EE 131A
Probability
Instructor: Lara Dolecek

Homework 7
Due: March 7, 2023
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Chapters 5.6-5.10 & 6.4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Let X and Y be independent random variables that are uniformly distributed in the interval $[0, 1]$.
   (a) Find the pdf of $A = X + Y$.
   (b) Find the pdf of $B = X - Y$.
   (c) Find the pdf of $C = XY$.
   (d) Find the covariance of A and B.
   (e) Find the covariance of A and C.

2. Let X be a Gaussian random variable with mean 0 and variance $\sigma^2 = 1$. We define three new random variables as $Y = aX + b$, $Z = X^2$, and $W = X^3$. We require that $a \neq 0$. For reference, $\mathbb{E}[X^3] = 0$, $\mathbb{E}[X^4] = 3\sigma^4$, $\mathbb{E}[X^5] = 0$, and $\mathbb{E}[X^6] = 15\sigma^6$.
   (a) Find the correlation coefficients for each pair (X, Y), (X, Z), and (X, W).
   (b) For the case of $a = 5$ and $b = 0$, randomly sample X 1000 times, use each X to get Y, and use a scatter plot to plot (X, Y) where the values of X is along the x-axis and the values of Y are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot.
   (c) For the case of $a = -4$ and $b = 0$, randomly sample X 1000 times, use each X to get Y, and use a scatter plot to plot (X, Y) where the values of X is along the x-axis and the values of Y are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot and explain the difference to part (b).
   (d) Now, randomly sample X 1000 times, use each X to get Z, and use a scatter plot to plot (X, Z) where the values of X is along the x-axis and the values of Z are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot and explain the difference to parts (b) and (c).
   (e) Now, randomly sample X 1000 times, use each X to get W, and use a scatter plot to plot (X, W) where the values of X is along the x-axis and the values of W are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot.

3. Let X and Y be two random variables with identical distributions. These two random variables are not necessarily independent. Answer the following questions given that $C = aX + bY$ and $D = aX - bY$.

   (a) Find $COV[C, D]$ in terms of the variances and covariances for X and Y

   (b) Find the relation between a and b if the random variables C and D are independent

4. Answer the following. **Show all your work.**

   (a) Consider two random variables X and Y. Prove that the correlation coefficient $\rho_{X,Y}$ satisfies $-1 \leq \rho_{X,Y} \leq 1$.

   (b) Let X be a random variable, and Y be another random variable given by $Y = aX + b$. What is the correlation coefficient between X and Y. Does the answer depend on the sign of a?

5. Let X and Y be jointly Gaussian random variables with $\mathbb{E}[Y] = 0$, $\sigma_X = 4$, $\sigma_Y = 3$ and $\mathbb{E}[X|Y] = \frac{4Y}{9} + 2$. Find the joint pdf of X and Y.

6. **Q3 from HW6**
   Let X and Y be two jointly continuous random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 6xy, & 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, \\ 0, & \text{otherwise}, \end{cases}$$

   (a) Find $f_X(x)$.

   (b) Find the conditional pdf of X given $Y = y$, $f_{X|Y}(x|y)$.

   (c) Find $E[X|Y = y]$, for $0 \leq y < 1$. What is $E[X|Y]$?

   (d) Let A be the event $\{X \geq \frac{1}{2}\}$. Find $P[A]$, $f_{X|A}(x)$, and $E[X|A]$.