ECE113, Fall 2022 | **Practice Final**
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ECE113 Practice Final Solutions

1. **Problem 1**
   Consider the arbitrary signal $x[n]$ whose DTFT is $X(\Omega)$. Find the DTFT of the following signals as a function of $X(\Omega)$.

   (a) $x_1[n] = nx[n-1]$

   (b) $x_2[n] = e^{j\frac{\pi n}{2}}(x[n] * x[n])$

   (c) $x_o[n]$, the odd part of $x[n]$

   Solutions: Use the DTFT table to solve these.

   (a) $X_1(\Omega) = j\frac{d}{d\Omega}(X(\Omega)e^{-j\Omega})$

   (b) $X_2(\Omega) = X^2(\Omega - \pi/2)$

   (c) $X_o(\Omega) = \frac{X(\Omega)-X(-\Omega)}{2}$

2. **Problem 2**

(a) Find an expression for the inverse discrete time Fourier transform (DTFT) of:

$$X(\Omega) = cos^2(\Omega)$$

(b) Find the DTFT of:

$$x[n] = \frac{sinc(n/4)}{4}\frac{sinc(n/2)}{2}$$

Solutions:

(a)

$$X(\Omega) = cos^2(\Omega) = \frac{1}{4}(e^{j\Omega} + e^{-j\Omega})^2 = \frac{1}{4}(e^{2j\Omega} + e^{-2j\Omega} + 2)$$

Then using DTFT pairs:

$$x[n] = \frac{1}{4}(\delta[n+2] + \delta[n-2] + 2\delta[n])$$

(b) Assume,

$$x_1[n] = \frac{sinc(n/4)}{4} \quad and \quad x_2[n] = \frac{sinc(n/2)}{2}$$

Then,

$$X_1(\Omega) = \begin{cases} 1, & |\Omega| \leq \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

$$X_2(\Omega) = \begin{cases} 1, & |\Omega| \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

Using the DTFT properties table, we see that multiplication of the 2 functions in the time domain results in convolution in the time domain:

$$X(\Omega) = \begin{cases} \frac{\Omega}{2\pi} + 3/8, & -3\pi/4 \leq \Omega \leq -\pi/4 \\ 1/4, & -\pi/4 \leq \Omega \leq \pi/4 \\ \frac{-\Omega}{2\pi} + 3/8, & \pi/4 \leq \Omega \leq 3\pi/4 \\ 0, & |\Omega| > 3\pi/4 \end{cases}$$

3. **Problem 3** (20 points)
   Let $x[n]$ be a signal with non-zero values from $n = 0, 1, \cdots, N-1$. Assume that $x[n] = 0$ for $n > N-1$ and for $n < 0$.
   Let $y_M[n]$ be an M length finite version of $x[n]$.

$$y_M[n] = \begin{cases} x[n], & 0 \leq n < N \\ 0, & N \leq n < M \end{cases}$$

Show that the M point DFT of y[n] satisfies

$$Y_M[k] = X\left(\frac{2\pi k}{M}\right), \text{ for } k = 0, 1, ..., M-1.$$

where $X(\Omega)$ is the DTFT of x[n].

Solutions: Let's take the DTFT of the signal x[n],

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}$$

and the DFT of the finite-duration signal $y_M[n]$,

$$Y_M[k] = \sum_{n=0}^{M-1} y_M[n]e^{-j2\pi \frac{k}{M} n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi \frac{k}{M} n}$$

Then, we can see that:

$$Y_M[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi \frac{k}{M} n} = X\left(\frac{2\pi k}{M}\right)$$

4. **Problem 4**
Assume $x[n] = cos(2\pi\frac{3}{10}n)$ is a $N = 10$ length signal. Similar to Problem 4, $y[n]$ is the zero-padded version of $x[n]$ where the length of $y[n]$ is $M = 20$.

   (a) What is the DFT of x[n]? Plot the magnitude and phase. (you should not be using a calculator here)

   (b) Describe how the DFT of $y[n]$ compares to the DFT of $x[n]$. Justify your reasoning.

   (c) Sketch the magnitude of the DFT of y[n]. (Just a sketch, you do not have to compute the DFT of y[n]).

   (d) We now have another signal, $g[n] = cos(2\pi\frac{3.14159}{10}n)$ which is the same length as $x[n]$. Can you easily get the DFT of $g[n]$ by hand as you did in part (a) without a calculator? Justify why you can or cannot.

   (e) Sketch the magnitude of the DFT of $g[n]$, $|G[k]|$ and comment on the features that make it look different from the magnitude spectrum of $|X[k]|$.

Solutions:

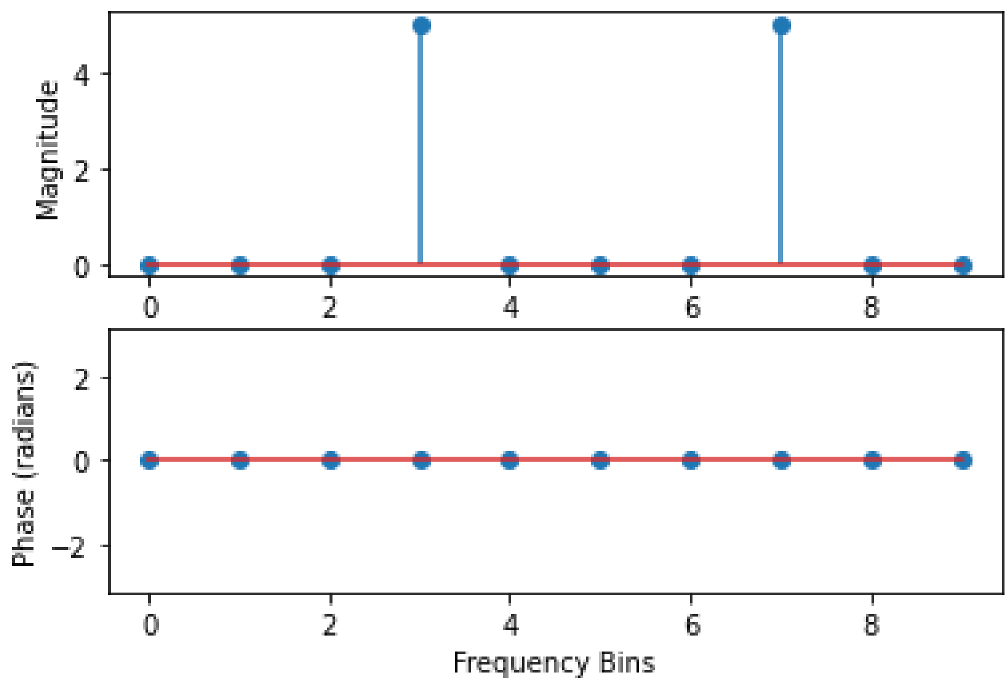
(a)

$$x[n] = cos(2\pi\frac{3}{10}n) = \frac{1}{2}\left(e^{j2\pi\frac{3}{10}n} + e^{-j2\pi\frac{3}{10}n}\right)$$

$$X[k] = \sum_{n=0}^{9}\frac{1}{2}\left(e^{j2\pi\frac{3}{10}n} + e^{-j2\pi\frac{3}{10}n}\right)e^{-j2\pi\frac{k}{10}n} = 5(\delta[k+3] + \delta[k-3])$$

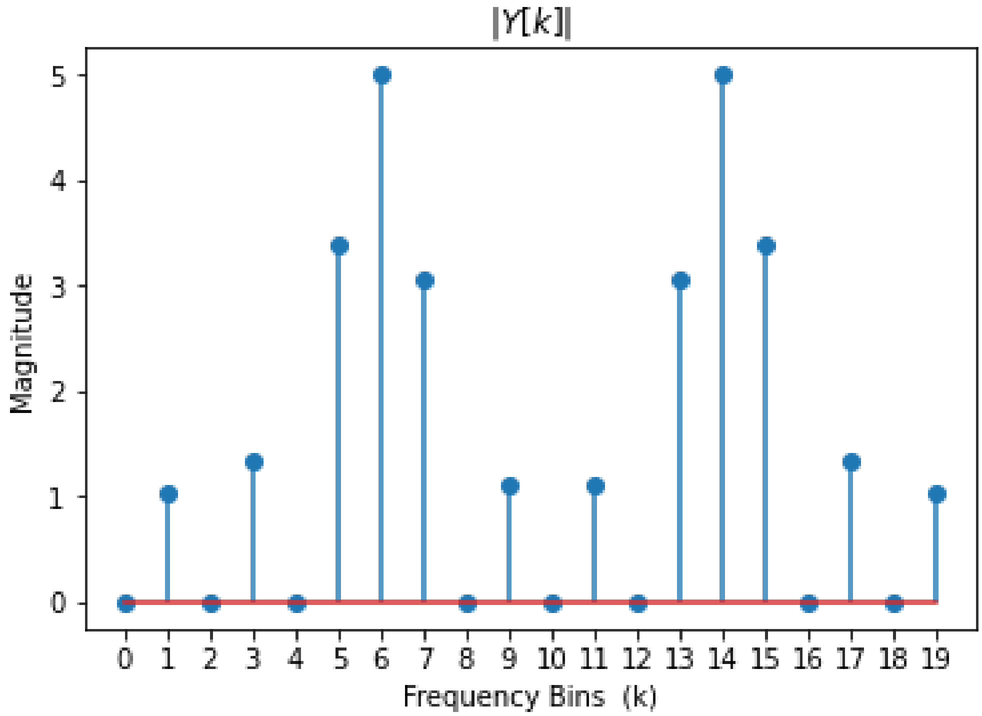
$$\text{But } \delta[k+3] = \delta[k+3-10] = \delta[k-7]$$

$$X[k] = [0,0,0,5,0,0,0,5,0,0]$$



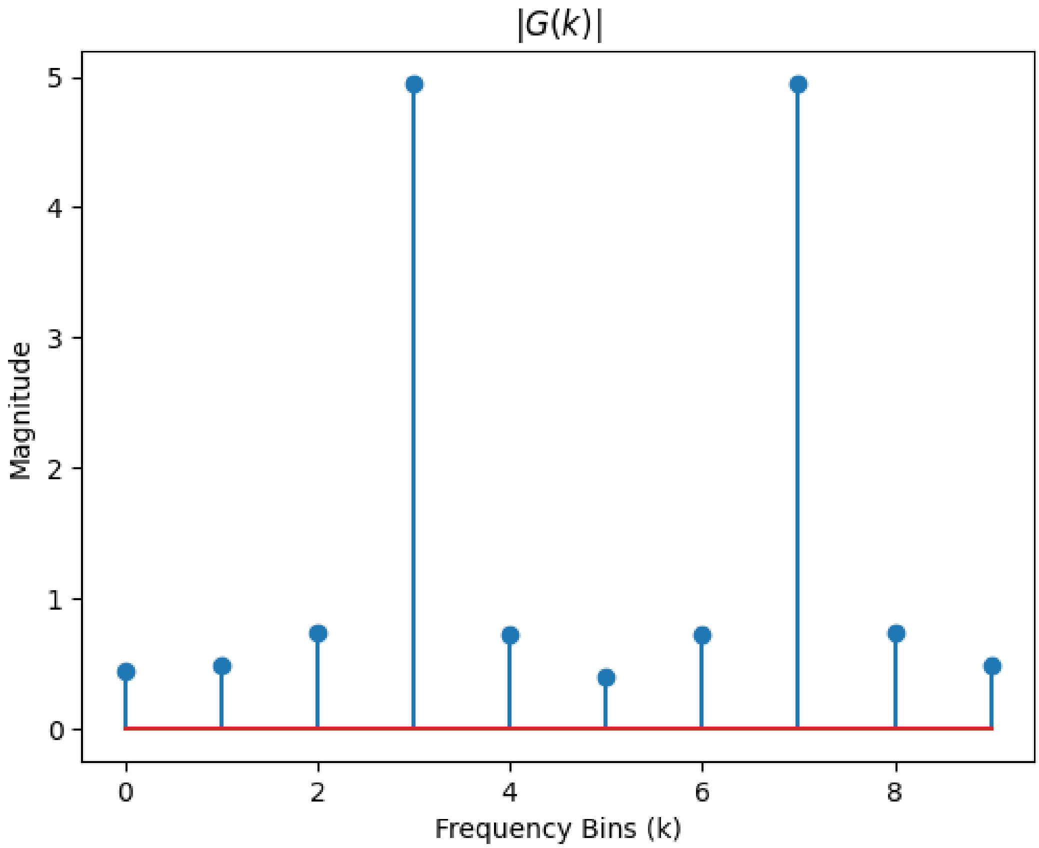
(b) We can predict what the DFT of $y[n]$ will look like by first thinking about its DTFT. The DTFT of an infinite length sinusoid has 2 delta functions. But, in this case, the sinusoid is constrained to 10 samples, with all other samples being zero. This is equivalent to the following infinite length signal: $p[n] = cos(2\pi\frac{3}{10}n)(u[n] - u[n-10])$. The DTFT of this signal, $P(\Omega)$, would be equivalent to $X(\Omega)$. Therefore, we just need to split $p[n]$ into two functions, $p_1[n] = cos(2\pi\frac{3}{10}n)$ and $p_2[n] = u[n] - u[n-10]$. Their multiplication in the time domain, results in convolution in the frequency domain. Since, $P_1(\Omega)$ is 2 delta functions and $P_2(\Omega)$ is a sinc function (in the DTFT table, it is the DTFT pair named 'Discrete-time pulse'), $P(\Omega)$ will look similar to two sinc functions centered at the locations of the 2 delta functions (due to the convolution of $P_1(\Omega)$ and $P_2(\Omega)$. Once we have understood this, we know that $y[n]$ is $p_1[n] \cdot p_2[n]$ for $n = 0, 1, 2..., M-1$. The DFT of $y[n]$ samples the DTFT, so we can predict that the DFT will look similar to two sinc functions centered at the locations of the original 2 delta functions.

(c) Plot Below. Anything that has leakage in the spectrum or looks like sinc functions interpolated with delta functions should get credit.



(d) The reason we cannot use the same trick we used in part (a) to evaluate the DFT of $g[n]$ is that the frequency of the cosine function is not an integer, $k = 3.14159$. (The fact that the frequency is $\pi = 3.14159$ has nothing to do with it) Since its not an integer frequency, none of the DFT's complex exponentials will be orthogonal to it and evaluate to zero. This is a sufficient answer for an exam, if you wanted to go above and beyond, you can talk about how this relates to what you derived in HW4 Practice Problems 1 (a) and 1 (b).

(e) We can think about this problem in a similar way to part (b). In reality we are sampling the DTFT of this signal which is a sinc function. However, since the frequency is not an integer frequency, the sinc function's zeros do not perfectly land on most of the frequencies we are sampling like in the DFT of part (a). The non-integer frequency makes the sinc function visible in the magnitude spectrum.


$|G(k)|$
5
4
3
2
1
0
Magnitude
0
2
4
6
8
Frequency Bins (k)

5. **Problem 5**
Consider the z transform of the signal $h[n]$:

$$H(z) = \frac{(1 - z^{-1})(1 + z^{-1})}{(1 - 0.9e^{\mathrm{j}\frac{5\pi}{6}} z^{-1})(1 - 0.9e^{-\mathrm{j}\frac{5\pi}{6}} z^{-1})(1 - 0.9e^{\mathrm{j}\frac{2\pi}{3}} z^{-1})(1 - 0.9e^{-\mathrm{j}\frac{2\pi}{3}} z^{-1})}$$

Plot the poles and zeros for $H(z)$ on the z-plane. Also, plot the approximate DTFT/DFT spectrum for the sequence $h[n]$.

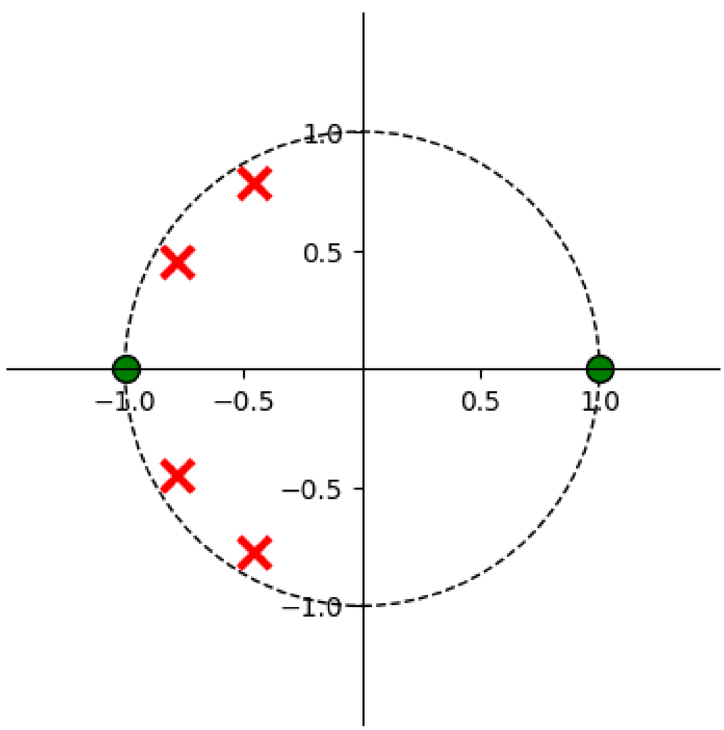
Solutions:

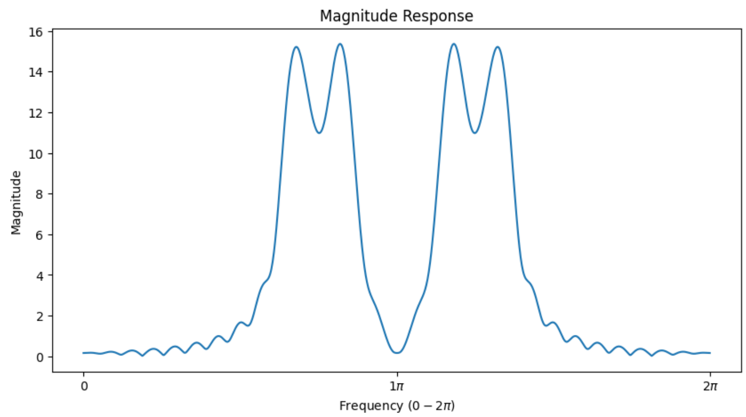
To find the poles, we find the roots of the terms in the denominator. Similarly, to find the zeros, we find the roots of the terms in the numerator.

From the lectures, we know that $z = a$ is a root for the term $(1 - az^{-1})$. Factoring this logic into our analysis of $H(z)$, we observe that the root of the numerator is $z = 1, z = -1$. This tells us that the **zeros** of the transfer function are at $z = \pm 1$.

Extending our analysis to the denominator, we obtain the following roots, i.e., our **poles**:

$$z = 0.9e^{\mathrm{j}\frac{5\pi}{6}}$$
$$z = 0.9e^{-\mathrm{j}\frac{5\pi}{6}}$$
$$z = 0.9e^{\mathrm{j}\frac{2\pi}{3}}$$
$$z = 0.9e^{-\mathrm{j}\frac{2\pi}{3}}$$




Magnitude Response
Magnitude
Frequency $(0-2\pi)$
0
1π
2π
16
14
12
10
8
6
4
2
0