ECE113, Fall 2022 **Homework #4 Solutions**

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Due Friday, 3 March 2023, by 11:59pm to Gradescope.
50 points total.

Note: Unless specified, you are free to use any of the properties of DFT that were taught in class. Also, all repeated derivations can be referenced with appropriate equation/result numbers.

1. (5 points) Determine the DFT of the following sequences:

   a.) $x[n] = sin(2\pi\frac{1}{8}n)$ for n = [0,1,...,7]

   b.) $y_1 = [0, -1, 0, 1]$

   c.) $y_2 = [j, 0, j, 1]$

   Solutions:

   a.)

$$X[k] = \sum_{n=0}^{7} sin(\frac{2\pi}{8}n)e^{-j2\pi k\frac{n}{8}} = \frac{1}{2}\sum_{n=0}^{7}(e^{j2\pi\frac{n}{8}} + e^{-j2\pi\frac{n}{8}})e^{-j2\pi k\frac{n}{8}} \tag{1}$$

$$X[k] = \frac{1}{2j}\sum_{n=0}^{7}(e^{-j2\pi\frac{k-1}{8}n} - e^{-j2\pi\frac{k+1}{8}n}) \tag{2}$$

$$X[k] = \begin{cases} 4, \text{ where } k = 1, 7 \\ 0, \text{ elsewhere} \end{cases} \tag{3}$$

$$X = [0, \frac{4}{j}, 0, 0, 0, 0, 0, \frac{-4}{j}] \tag{4}$$

   b.)

$$Y_1[k] = \sum_{n=0}^{3} y_1 e^{-j2\pi k\frac{n}{4}} = \frac{j}{2}\sum_{n=0}^{3}(e^{j\frac{2\pi}{4}n} - e^{-j\frac{2\pi}{4}n})e^{-j2\pi k\frac{n}{4}} \tag{5}$$

$$Y_1[k] = \frac{j}{2}\sum_{n=0}^{7}(e^{-j2\pi\frac{k-1}{4}n} - e^{-j2\pi\frac{k+1}{4}n}) \tag{6}$$

$$Y_1[k] = \begin{cases} 2j, \text{ where } k = 1 \\ -2j, \text{ where } k = 3 \\ 0, \text{ elsewhere} \end{cases} \tag{7}$$

$$Y_1 = [0, 2j, 0, -2j] \tag{8}$$

c.)

$$Y_2[k] = \sum_{n=0}^{3} y_2 e^{-j2\pi k \frac{n}{4}} \tag{9}$$

$Y_2[0] = j(1) + 0(1) + j(1) + 1(1) = 1 + 2j$
$Y_2[1] = j(1) + 0(-j) + j(-1) + 1(j) = j$
$Y_2[2] = j(1) + 0(-1) + j(1) + 1(-1) = -1 + 2j$
$Y_2[3] = j(1) + 0(j) + j(-1) + 1(-j) = -j$

$$Y_2 = [1 + 2j, j, -1 + 2j, -j] \tag{10}$$

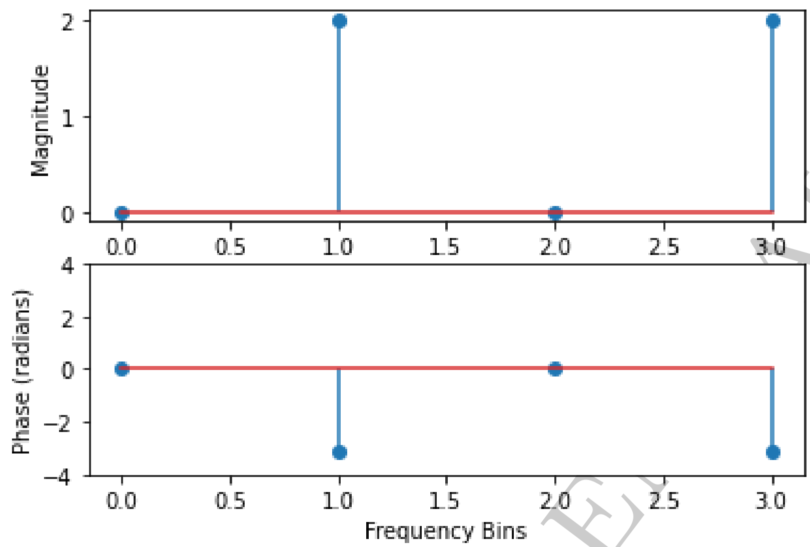
2. (5 points) Plot the the magnitude and phase of the function y_3 in the frequency domain. Where $y_3 = y_1 \otimes y_2$ which is the convolution of the function y_1 and y_2.

Solutions:

$$Y_3[k] = Y_1[k]Y_2[k] \tag{11}$$

$$Y_3 = [0\ (1+2j), 2j(j), 0\ (-1+2j), -2j(-j)] \tag{12}$$

$$Y_3 = [0, -2, 0, -2] \tag{13}$$



(Phase should be $-\pi$)

3. (40 points) Coding Assignment Instructions:

   (a) Find the hw4_coding.zip file on BruinLearn. Download and unzip it.

   (b) The zip contains three files. The main files you will be editing are "hw4.ipynb" and "dsp_toolbox.py". The last file is there for utility only.

   (c) Begin with the Jupyter notebook and follow through the sections linearly (some sections may depend on previous ones). The Jupyter notebook will ask you to implement empty functions in the "dsp_toolbox.py" file. Make sure you only make changes in the code where specifically asked of you by Python comments.

Solutions: Solutions for the coding section of this are located in "hw4_coding_solutions.zip".

Practice Problems

1. (a) Derive the DFT of a sinusoid with an arbitrary frequency, $x[n] = sin(2\pi \frac{f_o}{N} n)$. ($f_o$ can be any value and is not necessarily a rational number).

   (b) The solution to part (a.) looks familiar, as if it were the Fourier transform of the multiplication of two functions in the time domain. What are these functions? What is the connection between this derivation and the leaky DFT spectrums we saw in Part 3.1 of the "hw4_coding" assignment?

Solutions:

(a)

$$\sum_{n=0}^{N-1} sin(2\pi \frac{f_o}{N} n) e^{-j2\pi nk/N} = \frac{-j}{2} \sum_{n=0}^{N-1} (e^{j2\pi \frac{f_o}{N} n} - e^{-j2\pi \frac{f_o}{N} n}) e^{-j2\pi nk/N} \tag{14}$$

$$\frac{-j}{2} \sum_{n=0}^{N-1} (e^{j2\pi \frac{f_o}{N} n} - e^{-j2\pi \frac{f_o}{N} n}) e^{-j2\pi nk/N} = \frac{-j}{2} \sum_{n=0}^{N-1} e^{-j2\pi \frac{k-f_o}{N} n} - e^{-j2\pi \frac{k+f_o}{N} n} \tag{15}$$

Using geometric sums:

$$G_N(k - f_o) = \sum_{n=0}^{N-1} e^{-j2\pi \frac{k-f_o}{N} n} = \frac{a_o(1 - r^N)}{1 - r} = \frac{1 - (e^{-j2\pi \frac{k-f_o}{N}})^N}{1 - e^{-j2\pi \frac{k-f_o}{N}}} \tag{16}$$

$$\frac{1 - e^{-j2\pi(k-f_o)}}{1 - e^{-j2\pi \frac{k-f_o}{N}}} = \frac{e^{-j2\pi(k-f_o)/2}}{e^{-j2\pi \frac{k-f_o}{N}/2}} \cdot \frac{e^{j2\pi(k-f_o)/2} - e^{-j2\pi(k-f_o)/2}}{e^{j2\pi \frac{k-f_o}{N}/2} - e^{-j2\pi \frac{k-f_o}{N}/2}} \tag{17}$$

$$e^{-j2\pi(k-f_o)\frac{N-1}{2N}} \cdot \frac{sin(2\pi(k-f_o)/2)}{sin(2\pi \frac{k-f_o}{N}/2)} \tag{18}$$

Then plugging back in:

$$\frac{-j}{2} \sum_{n=0}^{N-1} e^{-j2\pi \frac{k-f_o}{N} n} - e^{-j2\pi \frac{k+f_o}{N} n} = \frac{-j}{2}(G_N(k - f_o) + G_N(k + f_o)) \tag{19}$$

(b) The fourier transform of a sinusoid multiplied by a rect function. When we take the DFT of a sinusoid whose frequency is not k/N (where k is an integer and N is the length of the sequence) we will not get 2 perfect delta functions. Instead we will begin to see the sinc function that is convolved with the delta functions that is expected in the DTFT domain but hidden in the discrete frequency domain when the sinusoid's frequency is k/N.

2. Evaluate the following expressions that are composed of the arbitrary sequence $x[n]$. Your final answer should be in a functional form with $X[k]$ being in the expression.

   (a) $DFT(DFT(DFT(DFT(x[n]))))$

   (b) $DFT(x[-n])$

   Solutions:

   (a) The $DFT(x[n]) = X[k]$ and the $IDFT(X[k]) = x[n]$. Then the $DFT(X[n])$ is:

$$DFT(X[n]) = \sum_{k=0}^{N-1} X[k]e^{-j2\pi nk/N} = N(\frac{1}{N}\sum_{n=0}^{N-1} X[n]e^{j2\pi k(N-n)/N}) \tag{20}$$

$$DFT(X[n]) = N\ IDFT(X[N-k\ mod\ N]) = N\ IDFT(X[-k]) = Nx[-n] \tag{21}$$

   Then we can evaluate 4 back to back DFTs as:

$$DFT(DFT(DFT(DFT(x[n])))) = DFT(DFT(Nx[-n])) = N^2x[n] \tag{22}$$

   (b)

$$DFT(x[-n]) = \sum_{n=0}^{N-1} x[-n]e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x[n]e^{j2\pi nk/N} = X[-k] \tag{23}$$