EE 131A | Homework 6
Probability | Due: February 28, 2023
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Chapters 4.6-4.9 & 5.1-5.5 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. The input X to a communication channel is "-1" or "1", with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$.The output of the channel Y is given by

$$Y = \begin{cases} X & \text{with probability; } 1-p-p_e \\ -X & \text{with probability; } p \\ 0 & \text{with probability; } p_e \end{cases}$$

   (a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

   (b) Find the probabilities for all values of (X, Y)

   (c) Find $P[X \neq Y]$, $P[Y = 0]$.

2. Suppose you are planning to build a fence for your backyard. You have a long piece of land with a fixed length L. You want to place two fence posts on opposite sides of the midpoint of the land. The positions of the fence posts are chosen randomly, such that one post is uniformly distributed over the first half of the land, and the other post is uniformly distributed over the second half of the land. What is the probability that the distance between the two fence posts is greater than one-third of the total length of the land?

   (Assume that placing one fence is independent of the other)

3. Let X and Y be two jointly continuous random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} 6xy, & 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, \\ 0, & \text{otherwise,} \end{cases}$$

   (a) Find $f_X(x)$.

   (b) Find the conditional pdf of X given $Y = y$, $f_{X|Y}(x|y)$.

   (c) Find $E[X|Y = y]$, for $0 \leq y < 1$. What is $E[X|Y]$?

   (d) Let A be the event $\{X \geq \frac{1}{2}\}$. Find $P[A]$, $f_{X|A}(x)$, and $E[X|A]$.

4. Answer the following. **Show all your work.**

   (a) Let X be a Poisson random variable with parameter λ. Show that $P(X \geq 2\lambda) \leq \frac{1}{\lambda}$.

(b) Let X be a standard normal random variable and $Q(x) = P(X \geq x)$. Use the Chernoff bound to show that $Q(x) \leq e^{-\frac{x^2}{2}}$, $x > 0$.

5. (Generating Jointly Gaussian Random Variables)

In this question, you will use MATLAB to generate zero mean, unit variance, uncorrelated (and hence independent) jointly Gaussian random variables using the *Box-Muller* method. The Box-Muller method involves the following steps:

- Generate U_1 and U_2, two independent random variables uniformly distributed in the unit interval $[0, 1]$.
- Generate random variables X_1 and X_2 by using the following transformation:

$$X_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2)$$

$$X_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$$

- X_1 and X_2 generated are independent, zero-mean, unit-variance Gaussian random variables.

(a) Generate 5000 samples of independent zero-mean, unit-variance Gaussian random variables X_1 and X_2 by the *Box-Muller* method (i.e by generating 5000 samples of U_1 and U_2 each and then using the transformation to get 5000 samples each of X_1 and X_2).

(b) Plot the histogram of 5000 samples of X_1 and compare it with the pdf of a zero-mean unit variance Gaussian random variable. Repeat the same for X_2.

(c) Plot the samples of X_1 vs X_2 in a scatter plot. Does the plot have a circular symmetry? Explain.