EE 131A
Probability
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Homework 8
Due: March 14, 2023
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Chapters 7.1 - 7.3 & 8 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Consider the jointly Gaussian random variables X and Y that have the following joint PDF:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right].$$

   (a) Prove that Y is a Gaussian random variable by deriving its marginal PDF, $f_Y(y)$. Find the mean and variance of Y.

   (b) Prove that $f_{X|Y}(x|y)$ corresponds to another Gaussian random variable by determining its closed form equation, then find its mean and variance.

2. Assume that $X_1, X_2, ..., X_n$ are independent random variables with possibly different distributions and let S_n be their sum. Let $m_k = E(X_k)$, $\sigma_k^2 = VAR(X_k)$, and $M_n = m_1 + m_2 + \cdots + m_n$. Assume that $\sigma_k^2 < R$ and $m_k < T$ for all k. Prove that, for any $\epsilon > 0$,

$$P(|\frac{S_n}{n} - \frac{M_n}{n}| < \epsilon) \to 1$$

   as $n \to \infty$ using Chebyshev's inequality.

3. *Application of CLT.*

   (a) A fair coin is tossed 100 times. Estimate the probability that the number of heads is between 40 and 60. Estimate the probability that the number is between 60 and 80.

   (b) Repeat part (a) for if we toss the coin 10000 times and for the intervals [4000,6000] and [6000,8000].

4. The sum of a list of 108 real numbers is to be computed. Suppose that the numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval (-0.5,0.5). Use the central limit theorem to estimate the probability that the absolute value of the total error in the sum of the 108 numbers exceeds 2.
   Hint: Assign a random variable to each of the rounding errors and use the CLT on their sum.

5. Suppose $X_1, \cdots, X_n$ are i.i.d random variables with mean $E(X) = \mu$ and variance $VAR(X) = \sigma^2$. *Sample variance* is defined as

$$V_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - M_n)^2,$$

where M_n is the sample mean. Show that the expected value of V_n is given by:

$$\mathbb{E}[V_n] = \frac{n-1}{n}\sigma^2$$

Hint: Manipulate V_n into the form:

$$V_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 - (M_n - \mu)^2$$

Provide an explanation as to why the sample variance is not an unbiased estimator; in other words, why do we have the factor $\frac{n-1}{n}$?