ECE113, Winter 2023
Digital Signal Processing
University of California, Los Angeles; Department of ECE

Homework #6 Solutions
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Due Friday, 17 March 2023, by 11:59pm to Gradescope.
50 points total.

1. (10 points) **Z-transform**
   Find the Z-transform of the following sequences and indicate their region of convergence. Additionally, check if the DTFT exists. For these problems, a is an arbitrary constant.

   (a) $x[n] = au[n]$
   (b) $x[n] = a^n u[n]$

   Solutions:

   (a) $X[z] = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$. This series converges only if the magnitude of az^{-1} is less than one. Then the z-transform and the associated region of convergence is:

   $$X[z] = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|.$$

   The DFT will only exist if $|a| < 1$.

   (b) $X[z] = \sum_{n=-\infty}^{\infty} au[n] z^{-n}$. This series converges only if the magnitude of z^{-1} is less than one. Then the z-transform and the associated region of convergence is:

   $$X[z] = \frac{a}{1 - z^{-1}} = \frac{az}{z-1}, \quad |z| > 1.$$

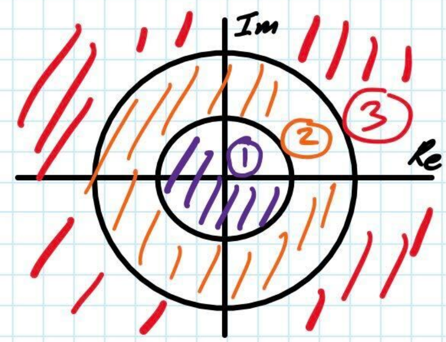
   The DFT does not exist since the ROC does not include the unit circle.

2. (10 points) **Z-transform with multiple ROCs**
Consider the z transform of, $g[n]$:

$$G(z) = \frac{(2z^2 - 0.8z + 0.5)(3z^2 + 2z + 3)}{(z^2 - 0.8z + 0.38)(z^2 + 4z + 5)}$$

There are 3 possible nonoverlapping regions of convergence (ROCs) of this z-transform. Discuss the type of inverse z-transform (left-sided, right-sided, or two-sided sequences) associated with each of the 3 ROCs. It is not necessary to compute the exact inverse transform).

Solutions:
To find the poles, we find the roots of the 2 terms in the denominator.
For the first term in the denominator, using the quadratic equation, the first root is $z_1 = 0.4 + j\sqrt{0.22}$ and the second root is $z_2 = 0.4 - j\sqrt{0.22}$. The magnitude of these roots is $|z_1| = |z_2| = \sqrt{0.38}$. Similarly, for the second term in the denominator, $z^2 + 4z + 5$, has the roots $z_3 = -2 + j$ and $z_4 = -2 - j$ whose magnitudes are $|z_3| = |z_4| = \sqrt{5}$.



1 ROC : $|z| < |z_1|$
Left-sided inverse Z-transform

2 ROC : $|z_1| < |z| < |z_3|$
Double-sided inverse z-transform

3 ROC : $|z_3| < |z|$
Right-sided inverse z-transform

3. (10 points) **Z-transform Properties**
Determine the z-transform and the corresponding ROC of the following sequences. Assume $|\beta| > |\alpha| > 0$.
Show the ROC on these plots.

(a) $x_1[n] = (\alpha^n + \beta^n)u[n+2]$

(b) $x_2[n] = \alpha^n u[-n-2] + \beta^n u[n-1]$

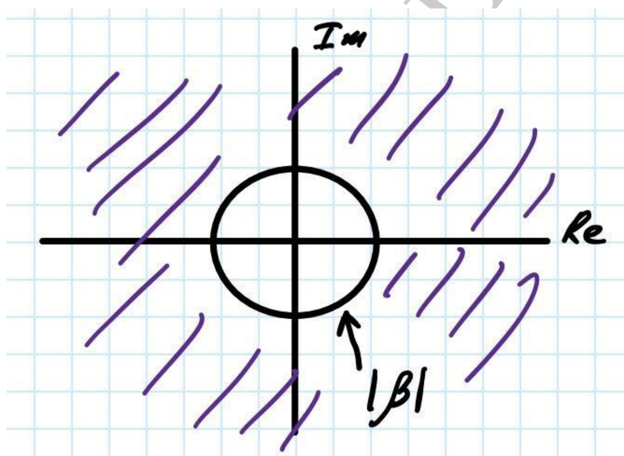
(c) $x_3[n] = \alpha^n u[n+1] + \beta^n u[-n-2]$

Solutions:

(a)

$$X_1(z) = \frac{\alpha^{-2}z^2}{1-\alpha z^{-1}} + \frac{\beta^{-2}z^2}{1-\beta z^{-1}} = \frac{\alpha^{-2}z^2 - \alpha^{-2}\beta z + \beta^{-2}z^2 - \alpha\beta^{-2}z}{(1-\alpha z^{-1})(1-\beta z^{-1})}$$

This will have poles at α and β and the ROC will then be: $|z| > |\beta|$.



(b)

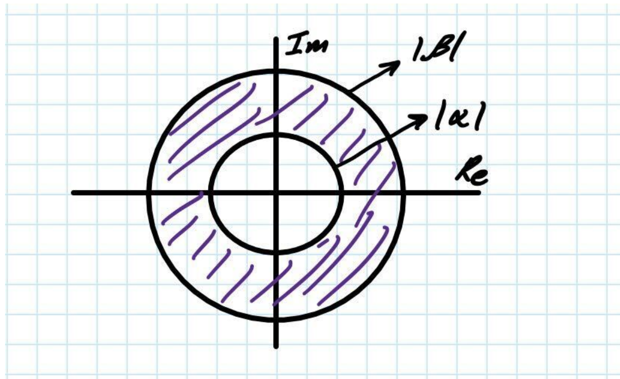
$$X_2(z) = \frac{-\alpha^{-1}z}{1-\alpha z^{-1}} + \frac{\beta z^{-1}}{1-\beta z^{-1}}$$

Since the first term has a ROC when $|z| < |\alpha|$ and the second term has a ROC when $|z| > |\beta|$, there is no intersection between the two terms' ROC. Therefore, there is no ROC and $X_2(z)$ does not exist.

(c)

$$X_3(z) = \frac{\alpha^{-1}z}{1-\alpha z^{-1}} + \frac{-\beta^{-1}z}{1-\beta z^{-1}}$$

Therefore, the ROC: $|\alpha| < |z| < |\beta|$.


Im
|β|
|α|
Re

4. (10 points) **Z-transform Block Diagram** The following causal system, in Figure 1, has the input $x[n]$ and the output $y[n]$. Where $G(z)$ is defined as:

$$G(z) = \alpha + z^{-1}, \quad \alpha > 1$$

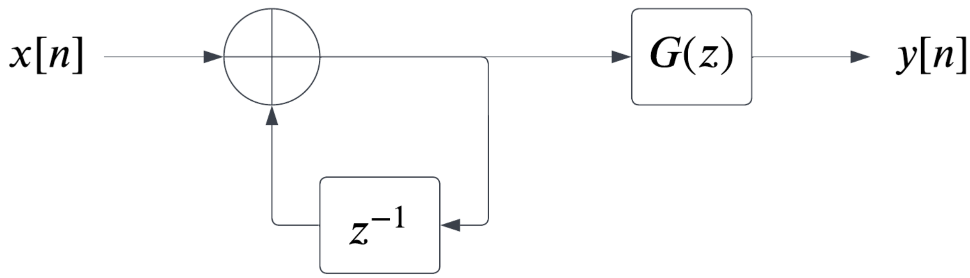


Figure 1: System Block Diagram

(a) What is the transfer function for this system?

(b) What is the impulse response?

(c) What is the region of convergence corresponding to the impulse response?

Solutions:

(a) For the feedback loop, let's assume its output is M(z). Then,

$$M(z) = X(z) + z^{-1}M(z) \longrightarrow \frac{M(z)}{X(z)} = \frac{1}{1 - z^{-1}}$$

Then the transfer function can be evaluated as,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1}}G(z) = \frac{\alpha + z^{-1}}{1 - z^{-1}}$$

(b) To get the impulse response, we apply the inverse Z-transform with the knowledge that this system is causal.

$$h[n] = \alpha u[n] + u[n-1]$$

(c) Since its causal and just simple unit step functions, ROC: $|z| > 1$.

5. (10 points) **Inverse Z-transform** Find the impulse response, $h[n]$ of the following transfer function $H(z)$ and its ROC:

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}, \quad |z| > \frac{1}{2}$$

Solutions:

First decompose into partial fractions:

$$H(z) = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - \frac{1}{3}z^{-1}}$$

Solving for A_1 and A_2 yields -9 and 10 respectively. Therefore,

$$H(z) = \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{10}{1 - \frac{1}{3}z^{-1}}$$

Then, using a Z-transform table with the associated ROC the impulse response would be:

$$h[n] = -9(\frac{1}{2})^n u[n] + 10(\frac{1}{3})^n u[n]$$