EE 131A
Probability
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Homework 7 Solutions
Due: March 7, 2023
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Chapters 5.6-5.10 & 6.4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Let X and Y be independent random variables that are uniformly distributed in the interval $[0, 1]$.

   (a) Find the pdf of $A = X + Y$.
   Solution: Let $F_U(u)$ and $f_U(u)$ be the cdf and pdf of the uniform random variable. Let $F_A(a)$ and $f_A(a)$ be the cdf and pdf of A. As such,

$$\begin{aligned} F_A(a) &= P(X + Y \leq a) \\ &= \int_{-\infty}^{\infty} P(X + y \leq a | Y = y) f_U(y) dy \\ &= \int_{-\infty}^{\infty} P(X \leq a - y | Y = y) f_U(y) dy \\ &= \int_{-\infty}^{\infty} F_U(a - y) f_U(y) dy \end{aligned}$$

   By taking the derivative with respect to a, we get

$$\begin{aligned} f_A(a) &= \frac{dF_A(a)}{da} \\ &= \int_{-\infty}^{\infty} f_U(a - y) f_U(y) dy \end{aligned}$$

   which is the convolution of the two probability distributions. We note that

$$f_U(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   and

$$f_U(a - u) = \begin{cases} 1 & 0 \leq a - u \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Therefore, the only region that $f_U(a-y)f_U(y)$ is non-zero is $\max(a-1, 0) \leq y \leq \min(a, 1)$. From here, we get

$$f_A(a) = \begin{cases} a & 0 \leq a \leq 1 \\ 2 - a & 1 \leq a \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find the pdf of $B = X - Y$.

Solution: We get a similar solution to part (a) except that the pdf of B is

$$f_B(b) = \int_{-\infty}^{\infty} f_U(b+y) f_U(y) dy.$$

As such, we get

$$f_B(b) = \begin{cases} 1+b & -1 \leq b \leq 0 \\ 1-b & 0 \leq b \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the pdf of $C = XY$.

Solution: As in part (a), we calculate the cdf of C to find the pdf. Let $F_C(c)$ and $f_C(c)$ be the cdf and pdf of C.

$$\begin{aligned} F_C(c) &= P(XY \leq c) \\ &= \int_{-\infty}^{\infty} P(Xy \leq c | Y = y) f_U(y) dy \\ &= \int_{-\infty}^{\infty} P(X \leq \frac{c}{y} | Y = y) f_U(y) dy \\ &= \int_{-\infty}^{\infty} F_U(\frac{c}{y}) f_U(y) dy \end{aligned}$$

We note that the third step is only valid because Y is non-negative so multiplying both sides by y does not change the sign.

By taking the derivative with respect to c, we get

$$\begin{aligned} f_C(c) &= \frac{dF_C(c)}{dc} \\ &= \int_{-\infty}^{\infty} f_U(\frac{c}{y}) f_U(y) \frac{1}{y} dy \end{aligned}$$

Consider the case when $0 \leq c \leq 1$ since all other cases must have zero for the pdf. As such,

$$\begin{aligned} f_C(c) &= \int_{-\infty}^{\infty} f_U(\frac{c}{y}) f_U(y) \frac{1}{y} dy \\ &= \int_{c}^{1} \frac{1}{y} dy \\ &= \ln(y)|_c^1 = \ln(1) - \ln(c) = -\ln(c) \end{aligned}$$

Hence, we get

$$f_C(c) = \begin{cases} -\ln(c) & 0 < c \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Another way to get this result is to calculate the cdf first and then differentiate. Again, consider the case when $0 \le c \le 1$ which results in

$$
\begin{aligned}
F_C(c) &= \int_{-\infty}^{\infty} F_U(\frac{c}{y}) f_U(y) dy \\
&= \int_0^c dy + \int_c^1 \frac{c}{y} dy \\
&= c - c\ln(c).
\end{aligned}
$$

By taking the derivative with respect to c, we get the same result.

(d) Find the covariance of A and B.
Solution:

$$
\begin{aligned}
Cov(A, B) &= \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B] \\
&= \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[(X+Y)]\mathbb{E}[(X-Y)] \\
&= \mathbb{E}[X^2 - Y^2] - \mathbb{E}[(X+Y)]\mathbb{E}[(X-Y)] \\
&= 0
\end{aligned}
$$

(e) Find the covariance of A and C.
Solution:

$$
\begin{aligned}
Cov(A, C) &= \mathbb{E}[AC] - \mathbb{E}[A]\mathbb{E}[C] \\
&= \mathbb{E}[(X+Y)XY] - \mathbb{E}[(X+Y)]\mathbb{E}[XY] \\
&= \mathbb{E}[X^2Y + XY^2] - \mathbb{E}[(X+Y)]\mathbb{E}[XY] \\
&= \mathbb{E}[X^2Y] + \mathbb{E}[XY^2] - \mathbb{E}[(X+Y)]\mathbb{E}[XY] \\
&= \mathbb{E}[X^2]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y] - \mathbb{E}[Y]^2\mathbb{E}[X] \\
&= \mathbb{E}[Y] \cdot (\mathbb{E}[X^2] - \mathbb{E}[X]^2) + \mathbb{E}[X] \cdot (\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) \\
&= \mathbb{E}[Y] \cdot Var(X) + \mathbb{E}[X] \cdot Var(Y) \\
&= \frac{1}{2} \cdot \frac{1}{12} + \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{12}
\end{aligned}
$$

2. Let X be a Gaussian random variable with mean 0 and variance $\sigma^2 = 1$. We define three new random variables as $Y = aX + b$, $Z = X^2$, and $W = X^3$. We require that $a \neq 0$. For reference, $\mathbb{E}[X^3] = 0$, $\mathbb{E}[X^4] = 3\sigma^4$, $\mathbb{E}[X^5] = 0$, and $\mathbb{E}[X^6] = 15\sigma^6$.

   (a) Find the correlation coefficients for each pair (X, Y), (X, Z), and (X, W).
   Solution: First, we calculate the expectation of each random variable which are $\mathbb{E}[X] = 0 = \mathbb{E}[Y] = \mathbb{E}[W]$ and $\mathbb{E}[Z] = \sigma^2$.

Next, we calculate the variances of each random variable:

$$\begin{aligned}
Var(X) &= \sigma^2 = 1 \\
Var(Y) &= Var(aX+b) = a^2 Var(X) = a^2 \\
Var(Z) &= Var(X^2) = \mathbb{E}[X^4] - \mathbb{E}[X^2]^2 \\
&= 3\sigma^4 - \sigma^4 = 2\sigma^4 = 2 \\
Var(W) &= Var(X^3) = \mathbb{E}[X^6] - \mathbb{E}[X^3]^2 \\
&= 15\sigma^6 = 15.
\end{aligned}$$

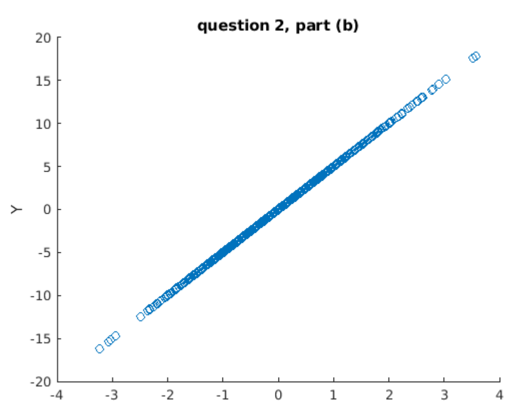
Now, we calculate the covariance of X to all the other terms.

$$\begin{aligned}
Cov(X,Y) &= \mathbb{E}[X(aX+b)] - \mathbb{E}[aX+b]\mathbb{E}[X] \\
&= a\mathbb{E}[X^2] - a\mathbb{E}[X]^2 = aVar(X) = a \\
Cov(X,Z) &= \mathbb{E}[XX^2] - \mathbb{E}[X]\mathbb{E}[X^2] \\
&= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] = 0 \\
Cov(X,W) &= \mathbb{E}[XX^3] - \mathbb{E}[X]\mathbb{E}[X^3] \\
&= \mathbb{E}[X^4] - \mathbb{E}[X]\mathbb{E}[X^3] = 3\sigma^4 = 3.
\end{aligned}$$

Finally, we can calculate the correlation coefficients:

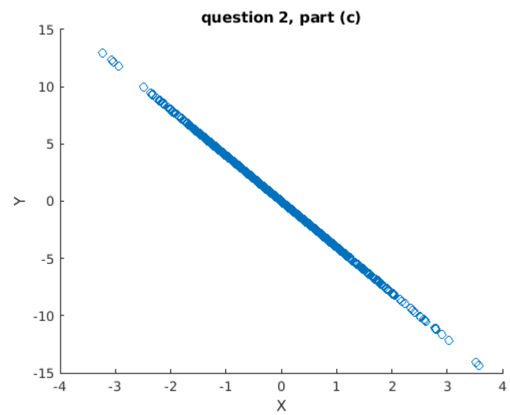
$$\begin{aligned}
\rho_{X,Y} &= \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \\
&= \frac{a}{\sqrt{a^2}} = \frac{a}{|a|} = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases} \\
\rho_{X,Z} &= \frac{0}{\sqrt{2\sigma^4}} = 0 \\
\rho_{X,W} &= \frac{3}{\sqrt{15}}
\end{aligned}$$

(b) For the case of $a = 5$ and $b = 0$, randomly sample X 1000 times, use each X to get Y, and use a scatter plot to plot (X, Y) where the values of X is along the x-axis and the values of Y are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot.
Solution:



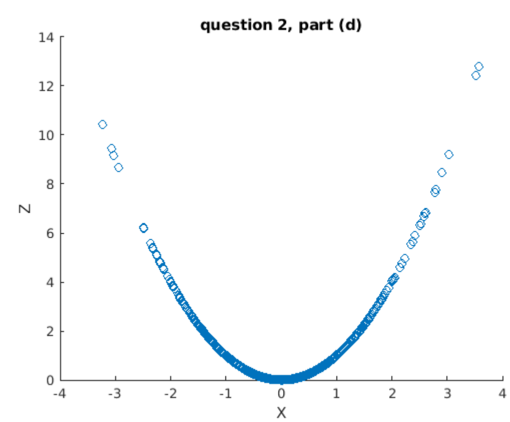
We can see that having a positive correlation coefficient implies that X and Y are positively correlated which means that when one increases, the other one is also likely to increase. Even more strongly, when the coefficient is 1, it means that X and Y actually have a linear relationship to each other. That is why you see a straight line in the figure.

(c) For the case of $a = -4$ and $b = 0$, randomly sample X 1000 times, use each X to get Y, and use a scatter plot to plot (X, Y) where the values of X is along the x-axis and the values of Y are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot and explain the difference to part (b).
Solution:



Similarly, we can see that having a negative correlation coefficient implies that X and Y are negativel correlated which means that when one increases, the other one is likely to decrease. Additionally,we see that when the coefficient is -1, X and Y also follow a linear relationship to each other only negatively this time.

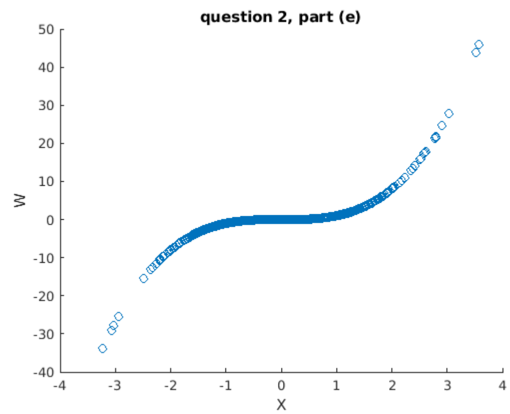
(d) Now, randomly sample X 1000 times, use each X to get Z, and use a scatter plot to plot (X, Z) where the values of X is along the x-axis and the values of Z are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot and explain the difference to parts (b) and (c).
Solution:



Now, we can see that the plot has Z both decreasing and increasing as X increases. This is what is implied by the correlation coefficient being 0. As such, when the coefficient is 0, Z can, on average, either decrease or increase as X increases.

(e) Now, randomly sample X 1000 times, use each X to get W, and use a scatter plot to plot (X, W) where the values of X is along the x-axis and the values of W are along the y-axis. Describe how the correlation coefficient relates to what you see in the scatter plot.

Solution:



Here we see that the correlation coefficient is positive but not 1. As such, we can clearly see a positive trend since W increases as X increases. Yet, the trend is clearly not linear which is why the correlation coefficient is not 1.

3. Let X and Y be two random variables with identical distributions. These two random variables are not necessarily independent. Answer the following questions given that $C = aX + bY$ and $D = aX - bY$.

(a) Find $COV[C, D]$ in terms of the variances and covariances of X and Y

Solution:

$$
\begin{aligned}
COV[C, D] &= COV[aX + bY, aX - bY] \\
&= E[(aX + bY)(aX - bY)] - E[aX + bY]E[aX - bY] \\
&= E[a^2X^2 - b^2Y^2] - (aE[X] + bE[Y])(aE[X] - bE[Y]) \\
&= a^2E[X^2] - b^2E[Y^2] - a^2E[X]^2 + b^2E[Y]^2 \\
&= a^2VAR[X] - b^2VAR[Y] \\
&= (a^2 - b^2) * VAR[X]
\end{aligned}
$$

(b) Find the relation between a and b if the random variables C and D are independent

Solution:

Since it is given that C and D are independent, therefore, $COV[C, D] = 0$.

$$
\begin{aligned}
COV[C, D] &= (a^2 - b^2) * VAR[X] \\
0 &= (a^2 - b^2) * VAR[X]
\end{aligned}
$$

Since $\text{Var}(X) > 0$ (since X is a non-constant random variable), we must have either $a + b = 0$ or $a - b = 0$. If $a + b = 0$, then $a = -b$, and if $a - b = 0$, then $a = b$. Therefore, $a = \pm b$

4. Answer the following. **Show all your work.**

   (a) Consider two random variables X and Y. Prove that the correlation coefficient $\rho_{X,Y}$ satisfies $-1 \leq \rho_{X,Y} \leq 1$.
   Solution:

$$\begin{aligned}
0 &\leq \mathbb{E}[(\frac{X - \mathbb{E}[X]}{\sigma_X} \pm \frac{Y - \mathbb{E}[Y]}{\sigma_Y})^2] \\
&= \mathbb{E}[(\frac{X - \mathbb{E}[X]}{\sigma_X})^2] \pm 2\mathbb{E}[\frac{(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])}{\sigma_X \sigma_Y}] + \mathbb{E}[(\frac{Y - \mathbb{E}[Y]}{\sigma_Y})^2] \\
&= 1 \pm 2\rho_{X,Y} + 1 = 2(1 \pm \rho_{X,Y}) \\
&\implies -1 \leq \rho_{X,Y} \leq 1
\end{aligned}$$

   (b) Let X be a random variable, and Y be another random variable given by $Y = aX + b$. What is the correlation coefficient between X and Y. Does the answer depend on the sign of a?
   Solution:
   The correlation coefficient is given by:

$$\rho_{X,Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}.$$

   We have that,

$$\begin{aligned}
E[XY] &= E[aX^2 + bX] = aE[X^2] + bE[X], \\
E[X]E[Y] &= E[X](aE[X] + b) = aE[X]^2 + bE[X], \\
COV[X, Y] &= E[XY] - E[X]E[Y] = aE[X^2] + bE[X] - aE[X]^2 - bE[X] = aVAR[X],
\end{aligned}$$

   Next, $\sigma_Y = \sqrt{VAR[aX + b]} = \sqrt{a^2 VAR[X]} = a\sigma_X$, if $a > 0$ and $-a\sigma_X$, if $a < 0$. and,

$$\sigma_X \sigma_Y = \pm a\sigma_X^2 = \pm aVAR[X].$$

   Thus, if $VAR[X] \neq 0$,

$$\rho_{X,Y} = \begin{cases} \frac{aVar[X]}{aVar[X]} = 1 \text{ if } a > 0 \\ \frac{aVar[X]}{-aVar[X]} = -1 \text{ if } a < 0 \end{cases}.$$

   Clearly, the sign of a does matter.

5. Let X and Y be jointly Gaussian random variables with $\mathbb{E}[Y] = 0$, $\sigma_X = 4$, $\sigma_Y = 3$ and $\mathbb{E}[X|Y] = \frac{4Y}{9} + 2$. Find the joint pdf of X and Y.

Solution:
Let m_x and m_y be the means of X and Y. By the definition of marginal distributions for jointly Gaussian random variables, we know that

$$\mathbb{E}[X|Y] = \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(y - m_y) + m_x.$$

As such,

$$\mathbb{E}[X|Y] = \frac{4Y}{9} + 2 \implies m_x = 2, \rho_{X,Y} = \frac{1}{3}.$$

Now we have all the parameters to find the joint pdf of X and Y which is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\exp[\frac{1}{2(1-\rho_{X,Y}^2)}((\frac{x-m_x}{\sigma_X})^2 - 2\rho_{X,Y}(\frac{x-m_x}{\sigma_X})(\frac{y-m_y}{\sigma_Y}) + (\frac{y-m_y}{\sigma_Y})^2)]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} \\ &= \frac{\exp[\frac{1}{2(\frac{8}{9})}((\frac{x-2}{4})^2 - \frac{(x-2)(y)}{18} + (\frac{y}{3})^2)]}{2\pi(12)\sqrt{\frac{8}{9}}} \\ &= \frac{\exp[\frac{9}{16}((\frac{x-2}{4})^2 - \frac{(x-2)y}{18} + (\frac{y}{3})^2)]}{16\pi\sqrt{2}} \end{aligned}$$

6. **Q3 from HW6**
Let X and Y be two jointly continuous random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} 6xy, & 0 \le x \le 1, 0 \le y \le \sqrt{x}, \\ 0, & \text{otherwise,} \end{cases}$$

(a) Find $f_X(x)$.
Solution:
For $0 \le x \le 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^{\sqrt{x}} 6xydy \\ &= 3x^2 \end{aligned}$$

Thus

$$f_X(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find the conditional pdf of X given $Y = y$, $f_{X|Y}(x|y)$.
Solution:

For $0 \le y \le 1$,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_{y^2}^{1} 6xydx \\ &= 3y(1-y^4) \end{aligned}$$

Thus

$$f_Y(y) = \begin{cases} 3y(1-y^4), & 0 \le y \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{2x}{1-y^4} & y^2 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(c) Find $E[X|Y=y]$, for $0 \le y < 1$. What is $E[X|Y]$?
Solution:
For $0 \le y < 1$

$$\begin{aligned} E[X|Y=y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx \\ &= \int_{y^2}^{1} x\frac{2x}{1-y^4}dx \\ &= \frac{2(1-y^6)}{3(1-y^4)} \end{aligned}$$

$E[X|Y] = \frac{2(1-Y^6)}{3(1-Y^4)}$

(d) Let A be the event $\{X \ge \frac{1}{2}\}$. Find $P[A]$, $f_{X|A}(x)$, and $E[X|A]$.
Solution:

$$P(A) = \int_A f_X(x)dx = \int_{\frac{1}{2}}^{1} 3x^2dx = \frac{7}{8}$$

$$f_{X|A}(x|A) = \begin{cases} \frac{f_X(x)}{P[A]} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|A}(x|A) = \begin{cases} \frac{24x^2}{7} & \text{if } \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$E[X|A] = \int_{\infty}^{\infty} x f_{X|A}(x|A)dx = \int_{\frac{1}{2}}^{1} x\frac{24x^2}{7}dx = \left[\frac{6x^4}{7}\right]_{\frac{1}{2}}^{1} = \frac{45}{56}$$

Appendix: Reference code for Q2

(a)

```
%% Part (b)
X = randn(1000,1);
a = 5;
Y = a*X;
figure()
scatter(X,Y)
xlabel("X")
ylabel("Y")
title("question 2, part (b)")
%% Part (c)
X = randn(1000,1);
a = -4;
Y = a*X;
figure()
scatter(X,Y)
xlabel("X")
ylabel("Y")
title("question 2, part (c)")

%% Part (d)
X = randn(10000,1);
Z = X.^2;
figure()
scatter(X,Z)
xlabel("X")
ylabel("Z")
title("question 2, part (d)")

%% Part (e)
X = randn(100,1);
W = X.^3;
figure()
scatter(X,W)
xlabel("X")
ylabel("W")
title("question 2, part (e)")

%%%
```