EE 131A	Homework 3 Solutions
Probability and Statistics	Friday, February 03, 2023
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Reading: Chapters 2.6, 3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Some parts of Texas are particularly tornado-prone. Suppose that in one metropolitan area, 25% of all home-owners are insured against tornado damage. Four home-owners are to be selected at random; let X denote the number among the four who have tornado insurance.

   (a) Find and sketch the pmf of X.

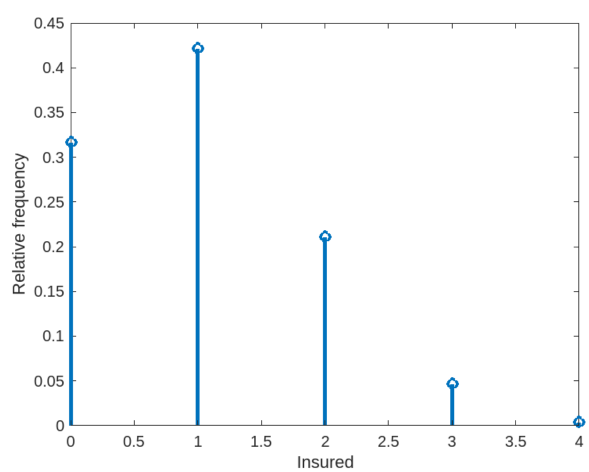
   (b) What is the most likely value for X?

   (c) What is the probability that at least two of the four selected have tornado insurance?

   Solution:

   (a) The pmf of X is given by

x	**Outcomes**	**P(x)**
0	FFFF	$(0.75)^4 = 0.3164$
1	FFFS,FFSF,FSFF,SFFF	$4[(0.75)^3(0.25)] = 0.4218$
2	FFSS,FSFS,SFFS,FSSF,SFSF,SSFF	$6[(0.75)^2(0.25)^2] = 0.2109$
3	FSSS, SFSS,SSFS,SSSF	$4[(0.75)(0.25)^3] = 0.0468$
4	SSSS	$(0.25)^4 = 0.0039$



   (b) $P(x)$ is largest for X = 1.

   (c) $P(X \geq 2) = P(2) + P(3) + P(4) = .2616$.
   You could also do this as: $P(X \geq 2) = 1 - P(X < 2)$.

2. Suppose X is a Binomial random variable with parameters $n = 4$, and p.

   (a) Express $E\left[sin\left(\pi X/2\right)\right]$ in terms of p.
   Solution:
   The PMF of X is given by:

$$p_X(x) = \binom{4}{x} p^x (1-p)^{4-x}, \ x = 0, 1, 2, 3, 4.$$

   As a result, we conclude that:

$$\begin{aligned} E\left[sin\left(\frac{\pi X}{2}\right)\right] &= \sum_{x=0}^{4} sin\left(\frac{\pi x}{2}\right) \binom{4}{x} p^x (1-p)^{4-x} \\ &= 0 + sin\left(\frac{\pi}{2}\right)\binom{4}{1} p(1-p)^3 + 0 + sin\left(\frac{3\pi}{2}\right)\binom{4}{3} p^3(1-p) + 0 \\ &= 4p(1-p)^3 - 4p^3(1-p) = 4p(1-p)(1-2p). \end{aligned}$$

   (b) Express $E\left[cos\left(\pi X/2\right)\right]$ in terms of p.
   Solution:
   Similar to part (a), we can see that:

$$\begin{aligned} E\left[cos\left(\frac{\pi X}{2}\right)\right] &= \sum_{x=0}^{4} cos\left(\frac{\pi x}{2}\right) \binom{4}{x} p^x (1-p)^{4-x} \\ &= cos\left(0\right)\binom{4}{0}(1-p)^4 + 0 + cos\left(\pi\right)\binom{4}{2} p^2(1-p)^2 \\ &+ 0 + cos\left(2\pi\right)\binom{4}{4} p^4 = (1-p)^2(1-2p-5p^2) + p^4. \end{aligned}$$

3. Two coins are simultaneously tossed until one of them comes up a head and the other a tail. The first coin comes up a head with probability p and the second with probability q. All tosses are assumed independent.

   (a) Find the PMF, the expected value, and the variance of the number of tosses.
   Solution: Let X be the number of tosses until one of them comes up as a head and the other as a tail. This experiment can be seen as a series of Beroulli trials with the success determined by the outcomes $\{HT, TH\}$. Thus X is a Geometric RV with probability of success

$$P(\{HT, TH\}) = p(1-q) + q(1-p).$$

   Thus the PMF of X is given by

$$P(X = k) = \left(1 - p(1-q) - q(1-p)\right)^{k-1}\left(p(1-q) + q(1-p)\right), \quad k = 1, 2, \ldots$$

   Therefore

$$E[X] = \frac{1}{p(1-q)+q(1-p)}$$

$$VAR(X) = \frac{1-p(1-q)-q(1-p)}{(p(1-q)+q(1-p))^2}.$$

(b) What is the probability that the last toss of the first coin is a head?
Solutions: The probability that the last toss of the first coin is a head is

$$P(\{HT\}|\{HT,TH\}) = \frac{P(\{HT\})}{P(\{HT,TH\})} = \frac{p(1-q)}{p(1-q)+q(1-p)}$$

4. Messages passed amongst neurons in the brain can be recorded using an external electrode, and they show up as a series of spikes of equal magnitude. These spikes can be modelled by a Poisson random variable with rate 6000 spikes per minute. The PMF of the Poisson RV is given by:

$$P(N=k) = \frac{\alpha^k e^{-\alpha}}{k!} \qquad \text{for k = 0, 1, 2, ...} \tag{1}$$

(a) What is the probability of recording no spikes in the first 100 milliseconds?

(b) What is the probability of recording between 5 to 10 spikes (both included) in the [300ms, 400ms] time range?

Leave your answers as an expression in terms of a sum of 1 or more exponentials.
Hint: The parameter α of the Poisson RV is the average number of event occurrences in a specified time interval, so utilize the given information to find out this parameter.
Solution:

(a) We know that the PMF of the Poisson RV is given by:

$$P(N=k) = \frac{\alpha^k e^{-\alpha}}{k!} \qquad \text{for k = 0, 1, 2, ...} \tag{2}$$

where the parameter α of the Poisson RV is the average number of event occurrences in a specified time interval.
Let N1 = Number of spikes in [0, 100ms] time range.
Thus, we want to compute P(N1 = 0).
As we are given the rate, and we know the time interval, we calculate α as:

$$\alpha = \text{rate} * \text{time} = \frac{6000}{60\text{sec}} * 0.1\text{sec} = 10 \tag{3}$$

Therefore, the required PMF is given by:

$$P(N1=0) = \frac{10^0 e^{-10}}{0!} = e^{-10} \tag{4}$$

(b) Let N2 = Number of spikes in [300, 400ms] time range.
Thus, we want to compute $P(5 \leq N2 \leq 10)$.
As before, we compute the parameter α as:

$$\alpha = \text{rate} * \text{time} = \frac{6000}{60\text{sec}} * (400 - 300) * 0.001\text{sec} = 10 \tag{5}$$

Thus N2 is a Poisson RV with parameter $\alpha = 10$. Finally,

$$\begin{aligned} P(N2 \in [5, 10]) &= P(N2 = 5) + P(N2 = 6) + P(N2 = 7) && (6) \\ &+P(N2 = 8) + P(N2 = 9) + P(N2 = 10). && (7) \\ &= \sum_{k=5}^{10} \frac{10^k e^{-10}}{k!} && (8) \end{aligned}$$

5. Use MATLAB to plot the pmfs of the following random variables. This question is to provide practice with MATLAB which you will use for the future project so do not provide hand-drawn plots.

(a) Discrete Uniform Random Variable on the set $\{1, 2, 3, 4\}$.

(b) Binomial Random Variable with parameters $n = 5$ and $p = 0.25$.

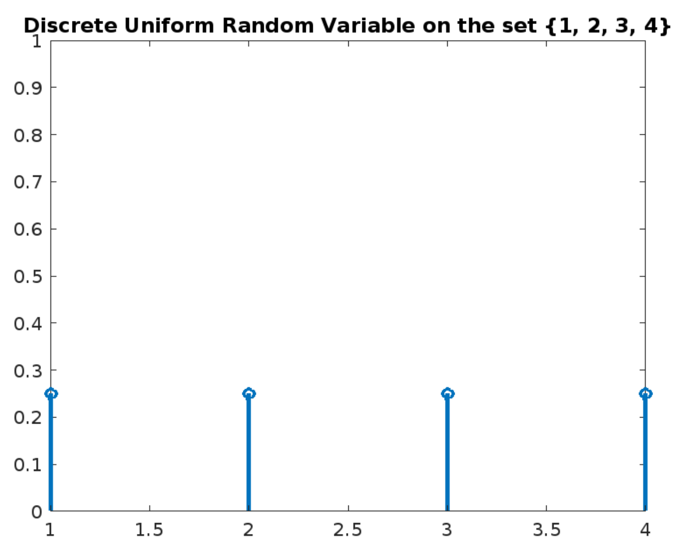
(c) Geometric Random Variable with $p = 0.5$ and plot the pmf for values less than or equal to 5.

Solution:

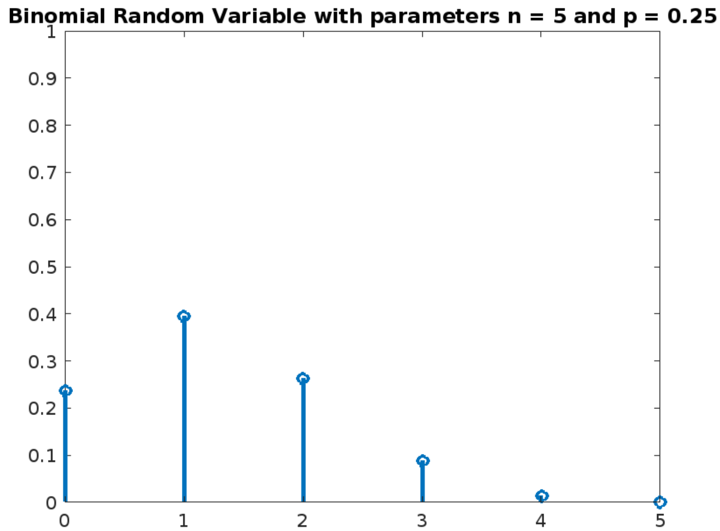
```
% (a)

x = 1:1:4;

y = 1 / size(x, 2) .* ones(size(x));

U = figure("Name", "Uniform");
stem(x, y, "LineWidth", 2);
ylim([0 1]);
title("Discrete Uniform Random Variable on the set \{1, 2, 3, 4\}");
% saveas(U, "Uniform", "png");

% (b)

n = 5;
p = 0.25;

x = 0:1:n;
```

```
        y = zeros(size(x));

        for k = 0 : n
            y(k + 1) = nchoosek(n, k) * p ^ k * (1 - p) ^ (n - k);
        end

        B = figure("Name", "Binomial");
        stem(x, y, "LineWidth", 2);
        ylim([0 1]);
        title("Binomial Random Variable with parameters n = 5 and p = 0.25");
        % saveas(B, "Binomial", "png");

        % (c)

        t = 5;
        p = 0.5;
        x = 1:t;

        y = p * (1 - p) .^ (x - 1);

        G = figure("Name", "Geometric");
        stem(x, y, "LineWidth", 2);
        ylim([0 1]);
        title("Geometric Random Variable with p = 0.5");
        % saveas(G, "Geometric", "png");
```

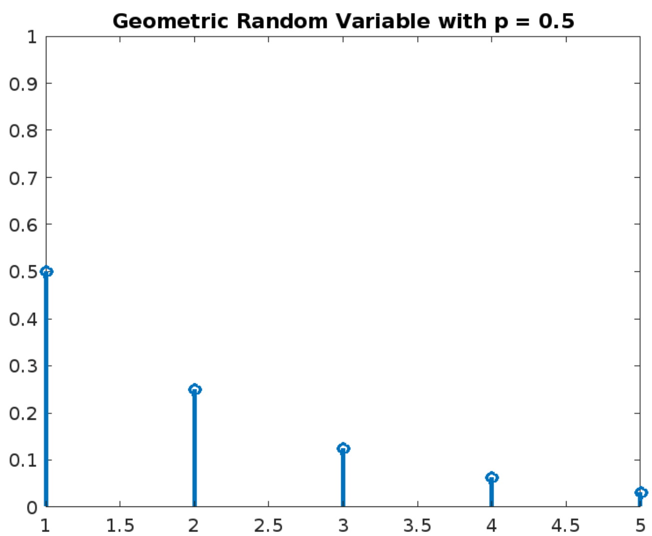
(a)



(b)


Binomial Random Variable with parameters n = 5 and p = 0.25
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5

(c)


Geometric Random Variable with p = 0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
1.5
2
2.5
3
3.5
4
4.5
5