ECE 113 CS

$*113$: $\text{sinc}(x) = \frac{\sin x}{x}$

Signals: 113 → Discrete: $x[n] = \cos(2\pi n)$ || $x[n] = \{1,2,3,4,5\}$, $-2 \le n \le 2$

Signal Manipulation: ① Arithmetic: ⊕, ⊗ ② Time scaling & scaling: $x[\alpha n + \beta]$

Basic Bitch Signals: $\delta[n] = \begin{cases}1, & n=0\\ 0, & ow\end{cases}$, $u[n] = \begin{cases}1, & n\ge0\\ 0, & ow\end{cases}$, $r[n] = \begin{cases}n, & n\ge0\\ 0, & ow\end{cases}$ | Sampling Holds in $\delta[n]$ | $r' = u$, $r'' = u' = \delta$ | Any Signals can be rep by δ: $x[n] = \{1,2,0,3\}$, $x[n] = \delta[n] + 2\delta[n-1]$ ← Canonical Basis

$\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2}$

$\sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{2j}$

~ denotes periodicity

Sinusoid: $\tilde{x}[n] = A\cos(\Omega_0 n + \theta)$: $\Omega_0 \in [-\pi, \pi]$, $\Omega = 2\pi F_0$; $F_0 \to$ Frequency; Normalized Ω: angular Freq, F_s: sampling freq | Complex: $x[n] = Ae^{j(2\pi f_0 n + \theta)}$, $e^{j\phi} = \cos\phi + j\sin\phi$

Fundamental Period: For $\tilde{x}[n]$, smallest N, such: $\tilde{x}[n+N] = x[n]$ | Extra constraint: N must be a ratio of rational #s

Even & odd signals: E: $x[n] = x[-n]$ | O: $x[n] = -x[-n]$ | $x_e[n] = \frac{x[n] + x[-n]}{2}$, $x_o[n] = \frac{x[n] - x[-n]}{2}$ | E.S.: $x^*[n] = x[-n]$, O.S.: $x^*[n] = -x[-n]$

Complex: $e^{j\pi} - 1 = 0$ Sym $= x^*[n] = x[-n]$, $= x^*[n] = -x[-n]$

$x^*[n] = |x[n]|e^{-j\angle x[n]}$

$x[n] = Re(x[n]) + jIm(x[n]) = |x[n]|e^{j\angle x[n]}$

$Re(x) = r\cos\phi$, $Im(x) = r\sin\phi$ | $x^* = Re(x) - jIm(x)$, $x^2 = xx^*$, $j = -\frac{1}{j}$ | $j = e^{j\pi/2}$, $-j = e^{-j\pi/2}$, $1 = e^{j\cdot 0}$, $-1 = e^{j\pi}$ *

$z = re^{j\phi}$ r: Mag, ϕ = Angle

$re^{j\phi} = r\cos\phi + rj\sin\phi$

$r = \sqrt{Re^2 + Im^2}$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

Constant Coeff Difference

$x \to [S] \to y$

$y = \sum_{k=0}^{N-1} b_k x[n-k]$

Feed Back → Needs init condit

$\sum_{k=0}^{M-1} a_k y[n-k] = \sum_{k=0}^{N-1} b_k x[n-k]$

Energy: $E_x = \sum_n |x[n]|^2$: Energy Signal if $0 < E_x < \infty$ → similar applies

Power: If $\tilde{x} \to P_x = \frac{1}{N}\sum_{n=0}^{N}|x[n]|^2$ else $P_x = \lim_{m\to\infty}\frac{1}{2m+1}\sum_{n=-m}^{m}|x[n]|^2$

Suppose $x_1[n] = 2\cos(.4\pi n)$, $x_2 = 1.5\cos(.48\pi n)$

$N_1 = \frac{k_1}{F_1} = \frac{2\pi k_1}{.4\pi} = 5k_1 = 5$, $N_2 = \frac{k_2}{F_2} = \frac{2\pi k_2}{.48} = \frac{25}{6}k_2 = 25$

Complexity: DFT: $O(N^2)$, FFT: $O(N\log N)$ | Loops → N, Bin Search → $N\log N$

Exp smoother: $y[n] = (1-\alpha)y[n-1] + \alpha x[n]$

Systems: $x \to [H] \to y$

Linearity: $H[ax + bx_0] = aH[x] + bH[x_0]$

Homogeneity: $H[ax] = aH[x]$

Superpos: $H[x + x_0] = H[x] + H[x_0]$

Time Invariance: $S[x[n-k]]$ Delay Inp $= y[n-k]$ Delay Out

Causality: No dependance on past values

Stability: BiBo — BiBO stable: $|x[n]| < B$ & $|y[n]| < C$

System Blocks: Unit Delay: [D], Additional Delay: $[z^{-n}]$, ⊗ation: $\triangleright k$, ⊕tion: ⊕

Convolution: $x * h = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

1) $x_1 * x_2 = x_2 * x_1$
2) $x * (h_0 + h_1) = x * h_0 + x * h_1$
3) $h[n] * \delta[n-k] = h[n-k]$

$x_1[n]$: $n_1 \le n \le n_1 + L_1$ | $L = L_1 + L_2 + 1$
$x_2[n]$: $n_2 \le n \le n_2 + L_2$
$x_1 * x_2$: $n_1 + n_2 \le n \le n_1 + n_2 + L_1 + L_2$

Thinking of signals with basis vectors.

$x[n] = \sum_k c_k \psi_k[n]$ | c_k: Basis coeff, ψ_k: Basis vector

Syn: Basis + coeff = vector

Anal: Basis + Signal = coeff?

Fourier Series $\Omega_k = k\Omega_0 = k\frac{2\pi}{N}$

Anal: $c_k = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk\Omega_0 n}$

Synthesis: $\tilde{x}[n] = \sum_k c_k e^{jk\Omega_0 n}$

$c_k^* = c_{-k}$

for $x \circledast y$: $e_k = LCM(N_x, N_y) \cdot c_k d_k$

Periodic Convolution: $\tilde{x}[n]$ w/ period N_x: c_k, $\tilde{y}[n]$ w/ period N_y: d_k

DT Fourier Transform: $X(\Omega) = \lim_{m\to\infty}\sum_{n=-m}^{m}x[n]e^{-jk\Omega_0 n}$, $\Omega_0 k \in \{-\pi, \pi\}$

Inverse DT Fourier Transform: $\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega)e^{j\Omega n}d\Omega$

Existance: $x[n]$ needs to be absolutely Summable $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$ or sq. Summable $\sum_{n=-\infty}^{\infty}|x[n]|^2 < \infty$

DTFT & DTFS bad — replace $\Omega_0 k$ with Ω

DFT → Solution | DFT as sampling: $X[k] = X(\Omega)|_{\Omega = \frac{2\pi}{N}k}$ | $\sum_{0}^{N-1} x[k]h[n-k]$ | DFT as basis for signal

$\omega_N = e^{-j\frac{2\pi}{N}}$, $X[k] = \sum x[n]\omega_N^{kn}$, $x[n] = \frac{1}{N}\sum X[k]\omega_N^{-kn}$

$X[k] = \sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}$

$x[n] = \frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi}{N}kn}$

Don't accidentally count 0 twice

Random Shit

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $a < 1$

$|X(\Omega)| = \pi\sum_{l=-\infty}^{\infty}\delta(\Omega - \Omega_o - 2\pi l) + \delta(\Omega + \Omega_o - 2\pi l)$

$\angle X(\Omega) = \begin{cases}-\frac{\pi}{2}, & \text{where } \Omega = +\Omega_o + 2\pi l,\ \forall\, l \in \mathbb{Z}\\ +\frac{\pi}{2}, & \text{where } \Omega = -\Omega_o + 2\pi l,\ \forall\, l \in \mathbb{Z}\end{cases}$

$\int u\,dv = uv - \int v\,du$

$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

$\sum_{a=a_1}^{a_2} a = \frac{(a_1 + a_2)(a_1 - a_2)}{2}$

$\sum_{n=N_0}^{N_1} r^n = \frac{r^{N_0} - r^{(N_1+1)}}{1-r}$

$(\{1,1,1,1,1,1,1\}) * (.9^n u[n])$ → $y[n] = \sum_{k=0}^{n} 0.9^k = \frac{1 - 0.9^{n+1}}{1 - 0.9}$

For $0 \le n < 7$, $u[n - 7 - k]$

For $n \ge 7$: $y[n] = \sum_{r=0}^{6} 0.9^{n-r} = 0.9^n\frac{1 - 0.9^{-7}}{1 - 0.9^{-1}}$

$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, $n \ge 1$

FT Pairs (102)

	Series	
	$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
DTFT	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$ | $a_1 = \{1\}$
Infer ←	$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$ | $a_1 = \frac{1}{2}$, $a_{-1} = \frac{1}{2}$
$\frac{\pi}{j}\sum_{-\infty}^{\infty}\delta(\Omega - \Omega_0 - 2\pi l) - \delta(\Omega + \Omega_0 - 2\pi l)$	$\sin(\omega_0 t)$	$\frac{\pi}{j}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$ | $a_1 = \frac{1}{2j}$, $a_{-1} = -\frac{1}{2j}$
	1	$2\pi\delta(\omega)$ | $a_0 = 1$
$e^{-j\Omega_0 n}$	$rect(t/T)$	$T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right)$
$\sum_{-\infty}^{\infty}2\pi\delta(\Omega + \Omega_0 - 2\pi l)$	$\delta(t - t_0)$	$e^{-j\omega t_0}$
	$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(a + j\omega)^n}$
$\frac{d}{d\Omega}(X(\Omega)e^{-j\Omega})$	$n \cdot x[n-1]$	

$\vec{X} = W\vec{x}$

$\begin{bmatrix}X[0]\\X[1]\\X[2]\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & \omega_N^1 & \omega_N^{N-1}\\1 & \omega_N^{N-1} & \omega_N^{(N-1)^2}\end{bmatrix}\begin{bmatrix}x[0]\\x[1]\\x[2]\end{bmatrix}$

& $\vec{x} = \frac{1}{N}W^*(\omega\vec{x})$

Bullshit: $\|\vec{x}\|_2 \to l_2$ Norm

$\arg\min_{\hat{x}[n]} \underbrace{\|x[n] - \hat{x}[n]\|_2^2}_{\text{fidelity}} + \lambda\underbrace{\|\hat{x}[n]\|_2^2}_{\text{regularization}}$

l_p Norm: $\sqrt{\sum_{i=1}^{N} x_i^2}$ → Distance metric

Periodic Convolution exists

Circular Convolution: $y[n] = \sum_{k=0}^{N-1} x[k]h[n-k]_{\text{mod } N}$ ← god fucking awful notation

FFT: Prop

$\omega_N = e^{-j\frac{2\pi}{N}}$

$X[K] = \sum_{n=0}^{N-1} x[n]\,\omega_N^{Kn}$

$\omega_N^2 = \omega_{N/2}$

$\omega_N^{2+\frac{N}{2}} = -\omega_N^2$

$\omega_N^{2N/2} = (-1)^2$

Implement

① Split $x[n]$ e&O

$X[K] = \sum_{m=0}^{\frac{N}{2}-1} x[2m]\,\omega_N^{K2m} + \sum_{m=0}^{N/2-1} x[2m+1]\,\omega_N^{K(2m+1)}$

$k < \frac{N}{2}: X_K = X_e[K] + \omega_N^K X_O K$

$X[k+\frac{N}{2}] = X_e[k+\frac{N}{2}] + \omega_N^{K+\frac{N}{2}} \cdots$

② Keep splitting

X_{ee} & X_{eo} & X_{oe} & X_{oo}

$N = 2^p \rightarrow$ splits

$\therefore p > \log N$

$\therefore N \cdot \log N$

zero pad to next power of 2

Z

$x[n] \xrightarrow{Z} X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$X(e^{j\Omega}) \overset{DTFT}{=} X(\Omega)$ | Convergance illegal

Existance: No infinity values

ROC is circular shaped

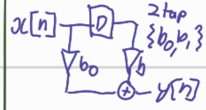
No poles

Causal signal ROC OUTside

Anti - - - - - - INSide

Mixed will have Ring

Random Shit:



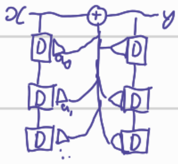
Recurse: $\sum_{K=0}^{N} a_K y[n-K] = \sum_{K=0}^{M} b_K y[n-K]$

$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum^{M} b_K z^{-K}}{\sum^{N} a_K z^{-K}}$

Poly: $\frac{b_0 + b_1 z^{-1} \cdots b_M z^{-M}}{a_0 + \cdots a_N z^{-N}}$

Factor: $\frac{\sim\sim}{(\sim)(\sim)(\sim)}$

Filter:



$H_2(z) = \frac{z^2}{2(z^2 - 1/4)^2}$

$H(z) = \frac{z}{z - \frac{1}{2}}$

$h_2[n] = \begin{cases} n h[n], & n \text{ eve} \\ 0, & n \text{ o} \end{cases}$

$x[n] = \begin{cases} \alpha^n u[n] & n \% K = 0 \\ \beta^n u[n] & ow \end{cases}$

$X(z) = \frac{1}{1-\alpha^K z^{-K}} + \frac{1}{1-\beta z^{-1}} - \frac{1}{1-\beta^K z^{-K}} \quad |z| > \max\{A, B\}$

Smelse

$z = 0, 1/2$

$p = \frac{1}{3} \pm \frac{i\sqrt{2}}{3}$

$y[n] - \frac{2}{3} y[n-1] + \frac{1}{3} y[n-2] = x[n] - \frac{1}{2} x[n-1]$

$x[n] \rightarrow X(\Omega)$

$x_1 = n x[n-1] \rightarrow X_1 = j \frac{d}{d\Omega} X(\Omega) e^{-j\Omega}$

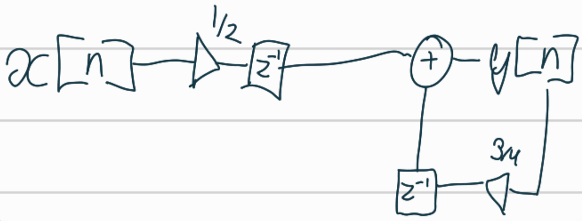
$x_2 = e^{j\frac{\pi n}{2}} (x[n] * x[n]) \quad X_2 = X(\Omega - \pi/2)^2$

$x_0 = \text{ord} \quad X_3 = \frac{X(\Omega) - X(-\Omega)}{2}$

need more poles than zeros for causal

Practice

$$y[n] = \frac{3}{4}y[n-1] + \frac{1}{2}x[n-1]$$



$$Y(z) = \frac{1}{2}X(z)z^{-1} + \frac{3}{4}(Y(z)z^{-1}$$

$$Y(z)\left(1 - \frac{3}{4}z^{-1}\right) = \frac{1}{2}X(z)z^{-1}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2}z^{-1}\left(1 - \frac{3}{4}z^{-1}\right)^{-1} = \frac{2}{4z-3}$$

FR

$$\frac{2}{4e^{j\omega}-3}$$

$$\frac{2}{4e^{j\frac{\pi}{2}}-3} > \frac{2}{5}$$