ECE113, Winter 2023 | **Homework #2**
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Due Friday, 3 Feb 2023, by 11:59pm to Gradescope.
50 points total.

1. (10 points) Determine the even and odd parts of the following real sequences:
   (a) $x_1[n] = u[n-3]$
   (b) $x_2[n] = \alpha^n u[n-1]$
   (c) $x_3[n] = n\alpha^n u[n+1]$
   (d) $x_4[n] = \alpha^{|n|}$

   Solutions:
   (a) $x_1[n] = u[n-3]$. Hence, $x_1[-n] = u[-n-3]$. Therefore,

$$x_{1,ev}[n] = \frac{1}{2}(u[n-3] + u[-n-3]) = \begin{cases} 1/2, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \\ 1/2, & n \leq -3, \end{cases}$$

$$x_{1,od}[n] = \frac{1}{2}(u[n-3] - u[-n-3]) = \begin{cases} 1/2, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \\ -1/2, & n \leq -3. \end{cases}$$

   (b) $x_2[n] = \alpha^n u[n-1]$. Hence, $x_2[-n] = \alpha^{-n} u[-n-1]$. Therefore,

$$x_{2,ev}[n] = \frac{1}{2}\left(\alpha^n u[n-1] + \alpha^{-n} u[-n-1]\right) = \begin{cases} \frac{1}{2}\alpha^n, & n \geq 1, \\ 0, & n = 0, \\ \frac{1}{2}\alpha^{-n}, & n \leq -1, \end{cases}$$

$$x_{2,od}[n] = \frac{1}{2}\left(\alpha^n u[n-1] - \alpha^{-n} u[-n-1]\right) = \begin{cases} \frac{1}{2}\alpha^n, & n \geq 1, \\ 0, & n = 0, \\ -\frac{1}{2}\alpha^{-n}, & n \leq -1. \end{cases}$$

   (c) $x_3[n] = n\alpha^n u[n+1]$. Hence, $x_3[-n] = -n\alpha^{-n} u[-n+1]$. Therefore,

$$x_{3,ev}[n] = \frac{1}{2}\left(n\alpha^n u[n+1] + (-n)\alpha^{-n} u[-n+1]\right) = \begin{cases} \frac{1}{2}n\alpha^n, & n \geq 2, \\ \frac{1}{2}\left(a - \frac{1}{a}\right), & |n| = 1 \\ 0, & n = 0, \\ -\frac{1}{2}n\alpha^{-n}, & n \leq -2 \end{cases}$$

$$x_{3,od}[n] = \frac{1}{2}\left(n\alpha^n u[n+1] - (-n)\alpha^{-n} u[-n+1]\right) = \begin{cases} \frac{1}{2}n\alpha^n, & n \geq 2, \\ \frac{1}{2}\left(a + \frac{1}{a}\right), & n = 1 \\ 0, & n = 0, \\ -\frac{1}{2}\left(a + \frac{1}{a}\right), & n = -1 \\ \frac{1}{2}n\alpha^{-n}, & n \leq 2, \end{cases}$$

(d) $x_4[n] = \alpha^{|n|}$. Hence, $x_4[-n] = \alpha^{|-n|} = \alpha^{|n|} = x_4[n]$. Therefore,

$$x_{4,e\nu}[n] = \frac{1}{2}\left(x_4[n] + x_4[-n]\right) = \frac{1}{2}\left(x_4[n] + x_4[n]\right) = x_4[n] = \alpha^{|n|}$$
$$x_{4,od}[n] = \frac{1}{2}\left(x_4[n] - x_4[-n]\right) = \frac{1}{2}\left(x_4[n] - x_4[n]\right) = 0.$$

2. (10 points) Answer True or False. In each case, either prove your answer or give a counter-example.

   (a) A power sequence is necessarily an energy sequence.

   (b) Every energy sequence has zero average power.

   (c) If $x[n]$ is an energy sequence then $x[n] \to 0$ as $n \to \infty$.

   (d) There does not exist a sequence with infinite average power.

Solutions:

(a) False.
$u[n]$ is not an energy sequence. However, it is a power sequence.

(b) True.
An energy sequence $x[n]$ must have

$$\lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = E < \infty$$

Hence, its average power is

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} E = 0$$

(c) True.
From calculus, we learned that a "necessary" condition for the convergence of a series, $\sum_{n=0}^{\infty} a[n]$, is that $\lim_{n\to\infty} a[n] = 0$. An energy sequence $x[n]$ has $\lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = E$. If we let

$$a[n] = \begin{cases} |x[0]|^2, & n = 0 \\ |x[-n]|^2 + |x[n]|^2, & otherwise \end{cases}$$

By using the "necessary" condition, we know

$$\lim_{n\to\infty} a[n] = \lim_{n\to\infty} (|x[-n]|^2 + |x[n]|^2) = 0$$

Hence, we must have $\lim_{n\to\infty} x[n] = 0$

(d) False.
Let $x[n] = n$, then $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} n^2$. Since $\sum_{n=-N}^{N} n^2 \geq N^2$,

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} n^2 \geq \lim_{N\to\infty} \frac{N^2}{2N+1} = \lim_{N\to\infty} \frac{2N}{2} \to \infty.$$

3. (10 points) System I is defined by $y[n] = \log(|x[n-1]|)$ and system II is defined by $y[n] = \exp(x[2n])$. Which of the following statements is correct?

   (a) Both systems are BIBO stable.

   (b) Both systems are unstable.

   (c) System I is unstable and system II is BIBO stable.

   (d) Both systems are time invariant.

Solutions:
The statement (c) is correct.
System I is unstable because we can give the following counter-example.
$x[n] = 10^{-|n|}$ is an input sequence bounded by 1, however, the output sequence is

$$y[n] = \log(|x[n-1]|) = -|n-1|$$

which is unbounded.
To show system II is BIBO stable, we let $x(n)$ be an arbitrary input sequence bounded by M, i.e. $|x[n]| < M$ for all n, then the output sequence $y[n] = \exp(x[2n])$ is also bounded by e^M. By definition, system II is BIBO stable.
System I is time-invariant, but system II is time-variant. To show system II is time-variant, we can compare the output of two time-shifted input sequences, for example

$$x_1[n] = \delta[n] \text{ and } x_2[n] = x_1[n-1] = \delta[n-1]$$

We can compute the two output sequences as

$$y_1[n] = 1 + (e-1)\delta[n] \text{ and } y_2[n] = 1$$

and we find $y_2[n] \neq y_1[n-1]$. Hence, by definition, system II is time-variant.

4. (10 points) Determine whether each of the following systems is linear or not, time-invariant or not, causal or not, BIBO stable or not, relaxed or not:

   (a) $y[n] = \ln(|x[n]| + 1)$

   (b) $y[n] = y[n-1] + x[n]$, $y[-1] = 0$

   (c) $y[n] = y[n-1] + x[n]$, $y[-1] = 1$

   (d) $y[n] = 2 + x[n]$

   Solutions:

   (a) $y[n] = \ln[|x[n]| + 1]$ is nonlinear, time-invariant, causal, BIBO stable, and relaxed.

   (b) $y[n] = y[n-1] + x[n]$,$y[-1] = 0$ is linear, time-invariant, causal, not stable (The input $x[n] = u[n]$ has output as $y[n] = n + 1$ for all $n \geq 0$. Hence $\lim_{n\to\infty} y[n] = \infty$ and the system is not BIBO stable), and relaxed

   (c) $y[n] = y[n-1]+x[n]$,$y[-1] = 1$ is non-linear, not time-invariant, causal, not stable (The input $x[n] = u[n]$ has output as $y[n] = n+2$ for all $n \geq 0$. Hence $\lim_{n\to\infty} y[n] = \infty$ and the system is not BIBO stable), and not relaxed

   (d) $y[n] = 2 + x[n]$ is nonlinear (generally adding a constant makes the system non-linear), time-invariant, causal, BIBO stable, and not relaxed.

   (For (b) and (c), please refer the notes for whether they are linear, time-invariant or relaxed.)

5. (10 points) Determine the conditions on the parameters of the following systems for stability:

(a) $h[n] = a^n u[-n]$.

(b) $h[n] = a^n(u[n] - u[n-100])$.

(c) $h[n] = r^n \sin[nw_0]u[n]$

(d) $h[n] = a^{|n|}$

(e) $h[n] = K(-1)^n u[n]$

Solutions:

(a) $h[n] = a^n u[-n]$ is stable if $|a| > 1$.

(b) $h[n] = a^n(u[n] - u[n-100])$ has finite support (i.e. a finite length sequence) and is stable if a is finite.

(c) $h[n] = r^n \sin[n\omega_0]u[n]$ is always stable for $|r| < 1$ and any ω_0. Likewise, if $\omega_0 = k\pi$ for some integer k, we get $h[n] = 0$, and consequently $h[n]$ is stable for any r.

(d) $h(n) = a^{|n|}$ is stable for $|a| < 1$.

(e) $h(n) = K(-1)^n u[n]$ is stable when $K = 0$. If $K \neq 0$, $h[n]$ is not absolutely summable, because

$$\sum_{n=-N}^{n=N} |h[n]| = |K|(N+1)$$

which is unbounded as $N \to \infty$