ECE113, Winter 2023 **Midterm**

Digital Signal Processing

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UCLA True Bruin academic integrity principles apply.

This exam is one-page open notes.
No laptop/tablet/phone or other devices.

8:10 am - 9:45 am Wednesday, 15 Feb 2022

State your assumptions and reasoning.
No credit without reasoning.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	_____	/ 20
Problem 2	_____	/ 20
Problem 3	_____	/ 10
Problem 4	_____	/ 20
Problem 5	_____	/ 10
Problem 6	_____	/ 20
Total	_____	/ 100 points

1. (20 points) **Digital Signal (Corresponds to Lecture 3 & Quiz 2)**
Now we have the following sequences:

$$x_1[n] = \cos(\frac{2\pi}{3}n + \frac{\pi}{6})$$
$$x_2[n] = 2\sin(\frac{\pi}{4}n)$$
$$x_3[n] = x_1[n] + x_2[n]$$

(a) (3 pts) What is the period for each signal?

(b) (3 pts) What is the energy for $x_3[n]$?

(c) (14 pts) What is the average-power for $x_3[n]$?

Hint: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$

Solutions:

(a) Let $x_1(n) = \cos(\frac{2\pi}{3}n + \frac{\pi}{6})$ with N_1 denoting its period, and let $x_2(n) = 2\sin(\frac{\pi}{4}n)$ with N_2 denoting its period. Then:

$$\frac{2\pi}{3}N_1 = 2\pi k, \quad N_1 = 3k \quad \rightarrow \quad N_1 = 3$$

$$\frac{\pi}{4}N_2 = 2\pi k, \quad N_2 = 8k \quad \rightarrow \quad N_2 = 8$$

Hence, period N can be obtained by

$$N = \frac{N_1 \cdot N_2}{GCD(N_1, N_2)} = \frac{3 \cdot 8}{GCD(3, 8)} = 24,$$

where $GCD(A, B)$ denotes the greatest common divisor of A and B.

(b) Let ε_p be the energy over 1 period of $x_3[n]$. Then:

$$\varepsilon_x = \sum_{n=-\infty}^{\infty} |x_3[n]|^2 = \sum_{k=-\infty}^{\infty} \varepsilon_p \rightarrow \infty$$

(c) For periodic sequences, the average power is precisely the energy over one period ε_p divided by the period N. Hence, $P_x = \varepsilon_p/24$. A method to find ε_p is to do a numerical computation for all 24 samples. However, below we show an analytical approach, which is usually preferred over numerical estimation:

$$\begin{aligned}
\varepsilon_p &= \sum_{n=0}^{23} \left| \cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{4}n\right) \right|^2 \\
&= \sum_{n=0}^{23} \cos^2\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right) + 4\sum_{n=0}^{23} \cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right) \sin\left(\frac{\pi}{4}n\right) + 4\sum_{n=0}^{23} \sin^2\left(\frac{\pi}{4}n\right)
\end{aligned}$$

It can be easily shown that the sum of all samples of a sinusoidal sequence over any integer multiple of periods is equal to zero. To evaluate ε_p, we convert our expressions into sinusoids using the following identities:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos x \sin y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$$

Hence,

$$\sum_{n=0}^{23} \cos^2\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right) = \sum_{n=0}^{23} \frac{1}{2}\left(1 + \cos\left(\frac{4\pi}{3}n + \frac{\pi}{3}\right)\right) = \sum_{n=0}^{23} \frac{1}{2} = 12$$

$$\sum_{n=0}^{23} \sin^2\left(\frac{\pi}{4}n\right) = \sum_{n=0}^{23} \frac{1}{2}\left(1 - \cos\left(\frac{\pi}{2}n\right)\right) = \sum_{n=0}^{23} \frac{1}{2} = 12$$

and

$$\sum_{n=0}^{23} \cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right) \sin\left(\frac{\pi}{4}n\right) = \sum_{n=0}^{23} \frac{1}{2}\left(\sin\left(\frac{11\pi}{12}n + \frac{\pi}{6}\right) - \sin\left(\frac{5\pi}{12}n + \frac{\pi}{6}\right)\right) = 0$$

The last equation uses the fact that whenever two sinusoids of different frequencies are multiplied together and summed over an entire period, the answer is always 0.
Therefore,

$$\varepsilon_p = 12 + 4 \times 12 = 60,$$

and

$$P_x = \frac{60}{24} = 2.5.$$

2. (20 points) **System (Corresponds to Lecture 3,4 & HW 2)**
An LTI discrete-time system has an impulse response $h[n] = u[n+1] - u[n-4]$, and as input the signal $x[n] = u[n] - u[n-(N+1)]$ for a positive integer N. The output of the system is denoted as $y[n]$.

(a) (5 pts) Derive input output relationship in the form of difference equation.

(b) (5 pts) If $N = 4$, without calculating $y[n]$, what is the length of the output $y[n]$? Explain your answer.

(c) (5 pts) Is the system stable? Why?

(d) (5 pts) Is the system causal? Why?

Solutions:

(a) The output of the system is defined as

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= (\delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]) * x[n] \\ &= x[n+1] + x[n] + x[n-1] + x[n-2] + x[n-3] \end{aligned}$$

(b) If two signals $x[n]$ and $h[n]$ are such that $x[n]$ only has non-zero samples in the range $N_{x1} \leq n \leq N_{x2}$ and $h[n]$ has non-zero samples only in the range $N_{h1} \leq n \leq N_{h2}$, then their convolution $y[n] = x[n] * h[n]$ can only have non-zero samples in the range $N_{h1} + N_{x1} \leq n \leq N_{h2} + N_{x2}$.

$$N_{h1} = -1; N_{h2} = 3; N_{x1} = 0; N_{x2} = 4$$

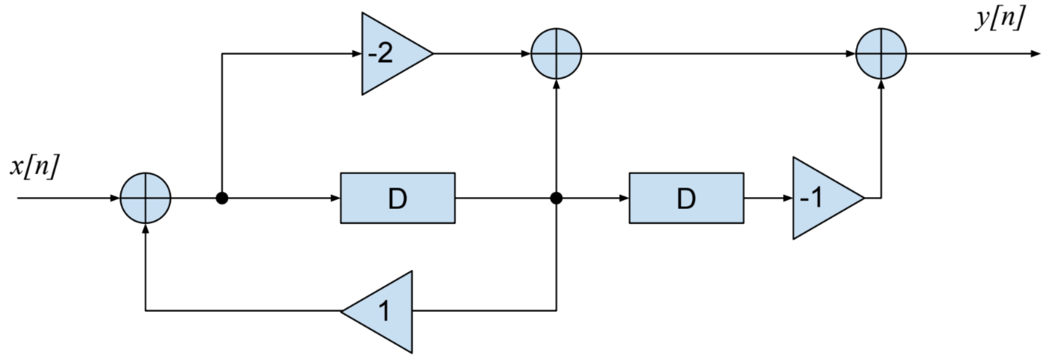
Therefore $-1 \leq n \leq 7$.

(c) The system is stable since

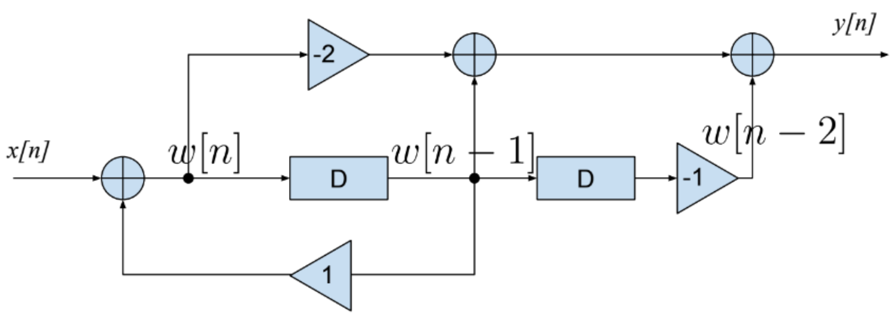
$$\sum_{n=-\infty}^{\infty} |h[n]| = 5 \leq \infty$$

(d) The system is not causal since $h[n]$ is not zero everywhere for $n < 0$. (or the current output $y[n]$ does not depend only on the current and past input samples.)

3. (10 points) **Graphical Representation (Corresponds to Lecture 4 & Quiz 4)** Consider the following block diagram representation of an LTI system. Derive the input-output equation.



Solution:



From the figure above Therefore,

$$w[n] = x[n] + w[n-1]$$

$$x[n] = w[n] - w[n-1] \tag{1}$$

Furthermore,

$$y[n] = -2w[n] + w[n-1] - w[n-2] \tag{2}$$

If we can write $y[n]$ as $y[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2]$ and $x[n]$ as $x[n] = a_0 w[n] + a_1 w[n-1]$, then the difference equation of the system can be written as:

$$a_0 y[n] + a_1 y[n-1] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

For this system using equations (1) and (2)

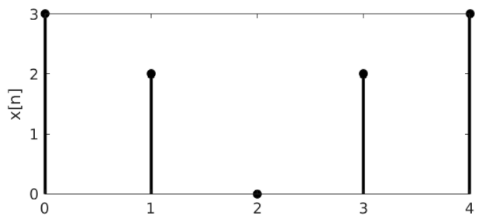
$$a_0 = 1; a_1 = -1$$
$$b_0 = -2; b_1 = 1; b_2 = -1$$

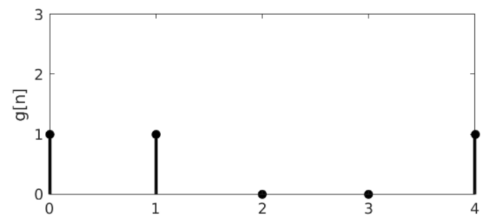
Therefore, the equation of this system is:

$$y[n] - y[n-1] = -2x[n] + x[n-1] - x[n-2].$$

4. (20 points) **DTFS (Corresponds to Lecture 6,7,8 & Quiz 6,7)**

Given two periodic signals $x[n]$ and $g[n]$ with one period shown below:





(a) (10 pts) Find the DTFS coefficients $\tilde{c}_k$ of $x[n]$ and $\tilde{d}_k$ of $g[n]$ respectively.

(b) (10 pts) Let $h[n]$ be a signal defined as $h[n] = x[n] \otimes g[n] \otimes x[n]$. Find the DTFS coefficient $\tilde{e}_k$ of $h[n]$. ($\otimes$ denotes periodic convolution)

Solution:

(a) Use the synthesis equation ($\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$) and analysis equation ($\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$) with the same approach in quiz 6.

(b) By the property of periodic convolution:

$$\tilde{h}[n] \overset{\text{DTFS}}{\longleftrightarrow} 25\tilde{c}_k\tilde{d}_k\tilde{c}_k$$

The DTFS coefficients:

k	$\tilde{d}_k$	$\tilde{c}_k$	$\tilde{e}_k$
0	0.6	2	60
1	0.326	0.5854 + 0.4253i	1.3090 + 4.0287i
2	−0.1236	−0.0854 − 0.2629i	0.1910 − 0.1388i
3	−0.1236	−0.0854 + 0.2629i	0.1910 + 0.1388i
4	0.3236	0.5854 − 0.4253i	1.3090 − 4.0287i

5. (10 points) **DTFS Properties (Corresponds to Lecture 7 CYUs)**

Consider a periodic signal $\tilde{x}[n]$ signal with fundamental period N (N is even) and DTFS coefficients $\tilde{c}_k$. Now we have:

$$y[n] = \tilde{x}[n] + \tilde{x}[n + \frac{N}{2}]$$

Prove that the DTFS coefficients $\tilde{d}_k$ of $y[n]$ is $2\tilde{c}_{2k}$.

Solution:

Firstly, we can easily prove that the fundamental period of $y[n]$ is $\frac{N}{2}$.

By the analysis equation, we can write out:

$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{c}_{2k} = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{4\pi}{N}kn}$$

$$2\tilde{c}_{2k} = \frac{2}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{4\pi}{N}kn} \tag{3}$$

Now we write out the DTFS coefficients for $y[n]$, with the period $\frac{N}{2}$:

$$\tilde{d}_k = \frac{2}{N} \sum_{n=0}^{(N/2)-1} (\tilde{x}[n] + \tilde{x}[n + \frac{N}{2}]) e^{-j\frac{4\pi}{N}kn} \tag{4}$$

You will find that $Eq.(3) = Eq.(4)$ because $\sum_{n=0}^{N-1} \tilde{x}[n] = \sum_{n=0}^{(N/2)-1} (\tilde{x}[n] + \tilde{x}[n + \frac{N}{2}])$.

Therefore, $\tilde{d}_k = 2\tilde{c}_{2k}$.

6. (20 points) **DTFT & IDTFT (Corresponds to Lecture 8,9 & Quiz 9,10)**

   (a) (10 pts) The values of the DTFT of the sequence $x[n] = \{a, b, c\}$ at the frequencies $\frac{3\pi}{2}$, 3π, and 6π are given by $3 - j$, 0, and 2, respectively. Determine the values of the samples a, b, c.

   (b) (10 pts) Find the IDTFT of:

$$X(\Omega) = \frac{2 + 0.8e^{-j\Omega}}{1 + 1.4e^{-j\Omega} + 0.48e^{-j2\Omega}}$$

   (Hint: consider $x[n] = \alpha^n u[n], |\alpha| < 1$)

Solution:

(a) $X(\Omega) = \sum_{n=0}^{2} x[n]e^{-j\Omega n} = a + be^{-j\Omega} + ce^{-j2\Omega}$. We get one such equation for each frequency Ω. This leads to a system of 3 equations in 3 variables:

$$\begin{bmatrix} 3-j \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & j & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \text{ which yields } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2-j \\ 1 \\ -1+j \end{bmatrix}$$

(b) Consider the right-sided exponential signal given in the hint and lineraity property, the DTFT of $x[n] = (A\alpha^n + B\beta^n)u[n]$ can be written as:

$$\begin{aligned} X(\Omega) &= \frac{A}{1 - \alpha e^{-j\Omega}} + \frac{B}{1 - \beta e^{-j\Omega}} \\ &= \frac{(A+B) - (A\beta + B\alpha)e^{-j\Omega}}{(1 - \alpha e^{-j\Omega})(1 - \beta e^{-j\Omega})} \\ &= \frac{(A+B) - (A\beta + B\alpha)e^{-j\Omega}}{1 - (\alpha + \beta)\alpha e^{-j\Omega} + \alpha\beta e^{-j2\Omega}} \end{aligned}$$

Lining up the terms in the denominator, we have $\alpha\beta = 0.48, \alpha + \beta = -1.4$. One solution is $\alpha = -0.8, \beta = -0.6$. Lining up the terms in the numerator, we have $A + B = 2, A\beta + B\alpha = -0.8$. The solution to this system of equations is $A = 4, B = -2$. Thus, $x[n] = (4(-0.8)^n - 2(-0.6)^n)\, u[n]$.