

Due Monday, 21 Nov 2022, by 11:59pm to Gradescope

Covers material up to lecture 15.

100 points total

1. (32 points) **Frequency Response**

- (a) (18 points) Consider the LTI system depicted in figure 1 whose response to an unknown input,
- $x(t)$
- , is

$$y(t) = (4e^{-t} - 4e^{-4t}) u(t)$$

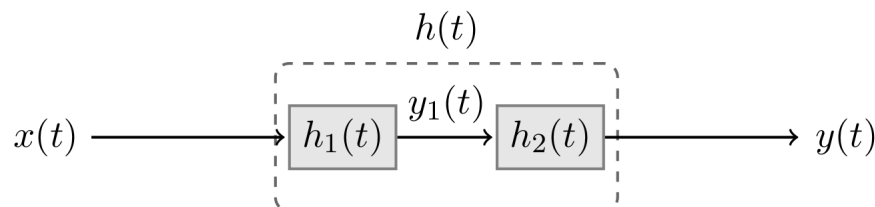


Figure 1: System for Problem 1(a)

We know that for the same unknown input  $x(t)$ , the intermediate signal,  $y_1(t)$ , is given by:

$$y_1(t) = 2e^{-t}u(t)$$

The overall LTI system is described by the following differential equation:

$$\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 3x(t)$$

- i. Find the frequency response,  $H(j\omega)$ , of the overall system  $h(t)$ .
  - ii. Find the frequency responses  $H_1(j\omega)$  of the first LTI system and  $H_2(j\omega)$  of the second LTI system.
  - iii. Find the impulse responses  $h(t)$ ,  $h_1(t)$  and  $h_2(t)$ .
- (b) (8 points) Consider the system shown in the figure 2, where the frequency response  $H(\omega)$  has magnitude and phase shown in figure 3.

If,

$$x_1(t) = \sin \left[ \omega_1 t + \frac{\pi}{4} \right]$$

$$x_2(t) = 2 \cos \left[ \omega_2 t - \frac{\pi}{3} \right]$$

where  $\omega_1 = \pi$  and  $\omega_2 = 2\pi$ , write an expression for  $y(t)$ .

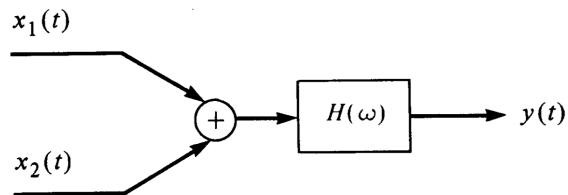


Figure 2: System for Problem 1(b)

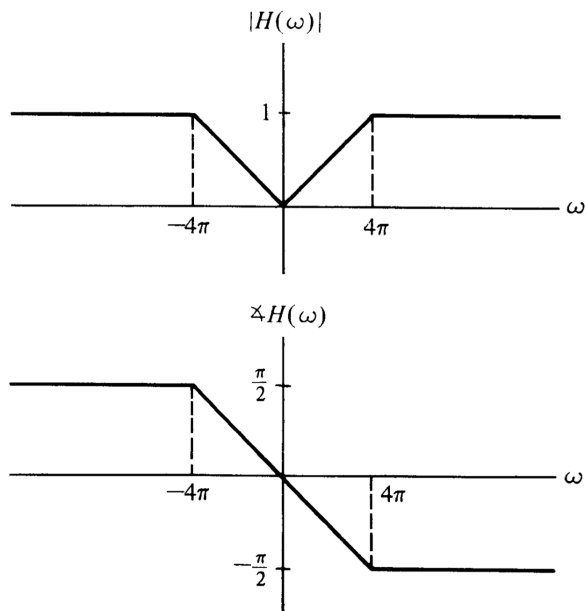


Figure 3: Frequency response for Problem 1(b)

- (c) (6 points) Consider an LTI filter with  $h(t)$  as impulse response. Its frequency response is given by:

$$H(j\omega) = \frac{1}{1 + j\omega}$$

- i. Find the frequency range in Hertz in which the magnitude of the system function exhibits 1 percent or less deviation from its value at  $\omega = 0$ . Determine its cutoff frequency  $\omega_c$ , where  $\omega_c$  is defined below:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H(j0)|$$

- ii. How can we transform  $h(t)$  so that the cutoff frequency of the new filter becomes  $5\omega_c$ ?

## 2. (18 points) **Filters**

- (a) (6 points) Consider an ideal low-pass filter  $h_{LP,1}(t)$  with frequency response  $H_{LP,1}(j\omega)$  depicted below in figure 4.

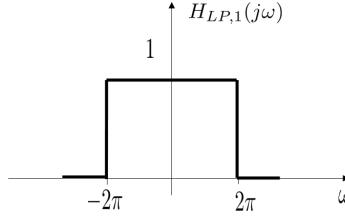


Figure 4: An ideal low pass filter

Using this filter, we construct the following new system:

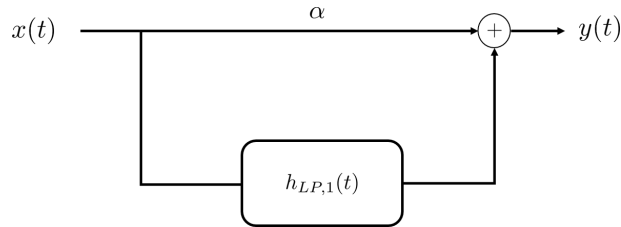


Figure 5: New system

We are given two choices for  $\alpha$ : 1 or -1. Which value should we choose so that the new system is a high-pass filter? Does the new filter have any phase in its frequency response?

- (b) (3 points) Why are the ideal filters non-realizable systems?
- (c) (5 points) We want to design a causal non-ideal low-pass filter  $h_{LP,2}(t)$ , using the following frequency response:

$$H_{LP,2}(j\omega) = \frac{k}{\beta + j\omega}$$

Find  $k$  and  $\beta$  so that  $H_{LP,2}(j0) = 1$  and its cutoff frequency is  $\omega_0 = 2\pi$  rad/s, (i.e., the magnitude of  $H_{LP,2}(j\omega)$  is  $1/\sqrt{2}$  for  $\omega = 2\pi$  rad/s).

- (d) (4 points) We again consider the system of part (a) where instead of the ideal low-pass filter, we are going to use the non-ideal low-pass filter of part (c).

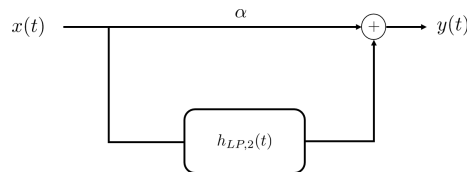
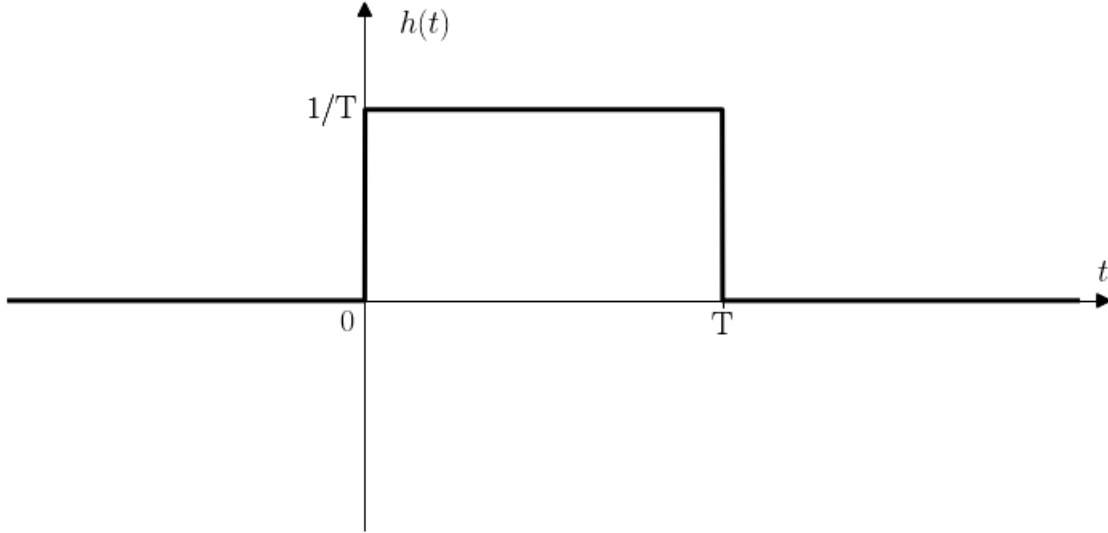


Figure 6: The system of part (a) with the non-ideal low pass filter

For the same value of  $\alpha$  you found in part (a), find the frequency response of the equivalent system. Explain if the new system behaves as a high-pass filter.

3. (25 points) **Moving average filters**

We now consider the moving average filter, also known as a “boxcar” filter, and is one of the most primitive filters in practice. Its impulse response is shown below:

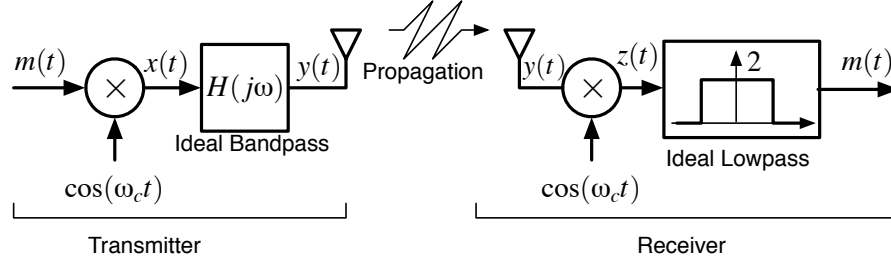


- (a) (6 points) What is the frequency response  $H(j\omega)$  of this filter?
- (b) (6 points) Sketch the amplitude response  $|H(j\omega)|$  of the filter. What happens to  $|H(j\omega)|$  as  $\omega \rightarrow \infty$ ?
- (c) (6 points) For non-constant **periodic signal**  $x(t)$  with what period does  $y(t) = x(t) * h(t) = C$  for some constant  $C$  (**Hint**: think about Fourier series and its Fourier transform.)?
- (d) (7 points) Suppose we have a baseband signal  $f(t)$  such that  $F(j\omega) \neq 0$  for  $-2\pi B < \omega < 2\pi B$ , and  $F(j\omega) = 0$  for  $|\omega| > 2\pi B$ . What is the maximum value of  $B$  (Hz) such that every frequency in  $f(t)$  is retained in the output (i.e.  $Y(j\omega) \neq 0$  for  $-2\pi B < \omega < 2\pi B$ )? Within this frequency range, what is the group delay, where group delay  $t_d(\omega)$  is defined as

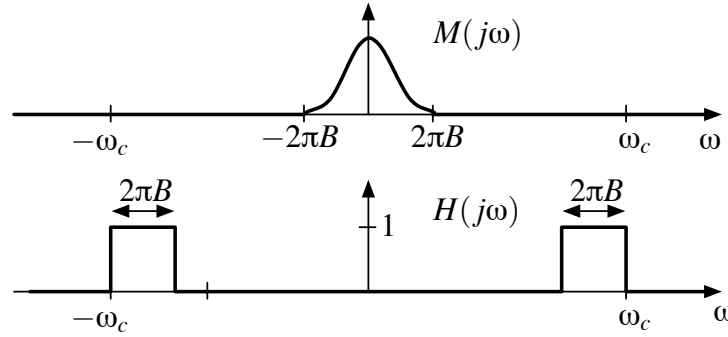
$$t_d(\omega) = -\frac{d}{d\omega} \angle H(j\omega)$$

4. (25 points) **Modulation and Demodulation**

(a) (10 points) Consider the communication system shown below:



The signal  $m(t)$  is first modulated by  $\cos(\omega_c t)$ , and then passed through an ideal bandpass filter. The spectrum of the input  $M(j\omega)$  and the frequency response of the ideal bandpass filter  $H(j\omega)$  are:



The modulated signal is  $x(t)$ , and the output of the ideal bandpass is  $y(t)$ . This signal is transmitted through a channel. We assume that this channel does not introduce distortion into  $y(t)$ . The received signal  $y(t)$  is then processed by a receiver. Sketch the signal spectrum at

- i. the output of the modulator, i.e.,  $X(j\omega)$ ,
- ii. the output of the ideal bandpass,  $Y(j\omega)$ , and
- iii. the output of the demodulator,  $Z(j\omega)$

Does this system recover  $m(t)$ ?

- (b) (15 points) In the first part of this problem, you have seen that to demodulate the received signal, we multiply  $y(t)$  by  $\cos(\omega_c t)$ , and then to recover  $m(t)$ , we low-pass filter the result. In this part, you will show that you can achieve the same effect with an ideal sampler. In other words, we assume instead the following block diagram of the receiver: where the ideal sampler is drawn as a switch that closes instantaneously every  $T$  seconds to acquire a new sample. Show that we can recover  $m(t)$  if the ideal sampler operates at a frequency  $\omega_c$  (i.e. samples at a rate of  $\omega_c/2\pi$  samples/s). Draw the spectrum of the signal right before the lowpass filter  $Z(j\omega)$ .

