23W-EC ENGR-131A-LEC-1 Homework 5

SANJIT SARDA

TOTAL POINTS

89 / 100

QUESTION 1

1 Question 1 **20 / 20**

✓ **- 0 pts** *All correct*

- 5 pts Missing plots in part b

- 6 pts Missing part a

- 8 pts Missing part b

- 6 pts Missing part c

QUESTION 2

2 Question 2 **14 / 20**

- 0 pts All correct

- 2 pts Partially correct part a

- 1 pts Missing condition on alpha in part a

✓ **- 2 pts** *Partially correct part b*

- 2 pts Partially correct part c

- 2 pts Partially correct part d

- 20 pts Missing solution

- 4 *Point adjustment*

Incomplete part d

QUESTION 3

3 Question 3 **10 / 10**

✓ **- 0 pts** *Correct*

- 5 pts Partially correct

QUESTION 4

4 Question 4 **25 / 25**

✓ **- 0 pts** *All correct*

- 2 pts Partially correct part a

- 2 pts Partially correct part b

- 2 pts Partially correct part c

- 2 pts Partially correct part d

- 2 pts Partially correct part e

- 25 pts Missing solution

QUESTION 5

5 Question 5 **20 / 25**

- 0 pts Correct

- 15 pts Partially correct

✓ **- 5 pts** *Calculation error*

- 20 pts Missing complete solution

- 25 pts Not attempted

ECE 131A

1) M → Geometric RV: p & X → Exp RV: λ

$$P_M(M=x) = (1-p)^{x-1}p$$

a) $$P(M>k) = \sum_{x=k}^{\infty}(1-p)^{x-1}p$$

$$= p\sum_{x=k}^{\infty}(1-p)^{x-1} = \frac{p(1-p)^k - (1-p)^{\infty}}{1-(1-p)} = \frac{\cancel{p}(1-p)^k}{\cancel{p}}$$
$$= (1-p)^k$$

$$P_X(X=x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & ow \end{cases}$$

$$\int_t^{\infty} \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda x}\Big|_t^{\infty}$$

$$= 1 - (1 - e^{-\lambda t}) = \boxed{e^{-\lambda t}}$$

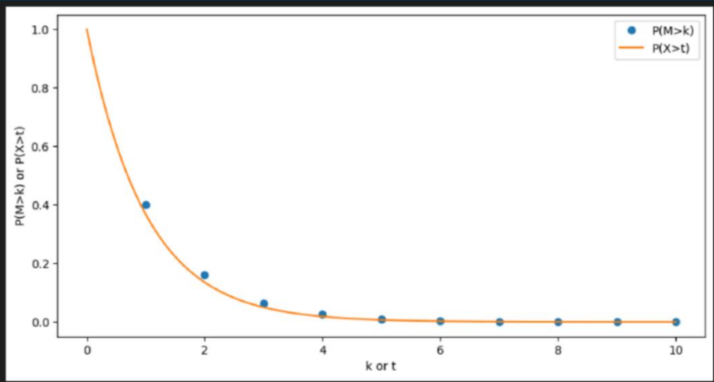
b)

```python
def M_cdf(x, p=0.6):
    return 1-(1-p)**x

def X_cdf(x, l=1):
    return 1-np.exp(-l*x)

k = np.arange(1, 11)
t = np.linspace(0, 10, 1000)

plt.figure(figsize=(10, 5))
plt.plot(k, 1-M_cdf(k), 'o', label='P(M>k)')
plt.plot(t, 1-X_cdf(t), label='P(X>t)')
plt.xlabel('k or t')
plt.ylabel('P(M>k) or P(X>t)')
plt.legend()
plt.show()
```

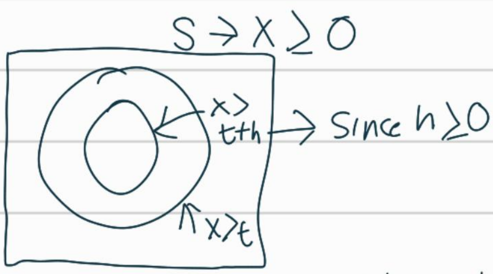


c) RV: X → satisfies memoryless property if for all t, & h ≥ 0,

$$P(X > t+h \mid X > t) = P(X > h) \;*$$

Prove for exponential RV.

$$\frac{P((X>t+h)\cap(X>t))}{P(X>t)}$$



$$= \frac{P(X>t+h)}{P(X>t)} = \frac{\int_{t+h}^{\infty}\lambda e^{-\lambda x}\,dx}{\int_t^{\infty}\lambda e^{-\lambda x}\,dx} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = \frac{e^{-\lambda t + -\lambda h}}{e^{-\lambda t}} = \frac{\cancel{e^{-\lambda t}}\,e^{-\lambda h}}{\cancel{e^{-\lambda t}}} = e^{-\lambda h}$$

$$* \; P(X>h) = \int_h^{\infty}\lambda e^{-\lambda x}\,dx = e^{-\lambda h}$$

$=$

∴ Memoryless Property! $P(X>t+h \mid X>t) = P(X>h)$ holds

1 Question 1 **20 / 20**

✓ **- 0 pts** *All correct*

- 5 pts Missing plots in part b

- 6 pts Missing part a

- 8 pts Missing part b

- 6 pts Missing part c

② $f(x;\alpha) = \begin{cases} \frac{k}{x^\alpha}, & x \geq 5 \\ 0, & ow \end{cases}$

$\frac{k}{x^3}$

$\swarrow \alpha > 1$

$\int_5^\infty \frac{k}{x^\alpha} dx = 1 \therefore -k \frac{1}{(\alpha-1)x^{\alpha-1}} \Big|_5^\infty = 1 = \frac{k \cdot x^{1-\alpha}}{1-\alpha} \Big|_5^\infty$

$\therefore 1 = -\frac{k\, 5^{1-\alpha}}{1-\alpha} \therefore k = \frac{\alpha-1}{5^{1-\alpha}}$

ⓐ $\alpha > 1$ is important

ⓑ CDF: $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \begin{cases} \int_5^x \frac{k}{x^\alpha}, & x \geq 5 \\ 0, & ow \end{cases}$

$$\therefore \boxed{F_X(x) = \begin{cases} \frac{1-5^{1}x^{1-1}}{5}, & x \geq 5 \\ 0, & ow \end{cases}}$$

ⓒ $E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_5^\infty x \frac{\alpha-1}{5^{1-\alpha}} \cdot x^{-\alpha} dx = \frac{\alpha-1}{5^{1-\alpha}} \int_5^\infty x^{1-\alpha} dx$

$= -\frac{x^{2-\alpha}}{\alpha-2}\Big|^\infty + \frac{x^{2-\alpha}}{\alpha-2}\Big|_5$

$= \boxed{\frac{5^{2-\alpha}}{\alpha-2}}$

ⓓ $\overset{y=x}{\ln(X/5)} \therefore P(X \leq x) = P(\ln(X/5) < y)$

$= P(X/5 < e^y)$

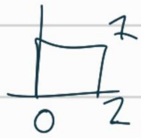
$= P(X < 5e^y)$

2 Question 2 **14 / 20**

- **- 0 pts** All correct
- **- 2 pts** Partially correct part a
- **- 1 pts** Missing condition on alpha in part a
- ✓ **- 2 pts** *Partially correct part b*
- **- 2 pts** Partially correct part c
- **- 2 pts** Partially correct part d
- **- 20 pts** Missing solution
- **- 4** *Point adjustment*

Incomplete part d

③ PDF of $X = -\ln(4-4U) \rightarrow U$ is Uniform$[0,1]$

$U =$ 

$\rightarrow f_U(u) = \begin{cases} 1, & 0 \le u \le 1 \\ 0, & ow \end{cases} \rightarrow F_U(u) = \begin{cases} u, & 0 \le u \le 1 \\ 0, & ow \end{cases} = P(U \le u)$

$X = -\ln(4-4U)$, $P(X \le x) = P(-\ln(4-4U) \le x)$

$= P(\ln(4-4U) \ge -x)$

$= P(4-4U \ge e^{-x})$

$= P(4-e^{-x} \ge 4U)$

$= P(U \le (4-e^{-x})/4)$

$= F_X(x) = \begin{cases} (4-e^{-x})/4, & -\ln 4 \le x \\ 0, & ow \end{cases}$

$\therefore f_X(x) = \frac{d}{dx} F_X(x)$

$= \begin{cases} (e^{-x})/4, & -\ln 4 \le x \\ 0, & ow \end{cases}$

3 Question 3 **10 / 10**

✓ **- 0 pts** *Correct*

- 5 pts Partially correct

(4) (a) $Y = \sum_y X_y$ Bernoulli

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$P_Y(y) = \binom{n}{y}(1-p)^{n-y} p^y$

$\Phi_Y(\omega) = \mathbb{E}[e^{j\omega Y}] = \sum_{y=0}^{n} e^{j\omega y} \cdot \binom{n}{y}(1-p)^{n-y} p^y = \sum_{y=0}^{n} \binom{n}{y}(1-p)^{n-y}(pe^{j\omega})^y$

$= (pe^{j\omega} + 1 - p)^n$

$p = \frac{\lambda}{n}$

(b) X = Poisson, $\therefore P_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$\Phi_X(\omega) = \mathbb{E}[e^{j\omega x}] = \sum_{x=0}^{\infty} e^{j\omega x} \frac{\lambda^x e^{-\lambda}}{x!} \quad e^{-\lambda}\sum_{x=0}^{\infty} \frac{(e^{j\omega}\lambda)^x}{x!} = e^{-\lambda} e^{\lambda e^{j\omega}}$

$= e^{\lambda(e^{j\omega}-1)}$

(c) $\Phi_Y(\omega)$ as $n \to \infty = \lim_{n\to\infty} (pe^{j\omega} + 1 - p)^n = \lim_{n\to\infty} \left(1 + \frac{\lambda}{n}(e^{j\omega}-1)\right)^n$

$= e^{\lambda(e^{j\omega}-1)}$

$\therefore \Phi_Y(\omega)_{n\to\infty} = \Phi_X(\omega) \quad \therefore$ Proved

(d) When approxing, $n \cdot p = \lambda$, Approximating only seems to be valid for really large values of n in the Binomial, or success is low.

(e) $P_{Bin}(X=1) = \frac{1}{10000}\left(\frac{9999}{10000}\right)^{1999}\binom{2000}{1} = 0.164$

$P_{poiss}(X=1) = \frac{e^{-1/5}\lambda^1}{1} \quad \frac{e^{-1/5}}{5} = 0.164$

4 Question 4 **25 / 25**

- ✓ **- 0 pts** *All correct*
- **- 2 pts** Partially correct part a
- **- 2 pts** Partially correct part b
- **- 2 pts** Partially correct part c
- **- 2 pts** Partially correct part d
- **- 2 pts** Partially correct part e
- **- 25 pts** Missing solution

⑤ Mean: μ, Var: σ^2

$$\Phi_X(\omega) = E[e^{j\omega X}] = \Phi[X] = \int_{-\infty}^{\infty} e^{j\omega x} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{j\omega x - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(2\mu j\omega\sigma^2 - \omega^2\sigma^4)} e^{\frac{x-K}{\sqrt{2}\sigma}} \cdot \frac{1}{\sqrt{2}\sigma} dx$$

$$v = \frac{x-K}{\sqrt{2}\sigma}$$

$$= \frac{e^4}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv$$

$$= e^{2\mu j\omega\sigma^2 - \omega^2\sigma^4}$$

5 Question 5 **20 / 25**

- **- 0 pts** Correct
- **- 15 pts** Partially correct
- ✓ **- 5 pts** *Calculation error*
- **- 20 pts** Missing complete solution
- **- 25 pts** Not attempted