23W-EC ENGR-131A-LEC-1 Homework 8

SANJIT SARDA

TOTAL POINTS

98 / 100

QUESTION 1

1 Question 1 **18 / 20**

✓ **- 0 pts** *Correct*

- 5 pts (a) incorrect

- 5 pts (b) incorrect

- 3 pts didn't state mean and variance for (b)

- 2 pts incorrect mean and/or variance for (b)

- 3 pts didn't state mean and variance for (a)

✓ **- 2 pts** *incorrect mean and/or variance for (a)*

- 20 pts missing

QUESTION 2

2 Question 2 **20 / 20**

✓ **- 0 pts** *Correct*

- 7 pts incorrect

- 10 pts little/no work and incorrect

- 20 pts missing

QUESTION 3

3 Question 3 **20 / 20**

✓ **- 0 pts** *Correct*

- 3 pts (a) (i) incorrect

- 3 pts (a) (ii) incorrect

- 3 pts (b) (i) incorrect

- 3 pts (b) (ii) incorrect

QUESTION 4

4 Question 4 **20 / 20**

✓ **- 0 pts** *Correct*

- 5 pts incorrect

- 10 pts incorrect and little/no work shown

- 20 pts missing

QUESTION 5

5 Question 5 **20 / 20**

✓ **- 0 pts** *Correct*

- 5 pts incorrect

- 10 pts incorrect and little/no work

- 20 pts missing

ECE 131A HW 8

① $f_{XY}(x,y) = f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right].$

ⓐ $f_Y(y)$

$$= \int_{-\infty}^{\infty} f_{XY}(x', y)\,dx'$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)} dx'$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)} dx$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)} dx$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{(1-\rho^2)}\frac{y^2}{\sigma_y^2}}}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}-\frac{x^2}{\sigma_x\sigma_y}-\frac{\rho^2y^2}{\sigma_x\sigma_y}+\frac{(x-\rho y)^2}{\sigma_x\sigma_y}\right)} dx$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{(1-\rho^2)}\frac{y^2}{\sigma_y^2}}}{\sigma_y\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}-\frac{x^2}{\sigma_x\sigma_y}-\frac{\rho^2y^2}{\sigma_x\sigma_y}+\frac{(x-\rho y)^2}{\sigma_x\sigma_y}\right)}}{\sigma_x\sqrt{2\pi(1-\rho^2)}} dx$$

$$= \frac{e^{-\frac{y^2}{2}\cdot\frac{1}{(1-\rho^2)}\left(\frac{1}{\sigma_y^2}-\frac{\rho^2}{\sigma_x\sigma_y}\right)}}{\sigma_y\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}-\frac{x^2}{\sigma_x\sigma_y}+\frac{(x-\rho y)^2}{\sigma_x\sigma_y}\right)}}{\sigma_x\sqrt{2\pi(1-\rho^2)}} dx$$

$$= \frac{e^{-\frac{y^2}{2\sigma_y^2}}}{\sigma_y\sqrt{2\pi}} \cdot 1 = \text{Gaussian} \rightarrow N(\overset{E[X]}{0}, \underset{Var[X]}{\sigma_y})$$

(b) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} =$

$$= \frac{e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\left(\sigma_y\sqrt{2\pi}\right)e^{y^2/2\sigma_y^2}$$

$$= \frac{e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)+\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma_x\,\sqrt{1-\rho^2}}$$

$$= \frac{e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}-\frac{y^2(1-\rho^2)}{\sigma_y^2}\right)}}{\sqrt{2\pi}\,\sigma_x\,\sqrt{1-\rho^2}}$$

$$= \frac{e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2\rho^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)}}{\sqrt{2\pi}\,\sigma_x\,\sqrt{1-\rho^2}} = \text{Gaussian} \rightarrow \mathcal{N}\Big(\overbrace{\rho y\frac{\sigma_x}{\sigma_y}}^{E[X]},\ \underbrace{(1-\rho^2)\sigma_x^2}_{Var[X]}\Big)$$

1 Question 1 **18 / 20**

✓ **- 0 pts** *Correct*

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- 3 pts didn't state mean and variance for (a)

✓ **- 2 pts** *incorrect mean and/or variance for (a)*

- 20 pts missing

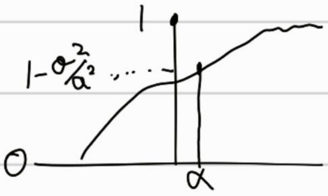
② $S_n = X_1, X_2 \ldots\ldots X_y = \sum_n X_n$

$m_K = E[X_K]$ $\quad \sigma_K^2 = Var[X_K]$ $\quad M_n = \sum_n m_n$

and

$\sigma^2 < R$ & $m_K < T$, $\therefore \lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - \frac{M_n}{n}\right| < \epsilon\right) = 1$, using Chebyshev:

$P(|x-\mu| > \alpha) \leq \frac{\sigma^2}{\alpha^2} \rightarrow$



$\rightarrow P(|x-\mu| < \alpha) \geq 1 - \frac{\sigma^2}{\alpha^2}$

$\therefore P(|x-\mu| < \alpha) + \frac{\sigma^2}{\alpha^2} \geq 1$

$P(|S_n - M_n| < \alpha) + \frac{Var[\sum X_K]}{\alpha^2} \geq 1$

$P(|S_n - M_n| < \alpha) + \frac{\sum Var[X_K]}{\alpha^2} \geq 1$

$P\left(\frac{1}{n}|S_n - M_n| < \frac{\alpha}{n}\right) + \frac{\sum Var[X_K]}{\alpha^2} \geq 1$

$\lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - \frac{M_n}{n}\right| < \epsilon\right) + \frac{\sum Var[X_K]}{\epsilon^2 n^2} \geq 1$

$\therefore P\left(\left|\frac{S_n}{n} - \frac{M_n}{n}\right| < \epsilon\right) + 0 \geq 1$

$\therefore P\left(\left|\frac{S_n}{n} - \frac{M_n}{n}\right| < \epsilon\right) \geq 1$

Since You cannot have $P > 1$,

$P\left(\left|\frac{S_n}{n} - \frac{M_n}{n}\right| < \epsilon\right) = 1$

2 Question 2 **20 / 20**

✓ **- 0 pts** *Correct*

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③ X = Fair coin Toss : $p = 1/2$

$\therefore E[X] = 1/2$

$\therefore Var[X] = (1/2)(1-1/2) = 1/4$

ⓐ $Y_1 = \sum_{0}^{100} X_i \quad \therefore E[Y_1] = 100E[X] = 50$

$Var[Y_1] = 100E[X] = 25$

$\therefore Y_1 \sim N(50, 25) \quad \therefore Z_1 = \frac{Y_1 - 50}{\sqrt{25}} \sim N(0,1)$

$\therefore P(40 \leq Y_1 \leq 60) = P\left(\frac{40-50}{5} \leq \frac{Y_1-50}{5} \leq \frac{60-50}{5}\right)$

$= P(-2 \leq Z_1 \leq 2)$

$= Q(2) - Q(2)$

$= 1 - Q(2) - Q(2)$

$= 1 - 2Q(2) \simeq .95$

$\therefore P(60 \leq Y_1 \leq 80) = P\left(\frac{60-50}{25} \leq \frac{Y_1-50}{25} \leq \frac{80-50}{25}\right)$

$= P(2 \leq Z_1 \leq 6)$

$= Q(2) - Q(6)$

$\simeq 0.02275$

ⓑ $Y_2 = \sum_{0}^{100} X_i \quad \therefore E[Y_2] = 10000E[X] = 5000$

$Var[Y_2] = 10000E[X] = 2500$

$\therefore Y_2 \sim N(5000, 2500) \quad \therefore Z_1 = \frac{Y_1 - 5000}{\sqrt{2500}} \sim N(0,1)$

$\therefore P(4000 < Y_1 \leq 6000) = P\left(\frac{4000-5000}{\sqrt{2500}} \leq \frac{Y_1-5000}{\sqrt{2500}} \leq \frac{6000-5000}{\sqrt{2500}}\right)$

$= P(-20 \leq Z_1 \leq 20)$

$= Q(-20) - Q(20)$

$= 1 - Q(20) - Q(20)$

$= 1 - 2Q(20) \simeq 1$

$\therefore P(6000 \leq Y_1 \leq 8000) = P\left(\frac{6000-5000}{\sqrt{2500}} \leq \frac{Y_1-5000}{\sqrt{2500}} \leq \frac{8000-5000}{\sqrt{2500}}\right)$

$= P(20 \leq Z_1 \leq 60)$

$= Q(20) - Q(60)$

$= 1 - Q(20) - Q(60)$

$\simeq 0$

3 Question 3 **20 / 20**

✓ **- 0 pts** *Correct*

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- 3 pts (b) (i) incorrect

- 3 pts (b) (ii) incorrect

④ We have 108 numbers in a list $L \to L_1, L_2, L_3 \cdots L_{108}$
These are represented by ints: $X \to X_1, X_2, X_3 \cdots X_{108}$

$\therefore$ Errors are $Y = X - L = U[-.5, .5] \to Y_1, Y_2 \cdots Y_{108}$
$W = \sum_i Y_i$ Sum of error: $E[W] = 108 \cdot E[Y_i] = 0$, $Var[W] = 108 \cdot Var[Y_i] = \frac{108}{12} = 9$

Applying CLT, $W \sim N(0, 27)$ $\therefore Z = \frac{W-0}{3} \sim N(0,1)$

$\therefore P(|W| > 2) = P(|\frac{W}{3}| > \frac{2}{3}) = 1 - P(|\frac{W}{3}| < \frac{2}{3}) = 1 - P(\frac{-2}{3} < Z < \frac{2}{3})$

$= 1 - P(\frac{-2}{3} < Z < \frac{2}{3}) = 1 - (Q(\frac{-2}{3}) - Q(\frac{2}{3})) = 1 - (1 - Q(\frac{2}{3}) - Q(\frac{2}{3}))$
$= 2Q(\frac{2}{3}) = 2Q(\frac{2}{3}) \sim 0.505$

4 Question 4 **20 / 20**

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⑤ $X_1 \cdots X_n \rightarrow \text{i.i.d} \rightarrow E(X) = \mu, \ Var(X) = \sigma^2$

$$V_n = \frac{1}{n}\sum_{i=1}^{n}(x_i - M_n)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu + \mu - M_n)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left((x_i-\mu)+(\mu-M_n)\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left((x_i-\mu)^2 + 2(x_i-\mu)(\mu-M_n) + (\mu-M_n)^2\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\mu)^2 + 2\sum_{i=1}^{n}(x_i-\mu)(\mu-M_n) + \sum_{i=1}^{n}(\mu-M_n)^2\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\mu)^2 + 2(\mu-M_n)\sum_{i=1}^{n}(x_i-\mu) + n(\mu-M_n)^2\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\mu)^2 + 2(\mu-M_n)\left(\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\mu\right) + n(\mu-M_n)^2\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\mu)^2 + 2(\mu-M_n)(M_n-\mu)n \quad + n(\mu-M_n)^2\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\mu)^2 + n\left((\mu-M_n)^2 - 2(\mu-M_n)^2\right)\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\mu)^2 - n(\mu-M_n)^2\right) = \frac{1}{n}\sum_{i=1}^{n}\left((x_i-\mu)^2 - (M_n-\mu)^2\right)$$

$$E[V_n] = \frac{1}{n}E\left[\sum_{i=1}^{n}(x_i-\mu)^2 - (M_n-\mu)^2\right] \qquad = \frac{1}{n}E\left[\sum_{i=1}^{n}(x_i-\mu)^2 - \sum_{i=1}^{n}(M_n-\mu)^2\right]$$

$$E[V_n] = \frac{1}{n}E\left[\sum_{i=1}^{n}(x_i-\mu)^2 - \sum_{i=1}^{n}(M_n-\mu)^2\right]$$

$$E[V_n] = \frac{1}{n}\left(E\left[\sum_{i=1}^{n}(x_i-\mu)^2\right] - E\left[\sum_{i=1}^{n}(M_n-\mu)^2\right]\right)$$

$$E[V_n] = \frac{1}{n}(n\sigma^2 - \sigma^2) = \frac{n-1}{n}\sigma^2$$

the $\frac{n-1}{n}$ term adds bias to the variance: σ^2
since it will be ever so slightly smaller
based on the number of samples.

5 Question 5 **20 / 20**

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- 5 pts incorrect

- 10 pts incorrect and little/no work

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ⓐ $f_Y(y) = \int_0^y f_{XY}(x,y)\,dx = \int_0^y \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\left(\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)\right)} dx$

$$f_X(x) = \frac{e^{-\frac{x^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2-\rho^2x^2}{2(1-\rho^2)}} dy$$

$$f_X(x) = \frac{e^{-\frac{x^2}{2}\frac{1}{(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} e^{\frac{\rho^2x^2}{2(1-\rho^2)}} dy$$

$$f_X(x) = \frac{e^{-\frac{x^2}{2}\frac{1}{(1-\rho^2)}+\frac{\rho^2x^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} dy$$

Bad

$$f_X(x) = \frac{e^{-\frac{x^2}{2}\frac{1}{(1-\rho^2)}+\frac{x^2}{2}\frac{\rho^2}{1-\rho^2}}}{\sqrt{2\pi(1-\rho^2)}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} dy$$

$$\therefore f_X(x) = \frac{e^{-\frac{x^2}{2}\left(\frac{1}{1-\rho^2}-\frac{\rho^2}{1-\rho^2}\right)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}} dy$$

$$\therefore f_X(x) = \frac{e^{-\frac{x^2}{2}\left(\frac{1-\rho^2}{1-\rho^2}\right)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}} dy$$

$$\therefore f_X(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}} dy$$

$$\therefore f_X(x) = \frac{e^{-x^2}}{\sqrt{2\pi}}$$

wide letter e

← tall lettere