EE 131A | Homework 5
Probability and Statistics | Due: Tuesday, February 21, 2023
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Reading: Chapter 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Let M be a geometric random variable with parameter p and let X be an exponential random variable with parameter λ.
   (a) Compute the tail probabilities $P(M > k)$ and $P(X > t)$ for the geometric and the exponential random variable where k is a positive integer and t is a non-negative real number.
   (b) Use Matlab to plot $P(M > k)$ as a function of k and $P(X > t)$ as a function of t. Use $p = 0.6$ and $\lambda = 1$. Compare the two plots.
   (c) A continuous random variable X is said to satisfy the memoryless property if for all $t, h \geq 0$, $P(X > t+h|X > t) = P(X > h)$. Prove that the exponential random variable satisfies the memoryless property.
   Recall that in Discussion 3, you proved that the geometric random variable satisfies the memoryless property.

2. On a special sale day, the UCLA store provides discounts for students only if the total sum of all their purchases ever made is at least \$500. Let X (in 100s of \$) be the total discount availed by a student. Assume X has the PDF:

$$f(x;\alpha) = \begin{cases} \frac{k}{x^\alpha} & x \geq 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   (a) Find the value of k. What restriction on α is necessary?
   (b) What is the CDF of X?
   (c) What is the expected discount availed by a randomly chosen student, if $\alpha > 2$?
   (d) Show that $\ln(X/5)$ has an exponential distribution with parameter $\alpha - 1$.

   Hint: k must be greater than 0.

3. Find the PDF of $X = -\ln(4-4U)$, where U is a continuous random variable, uniformly distributed on the $[0, 1]$ interval.

4. In this problem, you will show that the Poisson random variable is a good approximation for the Binomial random variable in the limit using characteristic functions:
   (a) Find the characteristic function of the Binomial random variable.
   Hint: The Binomial theorem states the following:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

(b) Find the characteristic function of the Poisson random variable.
Hint: We know from Taylor series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(c) As n approaches ∞, denote np as λ and use the fact that

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

to prove the question.

(d) What can you say about the value of n, p, and λ at this point? When is the Poisson random variable a good approximation for the Binomial random variable?
Hint: λ is the mean of the Poisson distribution: $\mathbb{E}[X] = \lambda$ where $X \sim Poisson(\lambda)$.

(e) Haemophilia is a rare genetic bleeding disorder that affects 1 in every 10,000 people in the world, where the person's blood does not clot by itself. In a sample of 2,000 people, what is the probability that exactly 1 person has haemophilia? Solve this question using the Binomial RV as well as its Poisson approximation and compare.

5. Find the characteristic function of a normal distribution with mean m and variance σ^2.
Hint:
   - Completion of square. Let $k = m + j\omega\sigma^2$, then $j\omega x - \frac{(x-m)^2}{2\sigma^2} = \frac{-(x-k)^2 + 2mj\omega\sigma^2 - \omega^2\sigma^4}{2\sigma^2}$.
   - $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.