ECE113, Winter 2023 **Homework #3.5 Solutions**

Digital Signal Processing Prof. A. Kadambi

University of California, Los Angeles; Department of ECE TA: S. Zhou, A. Vilesov

Due day, 13 Feb 2023, by 11:59am to Gradescope. **(Optional & Unscored)**
30 points total.

Note: Unless specified, you are free to use any of the properties of DTFT that were taught in class. Also, all repeated derivations can be referenced with appropriate equation/result numbers.

1. (10 points) Let $x[n] = 1 + e^{j\omega_0 n}$ and $y[n] = 1 + \frac{1}{2}e^{j4\omega_0 n} + \frac{1}{2}e^{j3\omega_0 n}$ be two signals with a fundamental period N, such that $\omega_0 = \frac{2\pi}{N}$.

   Find the DTFS coefficients of their product $z[n] = x[n]y[n]$, assuming $N = 3$.

   Solutions:

$$z[n] = 1 + \frac{1}{2}e^{j4\omega_0 n} + \frac{1}{2}e^{j3\omega_0 n} + e^{j\omega_0 n} + \frac{1}{2}e^{j5\omega_0 n} + \frac{1}{2}e^{j4\omega_0 n}$$
$$z[n] = 1 + e^{j\omega_0 n} + \frac{1}{2}e^{j3\omega_0 n} + e^{j4\omega_0 n} + \frac{1}{2}e^{j5\omega_0 n}$$

   The period of $z[n]$ is $N = 3$. We can then rewrite $z[n]$ as

$$z[n] = 1 + e^{j2\pi n/3} + \frac{1}{2}e^{j2\pi 3n/3} + e^{j2\pi 4n/3} + \frac{1}{2}e^{j2\pi 5n/3}$$
$$z[n] = 1 + e^{j2\pi n/3} + \frac{1}{2}e^{j2\pi 0n/3} + e^{j2\pi n/3} + \frac{1}{2}e^{j2\pi 2n/3}$$
$$z[n] = \frac{3}{2} + 2e^{j2\pi n/3} + \frac{1}{2}e^{j2\pi 2n/3}$$

   The DTFS basis functions for the signal with period $N = 3$ are $e^{j\frac{2\pi}{3}kn}$, for $k = 0, \dots, 2$. We can then conclude that :

$$\tilde{c}_0 = 3/2$$
$$\tilde{c}_1 = 2$$
$$\tilde{c}_2 = 1/2$$

2. (10 points) Let $x[n] = 3^n$.

   a.) Show that the DTFT does not exist.

   b.) Evaluate whether the DTFT exists for the following modifications to $x[n]$. If they exist, compute DTFT:

      i.) $x[n]\ u[n]$

      ii.) $x[-n]\ u[n]$

      iii.) $x[-|n|\ ]$

      iv.) $x[n]\ u[-n]$

Solutions:

a.) The DTFT of x[n] can be rewritten into 2 sums:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]u[n]e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} x[n]u[-(n+1)]e^{-j\Omega n} \tag{1}$$

$$X(\Omega) = \sum_{n=0}^{\infty} 3^n e^{-j\Omega n} + \sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (3e^{-j\Omega})^n + \sum_{n=-\infty}^{-1} (3e^{-j\Omega})^n \tag{2}$$

Since these are geometric sums we know that they converge when the absolute value of the ratio between adjacent terms is less than 1.

$$\sum_{n=0}^{\infty} a_o (r)^n = \frac{a_o}{1-r}, \text{when } |r| < 1. \tag{3}$$

The first sum has a ratio of $3e^{-j\Omega}$. Since $|3e^{-j\Omega}| \not< 1$ the geometric series does not converge and the DTFT does not exist. While it does not matter whether the second sum converges or not to show that the DTFT does not exist, solving its sum will give us insight in how we can quickly solve the problems in b.)

$$\sum_{n=-\infty}^{-1} (3e^{-j\Omega})^n = \sum_{n=1}^{\infty} (3e^{-j\Omega})^{-n} = \sum_{n=1}^{\infty} (\frac{1}{3}e^{j\Omega})^n = \frac{\frac{1}{3}e^{j\Omega}}{1 - \frac{1}{3}e^{j\Omega}} \tag{4}$$

b.) i. The DTFT of $x[n]\ u[n]$ is the same as the first sum in a.), which we have shown previously does not converge.

ii.) The DTFT $y[n] = x[-n]\ u[n]$ is similar to the second term in part a.).

$$Y(\Omega) = \sum_{n=0}^{\infty} 3^{-n} e^{-j\Omega n} = \sum_{n=0}^{\infty} (\frac{1}{3}e^{-j\Omega})^n = \frac{1}{1 - \frac{1}{3}e^{-j\Omega}} \tag{5}$$

iii.) The DTFT of $y[n] = x[-|n|\ ]$ is equal to:

$$Y(\Omega) = \sum_{n=0}^{\infty} 3^{-|n|} e^{-j\Omega n} + \sum_{n=-\infty}^{-1} 3^{-|n|} e^{-j\Omega n} = \sum_{n=0}^{\infty} 3^{-n} e^{-j\Omega n} + \sum_{n=1}^{\infty} 3^{-n} e^{j\Omega n} \tag{6}$$

Since the first sum is equivalent to the sum in equation (4) and the second sum is equivalent to the sum in equation (3):

$$Y(\Omega) = \frac{1}{1 - \frac{1}{3}e^{-j\Omega}} + \frac{\frac{1}{3}e^{j\Omega}}{1 - \frac{1}{3}e^{j\Omega}} \tag{7}$$

iv.) The DTFT of $y[n] = x[n]\ u[-n]$ is equal to:

$$Y(\Omega) = \sum_{n=-\infty}^{0} 3^{n} e^{-j\Omega n} = \sum_{n=0}^{\infty} 3^{-n} e^{j\Omega n} = \sum_{n=0}^{\infty} (\frac{1}{3} e^{j\Omega})^{n} = \frac{1}{1 - \frac{1}{3}e^{j\Omega}} \tag{8}$$

3. (10 points) Let $x[n] = sin(\Omega_o n)$.

   (a) Derive the DTFT of $x[n]$.

   (b) Write the real and imaginary part of $X(\Omega)$ as well as its magnitude and phase.

Solutions:

a.)

$$x[n] = sin(\Omega_o n) = \frac{1}{2j}(e^{j\Omega_o n} - e^{-j\Omega_o n}) \tag{9}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \frac{1}{2j}\sum_{n=-\infty}^{\infty}(e^{j\Omega_o n} - e^{-j\Omega_o n})e^{-j\Omega n} \tag{10}$$

$$X(\Omega) = \frac{1}{2j}\sum_{n=-\infty}^{\infty}(e^{-j(\Omega-\Omega_o)n} - e^{-j(\Omega+\Omega_o)n}) \tag{11}$$

We then need to show the DTFT of $e^{-j\Omega_o n}$. To do this, we guess that the DTFT of $e^{-j\Omega_o n}$ is $2\pi\delta(\Omega+\Omega_o)$ and show that this is correct by getting the original function back from the IDTFT. If $x_1[n] = e^{-j\Omega_o n}$ and its proposed DTFT is $X_1(\Omega) = 2\pi\delta(\Omega + \Omega_o)$, then we can see that our guess is correct by evaluating the following expression:

$$IDTFT(X_1(\Omega)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega + \Omega_o)e^{j\Omega n}d\Omega = x_1[n] = e^{-j\Omega_o n} \tag{12}$$

Then since the DTFT is 2π periodic in the frequency domain:

$$e^{-j\Omega_o n} \xrightarrow{\mathcal{F}} \sum_{l=-\infty}^{\infty} 2\pi\delta[\Omega + \Omega_o - 2\pi l] \tag{13}$$

Therefore, we can rewrite equation (11) as:

$$X(\Omega) = \frac{\pi}{j}\sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_o - 2\pi l) - \delta(\Omega + \Omega_o - 2\pi l) \tag{14}$$

b.) The imaginary and real parts can be extracted from the previous equation. $X(\Omega) = X_{Real}(\Omega) + jX_{Imag}(\Omega)$, then:

$$X_{Real}(\Omega) = 0, \forall\ \Omega \in \mathbf{R} \tag{15}$$

$$X_{Imag}(\Omega) = \begin{cases} -\pi, \text{ where } \Omega = +\Omega_o + 2\pi l,\ \forall\ l \in \mathbf{Z} \\ +\pi, \text{ where } \Omega = -\Omega_o + 2\pi l,\ \forall\ l \in \mathbf{Z} \end{cases} \tag{16}$$

For the magnitude and phase we convert from the rectangular form to the Euler form such that $X(\Omega) = X_{Real}(\Omega) + jX_{Imag}(\Omega) = |X(\Omega)|\ e^{-j\Theta(\Omega)}$:

$$|X(\Omega)| = \pi\sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_o - 2\pi l) + \delta(\Omega + \Omega_o - 2\pi l) \tag{17}$$

$$\angle X(\Omega) = \begin{cases} -\frac{\pi}{2}, \text{ where } \Omega = +\Omega_o + 2\pi l,\ \forall\ l \in \mathbf{Z} \\ +\frac{\pi}{2}, \text{ where } \Omega = -\Omega_o + 2\pi l,\ \forall\ l \in \mathbf{Z} \end{cases} \tag{18}$$