EC ENGR 131A
Probability and Statistics
Instructor: Lara Dolecek

Winter 2023
Thursday, February 16th, 2023
Exam 2 - Version 3

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

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Discussion Time and Day: Friday 5 ~~Friday~~
with Rushi

Problem	Score	Possible
1	25	25
2	25	25
3	25	25
4	8	25
Total	83	100

$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = \cancel{\ldots}$
$-x^{-2}$

1. (a) Find the characteristic function of the uniform continuous random variable, distributed uniformly on the interval $[-5, 5]$. *(10 points)*

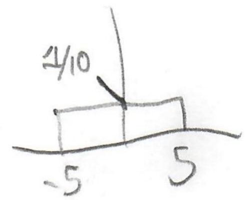
   (b) Find the mean of X by applying the moment theorem. *(15 points)*

Hint: $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ *(L'Hôpital's rule)*

Clearly show all steps.

$10 \cdot y = 1$
$y = 1/10$

ⓐ URV $\to [-5, 5]$



PDF: $f_X(x) = \begin{cases} 1/10, & -5 \le x \le 5 \\ 0, & \text{ow.} \end{cases}$

$\Phi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx = \int_{-5}^{5} e^{j\omega x} \cdot \frac{1}{10}\,dx$

$= \frac{1}{10} \cdot \frac{1}{j\omega} e^{j\omega x}\Big|_{-5}^{5} = \frac{1}{10j\omega}\left(e^{5j\omega} - e^{-5j\omega}\right)$ ✓

ⓑ $\frac{d\,\Phi_X(\omega)}{d\omega}\Big|_{\omega=0} = jE[X] = \cancel{\frac{1}{10j}(-\omega^{-2})\left(e^{5j\omega} - e^{-5j\omega}\right) + \frac{1}{10j\omega}\left(5je^{5j\omega} + 5je^{-5j\omega}\right)}\Big|_{\omega=0}$

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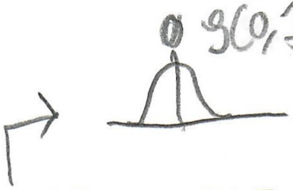
$\cancel{\lim_{\omega \to 0}\left(\frac{j}{10}\omega^{-2}\left(e^{5j\omega} - e^{-5j\omega}\right) + \frac{1}{10j\omega}(5e\right)}$

$\therefore jE[X] = \frac{d}{d\omega}\Big|_{\omega=0} \frac{e^{5j\omega}}{10j\omega} - \frac{d}{d\omega}\Big|_{\omega=0} \frac{e^{-5j\omega}}{10j\omega} = \frac{1}{2}\left(\frac{d}{d\omega}\Big|_{\omega=0}\left(\frac{e^{5j\omega}}{j\omega} - \frac{e^{-5j\omega}}{j\omega}\right)\right)$

$= \frac{1}{2}\lim_{\omega \to 0}\left(\frac{5je^{5j\omega} \cdot j\omega - e^{5j\omega} \cdot j}{(j\omega)^2} + \frac{5je^{-5j\omega}j\omega + je^{-5j\omega}}{(j\omega)^2}\right) = \frac{1}{2}\lim_{\omega \to 0}\left(\frac{j(5j\omega - 1)e^{5j\omega}}{(j\omega)^2} + \frac{j(5j\omega + 1)e^{-5j\omega}}{(j\omega)^2}\right)$

$= \frac{1}{2}\lim_{\omega \to 0}\left(\frac{1}{j\omega^2}\left((5j\omega - 1)e^{5j\omega} + (5j\omega + 1)e^{-5j\omega}\right)\right) = \frac{1}{2} \cdot \frac{1}{j \cdot 0} \cdot (-1 + 1) = \frac{0}{0} \therefore$

$= \frac{1}{2}\lim_{\omega \to 0}\left(\frac{d}{d\omega}\left(\frac{1}{j\omega^2}\left((5j\omega - 1)e^{5j\omega}\right) + (5j\omega + 1)e^{-5j\omega}\right)\right) = \boxed{0}$ ✓



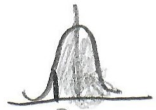
2. Suppose X is a standard Gaussian RV. Express the following functions in terms of the Q functions:

(a) $P(X > -3)$. *(7 points)*

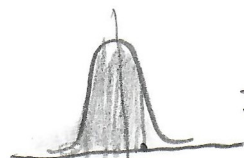
(b) $P(X \leq 2)$. *(7 points)*

(c) $P(-2 < X < 1)$. *(11 points)*

$\overset{\text{CDF}}{Q = 1 - F_X(x)}$

(a) 

$= Q(-3)$

(b) 

$= 1 - Q(2) = Q(-2)$

(c) 

$= \cancel{\Phi(1) - Q(-2) = 1 - Q(1) - Q(-2)}$

$\cancel{= Q(-1) - Q(-2)}$

$= \Phi(1) - \Phi(-2)$

$= 1 - Q(1) - 1 + Q(-2)$

$= -Q(1) + Q(-2)$

$= Q(-2) - Q(1)$

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$0-0=0 \quad x\,.5^3-.5^4=.125-.0625$

$=.0625$

3. Suppose that the continuous random variable X has the PDF

$$f_X(x) = \begin{cases} c(x^3 - x^4) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the PDF is valid. *(7 points)* 7

(b) Find the expected value of X. *(8 points)* 8

(c) Find the CDF of X. *(10 points)* 10

ⓐ From PDF, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\therefore \int_0^1 c(x^3-x^4)\,dx = 1$

$c\left(\int_0^1 x^3\,dx - \int_0^1 x^4\,dx\right)=1$

$c\left(\frac{1^4}{4}-\frac{1^5}{5}\right)=1 \therefore$

$\therefore c^{-1} = \frac{1}{4}-\frac{1}{5} = \frac{5-4}{20} = \frac{1}{20}$

$\therefore \boxed{c=20}$

Verify — Scratch ⓐ

$\int_0^1 20x^3\,dx - \int_0^1 20x^4\,dx$

$= 20\cdot\frac{x^4}{4}\Big|_0^1 - 20\,\frac{x^5}{5}\Big|_0^1$

$= 20\cdot\frac{1}{4} - 20\cdot\frac{1}{5}$

$= 5-4=1$ ☑

ⓑ $E[x] = \int_{-\infty}^{\infty} x\cdot f_X(x)\,dx$

$= \int_0^1 x\cdot(20(x^3-x^4))\,dx$

$= 20\int_0^1 (x^4-x^5)\,dx$

$= 20\left(\frac{x^5}{5}-\frac{x^6}{6}\right)\Big|_0^1$

$= 20\cdot\frac{1}{5} - 20\cdot\frac{1}{6}$

$= 4-\frac{10}{3} = \frac{12-10}{3} = \boxed{\frac{2}{3}}$

ⓒ $\frac{d}{dx}\,5x^4-4x^5$ Scratch

$= x^5-x^4 \quad -1$

which is positive

$\therefore$ increasing ✓

& @ 1, $5\cdot1-4\cdot1=1$

ⓒ CDF: $F_X(x) = \int_{-\infty}^{x} f_X\,dx$

$= \begin{cases} \int_{-\infty}^{x} 0, & x<0 \\ \int_0^x f_X(x)\,dx & 0\le x\le 1 \\ 1+\int_1^x f_X(x)\,dx, & x>1 \end{cases}$

$\int_0^x f_X(x)\,dx$

$= \int_0^x 20(x^3-x^4)\,dx = 20\left(\frac{x^4}{4}-\frac{x^5}{5}\right)\Big|_0^x$

$= 20\left(\frac{x^4}{4}-\frac{x^5}{5}\right)$

$= 5x^4-4x^5$

$\therefore$ ⓒ

$F_X(x)$

$= \begin{cases} 0, & x<0 \\ 5x^4-4x^5, & 0\le x\le 1 \\ 1, & x>1 \end{cases}$

$\log_2 2^{2\cdot 3} = 2^2 \cdot 2^3$

$\log_2(2^{2\cdot3}) = \log_2(2^2)\cdot\log_2(2^3)$

$= 6 \quad = 2 \cdot 3$

$e^{xy} = e^x e^y$

$\ln(e^{xy}) = \ln(x)\cdot\ln(y)$

4. Let X and Y be two independent exponential random variables with parameters $\lambda_1 = 4$ and $\lambda_2 = 5$, respectively. The random variable W is defined as $W = \min\{X, Y\}$.

(a) Find the PDF of W. *(15 points)* 8

(b) Find the expectation and variance of W. *(10 points)* 0

(a) $f_X(x) = \begin{cases} 4e^{-4x}, & x \ge 0 \\ 0, & ow \end{cases}$, $f_Y(y) = \begin{cases} 5e^{-5y}, & y \ge 0 \\ 0, & ow \end{cases}$

$W = \min(X, Y)$, $\therefore f_W(x,y) = \begin{cases} * \; 4e^{-4x}, & 4e^{-4x} < 5e^{-5y} \;\&\; x,y \ge 0 \\ ** \; 5e^{-5y}, & 5e^{-5y} \le 4e^{-4x} \;\; x\&y \ge 0 \\ 0, & x<0 \text{ or } y<0 \end{cases}$

$\therefore f_W(x,y) = \begin{cases} 4e^{-4x}, & x > \frac{5}{4}\ln\left(\frac{5}{4}\right)y \;\&\; x \ge 0, y \ge 0 \\ 5e^{-5y}, & y \ge \frac{4}{5}\ln\left(\frac{4}{5}\right)y \;\&\; x \ge 0, y \ge 0 \\ 0, & x<0 \text{ or } y<0 \end{cases}$

Solving for x & y in $*$ & $**$

$*$

$4e^{-4x} < 5e^{-5y}$

$e^{-4x} < \frac{5}{4}e^{-5y}$

$-4x < \ln\left(\frac{5}{4}e^{-5y}\right)$

$\therefore -4x < \ln\left(\frac{5}{4}\right)\cdot -5y$

$\therefore 4x > 5y \cdot \ln\left(\frac{5}{4}\right)$

$\therefore x > \frac{5}{4}\ln\left(\frac{5}{4}\right)y$

In $**$

$5e^{-5y} \le 4e^{-4x}$

$e^{-5y} \le \frac{4}{5}e^{-4x}$

$-5y \le \ln\left(\frac{4}{5}\right)\ln(e^{-4x})$

$5y \ge 4x\ln\left(\frac{4}{5}\right)$

$y \ge \frac{4}{5}\ln\left(\frac{4}{5}\right)x$

(b) $E[W] = \left(\int_{-\infty}^{\infty} x \cdot \right)$

Since they are independant,

$\Phi_Y(\omega) = E[e^{j\omega\{x,y\}}]$

$\Phi_Y(\omega) = \int$

$U =$

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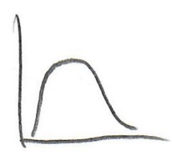
$$\int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

$$= \int_0^\infty e^{j\omega x}\cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{j\omega x} e^{-\lambda x}\,dx$$

$$\lambda \int_0^\infty e^{j\omega x-\lambda x}\,dx$$



$$= \lambda \int_0^\infty e^{x(j\omega-\lambda)}\,dx$$

$$= \cancel{\lambda (j\omega-\lambda) e^{x}}$$

$$\lambda \int_0^\infty e^{x} e^{j\omega-\lambda}\,dx$$

$$= \lambda e^{j\omega-\lambda} e^{x}$$

$$\lambda \cdot (j\omega-\lambda) e^{x(j\omega-\lambda)}\Big|_0^\infty \qquad \lambda$$

$$= \cancel{\lambda (j\omega-\lambda)\cdot 1}$$