EE 131A
Probability and Statistics
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Homework 1 Solutions
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Reading: Chapters 2.1 - 2.3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Four schools 1, 2, 3, and 4 are participating in a spelling bee competition. In the first round, 1 will play 2 and 3 will play 4. Then the two winners will face each other for the cup, and the two losers will also play. A possible outcome can be denoted by 1324 (1 beats 2 and 3 beats 4 in first-round games, and then 1 beats 3 and 2 beats 4).

   (a) List all outcomes in the sample space $\mathcal{S}$.
   Solution: $\mathcal{S} = \{$ 1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231$\}$. Notice that we cannot list all of the 4! possible ways as some of them violate the given information. It is not possible for 1 and 2 to be in the first or second pair of digits, nor for 3 and 4, since these teams would have played each other in the first round.

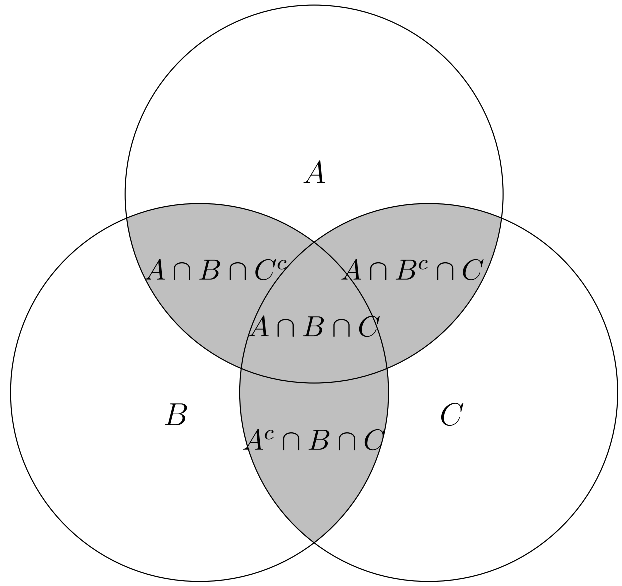
   (b) Let A denote the event that 1 wins the tournament. List outcomes in A.
   Solution: Event A contains the outcomes where 1 is first in the list: A = $\{$1324, 1342, 1423, 1432$\}$.

   (c) Let B denote the event that 2 gets into the final round. List outcomes in B.
   Solution: Event B contains the outcomes where 2 is first or second: B = $\{$ 2314, 2341, 2413, 2431, 3214, 3241, 4213, 4231 $\}$

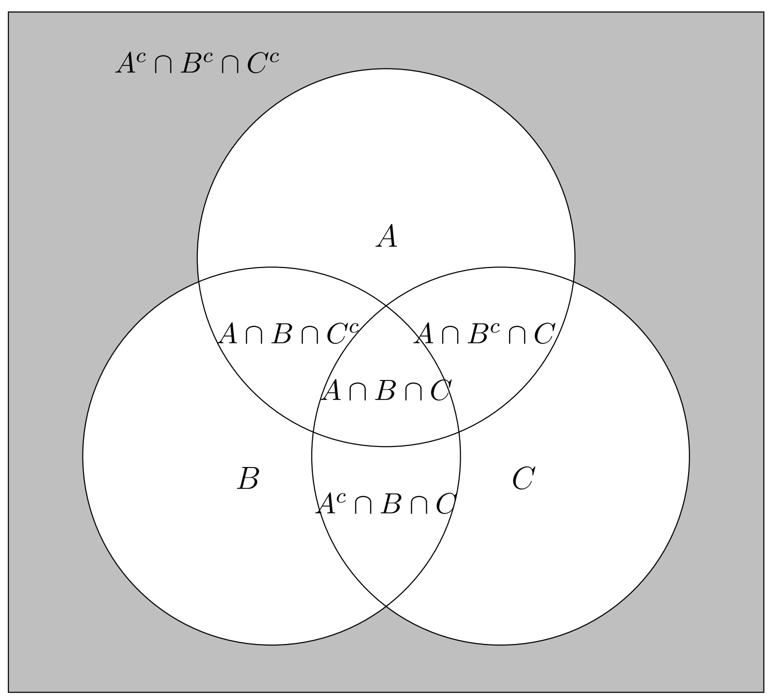
   (d) What are the outcomes in $A \cup B$ and in $A \cap B$? What are the outcomes in A^c?
   Solution: $A \cup B = \{$1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3214, 3241, 4213, 4231$\}$.
   $A \cap B = \phi$. This makes intuitive sense because events A and B can never occur together (only one of 1, 2 and will make it to the final round after defeating the other).
   $A^c = \{$2314, 2341, 2413, 2431, 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231$\}$

2. Let A, B, and C be three events. Find an expression and draw a Venn diagram for the following events:

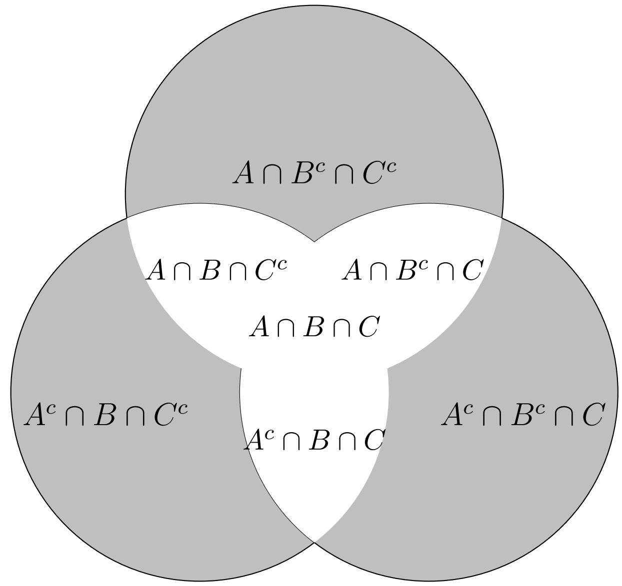
   (a) Two or more of the events occur.
   Solution: The event we want is $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$. Another solution is $(A \cap B) \cup (B \cap C) \cup (C \cap A)$. The Venn diagram is shown below:



(b) None of the events occur.
Solution: We want the event $A^c \cap B^c \cap C^c$. Another solution is $(A \cup B \cup C)^c$. The Venn diagram is:



(c) Exactly one of the three events occurs.
Solution: We want the event$(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$. The Venn diagram is:



3. Suppose A, B, and C are three events. Use the axioms of probability to prove the following:

   (a) $P(A \cap B) \geq P(A) + P(B) - 1$.
   Proof:We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   We also know that $P(A \cup B) \leq 1$
   Therefore, $P(A) + P(B) - P(A \cap B) \leq 1 \implies P(A \cap B) \geq P(A) + P(B) - 1$.

   (b) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.

   Proof: There are multiple ways to prove the above inequality. Here are two possible methods.

   We know that $P(A) \geq 0$. Also using axioms of probability, we know that $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$ for any events X and Y. Thus using $X = A$ and $Y = B \cup C$,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &\leq P(A) + P(B \cup C) \\ &= P(A) + P(B) + P(C) - P(B \cap C) \\ &\leq P(A) + P(B) + P(C) \end{aligned}$$

   Another method is using the following: when $A_1, A_2, \ldots, A_n$ are disjoint sets then $P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$. To use this axiom, let us create some auxiliary sets D, E, and F which are defined as $D = A$, $E = B \setminus A$, and $F = C \setminus (A \cup B)$. Note that D, E, and F are disjoint and that $P(A \cup B \cup C) = P(D \cup E \cup F)$. Additionally, we use corollary 7 from ALG to get $P(D) \leq P(A), P(E) \leq P(B)$, and $P(F) \leq P(C)$. Putting them all together, we get

$$\begin{aligned} P(A \cup B \cup C) &= P(D \cup E \cup F) \\ &= P(D) + P(E) + P(F) \\ &\leq P(A) + P(B) + P(C) \end{aligned}$$

4. Suppose we pick 2 distinct integers in the range from 1 to 100. How many ways are there to pick these 2 integers such that their product is a multiple of 11?
Solution: As long as one integer is a multiple of 11, the product of this integer and another integer will be a multiple of 11. Let S be the set of integers from 1 to 100. Let A be the set containing all multiples of 11, i.e., $A = \{11, 22, ..., 99\}$. There are 9 elements in A, and thus, there are 91 elements in A^c. To satisfy the condition, we can pick the 2 distinct integers from set A, or one integer from set A and the other integer from set A^c. Therefore, there are $\binom{9}{2} + \binom{9}{1}\binom{91}{1} = 855$ ways to pick 2 distinct integers such that their product is a multiple of 11.

5. Consider a spinning wheel that is divided into 10 parts that are numbered from 1 to 10. You spin the wheel 2 times, and each time, you are equally likely to land on any of the 10 numbers.

   (a) What is the probability that the sum of your spins is even?
   Solution: The sample space has 100 elements:
   $\mathcal{S}$ = { (1,1), (1,2),...,(1,10),
   (2,1), (2,2),...,(2,10),
   (3,1), (3,2),...,(3,10),
   (4,1), (4,2),...,(4,10),
   (5,1), (5,2),...,(5,10),
   (6,1), (6,2),...,(6,10),
   (7,1), (7,2),...,(7,10),
   (8,1), (8,2),...,(8,10),
   (9,1), (9,2),...,(9,10),
   (10,1), (10,2),...,(10,10)}
   For the sum of the spins to be even, we can either roll 2 even numbers, or roll 2 odd numbers. Noting that each row of $\mathcal{S}$ has exactly 5 such elements, the probability that the sum of our spins is even is $\frac{50}{100} = 0.5$.

   (b) What is the probability that at least one of the spins is odd?
   Solution: One solution is to simply count these elements. Another solution is to notice that P{At least one spin is odd} = 1 - P{Both spins are even}. Every row of $\mathcal{S}$ in which the 1st toss is even has 5 elements where both spins are even. Thus, P{Both spins are even} $= \frac{25}{100} = 0.25$, and so, the required answer is 0.75.

   (c) What is the probability that the absolute value of the difference between your spins is greater than or equal to 8?
   Solution: Let $\mathcal{A}$ be the event "The absolute value of the difference between your spins is greater than or equal to 8". Then $\mathcal{A}$ = {(1,9),(9,1),(1,10),(10,1),(2,10),(10,2)}. Thus the answer $= \frac{6}{100} = 0.06$.