EE 131A
Probability
Instructor: Lara Dolecek

Homework 4 Solutions
Due: February 9, 2023
TAs: Jayanth Shreekumar, Rushi Bhatt

Chapters 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. The battery lifetime of a Tesla Model Y vehicle is modeled by an exponential random variable, X, with a mean of 9 hours. You decide to rent a Tesla Model Y from Los Angeles and plan to drive it all the way up to Yellowstone National Park. It's been 9 hours since you started driving from Los Angeles and just made it to Salt Lake City. Given that you have already made it to Salt Lake City, what is the probability that you will be able to drive 9 more hours and make it to Yellowstone National Park (without recharging on the way)?

   Hint: Find $P(X \geq 18|X \geq 9)$

   Solution: Recall an exponential random variable X is expressed as:

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \lambda e^{-\lambda x} & \text{for } x \geq 0 \end{cases}$$

   with $E[X] = \frac{1}{\lambda}$
   Since for our case, the mean lifetime of the battery is 9 hrs ($= E[X]$); therefore $\lambda = \frac{1}{9}$
   Therefore X can be expressed as:

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{9} e^{-\frac{x}{9}} & \text{for } x \geq 0 \end{cases}$$

   $P(X \geq 18|X \geq 9)$ can be calculated by:

$$\begin{aligned} P(X \geq 18|X \geq 9) &= \frac{P(\{X \geq 18\} \cap \{X \geq 9\})}{P(X \geq 9)} \\ &= \frac{P(X \geq 18)}{P(X \geq 9)} \quad \text{for } x \geq 0 \\ &= \frac{\int_{18}^{\infty} f_X(x) \mathrm{d}x}{\int_{9}^{\infty} f_X(x) \mathrm{d}x} \\ &= \frac{\int_{18}^{\infty} \frac{1}{9} e^{-\frac{x}{9}} \mathrm{d}x}{\int_{9}^{\infty} \frac{1}{9} e^{-\frac{x}{9}} \mathrm{d}x} \\ &= \frac{\left[\frac{-9}{9} * e^{-\frac{x}{9}}\right]_{18}^{\infty}}{\left[\frac{-9}{9} * e^{-\frac{x}{9}}\right]_{9}^{\infty}} \\ &= \frac{e^{-2}}{e^{-1}} = e^{-1} = 0.36787 \end{aligned}$$

Alternate Solution: Since X is an exponential random variable, using the memoryless property:

$$\begin{aligned} P(X \geq 18|X \geq 9) &= P(X \geq 9) \\ &= \int_9^{\infty} \frac{1}{9} e^{-\frac{x}{9}} \mathrm{d}x \\ &= \Big[\frac{-9}{9} * e^{-\frac{x}{9}}\Big]_9^{\infty} \\ &= -\Big[e^{-\infty} - e^{-\frac{9}{9}}\Big] \\ &= e^{-1} = 0.36787 \end{aligned}$$

2. The PDF of a continuous random variable X is given by

$$f_X(x) = \begin{cases} 2c & 0 < x \leq 3 \\ c & 4 < x \leq 14 \\ 3c & 16 < x \leq 18 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

   (a) Find the numerical value of c.
   Solution: For a valid PDF, $f_X(x)$ must satisfy $\int_{-\infty}^{\infty} f_X(x) = 1$. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x) &= \int_0^3 2cdx + \int_4^{14} cdx + \int_{16}^{18} 3cdx \\ &= 6c + 10c + 6c = 1 \end{aligned}$$

   therefore, the numerical value of c is given by: $c = \frac{1}{22}$.

   (b) Compute $P(1 < X \leq 5)$.
   Solution:

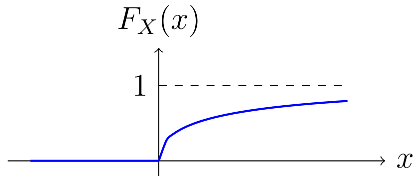
$$\begin{aligned} P(1 < X \leq 5) &= \int_1^5 f_X(x)dx \\ &= \int_1^3 2cdx + \int_3^4 (0)dx + \int_4^5 cdx \\ &= 5c = \frac{5}{22} \approx 0.2272 \end{aligned}$$

3. For $\beta > 0$ and $\lambda > 0$, the Weibull random variable X has cdf:

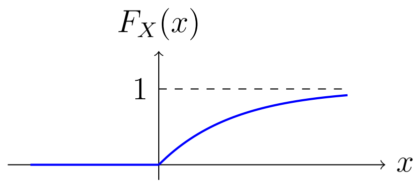
$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-(x/\lambda)^{\beta}} & \text{for } x \geq 0. \end{cases}$$

   (a) Plot the cdf of X for $\beta = 0.5$, 1, and 2. Take $\lambda = 1$ for all the three plots.
   Solution:
   The plots are shown below:

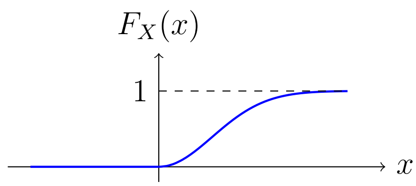
- $\beta = 0.5$



- $\beta = 1$



- $\beta = 2$



(b) Find the probability $P[k\lambda < X < (k+1)\lambda]$ and $P[X > k\lambda]$ for positive integer k.
Solution:

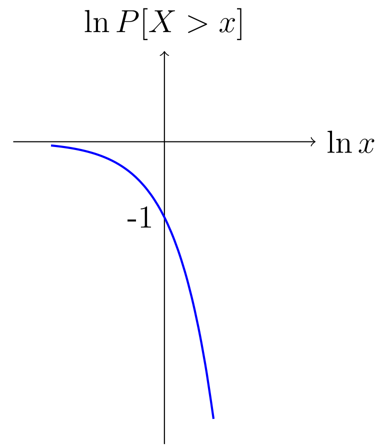
$$\begin{aligned}P[k\lambda < X < (k+1)\lambda] &= F_X((k+1)\lambda) - F_X(k\lambda) \\ &= (1 - e^{-((k+1)\lambda/\lambda)^\beta}) - (1 - e^{-(k\lambda/\lambda)^\beta}) \\ &= e^{-k^\beta} - e^{-(k+1)^\beta}\end{aligned}$$

$$P[X > k\lambda] = 1 - P[X \leq k\lambda] = 1 - F_X(k\lambda) = e^{-k^\beta}$$

(c) Plot $\ln P[X > x]$ vs. $\ln x$. Assume $\beta = 2$ and $\lambda = 1$.
Solution:
For $x \geq 0$,

$$\begin{aligned}\ln P[X > x] &= \ln(1 - P[X \leq x]) \\ &= \ln(1 - F_X(x)) \\ &= \ln e^{-(x/\lambda)^\beta} \\ &= -(x/\lambda)^\beta \\ &= -(e^{\ln x}/\lambda)^\beta\end{aligned}$$

For the following plot, $\beta = 2$, $\lambda = 1$, and $\ln P[X > x] = -e^{2\ln x}$.



4. In a game of curling, a person receives 10 points if their shot is within 1 inch of the target, 7 points if it is between 1 and 3 inches of the target, 5 points if it is between 3 and 5 inches of the target, and 3 points if it is between 5 and 7 inches of the target. Find the expected number of points scored if the distance from the point where the curling stone lands to the target is uniformly distributed between 0 and 10.
   Solution:
   We desire to calculate $E[P(D)]$, where $P(D)$ is the points scored when the distance to the target is D. This becomes

$$\begin{aligned} E[P(D)] &= \int_0^{10} P(D)f(D)dD \\ &= \frac{1}{10}\int_0^{10} P(D)dD \\ &= \frac{1}{10}\left(\int_0^1 10dD + \int_1^3 7dD + \int_3^5 5dD + \int_5^7 3dD + \int_7^{10} 0dD\right) \\ &= \frac{1}{10}(10(1) + 7(2) + 5(2) + 3(2) + 0(3)) \\ &= \frac{40}{10} \\ &= 4.0 \end{aligned}$$

5. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

   (a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what is the probability that he or she goes to destination A?
   Solution:
   Let X denote the time at which the passenger arrives at the station. $X \sim U(0, 60)$.

Then

$$\begin{aligned} P(\text{goes to A}) &= P(5 < X < 15) + P(20 < X < 30) \\ &\quad + P(35 < X < 45) + P(50 < X < 60) \\ &= \frac{40}{60} = \frac{2}{3} \end{aligned}$$

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10?

Solution:

Let X denote the time at which the passenger arrives at the station. $X \sim U(10, 70)$. Then

$$\begin{aligned} P(\text{goes to A}) &= P(10 < X < 15) + P(20 < X < 30) \\ &\quad + P(35 < X < 45) + P(50 < X < 60) + P(65 < X < 70) \\ &= \frac{40}{60} = \frac{2}{3} \end{aligned}$$

Appendix: Code for Reference

(a)
```
% Q 3(a)
x = 1:2000;
x = x/1000;

cdf_1 = 1 - exp(-sqrt(x));
cdf_2 = 1 - exp(-x);
cdf_3 = 1 - exp(-(x.^2));

subplot(311);
plot(x, cdf_1);
xlabel('X');
ylabel('CDF at B=0.5');
subplot(312);
plot(x, cdf_2);
xlabel('X');
ylabel('CDF at B=1');
subplot(313);
plot(x, cdf_3);
xlabel('X');
ylabel('CDF at B=2');
```

(b)
```
% Q 3 (c)
x = -2000:2000;
x = x/1000;

p = -exp(2*log(x));

plot(log(x), p);
xlim([-1.5 1.5]);
ylim([-2 0.2]);
xlabel('ln (x)');
ylabel('ln P[X > x]');
```