23W-EC ENGR-131A-LEC-1 Homework 7

SANJIT SARDA

TOTAL POINTS

88 / 100

QUESTION 1

1 1 **20 / 20**

- ✓ **- 0 pts** *Correct*
- **- 2 pts** Partially correct part a
- **- 2 pts** Partially correct part b
- **- 2 pts** Partially correct part c
- **- 2 pts** Partially correct part d
- **- 2 pts** Partially correct part e
- **- 20 pts** Missing

QUESTION 2

2 2 **20 / 20**

- ✓ **- 0 pts** *Correct*
- **- 2 pts** Partially correct part a
- **- 2 pts** Plot missing part b
- **- 2 pts** Plot missing part c
- **- 2 pts** Plot missing part d
- **- 2 pts** Plot missing part e
- **- 20 pts** Missing

QUESTION 3

3 3 **20 / 20**

- ✓ **- 0 pts** *Correct*
- **- 2 pts** Partially correct part a
- **- 2 pts** Partially correct part b
- **- 10 pts** Missing part a
- **- 10 pts** Missing part b

QUESTION 4

4 4 **20 / 20**

- ✓ **- 0 pts** *Correct*
- **- 2 pts** Partially correct part a
- **- 2 pts** Partially correct part b
- **- 10 pts** Missing part a
- **- 10 pts** Missing part b

QUESTION 5

5 5 **8 / 20**

- **- 0 pts** Correct
- **- 4 pts** Partially correct
- ✓ **- 12 pts** *Incorrect*
- **- 20 pts** Missing

QUESTION 6

6 Q3 from HW6 **0 / 0**

- ✓ **- 0 pts** *Check HW6 for score*
- **- 3 pts** (a) incorrect
- **- 3 pts** (b) incorrect
- **- 3 pts** (c) incorrect
- **- 3 pts** (d) incorrect

ECE 131A HW7

① X & Y are RVs : $U[0,1]$

$$F_X(x) = \begin{cases} 0 & x<0 \\ x & 0\le x\le 1 \\ 1 & x>1 \end{cases} \quad F_Y(y) = \begin{cases} 0 & y<0 \\ y & 0\le y\le 1 \\ 1 & y>1 \end{cases}$$

ⓐ $A = X+Y$

$F_A(a) = P(X+Y\le a) = P(X\le a-Y)$

$\therefore F_A(a) = \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} F_{XY}(x,y)\,dx\,dy$

$\therefore f_A(a) = \frac{d}{da}\int_{-\infty}^{\infty}\int_{-\infty}^{a-y} F_{XY}(x,y)\,dx\,dy$

$$\therefore f_A(a) = \int_{-\infty}^{\infty} F_X(a-y)F_Y(y)\,dy = \begin{cases} a, & 0\le a<1 \\ 2-a, & 1\le a\le 2 \\ 0, & \text{o.w} \end{cases}$$

ⓑ $B = X-Y$

$F_B(b) = P(X-Y\le b) = P(X\le b+Y)$

$\therefore F_B(b) = \int_{-\infty}^{\infty}\int_{-\infty}^{b+y} F_{XY}(x,y)\,dx\,dy$

$\therefore f_B(b) = \frac{d}{db}\int_{-\infty}^{\infty}\int_{-\infty}^{b+y} F_{XY}(x,y)\,dx\,dy$

$\therefore f_B(b) = \int_{-\infty}^{\infty} F_X(b+y)F_Y(y)\,dy$ let $v=-y$, $dv=-dy$

$$f_B(b) = \begin{cases} 1+b, & -1\le b<0 \\ 1-b, & 0\le b<1 \\ 0, & \text{o.w.} \end{cases}$$

ⓒ $C = XY$

$F_C(c) = P(XY\le c) = P(X\le c/Y)$

$F_C(c) = \int_{-\infty}^{\infty}\int_{-\infty}^{c/y} F_{XY}(x,y)\,dx\,dy$ $\qquad F' = f$

$f_C(c) = \int_{-\infty}^{\infty} F_X(c/y)F_Y(y)\,dy$

$$= \int_0^1 F_X(c/y)\cdot 1\cdot dy = \int_0^1 \begin{cases} c/y & 0\le c/y\le 1 \\ 0 & \text{ow} \end{cases} dy$$

$$= \begin{cases} \int_0^1 \{ c/y \quad 0\le c/y\le 1 \; dy \\ 0, \text{ ow} \end{cases} = \begin{cases} \int_0^1 \{ c/y \quad 0\le c\le y \; dy \\ 0 \text{ ow} \end{cases}$$

$$= \begin{cases} -\ln c & 0\le c\le 1 \\ 0, & \text{ow.} \end{cases}$$

(d) $COV(A,B) = E[(A-1)B]$

$= E[AB - B]$

$= E[X^2 - Y^2 - X + Y]$

$= E[X^2] - E[Y^2] - E[X] + E[Y]$

$= E[Y] - E[X]$

$= 1/2 - 1/2 = 0$

$E[X^2] = E[Y^2]$

$E[C] = \int_0^1 -c \ln c \, dc$

$= 1/4$

(e) $COV(A,C) = E[AC] - m_A m_C = E[AC] - m_C$

$= E[AC] - 1/4$

$= E[(X+Y)XY] - 1/4$

$= E[X^2Y + XY^2] - 1/4$

$= E[X^2Y] + E[XY^2] - 1/4$

$= \frac{1}{2}E[X^2] + \frac{1}{2}E[Y^2] - 1/4$

$= 1/3 - 1/4 = \frac{4}{12} - \frac{3}{12} = 1/12$

$X \perp\!\!\!\perp Y \therefore E[X^2Y] = E[X^2]E[Y]$

1 1 **20 / 20**

✓ **- 0 pts** *Correct*

- **- 2 pts** Partially correct part a
- **- 2 pts** Partially correct part b
- **- 2 pts** Partially correct part c
- **- 2 pts** Partially correct part d
- **- 2 pts** Partially correct part e
- **- 20 pts** Missing

② $X = \mathcal{N}(0,1)$, $Y = aX+b$, $a \neq 0$, $Z = X^2$, $W = X^3$

We have
$E[X^3]=0$
$E[X^4]=3\sigma^4$
$E[X^5]=0$
$E[X^6]=15\sigma^6$

$$E[Y] = E[aX+b] = aE[X]+b = b$$
$$E[Y^2] = E[a^2X^2+2abX+b^2] = a^2E[X^2]+2abE[X]+b^2$$
$$= a^2+b^2$$

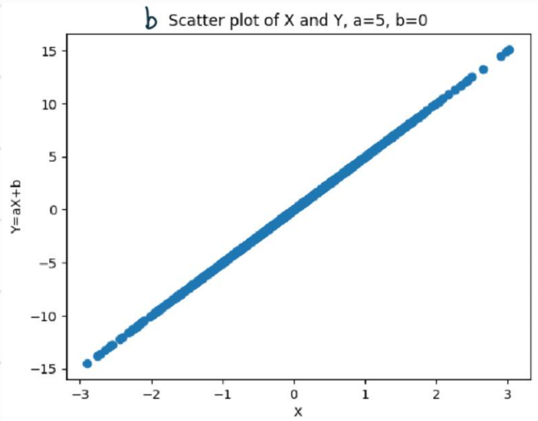
ⓐ $$\rho_{XY} = \frac{\mathrm{COV}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY]-E[X]E[Y]}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}} = \frac{E[XY]}{\sqrt{E[Y^2]-E[Y]^2}} = \frac{E[XY]}{\sqrt{a^2}}$$

$$= \frac{E[X(aX+b)]}{\sqrt{a^2}} = \frac{E[aX^2+bX]}{\sqrt{a^2}} = \frac{aE[X^2]+bE[X]}{\sqrt{a^2}} = \frac{a}{\sqrt{a^2}} = \mathrm{sign}(a)$$

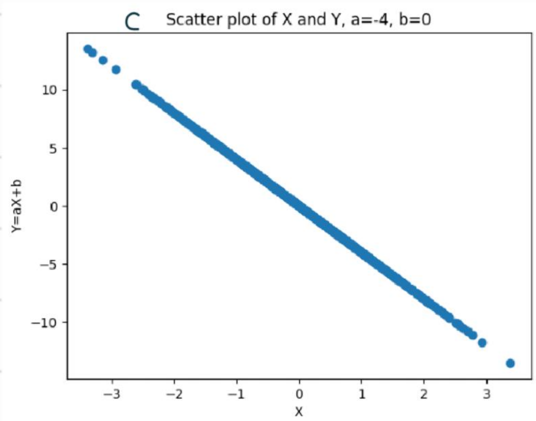
$$\rho_{XZ} = \frac{E[XZ]-E[X]E[Z]}{\sqrt{(E[X^2]-E[X]^2)(E[Z^2]-E[Z]^2)}} = \frac{E[XZ]}{\sqrt{E[Z^2]-E[Z]^2}} = \frac{E[X^3]}{\sqrt{E[X^4]-(E[X^2])^2}} = 0$$

$$\rho_{XW} = \frac{E[XW]-E[X]E[W]}{\sqrt{(E[X^2]-E[X]^2)(E[W^2]-E[W]^2)}} = \frac{E[XW]}{\sqrt{E[W^2]-E[W]^2}} = \frac{E[X^4]}{E[X^6]-E[X^3]^2}$$
$$= \frac{3\sigma^4}{\sqrt{15\sigma^6}} = \frac{3}{\sqrt{15}}$$

ⓑ

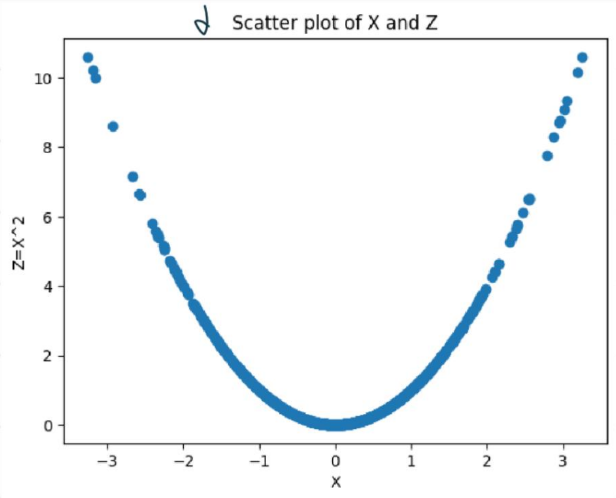


$\rho = \mathrm{sign}(+5) = +1$, from the image, we can see that Y is 5x and the ρ is ⊕, therefore there is an upward trend

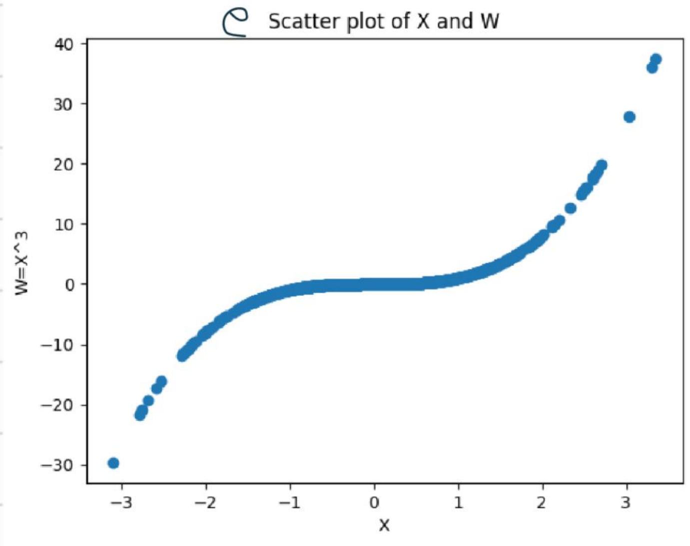


ⓒ $\rho = \mathrm{sign}(-4) = -1$ from the image we can see that $Y = -4x$ & there is a downward trend since ρ is ⊖. This is diff from b which had ρ ⊕ and an upward trend.

d



$\rho = 0$. From the image we see that $Y = X^2$, there is however no upward or downward trend. This is diff from b&c which are $+$ & $-$ ρ.



e $\rho = \frac{3}{\sqrt{15}}$. From the image we see that $Y = X^3$. ρ is $+$ and the plot trends upwards but does not linear fit.

2 2 **20 / 20**

✓ **- 0 pts** *Correct*

- 2 pts Partially correct part a

- 2 pts Plot missing part b

- 2 pts Plot missing part c

- 2 pts Plot missing part d

- 2 pts Plot missing part e

- 20 pts Missing

(3) X & Y are RVs with identical dists

$\therefore E[X^n] = E[Y^n]$, $\quad COV(X,Y) = E[XE[X]] - E[X]^2$

(a) $C = aX + bY \quad D = aX - bY$

$COV(C,D) = E[CD] - E[C]E[D]$

$= E[a^2X^2 - b^2Y^2] - E[aX+bY]E[aX-bY]$

$= a^2E[X^2] - b^2E[Y^2] - (aE[X] + bE[Y])(aE[X] - bE[Y])$

$= (a^2-b^2)(E[X^2]) - (aE[X])^2 + (bE[Y])^2$

$= (a^2-b^2)\ E[X^2] - (a^2-b^2)E[X]^2$

$= (a^2-b^2)(E[X^2] - E[X]^2) = (a^2-b^2)\cdot\sigma$, where $\sigma = Var(X) = Var(Y)$

(b) $C \perp\!\!\!\perp D$

$\therefore cov(C,D) = 0$, $\therefore (a^2-b^2)\sigma = 0$

$\therefore a^2 = b^2$

$\therefore$ If C & D are independant, $|a| = |b|$

3 3 **20 / 20**

✓ **- 0 pts** *Correct*

- 2 pts Partially correct part a

- 2 pts Partially correct part b

- 10 pts Missing part a

- 10 pts Missing part b

④ ⓐ RVs: X & Y

Prove: $|\rho_{XY}| \leq 1$

* σs are >0

$$\rho_{XY} = \frac{E[(X-E[X])(Y-E[Y])]}{\sigma_x \sigma_y}$$

$$\therefore |\rho_{XY}| = \frac{|E[(X-E[X])(X-E[Y])]|}{*\sqrt{\sigma_x \sigma_y}}$$

From Cauchy Schwarz,

$E[XY] \leq \sqrt{E[X^2]E[Y^2]}$

$$\therefore |\rho_{XY}| \leq \frac{\sqrt{E[(X-E[X])^2]E[(Y-E[Y])^2]}}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}}$$

$$\therefore |\rho_{XY}| \leq \frac{\sqrt{E[(X-E[X])^2] \quad E[(Y-E[Y])^2]}}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}}$$

$$\therefore |\rho_{XY}| \leq \frac{\sqrt{(E[X^2-2XE[X]+E[X]^2])(E[Y^2-2YE[Y]+E[Y]^2])}}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}}$$

$$\therefore |\rho_{XY}| \leq \frac{\sqrt{(E[X^2]-2E[X]E[E[X]]+E[E[X]^2])(E[Y^2]-2E[Y]E[E[Y]]+E[E[X]^2])}}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}}$$

$$\therefore |\rho_{XY}| \leq \frac{\sqrt{(E[X^2]-2(E[X])^2+E[X]^2)(E[Y^2]-2E[Y]^2+E[Y]^2)}}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y^2])}}$$

$$\therefore |\rho_{XY}| \leq \frac{\sqrt{(E[X^2]-E[X]^2}\sqrt{(E[Y^2]-E[Y]^2)}}{\sqrt{(E[X^2]-E[X]^2)}\sqrt{E[Y^2]-E[Y]^2}}$$

$\therefore |\rho_{XY}| \leq 1 \quad \therefore -1 \leq \rho_{XY} \leq 1$

ⓑ X is RV, $Y = aX+b$

$$\sigma_y = E[Y^2]-E[Y]^2$$
$$= \sqrt{a^2}\,\sigma_x$$

$$\rho_{XY} = \frac{E[XY]-E[X]E[Y]}{\sigma_x \sigma_y} = \frac{E[aX^2+bX]-E[X]\cdot E[aX+b]}{\sigma_x \sigma_y}$$

$$= \frac{aE[X^2]+bE[X]-aE[X]^2-bE[X]}{\sigma_x \sigma_y} = \frac{a(E[X^2]-E[X])}{\sigma_x \sigma_y}$$

$$= \frac{a\sigma_x}{\sigma_y} = \frac{a\cancel{\sigma_x}}{\sqrt{a^2}\cancel{\sigma_x}}$$

$\therefore$ it does depend on the sign of a.

$$= \frac{a}{\sqrt{a^2}} = \text{sign}(a)$$

4 4 **20 / 20**

✓ **- 0 pts** *Correct*

- 2 pts Partially correct part a

- 2 pts Partially correct part b

- 10 pts Missing part a

- 10 pts Missing part b

⑤ X & Y are jointly Gaussian: $E[Y]=0$ $\sigma_X=4$ $\sigma_Y=3$ $E[X|Y]=\frac{4Y}{9}+2$

Joint PDF: $f_{XY}(x,y) = \dfrac{E\left[\frac{1}{2(1-\rho_{XY}^2)}\left[\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho_{XY}\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right)+\left(\frac{y-m_Y}{\sigma_Y}\right)^2\right]\right]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}$

where $\sigma_X=4$, $\sigma_Y=3$, $m_Y=0$

To Find ρ_{XY} & m_X,

$E[Y^2]-(E[Y])^2=\sigma_Y^2$

$\therefore E[Y^2]=\sigma_Y+(E[Y])^2$

$=9+0$

$=9$

$E[X]=E[E[X|Y]]=E[4Y/9+2]=\frac{4}{9}E[Y]+2=2$

$\rho_{XY}=\dfrac{COV(X,Y)}{\sigma_X\sigma_Y}=\dfrac{E[XY]-E[X]E[Y]}{12}=\dfrac{E[E[XY|Y]]}{12}=\dfrac{E[YE[X|Y]]}{12}$

$=\frac{1}{12}E[Y\cdot\frac{4}{9}Y+2Y]=\frac{1}{12}\left(\frac{4}{9}E[Y^2]+2E[Y]\right)=\frac{1}{12}\left(\frac{4}{9}\cdot 9\right)=\frac{4}{12}=\frac{1}{3}$

$\therefore \rho_{XY}^2=\frac{1}{9}$

$\therefore \sigma_X=4, \sigma_Y=3, m_Y=0, m_X=2, \rho_{XY}=\frac{1}{3}$ & $\rho_{XY}^2=\frac{1}{9}$

5 5 **8 / 20**

- **- 0 pts** Correct
- **- 4 pts** Partially correct
- ✓ **- 12 pts** *Incorrect*
- **- 20 pts** Missing

⑥ $f_{XY}(x,y) = \begin{cases} 6xy, & 0 \le x \le 1, \quad 0 \le y \le \sqrt{x} \\ 0, & \text{o.w.} \end{cases}$

ⓐ $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\sqrt{x}} (6x)y\,dy = 3xy^2 \Big|_0^{\sqrt{x}} = 3x \cdot x = 3x^2$

ⓑ $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{f_{XY}(x,y)}{\int_{y^2}^1 6xy\,dx} = \dfrac{f_{XY}(x,y)}{3y(1^2-y^{2^2})} = \begin{cases} \frac{2x}{1-y^4}, & y^2 \le x \le 1 \\ 0, & \text{ow} \end{cases}$

ⓒ $E[X|Y=y]$

$= \int_{y^2}^1 x f(x|y)\,dx = \int_{y^2}^1 \frac{2x^2}{1-y^4}\,dx = \dfrac{2}{3(1-y^4)}$

$E[X|Y] = \int_{y^2}^1 \frac{2x^2}{1-y^3}\,dx = \dfrac{2}{3(1-y^4)}$

ⓓ A : event → $X \ge 1/2$

$P[A] = \int_{1/2}^1 \int_0^{\sqrt{x}} f_{XY}(x,y)\,dy\,dx = \int_{1/2}^1 \int_0^{\sqrt{x}} 6xy\,dy\,dx = \int_{1/2}^1 3x^2\,dx = x^3 \Big|_{1/2}^1 = 1 - 1/8$

$= 7/8$

$f_{X|A}(x) = P(X|A) = \dfrac{P(X \cap A)}{P(A)} = \dfrac{P(X)}{P(A)} = \begin{cases} \frac{48}{7}xy, & \frac{1}{2} \le x \le 1,\ 0 \le y \le \sqrt{x} \\ 0, & \text{ow} \end{cases}$

$E[X|A] = \int_{1/2}^1 \int_0^{\sqrt{x}} x f_{X|A}(x)\,dy\,dx = \int_{1/2}^1 \frac{24}{7}x^3\,dx = \frac{6}{7}x^4 \Big|_{1/2}^1 = \frac{6}{7} - \frac{6}{7}\cdot\frac{1}{16} = \frac{15}{\cancel{16}_8}\cdot\frac{\cancel{6}^3}{7} = \frac{45}{56}$

6 Q3 from HW6 0 / 0

✓ **- 0 pts** *Check HW6 for score*

- 3 pts (a) incorrect

- 3 pts (b) incorrect

- 3 pts (c) incorrect

- 3 pts (d) incorrect