EE 131A | Homework 2 Solution
Probability | January 17, 2023
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Chapters 2.4-2.5 & 3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Roll two fair dice independently. In terms of the possible outcomes, define the events:

$$\begin{aligned} A &= \{\text{First die is 1, 2 or 5}\} \\ B &= \{\text{Second die is 2, 3}\} \\ C &= \{\text{Sum of outcomes is 7}\} \end{aligned}$$

Are A, B, and C mutually independent? Hint: Three events A, B, and C are independent if all the four following constraints hold:

$$\begin{aligned} P(A \cap B) &= P(A)P(B), \\ P(A \cap C) &= P(A)P(C), \\ P(B \cap C) &= P(B)P(C), \\ P(A \cap B \cap C) &= P(A)P(B)P(C). \end{aligned}$$

Solution: Sample space is $S = \{(i,j) | 1 \le i, j \le 6\}$. And,

$$\begin{aligned} A &= \{(i,j) : i = 1,2,5 \text{ and } 1 \le j \le 6\}, \\ B &= \{(i,j) : j = 2,3 \text{ and } 1 \le i \le 6\}, \\ C &= \{(i,j) : i + j = 7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}. \end{aligned}$$

So, $|S| = 36$, $|A| = 18$, $|B| = 12$, and $|C| = 6$. Since the two fair dice rolls are independent, $P(A) = \frac{18}{36} = \frac{1}{2}$, $P(B) = \frac{12}{36} = \frac{1}{3}$, and $P(C) = \frac{6}{36} = \frac{1}{6}$.

$$\begin{aligned} A \cap B &= \{(1,2),(1,3),(2,2),(2,3),(5,2),(5,3)\}, \\ A \cap C &= \{(1,6),(2,5),(5,2)\}, \\ B \cap C &= \{(5,2),(4,3)\}, \\ A \cap B \cap C &= \{(5,2)\}. \end{aligned}$$

Hence, $P(A\cap B) = \frac{6}{36} = \frac{1}{6}$, $P(A\cap C) = \frac{3}{36} = \frac{1}{12}$, $P(B\cap C) = \frac{2}{36} = \frac{1}{18}$, $P(A\cap B\cap C) = \frac{1}{36}$. Three events A, B, and C are independent if all the four following constraints hold:

$$\begin{aligned} P(A \cap B) &= P(A)P(B), && (1) \\ P(A \cap C) &= P(A)P(C), && (2) \\ P(B \cap C) &= P(B)P(C), && (3) \\ P(A \cap B \cap C) &= P(A)P(B)P(C). && (4) \end{aligned}$$

Observe that (1), (2), (3), and (4) all hold true. So yes, A, B,and C are **mutually independent**.

2. Apple manufactures their M2 chips only in two locations: San Diego and Cupertino, with the latter producing 75% of their M2 chips. As per the data published by Apple, only 4% of all their manufactured M2 chips turn out to be defective. It is known that;

   P(M2 chip turns out to be defective given that it is manufactured in San Diego) = 5*P(M2 chip turns out to be defective given that it is manufactured in Cupertino).

   An M2 chip is randomly selected, and we find out that it works fine (not defective). Then what is the probability that it was manufactured in San Diego?

   Solution:
   Let D denote the event that a randomly selected M2 chip is defective, and S and, C denote the events that it was manufactured by San Diego and Cupertino facilities, respectively.

   We are given $P(C) = 0.75$, and since San Diego and Cupertino are the only manufacturing facilities, therefore $P(S) = 1 - P(C) = 0.25$. We also know $P(D) = 0.04$ signifying the overall defective rate.

   The goal of the problem is to find $P(S|ND)$, where ND denotes an event that a randomly selected M2 chip is not defective.

   Let $P(D|C) = x$, since we are given, $P(D|S) = 5 * P(D|C)$; therefore, $P(D|S) = 5x$

   Using law of total probability, $P(D)$ can be written as

$$P(D) = P(D|S)P(S) + P(D|C)P(C)$$
$$0.04 = 5x * 0.25 + x * 0.75$$
$$x = \frac{1}{50} = 0.02$$

   So, $P(D|C) = 0.02$ and $P(D|S) = 0.1$

   By the Bayes rule, we get,

$$P(S|ND) = \frac{P(ND|S) * P(S)}{P(ND)}$$

   Also we know that, $P(ND) = 1 - P(D) = 1 - 0.04 = 0.96$ and $P(ND|S) = 1 - P(D|S) = 1 - 0.1 = 0.9$

   Therefore,

$$P(S|ND) = \frac{0.9 * 0.25}{0.96} = \frac{15}{64} \approx 0.234375$$

   Alternate Solution:

   Let us assume Apple manufactures 100 M2 chips, out of which 4 are defective,

   So, San Diego manufactures a total of 25 chips out of which x chips are defective

   Cupertino manufactures a total of 75 chips out of which $4 - x$ chips are defective

It is given that P(M2 chip turns out to be defective given that it is manufactured in San Diego) = 5*P(M2 chip turns out to be defective given that it is manufactured in Cupertino)

Therefore,

$$\frac{\text{Defective Chips in San Diego}}{\text{Total Chips manufactured in San Diego}} = 5*\frac{\text{Defective Chips in Cupertino}}{\text{Total Chips manufactured in Cupertino}}$$

$$\frac{x}{25} = 5 * \frac{4-x}{75} \implies x = 2.5$$

Our goal is to find the probability that the chip was manufactured in San Diego, given that it is not defective (let Y),

$$Y = \frac{\text{Number of Chips manufactured in San Diego which are Not Defective}}{\text{Total Number of Chips Not Defective}}$$

$$Y = \frac{25-x}{96} = \frac{22.5}{96} \implies Y = \frac{15}{64} \approx 0.234375$$

3. There are two bags B_1 and B_2 containing a mix of Water-type and Fire-type Pokémon cards. B_1 contains 3 Water-type cards and 2 Fire-type cards, and B_2 contains only 1 Water-type card. A fair coin is tossed ($p = 0.5$), and based on the result of the toss, the following actions are taken:

   (a) If heads appear, then 1 card is randomly drawn from B_1 and put into B_2.

   (b) If tails appear, then 2 cards are randomly drawn from B_1 and put into B_2.

   A card is randomly drawn from B_2; it turns out to be a Water-type Pokémon card. What is the probability that heads appeared on the coin?

   Solution

   Let H be an event such that the result of the coin toss is head-side up, and T be an event that the result is tail-side up. Let W be an event that the card drawn from B_2 is a Water-type Pokémon card and F be an event that the card drawn from B_2 is a Fire-type Pokémon card.

   Our goal is to find $P(H|W)$

   Using Bayes theorem,

$$P(H|W) = \frac{P(W|H) * P(H)}{P(W)}$$

$$P(H|W) = \frac{P(W|H) * P(H)}{P(W|H) * P(H) + P(W|T) * P(T)} \tag{5}$$

   Since it is a fair coin toss, $P(H) = P(T) = 0.5$

Also,

$P(W|H) =$ Probability of drawing Water-type card from B_2 when heads appears on the coin

$$\begin{aligned} P(W|H) &= \frac{{}^3C_1}{{}^5C_1} * \frac{{}^2C_1}{{}^2C_1} \text{ (Water-type card transferred)} \\ &+ \frac{{}^2C_1}{{}^5C_1} * \frac{{}^1C_1}{{}^2C_1} \text{ (Fire-type card transferred)} \end{aligned} \tag{6}$$

$$P(W|H) = \frac{4}{5} = 0.8$$

Similarly,

$P(W|T) =$ Probability of drawing Water-type card from B_2 when tails appears on the coin

$$\begin{aligned} P(W|T) &= \frac{{}^3C_2}{{}^5C_2} * \frac{{}^3C_1}{{}^3C_1} \text{ (Two Water-type cards transferred)} \\ &+ \frac{{}^2C_2}{{}^5C_2} * \frac{{}^1C_1}{{}^3C_1} \text{ (Two Fire-type cards transferred)} \\ &+ \frac{{}^3C_1 * {}^2C_1}{{}^5C_2} * \frac{{}^2C_1}{{}^3C_1} \text{ (One Water and One Fire-type card transferred)} \end{aligned} \tag{7}$$

$$P(W|T) = \frac{22}{30} \approx 0.7333$$

Substituting values in Equation (5):

$$P(H|W) = \frac{\frac{4}{5} * 0.5}{\frac{4}{5} * 0.5 + \frac{22}{30} * 0.5} = \frac{12}{23} \approx 0.52174$$

4. Throw a pair of six-sided dice. Let X_1 be the number of dots on the resulting face of the first die and let X_2 be the number of dots on the resulting face of the second die. Let $Z = X_1 + X_2$ be the sum of the two dice rolls.

   (a) What is the pmf of Z?
   Solution:

$$P(Z = z) = \begin{cases} \frac{z-1}{36} & z \in \{2,3,4,5,6,7\} \\ \frac{13-z}{36} & z \in \{8,9,10,11,12\} \\ 0 & \text{otherwise} \end{cases}$$

   (b) Given that $Z = 10$, what is the probability that $X_1 = k$ for $k \in \{1,2,3,4,5,6\}$?
   Solution:
   By Bayes rule,

$$P(X_1 = k|Z = 10) = \frac{P(Z = 10|X_1 = k)P(X_1 = k)}{P(Z = 10)}.$$

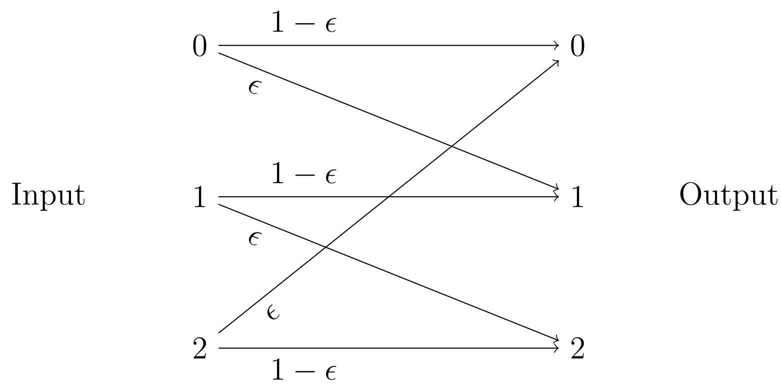
From part (a), we know $P(Z=10)=\frac{3}{36}$. Additionally, we know $P(X_1=k)=\frac{1}{6}$. Due to $Z=X_1+X_2$ and the independence of X_1 and X_2, we can write $P(Z=10|X_1=k)=P(X_2=10-X_1|X_1=k)=P(X_2=10-k)$. Note that

$$P(X_2=10-k)=\begin{cases}\frac{1}{6} & k\in\{4,5,6,7,8,9\}\\ 0 & \text{otherwise}\end{cases}.$$

Combining these results together, we get

$$P(X_1=k|Z=10)=\begin{cases}\frac{1}{3} & k\in\{4,5,6\}\\ 0 & \text{otherwise}\end{cases}$$

5. A ternary communication channel is shown in the figure. Assume that input symbols $0,1$, and 2 are chosen for transmission with probabilities $\frac{1}{6},\frac{1}{6}$, and $\frac{2}{3}$.



(a) Calculate the probability of each output.
Solution:

$$\begin{aligned}P(\text{Output }=0) &= P(\text{Output }=0|\text{Input }=0)P(\text{Input }=0)\\ &+P(\text{Output }=0|\text{Input }=1)P(\text{Input }=1)\\ &+P(\text{Output }=0|\text{Input }=2)P(\text{Input }=2)\\ &=\frac{1}{6}(1-\epsilon)+0\cdot 0+\frac{2}{3}\epsilon=\frac{1}{6}+\frac{1}{2}\epsilon=\frac{1+3\epsilon}{6}\end{aligned}$$

$$\begin{aligned}P(\text{Output }=1) &= P(\text{Output }=1|\text{Input }=0)P(\text{Input }=0)\\ &+P(\text{Output }=1|\text{Input }=1)P(\text{Input }=1)\\ &+P(\text{Output }=1|\text{Input }=2)P(\text{Input }=2)\\ &=\frac{1}{6}\epsilon+\frac{1}{6}(1-\epsilon)+0\cdot 0=\frac{1}{6}\end{aligned}$$

$$\begin{aligned}P(\text{Output }=2) &= P(\text{Output }=2|\text{Input }=0)P(\text{Input }=0)\\ &+P(\text{Output }=2|\text{Input }=1)P(\text{Input }=1)\\ &+P(\text{Output }=2|\text{Input }=2)P(\text{Input }=2)\\ &=0\cdot 0+\frac{1}{6}\epsilon+\frac{2}{3}(1-\epsilon)=\frac{2}{3}-\frac{1}{2}\epsilon=\frac{4-3\epsilon}{6}\end{aligned}$$

(b) Given that the output was 0, what is the probability that the input was 0? 1? 2?

Solution:

We want to figure out $P(\text{Input } = k|\text{Output } = 0)$. By Bayes rule, we get

$$P(\text{Input } = k|\text{Output } = 0) = \frac{P(\text{Output } = 0|\text{Input } = k)P(\text{Input } = k)}{P(\text{Output } = 0)}.$$

All the necessary terms were calculated in part (a) which gives us the result

$$\begin{aligned} P(\text{Input } = 0|\text{Output } = 0) &= \frac{\frac{1}{6}(1-\epsilon)}{\frac{1+3\epsilon}{6}} = \frac{1-\epsilon}{1+3\epsilon} \\ P(\text{Input } = 1|\text{Output } = 0) &= 0 \\ P(\text{Input } = 2|\text{Output } = 0) &= \frac{\frac{2}{3}\epsilon}{\frac{1+3\epsilon}{6}} = \frac{4\epsilon}{1+3\epsilon} \end{aligned}$$