EE 131A
Probability
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Homework 6 Solutions
Due: February 28, 2023
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Chapters 4.6-4.9 & 5.1-5.5 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. The input X to a communication channel is "-1" or "1", with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$.The output of the channel Y is given by

$$Y = \begin{cases} X & \text{with probability; } 1 - p - p_e \\ -X & \text{with probability; } p \\ 0 & \text{with probability; } p_e \end{cases}$$

   (a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).
   Solution:
   Input $X \in \{-1, 1\}$ and output $Y \in \{-1, 0, 1\}$, then the mapping from S to S_{XY} is all possible combinations of (X, Y) pairs:

$$S_{XY} = \{(-1, -1), (-1, 0), (-1, 1), (1, -1), (1, 0), (1, 1)\}$$

   (b) Find the probabilities for all values of (X, Y).
   Solution:
   The probabilities for all values of (X, Y) are given by:
   $P[X = -1, Y = -1] = (1 - p - p_e)/4$
   $P[X = -1, Y = 0] = p_e/4$
   $P[X = -1, Y = 1] = p/4$
   $P[X = 1, Y = -1] = 3p/4$
   $P[X = 1, Y = 0] = 3p_e/4$
   $P[X = 1, Y = 1] = 3(1 - p - p_e)/4$

   (c) Find $P[X \neq Y]$, $P[Y = 0]$.
   Solution:
   $P[X \neq Y] = p_e/4 + p/4 + 3p/4 + 3p_e/4 = p + p_e$
   $P[Y = 0] = p_e/4 + 3p_e/4 = p_e$

2. Suppose you are planning to build a fence for your backyard. You have a long piece of land with a fixed length L. You want to place two fence posts on opposite sides of the midpoint of the land. The positions of the fence posts are chosen randomly, such that one post is uniformly distributed over the first half of the land, and the other post is uniformly distributed over the second half of the land. What is the probability that the distance between the two fence posts is greater than one-third of the total length of the land?

   (Assume that placing one fence is independent of the other)

Solution: Let X and Y be random variables denoting the position of fence post 1 and fence post 2, respectively. Therefore, $X \sim U(0, \frac{L}{2})$ and $Y \sim U(\frac{L}{2}, L)$

$$f(x) = \frac{1}{\frac{L}{2} - 0} = \frac{2}{L} \qquad ; 0 \leq x \leq \frac{L}{2}$$

$$f(y) = \frac{1}{L - \frac{L}{2}} = \frac{2}{L} \qquad ; \frac{L}{2} \leq y \leq L$$

Since they are independent, we can express the joint pdf as:

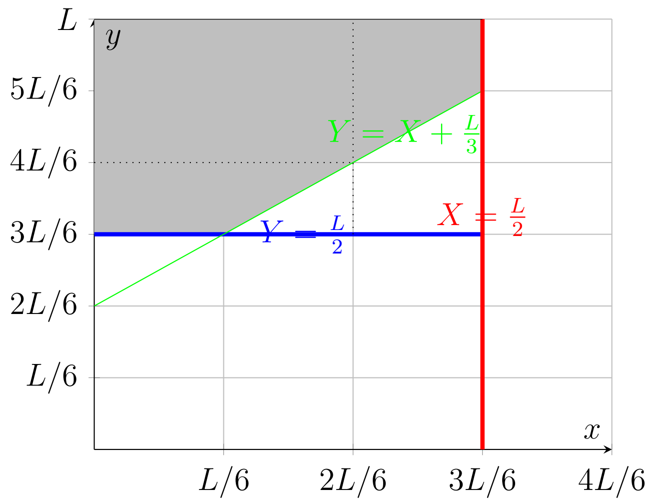
$$f(x, y) = f(x)f(y)$$

Therefore,

$$f(x, y) = \begin{cases} \frac{4}{L^2} & 0 \leq x \leq \frac{L}{2} \quad \& \quad \frac{L}{2} \leq y \leq L \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
P(Y - X > \frac{L}{3}) &= \int \int_{y-x>\frac{L}{3}} f(x, y) dy dx \\
&= \int_{x=0}^{\frac{L}{2}} \int_{y=\max\{x+\frac{L}{3}, \frac{L}{2}\}}^{L} f(x, y) dy dx \\
&= \int_0^{\frac{L}{6}} \int_{\frac{L}{2}}^{L} \frac{4}{L^2} dy dx + \int_{\frac{L}{6}}^{\frac{L}{2}} \int_{x+\frac{L}{3}}^{L} \frac{4}{L^2} dy dx \\
&= \int_0^{\frac{L}{6}} \frac{4}{L^2}(L - \frac{L}{2}) dx + \int_{\frac{L}{6}}^{\frac{L}{2}} \frac{4}{L^2}(L - \frac{L}{3} - x) dx \\
&= \int_0^{\frac{L}{6}} \frac{2}{L} dx + \int_{\frac{L}{6}}^{\frac{L}{2}} \frac{8}{3L} - \frac{4x}{L^2} dx \\
&= \frac{2}{L}\frac{L}{6} + \left[\frac{8x}{3L} - \frac{2x^2}{L^2}\right]_{\frac{L}{6}}^{\frac{L}{2}} \\
&= \frac{1}{3} + \frac{8}{3L}\frac{L}{3} - \frac{2}{L^2}(\frac{L^2}{4} - \frac{L^2}{36}) \\
&= \frac{1}{3} + \frac{8}{9} - \frac{4}{9} \\
&= \frac{7}{9}
\end{aligned}$$

Graphical Approach:



The area between the three curves shown can be calculated as follows:

$$\begin{aligned}
\text{Area between curves} &= \frac{1}{2}\left(\frac{L}{2} + \frac{2L}{6}\right)\frac{L}{6} + \frac{2L}{6}\frac{2L}{6} + \frac{1}{2}\left(\frac{L}{6} + \frac{2L}{6}\right)\frac{L}{6} \\
&= \frac{3L^2}{72} + \frac{8L^2}{72} + \frac{3L^2}{72} \\
&= \frac{14L^2}{72}
\end{aligned}$$

$$\begin{aligned}
\text{Probability} &= \frac{\text{Area between curves}}{\text{Total Area under X and Y}} \\
&= \frac{\frac{14L^2}{72}}{\frac{L}{2}\frac{L}{2}} \\
&= \frac{7}{9}
\end{aligned}$$

3. Let X and Y be two jointly continuous random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 6xy, & 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, \\ 0, & \text{otherwise}, \end{cases}$$

(a) Find $f_X(x)$.
Solution:
Will be released in HW7 solutions

(b) Find the conditional pdf of X given $Y = y$, $f_{X|Y}(x|y)$.
Solution:
Will be released in HW7 solutions

(c) Find $E[X|Y=y]$, for $0 \leq y < 1$. What is $E[X|Y]$?
Solution:
Will be released in HW7 solutions

(d) Let A be the event $\{X \geq \frac{1}{2}\}$. Find $P[A]$, $f_{X|A}(x)$, and $E[X|A]$.
Solution:
Will be released in HW7 solutions

4. Answer the following. **Show all your work.**

(a) Let X be a Poisson random variable with parameter λ. Show that $P(X \geq 2\lambda) \leq \frac{1}{\lambda}$.
Solution:
The Chebyshev inequality is given by:

$$P(|X-m| \geq a) \leq \frac{\sigma^2}{a^2}$$

For a Poisson random variable, $E(X) = \lambda$ and $\sigma^2 = Var(X) = \lambda$. So we have:

$$P(|X-\lambda| \geq \lambda) \leq \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

Since X is a Poisson RV, $X \geq 0$. Hence we have:

$$\frac{1}{\lambda} \geq P(|X-\lambda| \geq \lambda) \geq P(X-\lambda \geq \lambda) = P(X \geq 2\lambda)$$

(b) Let X be a standard normal random variable and $Q(x) = P(X \geq x)$. Use the Chernoff bound to show that $Q(x) \leq e^{-\frac{x^2}{2}}$, $x > 0$.
Solution:
The Chernoff bound of X is given by:

$$Q(x) = P(X \geq x) \leq e^{-sx} E[e^{sX}] \ \ , s > 0$$

Now as X is standard normal, $E[e^{sX}] = e^{\frac{s^2}{2}}$. (Recall: characteristic function for standard Gaussian is $E[e^{i\omega X}] = e^{-\frac{\omega^2}{2}}$. We get $E[e^{sX}]$ by substituting $\omega = \frac{s}{j}$.). Thus the Chernoff bound becomes:

$$Q(x) \leq e^{-sx+\frac{s^2}{2}}$$

The parameter s can be selected to minimize the upper bound. For $x > 0$, the RHS is minimized for $s = x$. Substituting $s = x$, we get

$$Q(x) \leq e^{-\frac{x^2}{2}}$$

5. (Generating Jointly Gaussian Random Variables)

In this question, you will use MATLAB to generate zero mean, unit variance, uncorrelated (and hence independent) jointly Gaussian random variables using the *Box-Muller* method. The Box-Muller method involves the following steps:

- Generate U_1 and U_2, two independent random variables uniformly distributed in the unit interval $[0, 1]$.
- Generate random variables X_1 and X_2 by using the following transformation:

$$X_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2)$$

$$X_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$$

- X_1 and X_2 generated are independent, zero-mean, unit-variance Gaussian random variables.

(a) Generate 5000 samples of independent zero-mean, unit-variance Gaussian random variables X_1 and X_2 by the *Box-Muller* method (i.e by generating 5000 samples of U_1 and U_2 each and then using the transformation to get 5000 samples each of X_1 and X_2).

(b) Plot the histogram of 5000 samples of X_1 and compare it with the pdf of a zero-mean unit variance Gaussian random variable. Repeat the same for X_2.

(c) Plot the samples of X_1 vs X_2 in a scatter plot. Does the plot have a circular symmetry? Explain.

Solution:

(a) See the MATLAB script for the Box-Muller method generation of X_1 and X_2.

(b) The Histogram of X_1 and X_2 are shown in Figure 2. It is clear that it closely matches the standard normal Gaussian pdf.

(c) Scatter plot is shown in Fig 3. It has circular symmetry. This is due to the fact that the joint pdf of X_1 and X_2 is given by

$$f_{X_1X_2}(x_1x_2) = \frac{1}{\sqrt{2\pi}}e^{\frac{-(x_1^2+x_2^2)}{2}}$$

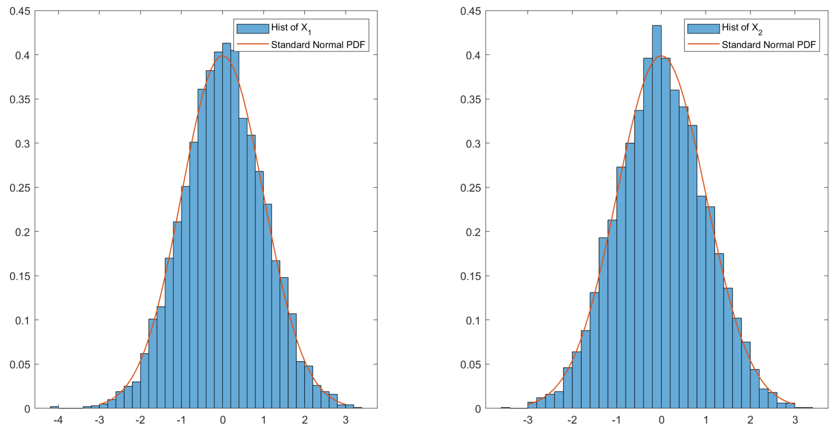


Figure 1: Histogram of X_1 and X_2

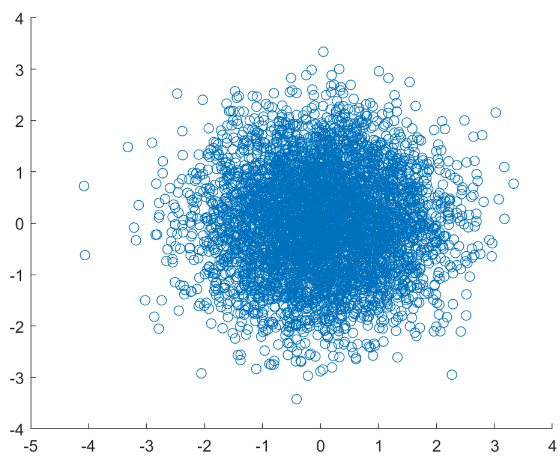


Figure 2: Scatter plot of X_1 vs X_2

Appendix: Code for Reference

(a)
```
% Q 5
U1 = rand(5000,1);
U2 = rand(5000,1);

X1 = sqrt(-2*log(U1)).*cos(2*pi.*U2);
X2 = sqrt(-2*log(U1)).*sin(2*pi.*U2);

pd = makedist('Normal');
x = -3:.1:3;
pdf_normal = pdf(pd,x);

```

```
figure
subplot(1,2,1)
histogram(X1,'Normalization','pdf');
hold on;
plot(x,pdf_normal,'LineWidth',1);
legend('Hist of X_1','Standard Normal PDF')

subplot(1,2,2)
histogram(X2,'Normalization','pdf');
hold on;
plot(x,pdf_normal,'LineWidth',1);
legend('Hist of X_2','Standard Normal PDF')


figure
scatter(X1,X2);
```