ECE113, Winter 2023 **Homework #3**

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Due Friday, 10 Feb 2023, by 11:59pm to Gradescope.
50 points total.

1. (10 points) Let $y[n]$ denote the linear convolution of the two sequences:
   $x[n] = \{2, -3, 4, 1\}, -1 \leq n \leq 2,$
   $h[n] = \{-3, 5, -6, 4\}, -2 \leq n \leq 1.$
   Determine the value of $y[-1]$ without computing the convolution sum.

   Solutions:
   $y[-1] = x[-1]h[0] + x[0]h[-1] + x[1]h[-2] = 2 \times (-6) + (-3) \times 5 + 4 \times (-3) = -39$

2. (10 points) Evaluate the linear convolution of each of the following sequences with itself:
   (a) $x_1[n] = \{1, -1, 1\}, -1 \leq n \leq 1,$
   (b) $x_2[n] = \{1, -1, 0, 1, -1\}, 0 \leq n \leq 4,$
   (c) $x_3[n] = \{-1, 2, 0, -2, 1\}, -3 \leq n \leq 1.$

   Solutions:
   (a) $\sum_{k=-\infty}^{\infty} x_1[k]x_1[n-k] = \left\{ \begin{array}{ccccc} 1 & -2 & 3 & -2 & 1 \end{array} \right\}, \quad -2 \leq n \leq 2$
   (b) $\sum_{k=-\infty}^{\infty} x_2[k]x_2[n-k] = \left\{ \begin{array}{ccccccccc} 1 & -2 & 1 & 2 & -4 & 2 & 1 & -2 & 1 \end{array} \right\}, \quad 0 \leq n \leq 8.$
   (c) $\sum_{k=-\infty}^{\infty} x_3[k]x_3[n-k] = \left\{ \begin{array}{ccccccccc} 1 & -4 & 4 & 4 & -10 & 4 & 4 & -4 & 1 \end{array} \right\}, \quad -6 \leq n \leq 2.$

3. (10 points) Determine the output of a LTI system with impulse response $h[n] = (\frac{1}{2})^n u[n]$ when excited by input $x[n] = 2^n u[-n]$.

   Solutions: The output is

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\ &= \sum_{k=-\infty}^{\infty} 2^k u[-k] (\tfrac{1}{2})^{n-k} u[n-k] \\ &= \begin{cases} \sum_{k=-\infty}^{n} 2^k (\frac{1}{2})^{n-k}, & n < 0 \\ \sum_{k=-\infty}^{0} 2^k (\frac{1}{2})^{n-k}, & n \geq 0 \end{cases} \end{aligned} \tag{1}$$

We can compute (we will use a variable change $m = -k$)

$$
\begin{aligned}
\sum_{k=-\infty}^{n} 2^k \left(\tfrac{1}{2}\right)^{n-k} &= \sum_{m=-n}^{\infty} 2^{-m} \left(\frac{1}{2}\right)^{n+m} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{m=-n}^{\infty} \left(\frac{1}{4}\right)^{m} \\
&= (\frac{1}{2})^n (\frac{1}{4})^{-n} \frac{1}{1-\frac{1}{4}} \\
&= \frac{4}{3} 2^n
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
\sum_{k=-\infty}^{0} 2^k \left(\tfrac{1}{2}\right)^{n-k} &= \sum_{m=0}^{\infty} 2^{-m} \left(\tfrac{1}{2}\right)^{n+m} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{m} \\
&= (\frac{1}{2})^n \frac{1}{1-\frac{1}{4}} \\
&= \frac{4}{3} 2^{-n}
\end{aligned}
\tag{3}
$$

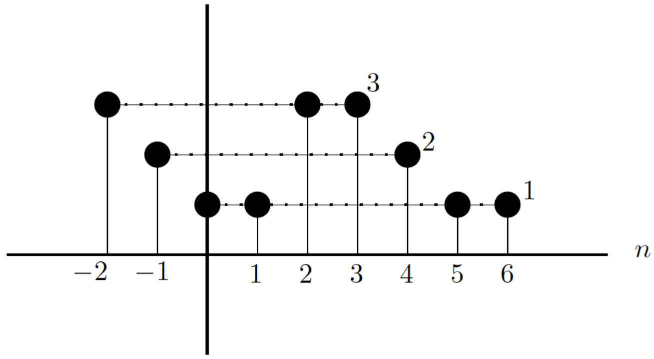
Hence,

$$
y(n) = \begin{cases} \frac{4}{3} 2^n, & n < 0 \\ \frac{4}{3} 2^{-n}, & n \geq 0 \end{cases}
$$

Or it can simplified as

$$
y(n) = \frac{4}{3} 2^{-|n|}
$$

4. (10 points) Consider the sequence $x[n]$ that is shown below. It is zero except at the specified time instants. The amplitudes of the non-zero samples are either 1, 2, or 3.



(i) Define the sequence $y[n] = u[n+1] - u[n-2]$. Compute the convolution $x[n] * y[n]$.

(ii) Define

$$h_1[n] = (\frac{1}{2})^n h[n] u[n]$$

where

$$h[n] = (\frac{1}{2}) x[n+2] - \frac{3}{2}\delta[n] + u[n-3]$$

Take $h_1[n]$ to be the impulse response of an LTI system. What would the response of the system be to the input sequence $(\frac{1}{3})^n u[n]$?

Solutions:

i. The sequence $y(n) = u[n+1] - u[n-2]$ has three nonzero samples at $n = -1, 0, 1$. i.e., $y[n] = \{1, \boxed{1}, 1\}$, where the boxed sample corresponds to the sample at $n = 0$. Let $z[n] = x[n] * y[n]$, since $x[n]$ is nonzero for $-2 \leq n \leq 6$ and $y[n]$ is nonzero for $-1 \leq n \leq 1$, then $z[n]$ is nonzero for $-3 \leq n \leq 7$. We can write $y[n]$ alternatively as

$$y[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

Then

$$z[n] = x[n+1] + x[n] + x[n-1]$$

which can be evaluated as follows:

$x[n+1]$	3	2	1	$\boxed{1}$	3	3	2	1	1		
$x[n]$		3	2	$\boxed{1}$	1	3	3	2	1	1	
$x[n-1]$			3	$\boxed{2}$	1	1	3	3	2	1	1
$z[n]$	3	5	6	$\boxed{4}$	5	7	8	6	4	2	1

Since $z[n] = 0$ for $n < -3$ And $n > 7$, we get

$$z[n] = \{3, 5, 6, \boxed{4}, 5, 7, 8, 6, 4, 2\}$$

ii. The samples of $h[n] = \frac{1}{2}x[n+2] - \frac{3}{2}\delta[n] + u[n-3]$ are calculated as follows:

$\frac{1}{2}x[n+2]$	3/2	1	1/2	1/2	$\boxed{3/2}$	3/2	1	1/2	1/2				
$-\frac{3}{2}\delta[n]$					$\boxed{-3/2}$								
$u[n-3]$								1	1	1	1	1	$\cdots$
$h[n]$	3/2	1	1/2	1/2	$\boxed{0}$	3/2	1	3/2	3/2	1	1	1	$\cdots$

and the sequence $h_1[n] = \left(\frac{1}{2}\right)^n h[n]u[n]$ is given by

$$h_1(n) = \begin{cases} 0 & n \leq 0 \\ 3/4 & n = 1 \\ 1/4 & n = 2 \\ 3/16 & n = 3 \\ 3/32 & n = 4 \\ \left(\frac{1}{2}\right)^n & n \geq 5 \end{cases}$$

which could be written as

$$h_1[n] = \frac{3}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{3}{16}\delta[n-3] + \frac{3}{32}\delta[n-4] + \left(\frac{1}{2}\right)^n u[n-5]$$

Then,

$$\begin{aligned} y[n] &= x[n] * h_1[n] \\ &= \left(\frac{1}{3}\right)^n u[n] * \left(\frac{3}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{3}{16}\delta[n-3] + \frac{3}{32}\delta[n-4] + \left(\frac{1}{2}\right)^n u[n-5]\right) \\ &= \frac{3}{4}\left(\frac{1}{3}\right)^{n-1} u[n-1] + \frac{1}{4}\left(\frac{1}{3}\right)^{n-2} u[n-2] + \frac{3}{16}\left(\frac{1}{3}\right)^{n-3} u[n-3] \\ &\quad + \frac{3}{32}\left(\frac{1}{3}\right)^{n-4} u[n-4] + \underbrace{\sum_{k=-\infty}^{\infty}\left(\frac{1}{3}\right)^k u[k]\left(\frac{1}{2}\right)^{n-k} u[n-k-5]}_{v[n]} \end{aligned}$$

The last term can be evaluated as follows

$$\begin{aligned} v[n] &= \sum_{k=-\infty}^{\infty}\left(\frac{1}{3}\right)^k u[k]\left(\frac{1}{2}\right)^{n-k} u[n-k-5] \\ &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n-5}\left(\frac{2}{3}\right)^k = \left(\frac{1}{2}\right)^n \frac{1}{1-2/3}\left(1-\left(\frac{2}{3}\right)^{n-4}\right) \\ &= 3\left(\frac{1}{2}\right)^n - \frac{243}{16}\left(\frac{1}{3}\right)^n \qquad \text{for } n \geq 5, \end{aligned}$$

and

$$v(n) = 0, \quad \text{otherwise.}$$

Therefore

$$\begin{aligned} y[n] &= \frac{3}{4}\left(\frac{1}{3}\right)^{n-1} u[n-1] + \frac{1}{4}\left(\frac{1}{3}\right)^{n-2} u[n-2] + \frac{3}{16}\left(\frac{1}{3}\right)^{n-3} u[n-3] \\ &\quad + \frac{3}{32}\left(\frac{1}{3}\right)^{n-4} u[n-4] + \left[3\left(\frac{1}{2}\right)^{n} - \frac{243}{16}\left(\frac{1}{3}\right)^{n}\right] u[n-5] \end{aligned}$$

5. (10 points) Let $y[n] = x[n] * h[n]$ with

$$x[n] = f[n]\left(u\left[n - n_1\right] - u\left[n - n_2\right]\right)$$

and

$$h[n] = g[n]\left(u\left[n - n_3\right] - u\left[n - n_4\right]\right)$$

where $f[n]$ and $g[n]$ are arbitrary functions, and $n_1 < n_2$ and $n_3 < n_4$. Therefore, $x[n]$ and $h[n]$ are pulse-like signals of finite duration $n_x = n_2 - n_1$ and $n_h = n_4 - n_3$, respectively.

(a) For what value of the index n does the first non-zero output element $y[n]$ occur?

(b) For what value of the index n does the last non-zero output element $y[n]$ occur?

(c) What is the duration n_y of the output sequence $y[n]$ in terms of n_x and n_h?

Solutions:

- a. We begin by writing $y(n)$ explicitly:

  $$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  Since $x[n]$ has all zero values for $n < n_1$ and $n \geq n_2$,

  $$y[n] = \sum_{k=n_1}^{n_2-1} x[k]h[n-k]$$

  If $h[n-k] = 0$ for all k where

  $$n_1 \leq k \leq n_2 - 1,$$

  then we have $y[n] = 0$. $h[n-k]$ can be written as

  $$h[n-k] = g[n-k](u[n-n_3-k] - u[n-n_4-k])$$

  which is zero when $n \leq n_3 + k - 1$ and $n \geq n_4 + k$. To satisfy the constraint $n_1 \leq k \leq n_2 - 1$ for all k, we need

  $$n \leq n_3 + \min(k) - 1 = n_3 + n_1 - 1$$

  Hence the first non-zero element is at $n_3 + n_1$.

- b. Based on the bounds above, we also have:

  $$n \geq n_4 + \max(k) = n_4 + n_2 - 1$$

  Hence the last non-zero element is at $n_4 + n_2 - 2$.

- c. $y[n]$ is non-zero when $n_3 + n_1 \leq n < n_4 + n_2 - 1$. The duration of the sequence $y[n]$ is

  $$(n_4 + n_2 - 1) - (n_3 + n_1) = n_h + n_x - 1$$