite it as

$$\sum_{k=1}^{\infty} 3^{2k}5^{1-k} = \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}} = \sum_{k=1}^{\infty} 9\left(\frac{9}{5}\right)^{k-1}$$

Since $r = \frac{9}{5} > 1$, the series diverges. ◀

Example 3 Find the rational number represented by the repeating decimal

$$0.784784784\ldots$$

TECHNOLOGY MASTERY

Computer algebra systems have commands for finding sums of convergent series. If you have a CAS, use it to compute the sums in Examples 2 and 3.

Solution. We can write

$$0.784784784\ldots = 0.784 + 0.000784 + 0.000000784 + \cdots$$

so the given decimal is the sum of a geometric series with $a = 0.784$ and $r = 0.001$. Thus,

$$0.784784784\ldots = \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{0.784}{0.999} = \frac{784}{999} \blacktriangleleft$$

Example 4 In each part, find all values of x for which the series converges, and find the sum of the series for those values of x.

$$\text{(a)} \sum_{k=0}^{\infty} x^k \qquad \text{(b)}\ 3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \cdots + \frac{3(-1)^k}{2^k}x^k + \cdots$$

Solution (a). The expanded form of the series is

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

The series is a geometric series with $a = 1$ and $r = x$, so it converges if $|x| < 1$ and diverges otherwise. When the series converges its sum is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Solution (b). This is a geometric series with $a = 3$ and $r = -x/2$. It converges if $|-x/2| < 1$, or equivalently, when $|x| < 2$. When the series converges its sum is

$$\sum_{k=0}^{\infty} 3\left(-\frac{x}{2}\right)^k = \frac{3}{1-\left(-\frac{x}{2}\right)} = \frac{6}{2+x} \blacktriangleleft$$

TELESCOPING SUMS

Example 5 Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

converges or diverges. If it converges, find the sum.

Solution. The nth partial sum of the series is

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We will begin by rewriting s_n in closed form. This can be accomplished by using the method of partial fractions to obtain (verify)

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

The sum in (10) is an example of a ***telescoping sum***. The name is derived from the fact that in simplifying the sum, one term in each parenthetical expression cancels one term in the next parenthetical expression, until the entire sum collapses (like a folding telescope) into just two terms.

from which we obtain the sum

$$\begin{aligned}
s_n &= \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\
&= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= 1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}\\
&= 1-\frac{1}{n+1} \qquad (10)
\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(1-\frac{1}{n+1}\right) = 1 \blacktriangleleft$$

HARMONIC SERIES

One of the most important of all diverging series is the ***harmonic series***,

$$\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

which arises in connection with the overtones produced by a vibrating musical string. It is not immediately evident that this series diverges. However, the divergence will become apparent when we examine the partial sums in detail. Because the terms in the series are all positive, the partial sums

$$s_1 = 1,\quad s_2 = 1+\frac{1}{2},\quad s_3 = 1+\frac{1}{2}+\frac{1}{3},\quad s_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4},\ldots$$

form a strictly increasing sequence

$$s_1 < s_2 < s_3 < \cdots < s_n < \cdots$$

(Figure 9.3.4*a*). Thus, by Theorem 9.2.3 we can prove divergence by demonstrating that there is no constant M that is greater than or equal to *every* partial sum. To this end, we will consider some selected partial sums, namely, $s_2, s_4, s_8, s_{16}, s_{32}, \ldots$. Note that the subscripts are successive powers of 2, so that these are the partial sums of the form s_{2^n} (Figure 9.3.4*b*). These partial sums satisfy the inequalities

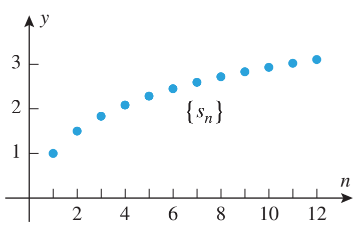
$$\begin{aligned}
s_2 &= 1+\tfrac{1}{2} > \tfrac{1}{2}+\tfrac{1}{2} = \tfrac{2}{2}\\
s_4 &= s_2+\tfrac{1}{3}+\tfrac{1}{4} > s_2+\left(\tfrac{1}{4}+\tfrac{1}{4}\right) = s_2+\tfrac{1}{2} > \tfrac{3}{2}\\
s_8 &= s_4+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8} > s_4+\left(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}\right) = s_4+\tfrac{1}{2} > \tfrac{4}{2}\\
s_{16} &= s_8+\tfrac{1}{9}+\tfrac{1}{10}+\tfrac{1}{11}+\tfrac{1}{12}+\tfrac{1}{13}+\tfrac{1}{14}+\tfrac{1}{15}+\tfrac{1}{16}\\
&> s_8+\left(\tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16}\right) = s_8+\tfrac{1}{2} > \tfrac{5}{2}\\
&\vdots\\
s_{2^n} &> \frac{n+1}{2}
\end{aligned}$$

If M is any constant, we can find a positive integer n such that $(n+1)/2 > M$. But for this n

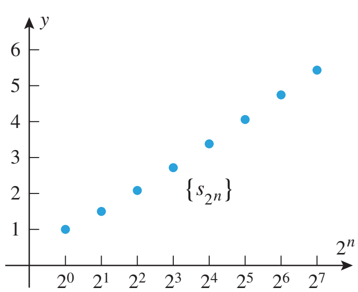
$$s_{2^n} > \frac{n+1}{2} > M$$

so that no constant M is greater than or equal to *every* partial sum of the harmonic series. This proves divergence.

This divergence proof, which predates the discovery of calculus, is due to a French bishop and teacher, Nicole Oresme (1323–1382). This series eventually attracted the interest of Johann and Jakob Bernoulli (p. 700) and led them to begin thinking about the general concept of convergence, which was a new idea at that time.



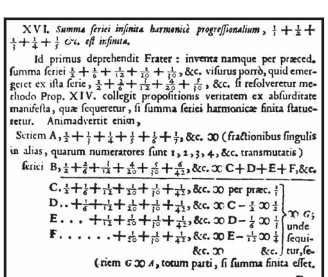
(*a*)



(*b*)

Partial sums for the harmonic series

Figure 9.3.4



XVI. *Summa seriei infinitæ harmonicè progressionalium*, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ *&c. est infinita.*

Id primus deprehendit Frater: inventa namque per præced. summa seriei $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c. visurus porrò, quid emergeret ex ista serie, $\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}$, &c. si resolveretur methodo Prop. XIV. collegit propositionis veritatem ex absurditate manifesta, quæ sequeretur, si summa seriei harmonicæ finita statueretur. Animadvertit enim,

Seriem A, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$, &c. ∞ (fractionibus singulis in alias, quarum numeratores sunt 1, 2, 3, 4, &c. transmutatis) seriei B, $\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}+\frac{6}{42}$, &c. ∞ C + D + E + F, &c.

C. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ per præc. $\frac{1}{1}$
D. . $+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ C $-\frac{1}{2}$ ∞ $\frac{1}{2}$
E. . . $+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ D $-\frac{1}{6}$ ∞ $\frac{1}{3}$
F. . . . . . $+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ E $-\frac{1}{12}$ ∞ $\frac{1}{4}$
&c. ∞ &c.

∞ G; unde sequitur, seriem G ∞ A, totum parti, si summa finita esset.

Ego

Courtesy Lilly Library, Indiana University

This is a proof of the divergence of the harmonic series, as it appeared in an appendix of Jakob Bernoulli's posthumous publication, Ars Conjectandi, *which appeared in 1713.*

✔QUICK CHECK EXERCISES 9.3 (See page 623 for answers.)

1. In mathematics, the terms "sequence" and "series" have different meanings: a ________ is a succession, whereas a ________ is a sum.

2. Consider the series
$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$
If $\{s_n\}$ is the sequence of partial sums for this series, then $s_1 =$ ________, $s_2 =$ ________, $s_3 =$ ________, $s_4 =$ ________, and $s_n =$ ________.

3. What does it mean to say that a series $\sum u_k$ *converges*?

4. A geometric series is a series of the form
$$\sum_{k=0}^{\infty} \underline{\qquad\qquad}$$
This series converges to ________ if ________. This series diverges if ________.

5. The harmonic series has the form
$$\sum_{k=1}^{\infty} \underline{\qquad\qquad}$$
Does the harmonic series converge or diverge?

EXERCISE SET 9.3 [C] CAS

1–2 In each part, find exact values for the first four partial sums, find a closed form for the nth partial sum, and determine whether the series converges by calculating the limit of the nth partial sum. If the series converges, then state its sum.

1. (a) $2 + \frac{2}{5} + \frac{2}{5^2} + \cdots + \frac{2}{5^{k-1}} + \cdots$
(b) $\frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \cdots + \frac{2^{k-1}}{4} + \cdots$
(c) $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(k+1)(k+2)} + \cdots$

2. (a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k$ (b) $\displaystyle\sum_{k=1}^{\infty} 4^{k-1}$ (c) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+3} - \frac{1}{k+4}\right)$

3–14 Determine whether the series converges, and if so find its sum.

3. $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1}$

4. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+2}$

5. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{7}{6^{k-1}}$

6. $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{3}{2}\right)^{k+1}$

7. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)}$

8. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$

9. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{9k^2+3k-2}$

10. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2-1}$

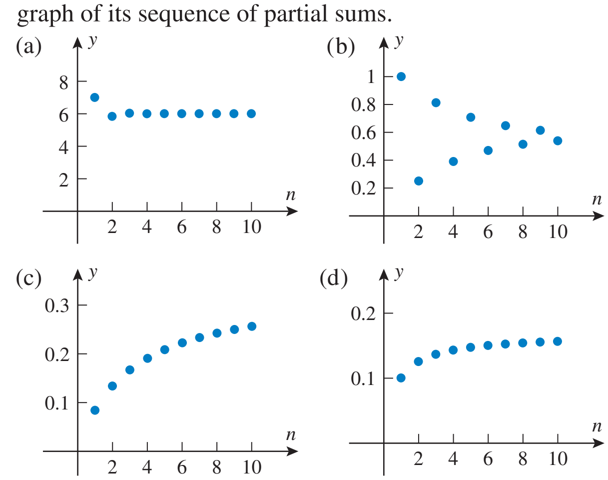
11. $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k-2}$

12. $\displaystyle\sum_{k=5}^{\infty}\left(\frac{e}{\pi}\right)^{k-1}$

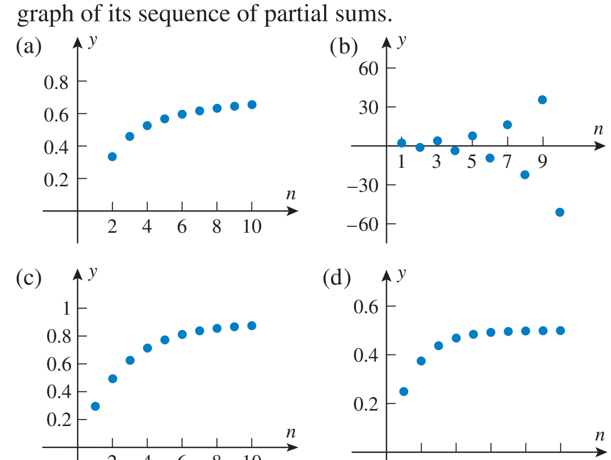
13. $\displaystyle\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}}$

14. $\displaystyle\sum_{k=1}^{\infty} 5^{3k}7^{1-k}$

15. Match a series from one of Exercises 3, 5, 7, or 9 with the graph of its sequence of partial sums.



16. Match a series from one of Exercises 4, 6, 8, or 10 with the graph of its sequence of partial sums.



17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. An infinite series converges if its sequence of terms converges.

18. The geometric series $a + ar + ar^2 + \cdots + ar^n + \cdots$ converges provided $|r| < 1$.

19. The harmonic series diverges.

20. An infinite series converges if its sequence of partial sums is bounded and monotone.

21–24 Express the repeating decimal as a fraction.

21. $0.9999\ldots$

22. $0.4444\ldots$

23. $5.373737\ldots$

24. $0.451141414\ldots$

25. Recall that a *terminating decimal* is a decimal whose digits are all 0 from some point on ($0.5 = 0.50000\ldots$, for example). Show that a decimal of the form $0.a_1a_2\ldots a_n9999\ldots$, where $a_n \neq 9$, can be expressed as a terminating decimal.

FOCUS ON CONCEPTS

26. The great Swiss mathematician Leonhard Euler (biography on p. 3) sometimes reached incorrect conclusions in his pioneering work on infinite series. For example, Euler deduced that

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

and

$$-1 = 1 + 2 + 4 + 8 + \cdots$$

by substituting $x = -1$ and $x = 2$ in the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

What was the problem with his reasoning?

27. A ball is dropped from a height of 10 m. Each time it strikes the ground it bounces vertically to a height that is $\frac{3}{4}$ of the preceding height. Find the total distance the ball will travel if it is assumed to bounce infinitely often.

28. The accompanying figure shows an "infinite staircase" constructed from cubes. Find the total volume of the staircase, given that the largest cube has a side of length 1 and each successive cube has a side whose length is half that of the preceding cube.

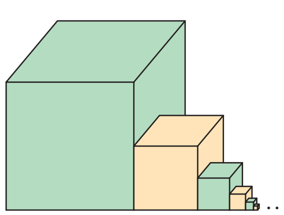


Figure Ex-28

29. In each part, find a closed form for the nth partial sum of the series, and determine whether the series converges. If so, find its sum.

(a) $\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{k}{k+1} + \cdots$

(b) $\ln\left(1 - \frac{1}{4}\right) + \ln\left(1 - \frac{1}{9}\right) + \ln\left(1 - \frac{1}{16}\right) + \cdots + \ln\left(1 - \frac{1}{(k+1)^2}\right) + \cdots$

30. Use geometric series to show that

(a) $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^k = \frac{1}{1+x}$ if $-1 < x < 1$

(b) $\displaystyle\sum_{k=0}^{\infty}(x-3)^k = \frac{1}{4-x}$ if $2 < x < 4$

(c) $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^{2k} = \frac{1}{1+x^2}$ if $-1 < x < 1$.

31. In each part, find all values of x for which the series converges, and find the sum of the series for those values of x.

(a) $x - x^3 + x^5 - x^7 + x^9 - \cdots$

(b) $\dfrac{1}{x^2} + \dfrac{2}{x^3} + \dfrac{4}{x^4} + \dfrac{8}{x^5} + \dfrac{16}{x^6} + \cdots$

(c) $e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \cdots$

32. Show that for all real values of x

$$\sin x - \frac{1}{2}\sin^2 x + \frac{1}{4}\sin^3 x - \frac{1}{8}\sin^4 x + \cdots = \frac{2\sin x}{2 + \sin x}$$

33. Let a_1 be any real number, and let $\{a_n\}$ be the sequence defined recursively by

$$a_{n+1} = \tfrac{1}{2}(a_n + 1)$$

Make a conjecture about the limit of the sequence, and confirm your conjecture by expressing a_n in terms of a_1 and taking the limit.

34. Show: $\displaystyle\sum_{k=1}^{\infty}\frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = 1.$

35. Show: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{3}{2}.$

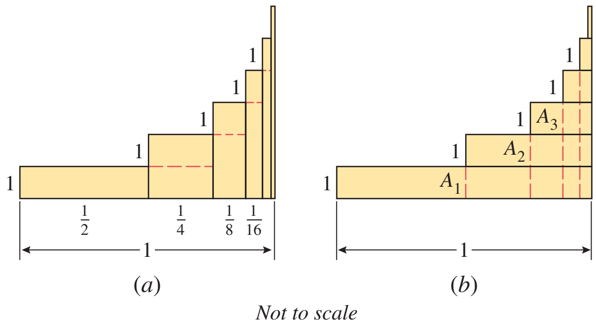
36. Show: $\dfrac{1}{1\cdot 3} + \dfrac{1}{2\cdot 4} + \dfrac{1}{3\cdot 5} + \cdots = \dfrac{3}{4}.$

37. Show: $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \cdots = \dfrac{1}{2}.$

38. In his *Treatise on the Configurations of Qualities and Motions* (written in the 1350s), the French Bishop of Lisieux, Nicole Oresme, used a geometric method to find the sum of the series

$$\sum_{k=1}^{\infty}\frac{k}{2^k} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

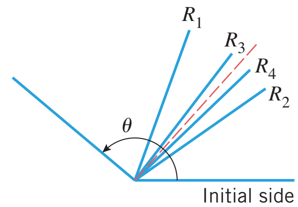
In part (*a*) of the accompanying figure, each term in the series is represented by the area of a rectangle, and in part (b) the configuration in part (a) has been divided into rectangles with areas $A_1, A_2, A_3, \ldots$. Find the sum $A_1 + A_2 + A_3 + \cdots$.



▲ **Figure Ex-38**

39. As shown in the accompanying figure, suppose that an angle θ is bisected using a straightedge and compass to produce ray R_1, then the angle between R_1 and the initial side is bisected to produce ray R_2. Thereafter, rays $R_3, R_4, R_5, \ldots$ are constructed in succession by bisecting the angle between the preceding two rays. Show that the sequence of angles that these rays make with the initial side has a limit of $\theta/3$.

Source: This problem is based on "Trisection of an Angle in an Infinite Number of Steps" by Eric Kincannon, which appeared in *The College Mathematics Journal*, Vol. 21, No. 5, November 1990.



◀ **Figure Ex-39**

40. [C] In each part, use a CAS to find the sum of the series if it converges, and then confirm the result by hand calculation.

(a) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}2^k 3^{2-k}$ (b) $\displaystyle\sum_{k=1}^{\infty}\frac{3^{3k}}{5^{k-1}}$ (c) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{4k^2-1}$

41. Writing Discuss the similarities and differences between what it means for a sequence to converge and what it means for a series to converge.

42. Writing Read about Zeno's dichotomy paradox in an appropriate reference work and relate the paradox in a setting that is familiar to you. Discuss a connection between the paradox and geometric series.

✔ QUICK CHECK ANSWERS 9.3

1. sequence; series **2.** $\frac{1}{2}; \frac{3}{4}; \frac{7}{8}; \frac{15}{16}; 1 - \frac{1}{2^n}$ **3.** The sequence of partial sums converges.
4. ar^k $(a \neq 0)$; $\frac{a}{1-r}$; $|r| < 1$; $|r| \geq 1$ **5.** $\frac{1}{k}$; diverge

9.4 CONVERGENCE TESTS

In the last section we showed how to find the sum of a series by finding a closed form for the nth partial sum and taking its limit. However, it is relatively rare that one can find a closed form for the nth partial sum of a series, so alternative methods are needed for finding the sum of a series. One possibility is to prove that the series converges, and then to approximate the sum by a partial sum with sufficiently many terms to achieve the desired degree of accuracy. In this section we will develop various tests that can be used to determine whether a given series converges or diverges.

THE DIVERGENCE TEST

In stating general results about convergence or divergence of series, it is convenient to use the notation $\sum u_k$ as a generic notation for a series, thus avoiding the issue of whether the sum begins with $k = 0$ or $k = 1$ or some other value. Indeed, we will see shortly that the starting index value is irrelevant to the issue of convergence. The kth term in an infinite series $\sum u_k$ is called the ***general term*** of the series. The following theorem establishes a relationship between the limit of the general term and the convergence properties of a series.

9.4.1 THEOREM (*The Divergence Test*)

(*a*) *If* $\lim_{k\to+\infty} u_k \neq 0$, *then the series* $\sum u_k$ *diverges.*

(*b*) *If* $\lim_{k\to+\infty} u_k = 0$, *then the series* $\sum u_k$ *may either converge or diverge.*

PROOF (*a*) To prove this result, it suffices to show that if the series converges, then $\lim_{k\to+\infty} u_k = 0$ (why?). We will prove this alternative form of (*a*).

Let us assume that the series converges. The general term u_k can be written as

$$u_k = s_k - s_{k-1} \tag{1}$$

where s_k is the sum of the terms through u_k and s_{k-1} is the sum of the terms through u_{k-1}. If S denotes the sum of the series, then $\lim_{k\to+\infty} s_k = S$, and since $(k-1)\to+\infty$ as $k\to+\infty$, we also have $\lim_{k\to+\infty} s_{k-1} = S$. Thus, from (1)

$$\lim_{k\to+\infty} u_k = \lim_{k\to+\infty}(s_k - s_{k-1}) = S - S = 0$$

PROOF (*b*) To prove this result, it suffices to produce both a convergent series and a divergent series for which $\lim_{k\to+\infty} u_k = 0$. The following series both have this property:

$$\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}+\cdots \quad \text{and} \quad 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\cdots$$

The first is a convergent geometric series and the second is the divergent harmonic series. ■

WARNING

The converse of Theorem 9.4.2 is false; that is, showing that

$$\lim_{k\to+\infty} u_k = 0$$

does not prove that $\sum u_k$ converges, since this property may hold for divergent as well as convergent series. This is illustrated in the proof of part (*b*) of Theorem 9.4.1.

The alternative form of part (*a*) given in the preceding proof is sufficiently important that we state it separately for future reference.

9.4.2 THEOREM *If the series* $\sum u_k$ *converges, then* $\lim_{k\to+\infty} u_k = 0$.

▶ Example 1 The series

$$\sum_{k=1}^{\infty}\frac{k}{k+1} = \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{k}{k+1}+\cdots$$

diverges since

$$\lim_{k\to+\infty}\frac{k}{k+1} = \lim_{k\to+\infty}\frac{1}{1+1/k} = 1 \neq 0 \;◀$$

ALGEBRAIC PROPERTIES OF INFINITE SERIES

For brevity, the proof of the following result is omitted.

See Exercises 27 and 28 for an exploration of what happens when $\sum u_k$ or $\sum v_k$ diverge.

9.4.3 THEOREM

(a) *If* $\sum u_k$ *and* $\sum v_k$ *are convergent series, then* $\sum(u_k + v_k)$ *and* $\sum(u_k - v_k)$ *are convergent series and the sums of these series are related by*

$$\sum_{k=1}^{\infty}(u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty}(u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) *If c is a nonzero constant, then the series* $\sum u_k$ *and* $\sum cu_k$ *both converge or both diverge. In the case of convergence, the sums are related by*

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

(c) *Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer* K, *the series*

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

WARNING

Do not read too much into part (*c*) of Theorem 9.4.3. Although convergence is not affected when finitely many terms are deleted from the beginning of a convergent series, the *sum* of the series is changed by the removal of those terms.

Example 2 Find the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

Solution. The series

$$\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

is a convergent geometric series $\left(a = \frac{3}{4}, r = \frac{1}{4}\right)$, and the series

$$\sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

is also a convergent geometric series $\left(a = 2, r = \frac{1}{5}\right)$. Thus, from Theorems 9.4.3(*a*) and 9.3.3 the given series converges and

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right) = \sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}}$$

$$= \frac{\frac{3}{4}}{1-\frac{1}{4}} - \frac{2}{1-\frac{1}{5}} = -\frac{3}{2} \blacktriangleleft$$

▶ Example 3 Determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{5}{k} = 5 + \frac{5}{2} + \frac{5}{3} + \cdots + \frac{5}{k} + \cdots \qquad \text{(b)} \sum_{k=10}^{\infty} \frac{1}{k} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$$

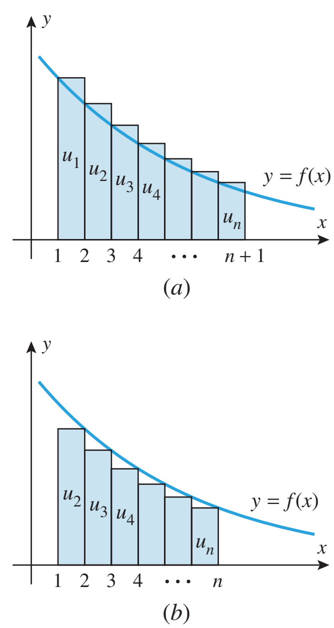
Solution. The first series is a constant times the divergent harmonic series, and hence diverges by part (*b*) of Theorem 9.4.3. The second series results by deleting the first nine terms from the divergent harmonic series, and hence diverges by part (*c*) of Theorem 9.4.3. ◀

THE INTEGRAL TEST

The expressions

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \int_1^{+\infty} \frac{1}{x^2}\, dx$$

are related in that the integrand in the improper integral results when the index k in the general term of the series is replaced by x and the limits of summation in the series are replaced by the corresponding limits of integration. The following theorem shows that there is a relationship between the convergence of the series and the integral.



▲ **Figure 9.4.1**

> **9.4.4 THEOREM (*The Integral Test*)** *Let $\sum u_k$ be a series with positive terms. If f is a function that is decreasing and continuous on an interval $[a, +\infty)$ and such that $u_k = f(k)$ for all $k \ge a$, then*
>
> $$\sum_{k=1}^{\infty} u_k \quad \textit{and} \quad \int_a^{+\infty} f(x)\, dx$$
>
> *both converge or both diverge.*

The proof of the integral test is deferred to the end of this section. However, the gist of the proof is captured in Figure 9.4.1: if the integral diverges, then so does the series (Figure 9.4.1*a*), and if the integral converges, then so does the series (Figure 9.4.1*b*).

▶ Example 4 Show that the integral test applies, and use the integral test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{k} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Solution (a). We already know that this is the divergent harmonic series, so the integral test will simply illustrate another way of establishing the divergence.

Note first that the series has positive terms, so the integral test is applicable. If we replace k by x in the general term $1/k$, we obtain the function $f(x) = 1/x$, which is decreasing and continuous for $x \ge 1$ (as required to apply the integral test with $a = 1$). Since

$$\int_1^{+\infty} \frac{1}{x}\, dx = \lim_{b \to +\infty} \int_1^b \frac{1}{x}\, dx = \lim_{b \to +\infty} [\ln b - \ln 1] = +\infty$$

the integral diverges and consequently so does the series.

WARNING

In part (b) of Example 4, do not erroneously conclude that the sum of the series is 1 because the value of the corresponding integral is 1. You can see that this is not so since the sum of the first two terms alone exceeds 1. Later, we will see that the sum of the series is actually $\pi^2/6$.

Solution (b). Note first that the series has positive terms, so the integral test is applicable. If we replace k by x in the general term $1/k^2$, we obtain the function $f(x) = 1/x^2$, which is decreasing and continuous for $x \geq 1$. Since

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to+\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to+\infty} \left[1 - \frac{1}{b}\right] = 1$$

the integral converges and consequently the series converges by the integral test with $a = 1$. ◀

p-SERIES

The series in Example 4 are special cases of a class of series called ***p-series*** or ***hyperharmonic series***. A p-series is an infinite series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

where $p > 0$. Examples of p-series are

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots \qquad \boxed{p = 1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \cdots \qquad \boxed{p = 2}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \cdots \qquad \boxed{p = \tfrac{1}{2}}$$

The following theorem tells when a p-series converges.

9.4.5 THEOREM (*Convergence of p-Series*)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.

PROOF To establish this result when $p \neq 1$, we will use the integral test.

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{b\to+\infty} \int_1^b x^{-p}\,dx = \lim_{b\to+\infty} \left.\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to+\infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

Assume first that $p > 1$. Then $1 - p < 0$, so $b^{1-p} \to 0$ as $b \to +\infty$. Thus, the integral converges [its value is $-1/(1-p)$] and consequently the series also converges.

Now assume that $0 < p < 1$. It follows that $1 - p > 0$ and $b^{1-p} \to +\infty$ as $b \to +\infty$, so the integral and the series diverge. The case $p = 1$ is the harmonic series, which was previously shown to diverge. ■

▶ Example 5

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

diverges since it is a p-series with $p = \frac{1}{3} < 1$. ◀

PROOF OF THE INTEGRAL TEST

Before we can prove the integral test, we need a basic result about convergence of series with *nonnegative* terms. If $u_1 + u_2 + u_3 + \cdots + u_k + \cdots$ is such a series, then its sequence of partial sums is increasing, that is,

$$s_1 \le s_2 \le s_3 \le \cdots \le s_n \le \cdots$$

Thus, from Theorem 9.2.3 the sequence of partial sums converges to a limit S if and only if it has some upper bound M, in which case $S \le M$. If no upper bound exists, then the sequence of partial sums diverges. Since convergence of the sequence of partial sums corresponds to convergence of the series, we have the following theorem.

9.4.6 THEOREM *If $\sum u_k$ is a series with nonnegative terms, and if there is a constant M such that*

$$s_n = u_1 + u_2 + \cdots + u_n \le M$$

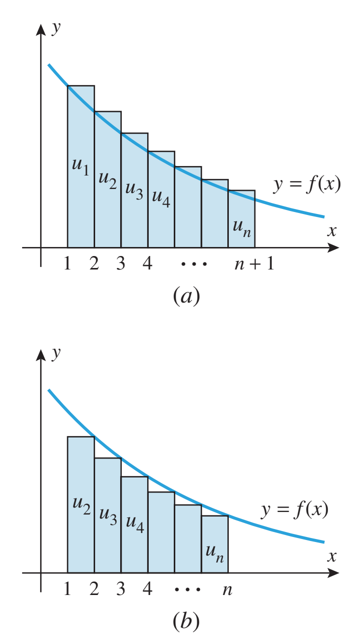
for every n, then the series converges and the sum S satisfies $S \le M$. If no such M exists, then the series diverges.

In words, this theorem implies that *a series with nonnegative terms converges if and only if its sequence of partial sums is bounded above.*

PROOF OF THEOREM 9.4.4 We need only show that the series converges when the integral converges and that the series diverges when the integral diverges. For simplicity, we will limit the proof to the case where $a = 1$. Assume that $f(x)$ satisfies the hypotheses of the theorem for $x \ge 1$. Since

$$f(1) = u_1,\ f(2) = u_2, \ldots,\ f(n) = u_n, \ldots$$

the values of $u_1, u_2, \ldots, u_n, \ldots$ can be interpreted as the areas of the rectangles shown in Figure 9.4.2.



▲ **Figure 9.4.2**

The following inequalities result by comparing the areas under the curve $y = f(x)$ to the areas of the rectangles in Figure 9.4.2 for $n > 1$:

$$\int_1^{n+1} f(x)\,dx < u_1 + u_2 + \cdots + u_n = s_n \qquad \text{Figure 9.4.2}a$$

$$s_n - u_1 = u_2 + u_3 + \cdots + u_n < \int_1^{n} f(x)\,dx \qquad \text{Figure 9.4.2}b$$

These inequalities can be combined as

$$\int_1^{n+1} f(x)\,dx < s_n < u_1 + \int_1^{n} f(x)\,dx \tag{2}$$

If the integral $\int_1^{+\infty} f(x)\,dx$ converges to a finite value L, then from the right-hand inequality in (2)

$$s_n < u_1 + \int_1^{n} f(x)\,dx < u_1 + \int_1^{+\infty} f(x)\,dx = u_1 + L$$

Thus, each partial sum is less than the finite constant $u_1 + L$, and the series converges by Theorem 9.4.6. On the other hand, if the integral $\int_1^{+\infty} f(x)\,dx$ diverges, then

$$\lim_{n\to+\infty} \int_1^{n+1} f(x)\,dx = +\infty$$

so that from the left-hand inequality in (2), $s_n \to +\infty$ as $n \to +\infty$. This implies that the series also diverges. ■

✔ QUICK CHECK EXERCISES 9.4 (See page 631 for answers.)

1. The divergence test says that if ________ $\neq 0$, then the series $\sum u_k$ diverges.
2. Given that
$$a_1 = 3, \quad \sum_{k=1}^{\infty} a_k = 1, \quad \text{and} \quad \sum_{k=1}^{\infty} b_k = 5$$
it follows that
$$\sum_{k=2}^{\infty} a_k = \underline{\qquad\qquad} \quad \text{and} \quad \sum_{k=1}^{\infty}(2a_k + b_k) = \underline{\qquad\qquad}$$
3. Since $\int_1^{+\infty}(1/\sqrt{x})\,dx = +\infty$, the ________ test applied to the series $\sum_{k=1}^{\infty}$ ________ shows that this series ________.
4. A p-series is a series of the form
$$\sum_{k=1}^{\infty} \underline{\qquad\qquad}$$
This series converges if ________. This series diverges if ________.

EXERCISE SET 9.4 Graphing Utility C CAS

1. Use Theorem 9.4.3 to find the sum of each series.
(a) $\left(\frac{1}{2} + \frac{1}{4}\right) + \left(\frac{1}{2^2} + \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{2^k} + \frac{1}{4^k}\right) + \cdots$
(b) $\sum_{k=1}^{\infty}\left(\frac{1}{5^k} - \frac{1}{k(k+1)}\right)$
2. Use Theorem 9.4.3 to find the sum of each series.
(a) $\sum_{k=2}^{\infty}\left[\frac{1}{k^2-1} - \frac{7}{10^{k-1}}\right]$ (b) $\sum_{k=1}^{\infty}\left[7^{-k}3^{k+1} - \frac{2^{k+1}}{5^k}\right]$

3–4 For each given p-series, identify p and determine whether the series converges. ■

3. (a) $\sum_{k=1}^{\infty} \frac{1}{k^3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (c) $\sum_{k=1}^{\infty} k^{-1}$ (d) $\sum_{k=1}^{\infty} k^{-2/3}$
4. (a) $\sum_{k=1}^{\infty} k^{-4/3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$ (c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^5}}$ (d) $\sum_{k=1}^{\infty} \frac{1}{k^{\pi}}$

5–6 Apply the divergence test and state what it tells you about the series. ■

5. (a) $\sum_{k=1}^{\infty} \frac{k^2+k+3}{2k^2+1}$ (b) $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k$
(c) $\sum_{k=1}^{\infty} \cos k\pi$ (d) $\sum_{k=1}^{\infty} \frac{1}{k!}$
6. (a) $\sum_{k=1}^{\infty} \frac{k}{e^k}$ (b) $\sum_{k=1}^{\infty} \ln k$
(c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (d) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}+3}$

7–8 Confirm that the integral test is applicable and use it to determine whether the series converges. ■

7. (a) $\sum_{k=1}^{\infty} \frac{1}{5k+2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$
8. (a) $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{(4+2k)^{3/2}}$

9–24 Determine whether the series converges. ■

9. $\sum_{k=1}^{\infty} \frac{1}{k+6}$
10. $\sum_{k=1}^{\infty} \frac{3}{5k}$
11. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$
12. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{e}}$
13. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$
14. $\sum_{k=3}^{\infty} \frac{\ln k}{k}$
15. $\sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$
16. $\sum_{k=1}^{\infty} ke^{-k^2}$
17. $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{-k}$
18. $\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+3}$
19. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$
20. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$
21. $\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$
22. $\sum_{k=1}^{\infty} k^2 e^{-k^3}$
23. $\sum_{k=5}^{\infty} 7k^{-1.01}$
24. $\sum_{k=1}^{\infty} \operatorname{sech}^2 k$

25–26 Use the integral test to investigate the relationship between the value of p and the convergence of the series. ■

25. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$
26. $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)[\ln(\ln k)]^p}$

FOCUS ON CONCEPTS

27. Suppose that the series $\sum u_k$ converges and the series $\sum v_k$ diverges. Show that the series $\sum(u_k + v_k)$ and $\sum(u_k - v_k)$ both diverge. [*Hint:* Assume that $\sum(u_k + v_k)$ converges and use Theorem 9.4.3 to obtain a contradiction.]

28. Find examples to show that if the series $\sum u_k$ and $\sum v_k$ both diverge, then the series $\sum(u_k + v_k)$ and $\sum(u_k - v_k)$ may either converge or diverge.

29–30 Use the results of Exercises 27 and 28, if needed, to determine whether each series converges or diverges. ■

29. (a) $\displaystyle\sum_{k=1}^{\infty}\left[\left(\frac{2}{3}\right)^{k-1} + \frac{1}{k}\right]$ (b) $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{3k+2} - \frac{1}{k^{3/2}}\right]$

30. (a) $\displaystyle\sum_{k=2}^{\infty}\left[\frac{1}{k(\ln k)^2} - \frac{1}{k^2}\right]$ (b) $\displaystyle\sum_{k=2}^{\infty}\left[ke^{-k^2} + \frac{1}{k\ln k}\right]$

31–34 True–False Determine whether the statement is true or false. Explain your answer. ■

31. If $\sum u_k$ converges to L, then $\sum(1/u_k)$ converges to $1/L$.

32. If $\sum cu_k$ diverges for some constant c, then $\sum u_k$ must diverge.

33. The integral test can be used to prove that a series diverges.

34. The series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{p^k}$ is a p-series.

C **35.** Use a CAS to confirm that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$$

and then use these results in each part to find the sum of the series.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{3k^2-1}{k^4}$ (b) $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k^2}$ (c) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{(k-1)^4}$

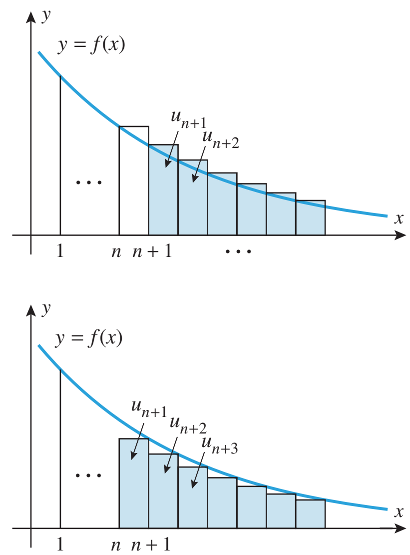
36–40 Exercise 36 will show how a partial sum can be used to obtain upper and lower bounds on the sum of a series when the hypotheses of the integral test are satisfied. This result will be needed in Exercises 37–40. ■

36. (a) Let $\sum_{k=1}^{\infty} u_k$ be a convergent series with positive terms, and let f be a function that is decreasing and continuous on $[n, +\infty)$ and such that $u_k = f(k)$ for $k \geq n$. Use an area argument and the accompanying figure to show that

$$\int_{n+1}^{+\infty} f(x)\,dx < \sum_{k=n+1}^{\infty} u_k < \int_{n}^{+\infty} f(x)\,dx$$

(b) Show that if S is the sum of the series $\sum_{k=1}^{\infty} u_k$ and s_n is the nth partial sum, then

$$s_n + \int_{n+1}^{+\infty} f(x)\,dx < S < s_n + \int_{n}^{+\infty} f(x)\,dx$$



◀ **Figure Ex-36**

37. (a) It was stated in Exercise 35 that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

Show that if s_n is the nth partial sum of this series, then

$$s_n + \frac{1}{n+1} < \frac{\pi^2}{6} < s_n + \frac{1}{n}$$

(b) Calculate s_3 exactly, and then use the result in part (a) to show that

$$\frac{29}{18} < \frac{\pi^2}{6} < \frac{61}{36}$$

(c) Use a calculating utility to confirm that the inequalities in part (b) are correct.

(d) Find upper and lower bounds on the error that results if the sum of the series is approximated by the 10th partial sum.

38. In each part, find upper and lower bounds on the error that results if the sum of the series is approximated by the 10th partial sum.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}$ (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2+1}$ (c) $\displaystyle\sum_{k=1}^{\infty}\frac{k}{e^k}$

39. It was stated in Exercise 35 that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$$

(a) Let s_n be the nth partial sum of the series above. Show that

$$s_n + \frac{1}{3(n+1)^3} < \frac{\pi^4}{90} < s_n + \frac{1}{3n^3}$$

(b) We can use a partial sum of the series to approximate $\pi^4/90$ to three decimal-place accuracy by capturing the sum of the series in an interval of length 0.001 (or less). Find the smallest value of n such that the interval containing $\pi^4/90$ in part (a) has a length of 0.001 or less.

(c) Approximate $\pi^4/90$ to three decimal places using the midpoint of an interval of width at most 0.001 that contains the sum of the series. Use a calculating utility to confirm that your answer is within 0.0005 of $\pi^4/90$.

40. We showed in Section 9.3 that the harmonic series $\sum_{k=1}^{\infty} 1/k$ diverges. Our objective in this problem is to demonstrate that although the partial sums of this series approach $+\infty$, they increase extremely slowly.

(a) Use inequality (2) to show that for $n \geq 2$

$$\ln(n+1) < s_n < 1 + \ln n$$

(b) Use the inequalities in part (a) to find upper and lower bounds on the sum of the first million terms in the series.

(c) Show that the sum of the first billion terms in the series is less than 22.

(d) Find a value of n so that the sum of the first n terms is greater than 100.

41. Use a graphing utility to confirm that the integral test applies to the series $\sum_{k=1}^{\infty} k^2 e^{-k}$, and then determine whether the series converges.

C **42.** (a) Show that the hypotheses of the integral test are satisfied by the series $\sum_{k=1}^{\infty} 1/(k^3+1)$.

(b) Use a CAS and the integral test to confirm that the series converges.

(c) Construct a table of partial sums for $n = 10, 20, 30, \ldots, 100$, showing at least six decimal places.

(d) Based on your table, make a conjecture about the sum of the series to three decimal-place accuracy.

(e) Use part (b) of Exercise 36 to check your conjecture.

✔ QUICK CHECK ANSWERS 9.4

1. $\lim_{k\to+\infty} u_k$ **2.** $-2; 7$ **3.** integral; $\dfrac{1}{\sqrt{k}}$; diverges **4.** $\dfrac{1}{k^p}$; $p > 1$; $0 < p \leq 1$

9.5 THE COMPARISON, RATIO, AND ROOT TESTS

In this section we will develop some more basic convergence tests for series with nonnegative terms. Later, we will use some of these tests to study the convergence of Taylor series.

THE COMPARISON TEST

We will begin with a test that is useful in its own right and is also the building block for other important convergence tests. The underlying idea of this test is to use the known convergence or divergence of a series to deduce the convergence or divergence of another series.

9.5.1 THEOREM (*The Comparison Test*) *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonnegative terms and suppose that*

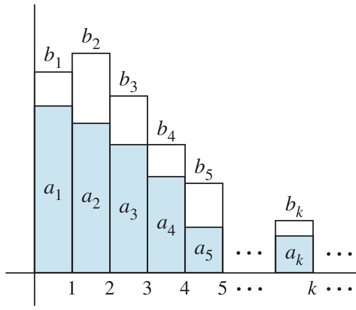
$$a_1 \leq b_1,\ a_2 \leq b_2,\ a_3 \leq b_3, \ldots, a_k \leq b_k, \ldots$$

(a) *If the "bigger series" Σb_k converges, then the "smaller series" Σa_k also converges.*

(b) *If the "smaller series" Σa_k diverges, then the "bigger series" Σb_k also diverges.*

It is not essential in Theorem 9.5.1 that the condition $a_k \leq b_k$ hold for all k, as stated; the conclusions of the theorem remain true if this condition is eventually true.

We have left the proof of this theorem for the exercises; however, it is easy to visualize why the theorem is true by interpreting the terms in the series as areas of rectangles



For each rectangle, a_k denotes the area of the blue portion and b_k denotes the combined area of the white and blue portions.

▲ **Figure 9.5.1**

(Figure 9.5.1). The comparison test states that if the total area $\sum b_k$ is finite, then the total area $\sum a_k$ must also be finite; and if the total area $\sum a_k$ is infinite, then the total area $\sum b_k$ must also be infinite.

USING THE COMPARISON TEST

There are two steps required for using the comparison test to determine whether a series $\sum u_k$ with positive terms converges:

Step 1. Guess at whether the series $\sum u_k$ converges or diverges.

Step 2. Find a series that proves the guess to be correct. That is, if we guess that $\sum u_k$ diverges, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if we guess that $\sum u_k$ converges, we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

In most cases, the series $\sum u_k$ being considered will have its general term u_k expressed as a fraction. To help with the guessing process in the first step, we have formulated two principles that are based on the form of the denominator for u_k. These principles sometimes *suggest* whether a series is likely to converge or diverge. We have called these "informal principles" because they are not intended as formal theorems. In fact, we will not guarantee that they *always* work. However, they work often enough to be useful.

9.5.2 INFORMAL PRINCIPLE *Constant terms in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.*

9.5.3 INFORMAL PRINCIPLE *If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.*

▶ **Example 1** Use the comparison test to determine whether the following series converge or diverge.

$$\text{(a)}\ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}} \qquad \text{(b)}\ \sum_{k=1}^{\infty} \frac{1}{2k^2+k}$$

Solution (a). According to Principle 9.5.2, we should be able to drop the constant in the denominator without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \tag{1}$$

which is a divergent p-series $\left(p = \frac{1}{2}\right)$. Thus, we will guess that the given series diverges and try to prove this by finding a divergent series that is "smaller" than the given series. However, series (1) does the trick since

$$\frac{1}{\sqrt{k}-\frac{1}{2}} > \frac{1}{\sqrt{k}} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series diverges.

Solution (b). According to Principle 9.5.3, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \tag{2}$$

which converges since it is a constant times a convergent p-series ($p = 2$). Thus, we will guess that the given series converges and try to prove this by finding a convergent series that is "bigger" than the given series. However, series (2) does the trick since

$$\frac{1}{2k^2 + k} < \frac{1}{2k^2} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series converges. ◄

THE LIMIT COMPARISON TEST

In the last example, Principles 9.5.2 and 9.5.3 provided the guess about convergence or divergence as well as the series needed to apply the comparison test. Unfortunately, it is not always so straightforward to find the series required for comparison, so we will now consider an alternative to the comparison test that is usually easier to apply. The proof is given in Appendix D.

9.5.4 THEOREM (*The Limit Comparison Test*) *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k}$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

The cases where $\rho = 0$ or $\rho = +\infty$ are discussed in the exercises (Exercise 56).

To use the limit comparison test we must again first guess at the convergence or divergence of $\sum a_k$ and then find a series $\sum b_k$ that supports our guess. The following example illustrates this principle.

▶ Example 2 Use the limit comparison test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + 1} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{2k^2 + k} \qquad \text{(c)} \sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$$

Solution (a). As in Example 1, Principle 9.5.2 suggests that the series is likely to behave like the divergent p-series (1). To prove that the given series diverges, we will apply the limit comparison test with

$$a_k = \frac{1}{\sqrt{k} + 1} \quad \text{and} \quad b_k = \frac{1}{\sqrt{k}}$$

We obtain

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{\sqrt{k}}{\sqrt{k} + 1} = \lim_{k \to +\infty} \frac{1}{1 + \dfrac{1}{\sqrt{k}}} = 1$$

Since ρ is finite and positive, it follows from Theorem 9.5.4 that the given series diverges.

Solution (b). As in Example 1, Principle 9.5.3 suggests that the series is likely to behave like the convergent series (2). To prove that the given series converges, we will apply the limit comparison test with

$$a_k = \frac{1}{2k^2+k} \quad \text{and} \quad b_k = \frac{1}{2k^2}$$

We obtain

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{2k^2}{2k^2+k} = \lim_{k\to+\infty} \frac{2}{2+\dfrac{1}{k}} = 1$$

Since ρ is finite and positive, it follows from Theorem 9.5.4 that the given series converges, which agrees with the conclusion reached in Example 1 using the comparison test.

Solution (c). From Principle 9.5.3, the series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{3k^3}{k^7} = \sum_{k=1}^{\infty} \frac{3}{k^4} \tag{3}$$

which converges since it is a constant times a convergent p-series. Thus, the given series is likely to converge. To prove this, we will apply the limit comparison test to series (3) and the given series. We obtain

$$\rho = \lim_{k\to+\infty} \frac{\dfrac{3k^3-2k^2+4}{k^7-k^3+2}}{\dfrac{3}{k^4}} = \lim_{k\to+\infty} \frac{3k^7-2k^6+4k^4}{3k^7-3k^3+6} = 1$$

Since ρ is finite and nonzero, it follows from Theorem 9.5.4 that the given series converges, since (3) converges. ◀

THE RATIO TEST

The comparison test and the limit comparison test hinge on first making a guess about convergence and then finding an appropriate series for comparison, both of which can be difficult tasks in cases where Principles 9.5.2 and 9.5.3 cannot be applied. In such cases the next test can often be used, since it works exclusively with the terms of the given series—it requires neither an initial guess about convergence nor the discovery of a series for comparison. Its proof is given in Appendix D.

9.5.5 THEOREM (*The Ratio Test*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}$$

(*a*) *If $\rho < 1$, the series converges.*

(*b*) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(*c*) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

▶ **Example 3** Each of the following series has positive terms, so the ratio test applies. In each part, use the ratio test to determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$ (d) $\displaystyle\sum_{k=3}^{\infty} \frac{(2k)!}{4^k}$ (e) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-1}$

Solution (a). The series converges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k\to+\infty} \frac{k!}{(k+1)!} = \lim_{k\to+\infty} \frac{1}{k+1} = 0 < 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \frac{1}{2} \lim_{k\to+\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

Solution (c). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k\to+\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to+\infty} \left(1 + \frac{1}{k}\right)^k = e > 1$$

See Formula (4) of Section 6.1

Solution (d). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \lim_{k\to+\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4}\right)$$

$$= \lim_{k\to+\infty} \left(\frac{(2k+2)(2k+1)(2k)!}{(2k)!} \cdot \frac{1}{4}\right) = \frac{1}{4} \lim_{k\to+\infty} (2k+2)(2k+1) = +\infty$$

Solution (e). The ratio test is of no help since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} = \lim_{k\to+\infty} \frac{2k-1}{2k+1} = 1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty} \frac{dx}{2x-1} = \lim_{b\to+\infty} \int_1^b \frac{dx}{2x-1} = \lim_{b\to+\infty} \frac{1}{2} \ln(2x-1)\Big]_1^b = +\infty$$

Both the comparison test and the limit comparison test would also have worked here (verify). ◀

THE ROOT TEST

In cases where it is difficult or inconvenient to find the limit required for the ratio test, the next test is sometimes useful. Since its proof is similar to the proof of the ratio test, we will omit it.

9.5.6 THEOREM (*The Root Test*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \sqrt[k]{u_k} = \lim_{k\to+\infty} (u_k)^{1/k}$$

(a) *If $\rho < 1$, the series converges.*

(b) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(c) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

▶ **Example 4** Use the root test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

Solution (a). The series diverges, since

$$\rho = \lim_{k\to+\infty} (u_k)^{1/k} = \lim_{k\to+\infty} \frac{4k-5}{2k+1} = 2 > 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty} (u_k)^{1/k} = \lim_{k\to+\infty} \frac{1}{\ln(k+1)} = 0 < 1 \blacktriangleleft$$

✔QUICK CHECK EXERCISES 9.5 *(See page 637 for answers.)*

1–4 Select between *converges* or *diverges* to fill the first blank.

1. The series
$$\sum_{k=1}^{\infty} \frac{2k^2+1}{2k^{8/3}-1}$$
________ by comparison with the p-series $\sum_{k=1}^{\infty}$ ________.

2. Since
$$\lim_{k\to+\infty} \frac{(k+1)^3/3^{k+1}}{k^3/3^k} = \lim_{k\to+\infty} \frac{\left(1+\dfrac{1}{k}\right)^3}{3} = \frac{1}{3}$$
the series $\sum_{k=1}^{\infty} k^3/3^k$ ________ by the ________ test.

3. Since
$$\lim_{k\to+\infty} \frac{(k+1)!/3^{k+1}}{k!/3^k} = \lim_{k\to+\infty} \frac{k+1}{3} = +\infty$$
the series $\sum_{k=1}^{\infty} k!/3^k$ ________ by the ________ test.

4. Since
$$\lim_{k\to+\infty} \left(\frac{1}{k^{k/2}}\right)^{1/k} = \lim_{k\to+\infty} \frac{1}{k^{1/2}} = 0$$
the series $\sum_{k=1}^{\infty} 1/k^{k/2}$ ________ by the ________ test.

EXERCISE SET 9.5

1–2 Make a guess about the convergence or divergence of the series, and confirm your guess using the comparison test.

1. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5k^2-k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k-\frac{1}{4}}$

2. (a) $\displaystyle\sum_{k=2}^{\infty} \frac{k+1}{k^2-k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^4+k}$

3. In each part, use the comparison test to show that the series converges.
(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k+5}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{5\sin^2 k}{k!}$

4. In each part, use the comparison test to show that the series diverges.
(a) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^{3/2}-\frac{1}{2}}$

5–10 Use the limit comparison test to determine whether the series converges.

5. $\displaystyle\sum_{k=1}^{\infty} \frac{4k^2-2k+6}{8k^7+k-8}$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{9k+6}$

7. $\displaystyle\sum_{k=1}^{\infty} \frac{5}{3^k+1}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{8k^2-3k}}$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$

11–16 Use the ratio test to determine whether the series converges. If the test is inconclusive, then say so.

11. $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k!}$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{4^k}{k^2}$

13. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5k}$

14. $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$

15. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^3}$

16. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

17–20 Use the root test to determine whether the series converges. If the test is inconclusive, then say so.

17. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3k+2}{2k-1}\right)^k$

18. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{100}\right)^k$

19. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{5^k}$

20. $\displaystyle\sum_{k=1}^{\infty} (1-e^{-k})^k$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. The limit comparison test decides convergence based on a limit of the quotient of consecutive terms in a series.

22. If $\lim_{k\to+\infty}(u_{k+1}/u_k) = 5$, then $\sum u_k$ diverges.

23. If $\lim_{k\to+\infty}(k^2u_k)=5$, then $\sum u_k$ converges.

24. The root test decides convergence based on a limit of kth roots of terms in the sequence of partial sums for a series.

25–49 Use any method to determine whether the series converges. ■

25. $\displaystyle\sum_{k=0}^{\infty}\frac{7^k}{k!}$ **26.** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2k+1}$ **27.** $\displaystyle\sum_{k=1}^{\infty}\frac{k^2}{5^k}$

28. $\displaystyle\sum_{k=1}^{\infty}\frac{k!10^k}{3^k}$ **29.** $\displaystyle\sum_{k=1}^{\infty}k^{50}e^{-k}$ **30.** $\displaystyle\sum_{k=1}^{\infty}\frac{k^2}{k^3+1}$

31. $\displaystyle\sum_{k=1}^{\infty}\frac{\sqrt{k}}{k^3+1}$ **32.** $\displaystyle\sum_{k=1}^{\infty}\frac{4}{2+3^k k}$

33. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k(k+1)}}$ **34.** $\displaystyle\sum_{k=1}^{\infty}\frac{2+(-1)^k}{5^k}$

35. $\displaystyle\sum_{k=1}^{\infty}\frac{2+\sqrt{k}}{(k+1)^3-1}$ **36.** $\displaystyle\sum_{k=1}^{\infty}\frac{4+|\cos x|}{k^3}$

37. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{1+\sqrt{k}}$ **38.** $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{k^k}$ **39.** $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{e^k}$

40. $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{e^{k^2}}$ **41.** $\displaystyle\sum_{k=0}^{\infty}\frac{(k+4)!}{4!k!4^k}$ **42.** $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k+1}\right)^{k^2}$

43. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{4+2^{-k}}$ **44.** $\displaystyle\sum_{k=1}^{\infty}\frac{\sqrt{k}\ln k}{k^3+1}$ **45.** $\displaystyle\sum_{k=1}^{\infty}\frac{\tan^{-1}k}{k^2}$

46. $\displaystyle\sum_{k=1}^{\infty}\frac{5^k+k}{k!+3}$ **47.** $\displaystyle\sum_{k=0}^{\infty}\frac{(k!)^2}{(2k)!}$ **48.** $\displaystyle\sum_{k=1}^{\infty}\frac{[\pi(k+1)]^k}{k^{k+1}}$

49. $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{3^k}$

50. For what positive values of α does the series $\sum_{k=1}^{\infty}(\alpha^k/k^\alpha)$ converge?

51–52 Find the general term of the series and use the ratio test to show that the series converges. ■

51. $1+\dfrac{1\cdot 2}{1\cdot 3}+\dfrac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\dfrac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\cdots$

52. $1+\dfrac{1\cdot 3}{3!}+\dfrac{1\cdot 3\cdot 5}{5!}+\dfrac{1\cdot 3\cdot 5\cdot 7}{7!}+\cdots$

53. Show that $\ln x<\sqrt{x}$ if $x>0$, and use this result to investigate the convergence of

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{k^2}$ (b) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{(\ln k)^2}$

FOCUS ON CONCEPTS

54. (a) Make a conjecture about the convergence of the series $\sum_{k=1}^{\infty}\sin(\pi/k)$ by considering the local linear approximation of $\sin x$ at $x=0$.
(b) Try to confirm your conjecture using the limit comparison test.

55. (a) We will see later that the polynomial $1-x^2/2$ is the "local quadratic" approximation for $\cos x$ at $x=0$. Make a conjecture about the convergence of the series

$$\sum_{k=1}^{\infty}\left[1-\cos\left(\frac{1}{k}\right)\right]$$

by considering this approximation.
(b) Try to confirm your conjecture using the limit comparison test.

56. Let $\sum a_k$ and $\sum b_k$ be series with positive terms. Prove:
(a) If $\lim_{k\to+\infty}(a_k/b_k)=0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(b) If $\lim_{k\to+\infty}(a_k/b_k)=+\infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

57. Use Theorem 9.4.6 to prove the comparison test (Theorem 9.5.1).

58. **Writing** What does the ratio test tell you about the convergence of a geometric series? Discuss similarities between geometric series and series to which the ratio test applies.

59. **Writing** Given an infinite series, discuss a strategy for deciding what convergence test to use.

✔ QUICK CHECK ANSWERS 9.5

1. diverges; $1/k^{2/3}$ **2.** converges; ratio **3.** diverges; ratio **4.** converges; root

9.6 ALTERNATING SERIES; ABSOLUTE AND CONDITIONAL CONVERGENCE

Up to now we have focused exclusively on series with nonnegative terms. In this section we will discuss series that contain both positive and negative terms.

ALTERNATING SERIES

Series whose terms alternate between positive and negative, called ***alternating series***, are of special importance. Some examples are

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$$\sum_{k=1}^{\infty}(-1)^{k}\frac{1}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

In general, an alternating series has one of the following two forms:

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots \tag{1}$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \cdots \tag{2}$$

where the a_k's are assumed to be positive in both cases.

The following theorem is the key result on convergence of alternating series.

9.6.1 THEOREM (***Alternating Series Test***) *An alternating series of either form* (1) *or form* (2) *converges if the following two conditions are satisfied:*

(*a*) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq \cdots$

(*b*) $\lim_{k\to+\infty} a_k = 0$

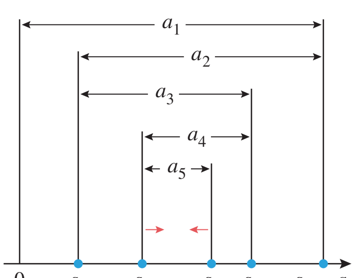


Figure 9.6.1

It is not essential for condition (*a*) in Theorem 9.6.1 to hold for all terms; an alternating series will converge if condition (*b*) is true and condition (*a*) holds eventually.

PROOF We will consider only alternating series of form (1). The idea of the proof is to show that if conditions (*a*) and (*b*) hold, then the sequences of even-numbered and odd-numbered partial sums converge to a common limit S. It will then follow from Theorem 9.1.4 that the entire sequence of partial sums converges to S.

Figure 9.6.1 shows how successive partial sums satisfying conditions (*a*) and (*b*) appear when plotted on a horizontal axis. The even-numbered partial sums

$$s_2, s_4, s_6, s_8, \ldots, s_{2n}, \ldots$$

form an increasing sequence bounded above by a_1, and the odd-numbered partial sums

$$s_1, s_3, s_5, \ldots, s_{2n-1}, \ldots$$

form a decreasing sequence bounded below by 0. Thus, by Theorems 9.2.3 and 9.2.4, the even-numbered partial sums converge to some limit S_E and the odd-numbered partial sums converge to some limit S_O. To complete the proof we must show that $S_E = S_O$. But the

If an alternating series violates condition (*b*) of the alternating series test, then the series must diverge by the divergence test (Theorem 9.4.1).

$(2n)$-th term in the series is $-a_{2n}$, so that $s_{2n} - s_{2n-1} = -a_{2n}$, which can be written as

$$s_{2n-1} = s_{2n} + a_{2n}$$

However, $2n \to +\infty$ and $2n - 1 \to +\infty$ as $n \to +\infty$, so that

$$S_O = \lim_{n\to+\infty} s_{2n-1} = \lim_{n\to+\infty} (s_{2n} + a_{2n}) = S_E + 0 = S_E$$

which completes the proof. ■

▶ Example 1 Use the alternating series test to show that the following series converge.

$$\text{(a)} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \qquad \text{(b)} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$$

The series in part (a) of Example 1 is called the ***alternating harmonic series***. Note that this series converges, whereas the harmonic series diverges.

Solution (a). The two conditions in the alternating series test are satisfied since

$$a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} a_k = \lim_{k\to+\infty} \frac{1}{k} = 0$$

Solution (b). The two conditions in the alternating series test are satisfied since

$$\frac{a_{k+1}}{a_k} = \frac{k+4}{(k+1)(k+2)} \cdot \frac{k(k+1)}{k+3} = \frac{k^2+4k}{k^2+5k+6} = \frac{k^2+4k}{(k^2+4k)+(k+6)} < 1$$

so

$$a_k > a_{k+1}$$

and

$$\lim_{k\to+\infty} a_k = \lim_{k\to+\infty} \frac{k+3}{k(k+1)} = \lim_{k\to+\infty} \frac{\frac{1}{k} + \frac{3}{k^2}}{1 + \frac{1}{k}} = 0 \blacktriangleleft$$

APPROXIMATING SUMS OF ALTERNATING SERIES

The following theorem is concerned with the error that results when the sum of an alternating series is approximated by a partial sum.

9.6.2 THEOREM *If an alternating series satisfies the hypotheses of the alternating series test, and if S is the sum of the series, then:*

(*a*) *S lies between any two successive partial sums; that is, either*

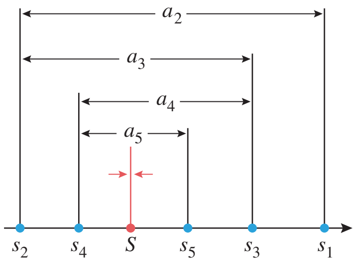
$$s_n \le S \le s_{n+1} \quad \text{or} \quad s_{n+1} \le S \le s_n \tag{3}$$

depending on which partial sum is larger.

(*b*) *If S is approximated by s_n, then the absolute error $|S - s_n|$ satisfies*

$$|S - s_n| \le a_{n+1} \tag{4}$$

Moreover, the sign of the error $S - s_n$ is the same as that of the coefficient of a_{n+1}.



▲ **Figure 9.6.2**

PROOF We will prove the theorem for series of form (1). Referring to Figure 9.6.2 and keeping in mind our observation in the proof of Theorem 9.6.1 that the odd-numbered partial sums form a decreasing sequence converging to S and the even-numbered partial sums form an increasing sequence converging to S, we see that successive partial sums oscillate from one side of S to the other in smaller and smaller steps with the odd-numbered partial sums being larger than S and the even-numbered partial sums being smaller than S. Thus, depending on whether n is even or odd, we have

$$s_n \le S \le s_{n+1} \quad \text{or} \quad s_{n+1} \le S \le s_n$$

which proves (3). Moreover, in either case we have

$$|S - s_n| \le |s_{n+1} - s_n| \tag{5}$$

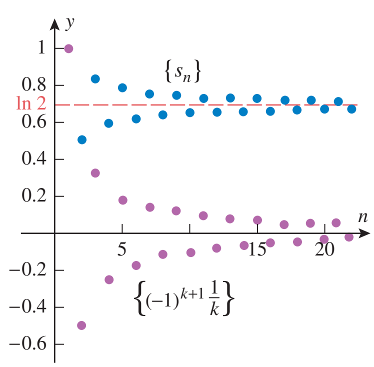
But $s_{n+1} - s_n = \pm a_{n+1}$ (the sign depending on whether n is even or odd). Thus, it follows from (5) that $|S - s_n| \le a_{n+1}$, which proves (4). Finally, since the odd-numbered partial sums are larger than S and the even-numbered partial sums are smaller than S, it follows that $S - s_n$ has the same sign as the coefficient of a_{n+1} (verify). ■

REMARK In words, inequality (4) states that for a series satisfying the hypotheses of the alternating series test, the magnitude of the error that results from approximating S by s_n is at most that of the first term that is *not* included in the partial sum. Also, note that if $a_1 > a_2 > \cdots > a_k > \cdots$, then inequality (4) can be strengthened to $|S - s_n| < a_{n+1}$.

▶ Example 2 Later in this chapter we will show that the sum of the alternating harmonic series is

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots$$

This is illustrated in Figure 9.6.3.



Graph of the sequences of terms and nth partial sums for the alternating harmonic series

▲ **Figure 9.6.3**

(a) Accepting this to be so, find an upper bound on the magnitude of the error that results if $\ln 2$ is approximated by the sum of the first eight terms in the series.

(b) Find a partial sum that approximates $\ln 2$ to one decimal-place accuracy (the nearest tenth).

Solution (a). It follows from the strengthened form of (4) that

$$|\ln 2 - s_8| < a_9 = \frac{1}{9} < 0.12 \tag{6}$$

As a check, let us compute s_8 exactly. We obtain

$$s_8 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = \frac{533}{840}$$

Thus, with the help of a calculator

$$|\ln 2 - s_8| = \left|\ln 2 - \frac{533}{840}\right| \approx 0.059$$

This shows that the error is well under the estimate provided by upper bound (6).

Solution (b). For one decimal-place accuracy, we must choose a value of n for which $|\ln 2 - s_n| \le 0.05$. However, it follows from the strengthened form of (4) that

$$|\ln 2 - s_n| < a_{n+1}$$

so it suffices to choose n so that $a_{n+1} \le 0.05$.

One way to find n is to use a calculating utility to obtain numerical values for a_1, a_2, $a_3, \ldots$ until you encounter the first value that is less than or equal to 0.05. If you do this, you will find that it is $a_{20} = 0.05$; this tells us that partial sum s_{19} will provide the desired accuracy. Another way to find n is to solve the inequality

$$\frac{1}{n+1} \leq 0.05$$

algebraically. We can do this by taking reciprocals, reversing the sense of the inequality, and then simplifying to obtain $n \geq 19$. Thus, s_{19} will provide the required accuracy, which is consistent with the previous result.

With the help of a calculating utility, the value of s_{19} is approximately $s_{19} \approx 0.7$ and the value of $\ln 2$ obtained directly is approximately $\ln 2 \approx 0.69$, which agrees with s_{19} when rounded to one decimal place. ◀

As Example 2 illustrates, the alternating harmonic series does not provide an efficient way to approximate $\ln 2$, since too many terms and hence too much computation is required to achieve reasonable accuracy. Later, we will develop better ways to approximate logarithms.

ABSOLUTE CONVERGENCE

The series

$$1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$$

does not fit in any of the categories studied so far—it has mixed signs but is not alternating. We will now develop some convergence tests that can be applied to such series.

9.6.3 DEFINITION A series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

is said to ***converge absolutely*** if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges and is said to ***diverge absolutely*** if the series of absolute values diverges.

▶ **Example 3** Determine whether the following series converge absolutely.

(a) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \cdots$ (b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Solution (a). The series of absolute values is the convergent geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$$

so the given series converges absolutely.

Solution (b). The series of absolute values is the divergent harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

so the given series diverges absolutely. ◀

It is important to distinguish between the notions of convergence and absolute convergence. For example, the series in part (b) of Example 3 converges, since it is the alternating harmonic series, yet we demonstrated that it does not converge absolutely. However, the following theorem shows that *if a series converges absolutely, then it converges*.

9.6.4 THEOREM *If the series*

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges, then so does the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

Theorem 9.6.4 provides a way of inferring convergence of a series with positive and negative terms from a related series with nonnegative terms (the series of absolute values). This is important because most of the convergence tests that we have developed apply only to series with nonnegative terms.

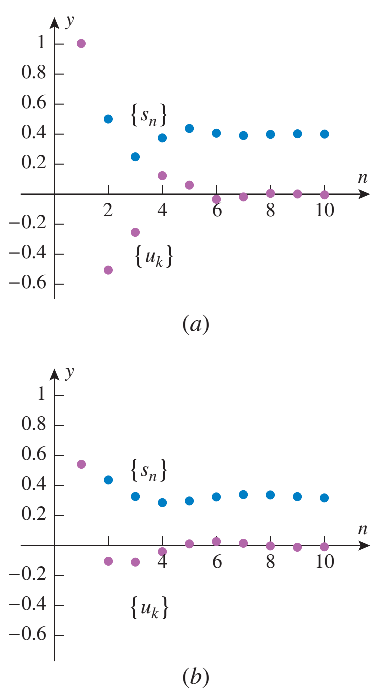
PROOF We will write the series $\sum u_k$ as

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} [(u_k + |u_k|) - |u_k|] \tag{7}$$

We are assuming that $\sum |u_k|$ converges, so that if we can show that $\sum (u_k + |u_k|)$ converges, then it will follow from (7) and Theorem 9.4.3(*a*) that $\sum u_k$ converges. However, the value of $u_k + |u_k|$ is either 0 or $2|u_k|$, depending on the sign of u_k. Thus, in all cases it is true that

$$0 \le u_k + |u_k| \le 2|u_k|$$

But $\sum 2|u_k|$ converges, since it is a constant times the convergent series $\sum |u_k|$; hence $\sum (u_k + |u_k|)$ converges by the comparison test. ■



Graphs of the sequences of terms and nth partial sums for the series in Example 4

▲ **Figure 9.6.4**

Example 4 Show that the following series converge.

$$\text{(a)}\ 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots \qquad \text{(b)}\ \sum_{k=1}^{\infty} \frac{\cos k}{k^2}$$

Solution (a). Observe that this is not an alternating series because the signs alternate in pairs after the first term. Thus, we have no convergence test that can be applied directly. However, we showed in Example 3(a) that the series converges absolutely, so Theorem 9.6.4 implies that it converges (Figure 9.6.4*a*).

Solution (b). With the help of a calculating utility, you will be able to verify that the signs of the terms in this series vary irregularly. Thus, we will test for absolute convergence. The series of absolute values is

$$\sum_{k=1}^{\infty} \left| \frac{\cos k}{k^2} \right|$$

However,

$$\left| \frac{\cos k}{k^2} \right| \le \frac{1}{k^2}$$

But $\sum 1/k^2$ is a convergent p-series ($p = 2$), so the series of absolute values converges by the comparison test. Thus, the given series converges absolutely and hence converges (Figure 9.6.4b). ◀

CONDITIONAL CONVERGENCE

Although Theorem 9.6.4 is a useful tool for series that converge absolutely, it provides no information about the convergence or divergence of a series that diverges absolutely. For example, consider the two series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots \tag{8}$$

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{k} - \cdots \tag{9}$$

Both of these series diverge absolutely, since in each case the series of absolute values is the divergent harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots$$

However, series (8) converges, since it is the alternating harmonic series, and series (9) diverges, since it is a constant times the divergent harmonic series. As a matter of terminology, a series that converges but diverges absolutely is said to ***converge conditionally*** (or to be ***conditionally convergent***). Thus, (8) is a conditionally convergent series.

▶ **Example 5** In Example 1(b) we used the alternating series test to show that the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$$

converges. Determine whether this series converges absolutely or converges conditionally.

Solution. We test the series for absolute convergence by examining the series of absolute values:

$$\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{k+3}{k(k+1)} \right| = \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)}$$

Principle 9.5.3 suggests that the series of absolute values should behave like the divergent p-series with $p = 1$. To prove that the series of absolute values diverges, we will apply the limit comparison test with

$$a_k = \frac{k+3}{k(k+1)} \quad \text{and} \quad b_k = \frac{1}{k}$$

We obtain

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{k(k+3)}{k(k+1)} = \lim_{k\to+\infty} \frac{k+3}{k+1} = 1$$

Since ρ is finite and positive, it follows from the limit comparison test that the series of absolute values diverges. Thus, the original series converges and also diverges absolutely, and so converges conditionally. ◀

THE RATIO TEST FOR ABSOLUTE CONVERGENCE

Although one cannot generally infer convergence or divergence of a series from absolute divergence, the following variation of the ratio test provides a way of deducing divergence from absolute divergence in certain situations. We omit the proof.

9.6.5 THEOREM (*Ratio Test for Absolute Convergence*) *Let $\sum u_k$ be a series with nonzero terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|}$$

(*a*) *If $\rho < 1$, then the series $\sum u_k$ converges absolutely and therefore converges.*

(*b*) *If $\rho > 1$ or if $\rho = +\infty$, then the series $\sum u_k$ diverges.*

(*c*) *If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.*

▶ Example 6 Use the ratio test for absolute convergence to determine whether the series converges.

$$\text{(a)}\ \sum_{k=1}^{\infty}(-1)^k\frac{2^k}{k!} \qquad \text{(b)}\ \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!}{3^k}$$

Solution (a). Taking the absolute value of the general term u_k we obtain

$$|u_k| = \left|(-1)^k\frac{2^k}{k!}\right| = \frac{2^k}{k!}$$

Thus,

$$\rho = \lim_{k\to+\infty}\frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty}\frac{2^{k+1}}{(k+1)!}\cdot\frac{k!}{2^k} = \lim_{k\to+\infty}\frac{2}{k+1} = 0 < 1$$

which implies that the series converges absolutely and therefore converges.

Solution (b). Taking the absolute value of the general term u_k we obtain

$$|u_k| = \left|(-1)^k\frac{(2k-1)!}{3^k}\right| = \frac{(2k-1)!}{3^k}$$

Thus,

$$\rho = \lim_{k\to+\infty}\frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty}\frac{[2(k+1)-1]!}{3^{k+1}}\cdot\frac{3^k}{(2k-1)!}$$

$$= \lim_{k\to+\infty}\frac{1}{3}\cdot\frac{(2k+1)!}{(2k-1)!} = \frac{1}{3}\lim_{k\to+\infty}(2k)(2k+1) = +\infty$$

which implies that the series diverges. ◀

SUMMARY OF CONVERGENCE TESTS

We conclude this section with a summary of convergence tests that can be used for reference. The skill of selecting a good test is developed through lots of practice. In some instances a test may be inconclusive, so another test must be tried.

Summary of Convergence Tests

NAME	STATEMENT	COMMENTS
Divergence Test (9.4.1)	If $\lim\limits_{k\to+\infty} u_k \neq 0$, then $\sum u_k$ diverges.	If $\lim\limits_{k\to+\infty} u_k = 0$, then $\sum u_k$ may or may not converge.
Integral Test (9.4.4)	Let $\sum u_k$ be a series with positive terms. If f is a function that is decreasing and continuous on an interval $[a, +\infty)$ and such that $u_k = f(k)$ for all $k \geq a$, then $$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{+\infty} f(x)\,dx$$ both converge or both diverge.	This test only applies to series that have positive terms. Try this test when $f(x)$ is easy to integrate.
Comparison Test (9.5.1)	Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonnegative terms such that $$a_1 \leq b_1,\ a_2 \leq b_2, \ldots, a_k \leq b_k, \ldots$$ If $\sum b_k$ converges, then $\sum a_k$ converges, and if $\sum a_k$ diverges, then $\sum b_k$ diverges.	This test only applies to series with nonnegative terms. Try this test as a last resort; other tests are often easier to apply.
Limit Comparison Test (9.5.4)	Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let $$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$$ If $0 < \rho < +\infty$, then both series converge or both diverge.	This is easier to apply than the comparison test, but still requires some skill in choosing the series $\sum b_k$ for comparison.
Ratio Test (9.5.5)	Let $\sum u_k$ be a series with positive terms and suppose that $$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}$$ (a) Series converges if $\rho < 1$. (b) Series diverges if $\rho > 1$ or $\rho = +\infty$. (c) The test is inconclusive if $\rho = 1$.	Try this test when u_k involves factorials or kth powers.
Root Test (9.5.6)	Let $\sum u_k$ be a series with positive terms and suppose that $$\rho = \lim_{k\to+\infty} \sqrt[k]{u_k}$$ (a) The series converges if $\rho < 1$. (b) The series diverges if $\rho > 1$ or $\rho = +\infty$. (c) The test is inconclusive if $\rho = 1$.	Try this test when u_k involves kth powers.
Alternating Series Test (9.6.1)	If $a_k > 0$ for $k = 1, 2, 3, \ldots$, then the series $$a_1 - a_2 + a_3 - a_4 + \cdots$$ $$-a_1 + a_2 - a_3 + a_4 - \cdots$$ converge if the following conditions hold: (a) $a_1 \geq a_2 \geq a_3 \geq \cdots$ (b) $\lim\limits_{k\to+\infty} a_k = 0$	This test applies only to alternating series.
Ratio Test for Absolute Convergence (9.6.5)	Let $\sum u_k$ be a series with nonzero terms and suppose that $$\rho = \lim_{k\to+\infty} \frac{\lvert u_{k+1}\rvert}{\lvert u_k\rvert}$$ (a) The series converges absolutely if $\rho < 1$. (b) The series diverges if $\rho > 1$ or $\rho = +\infty$. (c) The test is inconclusive if $\rho = 1$.	The series need not have positive terms and need not be alternating to use this test.

✔ QUICK CHECK EXERCISES 9.6 (See page 648 for answers.)

1. What characterizes an *alternating* series?
2. (a) The series
$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}$$
converges by the alternating series test since ________ and ________.
(b) If
$$S=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} \quad\text{and}\quad s_9=\sum_{k=1}^{9}\frac{(-1)^{k+1}}{k^2}$$
then $|S-s_9|<$ ________.
3. Classify each sequence as conditionally convergent, absolutely convergent, or divergent.
(a) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$: ________
(b) $\sum_{k=1}^{\infty}(-1)^{k}\frac{3k-1}{9k+15}$: ________
(c) $\sum_{k=1}^{\infty}(-1)^{k}\frac{1}{k(k+2)}$: ________
(d) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt[4]{k^3}}$: ________
4. Given that
$$\lim_{k\to+\infty}\frac{(k+1)^4/4^{k+1}}{k^4/4^k}=\lim_{k\to+\infty}\frac{\left(1+\frac{1}{k}\right)^4}{4}=\frac{1}{4}$$
is the series $\sum_{k=1}^{\infty}(-1)^k k^4/4^k$ conditionally convergent, absolutely convergent, or divergent?

EXERCISE SET 9.6 [C] CAS

1–2 Show that the series converges by confirming that it satisfies the hypotheses of the alternating series test (Theorem 9.6.1). ■

1. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k+1}$
2. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{3^k}$

3–6 Determine whether the alternating series converges; justify your answer. ■

3. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+1}{3k+1}$
4. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+1}{\sqrt{k}+1}$
5. $\sum_{k=1}^{\infty}(-1)^{k+1}e^{-k}$
6. $\sum_{k=3}^{\infty}(-1)^{k}\frac{\ln k}{k}$

7–12 Use the ratio test for absolute convergence (Theorem 9.6.5) to determine whether the series converges or diverges. If the test is inconclusive, say so. ■

7. $\sum_{k=1}^{\infty}\left(-\frac{3}{5}\right)^k$
8. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^k}{k!}$
9. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^k}{k^2}$
10. $\sum_{k=1}^{\infty}(-1)^{k}\frac{k}{5^k}$
11. $\sum_{k=1}^{\infty}(-1)^{k}\frac{k^3}{e^k}$
12. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^k}{k!}$

13–28 Classify each series as absolutely convergent, conditionally convergent, or divergent. ■

13. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{3k}$
14. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{4/3}}$
15. $\sum_{k=1}^{\infty}\frac{(-4)^k}{k^2}$
16. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}$
17. $\sum_{k=1}^{\infty}\frac{\cos k\pi}{k}$
18. $\sum_{k=3}^{\infty}\frac{(-1)^k\ln k}{k}$
19. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+2}{k(k+3)}$
20. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k^2}{k^3+1}$
21. $\sum_{k=1}^{\infty}\sin\frac{k\pi}{2}$
22. $\sum_{k=1}^{\infty}\frac{\sin k}{k^3}$
23. $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$
24. $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k(k+1)}}$
25. $\sum_{k=2}^{\infty}\left(-\frac{1}{\ln k}\right)^k$
26. $\sum_{k=1}^{\infty}\frac{k\cos k\pi}{k^2+1}$
27. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k!}{(2k-1)!}$
28. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^{2k-1}}{k^2+1}$

29–32 True–False Determine whether the statement is true or false. Explain your answer. ■

29. An alternating series is one whose terms alternate between even and odd.
30. If a series satisfies the hypothesis of the alternating series test, then the sequence of partial sums of the series oscillates between overestimates and underestimates for the sum of the series.
31. If a series converges, then either it converges absolutely or it converges conditionally.
32. If $\sum(u_k)^2$ converges, then $\sum u_k$ converges absolutely.

33–36 Each series satisfies the hypotheses of the alternating series test. For the stated value of n, find an upper bound on the absolute error that results if the sum of the series is approximated by the nth partial sum.

33. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k};\ n=7$ **34.** $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!};\ n=5$

35. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}};\ n=99$

36. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)\ln(k+1)};\ n=3$

37–40 Each series satisfies the hypotheses of the alternating series test. Find a value of n for which the nth partial sum is ensured to approximate the sum of the series to the stated accuracy.

37. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k};\ |\text{error}|<0.0001$

38. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!};\ |\text{error}|<0.00001$

39. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}};$ two decimal places

40. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)\ln(k+1)};$ one decimal place

41–42 Find an upper bound on the absolute error that results if s_{10} is used to approximate the sum of the given *geometric* series. Compute s_{10} rounded to four decimal places and compare this value with the exact sum of the series.

41. $\frac{3}{4}-\frac{3}{8}+\frac{3}{16}-\frac{3}{32}+\cdots$ **42.** $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\cdots$

43–46 Each series satisfies the hypotheses of the alternating series test. Approximate the sum of the series to two decimal-place accuracy.

43. $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots$ **44.** $1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots$

45. $\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots$

46. $\frac{1}{1^5+4\cdot 1}-\frac{1}{3^5+4\cdot 3}+\frac{1}{5^5+4\cdot 5}-\frac{1}{7^5+4\cdot 7}+\cdots$

FOCUS ON CONCEPTS

C **47.** The purpose of this exercise is to show that the error bound in part (*b*) of Theorem 9.6.2 can be overly conservative in certain cases.

(a) Use a CAS to confirm that
$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$
(b) Use the CAS to show that $|(\pi/4)-s_{25}|<10^{-2}$.

(c) According to the error bound in part (*b*) of Theorem 9.6.2, what value of n is required to ensure that $|(\pi/4)-s_n|<10^{-2}$?

48. Prove: If a series $\sum a_k$ converges absolutely, then the series $\sum a_k^2$ converges.

49. (a) Find examples to show that if $\sum a_k$ converges, then $\sum a_k^2$ may diverge or converge.

(b) Find examples to show that if $\sum a_k^2$ converges, then $\sum a_k$ may diverge or converge.

50. Let $\sum u_k$ be a series and define series $\sum p_k$ and $\sum q_k$ so that
$$p_k=\begin{cases}u_k, & u_k>0\\ 0, & u_k\le 0\end{cases}\quad\text{and}\quad q_k=\begin{cases}0, & u_k\ge 0\\ -u_k, & u_k<0\end{cases}$$
(a) Show that $\sum u_k$ converges absolutely if and only if $\sum p_k$ and $\sum q_k$ both converge.

(b) Show that if one of $\sum p_k$ or $\sum q_k$ converges and the other diverges, then $\sum u_k$ diverges.

(c) Show that if $\sum u_k$ converges conditionally, then both $\sum p_k$ and $\sum q_k$ diverge.

51. It can be proved that the terms of any conditionally convergent series can be rearranged to give either a divergent series or a conditionally convergent series whose sum is any given number S. For example, we stated in Example 2 that
$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$
Show that we can rearrange this series so that its sum is $\frac{1}{2}\ln 2$ by rewriting it as
$$\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\cdots$$
[*Hint:* Add the first two terms in each grouping.]

52–54 Exercise 51 illustrates that one of the nuances of "conditional" convergence is that the sum of a series that converges conditionally depends on the order that the terms of the series are summed. Absolutely convergent series are more dependable, however. It can be proved that any series that is constructed from an absolutely convergent series by rearranging the terms will also be absolutely convergent and has the same sum as the original series. Use this fact together with parts (*a*) and (*b*) of Theorem 9.4.3 in these exercises.

52. It was stated in Exercise 35 of Section 9.4 that
$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$
Use this to show that
$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

53. Use the series for $\pi^2/6$ given in the preceding exercise to show that
$$\frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

54. It was stated in Exercise 35 of Section 9.4 that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$

Use this to show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$$

55. Writing Consider the series

$$1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \cdots$$

Determine whether this series converges and use this series as an example in a discussion of the importance of hypotheses (a) and (b) of the alternating series test (Theorem 9.6.1).

56. Writing Discuss the ways that conditional convergence is "conditional." In particular, describe how one could rearrange the terms of a conditionally convergent series $\sum u_k$ so that the resulting series diverges, either to $+\infty$ or to $-\infty$. [*Hint:* See Exercise 50.]

✔ QUICK CHECK ANSWERS 9.6

1. Terms alternate between positive and negative. **2.** (a) $1 \geq \frac{1}{4} \geq \frac{1}{9} \geq \cdots \geq \frac{1}{k^2} \geq \frac{1}{(k+1)^2} \geq \cdots;\ \lim_{k\to+\infty} \frac{1}{k^2} = 0$ (b) $\frac{1}{100}$ **3.** (a) conditionally convergent (b) divergent (c) absolutely convergent (d) conditionally convergent **4.** absolutely convergent

9.7 MACLAURIN AND TAYLOR POLYNOMIALS

In a local linear approximation the tangent line to the graph of a function is used to obtain a linear approximation of the function near the point of tangency. In this section we will consider how one might improve on the accuracy of local linear approximations by using higher-order polynomials as approximating functions. We will also investigate the error associated with such approximations.

LOCAL QUADRATIC APPROXIMATIONS

Recall from Formula (1) in Section 2.9 that the local linear approximation of a function f at x_0 is

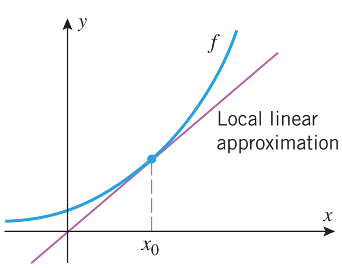
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{1}$$

In this formula, the approximating function

$$p(x) = f(x_0) + f'(x_0)(x - x_0)$$

is a first-degree polynomial satisfying $p(x_0) = f(x_0)$ and $p'(x_0) = f'(x_0)$ (verify). Thus, the local linear approximation of f at x_0 has the property that its value and the value of its first derivative match those of f at x_0.

If the graph of a function f has a pronounced "bend" at x_0, then we can expect that the accuracy of the local linear approximation of f at x_0 will decrease rapidly as we progress away from x_0 (Figure 9.7.1). One way to deal with this problem is to approximate the function f at x_0 by a polynomial p of degree 2 with the property that the value of p and the values of its first two derivatives match those of f at x_0. This ensures that the graphs of f and p not only have the same tangent line at x_0, but they also bend in the same direction at x_0 (both concave up or concave down). As a result, we can expect that the graph of p will remain close to the graph of f over a larger interval around x_0 than the graph of the local linear approximation. The polynomial p is called the ***local quadratic approximation of f at $x = x_0$***.



▲ **Figure 9.7.1**

To illustrate this idea, let us try to find a formula for the local quadratic approximation of a function f at $x = 0$. This approximation has the form

$$f(x) \approx c_0 + c_1x + c_2x^2 \tag{2}$$

where c_0, c_1, and c_2 must be chosen so that the values of

$$p(x) = c_0 + c_1x + c_2x^2$$

and its first two derivatives match those of f at 0. Thus, we want

$$p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0) \tag{3}$$

But the values of $p(0)$, $p'(0)$, and $p''(0)$ are as follows:

$$\begin{aligned} p(x) &= c_0 + c_1x + c_2x^2 & p(0) &= c_0 \\ p'(x) &= c_1 + 2c_2x & p'(0) &= c_1 \\ p''(x) &= 2c_2 & p''(0) &= 2c_2 \end{aligned}$$

Thus, it follows from (3) that

$$c_0 = f(0), \quad c_1 = f'(0), \quad c_2 = \frac{f''(0)}{2}$$

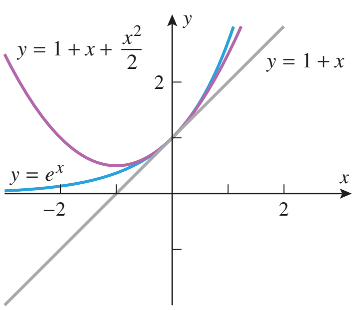
and substituting these in (2) yields the following formula for the local quadratic approximation of f at $x = 0$:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \tag{4}$$

Example 1 Find the local linear and quadratic approximations of e^x at $x = 0$, and graph e^x and the two approximations together.

Solution. If we let $f(x) = e^x$, then $f'(x) = f''(x) = e^x$; and hence

$$f(0) = f'(0) = f''(0) = e^0 = 1$$



▲ **Figure 9.7.2**

Thus, from (4) the local quadratic approximation of e^x at $x = 0$ is

$$e^x \approx 1 + x + \frac{x^2}{2}$$

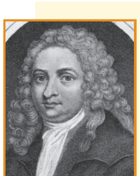
and the local linear approximation (which is the linear part of the local quadratic approximation) is

$$e^x \approx 1 + x$$

The graphs of e^x and the two approximations are shown in Figure 9.7.2. As expected, the local quadratic approximation is more accurate than the local linear approximation near $x = 0$. ◀

MACLAURIN POLYNOMIALS

It is natural to ask whether one can improve on the accuracy of a local quadratic approximation by using a polynomial of degree 3. Specifically, one might look for a polynomial of degree 3 with the property that its value and the values of its first three derivatives match



Colin Maclaurin (1698–1746) Scottish mathematician. Maclaurin's father, a minister, died when the boy was only six months old, and his mother when he was nine years old. He was then raised by an uncle who was also a minister. Maclaurin entered Glasgow University as a divinity student but switched to mathematics after one year. He received his Master's degree at age 17 and, in spite of his youth, began teaching at Marischal College in Aberdeen, Scotland. He met Isaac Newton during a visit to London in 1719 and from that time on became Newton's disciple. During that era, some of Newton's analytic methods were bitterly attacked by major mathematicians and much of Maclaurin's important mathematical work resulted from his efforts to defend Newton's ideas geometrically. Maclaurin's work, *A Treatise of Fluxions* (1742), was the first systematic formulation of Newton's methods. The treatise was so carefully done that it was a standard of mathematical rigor in calculus until the work of Cauchy in 1821. Maclaurin was also an outstanding experimentalist; he devised numerous ingenious mechanical devices, made important astronomical observations, performed actuarial computations for insurance societies, and helped to improve maps of the islands around Scotland.

[Image: © Bettmann/Corbis Images]

those of f at a point; and if this provides an improvement in accuracy, why not go on to polynomials of even higher degree? Thus, we are led to consider the following general problem.

9.7.1 PROBLEM Given a function f that can be differentiated n times at $x = x_0$, find a polynomial p of degree n with the property that the value of p and the values of its first n derivatives match those of f at x_0.

We will begin by solving this problem in the case where $x_0 = 0$. Thus, we want a polynomial

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n \tag{5}$$

such that

$$f(0) = p(0), \quad f'(0) = p'(0), \quad f''(0) = p''(0), \ldots, \quad f^{(n)}(0) = p^{(n)}(0) \tag{6}$$

But

$$\begin{aligned}
p(x) &= c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n \\
p'(x) &= c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} \\
p''(x) &= 2c_2 + 3 \cdot 2c_3x + \cdots + n(n-1)c_nx^{n-2} \\
p'''(x) &= 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_nx^{n-3} \\
&\ \vdots \\
p^{(n)}(x) &= n(n-1)(n-2)\cdots(1)c_n
\end{aligned}$$

Thus, to satisfy (6) we must have

$$\begin{aligned}
f(0) &= p(0) &&= c_0 \\
f'(0) &= p'(0) &&= c_1 \\
f''(0) &= p''(0) &&= 2c_2 = 2!c_2 \\
f'''(0) &= p'''(0) &&= 3 \cdot 2c_3 = 3!c_3 \\
&\ \vdots \\
f^{(n)}(0) &= p^{(n)}(0) &&= n(n-1)(n-2)\cdots(1)c_n = n!c_n
\end{aligned}$$

which yields the following values for the coefficients of $p(x)$:

$$c_0 = f(0), \quad c_1 = f'(0), \quad c_2 = \frac{f''(0)}{2!}, \quad c_3 = \frac{f'''(0)}{3!}, \ldots, \quad c_n = \frac{f^{(n)}(0)}{n!}$$

The polynomial that results by using these coefficients in (5) is called the *nth Maclaurin polynomial for f*.

Local linear approximations and local quadratic approximations at $x = 0$ of a function f are special cases of the MacLaurin polynomials for f. Verify that $f(x) \approx p_1(x)$ is the local linear approximation of f at $x = 0$, and $f(x) \approx p_2(x)$ is the local quadratic approximation at $x = 0$.

9.7.2 DEFINITION If f can be differentiated n times at 0, then we define the ***nth Maclaurin polynomial for f*** to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{7}$$

Note that the polynomial in (7) has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = 0$.