

Due Friday, 11 Nov 2022, by 11:59pm to Gradescope.

100 points total.

This homework covers questions relate to Fourier series and Fourier transform.

1. (18 points) **Fourier Series**

- (a) (6 points) When the periodic signal  $f(t)$  is real, we know that the fourier coefficients follow the following ,  
 $Re(c_k) = Re(c_{-k}), Im(c_k) = -Im(c_{-k}), c_k^* = c_{-k}, |c_k| = |c_{-k}|, \angle c_k = -\angle c_k^*$ .  
If  $f(t)$  is purely imaginary, how do the above relations change ?, provide mathematical justification .
- (b) (12 points) Suppose we are given the following information about a signal  $x(t)$ , please write down the expression of  $x(t)$ :
- $x(t)$  is real and even.
  - $x(t)$  is periodic with period  $T = 16$  and has Fourier coefficients  $a_k$ .
  - $a_k \neq 0$  if and only if  $|k| \leq 1$ .
  - $\int_0^4 |x(t) - (\sum_{l=0}^{\infty} a_{2l})|^2 dt = 2$ .
  - DC component of the signal is 5

2. (32 points) **Symmetry properties of Fourier transform**

- (a) (16 points) Determine which of the signals, whose Fourier transforms are depicted in Fig. 1, satisfy each of the following:
- $x(t)$  is even
  - $x(t)$  is odd
  - $x(t)$  is real
  - $x(t)$  is complex (neither real, nor pure imaginary)
  - $x(t)$  is real and even
  - $x(t)$  is imaginary and odd
  - $x(t)$  is imaginary and even
  - There exists a non-zero  $\omega_0$  ( $\omega_0$  can be positive or negative) such that  $e^{j\omega_0 t} x(t)$  is real and even
- (b) (8 points) Using the properties of Fourier transform, determine whether the assertions are true or false.
- The convolution of a real and even signal and a real and odd signal is odd.
  - The convolution of a imaginary and odd signal and the same signal time reversed is an real and odd signal.
- (c) (8 points) Show the following statements:

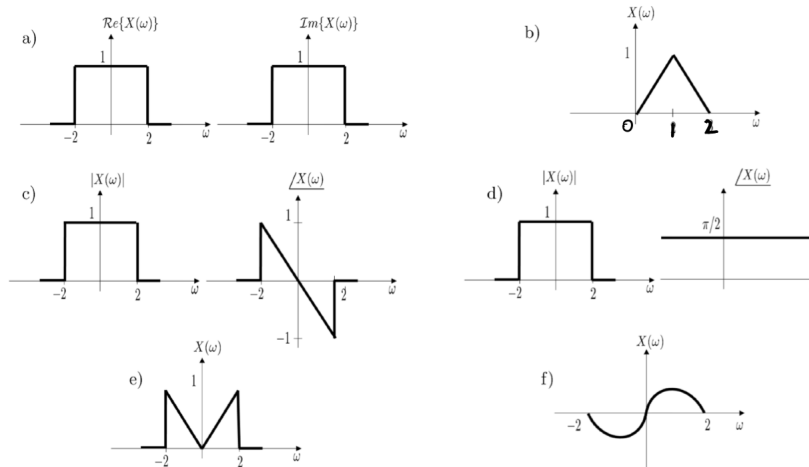


Figure 1: P2.a

- i. If  $x(t) = x^*(-t)$ , then  $X(j\omega)$  is real.
- ii. If  $x(t)$  is a real signal with  $X(j\omega)$  its Fourier transform, then the Fourier transforms  $X_e(j\omega)$  and  $X_o(j\omega)$  of the even and odd components of  $x(t)$  satisfy the following:

$$X_e(j\omega) = \text{Re}\{X(j\omega)\}$$

and

$$X_o(j\omega) = j\text{Im}\{X(j\omega)\}$$

### 3. (20 points) **Halloween Adventures with the Mystery Box**

On Halloween evening after ECE 102 lecture, you and your friend decide to head back home together. You are tired and are waiting for the pedestrian signal near Ronald Reagan Hospital, but it is taking forever... Suddenly your friend shouts “look what I found!” You see a weird box that has metallic contacts, a rugged structure with spider webs and a display monitor associated with it. Spooky!! You and your friend grow curious, and in the spirit of Halloween want to find out what this box all about. You decide to use a ideal signal generator from one of the labs in Engineering IV, and you observe the following.

- (a) (3 points) When you input a  $\delta(t)$ , the mystery box displays a mathematical expression of the mystery function =1  
 When you input a sine wave of frequency 5 Hz the mystery box displays a mathematical expression for mystery function  
 Mystery function =  $j\pi(\delta(\omega + 10\pi) - \delta(\omega - 10\pi))$  where  $\omega$  is in rad/s.

What operation that you learnt in ECE 102 do you think the Mystery Box is performing?

- (b) (10 points) Happy with your conclusions, you and your friend get coffee from nearest coffee shop, and want to explore the mystery box again. But oops!! You spill coffee on the mystery device and spoil only its display, while keeping the rest of its functionality intact.

Consider your signal generator inputs time-domain signal  $x(t)$  plotted below. Using the properties/definition of the Mystery function (let's call it  $X(j\omega)$ ), you should be able to compute all of the quantities required in each of the parts below, without explicitly looking at the Mystery function displayed. Indicate which properties you are using and clearly label your final answer for each part.

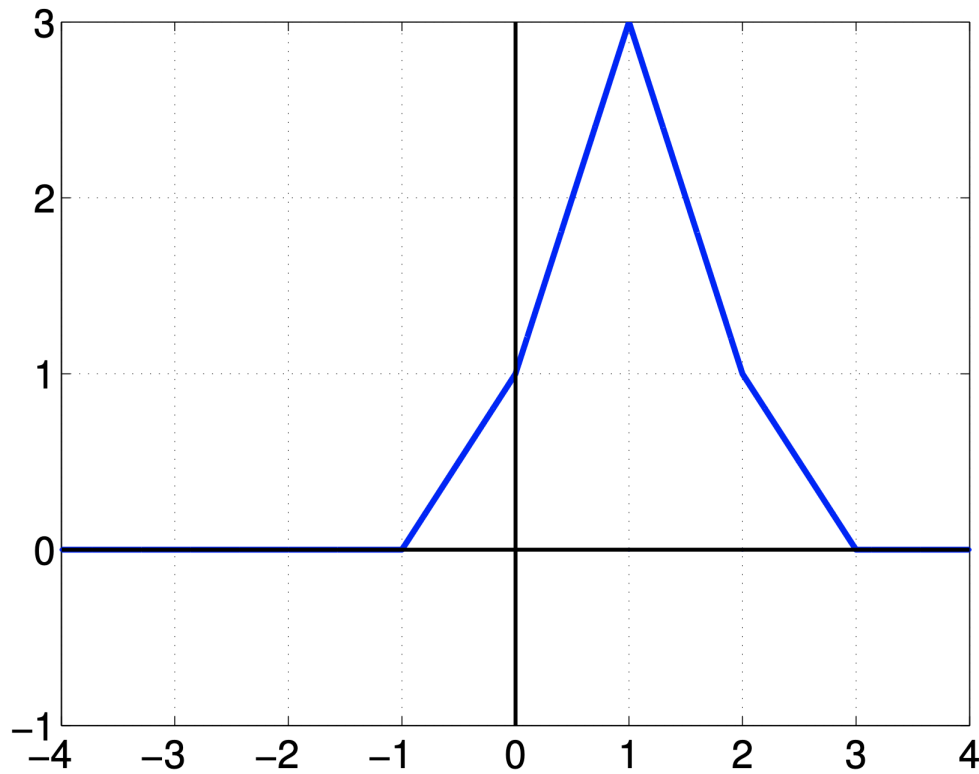


Figure 2:  $x(t)$

- i. Your friend claims that the value of  $X(0)$  is 6 is your friend right ?
- ii. Evaluate  $\int_{-\infty}^{+\infty} X(j\omega) d\omega$
- iii. Evaluate  $\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$
- iv. Compute the value of  $\text{Im}(X(j\omega))$  i.e Imaginary component of mystery function  
When input from signal generator is  $x(t+1) + x(-t+1)$  ?

- v. We know that for the input  $x(t)$  above mystery box outputs some  $X(j\omega)$ , express a new input  $x(t)$  where the output of the mystery box is  $\frac{1}{5}X(\frac{j\omega}{5})e^{j2\omega}$ .
- (c) Having explored the mystery box enough, you get more curious and feed  $x(t) = t^4$  to the mystery box, and you start hearing a rumbling sound. The sound grows with time and you started getting concerned, it is Halloween after all... and then suddenly \*smash\*!! You see that the mystery box crashes and stops working once and for all, ending your adventure for that day :(. Disheartened you and your friend wonder what just happened? Can you think of a reason why the mystery box may have crashed and stopped working when you supplied it with this particular  $x(t)$ ? What could you have done to avoid this crash? (7 points)

4. (30 points) **Fourier transform and its inverse**

- (a) (21 points) Find the Fourier transform of each of the signals given below:  
Hint: you may use Fourier Transforms derived in class.

i. (**optional**)  $x_1(t) = 2\text{rect}\left(\frac{-t-3}{2}\right)\cos(10\pi t)$

ii.  $x_2(t) = e^{(2+3j)t}u(-t+1)$

iii.  $x_3(t) = \begin{cases} 1 + \cos(\pi t), & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$

iv.  $x_4(t) = te^{-2t}u(t)$

- (b) (9 points) Two signals  $f_1(t)$  and  $f_2(t)$  are defined as

$$f_1(t) = \text{sinc}(2t)$$

$$f_2(t) = \text{sinc}(t)\cos(3.1\pi t)$$

Let the convolution of the two signals be

$$f(t) = (f_1 * f_2)(t)$$

- i. Find  $F(j\omega)$ , the Fourier transform of  $f(t)$ .  
ii. Find  $f(t)$ .